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PROJECTIVE DIFFERENTIAL GEOMETRY OF
CERTAIN SURFACES IN A SPACE
OF FOUR DIMENSIONS

THESIS FOR THE DEGREE OF M. A.

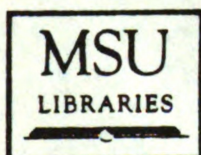
Paul L. Dressel

1934

Geometry, Projective
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PROJECTIVE DIFFERENTIAL GEOMETRY OF CERTAIN SURFACES
IN A SPACE OF FOUR DIMENSIONS

A Thesis
Submitted to the Faculty
of
MICHIGAN STATE COLLEGE
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In Partial Fulfillment of the
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of
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Paul L. Dressel

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PROJECTIVE DIFFERENTIAL GEOMETRY OF CERTAIN SURFACES IN A SPACE OF FOUR DIMENSIONS

1. INTRODUCTION

It is the purpose of this paper to study a surface in S_4 which sustains a conjugate net. The integrability conditions for the three differential equations defining the surface are derived. The differential equations are transformed by $x = \lambda \bar{x}$, $\bar{u} = \phi(u)$, $\bar{v} = \psi(v)$ and by suitable choice of λ , x_u and x_v are made to be covariant. A complete system of invariants is found.

Two covariant points in addition to x , x_u , x_v are found and a covariant local pentahedron of reference established. Canonical expansions for the coordinates of a point on the surface are determined. Certain nets on the surface are discussed. Finally several different hyperquadrics are found and briefly characterized.

2. THE DIFFERENTIAL EQUATIONS

Let the surface S_x be determined by the parametric equations

$$(1) \quad x_i = x_i(u, v) \quad , \quad i = 1, 2, 3, 4, 5.$$

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The parametric net is chosen to be the conjugate net on the surface. The five functions x_i are solutions of the system of differential equations

$$\begin{aligned}
 x_{uuu} &= \alpha x_{uu} + \beta x_{vv} + l x_u + m x_v + d x, \\
 x_{uuv} &= a x_{uu} + L x_u + M x_v + D x, \\
 (2) \quad x_{uv} &= a x_u + b x_v + c x, \\
 x_{uvv} &= b x_{vv} + L' x_u + M' x_v + D' x, \\
 x_{vvv} &= \gamma x_{uu} + \delta x_{vv} + l' x_u + m' x_v + d' x;
 \end{aligned}$$

wherein

$$\begin{aligned}
 L &= ab + c + a_u, & L' &= a^2 + a_v, \\
 M &= b^2 + b_u, & M' &= ab + c + b_v, \\
 D &= bc + c_u, & D' &= ac + c_v,
 \end{aligned}$$

the second and fourth being obtained from the third by differentiation. This system is completely integrable under the conditions

$$\begin{aligned}
 \alpha_v + \beta \gamma &= a_u + L, & \gamma d + \delta D' + m' c + d'_u &= b d' + L' c + D'_v, \\
 \beta \delta + \beta_v + m &= \beta a, & \gamma L + \delta L' + l'_u + a m' + d' &= b l' + L' a + L'_v, \\
 (3) \quad \alpha L + l_v + \beta l' &= L_u + a M + D, & \gamma m + \delta M' + m'_u &= b L' + M'_v + D', \\
 \alpha M + l b + m_v + d + \beta m' &= a m + b M + M_u, & \alpha \gamma + \gamma_u + l' &= b \gamma, \\
 \alpha D + c l + d_v + \beta d' &= a d + c M + D_u, & \beta \gamma + \delta_u &= b_v + M'.
 \end{aligned}$$

From the first and last of (3) we find readily that

$$(4) \quad \alpha_v + 2b_v = \delta_u + 2a_u.$$

Hence from (4) a function θ exists such that

$$(5) \quad \theta_u = \alpha + 2b, \quad \theta_v = \delta + 2a.$$

If the transformations

$$(6) \quad x = \lambda \bar{x},$$

$$(7) \quad \bar{u} = \phi(u), \quad \bar{v} = \psi(v),$$

are performed on system (2), the new coefficients

(denoted by $\bar{\alpha}, \bar{\beta}, \dots$ etc.) are respectively

$$(8) \quad \begin{aligned} \bar{\alpha} &= \alpha - 3 \frac{\lambda_u}{\lambda}, & \bar{a} &= a - \frac{\lambda_v}{\lambda}, \\ \bar{\beta} &= \beta, & \bar{b} &= b - \frac{\lambda_u}{\lambda}, \\ \bar{l} &= l + 2\alpha \frac{\lambda_u}{\lambda} - 3 \frac{\lambda_{uu}}{\lambda}, & \bar{c} &= c + b \frac{\lambda_v}{\lambda} + a \frac{\lambda_u}{\lambda} - \frac{\lambda_{uv}}{\lambda}, \\ \bar{m} &= m + 2\beta \frac{\lambda_v}{\lambda}, & \bar{\theta}_u &= \theta_u - 5 \frac{\lambda_u}{\lambda}, \\ \bar{d} &= d + m \frac{\lambda_v}{\lambda} + l \frac{\lambda_u}{\lambda} + \beta \frac{\lambda_{vv}}{\lambda} + \alpha \frac{\lambda_{uu}}{\lambda} - \frac{\lambda_{uuu}}{\lambda}; \end{aligned}$$

$$(9) \quad \begin{aligned} \bar{\alpha} &= \frac{1}{\phi_u} \left(\alpha - 3 \frac{\phi_{uu}}{\phi_u} \right), & \bar{a} &= \frac{a}{\psi_v}, \\ \bar{\beta} &= \beta \frac{\psi_v^2}{\phi_u^3}, & \bar{b} &= \frac{b}{\phi_u}, \\ \bar{l} &= \frac{1}{\phi_u^2} \left(l + \alpha \frac{\phi_{uu}}{\phi_u} - \frac{\phi_{uuu}}{\phi_u} \right), & \bar{c} &= \frac{c}{\phi_u \psi_v}, \\ \bar{m} &= \frac{1}{\phi_u^3} (m \psi_v + \beta \psi_{vv}), & \bar{\theta}_u &= \frac{1}{\phi_u} \left(\theta_u - 3 \frac{\phi_{uu}}{\phi_u} \right), \\ \bar{d} &= \frac{d}{\phi_u^3}, \end{aligned}$$

The remaining coefficients are obtainable from (8) and

(9) by the substitution S

$$(10) \quad \begin{pmatrix} u, \alpha, \beta, l, m, d, L, M, D, a, c, 1, 2, 4, \phi, \rho \\ v, \delta, \gamma, m', l', d', M', L', D', b, e, 1, 3, 5, \psi, \sigma \end{pmatrix}.$$

This substitution will also be of use later in obtaining like symmetrical expressions.

3. THE INVARIANTS OF THE SYSTEM OF DIFFERENTIAL EQUATIONS

Consider now an invariant* R of the surface such that $R \neq 0$. We shall find functions of the form

$$(11) \quad r = x_u + h p_u x, \quad s = x_v + h p_v x,$$

wherein

$$(12) \quad p_u = \theta_u + \frac{R_u}{R}, \quad p_v = \theta_v + \frac{R_v}{R},$$

such that r and s are covariant. If we examine the transforms of (11) under the transformations (3) and (7), we observe that these functions will be covariant if and only if

$$(13) \quad h = -\frac{1}{\xi}, \quad \bar{R} = \phi_u^3 \psi^3 R,$$

wherein $\bar{R}$ is the transform of R under (3) and (7).

We now require that r and s be transformed into x_u and x_v respectively. We find that for this purpose it is sufficient to choose

$$(14) \quad \lambda = c R^{\frac{1}{3}} \xi^{\frac{2}{3}}.$$

If we impose the condition that $\beta\gamma \neq 0$, we may choose

$$(15) \quad R = \frac{1}{(\beta\gamma)^3}.$$

The coefficients of the new system of defining differential equations for the choice (14) for λ and (15) for R may be readily computed. Of these coefficients, $\beta, \gamma, d, d', a, b, c, L, M'$ are found to be invariants.

* V.G. Grove, On Canonical Forms of Differential

Equations, Bulletin of the American Mathematical Society, August 1930. p. 333.

This canonical form is seen to be preserved by all transformations of the form

$$(16) \quad x = c\bar{x} \quad , \quad \bar{u} = \phi(u) \quad , \quad \bar{v} = \psi(v) \quad ,$$

wherein c is a non-zero constant, and ϕ and ψ are arbitrary functions of the indicated arguments.

4. THE COMPLETE SYSTEM OF INVARIANTS

Using the invariants $\beta, \gamma, d, d', a, b, c$ we now proceed to find the complete system of invariants. For this purpose introduce the notations

$$(17) \quad A = a^2\beta \quad , \quad B = b^2\gamma \quad , \quad C = \frac{c}{ab} \quad , \quad (ab \neq 0) \quad .$$

The transforms of $a, b, c, \beta, \dots$, by the transformation

(18) are

$$(18) \quad \begin{aligned} \bar{a} &= \frac{a}{\psi_v} \quad , \quad \bar{b} = \frac{b}{\phi_u} \quad , \quad \bar{d} = \frac{d}{\phi_u^3} \quad , \quad \bar{d}' = \frac{d'}{\psi_v^3} \quad , \\ \bar{A} &= \frac{A}{\phi_u^3} \quad , \quad \bar{B} = \frac{B}{\psi_v^3} \quad , \quad \bar{C} = C \quad . \end{aligned}$$

This gives seven invariants; and it is to be noted that if these invariants are given, $a, b, c, d, d', \beta, \gamma$ may be uniquely determined. We may note also, that if the known coefficients be disregarded, we have from the integrability conditions

$$m = f_1(d) \quad , \quad \alpha = f_2(d) \quad , \quad d' = f_3(\alpha) \quad , \quad m' = f_4(d) \quad ,$$

and by the choice of R

$$\begin{aligned} \alpha &= -2b + 3(\log \beta\gamma)_u \quad , \\ d &= -2a + 3(\log \beta\gamma)_v \quad . \end{aligned}$$

Hence we may state the theorem:

Given $a, b, c, \beta, \gamma, d, d'$, all the coefficients of (2) are determined, subject to the integrability conditions and to the particular choice of R . A surface sustaining a conjugate net is therefore uniquely determined in projective space of four dimensions if the invariants $a, b, c, \beta, \gamma, d, d'$ are given as functions of u and v subject to the integrability conditions.

Let us introduce two symbols of operation

$$(19) \quad U = \frac{1}{b} \frac{\partial}{\partial u}, \quad V = \frac{1}{a} \frac{\partial}{\partial v}, \quad a, b \neq 0.$$

Evidently the functions

$$\begin{aligned} V(A), \quad V(b), \quad V(d), \quad V(C), \\ U(B), \quad U(a), \quad U(d'), \quad U(C), \end{aligned}$$

are also invariants. In fact the transforms of these functions by (13) are

$$(20) \quad \begin{aligned} \bar{U}(\bar{B}) &= \frac{U(B)}{\psi_v^3}, & \bar{V}(\bar{A}) &= \frac{V(A)}{\phi_u^6}, \\ \bar{U}(\bar{a}) &= \frac{U(a)}{\psi_v}, & \bar{V}(\bar{b}) &= \frac{V(b)}{\phi_u}, \\ \bar{U}(\bar{d}') &= \frac{U(d')}{\psi_v^3}, & \bar{V}(\bar{d}) &= \frac{V(d)}{\phi_u^3}, \\ \bar{U}(\bar{C}) &= U(C), & \bar{V}(\bar{C}) &= V(C). \end{aligned}$$

Every repetition of this process also gives us an invariant.

From two invariants λ and μ for which

$$\bar{\lambda} = \frac{1}{\alpha_u^l} \lambda, \quad \bar{\mu} = \frac{1}{\alpha_u^m} \mu$$

another, its Wronskian, can always be formed.*

We find

$$\overline{\lambda^m \mu^{-l}} = \lambda^m \mu^{-l},$$

whence by logarithmic differentiation we find that

$$m \frac{\lambda_u}{\lambda} - l \frac{\mu_u}{\mu} = \frac{1}{\alpha_u} \left(m \frac{\lambda_u}{\lambda} - l \frac{\mu_u}{\mu} \right).$$

Hence the function

$$(21) \quad (\mu, \lambda_u) = m \mu \lambda_u - l \lambda \mu_u$$

is an invariant for which

$$(22) \quad \overline{(\mu, \lambda_u)} = \frac{(\mu, \lambda_u)}{\alpha_u^{l+m+1}}.$$

We shall speak of (μ, λ_u) as the Wronskian of μ and λ with respect to u .

By combining the Wronskian process with the operations U and V , it is clearly possible to deduce an infinity of invariants from the seven original ones. We now proceed to show that any invariant is a function of those obtained in this manner.

We have already shown that $a, b, c, d, d', \beta, \gamma$ may be expressed in terms of the seven invariants (18). Since all invariants are functions of $a, b, c, d, d', \beta, \gamma$ and their derivatives, any invariant may be expressed as a

* E.J. Wilczynski, Projective Differential Geometry of Curved Surfaces. First Memoir; Transactions of the American Mathematical Society, Vol. 8 (1907), p. 231. Hereafter referred to as Surfaces.

function of (13) and the derivatives of these quantities.

Introduce infinitesimal transformations by putting

$$(23) \quad \phi(u) = u + f(u) \delta t, \quad \psi(v) = v + g(v) \delta t.$$

Hence

$$(24) \quad \begin{aligned} \phi_u &= 1 + f_u \delta t, & \psi_u &= 0, \\ \phi_v &= 0, & \psi_v &= 1 + g_v \delta t, \\ \phi_u \psi_v - \phi_v \psi_u &= 1 + (f_u + g_v) \delta t. \end{aligned}$$

The infinitesimal transformations of (18) become

$$(25) \quad \begin{aligned} \delta a &= \bar{a} - a = \frac{a}{\psi_v} - a = a(1 - g_v) \delta t - a = -g_v a \delta t, \\ \delta b &= -f_u b \delta t, & \delta A &= -3f_u A \delta t, \\ \delta d &= -3f_u d \delta t, & \delta B &= -3g_v B \delta t, \\ \delta d' &= -3g_v d' \delta t, & \delta C &= 0. \end{aligned}$$

The infinitesimal transformation, δF , of any function F of the functions (18) being known, the infinitesimal transformations δF_u and δF_v of its first derivatives may be easily found. We find that

$$(26) \quad \begin{aligned} \delta F_u &= \frac{\partial \bar{F}}{\partial \bar{u}} - \frac{\partial F}{\partial u} = \frac{\partial (F + \delta F)}{\partial u} \frac{1}{1 + f_u \delta t} - \frac{\partial F}{\partial u} = \frac{\partial (\delta F)}{\partial u} - f_u F_u \delta t, \\ \delta F_v &= \frac{\partial \bar{F}}{\partial \bar{v}} - \frac{\partial F}{\partial v} = \frac{\partial (F + \delta F)}{\partial v} \frac{1}{1 + g_v \delta t} - \frac{\partial F}{\partial v} = \frac{\partial (\delta F)}{\partial v} - g_v F_v \delta t. \end{aligned}$$

Thus we find that

$$(27) \quad \begin{aligned} \delta b_u &= -(f_{uu} b + 2f_u b_u) \delta t, & \delta b_v &= -(g_v + f_u) b_v \delta t, \\ \delta a_u &= -(g_v + f_u) a_u \delta t, & \delta a_v &= -(g_{vv} a + 2g_v a_v) \delta t, \\ \delta d_u &= -(3f_{uu} d + 4f_u d_u) \delta t, & \delta d_v &= -(3f_u + g_v) d_v \delta t, \\ \delta d'_u &= -(3g_v + f_u) d'_u \delta t, & \delta d'_v &= -(3g_{vv} d' + 4g_v d'_v) \delta t, \\ \delta A_u &= -(3f_{uu} A + 4f_u A_u) \delta t, & \delta A_v &= -(3f_u + g_v) A_v \delta t, \\ \delta B_u &= -(f_{uu} C + 2f_u C_u) \delta t, & \delta B_v &= -(3g_{vv} B + 4g_v B_v) \delta t, \\ \delta C_u &= -f_u C_u \delta t, & \delta C_v &= -g_v C_v \delta t. \end{aligned}$$

All absolute invariants π which involve only (13)

and their first derivatives must, therefore, satisfy the system of partial differential equations obtained by forming the infinitesimal transformation $\delta \pi$ of an arbitrary function π of these arguments and equating to zero the coefficients of f_{uu}, g_{vv}, f_u, g_v . If we are looking only for relative invariants, we may dispense with the latter two equations. We omit the proof of this and also certain other necessary discussion since it may be found in Wilczynski's memoir in exactly the form needed here.*

Form the infinitesimal transformation $\delta \pi$ of an arbitrary function π of (18) and their derivatives

$$\begin{aligned}
 \delta \pi = & \frac{\partial \pi}{\partial a} \delta a + \frac{\partial \pi}{\partial b} \delta b + \frac{\partial \pi}{\partial d} \delta d + \frac{\partial \pi}{\partial d'} \delta d' + \frac{\partial \pi}{\partial A} \delta A + \frac{\partial \pi}{\partial B} \delta B \\
 & + \frac{\partial \pi}{\partial a_u} \delta a_u + \frac{\partial \pi}{\partial b_u} \delta b_u + \frac{\partial \pi}{\partial d_u} \delta d_u + \frac{\partial \pi}{\partial d'_u} \delta d'_u + \frac{\partial \pi}{\partial A_u} \delta A_u + \frac{\partial \pi}{\partial B_u} \delta B_u \\
 & + \frac{\partial \pi}{\partial C_u} \delta C_u + \frac{\partial \pi}{\partial a_v} \delta a_v + \frac{\partial \pi}{\partial b_v} \delta b_v + \frac{\partial \pi}{\partial d_v} \delta d_v + \frac{\partial \pi}{\partial d'_v} \delta d'_v \\
 & + \frac{\partial \pi}{\partial A_v} \delta A_v + \frac{\partial \pi}{\partial B_v} \delta B_v + \frac{\partial \pi}{\partial C_v} \delta C_v, \\
 (28) = & \left(-g_{va} \frac{\partial \pi}{\partial a} - f_{ub} \frac{\partial \pi}{\partial b} - 3f_{ud} \frac{\partial \pi}{\partial d} - 3g_{vd'} \frac{\partial \pi}{\partial d'} - 3f_{uA} \frac{\partial \pi}{\partial A} - 3g_v B \frac{\partial \pi}{\partial B} \right. \\
 & - f_{uu} b \frac{\partial \pi}{\partial b_u} - 2f_{ub} \frac{\partial \pi}{\partial b_u} - a_u g_v \frac{\partial \pi}{\partial a_u} - f_{ua} \frac{\partial \pi}{\partial a_u} - 3f_{uud} \frac{\partial \pi}{\partial d_u} - 4f_{ud} \frac{\partial \pi}{\partial d_u} \\
 & - 3g_v d'_u \frac{\partial \pi}{\partial d'_u} - f_{ud'} \frac{\partial \pi}{\partial d'_u} - 3f_{uA} \frac{\partial \pi}{\partial A_u} - 4f_{uA} \frac{\partial \pi}{\partial A_u} - 3g_v B_u \frac{\partial \pi}{\partial B_u} - f_u B_u \frac{\partial \pi}{\partial B_u} \\
 & - 2f_{uC_u} \frac{\partial \pi}{\partial C_u} - g_{vv} a \frac{\partial \pi}{\partial a_v} - 2g_{va} \frac{\partial \pi}{\partial a_v} - g_{vb} \frac{\partial \pi}{\partial b_v} - f_{ub} \frac{\partial \pi}{\partial b_v} - 3f_{ud} \frac{\partial \pi}{\partial d_v} \\
 & - g_v d'_v \frac{\partial \pi}{\partial d'_v} - 3g_{vd'} \frac{\partial \pi}{\partial d'_v} - 4g_{vd'} \frac{\partial \pi}{\partial d'_v} - 3f_{uA_v} \frac{\partial \pi}{\partial A_v} - g_v A_v \frac{\partial \pi}{\partial A_v} \\
 & \left. - 3g_v B \frac{\partial \pi}{\partial B_v} - 4g_v B_v \frac{\partial \pi}{\partial B_v} - g_v C_v \frac{\partial \pi}{\partial C_v} \right) \delta t.
 \end{aligned}$$

* E.J. Wilczynski, Surfaces. pp. 254, 255.

Equating the coefficients of f_{uu} and g_{vv} to zero we obtain

$$(29) \quad \begin{aligned} b \frac{\partial \theta}{\partial b_u} + 3d \frac{\partial \theta}{\partial d_u} + 3A \frac{\partial \theta}{\partial A_u} &= 0, \\ \alpha \frac{\partial \theta}{\partial \alpha_v} + 3d' \frac{\partial \theta}{\partial d'_v} + 3B \frac{\partial \theta}{\partial B_v} &= 0. \end{aligned}$$

All relative invariants involving only (13) and their first derivatives are isobaric solutions of the complete system of two equations (29) which involve twenty-one independent variables. It has therefore nineteen independent solutions. These may be selected in the form

$$(30) \quad \begin{aligned} &\alpha, b, d, d', A, B, C, \nabla(b), \nabla(d), \nabla(A), \nabla(C), U(\alpha), \\ &U(d'), U(B), (\alpha, d'_v), (b, d_u), (B, d'_v), (A, d_u), (A, C_u). \end{aligned}$$

These are clearly independent.

All relative invariants which involve also the second derivatives of (13) will be isobaric solutions of a complete system of four independent equations in forty-two variables. Hence there are thirty-eight independent solutions. We shall not record them here.

If this process be continued, it can be shown by mathematical induction that there would be $7k+5$ invariants involving the k th derivatives of the invariants (13). Hence the total number of invariants of order less than or equal to n is

$$(31) \quad \sum_{k=0}^n (7k+5) = \frac{1}{2}(7n^2 + 17n + 14).$$

5. A COVARIANT PENTAHEDRON

As shown earlier, by the selection of λ we have made x_u and x_v covariants. We have therefore the three covariant points x, x_u, x_v . Two more points are required to determine a covariant pentahedron of reference. These we proceed to determine.

The covariant point

$$(32) \quad \varrho = \beta x_{vv} + m x_v$$

is the point in which the tangent to the curve $u=c$ through x_v pierces the osculating S_3 to the curve $v=c$ at x . A similar characterization may be made for the point

$$(33) \quad \sigma = \gamma x_{uu} + l' x_u$$

These characterizations may be easily verified by noticing that from equations (2)

$$(34) \quad \begin{aligned} x_{uuu} - \alpha x_{uu} - l x_u - dx &= \beta x_{vv} + m x_v \\ x_{vvv} - \delta x_{vv} - m' x_v - d' x &= \gamma x_{uu} + l' x_u \end{aligned}$$

The points $x, x_u, x_v, \varrho, \sigma$ therefore, are covariant points and form the vertices of a covariant pentahedron of reference.

6. POWER SERIES EXPANSIONS

The coordinates of any point X near a point P_x on

a surface S can be represented by a Taylor's expansion of the form

$$\mathbf{X} = \mathbf{x} + x_u \Delta u + x_v \Delta v + (x_{uu} \Delta u^2 + 2x_{uv} \Delta u \Delta v + x_{vv} \Delta v^2)/2 + \dots.$$

By means of equations (2), the equations obtained therefrom by differentiation together with the integrability conditions, and equations (32), (33) it is possible to express every derivative of $\mathbf{x}$ uniquely as a linear combination of $\mathbf{x}, x_u, x_v, \varrho, \sigma$. Hence $\mathbf{X}$ may be expressed uniquely in the form

$$\mathbf{X} = x_1 \mathbf{x} + x_2 x_u + x_3 x_v + x_4 \varrho + x_5 \sigma,$$

wherein $x_1, \dots, x_5$ are power series in $\Delta u, \Delta v$. The functions $x_1, \dots, x_5$ are coordinates of the point $\mathbf{X}$, the pentahedron of reference being $\mathbf{x}, x_u, x_v, \varrho, \sigma$. Performing the calculations indicated, we find that

$$\begin{aligned} x_1 &= 1 + c \Delta u \Delta v + \frac{1}{2} (D \Delta u^2 \Delta v + D' \Delta u \Delta v^2) + \dots, \\ x_2 &= \Delta u - \frac{\alpha'}{2\beta} \Delta u^2 + a \Delta u \Delta v + \frac{1}{6} \left[\left(L - \frac{\alpha \beta'}{\beta} \right) \Delta u^3 + 3 \left(L - \frac{\alpha \beta'}{\beta} \right) \Delta u^2 \Delta v + 3 L' \Delta u \Delta v^2 \right] + \dots, \\ x_3 &= \Delta v - \frac{m}{2\beta} \Delta v^2 + b \Delta u \Delta v + \frac{1}{6} \left[\left(m' - \frac{\delta m}{\beta} \right) \Delta v^3 + 3 \left(m' - \frac{\delta m}{\beta} \right) \Delta u \Delta v^2 + 3 M \Delta u^2 \Delta v \right] + \dots, \\ x_4 &= \frac{1}{2\beta} \Delta u^2 + \frac{1}{6} \left(\frac{\alpha}{\beta} \Delta u^3 + 3 \frac{\alpha}{\beta} \Delta u^2 \Delta v + \Delta v^3 \right) + \dots, \\ x_5 &= \frac{1}{2\beta} \Delta v^2 + \frac{1}{6} \left(\Delta u^3 + 3 \frac{b}{\beta} \Delta u \Delta v^2 + \frac{\delta}{\beta} \Delta v^3 \right) + \dots. \end{aligned} \quad (35)$$

Introducing non-homogeneous coordinates by the definitions

$$(36) \quad x = \frac{x_2}{x_1}, \quad y = \frac{y_3}{y_1}, \quad z = \frac{y_4}{y_1}, \quad w = \frac{y_5}{y_1},$$

we find by use of (35) the following expansions

$$\begin{aligned}
 (37) \quad x &= \Delta u - \frac{\ell'}{2\gamma} \Delta u^2 + a \Delta u \Delta v + \frac{1}{6} \left(\ell - \frac{a\ell'}{\gamma} \right) \Delta u^3 + \frac{1}{2} \left(\ell - \frac{a\ell'}{\gamma} - 2c \right) \Delta u^2 \Delta v \\
 &\quad + \frac{1}{2} \Delta u \Delta v^2 + \dots, \\
 y &= \Delta v - \frac{m}{2\beta} \Delta v^2 + b \Delta u \Delta v + \frac{M}{2} \Delta u^2 \Delta v + \frac{1}{2} \left(M' - \frac{bm}{\beta} - 2c \right) \Delta u \Delta v^2 \\
 &\quad + \frac{1}{6} \left(m' - \frac{\delta m}{\beta} \right) \Delta v^3 + \dots, \\
 z &= \frac{1}{2\gamma} \Delta u^2 + \frac{\alpha}{6\gamma} \Delta u^3 + \frac{a}{2\gamma} \Delta u^2 \Delta v + \frac{1}{6} \Delta v^3 + \dots, \\
 w &= \frac{1}{2\beta} \Delta v^2 + \frac{1}{6} \Delta u^3 + \frac{b}{2\beta} \Delta u \Delta v^2 + \frac{\delta}{6\beta} \Delta v^3 + \dots.
 \end{aligned}$$

From the expansions (37) it is possible to compute expansions for z and w as power series in x and y by setting z or w equal to a power series in x, y with undetermined coefficients, and then demanding that the expansions (37) satisfy the equation identically in $\Delta u, \Delta v$ to terms of any desired order. By this method we obtain

$$\begin{aligned}
 (38) \quad z &= \frac{1}{2\gamma} x^2 + \frac{1}{2\gamma} \left(\frac{\alpha}{3} + \frac{\ell'}{\gamma} \right) x^3 - \frac{a}{2\gamma} x^2 y + \frac{1}{6} y^3 + \dots, \\
 w &= \frac{1}{2\beta} y^2 + \frac{1}{6} x^3 - \frac{b}{2\beta} x y^2 + \frac{1}{2\beta} \left(\frac{\delta}{3} + \frac{m}{\beta} \right) y^3 + \dots.
 \end{aligned}$$

The expansion for z represents the S_3 which is the projection of the original surface on the hyperplane $w=0$. A similar statement is true of the expansion for w . The following theorem may now be stated. The study of the surfaces represented by the differential equations (2), in the neighborhood of the point X is

equivalent to the study of the series (38).

The series (38) are similar to the power series used for the study of an analytic curve in S_3 . The latter are of the form*

$$y = a_0 + a_1x + a_2x^2 + \dots,$$

$$z = c_0 + c_1x + c_2x^2 + \dots.$$

The difference between the two of course lies in the number of variables.

Referring once again to the series (38), it should be noticed that the tangent plane $x_4 = x_5 = 0$ intersects the surface in a curve with a cusp at the point. A series for y in fractional powers of x could be obtained to further the study of this curve. However, we shall not continue that study here.

7. CONCERNING NETS WITH AXES

Consider two distinct curves C_λ, C_μ defined respectively by the differential equations

$$\begin{aligned} dv - \lambda du &= 0, \\ dv - \mu du &= 0. \end{aligned} \tag{39}$$

* E.P. Lane, Projective Differential Geometry of Curves and Surfaces. The University of Chicago Press (1932), p. 32.

The osculating plane to C_λ is defined by the points

$$(40) \quad x, x' = x_u + \lambda x_v, x'' = x_{uu} + \lambda^2 x_{vv} + 2a\lambda x_u + (\lambda' + 2b\lambda)x_v + 2c\lambda x.$$

The local coordinates of these points, using the pentahedron $x, x_u, x_v, x_{uu}, x_{vv}$ are respectively

$$(41) \quad (1, 0, 0, 0, 0), (0, 1, \lambda, 0, 0), (2c\lambda, 2a\lambda, 2b\lambda, 1, \lambda^2).$$

If the hyperplane

$$\sum_{i=1}^5 \xi_i x_i = 0$$

passes through these three points, its equation must be

$$(42) \quad \xi_3 [-\lambda x_2 + x_3 + (2a\lambda^2 - \lambda' - 2b\lambda)x_4] - \xi_5 (\lambda^2 x_4 - x_5) = 0.$$

The two hyperplanes

$$(43) \quad \begin{aligned} \lambda x_2 - x_3 + (\lambda' + 2b\lambda - 2a\lambda^2)x_4 &= 0, \\ \lambda^2 x_4 - x_5 &= 0, \end{aligned}$$

therefore intersect in the osculating plane to C_λ at

x . Hence equations (43) are the equations of the osculating plane to C_λ at x . Similarly the osculating plane to C_μ at x has the equations

$$(44) \quad \begin{aligned} \mu x_2 - x_3 + (\mu' + 2b\mu - 2a\mu^2)x_4 &= 0, \\ \mu^2 x_4 - x_5 &= 0. \end{aligned}$$

If we require that the two planes (43) and (44) intersect in a line we find that

$$\lambda^2 = \mu^2.$$

Since C_λ and C_μ are distinct it follows that

$$\lambda = -\mu.$$

Rewriting the equations of the two planes under this

condition they are

$$(45) \quad \begin{aligned} \lambda x_2 - x_3 + (\lambda_u + \lambda \lambda_v + 2b\lambda - 2a\lambda^2)x_4 &= 0, \\ \lambda^2 x_4 - x_5 &= 0 \quad ; \end{aligned}$$

$$(45a) \quad \begin{aligned} \lambda x_2 + x_3 + (\lambda_u - \lambda \lambda_v + 2b\lambda + 2a\lambda^2)x_4 &= 0, \\ \lambda^2 x_4 - x_5 &= 0. \end{aligned}$$

These planes intersect in the line joining X to the point

$$(46) \quad Y = -\left(2b + \frac{\lambda_u}{\lambda}\right)x_u + \lambda^2\left(\frac{\lambda_v}{\lambda} - 2a\right)x_v + x_{uu} + \lambda^2 x_{vv}.$$

For every net on the surface of the form

$$(47) \quad dv^2 - \lambda^2 du^2 = 0$$

through X there exists a unique line through X and the point determined from (40) by substituting therein the particular value of λ which determines the net.

As an example we may take the case in which $\lambda = \frac{b}{a}$

The net (47) becomes

$$(48) \quad a^2 dv^2 - b^2 du^2 = 0$$

and the point (46) becomes

$$(49) \quad Y = -\left(2b + \frac{b_u}{b} - \frac{a_u}{a}\right)x_u + \frac{b^2}{a^2}\left(\frac{b_v}{b} - \frac{a_v}{a} - 2a\right)x_v + x_{uu} + \frac{b^2}{a^2}x_{vv}.$$

The net (48) may be characterized as follows:

The point x_{uv} is the point of intersection of the tangents to $u=c$ on the surface generated by x_u , and the tangent to $v=c$ on the surface generated by x_v .

The tangent to the curve whose differential equation is

$$a dv - b du = 0$$

passes through the point x_{uv} . The tangent to the curve defined by

$$a dv + b du = 0$$

is the harmonic conjugate of the tangent to the curve defined by

$$a dv - b du = 0$$

with respect to the tangents to the parametric curves.

Consider the pencil of nets

$$(50) \quad dv^2 - \lambda^2 h^2 du^2 = 0.$$

The curves of this net are the integral curves of the differential equation

$$(51) \quad u' = \frac{\lambda u}{\lambda} u + \frac{\lambda v}{\lambda} u^2, \quad u = \frac{dv}{du}.$$

The osculating planes to the curves of this pencil intersect in the line joining x to

$$(52) \quad \gamma_h = \lambda^2 h^2 x_{vv} + x_{uu} + \lambda^2 h^2 \left(\frac{\lambda v}{\lambda} - 2a \right) x_v - \left(2b + \frac{\lambda u}{\lambda} \right) x_u.$$

The local coordinates of the point γ_h are

$$(53) \quad \begin{aligned} x_1 &= 0, & x_4 &= 1, \\ x_2 &= -\left(2b + \frac{\lambda u}{\lambda} \right), & x_5 &= \lambda^2 h^2. \\ x_3 &= \lambda^2 h^2 \left(-2a + \frac{\lambda v}{\lambda} \right), \end{aligned}$$

It is possible to eliminate h from these local coordinates in two different ways, thus obtaining

$$(54) \quad \begin{aligned} x_3 - \left(\frac{\lambda v}{\lambda} - 2a \right) x_5 &= 0, \\ x_2 + \left(\frac{\lambda u}{\lambda} + 2b \right) x_4 &= 0. \end{aligned}$$

Each of these equations represents a hyperplane. The

two equations taken together represent the intersection of the hyperplanes, this intersection being a plane through the point x . It follows, therefore, that the locus of the axis of the nets of a pencil is a plane through x .

Such a system of curves is of the type called by Bompiani a two-axial system. The plane (54) is the plane of which the given system of curves is the two-axial system.

The envelope of the osculating planes to all the integral curves of (51) is the cubic cone*

$$(55) \quad \left[x_3 - \left(\frac{\lambda_3}{\lambda} - 2a \right) x_5 \right]^2 x_4 = \left[x_2 + \left(\frac{\lambda_4}{\lambda} + 2b \right) x_4 \right]^2 x_5 .$$

The tangent plane

$$x_4 = x_5 = 0$$

is a simple directrix plane of (55), while the plane determined by equations (54) is a double directrix plane.

8. A PROJECTIVELY INVARIANT INTEGRAL

The differential form $\beta \gamma du dv$ is easily shown to be absolutely invariant under the transformation

* E. Bompiani, Projective Differential Geometry of Hyper-space. Lectures given in the University of Chicago, Summer 1930. Part IX. Hereafter referred to as Lectures.

$x = \lambda \bar{x}$, $\bar{u} = \phi(u)$, $\bar{v} = \psi(v)$. Therefore the integral

$$(56) \quad \int (\beta \gamma v')^{\frac{1}{2}} du$$

calculated along a curve is invariant. We may call this integral the projective arc-length of the curve along which it is calculated. We now make use of Euler's equation for the extremals of an integral of the form

$$\int \phi(u, v, v') du,$$

namely,

$$(57) \quad \phi_{v'v'} v'' = \phi_v - \phi_{uv'} - \phi_{vv'} v',$$

and find the differential equation of the extremals of the projective arc length to be

$$(58) \quad v'' = \left(\frac{\beta_u}{\beta} + \frac{\gamma_u}{\gamma} \right) v' - \left(\frac{\beta_v}{\beta} + \frac{\gamma_v}{\gamma} \right) v'^2.$$

This is an equation similar to (51), both of them belonging to the general form

$$(59) \quad v'' = A + B v' + C v'^2 + D v'^3.$$

Evidently for (52)

$$A = 0, B = (\log \beta \gamma)_u, C = -(\log \beta \gamma)_v, D = 0.$$

It has been shown by Bompiani* that on all surfaces sustaining a conjugate net, equation (52) represents a two-axial system of curves. By a little computation it is found that the plane which is the locus of the lines of intersection of the osculating planes to

* E. Bompiani, Lectures, Part IX.

the extremals of $\int (\beta \gamma v)^{\frac{1}{2}} du$ is the plane determined by the points

$$(60) \quad x, [2a + (\log \beta \gamma)_v] x_v + x_{vv}, [2b + (\log \beta \gamma)_u] x_u + x_{uu}.$$

If we call the extremals of $\int (\beta \gamma v)^{\frac{1}{2}} du$ projective geodesics, and the plane determined by (60) the projective normal plane, we may state the theorem:

The osculating planes of all the projective geodesics through x , intersect the projective normal plane determined by the points (60) in lines; and, moreover, this plane is unique for each point x .

9. ASSOCIATED HYPERQUADRICS

The equation of the most general non-singular hyperquadric is

$$(61) \quad a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + a_{44} x_4^2 + a_{55} x_5^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{14} x_1 x_4 + a_{15} x_1 x_5 + a_{23} x_2 x_3 + a_{24} x_2 x_4 + a_{25} x_2 x_5 + a_{34} x_3 x_4 + a_{35} x_3 x_5 + a_{45} x_4 x_5 = 0, \quad |a_{ik}| \neq 0.$$

If we demand that the series (35) satisfy this equation identically in $\Delta u, \Delta v$ as far as terms of the fourth degree we obtain the ∞^4 hyperquadrics

$$(62) \quad \begin{aligned} & a_{22} [x_2^2 - 2\gamma x_1 x_4 + 2(\frac{\alpha}{3} + l') x_2 x_4 - 2\gamma x_3 x_4 + \frac{2}{3} \beta \gamma x_3 x_5] \\ & + a_{33} [x_3^2 - 2\beta x_1 x_5 + 2(\frac{\beta}{3} + m) x_3 x_5 - 2b\beta x_2 x_5 + \frac{2}{3} \beta \gamma x_2 x_4] \\ & + a_{44} x_4^2 + a_{55} x_5^2 + 2a_{45} x_4 x_5 = 0. \end{aligned}$$

These hyperquadrics have four point contact with the surface at the point x . The tangent plane

$x_4 = x_5 = 0$ intersects this four point quadric in the lines

$$(63) \quad a_{22}x_2^2 + a_{33}x_3^2 = 0 = x_4 = x_5 \quad .$$

We have therefore the theorem: The four point hyperquadrics of S at P_x intersect the tangent plane to S at P_x in lines paired in involution, with the conjugate tangents as double lines of the involution.

Consider a curve $u=c$ on S through P_x . At P_x and two neighboring points P_1, P_2 on the curve construct the osculating planes of the curves $v=c$. Now let P_1, P_2 approach P_x independently. Does there exist a hyperquadric such that these three osculating planes are plane generators?

The osculating plane to the curve $u=c$ through X near P_x is determined by the three points

$$X = x + x_v \Delta v + x_{vv} \frac{\Delta v^2}{2} + \dots \quad ,$$

$$X_u = x_u + x_{uv} \Delta v + x_{uuv} \frac{\Delta v^2}{2} + \dots \quad ,$$

$$X_{uu} = x_{uu} + x_{uuv} \Delta v + x_{uuvv} \frac{\Delta v^2}{2} + \dots \quad .$$

Any point Y in the osculating plane may be defined by a linear combination of the form

$$(64) \quad Y = X + kX_u + hX_{uu} \quad .$$

By use of equations (2), (32), (33), Y can be expressed in the form

$$Y = x_1 x + x_2 x_u + x_3 x_v + x_4 e + x_5 \sigma$$

where the local coordinates $x_1, \dots, x_5$ of the point

Y are given by

$$\begin{aligned}
 x_1 &= 1 + k\epsilon\Delta v + hD\Delta v + \frac{hD'}{2}\Delta v^2 + \frac{h\alpha D}{2}\Delta v^2 + \frac{h\epsilon L}{2}\Delta v^2 + \frac{hD_v}{2}\Delta v^2 + \dots, \\
 x_2 &= K - \frac{h\ell'}{\gamma} + ka\Delta v - \frac{ha\ell'}{\gamma}\Delta v + hL\Delta v + \frac{KL'}{2}\Delta v^2 - \frac{ha^2\ell'}{2\gamma}\Delta v^2 + haL\Delta v^2 \\
 &\quad - \frac{ha_v\ell'}{2\gamma}\Delta v^2 + \frac{hL_v}{2}\Delta v^2 + \dots, \\
 (65) \quad x_3 &= \Delta v + kb\Delta v + hM\Delta v - \frac{hM}{2\beta}\Delta v^2 - \frac{kbM}{2\beta}\Delta v^2 + \frac{KM'}{2}\Delta v^2 + \frac{haM}{2}\Delta v^2 \\
 &\quad + \frac{hbL}{2}\Delta v^2 - \frac{hM_v}{2\beta}\Delta v^2 + \frac{hM_v}{2}\Delta v^2 + \frac{hD}{2}\Delta v^2 + \dots, \\
 x_4 &= h\left(\frac{1}{\gamma} + \frac{a}{\gamma}\Delta v + \frac{a^2}{2\gamma}\Delta v^2 + \frac{a_v}{2\gamma}\Delta v^2\right) + \dots, \\
 x_5 &= \frac{1}{2\beta}\Delta v^2 + \frac{kb}{2\beta}\Delta v^2 + \frac{hM}{2\beta}\Delta v^2 + \dots.
 \end{aligned}$$

Demanding that the equation of a general hyperquadric be satisfied by the series (58) for $x_1, \dots, x_5$ identically in h, k and identically in Δv as far as the terms of third degree we find that such a hyperquadric exists if and only if

$$MY = 0$$

Since $\gamma \neq 0$, M must be zero and hence

$$(1) \quad b = \frac{1}{a + \epsilon_1}, \quad (b \neq 0) \quad (2) \quad b \equiv 0.$$

For such points the hyperquadric has the equation

$$(66) \quad a_{33} [x_3^2 - 2\beta x_5 (x_1 + bx_2 + b\ell'x_4)] + a_{35}x_3x_5 + a_{55}x_5^2 = 0.$$

Corresponding to (66), there exists at points for which

$$(1) \quad a = \frac{1}{v + \epsilon_2}, \quad (a \neq 0) \quad (2) \quad a \equiv 0,$$

the hyperquadric

$$(67) \quad b_{22} [x_2^2 - 2\gamma x_4 (x_1 + \alpha x_3 + \alpha m x_5)] + b_{24} x_2 x_4 + b_{44} x_4^2 = 0 .$$

Consider the hyperquadrics of the family of hyperquadrics determined by the osculating planes to a curve C at x , and $k-1$ neighboring points X_i ($i=1, \dots, k-1$) as these points approach x independently along the curve. We shall say that such hyperquadrics have k plane contact with the curve C at x . We shall now determine all of the hyperquadrics which have three plane contact with the curves of the conjugate net on our surface.

The osculating plane to the curve $u=c$ at X near x is determined by the points X, X_v, X_{vv} whose coordinates can be represented respectively by the series

$$\begin{aligned} X &= x + x_v \Delta v + x_{vv} \frac{\Delta v^2}{2} + \dots , \\ X_v &= x_v + x_{vv} \Delta v + x_{vvv} \frac{\Delta v^2}{2} + \dots , \\ X_{vv} &= x_{vv} + x_{vvv} \Delta v + x_{vvvv} \frac{\Delta v^2}{2} + \dots . \end{aligned}$$

Any point Y in the osculating plane may be defined by a linear combination of the form

$$Y = X + h X_v + k X_{vv} .$$

If this be expressed in the form

$$Y = x_1 x + x_2 x_u + x_3 x_v + x_4 e + x_5 \sigma$$

the local coordinates $x_1, \dots, x_5$ of $\mathbf{Y}$ are given by

$$\begin{aligned}
 x_1 &= 1 + kd'\Delta v + (hd' + k\delta d' + k\ell'c + kd'_v) \frac{\Delta v^2}{2} + \dots, \\
 x_2 &= K(\gamma L - \gamma_v \ell' + \ell'_v) \frac{\Delta v^2}{2} + \dots, \\
 (68) \quad x_3 &= h - \frac{km}{\beta} + \left(1 - \frac{hm}{\beta} - \frac{k\delta m}{\beta} + km'\right) \Delta v + (hm' - m - hm\delta + KM\gamma - K\delta^2 m + K\delta m') \frac{\Delta v^2}{2} + \dots, \\
 x_4 &= K\Delta v + (h + Ka + \frac{K\gamma_v}{\gamma} + k\delta) \frac{\Delta v^2}{2} + \dots, \\
 x_5 &= \frac{K}{\beta} + \left(\frac{h + k\delta}{\beta}\right) \Delta v + \frac{1}{\beta} \left(1 + h\delta + k\delta^2 + k\delta_v + km'\right) \frac{\Delta v^2}{2} + \dots.
 \end{aligned}$$

Demanding that the equation of the general hyperquadric be satisfied by the series (68) for $x_1, \dots, x_5$ identically in h, K and identically in Δv as far as terms of the third degree, we obtain the hyperquadrics Q_i , whose equation is

$$(69) \quad Q_i = a_{22}x_2^2 + a_{24}x_2x_4 + a_{25}x_2x_5 + \frac{1}{2\beta}(\gamma_v \ell' - \gamma L - \ell'_v) a_{25}x_4^2 = 0.$$

The equation of the corresponding hyperquadrics for the curve $v=c$ is obtainable from (69) by the transformation (10). There are, therefore, a double infinity of quadrics having three plane contact with the curves of a conjugate net on a surface in S_4 .

Consider the conic

$$(70) \quad Q_i = x_1 = x_2 = 0.$$

Evidently the line $e\sigma$ is tangent to the conic (70) at the point σ .

If we impose the condition that the tangent plane to the surface at x be a plane generator of (69) we find that $a_{22}=0$. Hence the equations of all

hyperquadrics having three plane contact with $u=c$ at x and having the tangent plane at x as a plane generator are

$$(71) \quad \bar{Q}_1 = a_{24} x_2 x_4 + a_{25} x_2 x_5 + A a_{25} x_4^2 = 0,$$

$$A = \frac{1}{2\beta} \left(\frac{\delta_v}{\gamma} l' - \gamma L - l'_v \right).$$

Such hyperquadrics form a pencil.

By use of transformation (10) the hyperquadrics $\bar{Q}_2$, corresponding to (64), for the curve $v=c$ are found to be

$$(72) \quad b_{35} x_3 x_5 + b_{34} x_3 x_4 + B a_{34} x_5^2 = 0,$$

$$B = \frac{1}{2\gamma} \left(\frac{\beta_u}{\beta} m - \beta M - m_u \right).$$

The intersections of the hyperquadrics (64) and (65) are the cubic cones

$$(73) \quad \begin{aligned} x_2 &= -A a_{25} t^2 (b_{35} t + b_{34}), & x_4 &= t (a_{24} t + a_{25}) (b_{35} t + b_{34}), \\ x_3 &= -B b_{34} (a_{24} t + a_{25}), & x_5 &= (a_{24} t + a_{25}) (b_{35} t + b_{34}). \end{aligned}$$

wherein t is a parameter. The vertex of these cones is at the point x .

The intersection of the hyperplane $x_1=0$ with the cubic cone (73) is a twisted cubic. Any point P on this cubic is on $\bar{Q}_1$. Hence the plane determined by the point P and the tangent to $u=c$ is a plane generator of $\bar{Q}_1$. These plane generators intersect in a line, namely the line joining x to P . It follows that the locus of the lines of intersection

of the plane generators of $\bar{Q}_1, \bar{Q}_2$ which intersect in a line, is a cubic cone with vertex at x .

In order to characterize the single hyperquadric to which (63) reduces if $a_{22} = a_{24} = 0$,

$$(74) \quad x_2 x_5 + \frac{1}{2\beta} \left(\frac{\gamma}{\gamma'} l' - \gamma L - l'_v \right) x_4^2 = 0 ,$$

we note that the polar hyperplane, with respect to the hyperquadric Q_1 , of any point in the tangent plane $x_4 = x_5 = 0$ is the hyperplane

$$(75) \quad 2a_{22}x_2 + a_{24}x_4 + a_{25}x_5 = 0 .$$

Hence this hyperplane (75) may be called the polar hyperplane with respect to the hyperquadric Q_1 of the tangent plane $x_4 = x_5 = 0$.

The hyperquadric (67) may be characterized by the following:

- (1) It has three plane contact with $u=c$ at x .
- (2) It has the tangent plane as plane generator.
- (3) It has the plane $x_5=0$ as the polar hyperplane of the tangent plane.

Similar statements may be made for the associated quadric which may be obtained from the equation (74) by the transformation (10).

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