BOUNDARY METHOD FOR THE DETERMINATION OF STRESS COMPONENTS IN SOLID CIRCULAR PLATES

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Freidrich G. K. Grohé

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This is to certify that the

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Charles O. Harris
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BOUNDARY METHOD

FOR THE DETERMINATION OF STRESS COMPONENTS

IN SOLID CIRCULAR PLATES

By

Friedrich G. K. Grohé

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I also take the opportunity to express my indebtedness to Dr. Ing. K. Kloeppel, Professor of Advanced Structural Theory and Steel Structures, at Darmstadt Institute of Technology, Germany. In his outstanding lectures, he made me acquainted with the determination of boundary values of a stress function, a knowledge which actually inspired me to undertake this study.

NOTATIONS

x, y	Rectangular coordinates.
r, ϑ	Polar coordinates
P	Single concentrated load.
q	Intensity of a continuously distributed load.
R	Resultant of all external forces applied on the boundary between a starting point 0 and a point of reference k.
R_x, R_y	Components of R in the direction of the coordinate axes x, y.
R_s	Component of R parallel to the tangent at a boundary point k.
ρ	Radius of the boundary of a circular plate.
$k = f(\vartheta)$	Numbering of particular points of a grid lying on concentric circles.
$l = f(r)$	Numbering of particular points of a grid lying on rays from the center to the boundary.
X, Y	Components of a distributed boundary force per unit length of the boundary.
σ_x, σ_y	Normal components of stress parallel to x- and y-axes.
$\sigma_r, \sigma_\vartheta$	Radial and tangential normal stresses in polar coordinates.
τ_{xy}	Shearing stress component in rectangular coordinates.
$\tau_{r\vartheta}$	Shearing stress in polar coordinates.
$F(x, y)$	Airy stress function in rectangular coordinates.
$F(r, \vartheta)$	Airy stress function in polar coordinates.
F_k	Boundary value of the stress function at point k.
G_k	Boundary value of $G = r\frac{\partial F}{\partial r}$ at point k.
F_k^l	Value of $F(r, \vartheta)$ at a point of a grid as determined by k and l.
F_0	Value of $F(r, \vartheta)$ at the center point of a circular plate.
F_k^a	Extrapolated value of $F(r, \vartheta)$ at a point outside the plate.

CONTENTS

1.) SYNOPSIS

This thesis gives a method for the determination of stress components in solid circular plates under any kind of boundary forces which lie entirely in the plane of the plate. There are no forces applied inside the plate. The boundary values of the stress function are determined from the external forces and then used for the evaluation of the Fourier coefficients of a trigonometric stress function.

The same principle is used for an approximate method leading to approximate numerical values of the stress function at certain points inside the plate. The stress components at those points are then determined from these numerical values using finite differences instead of differentials.

2.) THE PROBLEM

The object of this thesis is the determination of the stresses in a solid circular plate subjected to arbitrary boundary loads in the plane of the plate. No loads act inside the plate; body forces are considered to be absent.

The thickness of the plate is taken as unity. The restrictions for the thickness are the same as in other two-dimensional problems of elasticity. For the case of single concentrated loads on the boundary, the results are true only if the plate is thin and the loads lie entirely in the centerplane of the plate. For the case of line loads uniformly distributed over the entire thickness, the thickness of the plate is not restricted.

3.) JUSTIFICATION OF THE STUDY

In the present time principles of higher mechanics, which were considered merely "academic cases" until recently, are going to be more and more introduced into practical design. In many countries, post-war shortage of structural materials obliges the designer to determine stresses and deformations more exactly in order to create the most efficient structure with a minimum of material.

Stress functions play an important role in this development. While there are a large number of functions which satisfy the compatibility equation, the problem is to bring those functions in agreement with the boundary conditions which are, of course, different for every individual case, depending upon the shape of the body or plate and the load conditions.

In recent years much work has been done in giving solutions for problems in rectangular coordinates. However, there seems to be a lack of general methods in polar coordinates which could enable the non-expert on elasticity to find stress functions for any loading condition. For a few special cases stress functions are given, for other cases only formulas for the determination of stress components have been devived. It is significant, for instance, that neither Timoshenko nor Frocht give a stress function for the case of two single concentrated loads acting on the diameter of a circular plate, but restrict themselves to formulas for the stress components gained by superposition of three different cases of loading. The stress function itself obtained in the same way by superposition and coordinate transformation would be so complicated that it is practically no more differentiable.

In order to have a method of general applicability, a simple relation between boundary conditions and stress functions should be found which would allow us to evaluate certain unknown coefficients of the stress function under any kind of loading.

4.) PRELIMINARY DESCRIPTION OF THE BOUNDARY METHOD

The boundary method is intended to be a method of general applicability. Boundary values of the stress function are obtained from the external forces as shown in Section 6. This gives the desired relation between boundary conditions and stress function as mentioned in the foregoing section. The further procedure leading to the stress components inside the plate is developed in three different modifications.

First Modification:

The boundary values of the stress function are represented by a continuous function around the boundary which is expanded in a Fourier series. The Fourier coefficients are determined in the usual way by integration around the boundary. A second biharmonic trigonometric series is assumed as stress function whose Fourier coefficients are obtained through comparison with the known coefficients of the expansion for the boundary values. The trigonometric stress function is then differentiated as usual for the determination of the stress components.

Second Modification:

The boundary values of the stress function cannot be expressed in one or two functions with continuous derivatives around the boundary. In order to obtain the Fourier coefficients, it is then useful to substitute the integration around the boundary by a finite summation using numerical boundary values. If certain formulas are used, which will be found in Section 8, the approximate coefficients of the trigonometric stress function are immediately obtained.

Third Modification:

This modification is an attempt to establish a purely algebraic method, avoiding stress functions, integrations, and partial differentiations in its practical application. A grid is laid over the circular plate. It is advisable to use a standard grid for which constant coefficients have been already evaluated. Using the boundary values of the stress function and a formula given in Section 9, one obtains numerical values of the stress function at the grid points inside the plate. From these values, stress components are determined by taking finite differences instead of differentials.

5.) THE AIRY STRESS FUNCTION

As an introduction to the mathematical part, the conditions for the existence of the Airy stress function may be mentioned briefly.

From elasticity it is known that a function $F(x, y, z)$ (called the Airy stress function), which satisfies certain conditions, enables us to determine the stress components at any point of a body which is under external loads. Assuming that body forces are absent, these conditions are for two dimensional problems as follows:

a) Differential Equations of Equilibrium:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned} \tag{1}$$

in which $\sigma_x, \sigma_y,$ and τ_{xy} are, respectively, the normal components of stress parallel to x and y axes and the shearing stress component in rectangular coordinates (Fig. 1).

b) Compatibility Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\sigma_x + \sigma_y) = 0 \tag{2}$$

c) Boundary Conditions:

$$\begin{aligned} \sigma_x \cos(nx) + \tau_{xy} \cos(ny) &= X \\ \tau_{xy} \cos(nx) + \sigma_y \cos(ny) &= Y \end{aligned} \tag{3}$$

in which (nx) and (ny) are, respectively, the angles of the normal to the boundary with the x - and y - axis (Fig. 2).

The stress function $F(x, y)$ is defined in such a way that the stress components are determined by the following equations:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} ; \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} ; \quad \tau_{xy} = - \frac{\partial^2 F}{\partial x \partial y} \tag{4}$$

Eqs. (4) satisfy the differential equations of equilibrium (1). Substituting Eqs. (4) in the compatibility equations (2), it is seen that the stress function $F(x,y)$ must also satisfy the two dimensional bipotential equation:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{5}$$

or briefly: $\Delta\Delta F = 0$

As a result, a two dimensional problem is solved if a stress function $F(x,y)$ can be found which satisfies $\Delta\Delta F = 0$ and the boundary conditions.

In <u>polar coordinates</u>, the stress components are obtained from $F(r,\theta)$ by:

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\\ \sigma_\theta &= \frac{\partial^2 F}{\partial r^2}\\ \tau_{r\theta} &= \frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial \theta}\end{aligned} \tag{6}$$

in which σ_r, σ_θ, and $\tau_{r\theta}$ are, respectively, the radial and tangential normal stresses and the shearing stress in polar coordinates (Fig. 3).

The compatibility equation $\Delta\Delta F = 0$ is as follows:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \tag{7}$$

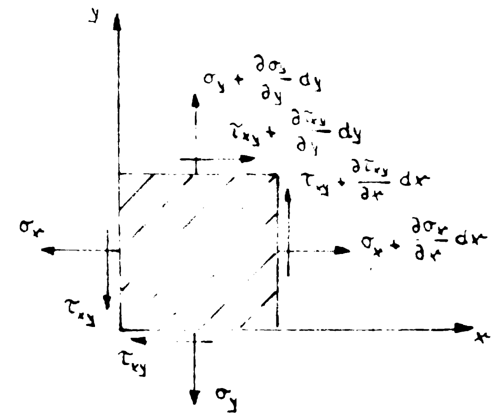


FIG. 1: EQUILIBRIUM OF A BODY ELEMENT IN RECTANG. COORDINATES

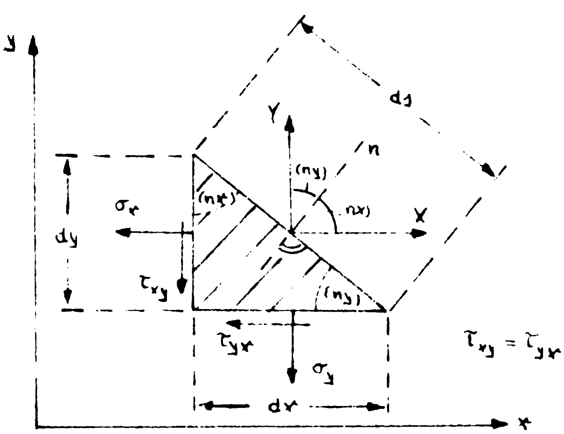


FIG. 2: EQUILIBRIUM OF A BOUNDARY ELEMENT

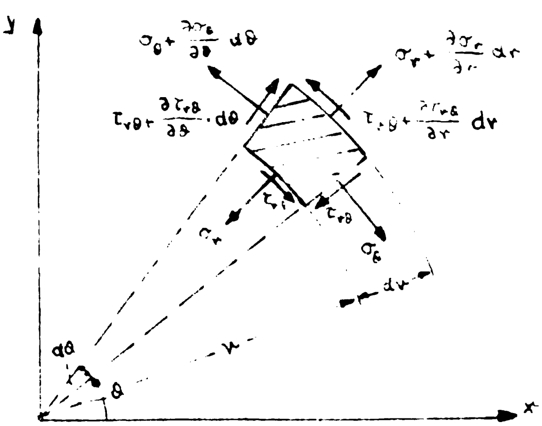


FIG. 3: EQUILIBRIUM OF A BODY ELEMENT IN POLAR COORDINATES

6.) BOUNDARY VALUES OF THE STRESS FUNCTION

As already mentioned in the acknowledgment, the relation between external forces and the boundary values of a stress function was not discovered by the writer but the idea was given to him by Professor Dr. Ing. K. Kloeppel. Although the idea is therefore not original, a complete derivation may be given since the relation does not seem to be generally known.

Equilibrium of a Boundary Element

Fig. 2 shows a boundary element of any plate under the action of external forces with the resultant components X and Y parallel to the coordinate axes. X and Y are forces per unit of length.

The conditions of equilibrium are:

$$\Sigma X = 0: \quad \sigma_x\, ds \cos(nx) + \tau_{yx}\, ds \cos(ny) = X \cdot ds$$

$$\Sigma Y = 0: \quad \tau_{xy}\, ds \cos(nx) + \sigma_y\, ds \cos(ny) = Y \cdot ds$$

$$\Sigma M = 0: \quad \tau_{xy} = \tau_{yx}$$

These are essentially the boundary conditions, Eqs. (3). With,

$$\cos(nx) = \frac{dy}{ds}\ ; \qquad \cos(ny) = -\frac{dx}{ds}$$

we may also write,

$$\begin{aligned} \sigma_x\, dy - \tau_{xy} \cdot dx &= X \cdot ds \\ \tau_{xy}\, dy - \sigma_y \cdot dx &= Y \cdot ds \end{aligned} \qquad (8)$$

Introduction of the Stress Function

The relation between stress components and stress function is,

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}\ ; \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}\ ; \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \qquad (4)$$

By inserting Eqs. (4) in Eqs. (8), the boundary conditions can be written in terms of the stress function $F(x, y)$ as follows:

$$\frac{\partial^2 F}{\partial y^2}\,dy + \frac{\partial^2 F}{\partial x \partial y}\,dx = X \cdot ds$$
$$-\frac{\partial^2 F}{\partial x \partial y}\,dy - \frac{\partial^2 F}{\partial x^2}\,dx = Y \cdot ds \tag{9}$$

The total differential of $F(x, y)$ is,

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy \quad \text{, and,}$$
$$F = \int dF = \int \frac{\partial F}{\partial x}\,dx + \int \frac{\partial F}{\partial y}\,dy \tag{10}$$

Eq. (10) may be partially differentiated with respect to x and y:

$$\frac{\partial F}{\partial x} = \int \frac{\partial^2 F}{\partial x^2}\,dx + \int \frac{\partial^2 F}{\partial x \partial y}\,dy$$
$$\frac{\partial F}{\partial y} = \int \frac{\partial^2 F}{\partial x \partial y}\,dx + \int \frac{\partial^2 F}{\partial y^2}\,dy \tag{11}$$

A comparison of the Eqs. (9) and (11) leads to the following relation:

$$\frac{\partial F}{\partial y} = \int X\,ds \;; \qquad \frac{\partial F}{\partial x} = -\int Y\,ds$$

If the integration of these line integrals is performed along the boundary from a certain starting point 0 to a point of reference k , we obtain:

$$\frac{\partial F}{\partial y_k} = \int_0^k X \cdot ds = R_x$$
$$\frac{\partial F}{\partial x_k} = -\int_0^k Y\,ds = -R_y \tag{12}$$

where R_x and R_y are the components, parallel to the x- and y- axis, respectively, of the resultant R of all external forces acting on the boundary between the points 0 and k (Fig. 4).

<u>Boundary Value of the Stress Function at Point k :</u>

If Eq. (10) is integrated by parts, we obtain,

$$F_k = \int_{x_0}^{x_k} \frac{\partial F}{\partial x}\,dx + \int_{y_0}^{y_k} \frac{\partial F}{\partial y}\,dy$$
$$= \frac{\partial F}{\partial x_k}(x_k - x_0) - \int_{x_0}^{x_k} \frac{\partial^2 F}{\partial x^2}\,x\,dx + \frac{\partial F}{\partial y_k}(y_k - y_0) - \int_{y_0}^{y_k} \frac{\partial^2 F}{\partial y^2}\,y\,dy$$

Considering Eqs. (12) and (4) we can write,

$$F_k = -R_y(x_k - x_o) - \int_{x_o}^{x_k} \sigma_y x\, dx + R_x(y_k - y_o) - \int_{y_o}^{y_k} \sigma_x y\, dy$$

The two integrals represent moments of the resultants R_y and R_x about Point O. If the point on the boundary where R_y and R_x act is designated by r, with coordinates x_r and y_r, we may write:

$$F_k = -R_y(x_k - x_o) + R_y(x_r - x_o) + R_x(y_k - y_o) - R_x(y_r - y_o)$$

Considering that point r lies between o and k on the boundary there results:

$$F_k = R_x(y_k - y_r) - R_y(x_k - x_r)$$

$$F_k = R_x \cdot y_{kr} - R_y \cdot x_{kr} = R \cdot r_k \tag{13}$$

where r_x is the moment arm of the resultant R, acting at point r, with respect to k. Fig. 4 shows that the moment is positive if taken in counterclockwise direction.

<u>Geometric Interpretation of the Stress Function</u>

Geometrically $F(x,y)$ is the equation of a surface. If we erect ordinates at every point of the plate, normal to its plane, the length of which designates the value of the stress function at this very point, then the endpoints of all these ordinates will form an imaginary <u>Airy stress surface</u>. Hence, we may say that the boundary moment as defined by Eq.(13) gives the boundary ordinate of the stress surface at point k.

In order to determine the Fourier coefficients of a trigonometric stress function, a second relation between boundary conditions and stress function is required. We use the first partial derivative of the stress function with respect to the normal to the boundary, e.g. $\frac{\partial F(x,y)}{\partial n_k}$. In terms of the geometric interpretation, this means the slope of the stress surface normal to the boundary.

Boundary Derivative of the Stress Function at Point k:

Figure 5 shows the introduction of a new, movable coordinate system n,t whose origin moves along the boundary such that the n-axis is always normal to the boundary. The partial differentiation of F(x,y), with respect to the normal, gives:

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial x}\frac{dx}{dn} + \frac{\partial F}{\partial y}\frac{dy}{dn} \tag{14}$$

From Fig. 5: $\frac{-dx}{-dn} = \cos(ys)$; $\frac{-dy}{-dn} = \cos(xs)$

where (ys) and (xs) are, respectively, the angle of the tangent at the boundary point k with the y - and x - axis.

Inserted in Eq. (14), we obtain,

$$\frac{\partial F}{\partial n} = \frac{\partial F}{\partial x}\cdot\cos(ys) + \frac{\partial F}{\partial y}\cdot\cos(xs)$$

and with Eqs. (12):

$$\frac{\partial F}{\partial n_k} = -R_y\cos(ys) + R_x\cos(xs)$$

It is seen from Figure 6 that this equation represents the difference between those components of R_x and R_y which are parallel to the tangent at the boundary point k. But this is the negative component R_s of the resultant R parallel to the tangent at point k. It can therefore be written:

$$\frac{\partial F}{\partial n_k} = R_x\cos(xs) - R_y\cos(ys) = -R_s \tag{15}$$

Arbitrary Selection of the Starting Point 0

The starting point 0 on the boundary can be chosen arbitrarily since its position has no influence on the magnitude of the stress components although we obtain boundary values of the stress function differing by a

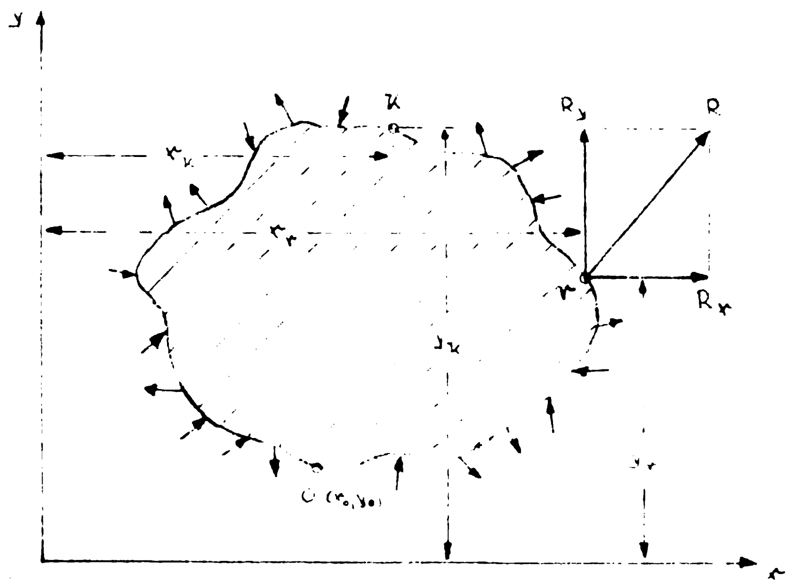


FIG.4: BOUNDARY VALUE AT POINT k

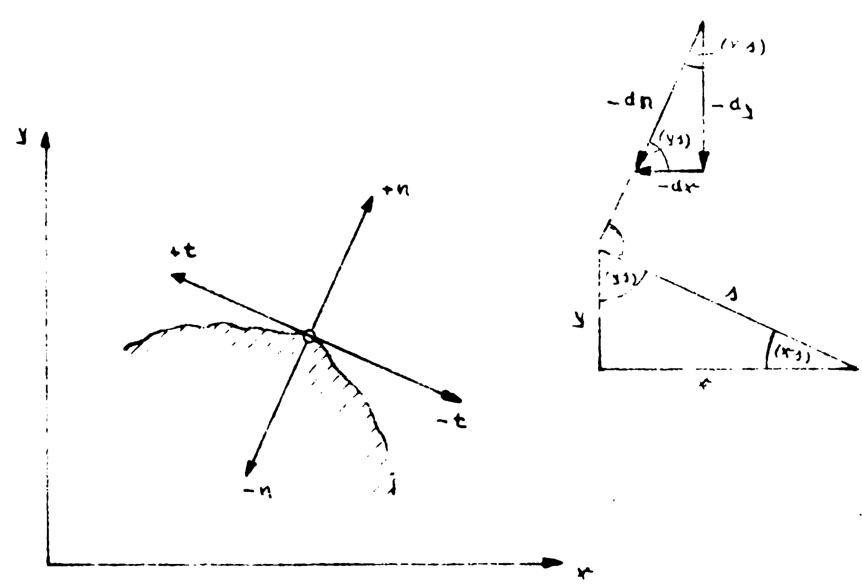


FIG.5: MOVABLE COORDINATE SYSTEM n,t

linear function in x and y for different starting points.

To prove this, we may consider the case when the starting point in Figure 7 is moved along the boundary to the left for the curve element ds. Then a new additional resultant of the boundary forces acting along ds originates whose components are:

$$dR_x = dX \cdot ds = \text{const.} \; ; \qquad dR_y = dY \cdot ds = \text{const.}$$

The boundary value of the stress function at any point k is increased by the statical moment of these constant forces about point k. The increment is linear and of the form

$$dF_k = A \cdot x + B \cdot y$$

It is seen immediately that these terms will vanish if F(x,y) is differentiated according to Eqs. (4) to obtain the stress components.

Summary

The boundary value of the stress function at any boundary point k is given by the statical moment of the resultant R with respect to that point (Eq. (13)). R is defined as the resultant of all external loads on the boundary between a starting point 0 and the point of reference, k, and acts at the boundary point r.

The starting point 0 may be chosen arbitrarily since a variation in its position leaves the magnitude of the stress components unaltered.

If the resultant R, acting at point r, is divided into two components so that the one is parallel to the boundary tangent at point k (R_s), while the other is parallel to the boundary normal at the same point k, then the negative former component, R_s, is equal to the first partial derivative of F(x,y) with respect to the normal to the boundary at point k, e.g. $\frac{\partial F}{\partial n_k} = - R_s$.   See Figure 7.

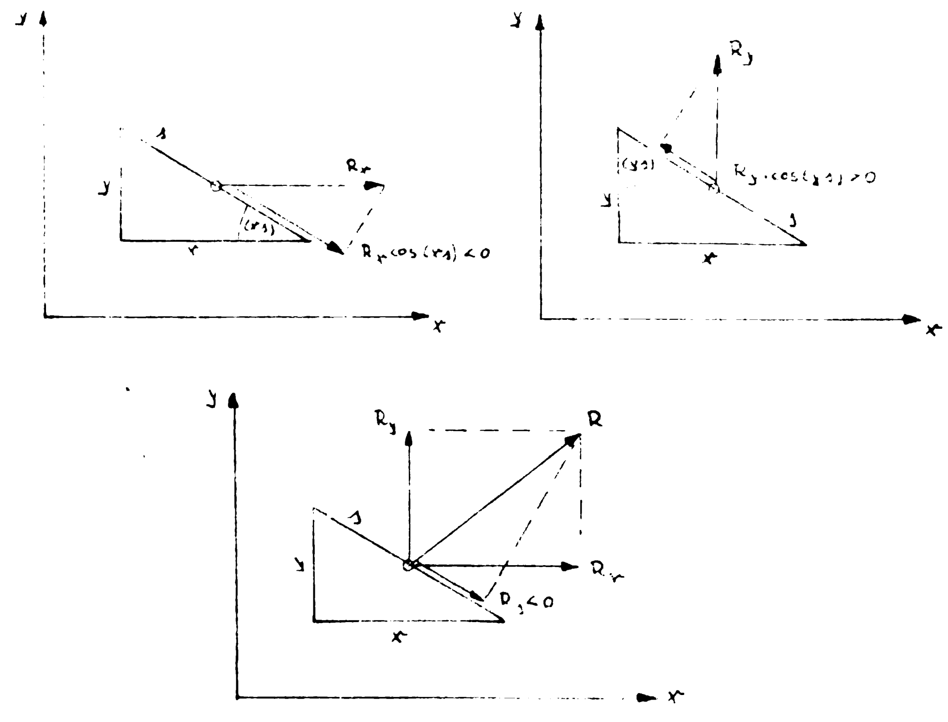


FIG. [illegible] BOUNDARY DERIVATIVE AT POINT r

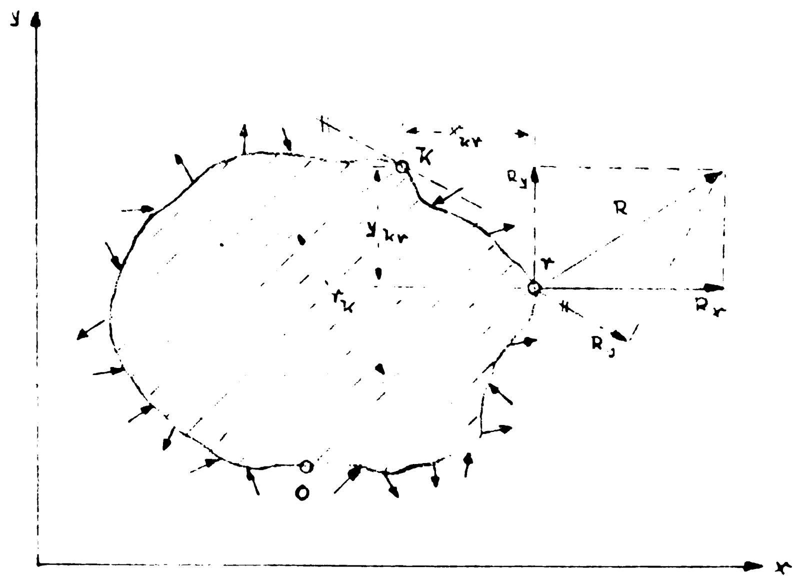


FIG. [illegible] UNKNOWN VALUE AND BOUNDARY DERIVATIVE AT POINT r

7.) TRIGONOMETRIC STRESS FUNCTION WITH FOURIER COEFFICIENTS OBTAINED BY INTEGRATION

For the two-dimensional problem of a circular plate, we assume a most general stress function in the form of a trigonometric series,

$$F(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos n\theta + \sum_{n=1}^{\infty} b_n r^n \sin n\theta + \\ + \frac{c_0}{2} r^2 + \sum_{n=1}^{\infty} c_n r^{n+2} \cos n\theta + \sum_{n=1}^{\infty} d_n r^{n+2} \sin n\theta \qquad (16)$$

This function is known to be biharmonic and hence satisfies the compatibility equation (7). By differentiating equation (16) partially with respect to r and multiplying this expression by r, we obtain:

$$G(r,\theta) = r\frac{\partial F}{\partial r} = \sum_{n=1}^{\infty} a_n n r^n \cos n\theta + \sum_{n=1}^{\infty} b_n n r^n \sin n\theta + \\ + c_0 r^2 + \sum_{n=1}^{\infty} c_n (n+2) r^{n+2} \cos n\theta + \sum_{n=1}^{\infty} d_n (n+2) r^{n+2} \sin n\theta \qquad (17)$$

To obtain the stress function for a certain case of loading, the unknown coefficients a_0, a_n, b_n, c_0, c_n, and d_n must be determined; e. g., the assumed expression for the stress function must be brought in agreement with the boundary conditions. This shall be done by means of the boundary values of the stress function. The application of the results of section 6 to this problem may be studied in two examples.

Example 1.

Given a solid circular plate with two single concentrated loads P acting on the vertical diameter as shown in Figure 8. The radius of the plate is ρ .

Boundary Values of the Stress Function

We recall from section 6 that the boundary value of the stress function at any point k is obtained by taking the statical moment of the resultant R = R(θ) with respect to that point k.

The point ($\rho, -\frac{\pi}{2}$) may be chosen as starting point 0 (Fig. 8). The load P applied at that point is then split into two forces, each of magnitude $\frac{P}{2}$, which are considered to act in an infinitely small distance to the left and right, respectively, of point ($\rho, -\frac{\pi}{2}$). This is done in order to obtain boundary values which are symmetric to both, the x - and y - axis.

The resultant R is then equal to $\frac{P}{2}$ and constant at every boundary point. The moment of R about any boundary point k is clockwise and therefore negative according to our sign convention. Noting that the moment arm is $\rho\,|\cos\theta|$, the boundary values of the stress function are represented by the following equation:

$$F(\rho,\theta) = -\frac{P\rho}{2}\,|\cos\theta| = N\cdot f(\theta) \tag{18}$$

where $N = -\frac{P\rho}{2}$ = const. and $f(\theta) = |\cos\theta|$

Boundary Derivatives of the Stress Function

From section 6, it is known that the slope of the stress surface, $\frac{\partial F}{\partial \nu}$, is given by the tangential component R_s of the resultant R. In our problem:

$$-R_s = \frac{\partial F}{\partial \nu} = -\frac{P}{2}\,|\cos\theta|$$

as seen from Figure 8. The expression G(ρ,θ), corresponding to Eq. (17), is then:

$$G(\rho,\theta) = \rho\,\frac{\partial F}{\partial \nu} = -\frac{P\rho}{2}\,|\cos\theta| = F(\rho,\theta) \tag{19}$$

Series Expansion of Boundary Values:

The expression for the boundary values, Eq. (17), may be expanded in a trigonometric series. Figure 9 shows the function

$$f(\theta) = |\cos\theta|$$

which is an even function with the period π. Hence there will be only even cosine terms in the Fourier expansion which is expressed by introducing $2n$ instead of n,

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_{2n} \cdot \cos 2n\theta \tag{20}$$

The coefficients $\frac{A_0}{2}$ and A_{2n} are determined as usual:

$$\frac{A_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \cos\theta\, d\theta$$

$$\frac{A_0}{2} = \frac{2}{\pi}$$

$$A_{2n} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos 2n\theta\, d\theta$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \cos\theta \cos 2n\theta\, d\theta$$

$$= \frac{2}{\pi}\left[\frac{\sin(2n-1)\theta}{2n-1} + \frac{\sin(2n+1)\theta}{2n+1}\right]_0^{\pi/2}$$

$$A_{2n} = \frac{2}{\pi}\,\frac{-2(-1)^n}{4n^2-1}$$

Inserting the evaluated coefficients in Eq. (20), we have,

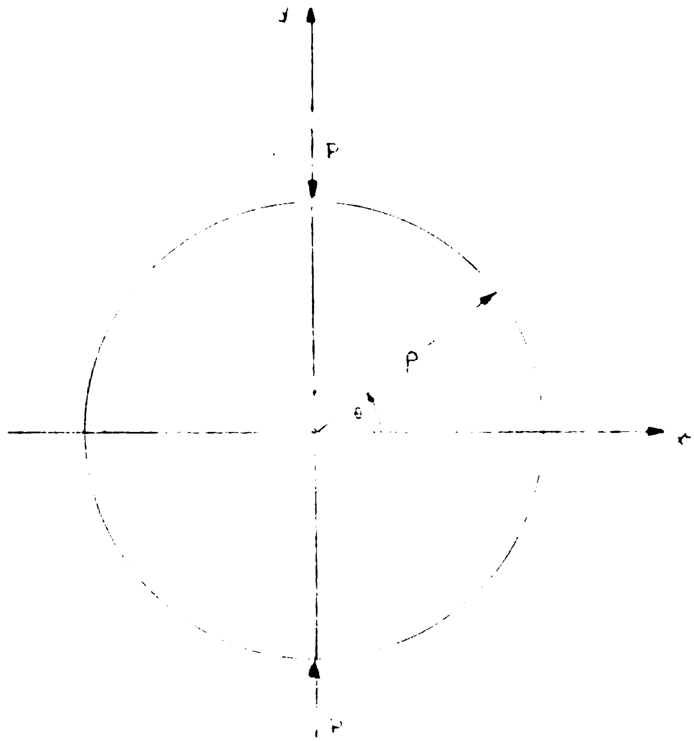
$$f(\theta) = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta\right]$$

With Eq. (18), the trigonometric expansion for the boundary values of the stress function can be written:

$$F(\rho,\theta) = N\cdot f(\theta) = -\frac{2P\rho}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta\right] \tag{21}$$

From Eq. (19) there follows:

$$G(\rho,\theta) = F(\rho,\theta) = -\frac{2P\rho}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta\right] \tag{22}$$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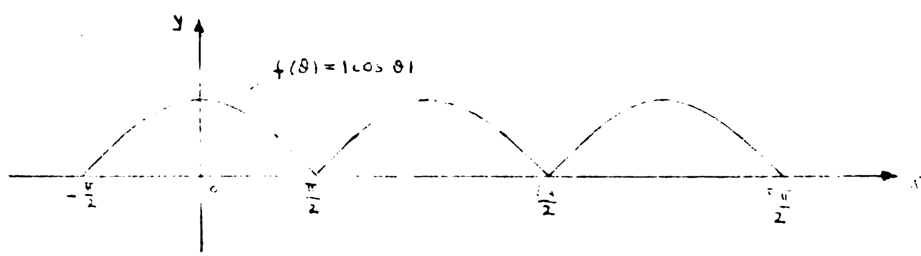

y
f(θ) = |cos θ|
0

[illegible]

<u>Determination of the Fourier Coefficients of Eq. (16):</u>

The unknown coefficients of Eq. (16) can now be determined from the condition that for all boundary points, Eqs. (16) and (17) must be identical with Eqs. (21) and (22), respectively. For the boundary, $r = \rho$, Eqs. (16) and (17) read as follows,

$$F(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \rho^n \cos n\theta + \sum_{n=1}^{\infty} b_n \rho^n \sin n\theta + \\ + \frac{c_0}{2}\rho^2 + \sum_{n=1}^{\infty} c_n \rho^{n+2} \cos n\theta + \sum_{n=1}^{\infty} d_n \rho^{n+2} \sin n\theta \qquad (23)$$

$$G(\rho,\theta) = \sum_{n=1}^{\infty} a_n n \rho^n \cos n\theta + \sum_{n=1}^{\infty} b_n n \rho^n \sin n\theta + \\ + c_0 \rho^2 + \sum_{n=1}^{\infty} c_n (n+2) \rho^{n+2} \cos n\theta + \sum_{n=1}^{\infty} d_n (n+2) \rho^{n+2} \sin n\theta \qquad (24)$$

If we compare the coefficients of Eq. (21) with those of Eq. (23) and the coefficients of Eq. (22) with those of Eq. (24), there results:

$$a_0 = -\frac{P\rho}{\pi}; \qquad a_{2n} = \frac{P\rho}{\pi} \cdot \frac{(-1)^n}{\rho^{2n}(2n-1)}$$

$$c_0 = -\frac{P}{\rho\pi}; \qquad c_{2n} = -\frac{P\rho}{\pi} \cdot \frac{(-1)^n}{\rho^{2n+2}(2n+1)}$$

$$a_n = c_n = 0 \quad \text{for all odd } n$$

$$b_n = d_n = 0 \quad \text{for all } n$$

With these values of the Fourier coefficients, Eq. (16) can be written,

$$F(r,\theta) = -\frac{1}{2}\frac{P\rho}{\pi} + \sum_{n=1}^{\infty} \frac{P\rho}{\pi} \cdot \frac{(-1)^n}{\rho^{2n}(2n-1)} r^{2n} \cos 2n\theta - \\ - \frac{1}{2}\frac{P}{\rho\pi} r^2 - \sum_{n=1}^{\infty} \frac{P\rho}{\pi} \cdot \frac{(-1)^n}{\rho^{2n+2}(2n+1)} r^{2n+2} \cos 2n\theta$$

from which,

$$F(r,\theta) = -\frac{p\rho}{\pi}\left\{ \frac{1}{2}\left[1+\left(\frac{r}{\rho}\right)^2\right] + \right. \tag{25a}$$
$$\left. + \sum_{n=1}^{\infty}(-1)^n \left[\frac{1}{2n+1}\left(\frac{r}{\rho}\right)^2 - \frac{1}{2n-1}\right]\left(\frac{r}{\rho}\right)^{2n}\cos 2n\theta \right\}$$

or:

$$F(r,\theta) = -\frac{p}{\rho\pi}\left\{ \frac{\rho^2+r^2}{2} - \right. \tag{25b}$$
$$\left. - \sum_{n=1}^{\infty}(-1)^n \frac{\rho^2(2n+1)-r^2(2n-1)}{4n^2-1}\left(\frac{r}{\rho}\right)^{2n}\cos 2n\theta \right\}$$

The stress components, in polar coordinates, are then obtained by differentiation according to Eqs. (6).

Example 2.

Next, the case shown in Figure 10 may be considered. Two distributed loads of intensity q and radial direction are applied on the boundary over an arc length $2\rho\beta$ where β is taken to the right and left of the vertical diameter.

Series Expansion of Boundary Values:

The series expansion for this case can be obtained directly from the expansion in Example 1. We consider the single concentrated load P of Example 1 as a load element of the distributed load q. If this single load element is moved along the boundary of the circular plate for an angle $+\alpha$ (in a counterclockwise direction), the boundary values of the stress function are altered by this angle due to the new position of q, (Fig. 11). Eq. (21) must then be written in the form,

$$dF(\rho,\theta) = -\frac{2q\rho}{\pi}\left[\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n(\theta+\alpha)\right] \tag{26}$$

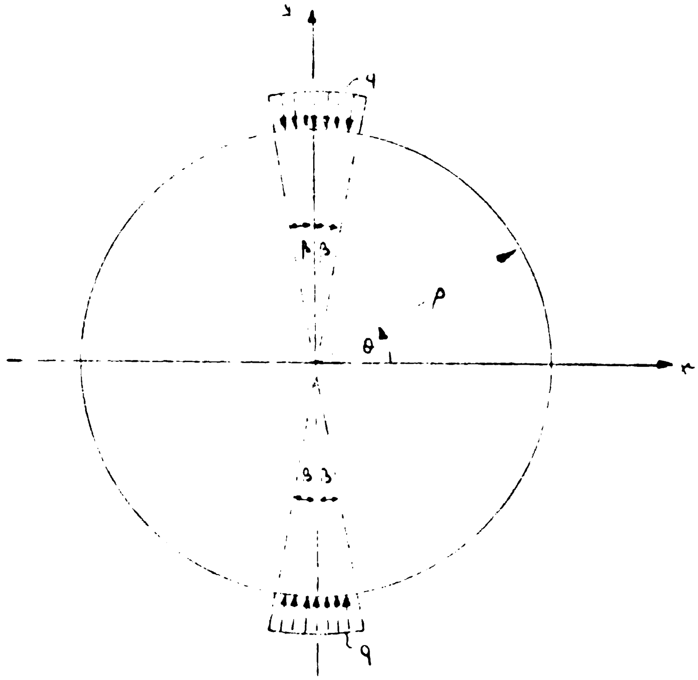
where $dF(\rho,\theta)$ is the element of the stress function on the boundary due to the load element q. It is seen that Eq. (26) becomes identical with Eq. (21) if α approaches zero.

The total stress function $F(\rho,\theta)$ for the boundary values is then obtained by integrating Eq. (26) over the angle on which q is applied:

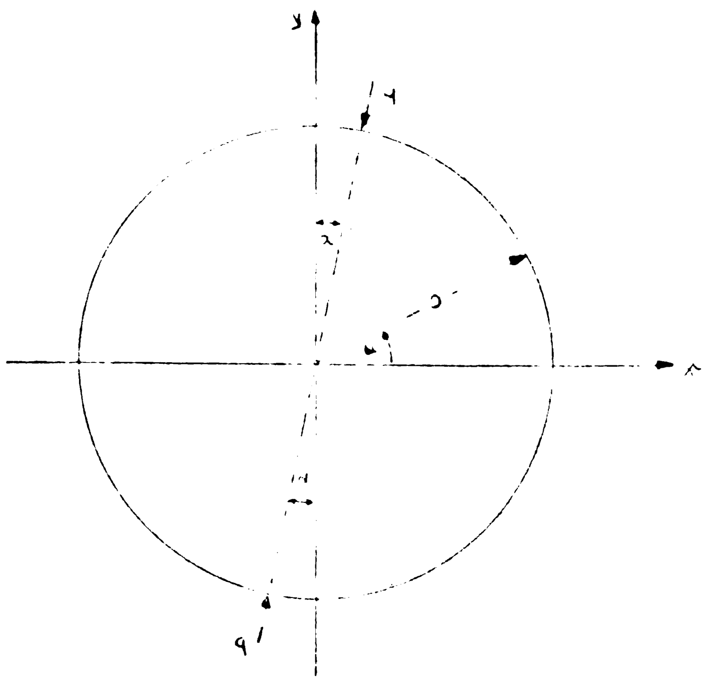
$$F(\rho,\theta) = \int_{-\beta}^{+\beta} dF(\rho,\theta) = -\frac{2q\rho^2}{\pi}\int_{-\beta}^{+\beta}\left[\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n(\theta+\alpha)\right]d\alpha$$

$$= -\frac{2q\rho^2}{\pi}\left\{\beta-\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\left[\frac{\sin 2n(\theta+\alpha)}{2n}\right]_{-\beta}^{+\beta}\right\}$$

$$F(\rho,\theta) = -\frac{4q\rho^2\beta}{\pi}\left[\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta\cdot\frac{\sin 2n\beta}{2n\beta}\right] \tag{27}$$



[illegible] PLATE WITH [illegible]TED LOAD q



F[illegible] ELEMENT q [illegible] CIRCULAR [illegible]

If we denote the total force exerted by the distributed load by P, then,

$$P = 2q\rho\beta$$

Substituting this expression in Eq. (27), we obtain,

$$F(\rho,\theta) = -\frac{P\rho}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta \cdot \frac{\sin 2n\beta}{2n\beta}\right]$$

which is the same as Eq. (21) with the exception of a new constant factor, $\frac{\sin 2n\beta}{2n\beta}$.

If β approaches zero, then,

$$\lim_{\beta\to 0}\frac{\sin 2n\beta}{2n\beta} = 1$$

and the expression becomes identical with Eq. (21).

It is seen that in this case again, $G(\rho,\theta)$ is identical with $F(\rho,\theta)$,

$$G(\rho,\theta) = \rho\frac{\partial F}{\partial r} = -\frac{4q\rho^2\beta}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos 2n\theta \cdot \frac{\sin 2n\beta}{2n\beta}\right] \tag{28}$$

__Fourier Coefficients of the Stress Function $F(r,\theta)$__

The Fourier coefficients of $F(r,\theta)$ are determined in the same way as those of Example 1. The general expressions for $F(r,\theta)$ and $G(r,\theta)$ on the boundary are, respectively,

$$\begin{aligned} F(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\rho^n\cos n\theta + \sum_{n=1}^{\infty} b_n\rho^n\sin n\theta + \\ + \frac{c_0}{2}\rho^2 + \sum_{n=1}^{\infty} c_n\rho^{n+2}\cos n\theta + \sum_{n=1}^{\infty} d_n\rho^{n+2}\sin n\theta \end{aligned} \tag{23}$$

$$\begin{aligned} G(\rho,\theta) = \sum_{n=1}^{\infty} a_n n\rho^n\cos n\theta + \sum_{n=1}^{\infty} b_n n\rho^n\sin n\theta + \\ + c_0\rho^2 + \sum_{n=1}^{\infty} c_n(n+2)\rho^{n+2}\cos n\theta + \sum_{n=1}^{\infty} d_n(n+2)\rho^{n+2}\sin n\theta \end{aligned} \tag{24}$$

The comparison of the coefficients of Eqs. (27) and (23) as well as of Eqs. (28) and (24) leads to the following result:

$$
\begin{aligned}
a_0 &= -\frac{2q\rho^2\beta}{\pi} \\
a_n &= c_n = 0 \quad \text{if } n \text{ is odd} \\
a_{2n} &= \frac{q\rho^2}{\pi\rho^{2n}} \cdot \frac{(-1)^n}{n(2n-1)} \cdot \sin 2n\beta \\
b_n &= d_n = 0 \quad \text{for all } n \\
c_0 &= -\frac{2q\beta}{\pi} \\
c_{2n} &= -\frac{q}{\pi\rho^{2n}} \cdot \frac{(-1)^n}{n(2n+1)} \cdot \sin 2n\beta
\end{aligned}
\tag{29}
$$

If these evaluated coefficients are inserted in Eq. (16), we obtain the following expression for the stress function $F(r,\theta)$:

$$
F(r,\theta) = -\frac{q\rho^2}{\pi}\left\{\beta\left[1+\left(\frac{r}{\rho}\right)^2\right] - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\left(\frac{r}{\rho}\right)^2\right]\left(\frac{r}{\rho}\right)^{2n}\cdot\cos 2n\theta\cdot\sin 2n\beta\right\}
\tag{30a}
$$

or:

$$
F(r,\theta) = -\frac{q}{\pi}\left\{\beta\left[\rho^2+r^2\right] - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\rho^2}{2n-1}-\frac{r^2}{2n+1}\right]\left(\frac{r}{\rho}\right)^{2n}\cos 2n\theta\cdot\sin 2n\beta\right\}
\tag{30b}
$$

It is seen that Eqs. (30a) and (30b) can be also obtained directly after the relation between example 1 and 2 has been established with Eq. (27). Then in Eqs. (25a) and (25b) there is substituted:

$$P = 2q\rho\beta$$

and the summation term is multiplied by $\frac{\sin 2n\beta}{2n\beta}$.

<u>Stress Components</u>

If Eq. (13) is differentiated according to Eq. (6), the following expressions for the stress components are obtained:

$$\begin{aligned}
\sigma_r &= c_0 - \Big\{ \sum_{n=1}^{\infty} a_n n(n-1) r^{n-2} \cos n\theta + \sum_{n=1}^{\infty} b_n n(n-1) r^{n-2} \sin n\theta + \\
&\quad + \sum_{n=1}^{\infty} c_n (n-2)(n+1) r^n \cos n\theta + \sum_{n=1}^{\infty} d_n (n-2)(n+1) r^n \sin n\theta \Big\} \\
\sigma_\theta &= c_0 + \sum_{n=1}^{\infty} a_n n(n-1) r^{n-2} \cos n\theta + \sum_{n=1}^{\infty} b_n n(n-1) r^{n-2} \sin n\theta + \\
&\quad + \sum_{n=1}^{\infty} c_n (n+2)(n+1) r^n \cos n\theta + \sum_{n=1}^{\infty} d_n (n+2)(n+1) r^n \sin n\theta \\
\tau_{r\theta} &= \sum_{n=1}^{\infty} a_n n(n-1) r^{n-2} \sin n\theta - \sum_{n=1}^{\infty} b_n n(n-1) r^{n-2} \cos n\theta + \\
&\quad + \sum_{n=1}^{\infty} c_n n(n+1) r^n \sin n\theta - \sum_{n=1}^{\infty} d_n n(n+1) r^n \cos n\theta
\end{aligned} \tag{31}$$

For our particular example where only even cosine terms appear in the trigonometric stress function, these equations become:

$$\begin{aligned}
\sigma_r &= c_0 - \Big\{ \sum_{n=1}^{\infty} a_{2n} 2n(2n-1) r^{2n-2} \cos 2n\theta + \\
&\quad + \sum_{n=1}^{\infty} c_{2n} (2n-2)(2n+1) r^{2n} \cos 2n\theta \Big\} \\
\sigma_\theta &= c_0 + \sum_{n=1}^{\infty} a_{2n} 2n(2n-1) r^{2n-2} \cos 2n\theta + \\
&\quad + \sum_{n=1}^{\infty} c_{2n} (2n+2)(2n+1) r^{2n} \cos 2n\theta \\
\tau_{r\theta} &= \sum_{n=1}^{\infty} a_{2n} 2n(2n-1) r^{2n-2} \sin 2n\theta + \sum_{n=1}^{\infty} c_{2n} 2n(2n+1) r^{2n} \sin 2n\theta
\end{aligned} \tag{32}$$

Numerical values of the stress components at any point (r,θ) for the case of loading shown in Fig. 10 are then obtained if the Fourier coefficients are evaluated according to Eqs. (29) and the coordinates of (r,θ) are inserted in Eqs. (32).

8.) TRIGONOMETRIC STRESS FUNCTION WITH FOURIER COEFFICIENTS OBTAINED BY FINITE SUMMATION

Under general loading conditions, it is often not possible to find a function $F(\rho,\vartheta)$ (expressing the boundary values of the stress function) whose derivative with respect to ϑ is continuous from zero to 2π. The integration for obtaining the Fourier coefficients must then be split into several intervals, the endpoints of which are determined by the points of discontinuity of $\frac{\partial F}{\partial \vartheta}$. In such a case, it may be useful to employ the following approximate method which replaces the integration by a finite summation around the boundary.

Expression for the Boundary Values of the Stress Function

As seen in section 6, the numerical value of the stress function at any boundary point can be found from the external loads. In the following derivation, we shall consider those numerical values at selected points k which are spaced at equal intervals $\Delta\vartheta$ around the boundary, π being an integral multiple of $\Delta\vartheta$. The total number of points k taken on a semicircle is denoted by m. The polar coordinates of such a point k are then $(\rho, \frac{k\pi}{m})$. An example with $m = 6$, $\Delta\vartheta = \frac{\pi}{6}$ is shown in Fig. 12.

We may further define F_k as the numerical value of $F(r,\vartheta)$ at k. Correspondingly, G_k is the numerical value of $G(r,\vartheta) = r\frac{\partial F}{\partial r}$ at point k.

Knowing the values F_k and G_k for all points k, k = 1, 2, 3 ····2m-1, 2m, we may express those values in the form of two finite trigonometric series,

$$F\left(\rho, \frac{k\pi}{m}\right) = \frac{A_0}{2} + \sum_{n=1}^{m-1} A_n \cos\frac{nk\pi}{m} + \frac{A_m}{2}\cos k\pi + \sum_{n=1}^{m} B_n \sin\frac{nk\pi}{m} \tag{33}$$

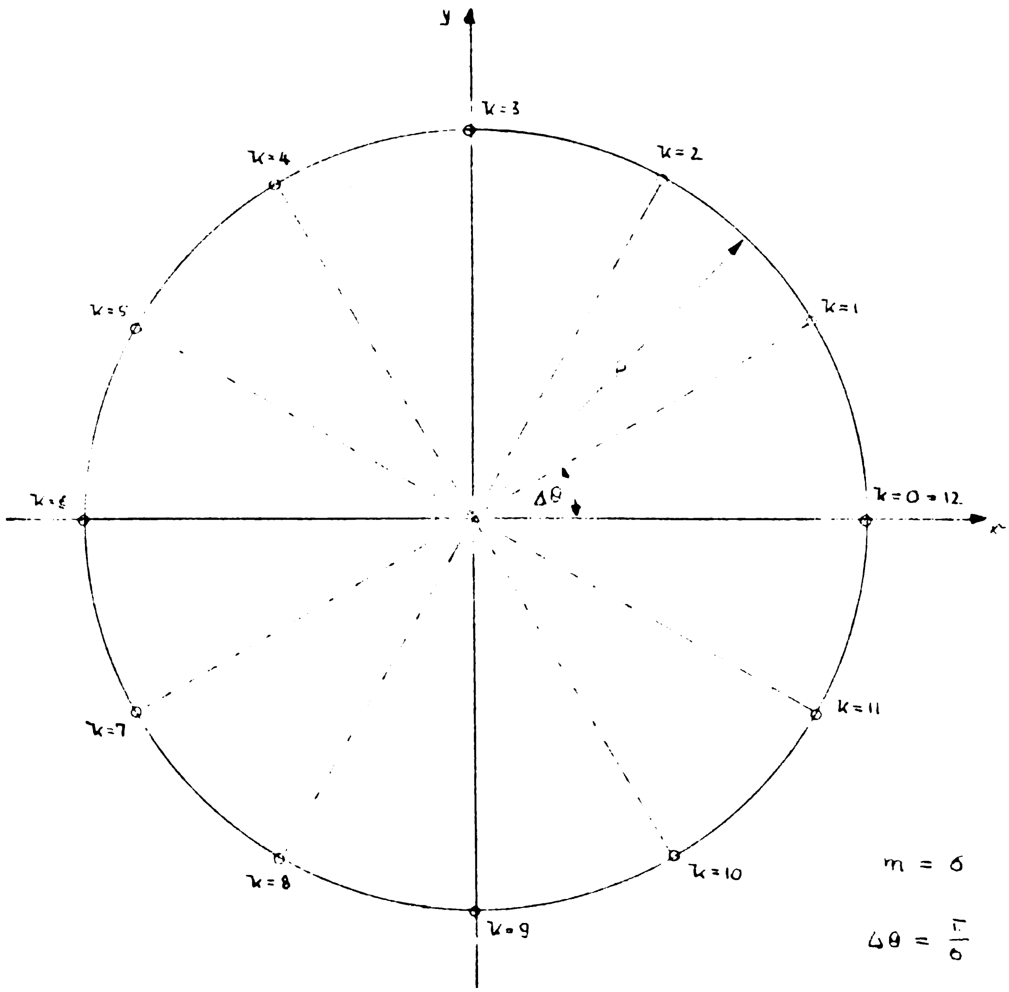


FIG.(2): EXAMPLE FOR A SET OF BOUNDARY POINTS K

$$G(\rho, \frac{k\pi}{m}) = \frac{C_0}{2} + \sum_{n=1}^{m-1} C_n \cos \frac{nk\pi}{m} + \frac{C_m}{2} \cos k\pi + \sum_{n=1}^{m} D_n \sin \frac{nk\pi}{m} \tag{34}$$

Since we break the series up after a finite number of terms, we must take the last terms, A_m and C_m, only half. The reason for this is the same as for taking the half of the first term, A_0. If we take A_m and C_m, instead of $\frac{A_m}{2}$ and $\frac{C_m}{2}$, the point ($\rho, k\pi$) would have a double weight. The proof may be found by writing the series

$$\frac{A_0}{2} + \sum_{n=1}^{m-1} A_n \cos \frac{nk\pi}{m} + \frac{A_m}{2} \cos k\pi$$

in terms of $e^{in\alpha}$ and $e^{-in\alpha}$.

Since we do not know the fumtions $F(\rho,\theta)$ and $G(\rho,\theta)$ but only the numerical values F_k and G_k at a finite number of points, we cannot use the usual method of integrating around the boundary in order to determine the Fourier coefficients. As an approximation, we take a finite summation of the values F_k and G_k. The formulas for the determination of the Fourier coefficients are then as follows,

$$\begin{aligned} A_0 &= \frac{1}{m} \sum_{k=1}^{2m} F_k \\ A_n &= \frac{1}{m} \sum_{k=1}^{2m} F_k \cdot \cos \frac{nk\pi}{m} \\ B_n &= \frac{1}{m} \sum_{k=1}^{2m} F_k \cdot \sin \frac{nk\pi}{m} \end{aligned} \tag{35}$$

$$\begin{aligned} C_0 &= \frac{1}{m} \sum_{k=1}^{2m} G_k \\ C_n &= \frac{1}{m} \sum_{k=1}^{2m} G_k \cdot \cos \frac{nk\pi}{m} \\ D_n &= \frac{1}{m} \sum_{k=1}^{2m} G_k \cdot \sin \frac{nk\pi}{m} \end{aligned} \tag{36}$$

Fourier Coefficients of the General Stress Function $F(r, \theta)$

We assume again a most general stress function $F(r, \theta)$, this time in the form of a finite trigonometric series,

$$F(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{m-1} a_n r^n \cos n\theta + \frac{a_m}{2} r^m \cos m\theta + \sum_{n=1}^{m} b_n r^n \sin n\theta + \\ + \frac{c_0}{2} r^2 + \sum_{n=1}^{m-1} c_n r^{n+2} \cos n\theta + \frac{c_m}{2} r^{m+2} \cos m\theta + \sum_{n=1}^{m} d_n r^{n+2} \sin n\theta \tag{37}$$

The corresponding expression for $G(r, \theta) = r \frac{\partial F}{\partial r}$ is ,

$$G(r,\theta) = \sum_{n=1}^{m-1} a_n n r^n \cos n\theta + \frac{a_m}{2} m r^m \cos m\theta + \sum_{n=1}^{m} b_n n r^n \sin n\theta + \\ + c_0 r^2 + \sum_{n=1}^{m-1} c_n (n+2) r^{n+2} \cos n\theta + \frac{c_m}{2} (m+2) r^{m+2} \cos m\theta + \\ + \sum_{n=1}^{m} d_n (n+2) r^{n+2} \sin n\theta \tag{38}$$

For the boundary points k, these equations read as follows,

$$F\left(\rho, \frac{k\pi}{m}\right) = \frac{a_0}{2} + \sum_{n=1}^{m-1} a_n \rho^n \cos \frac{nk\pi}{m} + \frac{a_m}{2} \rho^m \cos k\pi + \sum_{n=1}^{m} b_n \rho^n \sin \frac{nk\pi}{m} + \\ + \frac{c_0}{2} \rho^2 + \sum_{n=1}^{m-1} c_n \rho^{n+2} \cos \frac{nk\pi}{m} + \frac{c_m}{2} \rho^{m+2} \cos k\pi + \sum_{n=1}^{m} d_n \rho^{n+2} \sin \frac{nk\pi}{m} \tag{39}$$

$$G\left(\rho, \frac{k\pi}{m}\right) = \sum_{n=1}^{m-1} a_n n \rho^n \cos \frac{nk\pi}{m} + \frac{a_m}{2} m \rho^m \cos k\pi + \sum_{n=1}^{m} b_n n \rho^n \sin \frac{nk\pi}{m} + \\ + c_0 \rho^2 + \sum_{n=1}^{m-1} c_n (n+2) \rho^{n+2} \cos \frac{nk\pi}{m} + \frac{a_m}{2} (m+2) \rho^{m+2} \cos k\pi + \\ + \sum_{n=1}^{m} d_n (n+2) \rho^{n+2} \sin \frac{nk\pi}{m} \tag{40}$$

Eqs. (33) and (34) must be identical with Eqs. (39) and (40), respectively. By comparing the coefficients of the individual terms, the following relations between the Fourier coefficients are obtained,

$$a_0 = \frac{2A_0 - C_0}{2} \qquad c_0 = \frac{C_0}{2\rho^2} \tag{41}$$

$$a_n = \frac{(n+2)A_n - C_n}{2\rho^n} \qquad c_n = \frac{-nA_n + C_n}{2\rho^{n+2}}$$

$$b_n = \frac{(n+2)B_n - D_n}{2\rho^n} \qquad d_n = \frac{-nB_n + D_n}{2\rho^{n+2}}$$

The coefficients of the general stress function are here expressed in terms of the coefficients of the special stress function for the boundary values and its first derivative, Eqs. (33) and (34). But the latter coefficients can be evaluated by using the known boundary values F_k and G_k as seen from Eqs. (35) and (36). Hence, the Fourier coefficients of the general stress function, Eq. (37) are also known. If we substitute Eqs. (35) and (36) in Eqs. (41), we obtain,

$$\begin{aligned}
a_0 &= \frac{1}{2m}\sum_{k=1}^{2m}\left[2F_k - G_k\right] \\
a_n &= \frac{1}{2m\rho^n}\sum_{k=1}^{2m}\left[(n+2)F_k - G_k\right]\cos\frac{nk\pi}{m} \\
b_n &= \frac{1}{2m\rho^n}\sum_{k=1}^{2m}\left[(n+2)F_k - G_k\right]\sin\frac{nk\pi}{m} \\
c_0 &= \frac{1}{2m\rho^2}\sum_{k=1}^{2m} G_k \\
c_n &= \frac{1}{2m\rho^{n+2}}\sum_{k=1}^{2m}\left[G_k - nF_k\right]\cos\frac{nk\pi}{m} \\
d_n &= \frac{1}{2m\rho^{n+2}}\sum_{k=1}^{2m}\left[G_k - nF_k\right]\sin\frac{nk\pi}{m}
\end{aligned} \tag{42}$$

It is seen that the Fourier coefficients a_m and c_m in Eqs. (37) and (39) are merely a special case of the coefficients a_n and c_n. They follow immediately from the expressions for a_n and c_n, Eqs. (42) by setting n=m.

As a result, we have obtained a trigonometric stress function, the

Fourier coefficients of which are approximately known. The determination of the stress components follows the same procedures as previously described in section 7 (see Eqs. (31).

9.) NUMERICAL METHOD FOR AN APPROXIMATE DETERMINATION OF STRESS COMPONENTS

As mentioned already in the preliminary description, Section 4, this third modification of the boundary method is an attempt to establish a purely numerical method in which neither stress functions nor differentiations nor integration appear in the practical application. Approximate stress components are to be obtained by mere algebraic means.

Expression for the General Stress Function $F(r,\theta)$:

The derivation of this method is based on the same assumptions as made in Section 8. The general stress function is again assumed to be,

$$F(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{m-1} a_n r^n \cos n\theta + \frac{a_m}{2} r^m \cos m\theta + \sum_{n=1}^{m} b_n r^n \sin n\theta + \tag{37}$$

$$+ \frac{c_0}{2} r^2 + \sum_{n=1}^{m-2} c_n r^{n+2} \cos n\theta + \frac{c_m}{2} r^{m+2} \cos m\theta + \sum_{n=1}^{m} d_n r^{n+2} \sin n\theta$$

The Fourier coefficients of Eq. (37) are determined by,

$$\begin{aligned}
a_0 &= \frac{1}{2m} \sum_{k=1}^{2m} [2F_k - G_k] \\
a_n &= \frac{1}{2m\rho^n} \sum_{k=1}^{2m} [(n+2)F_k - G_k] \cos \frac{nk\pi}{m} \\
b_n &= \frac{1}{2m\rho^n} \sum_{k=1}^{2m} [(n+2)F_k - G_k] \sin \frac{nk\pi}{m} \\
\\
c_0 &= \frac{1}{2m\rho^2} \sum_{k=1}^{2m} G_k \\
c_n &= \frac{1}{2m\rho^{n+2}} \sum_{k=1}^{2m} [G_k - nF_k] \cos \frac{nk\pi}{m} \\
d_n &= \frac{1}{2m\rho^{n+2}} \sum_{k=1}^{2m} [G_k - nF_k] \sin \frac{nk\pi}{m}
\end{aligned} \tag{42}$$

in which F_k and G_k are, respectively, the known numerical values of the stress function and the product of r times the first partial derivative of the stress function in a direction normal to the boundary at any boundary point k. In contrast to the development of section 8, however, Eqs. (42) may not be evaluated but inserted in Eq. (37):

$$
\begin{aligned}
F(r,\theta) = \frac{1}{2m}\Big\{ & \sum_{k=1}^{2m} F_k + \sum_{k=1}^{2m} 2F_k\left(\frac{r}{\rho}\right)\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\cos\theta\cos\frac{k\pi}{m} + \sum_{k=1}^{2m} 2F_k\left(\frac{r}{\rho}\right)^2\left[1+\frac{2}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\cdot \\
& \cdot\cos 2\theta\cos\frac{2k\pi}{m} + \cdots\cdots + F_k\left(\frac{r}{\rho}\right)^m\left[1+\frac{m}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\cos m\theta\cos k\pi + \\
& + \sum_{k=1}^{2m} 2F_k\left(\frac{r}{\rho}\right)\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\sin\theta\sin\frac{k\pi}{m} + \sum_{k=1}^{2m} 2F_k\left(\frac{r}{\rho}\right)^2\left[1+\frac{2}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\cdot \\
& \cdot\sin 2\theta\sin\frac{2k\pi}{m} + \cdots\cdots + 2F_k\left(\frac{r}{\rho}\right)^{m-1}\left[1+\frac{m-1}{2}\left(1-\frac{r^2}{\rho^2}\right)\right]\sin(m-1)\theta\sin\frac{(m-1)k\pi}{m}\Big\} - \\
- \frac{1}{2m}\Big\{ & \sum_{k=1}^{2m}\frac{G_k}{2}\left[1-\frac{r^2}{\rho^2}\right] + \sum_{k=1}^{2m} G_k\left(\frac{r}{\rho}\right)\left[1-\frac{r^2}{\rho^2}\right]\cos\theta\cos\frac{k\pi}{m} + \sum_{k=1}^{2m} G_k\left(\frac{r}{\rho}\right)^2\left[1-\frac{r^2}{\rho^2}\right]\cdot \\
& \cdot\cos 2\theta\cos\frac{2k\pi}{m} + \cdots\cdots + \frac{G_k}{2}\left(\frac{r}{\rho}\right)^m\left[1-\frac{r^2}{\rho^2}\right]\cos m\theta\cos k\pi + \\
& + \sum_{k=1}^{2m} G_k\left(\frac{r}{\rho}\right)\left[1-\frac{r^2}{\rho^2}\right]\sin\theta\sin\frac{k\pi}{m} + \sum_{k=1}^{2m} G_k\left(\frac{r}{\rho}\right)^2\left[1-\frac{r^2}{\rho^2}\right]\sin 2\theta\sin\frac{2k\pi}{m} + \\
& + \cdots\cdots + G_k\left(\frac{r}{\rho}\right)^{m-1}\left[1-\frac{r^2}{\rho^2}\right]\sin(m-1)\theta\sin\frac{(m-1)k\pi}{m}\Big\}
\end{aligned}
$$

If we note that,

$$\cos n\theta\cos\frac{nk\pi}{m} + \sin n\theta\sin\frac{nk\pi}{m} = \cos n\left(\theta-\frac{k\pi}{m}\right)$$

and if we simplify the resulting expression, the equation for $F(r,\theta)$ becomes,

$$
\begin{aligned}
F(r,\theta) = \frac{1}{2m}\Big\{ & \sum_{k=1}^{2m} F_k\Big[1+\sum_{n=1}^{m-1} 2\left(\frac{r}{\rho}\right)^n\left\{1+\frac{n}{2}\left(1-\frac{r^2}{\rho^2}\right)\right\}\cos n\left(\theta-\frac{k\pi}{m}\right) + \left(\frac{r}{\rho}\right)^m\left\{1+\frac{m}{2}\left(1-\frac{r^2}{\rho^2}\right)\right\}\cdot \\
& \cdot\cos m\left(\theta-\frac{k\pi}{m}\right)\Big] - \qquad\qquad (43) \\
& - \sum_{k=1}^{2m} G_k\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\Big[1+\sum_{n=1}^{m-1} 2\left(\frac{r}{\rho}\right)^n\cos n\left(\theta-\frac{k\pi}{m}\right) + \left(\frac{r}{\rho}\right)^m\cos m\left(\theta-\frac{k\pi}{m}\right)\Big]\Big\}
\end{aligned}
$$

Reduction of Eq. (43) to a Single Summation:

The summation over n in Eq. (43) shall now be replaced by a simpler expression. For this purpose, we introduce two new functions φ_k and ψ_k . Eq. (43) can then be written in the form,

$$F(r,\theta) = \frac{1}{2m}\left\{ \sum_{k=1}^{2m} F_k\left[\varphi_k + \psi_k\right] - \sum_{k=1}^{2m} G_k \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\varphi_k \right\} \tag{44}$$

in which,

$$\varphi_k\left(r, \theta-\frac{k\pi}{m}\right) = 1 + \sum_{n=1}^{m-1} 2\left(\frac{r}{\rho}\right)^n \cos n\left(\theta-\frac{k\pi}{m}\right) + \left(\frac{r}{\rho}\right)^m \cos m\left(\theta-\frac{k\pi}{m}\right) \tag{45}$$

and ,

$$\psi_k\left(r, \theta-\frac{k\pi}{m}\right) = \sum_{n=1}^{m-1} 2\left(\frac{r}{\rho}\right)^n \frac{n}{2}\left(1-\frac{r^2}{\rho^2}\right)\cos n\left(\theta-\frac{k\pi}{m}\right) + \left(\frac{r}{\rho}\right)^m \frac{m}{2}\left(1-\frac{r^2}{\rho^2}\right)\cos m\left(\theta-\frac{k\pi}{m}\right) \tag{46}$$

If we differentiate Eq. (45) with respect to r, and multiply it by r, we obtain,

$$r\frac{\partial \varphi_k}{\partial r} = \sum_{n=1}^{m-1} 2n\left(\frac{r}{\rho}\right)^n \cos n\left(\theta-\frac{k\pi}{m}\right) + m\left(\frac{r}{\rho}\right)^m \cos m\left(\theta-\frac{k\pi}{m}\right)$$

Comparing this result with Eq. (46), we may write,

$$\psi_k\left(r, \theta-\frac{k\pi}{m}\right) = \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r\frac{\partial \varphi_k}{\partial r}$$

Hence we can express Eq. (44) as a function of φ_k only,

$$F(r,\theta) = \frac{1}{2m}\left\{ \sum_{k=1}^{2m} F_k\left[\varphi_k + \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r\frac{\partial \varphi_k}{\partial r}\right] - \sum_{k=1}^{2m} G_k \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\varphi_k \right\}$$

or in the form,

$$F(r,\theta) = \frac{1}{2m}\left\{\sum_{k=1}^{2m} F_k\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)r\frac{\partial}{\partial r}\right]\varphi_k - \sum_{k=1}^{2m} G_k\,\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\varphi_k\right\} \tag{47}$$

Consider now Eq. (45). In order to simplify writing, we may introduce the function,

$$N_k(r,\alpha) = 1+\sum_{n=1}^{m-1} 2r^n\cos n\alpha + r^m\cos m\alpha \tag{48}$$

which is of the same type as Eq. (45).

By using the familiar relations,

$$\begin{aligned} r^n e^{in\alpha} &= r^n(\cos n\alpha + i\sin n\alpha) \\ r^n e^{-in\alpha} &= r^n(\cos n\alpha - i\sin n\alpha) \end{aligned} \tag{49}$$

we find that,

$$2r^n\cos n\alpha = r^n\left(e^{in\alpha}+e^{-in\alpha}\right)$$

and hence,

$$N_k(r,\alpha) = 1+\sum_{n=1}^{m-1} r^n\left(e^{in\alpha}+e^{-in\alpha}\right) + \frac{r^m}{2}\left(e^{im\alpha}+e^{-im\alpha}\right) \tag{50}$$

Consider now the series,

$$\begin{aligned} &\frac{1}{2}+x+x^2+x^3+\cdots\cdots\cdots\cdots+x^{m-1}+\frac{1}{2}x^m \\ &= \frac{1+x}{2}\left[1+x+x^2+x^3+\cdots\cdots\cdots\cdots+x^{m-1}\right] \end{aligned} \tag{51}$$

The sum of the series on the right side is known to be,

$$S = \frac{1+x}{2} \cdot \frac{1-x^m}{1-x} \tag{52}$$

If we substitute in Eq. (51) first $x = re^{i\alpha}$ and then secondly $x = re^{-i\alpha}$ and add both series, we obtain,

$$1 + r(e^{i\alpha} + e^{-i\alpha}) + r^2(e^{i2\alpha} + e^{-i2\alpha}) + \cdots\cdots + r^{m-1}(e^{i(m-1)\alpha} + e^{-i(m-1)\alpha}) +$$

$$+ \frac{r^m}{2}(e^{im\alpha} + e^{-im\alpha})$$

$$= 1 + \sum_{n=1}^{m-1} r^n (e^{in\alpha} + e^{-in\alpha}) + \frac{r^m}{2}(e^{im\alpha} + e^{-im\alpha}) = N_k(r, \alpha)$$

Using Eq. (52), the sum of these two series is,

$$S_1 + S_2 = \frac{1+re^{i\alpha}}{2} \cdot \frac{1-r^m e^{im\alpha}}{1-re^{i\alpha}} + \frac{1+re^{-i\alpha}}{2} \cdot \frac{1-r^m e^{-im\alpha}}{1-re^{-i\alpha}}$$

$$= N_k(r, \alpha)$$

This can also be written in the form,

$$N_k(r,\alpha) = \frac{1}{2}\left[\frac{1+re^{i\alpha}}{1-re^{i\alpha}} + \frac{1+re^{-i\alpha}}{1-re^{-i\alpha}}\right] - \frac{r^m}{2}\left[e^{im\alpha}\frac{1+re^{i\alpha}}{1-re^{+i\alpha}} - e^{-im\alpha}\frac{1+re^{-i\alpha}}{1-re^{-i\alpha}}\right]$$

Substituting Eq. (49) and simplifying the resulting expression, we obtain,

$$N_k(r,\alpha) = \frac{(1-r^2)(1-r^m \cos m\alpha)}{1-2r\cos\alpha + r^2} + \frac{r^{m+1}\, 2\sin\alpha \sin m\alpha}{1-2r\cos\alpha + r^2}$$

$$N_k(r,\alpha) = \frac{(1-r^2)(1-r^m \cos m\alpha) + r^{m+1}\, 2\sin\alpha \sin m\alpha}{1-2r\cos\alpha + r^2} \tag{53}$$

Recalling that $N_k(r,\alpha)$ represents essentially the function φ_k, we may write,

$$\varphi_k\left(r, \theta - \frac{k\pi}{m}\right) = \frac{\left(1-\frac{r^2}{\rho^2}\right)\left[1-\left(\frac{r}{\rho}\right)^m \cos m\left(\theta - \frac{k\pi}{m}\right)\right] + \left(\frac{r}{\rho}\right)^{m+1} 2 \sin\left(\theta - \frac{k\pi}{m}\right) \sin m\left(\theta - \frac{k\pi}{m}\right)}{1 - 2\left(\frac{r}{\rho}\right)\cos\left(\theta - \frac{k\pi}{m}\right) + \left(\frac{r}{\rho}\right)^2} \tag{54}$$

With Eqs. (47) and (54), we have arrived at a simpler expression for $F(r,\theta)$ than Eq. (43) was. The double summation over k and n has been reduced to a single summation over k. However, the resulting expressions are still too complicated for practical purposes and may therefore be simplified further.

Practical Aspects:

The purpose of this numerical method is to find the numerical values of the general stress function $F(r, \theta)$ at certain selected points inside the circular plate. A grid may be laid over the plate as shown in Figure 13. We are then interested in the numerical values at the grid points, e. g. in the values $F\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$ where $\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$ indicates the position of the grid points on the plate. The notations l, n, k, and m may be understood from Figure 13. Later we shall also use the short notation F_k^l to give the position of the point on the plate.

If we use a grid with constant angle $\Delta\theta$ between the rays, the equation (54) for φ_k can be reduced to the case in which $\theta = 0$. Then, in Eq. (54)

$$\cos m\left(\theta - \frac{k\pi}{m}\right) = \cos k\pi = (-1)^k$$

$$\left(\frac{r}{\rho}\right)^{m+1} 2 \sin\left(\theta - \frac{k\pi}{m}\right) \sin m\left(\theta - \frac{k\pi}{m}\right) = 0$$

$$\cos\left(\theta - \frac{k\pi}{m}\right) = \cos \frac{k\pi}{m}$$

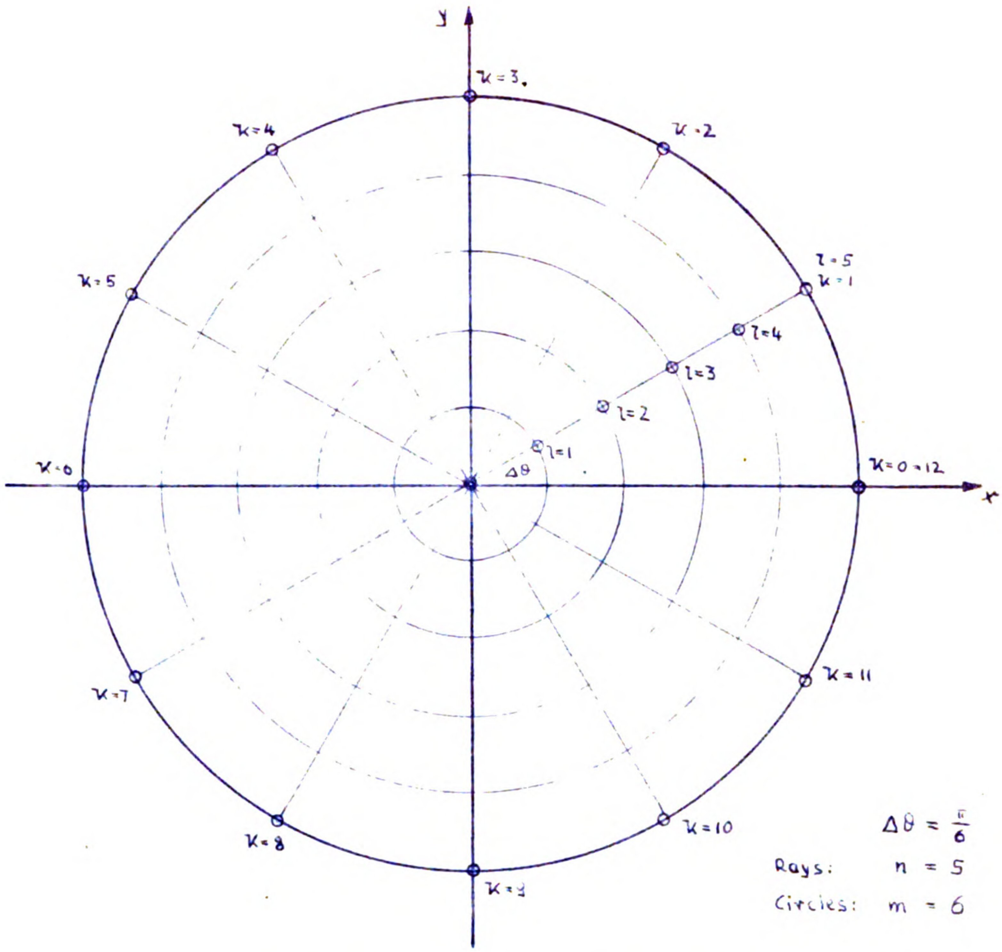


FIG. 13: EXAMPLE FOR A GRID OF A CIRCULAR PLATE

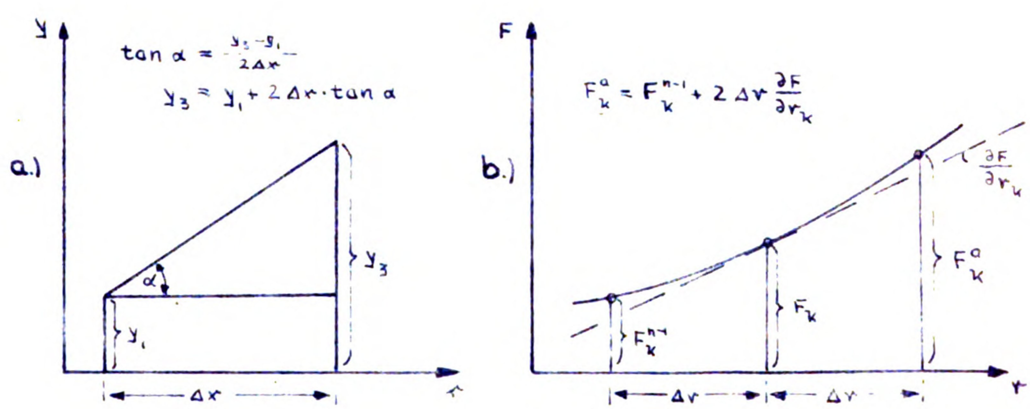


FIG. 14: VALUES OF THE STRESS FUNCTION OUTSIDE THE PLATE

Eq. (54) can then be written in the form,

$$\varphi_k\left(\frac{l\rho}{n}, -\frac{k\pi}{m}\right) = \frac{\left(1-\frac{r^2}{\rho^2}\right)\left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]}{1-2\left(\frac{r}{\rho}\right)\cos\frac{k\pi}{m}+\left(\frac{r}{\rho}\right)^2} \tag{55}$$

This is strictly valid only for the values $F\left(\frac{l\rho}{n}, 0\right)$, e.g., for the values at the points lying on the ray $\theta = 0$. However, since we have a summation around the boundary, according to Eq. (47), we can use the same set of values φ_k, Eq. (55), for all $F\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$ if we "rotate" the values φ_k in clockwise direction for the same number of angles $\Delta\theta$ as k in $F\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$ amounts to. An example, using Eq. (47) may illustrate the procedure:

<u>$k=0$:</u>

$$F\left(\frac{l\rho}{n}, 0\right) = \frac{1}{2m}\left\{\sum_{k=1}^{2m} F_k\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r\frac{\partial}{\partial r}\right]\varphi_k - \right.$$
$$\left. - \sum_{k=1}^{2m} G_k \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\varphi_k\right\}$$

<u>$k=1$:</u>

$$F\left(\frac{l\rho}{n}, \frac{1\pi}{m}\right) = \frac{1}{2m}\left\{\sum_{k=1}^{2m} F_k\left[\cdots\cdots\right]\varphi_{k-1} - \sum_{k=1}^{2m} G_k \cdots\cdots \varphi_{k-1}\right\}$$

<u>$k=5$:</u>

$$F\left(\frac{l\rho}{n}, \frac{5\pi}{m}\right) = \frac{1}{2m}\left\{\sum_{k=1}^{2m} F_k\left[\cdots\cdots\right]\varphi_{k-5} - \sum_{k=1}^{2m} G_k \cdots\cdots \varphi_{k-5}\right\}$$

and so forth. By observing this rule, the simple expression for φ_k, Eq. (55), can be used for determination of all $F\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$.

<u>Further Simplification of Eqs. (47) and (55):</u>

Eq. (55) can also be written in the form,

$$\varphi_k = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right] \frac{\frac{1}{2}\left(\frac{\rho}{r}-\frac{r}{\rho}\right)}{\frac{1}{2}\left(\frac{\rho}{r}+\frac{r}{\rho}\right)-\cos\frac{k\pi}{m}} \tag{56}$$
$$= \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]\cdot U_k$$

where,

$$U_k = \frac{\frac{1}{2}\left(\frac{\rho}{r} - \frac{r}{\rho}\right)}{\frac{1}{2}\left(\frac{\rho}{r} + \frac{r}{\rho}\right) - \cos\frac{k\pi}{m}} \tag{57}$$

Considering now the first partial derivative of φ_k with respect to r which appears in Eq. (47), we obtain from Eq. (55),

$$r\frac{\partial\varphi_k}{\partial r} = \frac{-\left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]2\left(\frac{r}{\rho}\right)^2}{1-2\left(\frac{r}{\rho}\right)\cos\frac{k\pi}{m} + \left(\frac{r}{\rho}\right)^2} + \frac{\left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]\left(1-\frac{r^2}{\rho^2}\right)\cdot 2\left(\frac{r}{\rho}\right)\left[\cos\frac{k\pi}{m} - \left(\frac{r}{\rho}\right)\right]}{\left[1-2\left(\frac{r}{\rho}\right)\cos\frac{k\pi}{m} + \left(\frac{r}{\rho}\right)^2\right]^2} - $$

$$- \frac{m(-1)^k\left(\frac{r}{\rho}\right)^m\left(1-\frac{r^2}{\rho^2}\right)}{1-2\left(\frac{r}{\rho}\right)\cos\frac{k\pi}{m} + \left(\frac{r}{\rho}\right)^2}$$

This equation can be brought by algebraic means to the following form,

$$r\frac{\partial\varphi_k}{\partial r} = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]\left[U_k^2 - \frac{\frac{1}{2}\left(\frac{\rho}{r}+\frac{r}{\rho}\right)}{\frac{1}{2}\left(\frac{\rho}{r}+\frac{r}{\rho}\right) - \cos\frac{k\pi}{m}} - \frac{m(-1)^k\left(\frac{r}{\rho}\right)^m}{1-(-1)^k\left(\frac{r}{\rho}\right)^m}\cdot U_k\right] \tag{58}$$

where U_k is the same as defined by Eq. (57). In Eq. (47), $r\frac{\partial\varphi_k}{\partial r}$ is multiplied by $\frac{1}{2}(1 - \frac{r^2}{\rho^2})$. Using Eq. (58), this product gives,

$$\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)r\frac{\partial\varphi_k}{\partial r} = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]\left[\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)U_k^2 - \frac{\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\frac{1}{2}\left(\frac{\rho}{r}+\frac{r}{\rho}\right)}{\frac{1}{2}\left(\frac{\rho}{r}+\frac{r}{\rho}\right) - \cos\frac{k\pi}{m}} - \right.$$

$$\left. -\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\frac{m(-1)^k\left(\frac{r}{\rho}\right)^m}{1-(-1)^k\left(\frac{r}{\rho}\right)^m}\cdot U_k\right]$$

and by contracting and rearranging,

$$\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)r\frac{\partial\varphi_k}{\partial r} = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right]\cdot U_k\cdot$$

$$\cdot\left[\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\left\{U_k - \frac{m(-1)^k\left(\frac{r}{\rho}\right)^m}{1-(-1)^k\left(\frac{r}{\rho}\right)^m}\right\} - \frac{1}{2}\left(1+\frac{r^2}{\rho^2}\right)\right] \tag{59}$$

Considering now the term,

$$\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r \frac{\partial}{\partial r}\right] \varphi_k \qquad \text{in eq. (47)}$$

we can write,

$$\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r \frac{\partial}{\partial r}\right] \varphi_k = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right] U_k + \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r \frac{\partial \varphi_k}{\partial r}$$

If we substitute Eq. (59) for the last term on the right-hand side of this equation, and contract the resulting expression, we obtain,

$$\left[1+\frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) r \frac{\partial}{\partial r}\right] \varphi_k = \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right] \cdot \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) U_k \left[U_k + 1 - \frac{m(-1)^k\left(\frac{r}{\rho}\right)^m}{1-(-1)^k\left(\frac{r}{\rho}\right)^m}\right] \qquad (60)$$

Using Eqs. (60) and (56), Eq. (47) for the stress function at the grid points becomes,

$$F\left(\frac{l\rho}{n},0\right) = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k \left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right] \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right) U_k \left[U_k + 1 - \frac{m(-1)^k\left(\frac{r}{\rho}\right)^m}{1-(-1)^k\left(\frac{r}{\rho}\right)^m}\right] - \right.$$
$$\left. - G_k \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\left[1-(-1)^k\left(\frac{r}{\rho}\right)^m\right] \cdot U_k \right\} \qquad (61)$$

We now introduce a new notation and can write Eq. (61) in the form,

$$F\left(\frac{l\rho}{n},0\right) = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k \cdot V_k \cdot W_k - G_k \cdot V_k \right\} \qquad (62)$$

in which,

$$U_k = \frac{\frac{1}{2}\left(\frac{\rho}{r} - \frac{r}{\rho}\right)}{\frac{1}{2}\left(\frac{\rho}{r} + \frac{r}{\rho}\right) - \cos\frac{k\pi}{m}} \tag{57}$$

$$V_k = \frac{1}{2}\left(1 - \frac{r^2}{\rho^2}\right)\left[1 - (-1)^k \left(\frac{r}{\rho}\right)^m\right] \cdot U_k \tag{63}$$

$$W_k = U_k + 1 - \frac{m(-1)^k \left(\frac{r}{\rho}\right)^m}{1 - (-1)^k \left(\frac{r}{\rho}\right)^m} \tag{64}$$

If we set,

$$Z_k = V_k \cdot W_k \tag{65}$$

we arrive at the following final expression for the stress function at the point $\left(\frac{2\rho}{n}, 0\right)$:

$$F\left(\frac{2\rho}{n}, 0\right) = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k \cdot Z_k - G_k \cdot V_k \right\} \tag{66}$$

The stress function for other points than those lying on the ray $\theta = 0$ are obtained by rotating the values of Z_k and V_k in clockwise direction as mentioned above in connection with φ_k . For instance,

$$\underline{k=1:} \qquad F\left(\frac{2\rho}{n}, \frac{\pi}{m}\right) = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k \cdot Z_{k-1} - G_k \cdot V_{k-1} \right\}$$

$$\underline{k=3:} \qquad F\left(\frac{2\pi}{m}, \frac{3\pi}{m}\right) = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k \cdot Z_{k-3} - G_k \cdot V_{k-3} \right\} \tag{66a}$$

and so forth.

Value of the Stress Function at the Center of the Plate:

The numerical value of the stress function at the center of the plate is given by,

$$F_0 = \frac{1}{2m} \sum_{k=1}^{2m} \left\{ F_k - \frac{1}{2} G_k \right\} \tag{67}$$

This follows immediately from the Eqs. (57), (63) to (66) by setting r=0. In this case,

$$U_k = 1 \;; \quad V_k = \frac{1}{2} \;; \quad W_k = 2 \;; \quad Z_k = 1$$

Values of the Stress Function Outside the Plate:

In order to determine the stress components on the boundary by means of finite differences, we need the numerical values of the stress function on a circle outside the plate which is drawn at the same distance Δr from the boundary circle as the other concentric circles of the grid inside the plate are spaced.

If we know the value of the stress function at a grid point on the circle next to the boundary as well as the slope of the stress surface on the corresponding boundary point, the value of the stress function on the circle outside the plate, which lies on the same ray, can be determined approximately by extrapolation.

From Figure 14a, it is seen that the tangent of the angle α is given by,

$$\tan \alpha = \frac{y_3 - y_1}{2\Delta r}$$

It follows for the ordinate y_3,

$$y_3 = y_1 + 2\Delta r \tan \alpha$$

The tangent α represents the slope of the function at a certain point. For the case of a stress surface representing the stress function $F(r,\vartheta)$, the tan α corresponds with the slope of the stress surface in a direction normal to the boundary. We have seen in section 6 that this slope can be obtained from the external forces for all boundary points. Hence, for points lying on the same ray of the grid, the value of the stress function at a grid point on the circle outside the plate is given by,

$$F_k^a = F_k^{n-1} + 2\,\Delta r\,\frac{\partial F}{\partial r_k} \tag{68}$$

in which F_k^{n-1} is the value of the stress function on the grid circle inside the plate and next to its boundary and $\frac{\partial F}{\partial r_k}$ is the slope of the stress surface at the corresponding boundary point k. Figure 14b shows a cross section of the imaginary stress surface on the boundary.

Noting that

$$\frac{\partial F}{\partial r_k} = \frac{G_k}{\rho} \quad \text{and} \quad \Delta r = \frac{\rho}{n}$$

where n is the number of concentric circles of a certain grid chosen, Eq. (68) becomes,

$$F_k^a = F_k^{n-1} + \frac{2}{n}\,G_k \tag{69}$$

<u>Practical Application of the Derived Formulas:</u>

It is seen that U_k, V_k, W_k, and Z_k are constant for a certain grid and independent of the individual problem as far as the size of the circular plate and the loading conditions are concerned. Using Eqs. (57), (63), (64), and (65), these constants may be evaluated for one, two, or more standard grids taking a different number of points for each of the standard grids. In solving an individual problem, it is advisable to use only standard grids for which the coefficients have already been evaluated and tabulated. In this case, it is only necessary to determine the boundary values of the stress function, F_k and G_k, from the external forces (Section 6) for the corresponding grid points, and to perform the multiplication and summation according to Eq. (66).

The approximate determination of the stress components from these numerical values of the stress function is shown in the following section.

10.) APPROXIMATE DETERMINATION OF STRESS COMPONENTS BY FINITE DIFFERENCES

Using the numerical method, Section 9, only numerical values of the stress function at selected points are obtained. Hence, we cannot determine the stress components by differentiation according to the following equations,

$$
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \vartheta^2} \\
\sigma_\vartheta &= \frac{\partial^2 F}{\partial r^2} \\
\tau_{r\vartheta} &= \frac{1}{r^2}\frac{\partial F}{\partial \vartheta} - \frac{1}{r}\frac{\partial^2 F}{\partial r \partial \vartheta}
\end{aligned}
\tag{6}
$$

The differentials in Eqs. (6) must rather be replaced by finite differences. To determine the stress components at the point $\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$, we use the numerical values of the stress function at that point and in its neighborhood.

The first derivative of a function at a point $\left(\frac{l\rho}{n}, \frac{k\pi}{m}\right)$, or briefly $\binom{l}{k}$, can be expressed approximately by the difference of the values of that function at two neighboring points, divided by the distance between those two points. Thus we obtain the following relations, (See Figure 15):

$$
\begin{aligned}
\frac{\partial F}{\partial r} &\approx \frac{F_k^{l+1} - F_k^{l-1}}{2\Delta r} \\
\frac{\partial^2 F}{\partial r^2} &\approx \frac{F_k^{l+1} - 2F_k^l + F_k^{l-1}}{\Delta r^2} \\
\frac{\partial F}{\partial \vartheta} &\approx \frac{F_{k+1}^l - F_{k-1}^l}{2\Delta\vartheta} \\
\frac{\partial^2 F}{\partial \vartheta^2} &\approx \frac{F_{k+1}^l - 2F_k^l + F_{k-1}^l}{\Delta\vartheta^2} \\
\frac{\partial^2 F}{\partial r \partial \vartheta} &\approx \frac{F_{k+1}^{l+1} - F_{k+1}^{l-1} - F_{k-1}^{l+1} + F_{k-1}^{l-1}}{4\Delta r \Delta\vartheta}
\end{aligned}
\tag{70}
$$

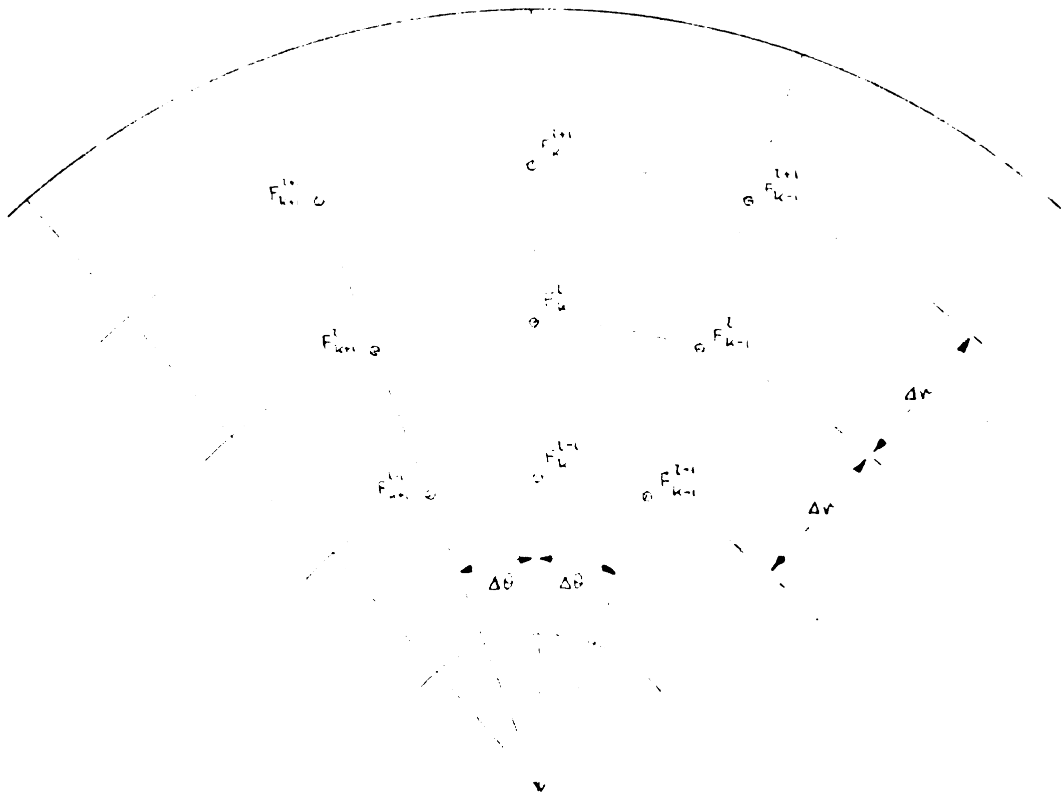


FIG. [illegible] GRID POINTS FOR THE DETERMINATION OF STRESS COMPONENTS AT POINT $(k)^{l}$ BY USING FINITE DIFFERENCE.

If we replace the differentials in Eqs. (6) by the expressions (70), we obtain,

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\,\frac{F_k^{l+1}-F_k^{l-1}}{2\Delta r} + \frac{1}{r^2}\,\frac{F_{k+1}^{l}-2F_k^{l}+F_{k-1}^{l}}{\Delta\theta^2} \\
\sigma_\theta &= \frac{F_k^{l+1}-2F_k^{l}+F_k^{l-1}}{\Delta r^2} \\
\tau_{r\theta} &= \frac{1}{r^2}\,\frac{F_{k+1}^{l}-F_{k-1}^{l}}{2\Delta\theta} - \frac{1}{r}\,\frac{F_{k+1}^{l+1}-F_{k+1}^{l-1}-F_{k-1}^{l+1}+F_{k-1}^{l-1}}{4\,\Delta r\,\Delta\theta}
\end{aligned} \tag{71}$$

Noting that,

$$\Delta r = \frac{\rho}{n}\,; \qquad r = \frac{l\rho}{n}\,; \qquad \Delta\theta = \frac{\pi}{m} \tag{72}$$

Eqs. (71) become,

$$\begin{aligned}
\sigma_r &= \frac{n^2}{2l\rho^2}\left[F_k^{l+1}-F_k^{l-1}+\frac{2m^2}{\pi^2 l}\left(F_{k+1}^{l}-2F_k^{l}+F_{k-1}^{l}\right)\right] \\
\sigma_\theta &= \frac{n^2}{\rho^2}\left[F_k^{l+1}-2F_k^{l}+F_k^{l-1}\right] \\
\tau_{r\theta} &= \frac{mn^2}{2\pi l^2\rho^2}\left[F_{k+1}^{l}-F_{k-1}^{l}-\frac{l}{2}\left(F_{k+1}^{l+1}-F_{k+1}^{l-1}-F_{k-1}^{l+1}+F_{k-1}^{l-1}\right)\right]
\end{aligned} \tag{73}$$

<u>Stress</u> <u>Components</u> <u>for</u> <u>a</u> <u>Particular</u> <u>Standard</u> <u>Grid</u>:

For a standard grid with m = 12, n = 10 (Figure 17), Eqs. (73) become,

$$\begin{aligned}
\sigma_r &= \frac{50}{l\rho^2}\left[F_k^{l+1}-F_k^{l-1}+\frac{288}{\pi^2 l}\left(F_{k+1}^{l}-2F_k^{l}+F_{k-1}^{l}\right)\right] \\
\sigma_\theta &= \frac{100}{\rho^2}\left[F_k^{l+1}-2F_k^{l}+F_k^{l-1}\right] \\
\tau_{r\theta} &= \frac{600}{\pi l^2\rho^2}\left[F_{k+1}^{l}-F_{k-1}^{l}-\frac{l}{2}\left(F_{k+1}^{l+1}-F_{k+1}^{l-1}-F_{k-1}^{l+1}+F_{k-1}^{l-1}\right)\right]
\end{aligned} \tag{74}$$

11.) NUMERICAL EXAMPLE

The case of a circular plate, shown in Figure 10, may now be investigated. Two uniformly distributed loads of the intensity q act on an arch length of $2\rho\beta$ on the boundary. Let the angle β be $10^\circ = \frac{\pi}{18}$. Using the boundary method, the variation of the radial and tangential stress on the horizontal and vertical diameter, respectively, is to be determined. Since the loading is symmetric with respect to the x- axis as well as with respect to the y-axis, it is sufficient to find the stress variation for one-half of each diameter.

(a) Stress Components by Using the First Modification Developed in Section 7.

In example 2, Section 7, the radial and tangential stress components were found to be,

$$\sigma_r = c_0 - \Big\{ \sum_{n=1}^{\infty} a_{2n}\, 2n\,(2n-1)\, r^{2n-2} \cos 2n\theta + \sum_{n=1}^{\infty} c_{2n}\,(2n-2)\,(2n+1)\, r^{2n} \cos 2n\theta \Big\}$$

$$\sigma_\theta = c_0 + \sum_{n=1}^{\infty} a_{2n}\, 2n\,(2n-1)\, r^{2n-2} \cos 2n\theta + \sum_{n=1}^{\infty} c_{2n}\,(2n+2)\,(2n+1)\, r^{2n} \cos 2n\theta \qquad (32)$$

in which,

$$c_0 = -\frac{2q\beta}{\pi}$$

$$a_{2n} = (-1)^n \frac{q \sin 2n\beta}{n(2n-1)\,\pi\,\rho^{2n-2}} \qquad (29)$$

$$c_{2n} = -(-1)^n \frac{q \sin 2n\beta}{n(2n+1)\,\pi\,\rho^{2n}}$$

Inserting Eqs. (29) in Eqs. (32), we obtain,

$$\sigma_r = -\frac{2q}{\pi}\left\{\beta + \sum_{n=1}^{\infty}(-1)^n \sin 2n\beta \left(\frac{r}{\rho}\right)^{2n-2}\cos 2n\theta - \right.$$
$$\left. - \sum_{n=1}^{\infty}(-1)^n \frac{n-1}{n}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\cos 2n\theta\right\} \qquad (73)$$

$$\sigma_\theta = -\frac{2q}{\pi}\left\{\beta - \sum_{n=1}^{\infty}(-1)^n \sin 2n\beta \left(\frac{r}{\rho}\right)^{2n-2}\cos 2n\theta + \right.$$
$$\left. + \sum_{n=1}^{\infty}(-1)^n \frac{n+1}{2}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\cos 2n\theta\right\}$$

For the right half of the horizontal diameter ($\theta=0$), this reduces to,

$$\sigma_r = -\frac{2q}{\pi}\left\{\beta + \sum_{n=1}^{\infty}(-1)^n \sin 2n\beta\left(\frac{r}{\rho}\right)^{2n-2} - \sum_{n=1}^{\infty}(-1)^n\frac{n-1}{n}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\right\}$$

$$\sigma_\theta = -\frac{2q}{\pi}\left\{\beta - \sum_{n=1}^{\infty}(-1)^n \sin 2n\beta\left(\frac{r}{\rho}\right)^{2n-2} + \sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\right\} \qquad (74)$$

In calculating the stress variation on one-half of the vertical diameter, we note that for $\theta = \frac{\pi}{2}$,

$$\cos 2n\theta = \cos n\pi = (-1)^n$$

Eq. (73) then becomes,

$$\sigma_r = -\frac{2q}{\pi}\left\{\beta + \sum_{n=1}^{\infty}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n-2} - \sum_{n=1}^{\infty}\frac{n-1}{n}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\right\}$$

$$\sigma_\theta = -\frac{2q}{\pi}\left\{\beta - \sum_{n=1}^{\infty}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n-2} + \sum_{n=1}^{\infty}\frac{n+1}{n}\sin 2n\beta\left(\frac{r}{\rho}\right)^{2n}\right\} \qquad (75)$$

The results of the evaluation of Eqs. (74) and (75) are given in the first columns of Tables 6 to 9 and are shown in Diagrams 1 and 2.

The stress components are determined for the following points on the horizontal and vertical diameter, respectively,

$$r = 0.1\rho\ ,\ 0.2\rho\ ,\ 0.3\rho\ , \cdots\cdots\ 0.9\rho$$

For this computation, only the first ten terms of the infinite series in Eqs. (74) and (75) were used. It is seen that the convergence of these series depends primarily on the ratios $(\frac{r}{\rho})^{2n-2}$ and $(\frac{r}{\rho})^{2n}$. While the convergence is very rapid for points near the origin, it becomes slower and slower towards the boundary. On the boundary itself, the series are divergent.

The divergence of the series on the boundary can be avoided by contracting the two summation expressions in Eq. (73). Then, for each stress component, two new series originate, one of which converges also on the boundary while the other can be replaced by a respective formula for the sum of an infinite and convergent series.

Although this is not of practical importance, the computation of the stress components has been carried out to the fifth decimal in order to supply a good comparison with the values obtained by using the other modifications of the boundary method.

(b) Stress Components by Using the Second Modification Developed in Section 8

Figure 16 shows the loading and the boundary points at which the boundary values of the stress function are used for the determination of the Fourier coefficients. Thirty-six points k are spaced at equal intervals $\Delta\theta = 10^{\circ}$ around the boundary.

Boundary Values:

Taking the starting point at (ρ, o), the boundary values are given by (See section 6 for method),

$$
\begin{array}{ll}
F_k = G_k = 0 & \text{if } 0 \leqq k \leqq 8 \\
F_k = G_k = -q\beta\rho^2 \sin 5^{\circ} & \text{if } k = 9 \\
F_k = G_k = -2q\beta\rho^2 |\cos\theta| & \text{if } 10 \leqq k \leqq 26 \\
F_k = G_k = -q\beta\rho^2 \sin 5^{\circ} & \text{if } k = 27 \\
F_k = G_k = 0 & \text{if } 28 \leqq k \leqq 36
\end{array}
$$

The evaluation leads to the following results,

k	$F_k = G_k$
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	-0.04 358 C
10	-0.17 365 C
11	-0.34 202 C
12	-0.50 000 C
13	-0.64 279 C
14	-0.76 604 C
15	-0.86 603 C
16	-0.93 969 C
17	-0.98 481 C
18	-1.00 000 C

k	$F_k = G_k$
19	-0.98 481 C
20	-0.93 969 C
21	-0.86 603 C
22	-0.76 604 C
23	-0.64 279 C
24	-0.50 000 C
25	-0.34 202 C
26	-0.17 365 C
27	-0.04 358 C
28	0
29	0
30	0
31	0
32	0
33	0
34	0
35	0
36	0

in which $C = \rho^2 \frac{\pi}{9} q$. It is seen that the boundary values are symmetrical with respect to the x - axis.

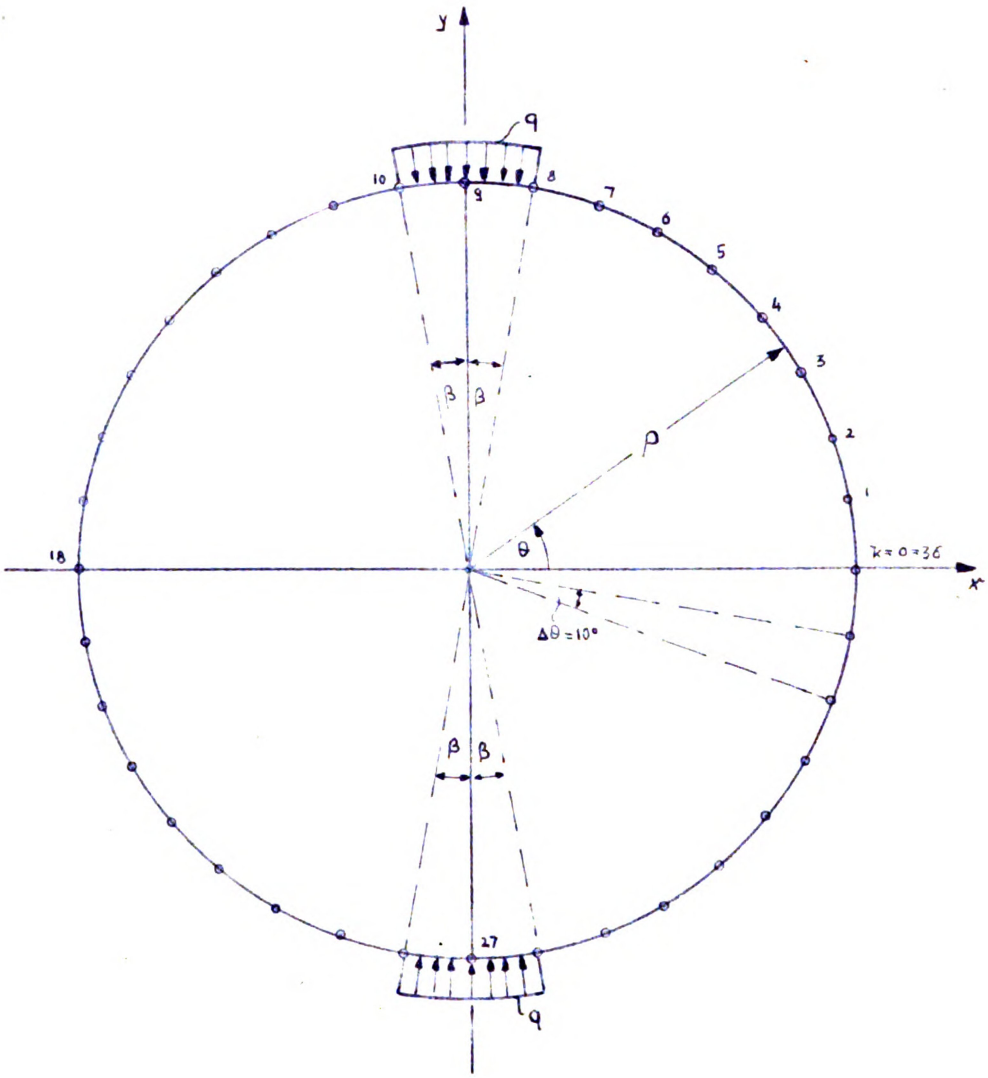


FIG. 16: PLATE WITH SET OF BOUNDARY POINTS FOR THE NUMERICAL EXAMPLE (SECOND MODIFICATION)

<u>Fourier Coefficients of the Stress Function:</u>

The Fourier coefficients of the stress function, Eq. (37), are then determined according to Eqs. (42). It is seen immediately that all sine terms drop out since the sine is antimetric with respect to the x - axis, while the boundary values are symmetric with respect to the same axis. Hence,

$$b_n = 0 \quad ; \quad d_n = 0$$

Noting further that $F_k = G_k$ and $m = 18$, Eqs. (42) reduce to,

$$\begin{aligned} a_o &= \frac{1}{36} \sum_{k=1}^{36} F_k & a_n &= \frac{1}{36\rho^n} \sum_{k=1}^{36} (n+1)\, F_k \cos \frac{nk\pi}{18} \\ c_o &= \frac{1}{36\rho^2} \sum_{k=1}^{36} F_k & c_n &= -\frac{1}{36\rho^{n+2}} \sum_{k=1}^{36} (n-1)\, F_k \cos \frac{nk\pi}{18} \end{aligned} \tag{76}$$

The evaluation of equations (76) leads to the following results,

n	$a_n \cdot \rho^n$
0	-0.31 992 C
1	0.50 000 C
2	-0.31 349 C
4	0.09 812 C
6	-0.05 270 C
8	0.03 173 C
10	-0.01 917 C
12	0.01 075 C
14	-0.00 502 C
16	0.00 146 C
18	-0.00 016 C

n	$c_n \cdot \rho^{n+2}$
0	-0.31 992 C
1	0
2	0.10 450 C
4	-0.05 887 C
6	0.03 763 C
8	-0.02 468 C
10	0.01 568 C
12	-0.00 910 C
14	0.00 435 C
16	-0.00 129 C
18	0.00 014 C

All add a_n's and c_n's except a_1 vanish.

<u>Stress Components:</u>

From Eqs. (31) which give an expression for the stress components, it is seen that the term with the coefficient a_1 vanishes. Hence, we have again only even cosine terms for the determination of the stresses, and the equations become as follows,

$$
\begin{aligned}
\sigma_r = c_0 - \Big\{ & \sum_{n=1}^{\frac{1}{2}(m-2)} a_{2n}\, 2n\,(2n-1)\, r^{2n-2} \cos 2n\theta + \frac{a_m}{2}\, m\,(m-1)\, r^{m-2} \cos m\theta + \\
& + \sum_{n=1}^{\frac{1}{2}(m-2)} c_{2n}\, 2(n-1)(2n+1)\, r^{2n} \cos 2n\theta + \frac{c_m}{2}\,(m-2)(m+1)\, r^{m} \cos m\theta \Big\}
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
\sigma_\theta = c_0 + & \sum_{n=1}^{\frac{1}{2}(m-2)} a_{2n}\, 2n\,(2n-1)\, r^{2n-2} \cos 2n\theta + \frac{a_m}{2}\, m\,(m-1)\, r^{m-2} \cos m\theta + \\
& + \sum_{n=1}^{\frac{1}{2}(m-2)} c_{2n}\, 2(n+1)(2n+1)\, r^{2n} \cos 2n\theta + \frac{c_m}{2}\,(m+2)(m+1)\, r^{m} \cos m\theta
\end{aligned}
$$

In order to obtain a convenient expression, we introduce a new notation,

$$
\begin{aligned}
c_0 &= \frac{\bar{c}_0}{\rho^2}\, C &&= \bar{c}_0\, \frac{\pi}{8} q \\
a_{2n} &= \frac{\bar{a}_{2n}}{\rho^{2n}}\, C &&= \bar{a}_{2n}\, \frac{1}{\rho^{2n-2}}\, \frac{\pi}{8} q \\
a_m &= \frac{\bar{a}_m}{\rho^m}\, C &&= \bar{a}_m\, \frac{1}{\rho^{m-2}}\, \frac{\pi}{8} q \\
c_{2n} &= \frac{\bar{c}_{2n}}{\rho^{2n+2}}\, C &&= \bar{c}_{2n}\, \frac{1}{\rho^{2n}}\, \frac{\pi}{8} q \\
c_m &= \frac{\bar{c}_m}{\rho^{m+2}}\, C &&= \bar{c}_m\, \frac{1}{\rho^m}\, \frac{\pi}{8} q
\end{aligned}
$$

where $C = \rho^2 \frac{\pi}{8} q$. The new coefficients $\bar{c}_0, \bar{a}_{2n}$, etc. represent merely the numerical part of the coefficients c_0, a_{2n} , etc. Taking the constant factor $\frac{\pi}{8} q$ in front, we can now write Eqs. (77) in the form,

$$\sigma_r = \frac{\pi}{9} q \left\{ \bar{c}_0 - \sum_{n=1}^{\frac{1}{2}(m-2)} \bar{a}_{2n}\, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} \cos 2n\theta - \frac{\bar{a}_m}{2}\, m(m-1) \left(\frac{r}{\rho}\right)^{m-2} \cos m\theta - \sum_{n=1}^{\frac{1}{2}(m-2)} \bar{c}_{2n}\, 2(n-1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} \cos 2n\theta - \frac{\bar{c}_m}{2} (m-2)(m+1) \left(\frac{r}{\rho}\right)^{m} \cos m\theta \right\} \tag{78}$$

$$\sigma_\theta = \frac{\pi}{9} q \left\{ \bar{c}_0 + \sum_{n=1}^{\frac{1}{2}(m-2)} \bar{a}_{2n}\, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} \cos 2n\theta + \frac{\bar{a}_m}{2}\, m(m-1) \left(\frac{r}{\rho}\right)^{m-2} \cos m\theta + \sum_{n=1}^{\frac{1}{2}(m-2)} \bar{c}_{2n}\, 2(n+1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} \cos 2n\theta + \frac{\bar{c}_m}{2} (m+2)(m+1) \left(\frac{r}{\rho}\right)^{m} \cos m\theta \right\}$$

The equations for the stress variation in the right half of the horizontal diameter ($\theta = 0$) are then (with $m = 18$),

$$\sigma_r = \frac{\pi}{9} q \left\{ \bar{c}_0 - \sum_{n=1}^{8} \bar{a}_{2n}\, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} - \frac{\bar{a}_{18}}{2} \cdot 306 \left(\frac{r}{\rho}\right)^{16} - \sum_{n=1}^{8} \bar{c}_{2n}\, 2(n-1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} - \frac{\bar{c}_{18}}{2} \cdot 304 \left(\frac{r}{\rho}\right)^{18} \right\}$$

$$\sigma_\theta = \frac{\pi}{9} q \left\{ \bar{c}_0 + \sum_{n=1}^{8} \bar{a}_{2n}\, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} + \frac{\bar{a}_{18}}{2} \cdot 306 \left(\frac{r}{\rho}\right)^{16} + \sum_{n=1}^{8} \bar{c}_{2n}\, 2(n+1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} + \frac{\bar{c}_{18}}{2} \cdot 380 \left(\frac{r}{\rho}\right)^{18} \right\} \tag{79}$$

The corresponding equations for the stresses in the upper half of the vertical diameter ($\theta = \frac{\pi}{2}$) are as follows,

$$\sigma_r = \frac{\pi}{9} q \left\{ \bar{c}_0 - \sum_{n=1}^{8} (-1)^n \bar{a}_{2n} \, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} - \frac{\bar{a}_{18}}{2} \, 306 \left(\frac{r}{\rho}\right)^{16} - \right.$$

$$\left. - \sum_{n=1}^{8} (-1)^n \bar{c}_{2n} \, 2(n-1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} - \frac{\bar{c}_{18}}{2} \cdot 304 \left(\frac{r}{\rho}\right)^{18} \right\}$$

$$\sigma_\theta = \frac{\pi}{9} q \left\{ \bar{c}_0 + \sum_{n=1}^{8} (-1)^n \bar{a}_{2n} \, 2n(2n-1) \left(\frac{r}{\rho}\right)^{2n-2} + \frac{\bar{a}_{18}}{2} \cdot 306 \left(\frac{r}{\rho}\right)^{16} + \right. \qquad (80)$$

$$\left. + \sum_{n=1}^{8} (-1)^n \bar{c}_{2n} \, 2(n+1)(2n+1) \left(\frac{r}{\rho}\right)^{2n} + \frac{\bar{c}_{18}}{2} \cdot 380 \left(\frac{r}{\rho}\right)^{18} \right\}$$

Eqs. (79) and (80) are used for the evaluation of the stress components σ_r and σ_θ on the horizontal and vertical diameter, respectively. The results are given in the second columns of Tables 6 to 9 and are shown in Diagrams 1 and 2.

(c) Stress Components by Using the Numerical Method Developed in Section 9

Figure 17 shows the loading conditions and the grid used for the numerical example. The grid consists of 24 rays spaced at equal intervals $\Delta\theta = 15°$ and of 10 concentric circles. Hence,

$$m = 12; \quad n = 10; \quad \Delta\theta = 15° = \frac{\pi}{12}$$

In addition, we have an eleventh circle outside the boundary of the plate.

Boundary Values:

The boundary values are obtained in the same way as in Section 11, b. If $(\rho, 0)$ is the starting point, these values are given by,

$$F_k = G_k = 0 \qquad \text{if } 0 \leq k \leq 5$$

$$F_k = G_k = -q\beta\rho^2 \sin 5° \qquad \text{if } k = 6$$

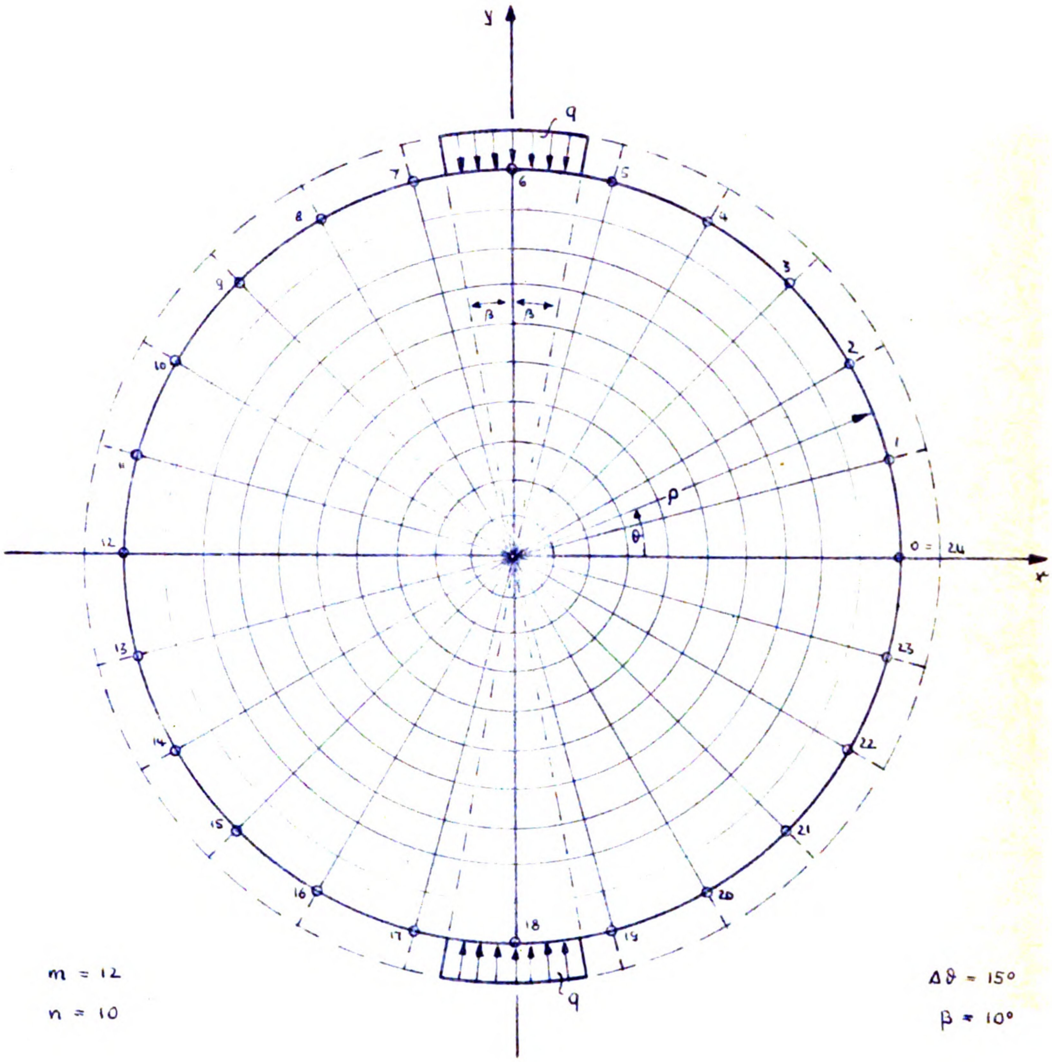


FIG. 17: PLATE WITH A STANDARD GRID FOR THE NUMERICAL EXAMPLE (THIRD MODIFICATION)

$$F_k = G_k = -2q\beta\rho^2 |\cos\theta| \quad \text{if} \quad 7 \leqq k \leqq 17$$

$$F_k = G_k = -q\beta\rho^2 \sin 5^\circ \quad \text{if} \quad k = 18$$

$$F_k = G_k = 0 \quad \text{if} \quad 19 \leqq k \leqq 24$$

This gives the following numerical values,

k	$F_k = G_k$	k	$F_k = G_k$
0	0	13	−0.9659 2583 C
1	0	14	−0.8660 2540 C
2	0	15	−0.7071 0678 C
3	0	16	−0.5000 0000 C
4	0	17	−0.2588 1905 C
5	0	18	−0.0435 7787 C
6	−0.0435 7787 C	19	0
7	−0.2588 1905 C	20	0
8	−0.5000 0000 C	21	0
9	−0.7071 0678 C	22	0
10	−0.8660 2540 C	23	0
11	−0.9659 2583 C	24	0
12	−1.0000 0000 C		

where $\quad C = \rho^2 \frac{\pi}{9} q$

<u>Evaluation of the Grid Constants:</u>

The grid constants U_k, V_k, W_k, and Z_k are evaluated according to Eqs. (57), (63), (64), and (65). The results are found in Tables 1 to 4.

<u>Values of the Stress Function Inside the Plate:</u>

The numerical values of the stress function at the grid points inside the plate are determined by means of Eqs. (66) and (66a). Since we are only interested in the stress variations on the horizontal and vertical diameters, the values of the stress function have not been calculated for all grid points but only for those which we need in the difference equations, (74). These values are given in Table 5.

The value of the stress function at the center of the plate is determined from Eq. (67). We obtain,

$$F_0 = -0.1600\ 6062\ c$$

Values of the Stress Function Outside the Plate:

The numerical values of the stress function on the additional circle outside the plate are obtained from Eq. (69) which becomes, for our example with n = 10,

$$F_k^a = F_k^9 + 0.2\, G_k \tag{81}$$

The result is also found in Table 5.

It should be mentioned that these values obtained from Eq. (81) are only rough approximations since they are gained from linear extrapolation as described in Section 9. However, they are only needed for the difference equations (74) if those stress components on the boundary are required which cannot be seen immediately from the boundary conditions. In our example, for instance, this would be the case with the tangential stress σ_θ at the point ($\rho, \frac{\pi}{2}$). However, one should remember that those stresses obtained by using values F_k^a are likely to be inaccurate.

Stress Components:

The stress components at the grid points are determined according to the difference equations, (74). The results are given in the third column of Tables 6 to 9 and are shown in Diagrams 1 and 2.

12. DISCUSSION OF THE RESULTS

Stress Components by Using Frocht's Formulas:

In order to obtain an estimate of the accuracy of the boundary method, it is desirable to compare the results obtained in the numerical example, Section 11, with the results of a similar problem gained by a well established and generally accepted method.

Frocht has investigated the case of a circular plate loaded with two single concentrated loads acting on the vertical diameter (See Figure 8). He arrives at the following formulas for the stress components on the horizontal and vertical diameter (See Photoelasticity, Vol. II, p. 127):

Horizontal Diameter:

$$\sigma_x = \frac{2P}{\pi t d}\left[\frac{d^2-4x^2}{d^2+4x^2}\right]$$

$$\sigma_y = -\frac{2P}{\pi t d}\left[\frac{4d^4}{(d^2+4x^2)^2}-1\right] \tag{82}$$

Vertical Diameter:

$$\sigma_x = \frac{2P}{\pi t d} = \text{const.}$$

$$\sigma_y = -\frac{2P}{\pi t}\left[\frac{2}{d-2y}+\frac{2}{d+2y}-\frac{1}{d}\right] \tag{83}$$

in which t is the thickness of the plate, d the diameter, and x, y the coordinates of the points on the horizontal and vertical diameter, respectively.

These formulas are obtained from three different superpositions. Each one of the forces P, acting on the edge of a semi-infinite plate, produces a radial stress distribution with respect to the point of application of the respective load. The superposition of these two cases results in uniform compression on the boundary of a circular plate. The third superposition, uniform tension of the same intensity, removes the uniform compression and thus satisfies the boundary conditions.

The formulas gained from superposition of these well known cases give accepted values for the stress components and will be used for a comparison with the boundary method. The loading condition in our numerical example (Figure 10) is slightly different. However, the principle of Saint Venant states that if forces acting on a small portion of the surface of an elastic body are replaced by another statically equivalent system of forces on the same portion of the surface, this redistribution of loading has a negligible effect on the stresses at distances which are large as compared with the linear dimensions of the surface on which the forces are changed. Hence, we may expect essentially the same stress distribution on the horizontal diameter as well as on the portion of the vertical diameter close to the center.

If we take Frocht's load P as of the same magnitude as the resultant of a uniformly distributed load,

$$P = 2\rho\beta q$$

and if we further assume the thickness t as unity, we obtain the stress variation on the horizontal and vertical diameter for Frocht's case of loading (Figure 8) from Eqs. (82) and (83). The results are given in the fourth column of Tables 6 to 9 and are shown in Diagrams 1 and 2.

<u>Comparison of Frocht's Values with Those Obtained from the Boundary Method:</u>

Tables 6 to 9 and Diagrams 1 and 2 allow a comparison between Frocht's values and the values obtained by using the three modifications of the boundary method. It is seen that everywhere where the principle of Saint Venant holds true, e.g. on the horizontal diameter as well as on the vertical diameter close to the center of the plate, all four values are in close agreement. On the horizontal diameter, the four different sets of values could not be plotted as four distinct curves since deviations mostly appear in the third decimal only. Even if these numerical values have still to be multiplied by the load intensity q, the difference is immaterial.

On the vertical diameter, the boundary method leads to finite values for the radial stress on the boundary while Frocht's value approaches infinity. But this is consistent with the different loading conditions. Frocht uses a single concentrated load, and the radial stress at its point of application is equal to infinity, whereas in our example, using a uniformly distributed load, the radial stress at the boundary point of the vertical diameter equals the load intensity q. Modifications I and II furnish values which are too large. The error in modification II amounts to 14% at that point. Modification I would probably lead to a correct value if ten or twenty more terms of the series had been considered in the computation. It was already mentioned that the corresponding series shows a slow convergence for points close to the boundary. At the same point, the third modification gives a value which is smaller than the correct value by 11%. The inaccuracy of modifications II and III on the boundary is due to the fact that the number of boundary points which were considered for a representation of the boundary values of the stress function was not sufficient. If a higher degree of accuracy is required, it is necessary to take one boundary point on each side of the y-axis so that this point lies under the distributed load applied to that part of the boundary.

In our example, for instance, we would have to space the boundary points at a constant angle of $\Delta\theta = 5^\circ$.

On the vertical diameter, the boundary method shows a change in sign for the tangential stress near the boundary, while Frocht gives constant tension. This may again be understood from the different boundary conditions. It seems natural that a uniformly distributed load in radial direction produces a compressive tangential stress at the point of application of its resultant and in the neighborhood of this point.

<u>Comments on the First Modification of the Boundary Method:</u>

The practical application of the first modification of the boundary method seems to be restricted to some special cases where the function expressing the boundary values of the stress function has a continuous first derivative with respect to θ . In those cases, this modification leads to correct values of the stress components provided the computation is carried to a sufficient number of decimals and considers a sufficient number of terms of the infinite series. The disadvantage is that this series is likely to converge very slowly at points close to the boundary. Although ten terms were taken into consideration in our numerical example, the radial stress on the vertical diameter at point $1 = 9$ is obtained as -1.02 168 q while in reality it should be below -1.00 000 q, since -1.00 000 q is the value on the boundary.

In order to avoid the disadvantage of infinite series with their slow convergence, or even divergence on the boundary, Dr. Frame has developed the following formula which should be mentioned in this connection. Its derivation is not given.

$$F(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi}(F\cdot U)\,d\theta - \frac{1}{2}\left(1-\frac{r^2}{\rho^2}\right)\rho\frac{\partial}{\partial\rho}\,\frac{1}{2\pi}\int_0^{2\pi}(F\cdot U)\,d\theta \qquad (84)$$

where F is the function representing the boundary values of the stress function, and U is as follows,

$$U = \frac{1-\left(\frac{r}{\rho}\right)^2}{1-2\left(\frac{r}{\rho}\right)\cos(\theta-\varphi)+\left(\frac{r}{\rho}\right)^2} \tag{85}$$

Eq. (84) can be obtained from our former Eqs. (57), (63), (64), (65), and (66) by letting m approach infinity and by proving that U is a harmonic function, while V, Eq. (63), and Z , Eq. (65), are both biharmonic.

Using Eq. (84), Dr. Frame has shown that the stress function for the case of two single concentrated loads acting on the vertical diameter (Figure 8) is as follows,

$$F(r,\theta) = -\frac{Pr}{\pi}\left[A \cdot \arctan\frac{A}{S'} + S'\right] \tag{86}$$

in which $\qquad A = \cos\theta$

and $\qquad S = \frac{1}{2}\left(\frac{\rho}{r}-\frac{r}{\rho}\right)$

However, considerable difficulties are encountered in integrating Eq. (86) to obtain the case of a uniformly distributed load.(Figure 10.)

In summary, the first modification of the boundary method leads to correct values for the stress components provided the loading conditions are such that the resulting expressions are practically integrable and differentiable. However, this modification does not satisfy the too basic demands we have made for this study: it is not a procedure of general applicability to any kind of loading, and its performance is likely to require some special mathematical knowledge as to integration and differentiation of complicated expressions and to the theory of infinite series.

<u>Comments on the Second Modification of the Boundary Method</u>:

The second modification gives a good approximation to the correct values if a sufficient number of boundary points k are taken. The results will then be accurate enough for all practical purposes. As compared with the third modification, the advantage of this method is that the stress components can be calculated at any point of the plate, not just at certain selected grid points. Moreover, we obtain an approximate stress function in the form of a finite trigonometric series which can also be used for the determination of deformations of the plate under the applied loads.

The calculation of the Fourier coefficients according to Eqs. (42) and of the stress components according to Eqs. (73), consists essentially of summing a finite number of products. In order to cut down the time of computation to a reasonable amount, it is desirable to use a modern electric calculator which can perform positive and negative accumulative multiplications. The use of a slide rule would not only increase the amount of time required, but also lead to useless results since the computation should be carried at least to the fourth decimal. This is necessary because in the course of this computation, small quantities are sometimes multiplied by large factors, and numbers differing only by a very small quantity are subtracted from each other. If, however, a calculator is used, the consideration of several decimals scarcely increases the amount of work.

In summary, the second modification satisfies both of our demands. It is a method of general applicability to any kind of loading, and its procedure is such that every engineer should be able to perform the computations.

Comments on the Third Modification of the Boundary Method:

The third modification of the boundary method also gives approximations close enough to the correct values to satisfy all requirements of practical engineering, provided a grid with a sufficient number of points is used. This is necessary not only to obtain a good representation of the stress function on the boundary, but also to keep the inaccuracy arising from the use of finite differences inside the plate within acceptable limits.

The larger amount of work involved in this modification is required for the evaluation of the grid constants U_k, V_k, W_k, and Z_k. However, this work has to be done only once for a certain grid, and the same constants can be used for any case of loading provided the same grid is always chosen. For practical application, it is therefore suggested that for all computations only two or three or four "standard grids" be used, for which the grid constants have been determined once and collected in tables. An example for such a standard grid with $m = 12$, $n = 10$, $\Delta\vartheta = \frac{\pi}{12} = 15°$ is shown in Figure 17. This grid was used in the numerical example, Section 11, c, and its grid constants are given in Tables 1 to 4. Other possible standard grids could be:

$$m = 18, \quad n = 15, \quad \Delta\vartheta = \frac{\pi}{18} = 10^{\circ}$$

$$m = 36, \quad n = 25, \quad \Delta\vartheta = \frac{\pi}{36} = 5^{\circ}$$

Using one of those standard grids, the grid constants of which are already known, the work to be done reduces to the determination of the boundary values and boundary derivations of the stress function according to Section 6, the determination of the numerical values of the stress function inside and outside the plate according to Eqs. (66), (66a), and (69), and to the calculation of the stress components by means of the difference equations(73). Here also an electric calculator with provision for positive and negative accumlative multiplication is desirable. Using such a calculator as well as prepared charts which contain the constants of the standard grid to be used, the computation can be performed within a reasonable period of time.

The main feature of this numerical method is the elimination of all mathematical operation except algebraic ones. After having determined the boundary values and boundary derivatives, almost all the work can be done by any person who knows how to use an electric calculator, if this person is given simple instructions, and if prepared charts are used which lead the computer automatically from step to step.

A further advantage of this method should be emphasized. The writer knows about approximate methods for rectangular coordinates where the numerical values of the stress function at the points of a rectangular grid are obtained in the form of a system of n linear equations with n unknowns which may be solved by the Gauss algorithmus, by different matrix methods, or by relaxation methods. However, the solution of such a system of linear equations is always troublesome, and the number of grid points taken is restricted by the possible number of equations in this system. If there are more than, say 20 or 30, equations, it is not feasible to solve them. At least, the effort would not be justified by the result. This is a severe restriction on the accuracy of those numerical methods. Moreover, if one is only interested in the stress components at certain cross-sections or in certain portions of the plate, one has nevertheless to solve the whole system of linear equations, thus obtaining also the numerical values of the stress function of those points which are of no interest. The third modification of the boundary method, however, permits the calculation of the value of the stress function at any point inside the plate independent of the values at other points. Unnecessary work is therefore avoided in this method.

In summary, the third modification of the boundary method satisfies our two basic demands. It is not only a method of general applicability within the scope of this study, but is also of reasonable simplicity in its practical application.

Further Aspects and Possibilities:

The writer has considered this study as an interesting example of the power of practical mathematical methods in engineering. However, this investigation represents only a small step in the direction indicated. At this point, many questions relative to extended applications of the boundary method arise which could not be studies due to lack of time. The most important ones may be the possible extension of the boundary method to ring problems and the determination of deformations from numerical values of stress function obtained by using the third modification.

BIBLIOGRAPHY

Theory of Elasticity, S. Timoshenko, McGraw-Hill Book Company, Inc., New York, 1934.

Photoelasticity II, M. M. Frocht, John Wiley and Sons, Inc., New York, 1948.

TABLES AND DIAGRAMS

TABLE 1

VALUES OF U_k^l FOR STANDARD GRID

l	$\frac{r}{\rho}$	U_0^l	U_1^l	U_2^l	U_3^l	U_4^l	U_5^l	U_6^l	U_7^l	U_8^l	U_9^l	U_{10}^l	U_{11}^l	U_{12}^l	$\frac{r}{\rho}$	l
1	0.1	1.2222 2222	1.2120 2500	1.1830 8557	1.1397 9316	1.0879 1208	1.0331 4820	0.9801 9801	0.9324 1075	0.8918 9189	0.8598 0687	0.8367 1040	0.8228 1599	0.8181 8181	0.1	1
2	0.2	1.5000 0000	1.4687 2158	1.3841 0331	1.2679 0036	1.1428 5714	1.0251 2366	0.9230 7692	0.8395 0748	0.7741 9354	0.7257 0986	0.6924 3577	0.6730 3699	0.6666 6667	0.2	2
3	0.3	1.8571 4285	1.7827 5991	1.5954 1429	1.3669 0834	1.1518 9873	0.9735 6548	0.8348 6239	0.7307 5264	0.6546 7626	0.6009 5199	0.5653 5250	0.5450 5526	0.5384 6154	0.3	3
4	0.4	2.3333 3333	2.1690 8909	1.7980 2341	1.4133 9290	1.1052 6316	0.8814 7817	0.7241 3793	0.6144 5944	0.5384 6154	0.4867 6311	0.4533 6290	0.4346 1599	0.4285 7143	0.4	4
5	0.5	3.0000 0000	2.6401 5556	1.9532 5420	1.3814 8714	1.0000 0000	0.7566 7314	0.6000 0000	0.4970 7750	0.4285 7143	0.3832 1874	0.3544 3809	0.3384 5898	0.3333 3333	0.5	5
6	0.6	4.0000 0008	3.1858 3883	1.9952 0205	1.2512 9073	0.8421 0526	0.6098 6235	0.4705 8823	0.3830 9982	0.3265 3061	0.2897 8576	0.2667 5220	0.2540 5788	0.2500 0000	0.6	6
7	0.7	5.6666 6698	3.7036 0059	1.8374 1118	1.0198 8697	0.6455 6962	0.4522 6665	0.3422 8188	0.2753 2643	0.2328 7671	0.2056 4935	0.1887 1865	0.1794 3239	0.1764 7059	0.7	7
8	0.8	9.0000 0000	3.8087 7124	1.4144 3071	0.7077 8483	0.4285 7143	0.2936 6431	0.2195 1220	0.1752 5834	0.1475 4098	0.1298 9961	0.1189 8307	0.1130 1275	0.1111 1111	0.8	8
9	0.9	19.0000 1800	2.6635 4519	0.7565 0711	0.3536 8063	0.2087 9120	0.1413 5582	0.1048 7237	0.0834 8440	0.0701 1070	0.0616 3244	0.0563 9914	0.0535 4124	0.0526 3158	0.9	9
		U_{24}^l	U_{23}^l	U_{22}^l	U_{21}^l	U_{20}^l	U_{19}^l	U_{18}^l	U_{17}^l	U_{16}^l	U_{15}^l	U_{14}^l	U_{13}^l			

TABLE 2

VALUES OF W_k^l FOR STANDARD GRID

l	$\frac{r}{\rho}$	W_0^l	W_1^l	W_2^l	W_3^l	W_4^l	W_5^l	W_6^l	W_7^l	W_8^l	W_9^l	W_{10}^l	W_{11}^l	W_{12}^l	$\frac{r}{\rho}$	l
1	0.1	2.2222 2222	2.2120 2500	2.1830 8557	2.1397 9316	2.0879 1208	2.0331 4820	1.9801 9801	1.9324 1075	1.8918 9189	1.8598 0687	1.8367 1040	1.8228 1599	1.8181 8181	0.1	1
2	0.2	2.4999 9995	2.4687 2163	2.3841 0326	2.2679 0041	2.1428 5708	2.0251 2371	1.9230 7687	1.8395 0753	1.7741 9349	1.7257 0991	1.6924 3572	1.6730 3704	1.6666 6662	0.2	2
3	0.3	2.8571 3647	2.7827 6629	2.5954 0791	2.3669 1472	2.1518 9235	1.9735 7186	1.8348 5601	1.7307 5902	1.6546 6988	1.6009 5837	1.5653 4612	1.5450 6164	1.5384 5516	0.3	3
4	0.4	3.3331 3200	3.1692 9042	2.7978 2208	2.4135 9423	2.1050 6183	1.8816 7950	1.7239 3660	1.6146 6077	1.5382 6021	1.4869 6444	1.4531 6157	1.4348 1732	1.4283 7010	0.4	4
5	0.5	3.9970 6959	3.6430 8453	2.9603 2379	2.3844 1611	1.9970 6959	1.7596 0211	1.5970 6959	1.5000 0647	1.4256 4102	1.3861 4771	1.3515 0768	1.3413 8795	1.3304 0292	0.5	5
6	0.6	4.9738 2171	4.2119 0358	2.9690 2368	2.2773 5538	1.8159 2689	1.6359 2700	1.4444 0986	1.4091 6447	1.3003 5224	1.3158 5041	1.2405 7383	1.2801 2253	1.2238 2163	0.6	6
7	0.7	6.4982 4025	4.8674 2849	2.6689 8445	2.1837 2487	1.4771 4289	1.6160 9455	1.1738 5515	1.4391 5433	1.0644 4998	1.3694 7725	1.0202 9192	1.3432 6029	1.0080 4386	0.7	7
8	0.8	9.1145 1630	5.5803 8038	1.5289 4701	2.4793 9397	0.5430 8773	2.0652 7345	0.3340 2850	1.9468 6748	0.2620 5728	1.9015 0875	0.2334 9837	1.8846 2189	0.2256 2741	0.8	8
9	0.9	15.2769 2162	6.3063 0583	-2.9665 8927	3.9964 4137	-3.5143 0518	3.7841 1656	-3.6181 2401	3.7262 4514	-3.6529 8568	3.7043 9318	-3.6666 9724	3.6963 0198	-3.6704 6480	0.9	9
		W_{24}^l	W_{23}^l	W_{22}^l	W_{21}^l	W_{20}^l	W_{19}^l	W_{18}^l	W_{17}^l	W_{16}^l	W_{15}^l	W_{14}^l	W_{13}^l			

TABLE 3

VALUES OF V^l_k FOR STANDARD GRID

l	r/ρ	V^l_0	V^l_1	V^l_2	V^l_3	V^l_4	V^l_5	V^l_6	V^l_7	V^l_8	V^l_9	V^l_{10}	V^l_{11}	V^l_{12}	r/ρ	l
1	0.1	0.6050 0000	0.5999 5238	0.5856 2736	0.5641 9761	0.5385 1648	0.5114 0836	0.4851 9801	0.4615 4332	0.4414 8649	0.4256 0440	0.4141 7165	0.4072 9392	0.4050 0000	0.1	1
2	0.2	0.7200 0000	0.7049 8636	0.6643 6959	0.6085 9217	0.5485 7143	0.4920 5936	0.4430 7692	0.4029 6359	0.3716 1290	0.3483 4073	0.3323 6917	0.3230 5776	0.3200 0000	0.2	2
3	0.3	0.8449 9955	0.8111 5619	0.7259 1312	0.6219 4362	0.5241 1365	0.4429 7253	0.3798 6249	0.3324 9263	0.2978 7754	0.2734 3330	0.2572 3525	0.2480 0027	0.2449 9987	0.3	3
4	0.4	0.9799 8355	0.9110 3271	0.7551 5716	0.5936 3498	0.4642 0274	0.3702 2705	0.3041 3283	0.2580 7730	0.2261 5005	0.2044 4394	0.1904 0922	0.1825 4178	0.1799 9698	0.4	4
5	0.5	1.1240 5035	0.9903 0004	0.7318 5202	0.5181 8415	0.3746 8345	0.2838 2170	0.2248 1007	0.1864 4957	0.1605 7862	0.1437 4211	0.1328 0209	0.1269 5310	0.1248 9448	0.5	5
6	0.6	1.2772 1375	1.0216 8762	0.6370 7486	0.4012 8465	0.2688 8710	0.1955 8076	0.1502 6044	0.1228 5880	0.1042 6234	0.0929 3330	0.0851 7489	0.0814 7549	0.0798 2586	0.6	6
7	0.7	1.4249 9941	0.9574 9012	0.4620 5465	0.2636 7348	0.1623 4170	0.1169 2428	0.0860 7379	0.0711 8001	0.0585 6162	0.0531 6643	0.0474 5717	0.0463 8857	0.0443 7714	0.7	7
8	0.8	1.5086 7441	0.7326 9146	0.2371 0171	0.1361 5622	0.0718 4164	0.0564 9206	0.0367 9694	0.0337 1436	0.0247 3237	0.0249 8872	0.0199 4519	0.0217 4021	0.0186 2561	0.8	8
9	0.9	1.2352 1584	0.3245 0187	0.0515 7048	0.0430 8920	0.0142 3313	0.0172 2149	0.0071 5588	0.0101 7097	0.0047 7939	0.0075 0873	0.0038 4468	0.0065 2297	0.0035 8785	0.9	9
		V^l_{24}	V^l_{23}	V^l_{22}	V^l_{21}	V^l_{20}	V^l_{19}	V^l_{18}	V^l_{17}	V^l_{16}	V^l_{15}	V^l_{14}	V^l_{13}			

TABLE 4

VALUES OF Z^l_k FOR STANDARD GRID

l	r/ρ	Z^l_0	Z^l_1	Z^l_2	Z^l_3	Z^l_4	Z^l_5	Z^l_6	Z^l_7	Z^l_8	Z^l_9	Z^l_{10}	Z^l_{11}	Z^l_{12}	r/ρ	l
1	0.1	1.3444 4444	1.3271 0966	1.2784 7464	1.2072 6619	1.1243 7506	1.0397 6899	0.9607 8813	0.8918 9127	0.8352 4471	0.7915 4199	0.7607 1338	0.7424 2187	0.7363 6363	0.1	1
2	0.2	1.7999 9996	1.7404 1508	1.5839 2571	1.3802 2643	1.1755 1018	0.9964 8108	0.8520 7100	0.7412 5456	0.6593 1319	0.6011 3505	0.5625 1346	0.5404 8760	0.5333 3332	0.2	2
3	0.3	2.4142 7903	2.2572 5810	1.8840 4065	1.4720 8751	1.1278 3615	0.8742 3812	0.6969 9242	0.5754 6462	0.4928 8899	0.4377 5533	0.4026 6220	0.3831 7570	0.3769 2131	0.3	3
4	0.4	3.2664 1453	2.8873 2724	2.1127 9538	1.4327 9396	0.9771 7547	0.6966 4865	0.5243 0572	0.4167 0729	0.3478 7762	0.3040 0087	0.2766 9536	0.2619 1411	0.2571 0230	0.4	4
5	0.5	4.4929 0747	3.6077 4676	2.1592 0043	1.2355 6664	0.7482 6892	0.4994 1328	0.3590 3732	0.2796 7556	0.2289 2746	0.1992 4780	0.1794 8304	0.1702 9336	0.1661 5998	0.5	5
6	0.6	6.3526 3348	4.3032 4974	1.8914 9035	0.9138 6776	0.4882 7932	0.3199 5585	0.2170 3766	0.1731 2826	0.1355 7777	0.1222 8632	0.1056 6574	0.1042 9861	0.0976 9261	0.6	6
7	0.7	9.2599 8852	4.6605 1469	1.2332 1668	0.5757 9034	0.2398 0189	0.1889 6069	0.1010 3816	0.1024 3902	0.0623 3592	0.0728 1022	0.0484 2017	0.0623 1192	0.0447 3410	0.7	7
8	0.8	13.7508 3750	4.0886 9708	0.3625 1595	0.3375 8491	0.0390 1631	0.1166 7155	0.0122 9123	0.0656 3739	0.0064 8130	0.0475 1627	0.0046 5719	0.0409 7208	0.0042 0245	0.8	8
9	0.9	19.7869 1087	2.0464 0809	−0.1529 8843	0.1722 0346	−0.0500 1956	0.0651 6813	−0.0258 9086	0.0378 9953	−0.0174 5904	0.0278 1529	−0.0140 9728	0.0241 1087	−0.0131 6908	0.9	9
		Z^l_{24}	Z^l_{23}	Z^l_{22}	Z^l_{21}	Z^l_{20}	Z^l_{19}	Z^l_{18}	Z^l_{17}	Z^l_{16}	Z^l_{15}	Z^l_{14}	Z^l_{13}			

TABLE 5

NUMERICAL VALUES OF THE STRESS FUNCTION (EXAMPLE)

k	θ	F_k^1	F_k^2	F_k^3	F_k^4	F_k^5	F_k^6	F_k^7	F_k^8	F_k^9	$F_k^{10} = F_k$	$F_k^{11} = F_k^0$	θ	k
0	0°	−0.1147 7066 c	−0.0786 6504 c	−0.0510 7318 c	−0.0309 8823 c	−0.0171 9367 c	−0.0084 2155 c	−0.0034 0943 c	−0.0010 1028 c	−0.0001 7068 c	0	−0.0001 7068 c	0°	0
1	15°	−0.1160 6130 c	−0.0804 9074 c	−0.0528 6471 c	−0.0324 2955 c	−0.0181 4523 c	−0.0088 3759 c	−0.0035 8970 c	−0.0009 6478 c	−0.0000 6181 c	0	−0.0000 6181 c	15°	1
5	75°	−0.1460 1422 c	−0.1298 0858 c	−0.1116 1013 c	−0.0917 3812 c	−0.0707 1950 c	−0.0495 3196 c	−0.0295 7824 c	−0.0130 9878 c	−0.0027 9957 c	0	−0.0027 9957 c	75°	5
6	90°	−0.1585 3239 c	−0.1539 5615 c	−0.1463 5958 c	−0.1357 9756 c	−0.1223 3524 c	−0.1062 6843 c	−0.0879 0526 c	−0.0683 3682 c	−0.0506 1938 c	−0.0435 7787 c	−0.0593 3495 c	90°	6
7	105°	−0.1718 9612 c	−0.1815 7239 c	−0.1892 5626 c	−0.1952 6574 c	−0.2000 9001 c	−0.2048 2340 c	−0.2107 5576 c	−0.2201 4861 c	−0.2357 3715 c	−0.2588 1905 c	−0.2875 0033 c	105°	7
11	165°	−0.2126 5388 c	−0.2736 7591 c	−0.3426 4245 c	−0.4187 8546 c	−0.5009 6255 c	−0.5884 9312 c	−0.6797 3785 c	−0.7736 8525 c	−0.8693 9670 c	−0.9659 2583 c	−1.0625 8188 c	165°	11
12	180°	−0.2147 7067 c	−0.2786 6504 c	−0.3510 7318 c	−0.4309 8823 c	−0.5170 4294 c	−0.6084 2158 c	−0.7034 0951 c	−0.8014 0605 c	−0.9001 7238 c	−1.0000 0000 c	−1.1001 7238 c	180°	12
13	195°	−0.2126 5388 c	−0.2736 7591 c	−0.3426 4245 c	−0.4187 8546 c	−0.5009 6255 c	−0.5884 9312 c	−0.6797 3785 c	−0.7736 8525 c	−0.8693 9670 c	−0.9659 2583 c	−1.0625 8188 c	195°	13
17	255°	−0.1718 9612 c	−0.1815 7239 c	−0.1892 5626 c	−0.1952 6574 c	−0.2000 9001 c	−0.2048 2340 c	−0.2107 5576 c	−0.2201 4861 c	−0.2357 3715 c	−0.2588 1905 c	−0.2875 0033 c	255°	17
18	270°	−0.1585 3239 c	−0.1539 5615 c	−0.1463 5958 c	−6.1357 9756 c	−0.1223 3524 c	−0.1062 6843 c	−0.0879 0526 c	−0.0683 3682 c	−0.0506 1938 c	−0.0435 7787 c	−0.0593 3492 c	280°	18
19	285°	−0.1460 1422 c	−0.1298 0858 c	−0.1116 1013 c	−0.0917 3812 c	−0.0707 1950 c	−0.0495 3196 c	−0.0295 7824 c	−0.0130 9878 c	−0.0027 9957 c	0	−0.0027 9957 c	295°	19
23	345°	−0.1160 6130 c	−0.0804 9074 c	−0.0528 6471 c	−0.0324 2955 c	−0.0181 4523 c	−0.0089 3759 c	−0.0035 8970 c	−0.0009 6478 c	−0.0000 6181 c	0	−0.0000 6181 c	345°	23
24	360°	−0.1147 7066 c	−0.0786 6504 c	−0.0510 7318 c	−0.0309 8823 c	−0.0171 9367 c	−0.0084 2155 c	−0.0034 0943 c	−0.0018 4362 c	−0.0010 0402 c	0	−0.0010 0402 c	360°	24

VALUE AT THE CENTER OF THE PLATE : $F(0) = -0.1600\ 6062\ c$

$$c = \rho^2 \frac{\pi}{9} \cdot q$$

TABLE 6

RADIAL STRESS σ_r ON THE HORIZONTAL DIAMETER

i	[illegible]	Modification [illegible]	Modification [illegible]	Modification [illegible]	[illegible]	[illegible]	i
0	0				+ 0.11 111 q	0	0
1	0.1	+ 0.10 254 q	+ 0.10 308 q	+ 0.10 [illegible] q	+ 0.10 [illegible] q	0.1	1
2	0.2	+ 0.0[illegible] 060 q	+ 0.0[illegible] 183 q	+ 0.0[illegible] 057 q	+ 0.0[illegible] 467 q	0.2	2
3	0.3	+ 0.07 547 q	+ 0.07 282 q	+ 0.07 461 q	+ 0.07 744 q	0.3	3
4	0.4	+ 0.05 707 q	+ 0.05 [illegible] q	+ 0.05 607 q	+ 0.05 825 q	0.4	4
5	0.5	+ 0.03 246 q	+ 0.03 758 q	+ 0.04 000 q	+ 0.04 [illegible] q	0.5	5
6	0.6	+ 0.02 446 q	+ 0.02 462 q	+ 0.02 [illegible] q	+ 0.02 461 q	0.6	6
7	0.7	+ 0.01 514 q	+ 0.01 032 q	+ 0.01 473 q	+ 0.01 302 q	0.7	7
8	0.8	+ 0.00 240 q	+ 0.00 556 q	+ 0.00 773 q	+ 0.00 540 q	0.8	8
9	0.9	+ 0.00 [illegible] q	+ 0.00 147 q	+ 0.00 347 q	+ 0.00 122 q	0.9	9
10	1		+ 0.00 050 q	0	0	1	10

TABLE 7

TANGENTIAL STRESS σ_t ON THE HORIZONTAL DIAMETER

i	[illegible]	Modification [illegible]	Modification [illegible]	Modification [illegible]	[illegible]	[illegible]	i
0	0			− 0.32 [illegible] q	− 0.33 333 q	0	0
1	0.1	− 0.32 042 q	− 0.32 204 q	− 0.32 [illegible] q	− 0.32 457 q	0.1	1
2	0.2	− 0.28 685 q	− 0.28 835 q	− 0.28 [illegible] q	− 0.2[illegible] 930 q	0.2	2
3	0.3	− 0.26 136 q	− 0.26 269 q	− 0.26 204 q	− 0.26 237 q	0.3	3
4	0.4	− 0.21 877 q	− 0.21 977 q	− 0.21 908 q	− 0.21 918 q	0.4	4
5	0.5	− 0.17 366 q	− 0.17 243 q	− 0.17 532 q	− 0.17 333 q	0.5	5
6	0.6	− 0.12 886 q	− 0.13 055 q	− 0.13 125 q	− 0.12 918 q	0.6	6
7	0.7	− 0.08 984 q	− 0.08 099 q	− 0.08 121 q	− 0.08 908 q	0.7	7
8	0.8	− 0.05 537 q	− 0.05 503 q	− 0.05 444 q	− 0.05 413 q	0.8	8
9	0.9	− 0.02 735 q	− 0.02 518 q	− 0.02 550 q	− 0.02 455 q	0.9	9
10	1		− 0.00 147 q	− 0.00 [illegible] q	0	1	10

TABLE 8

RADIAL STRESS σ_r ON THE VERTICAL DIAMETER

l	r/ρ	MODIFICATION I	MODIFICATION II	MODIFICATION III	FROCHT	r/ρ	l
0	0				−0.33 333 q	0	0
1	0.1	−0.33 295 q	−0.33 463 q	−0.32 410 q	−0.33 784 q	0.1	1
2	0.2	−0.34 537 q	−0.34 752 q	−0.33 542 q	−0.35 135 q	0.2	2
3	0.3	−0.36 869 q	−0.37 051 q	−0.35 540 q	−0.37 723 q	0.3	3
4	0.4	−0.40 438 q	−0.40 627 q	−0.38 565 q	−0.41 798 q	0.4	4
5	0.5	−0.45 751 q	−0.45 937 q	−0.42 942 q	−0.48 148 q	0.5	5
6	0.6	−0.53 586 q	−0.53 717 q	−0.43 146 q	−0.58 332 q	0.6	6
7	0.7	−0.65 114 q	−0.65 038 q	−0.51 037 q	−0.76 034 q	0.7	7
8	0.8	−0.81 606 q	−0.81 004 q	−0.63 716 q	−1.12 345 q	0.8	8
9	0.9	−1.02 168 q	−1.00 809 q	−0.61 526 q	−2.22 805 q	0.9	9
10	1		−1.13 934 q	−0.88 343 q	∞	1	10

TABLE 9

TANGENTIAL STRESS σ_θ ON THE VERTICAL DIAMETER

l	r/ρ	MODIFICATION I	MODIFICATION II	MODIFICATION III	FROCHT	r/ρ	l
0	0			+0.10 663 q	+0.11 111 q	0	0
1	0.1	+0.10 637 q	+0.10 681 q	+0.10 642 q	+0.11 111 q	0.1	1
2	0.2	+0.10 547 q	+0.10 600 q	+0.10 543 q	+0.11 111 q	0.2	2
3	0.3	+0.10 367 q	+0.10 415 q	+0.10 351 q	+0.11 111 q	0.3	3
4	0.4	+0.10 026 q	+0.10 063 q	+0.10 124 q	+0.11 111 q	0.4	4
5	0.5	+0.09 358 q	+0.09 368 q	+0.09 091 q	+0.11 111 q	0.5	5
6	0.6	+0.07 933 q	+0.07 873 q	+0.08 016 q	+0.11 111 q	0.6	6
7	0.7	+0.04 533 q	+0.04 294 q	+0.04 207 q	+0.11 111 q	0.7	7
8	0.8	−0.04 715 q	−0.05 231 q	−0.06 451 q	+0.11 111 q	0.8	8
9	0.9	−0.32 294 q	−0.32 579 q	−0.37 255 q	+0.11 111 q	0.9	9
10	1		−0.91 620 q	−0.73 582 q	+0.11 111 q	1	10

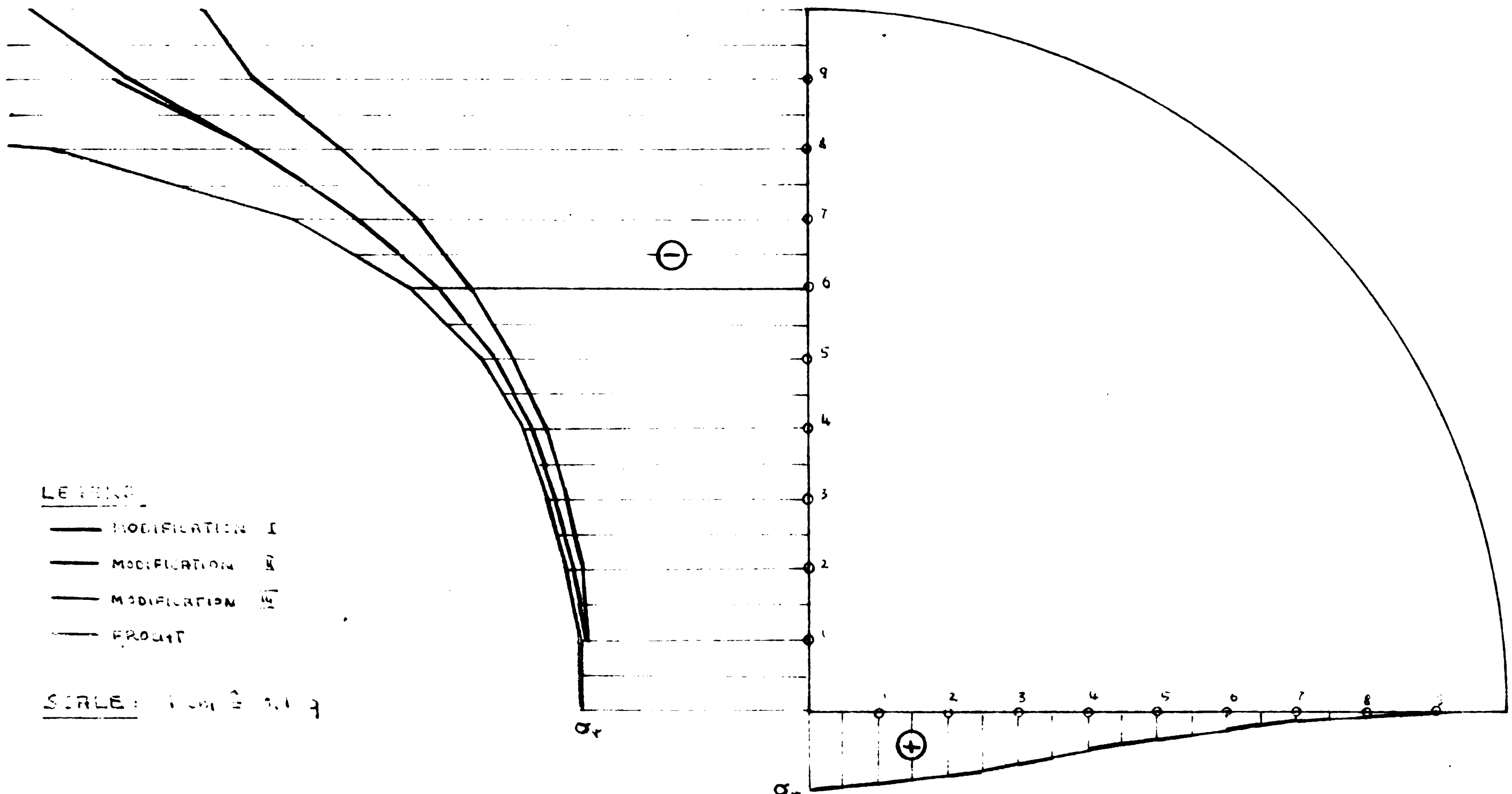


DIAGRAM 1: RADIAL STRESS VARIATION

ON THE HORIZONTAL AND VERTICAL DIAMETER

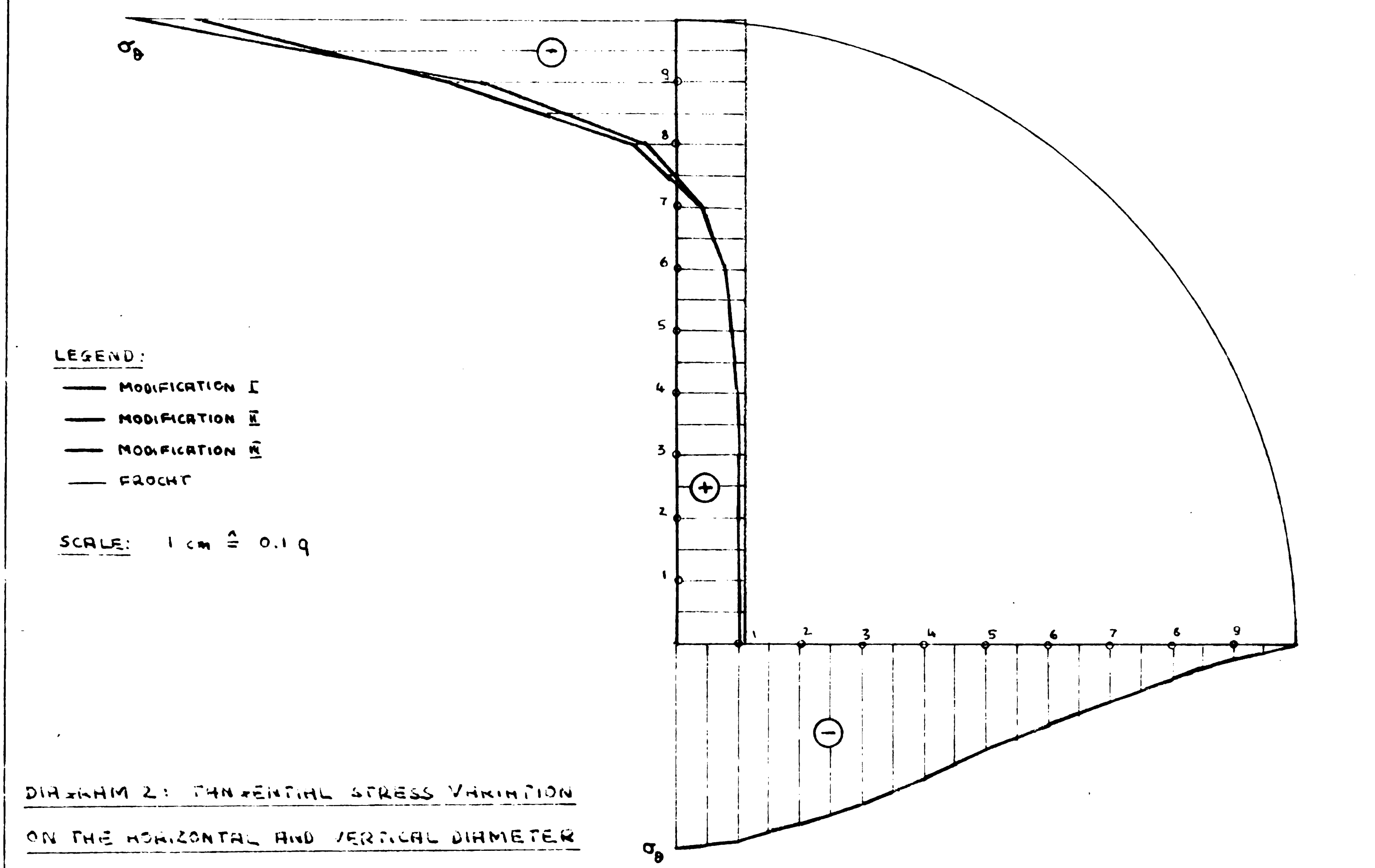


DIAGRAM 2: TANGENTIAL STRESS VARIATION ON THE HORIZONTAL AND VERTICAL DIAMETER