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ON THE BUCKLING OF RECTANGULAR PLATES WITH INTERNAL POINT SUPPORT

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James Andrew Gusack

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This is to certify that the

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WITH INTERNAL POINT SUPPORT

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1. INTRODUCTION

Statement of the Problem

The problem considered in this paper is to determine the effect of an internal point support on the critical elastic buckling load of a rectangular thin plate under various edge support conditions.

Critical Buckling Load

A plate subjected to an edge load in its plane is said to be on the verge of buckling when the plate is in a condition of neutral equilibrium. In this condition the edge load may produce either strain in the initial flat equilibrium configuration or, more important, a laterally bent equilibrium configuration. This second configuration is called a buckling mode. The edge load necessary to produce a condition of neutral equilibrium in a plate is called the critical buckling load. A buckling mode corresponding to a higher critical buckling load is possible, but in practice the plate will buckle in the first mode corresponding to the lowest critical buckling load unless constrained.

Method of Solution of the Problem

The extended Ritz method was chosen for the solution because an explicit solution of the buckling differential equation of the plate was not available with the additional restraint of the point support. A discussion of this method is given in the General Theory, Chapter 2.

Assumptions

The usual classical theory assumptions are made:

a. The material is homogeneous, isotropic, and follows Hooke's Law.

b. Normals to the undeformed middle plane of the plate remain straight and normal to the deformed middle surface.

c. The cross section thickness is constant and small compared to the length and width of the plate.

d. The plate is loaded in plane stress before buckling.

Applications of the Theory

Two applications of the General Theory of Chapter 2 are made in this paper. The rectangular plate with simply supported edges and an arbitrary point support is discussed in Chapter 3. The rectangular plate with clamped edges and an arbitrary point support is discussed in Chapter 4.

2. GENERAL THEORY

Stationary Potential Energy

The total potential energy of a mechanical system is said to be stationary for a given equilibrium configuration of the system if the first order change in the total potential energy is zero for any arbitrary small displacements from the given configuration. The Theorem of Stationary Potential Energy[1] states that at an equilibrium configuration of a system, the total potential energy is stationary.

Let $U = V + U_w$ (1)

where V is the strain energy of bending of the plate in the buckled configuration and the symbol U_w is the change of potential energy of the external loads when the plate buckles into the buckled configuration. The Theorem of Stationary Potential Energy requires that U be stationary for any buckled equilibrium configuration of the plate.

The Extended Ritz Method

The extended Ritz method[2] is used to solve the plate buckling problem. In this method the lateral deflection

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1. Friedrich Bleich, Buckling Strength of Metal Structures, McGraw-Hill, New York: 1952. pp. 70,71.
2. Ibid. pp. 77-81.

of the plate is expressed as a sum of suitably chosen coordinate functions.

For the rectangular plate

$$0 \leq x \leq a \;, \quad 0 \leq y \leq b \;,$$

we choose

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \, \phi_m(x) \cdot \theta_n(y) \qquad (2)$$

where the functions $\phi_m(x)$ are the complete set of eigenfunctions of a beam with no internal support subject to end conditions at $x = 0$, $x = a$ which are the same as the end conditions of the rectangular plate under consideration. The functions $\theta_n(y)$ are the complete set of eigenfunctions of a beam with no internal support subject to end conditions at $y = 0$, $y = b$ which are the same as the end conditions of the rectangular plate under consideration. It is known that any arbitrary deflection configuration of a rectangular plate can be represented by an infinite double series of eigenfunction products of the form chosen.[3] The coefficients A_{mn} are to be determined so that the constraint condition of the internal point support is satisfied, and the total potential energy of the system is stationary. The constraint condition for an internal point support at an arbitrary point with coordinates (ξ, η) is

$$w(\xi, \eta) = 0 \qquad (3)$$

3. R. Courant and D. Hilbert, Methoden Der Mathematischen Physik, Vol. 1, Berlin: Springer, 1931. p. 47.

The problem of making the expression for the total potential energy of the system stationary and simultaneously satisfying the subsidiary conditions of constraint can be solved by the Lagrange multiplier method.[4] In this method, the expression

$$\bar{U} = V + U_w - \lambda\, W(\xi, \eta) \tag{4}$$

is introduced. The parameters A_{mn} and the values λ that make $\bar{U}$ stationary also make U [Eq. (1)] stationary and satisfy the subsidiary constraint condition. The necessary conditions for $\bar{U}$ to be stationary are[5]

$$W(\xi, \eta) = 0 \qquad \text{and} \qquad \frac{\partial \bar{U}}{\partial A_{mn}} = 0 \,. \tag{5}$$

For the applications to be made, V and U_w [in Eq. (4)] are given by certain double integrals over the plate. When the series for W [Eq. (2)] is substituted in these integrals, and the integrals are evaluated, V and U_w are obtained as quadratic expressions in the coefficients A_{mn}.

In practice, if N coefficients are to be determined, the N Equations (6) together with Equation (3) form a system of N+1 linear homogeneous algebraic equations for the N coefficients and the multiplier λ. The solution of this system is obtained only up to an undetermined constant multiplier. Hence the shape of the deflected equilibrium configuration is determined, but not its amplitude.

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4. F. Bleich, Op. Cit. pp. 77-81.

5. I. S. Sokolnikoff and E. S. Sokolnikoff, Higher Mathematics for Engineers and Physicists, McGraw-Hill, New York: 1941. pp. 163-167.

In the following Chapter the buckling problem of a simply supported rectangular plate with lateral point support is solved using the method discussed in this Chapter.

3. SIMPLY SUPPORTED PLATE

Application of the General Theory

A rectangular plate with simply supported edges on four sides and an arbitrary point support with coordinates (ξ, η) is considered.

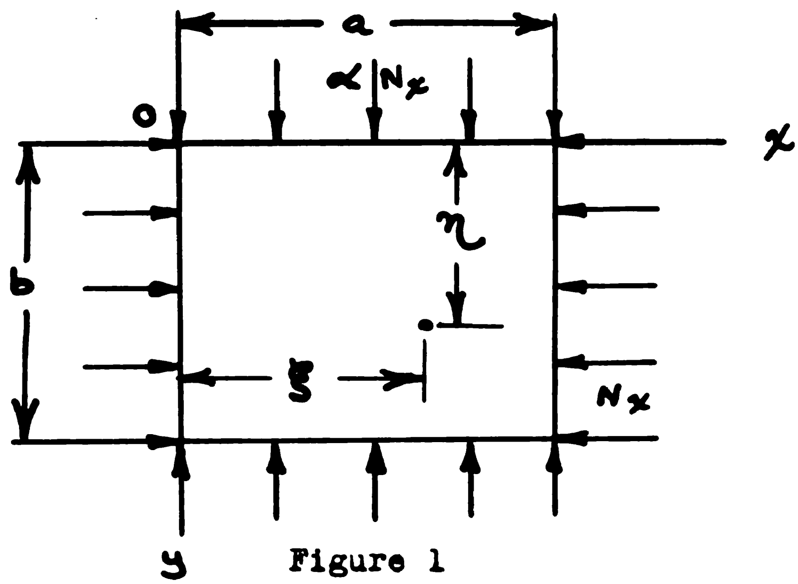

Figure 1

In Figure 1

a = length of the plate in the x-direction,

b = length of the plate in the y-direction,

N_x = uniform compressive force per unit length acting in the plane of the plate on the edges $x = 0$, $x = a$, and

αN_x = uniform compressive force per unit length acting in the plane of the plate on the edges $y = 0$, $y = b$, where α is a dimensionless constant.

The series expression [Eq.(2)] for the deflection of the plate is then

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \tag{6}$$

where the functions $\sin \frac{m\pi x}{a}$ are the complete set of eigenfunctions of a simply supported beam of length a,[6] and the functions $\sin \frac{n\pi y}{b}$ are the complete set of eigenfunctions of a simply supported beam of length b. The coefficients A_{mn} are the set of parameters to be determined.

The strain energy of bending of a rectangular plate is[7]

$$V = \frac{1}{2} D \int_0^a \int_0^b \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx\,dy \tag{7}$$

where

$$D = \frac{E h^3}{12(1-\nu^2)} \tag{8}$$

is the flexural rigidity of the plate. In Equation (8)

E = modulus of elasticity,

h = thickness of the plate, and

ν = Poisson's ratio.

We assume that the limited bending that occurs when the plate enters the buckled equilibrium configuration takes place

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6. F.B. Hildebrand, Advanced Calculus for Engineers, Prentice-Hall, New York: 1949. p.215.

7. S. Timoshenko, Theory of Elastic Stability, McGraw-Hill, New York: 1936. pp. 305-307.

with negligible stretch or compression of the middle plane of the plate.[8] If we then take the datum configuration of zero potential energy to be the flat configuration of the plate just before buckling occurs, the quantity V is the total strain energy in the buckled configuration.

The appropriate derivatives of W [Eq.(6)] are taken and substituted in V [Eq.(7)]. If we observe the orthogonality of the eigenfunctions,[9] namely

$$\left.\begin{array}{l} \int_0^a \sin\frac{m\pi x}{a}\cdot\sin\frac{m_1\pi x}{a} = \\ \\ \int_0^a \cos\frac{m\pi x}{a}\cdot\cos\frac{m_1\pi x}{a} = \end{array}\right\} \begin{array}{l} 0 \text{ for } m \neq m_1 \\ \frac{a}{2} \text{ for } m = m_1 \end{array},$$

V is found to be

$$V = \frac{\pi^2}{8}\frac{b}{a}\cdot\frac{\pi^2 D}{a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2\left[m^2+\left(\frac{a}{b}\right)^2 n^2\right]^2. \quad (9)$$

We also assume that the edge loads do not change during the buckling of the plate. Then the work done by the external compressive forces N_x and αN_x during buckling[10] is

$$\frac{1}{2}N_x\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2+\alpha\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy.$$

The change in the potential energy of the external loads during buckling is then

$$U_w = -\frac{1}{2}N_x\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^2+\alpha\left(\frac{\partial w}{\partial y}\right)^2\right]dx\,dy. \quad (10)$$

8. S.Timoshenko, Op.Cit.,p.325.
9. Friedrich Bleich, Op.Cit.,pp.65-69.
10. S.Timoshenko, Op.Cit.,pp.308-314.

The appropriate derivatives of W [Eq.(6)] are taken and substituted in U_w [Eq.(10)]. The expression for U_w when integrated is

$$U_w = -\frac{\pi^2 b}{8\,a} \cdot N_x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left[m^2 + \alpha\left(\frac{a}{b}\right)^2 n^2\right]. \tag{11}$$

The extended Ritz method as discussed in Chapter 2 is now applied. The expressions found for V and U_w are substituted in $\bar{U}$ [Eq.(4)]. Then

$$\begin{aligned} \bar{U} = {} & \frac{\pi^2 b}{8\,a} \cdot N_o \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left[m^2 + \left(\frac{a}{b}\right)^2 n^2\right]^2 \\ & -\frac{\pi^2 b}{8\,a} \cdot N_x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left[m^2 + \alpha\left(\frac{a}{b}\right)^2 n^2\right] \\ & -\lambda \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}, \end{aligned} \tag{12}$$

where $N_o = \frac{\pi^2 D}{a^2}$ (load per unit width), the Euler critical load for a column with flexural rigidity equal to D .

The necessary conditions[11] for $\bar{U}$ to be stationary are

$$\frac{\partial \bar{U}}{\partial A_{mn}} = 0 \qquad \text{and} \qquad w(\xi, \eta) = 0 .$$

The derivatives of $\bar{U}$ [Eq.(12)] with respect to the parameters A_{mn} produce the equations

$$\begin{aligned} \frac{\partial \bar{U}}{\partial A_{mn}} = {} & \frac{\pi^2 b}{4\,a} \cdot N_o\, A_{mn} \left[m^2 + \left(\frac{a}{b}\right)^2 n^2\right]^2 \\ & -\frac{\pi^2 b}{4\,a} \cdot N_x A_{mn} \left[m^2 + \alpha\left(\frac{a}{b}\right)^2 n^2\right]^2 \\ & -\lambda \sin\frac{m\pi\xi}{a} \cdot \sin\frac{n\pi\eta}{b} = 0 . \end{aligned} \tag{13}$$

11. See Lagrange's multiplier method in Chapter 2.

In these equations the multiplier λ may be zero or different from zero. If λ is zero the Equations (13) require that for each possible combination of the two indices m,n, either A_{mn} is zero or

$$\left[m^2+\left(\frac{a}{b}\right)^2 n^2\right]^2-\frac{N_x}{N_o}\left[m^2+\alpha\left(\frac{a}{b}\right)^2 n^2\right]=0. \qquad (14)$$

If the load ratio α and the side ratio $\frac{a}{b}$ have been chosen in advance, the critical value N_x is then determined by Equation (14) for one choice of m,n. The choice made is that which gives the lowest critical value of N_x and still satisfies the constraint condition $w(\xi,\eta)=0$. All of the other coefficients A_{mn} except the one corresponding to this choice of m,n must be zero in order that Equations (13) can all be satisfied. Thus only one term of the series for w [Eq. (6)] survives when λ is zero. Physically this means that the internal point support is located on a nodal point of one of the buckling modes of a simply supported plate. In fact an expression similar to Equation (14) is given by S. Timoshenko[12] to find the critical buckling load for a simply supported plate.

When λ is not zero, Equations (13) require that for each choice of the indices m,n,

$$A_{mn}=\frac{\frac{4\lambda}{\pi^2 N_o}\cdot\frac{a}{b}\left[\sin\frac{m\pi\xi}{a}\cdot\sin\frac{n\pi\eta}{b}\right]}{\left[m^2+\left(\frac{a}{b}\right)^2 n^2\right]^2-k\left[m^2+\alpha\left(\frac{a}{b}\right)^2 n^2\right]}, \qquad (15)$$

12. S. Timoshenko, Op.Cit., p. 318.

where $\mu = \frac{N_x}{N_0}$, a dimensionless load parameter. The other necessary condition is that $w(\xi,\eta) = 0$. Substitution of A_{mn} [Eq. (15)] in the expression for w [Eq. (6)] at the point (ξ,η) results in the characteristic equation,

$$\frac{4\lambda}{\pi^2 N_0}\cdot\frac{a}{b}\cdot\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\sin^2\frac{m\pi\xi}{a}\cdot\sin^2\frac{n\pi\eta}{b}}{\left[m^2+\left(\frac{a}{b}\right)^2n^2\right]^2-\mu\left[m^2+\alpha\left(\frac{a}{b}\right)^2n^2\right]}\right\}=0. \quad (16)$$

Since the characteristic equation was obtained for the condition that λ is not zero, and the coefficients

$$\frac{4}{\pi^2 N_0}\cdot\frac{a}{b} \neq 0,$$

then

$$\psi(\mu) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\sin^2\frac{m\pi\xi}{a}\cdot\sin^2\frac{n\pi\eta}{b}}{\left[m^2+\left(\frac{a}{b}\right)^2n^2\right]^2-\mu\left[m^2+\alpha\left(\frac{a}{b}\right)^2n^2\right]}\right\} \quad (17)$$

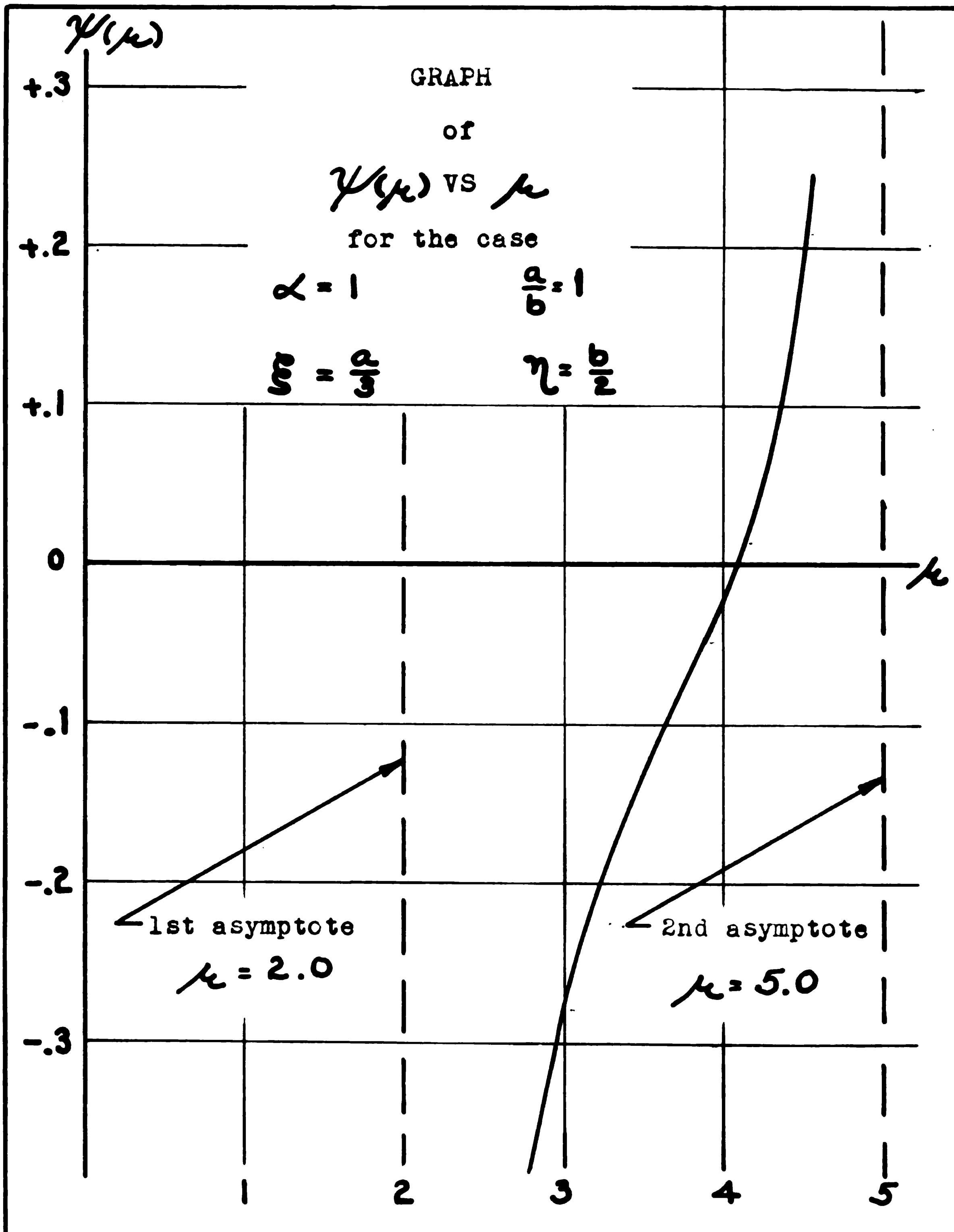
must equal zero for each critical buckling value of μ. If the load ratio α, the point support location, and the side ratio $\frac{a}{b}$ have been chosen in advance, the values of μ that satisfy $\psi(\mu)=0$ are the critical buckling dimensionless load parameters of that simply supported rectangular plate with a point support at (ξ,η).

Graphical Solution of the Characteristic Equation

A graphical method can be used to solve $\psi(\mu) = 0$. After the load ratio α , the side ratio $\frac{a}{b}$, and the coordinates of the point support (ξ, η) are selected, a graph is constructed with values of μ on the abscissa axis and the values of the sum $\psi(\mu)$ on the ordinate axis (See Fig.2). This graph has a vertical asymptote at each value of μ corresponding to a buckling mode of a simply supported plate with no point support. Since there is no root of the Equation $\psi(\mu) = 0$ before the first asymptote (The terms are all positive .), the lowest critical value of μ for the point supported plate where $\psi(\mu) = 0$, will lie between the first two asymptotes. The accuracy of the determination of the root (where $\psi(\mu)$ crosses the μ - axis) can be improved by numerical interpolation, using a suitable number of terms of the sum $\psi(\mu)$.

Example of the Numerical Solution

Let us consider as an example the problem of a simply supported square plate with equal loads in the two directions ($\alpha = 1$ and $\frac{a}{b} = 1$) and a point support at ($\xi = \frac{a}{3}, \eta = \frac{b}{2}$). The terms of $\psi(\mu)$ can be represented in the doubly infinite array (See Fig. 3),



$\psi(\mu)$
+.3
+.2
+.1
0
-.1
-.2
-.3
GRAPH
of
$\psi(\mu)$ VS μ
for the case
$\alpha = 1$
$\frac{a}{b} = 1$
$\xi = \frac{a}{3}$
$\eta = \frac{b}{2}$
μ
1st asymptote
$\mu = 2.0$
2nd asymptote
$\mu = 5.0$
1
2
3
4
5

Figure 2.

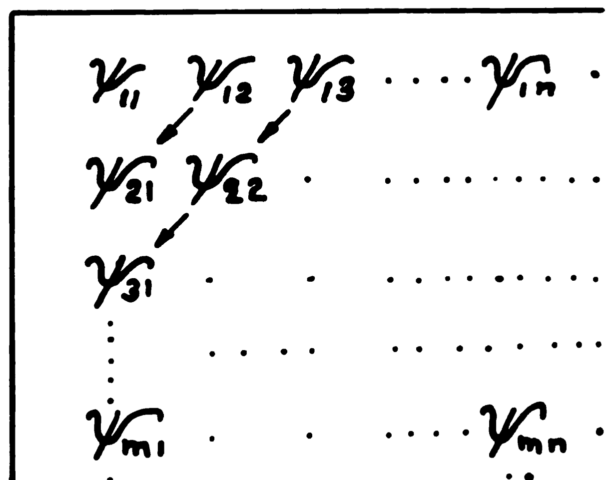


Figure 3

where ψ_{mn} is the term corresponding to the summation indices m,n in the sum [Eq.(17)]. In practice a finite number of terms of the infinite array are summed to evaluate $\psi(\lambda)$, starting at the upper left hand corner and moving diagonally through the array as shown by the arrows.

For the particular square plate of this example the first six non-vanishing terms of the sum are

$$\frac{.75}{4-2\lambda} + \frac{.75}{25-5\lambda} + \frac{.75}{100-10\lambda} + \frac{.75}{169-13\lambda} + \frac{.75}{289-17\lambda} + \frac{.75}{676-26\lambda}.$$

The first asymptotes of the graph $\psi(\lambda)$ are found by equating the denominators of the doubly infinite array (Fig. 3) to zero. The first two asymptotes are $\lambda = 2$ and $\lambda = 5$ (See Fig. 2). The term ψ_{11} and the term that will make the largest contribution to the sum $\psi(\lambda)$ for a value of λ between the asymptotes are now summed and equated to zero.

$$\psi_{11} + \psi_{21} = \frac{.75}{4-2\mu} + \frac{.75}{25-5\mu} = 0 .$$

This equation is now solved for the first approximate value of μ, which is $\mu = 4.14$. The first approximate value of μ is now inserted in the first six contributory terms of $\psi(\mu)$ and the sum taken. The sum of $\psi(4.14)$ is $+.015$. Since this sum is positive, a value of μ less than $\mu = 4.14$ is chosen. We choose $\mu = 3.85$ arbitrarily and sum the first six terms again with this value. The sum of $\psi(3.85)$ is $-.049$. Since this sum is negative, the critical value of μ lies within the interval $3.85 < \mu < 4.14$. Further numerical interpolation will give the value of μ with greater accuracy. The critical value of μ for this example is approximately $\mu = 4.05$ (See Fig. 2).

Critical values of μ for other values of load ratios α, side ratios $\frac{a}{b}$, and point support locations (ξ, η) for the simply supported plate are tabulated in Table I.

The lowest critical value of μ occurs for λ not zero except for the case that the point support is at the middle of the plate ($\xi = \frac{a}{2}, \eta = \frac{b}{2}$). The values tabulated were therefore all obtained by solving Equation (16), except the values for the simply supported plate with a point support in the middle of the plate. In these cases the lowest critical value of μ occurs for $\lambda = 0$, and the denominator of the Equation (16) was equated to zero to find the lowest value.

TABLE I

CRITICAL VALUES OF μ
FOR THE SIMPLY SUPPORTED PLATE

ξ	η	$\frac{a}{b}$	α	μ
$\frac{a}{2}$	$\frac{b}{2}$	1	0	6.25
$\frac{a}{2}$	$\frac{b}{2}$	2	0	16.00
$\frac{a}{2}$	$\frac{b}{2}$	$\frac{1}{2}$	0	4.00
$\frac{a}{2}$	$\frac{b}{2}$	1	1	5.00
$\frac{a}{2}$	$\frac{b}{2}$	2	1	8.00
$\frac{a}{2}$	$\frac{b}{2}$	$\frac{1}{2}$	1	2.00
$\frac{a}{3}$	$\frac{b}{2}$	1	0	5.75
$\frac{a}{3}$	$\frac{b}{2}$	1	1	4.05

4. CLAMPED PLATE

Application of the General Theory

A rectangular plate with clamped edges on four sides and an arbitrary point support with coordinates (ξ, η) is considered.

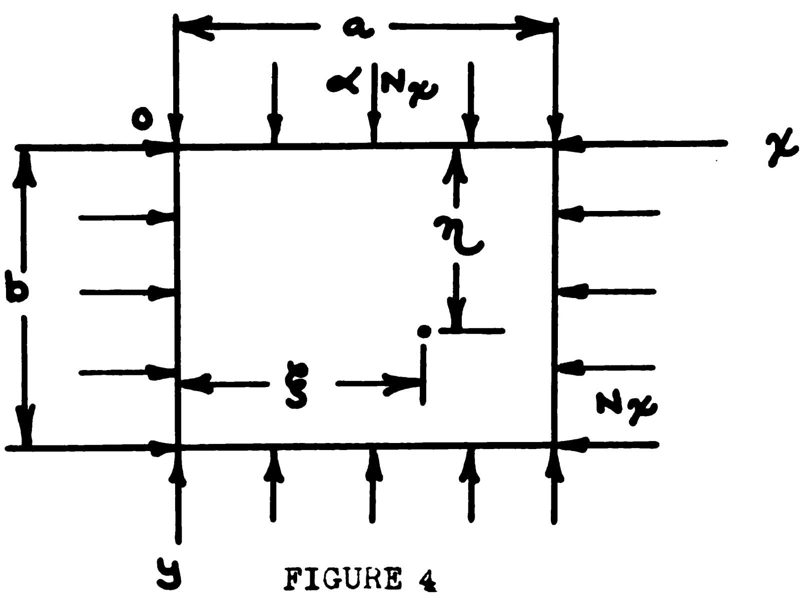


FIGURE 4

In Figure 4

a = length of the plate in the x - direction,

b = length of the plate in the y - direction,

N_x = uniform compressive force per unit length acting in the plane of the plate on the edges $x=0, x=a$,

αN_x = uniform compressive force per unit length acting in the plane of the plate on the edges $y=0, y=b$ where α is a dimensionless constant.

The series expression Eq.(2) for the deflection of this plate is

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big[A_{mn}\, f_m \cdot g_n + B_{mn}\, F_m \cdot G_n$$

$$+ C_{mn}\, f_m \cdot G_n + E_{mn}\, F_m \cdot g_n \Big], \tag{18}$$

where the functions

$$f_m = \left(1 - \cos \frac{2m\pi x}{a}\right) \tag{19}$$

and

$$F_m = \left(\frac{u_m \frac{x}{a} - \sin u_m \frac{x}{a}}{u_m - \sin u_m}\right) - \left(\frac{1 - \cos u_m \frac{x}{a}}{1 - \cos u_m}\right) \tag{20}$$

are the complete set of eigenfunctions of a clamped beam of length a.[13] The functions

$$g_n = \left(1 - \cos \frac{2n\pi y}{b}\right) \tag{21}$$

and

$$G_n = \left(\frac{u_n \frac{y}{b} - \sin u_n \frac{y}{b}}{u_n - \sin u_n}\right) - \left(\frac{1 - \cos u_n \frac{y}{b}}{1 - \cos u_n}\right) \tag{22}$$

are the complete set of eigenfunctions of a clamped beam of length b. The numbers u_m and u_n respectively in the functions F_m and G_n are the positive roots[14] of the transcendental equation

$$\tan \frac{u}{2} = \frac{u}{2} .$$

The coefficients A_{mn}, B_{mn}, C_{mn}, and E_{mn} are the multiple set of parameters to be determined.

The same expression for the strain energy of bending [Eq.(7)] is used as was used in Chapter 3. We observe the

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13. F.B. Hildebrand, Op.Cit. p. 217.
14. See Table II.

orthogonality properties of the eigenfunctions,[15] namely

$$\left.\begin{aligned}&\int_0^a f_m \cdot f_{m_1}\,dx\\ &\int_0^a f'_m \cdot f'_{m_1}\,dx\\ &\int_0^a f''_{m_1} \cdot f''_{m_1}\,dx\end{aligned}\right\}\begin{aligned}&= 0 \text{ for } m \neq m_1\\ &\neq 0 \text{ for } m = m_1\end{aligned}$$

A fourth orthogonality property,

$$\int_0^a f_m \cdot f''_{m_1}\,dx \qquad \begin{aligned}&= 0 \text{ for } m \neq m_1\\ &\neq 0 \text{ for } m = m_1\end{aligned}$$

is obtained by integration by parts as follows:

$$\int_0^a f_m \cdot f''_{m_1}\,dx = (f_m \cdot f'_{m_1})\Big|_0^a - \int_0^a f'_m \cdot f'_{m_1}\,dx .$$

The integrated part vanishes because of the end conditions at $x=0$, $x=a$. Since the remaining integral is the same as the second orthogonality property listed above, the fourth orthogonality property is proved. The same orthogonality properties of the eigenfunctions are true if f_m and f_{m_1} are replaced by F_m and F_{m_1}. If, however, only one of the two functions is replaced (f_m by F_m or f_{m_1} by F_{m_1}, but not both), then the integrals vanish whether or not $m = m_1$. Similar orthogonality properties are true for the eigenfunctions g_n and G_n .

15. Friedrich Bleich, Op.Cit, pp.65-69.

With these properties in mind, we first evaluate the contribution of the curved bracket

$$\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 dx\, dy$$

in Equation (7). This expression can be expanded to

$$\int_0^a \int_0^b \left\{ \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2\, \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right\} dx\, dy .$$

The only terms that will not vanish due to orthogonality when the appropriate derivatives of w [Eq.(18)] are substituted in the expressions above are found to be

$$\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx\,dy = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^a \int_0^b \Big\{ A_{mn}^2 (f_m'')^2 \cdot g_n^2 + B_{mn}^2 (F_m'')^2 \cdot G_n^2 + C_{mn}^2 (f_m'')^2 \cdot G_n^2 + E_{mn}^2 (F_m'')^2 g_n^2 \Big\} dx\,dy ,$$

$$2 \int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \right) dx\,dy = 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^a \int_0^b \Big\{ A_{mn}^2 f_m f_m'' g_n g_n'' + B_{mn}^2 F_m F_m'' G_n G_n'' + C_{mn}^2 f_m f_m'' G_n G_n'' + E_{mn}^2 F_m F_m'' g_n g_n'' \Big\} dx\,dy ,$$

$$\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial y^2} \right)^2 dx\,dy = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^a \int_0^b \Big\{ A_{mn}^2 f_m^2 (g_n'')^2 + B_{mn}^2 F_m^2 (G_n'')^2 + C_{mn}^2 f_m^2 (G_n'')^2 + E_{mn} F_m^2 (g_n'')^2 \Big\} dx\,dy$$

Similar expressions for the square bracket of Equation (7) are also found. The appropriate derivatives of the eigen-functions f_m, F_m, g_n, and G_n are calculated from Equations (19,20,21, and 22), and substituted in Equation (7). We find that the square bracket vanishes and the curved bracket when integrated produces

$$V = \frac{1}{2} N_o \frac{b}{a} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big\{ 4\pi^2 A_{mn}^2 \left(3m^4 + 2\left(\frac{a}{b}\right)^2 m^2 n^2 + 3\left(\frac{a}{b}\right)^4 n^4\right) +$$

$$\frac{B_{mn}^2}{\pi^2} \left(H_m \cdot J_n + 2\left(\frac{a}{b}\right)^2 K_m \cdot L_n + \left(\frac{a}{b}\right)^4 P_m \cdot Q_n\right) +$$

$$C_{mn}^2 \left(8m^4 \pi^2 \cdot J_n + 4\left(\frac{a}{b}\right)^2 m^2 \cdot L_n + \frac{3}{2}\left(\frac{a}{b}\right)^4 \frac{Q_n}{\pi^2}\right) +$$

$$E_{mn}^2 \left(\frac{3}{2} \frac{H_m}{\pi^2} + 4\left(\frac{a}{b}\right)^2 n^2 \cdot K_m + 8\left(\frac{a}{b}\right)^4 n^4 \pi^2 \cdot P_m\right) \Big\} , \quad (23)$$

where

$$N_o = \frac{\pi^2 D}{a^2} ,$$

$$H_m = u_m^3 \left[\frac{\left(\frac{u_m}{2} - \frac{\sin 2u_m}{4}\right)}{(u_m - \sin u_m)^2} - \frac{\sin^2 u_m}{(u_m - \sin u_m)(1 - \cos u_m)} + \frac{\left(\frac{u_m}{2} + \frac{\sin 2u_m}{4}\right)}{(1 - \cos u_m)^2} \right] ,$$

$$P_m = \frac{1}{u_m}\left[\frac{\left(\frac{u_m^3}{3} - 2\sin u_m + 2u_m\cos u_m + \frac{u_m}{2} - \frac{\sin 2u_m}{4}\right)}{(u_m - \sin u_m)^2} + \right.$$

$$\frac{(-u_m^2 + 2u_m \sin u_m - \sin^2 u_m)}{(u_m - \sin u_m)(1 - \cos u_m)} +$$

$$\left.\frac{\left(\frac{3}{2}u_m - 2\sin u_m + \frac{1}{4}\sin 2u_m\right)}{(1-\cos u_m)^2}\right] \text{, and}$$

$$K_m = u_m\left[\frac{\left(\frac{3}{2}u_m - 2\sin u_m + \frac{1}{4}\sin 2u_m\right)}{(u_m - \sin u_m)^2} + \right.$$

$$\frac{(2\cos u_m - 2 + \sin^2 u_m)}{(u_m - \sin u_m)(1-\cos u_m)} +$$

$$\left.\frac{\left(\frac{u_m}{2} - \frac{1}{4}\sin 2u_m\right)}{(1-\cos u_m)^2}\right] .$$

Note that $Q_n = H_n$, $J_n = P_n$, and $L_n = K_n$. Different symbols have been used to avoid confusion when one of the quantities is evaluated for an index m, and the other one is evaluated for a different index n. These quantities have been evaluated for m,n = 1,2, and 3, and are tabulated in Table III.

The same expression U_w for the change in the potential energy of the external loads during buckling is used as in Chapter 3 [Eq.(10)]. The appropriate derivatives of w [Eq.(18)] are taken and substituted in U_w. The expression for U_w

when integrated is

$$U_w = -\frac{1}{2}\frac{\pi^2 b}{a}\cdot N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{3A_{mn}^2\left(m^2+\alpha\left(\tfrac{a}{b}\right)^2 n^2\right)+$$

$$\frac{B_{mn}^2}{\pi^2}\left(K_m\cdot J_n+\alpha\left(\tfrac{a}{b}\right)^2 L_n P_m\right)+$$

$$C_{mn}^2\left(2m^2\cdot J_n+\alpha\left(\tfrac{a}{b}\right)^2\frac{3}{2\pi^2}L_n\right)+$$

$$E_{mn}^2\left(\frac{3}{2\pi^2}K_m+\alpha\left(\tfrac{a}{b}\right)^2\cdot 2n^2\cdot P_m\right)\Big\}, \tag{24}$$

where K_m, J_n, L_n, and P_m are the quantities defined following Equation (23).

The extended Ritz method as discussed in Chapter 2 is now applied. The expressions found for V [Eq.(23)] and U_w [Eq.(24)] are substituted in $\bar{U}$ [Eq.(4)].

The necessary conditions[16] for $\bar{U}$ to be stationary are

$$\frac{\partial\bar{U}}{\partial A_{mn}}=0,\quad \frac{\partial\bar{U}}{\partial B_{mn}}=0,\quad \frac{\partial\bar{U}}{\partial C_{mn}}=0,\quad \frac{\partial\bar{U}}{\partial E_{mn}}=0$$

and

$$w(\xi,\eta)=0.$$

The derivatives of $\bar{U}$ with respect to A_{mn} are

$$\frac{\partial\bar{U}}{\partial A_{mn}} = A_{mn}\;\frac{b}{a}N_0\pi^2\Big[4\left(3m^4+2\left(\tfrac{a}{b}\right)^2 m^2n^2+\right.$$

$$\left.3\left(\tfrac{a}{b}\right)^4 n^4\right)-3\mu\left(m^2+\alpha\left(\tfrac{a}{b}\right)^2 n^2\right)\Big]-\lambda\cdot f_m(\xi)\cdot g_n(\eta)=0. \tag{25}$$

16. See Lagrange's multiplier method, Chapter 2.

The derivatives of $\bar{U}$ with respect to B_{mn} are

$$\frac{\partial \bar{U}}{\partial B_{mn}} = B_{mn}\frac{b}{a}N_0\left[\left(H_m \cdot J_n + 2\left(\frac{a}{b}\right)^2 K_m \cdot L_n + \left(\frac{a}{b}\right)^4 P_m Q_n\right) - \pi^2\mu\left(K_m \cdot J_n + \alpha\left(\frac{a}{b}\right)^2 P_m \cdot L_n\right)\right] - \pi^2\lambda \cdot F_m(\xi)\cdot G_n(\eta) = 0. \qquad (26)$$

The derivatives of $\bar{U}$ with respect to C_{mn} are

$$\frac{\partial \bar{U}}{C_{mn}} = C_{mn}\frac{b}{a}N_0\left[\left(8\pi^4 m^4 \cdot J_n + 4\left(\frac{a}{b}\right)^2\pi^2 m^2 \cdot L_n + \frac{3}{2}\left(\frac{a}{b}\right)^4 Q_n\right) - \pi^2\mu\left(2\pi^2 m^2 J_n + \alpha\frac{3}{2}\left(\frac{a}{b}\right)^2 L_n\right)\right] - \pi^2\lambda \cdot f_m(\xi)\cdot G_n(\eta) = 0. \qquad (27)$$

The derivatives of $\bar{U}$ with respect to E_{mn} are

$$\frac{\partial \bar{U}}{\partial E_{mn}} = E_{mn}\frac{b}{a}N_0\left[\left(\frac{3}{2}H_m + 4\left(\frac{a}{b}\right)^2\pi^2 n^2 \cdot K_m + 8\left(\frac{a}{b}\right)^4\pi^4 n^4 \cdot P_m\right) - \pi^2\mu\left(\frac{3}{2}K_m + 2\alpha\left(\frac{a}{b}\right)^2\pi^2 n^2 \cdot P_m\right)\right] - \pi^2\lambda\, F_m(\xi)\cdot g_n(\eta) = 0. \qquad (28)$$

In Equations (25,26,27, and 28), λ may be zero or different from zero. If λ is zero, the critical value of is determined from one of the Equations (25,26,27, and 28) in a manner similar to that for the case of the simply supported plate of Chapter 3. When the lowest critical value of μ has been chosen in this manner, only one term of the series for w [Eq.(18)] survives. The other coefficients of that set and all three of the remaining sets of coefficients must be equal to zero in order that Equations (25,26,27, and 28) can all be satisfied for the lowest critical buckling value of μ chosen. Physically this means that the internal point support is located at a nodal point of one of the buckling modes of a clamped plate with no internal support.

For the case that λ is different from zero, Equations (25,26,27, and 28) require that for each choice of the indices m,n,

$$A_{mn} = \frac{\dfrac{\lambda\, a}{\pi^2\, b\, N_o}\cdot\left(f_m(\xi)\cdot g_n(\eta)\right)}{\left[4\left(3m^4 + 2\left(\frac{a}{b}\right)^2 m^2 n^2 + 3\left(\frac{a}{b}\right)^4 n^4\right) - 3\mu\left(m^2 + \alpha\left(\frac{a}{b}\right)^2 n^2\right)\right]},$$

$$B_{mn} = \frac{\dfrac{\lambda a \pi^2}{b N_o}\cdot\left(F_m(\xi)\cdot G_n(\eta)\right)}{\left[H_m\cdot J_n + 2\left(\frac{a}{b}\right)^2 K_m\cdot L_n + \left(\frac{a}{b}\right)^4 P_m\cdot Q_n\right] - \pi^2\mu\left[K_m\cdot J_n + \alpha\left(\frac{a}{b}\right)^2 P_m\cdot L_n\right]},$$

$$C_{mn} = \frac{\dfrac{\lambda a \pi^2}{b N_o}\cdot\left(f_m(\xi)\cdot G_n(\eta)\right)}{\left[8\pi^4 m^4\cdot J_n + 4\left(\frac{a}{b}\right)^2 \pi^2 m^2\cdot L_n + \frac{3}{2}\left(\frac{a}{b}\right)^4\cdot Q_n\right] - \pi^2\mu\left[2\pi^2 m^2\cdot J_n + \alpha\,\frac{3}{2}\left(\frac{a}{b}\right)^2 L_n\right]}, \text{ and}$$

$$E_{mn} = \frac{\dfrac{\lambda a \pi^2}{b N_o}\cdot\left(F_m(\xi)\cdot g_n(\eta)\right)}{\left[\frac{3}{2} H_m + 4\left(\frac{a}{b}\right)^2 \pi^2 n^2 K_m + 8\left(\frac{a}{b}\right)^4 \pi^4 n^4\cdot P_m\right] - \pi^2\mu\left[\frac{3}{2} K_m + \alpha\, 2\left(\frac{a}{b}\right)^2 \pi^2 n^2\cdot P_m\right]}.$$

The other necessary condition for a stationary value of $\bar{U}$ is $w(\xi,\eta)=0$. Substitution of the expressions for Amn, Bmn, Cmn, and Emn in this necessary condition results in the characteristic equation

$$\frac{\lambda \pi^2 a}{N_o b} \cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \frac{\left(f_m^2(\xi) \cdot g_n^2(\eta) \right)}{\pi^4 \left[4\left(3m^4 + 2\left(\frac{a}{b}\right)^2 m^2 n^2 + 3\left(\frac{a}{b}\right)^2 n^4 \right) - 3\mu \left(m^2 + \alpha \left(\frac{a}{b}\right)^2 n^2 \right) \right]} + \right.$$

$$\frac{F_m^2(\xi) \cdot G_n^2(\eta)}{\left[H_m \cdot J_n + 2\left(\frac{a}{b}\right)^2 K_m \cdot L_n + \left(\frac{a}{b}\right)^4 P_m \cdot Q_n \right] - \pi^2 \mu \left[K_m \cdot J_n + \alpha \left(\frac{a}{b}\right)^2 P_m \cdot L_n \right]} +$$

$$\frac{f_m^2(\xi) \cdot G_n^2(\eta)}{\left[8\pi^4 m^4 \cdot J_n + 4\left(\frac{a}{b}\right)^2 \pi^2 m^2 \cdot L_n + \frac{3}{2}\left(\frac{a}{b}\right)^4 \cdot Q_n \right] - \pi^2 \mu \left[2\pi^2 m^2 \cdot J_n + \alpha \frac{3}{2} \left(\frac{a}{b}\right)^2 L_n \right]} +$$

$$\left. \frac{F_m^2(\xi) \cdot g_n^2(\eta)}{\left[\frac{3}{2} H_m + 4\left(\frac{a}{b}\right)^2 \pi^2 n^2 \cdot K_m + 8\left(\frac{a}{b}\right)^4 \pi^4 n^4 \cdot P_m \right] - \pi^2 \mu \left[\frac{3}{2} K_m + \alpha \cdot 2\left(\frac{a}{b}\right)^2 \pi^2 n^2 \cdot P_m \right]} \right\} = 0 \quad (29)$$

Let $\psi(\mu)$ equal the doubly infinite sum in Equation (29). Then the values of μ that satisfy the equation

$$\psi(\mu) = 0 \tag{30}$$

will be the critical values of μ for the case that λ is different than zero.

Solution of the Characteristic Equation

The characteristic equation of this Chapter is solved by a numerical interpolation method similar to the method of solution of the characteristic equation of Chapter 3. A graph of $\psi(\mu)$ versus μ for a specific clamped plate will have vertical asymptotes at values of μ corresponding to the critical buckling values of μ for that clamped plate with no internal point support. These values are found for the case that λ is zero. The lowest solution of the characteristic equation will lie in the interval between the first two asymptotes. A value of μ within this interval is arbitrarily selected and is substituted in a finite number of terms of each of the four infinite arrays. After the sign of the sum is found, a second value of μ is selected, lower than the first choice if the sum is positive, and higher than the first choice if the sum is negative. When two values of μ have been found to give opposite signs for the sum, the root of $\psi(\mu) = 0$ is found by repeated numerical interpolation.

Several critical values of μ for various load ratios α, side ratios $\frac{a}{b}$, and point support locations (ξ, η) are tabulated in Table IV.

The lowest critical value of μ occurs for λ not zero except in the case that the point support is at the middle of the plate ($\xi = \frac{a}{2}$, $\eta = \frac{b}{2}$). The values tabulated were therefore all obtained by solving Equation (30), except for the case of the middle support. In this case the lowest critical value of μ occurs for $\lambda = 0$, and one of the denominators of a term in Equations (25,26,27, and 28) was equated to zero to find the lowest value.

TABLE II

ROOTS OF THE
TRANSCENDENTAL EQUATION

u_m	
u_1	8.9868
u_2	15.4506
u_3	21.8082
u_4	28.1324
u_5	34.4416

TABLE III

EVALUTED CONSTANTS
OF THE CHARACTERISTIC EQUATION

m	H_m Q_m	K_m L_m	P_m J_m
1	896.	11.12	.230
2	7380.	30.9	.200
3	28250.	60.3	.211

TABLE IV

CRITICAL VALUES OF
FOR THE CLAMPED PLATE

ξ	η	$\frac{a}{b}$	α	μ
$\frac{a}{2}$	$\frac{b}{2}$	1	0	11.90
$\frac{a}{2}$	$\frac{b}{2}$	1	1	9.35

5. SUMMARY AND CONCLUSIONS

The problem considered in this paper was to determine the effect of an internal point support on the critical elastic buckling load of a rectangular thin plate under various edge support conditions. The extended Ritz energy method was used in the solution. The general theory of this solution is discussed in Chapter 2. Two applications of the theory, namely the simply supported plate in Chapter 3 and the clamped plate in Chapter 4, were made.

The lowest critical buckling load for a rectangular plate with a point support not in the middle of the plate was shown to lie between the first and second critical buckling load of the plate with no internal point support. The lowest critical load for a plate with a point support in the middle of the plate was shown to be the second critical buckling load of that plate with no internal point support. Therefore the most effective location of the point support is at the middle of the plate.

The critical values of μ (a dimensionless load parameter) can be used for plates of any material that conforms to the assumptions made in Chapter 1, namely that the material is homogeneous, isotropic, and follows Hooke's law. The specific material properties are only introduced when the critical buckling load N_x is evaluated using the dimensionless parameter μ.

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