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TOPOLOGICAL ENTROPY OF THE LOZI FAMILY

By

Izzet Burak Yildiz

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ABSTRACT

TOPOLOGICAL ENTROPY OF THE LOZI FAMILY

By

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We study the topological entropy of a two dimensional map, called the Lozi map. The Lozi map is a piecewise-affine analog of the Henon map, one of the most studied examples in dynamical systems. In this area, it is extremely important to understand the complexity of a given system and topological entropy is a nonnegative number which measures this complexity. We investigate how the complexity of the Lozi map changes depending on the parameters.

In particular, we study the monotonicity and discontinuity properties of topological entropy of the Lozi maps. In 1997, Y. Ishii and D. Sands showed the monotonicity of the Lozi family $\mathcal{L}_{a,b}$ in a C^1 neighborhood of a-axis in the a-b parameter space. We show the monotonicity of the entropy in the vertical direction around $a = 2$ and in some other directions for $1 < a \leq 2$. Also we give some rigorous and numerical results for the parameters at which the Lozi family has zero entropy.

Moreover, in 2009, J. Buzzi showed that the entropy map $f \to h_{top}(f)$ is lower semi-continuous for all piecewise affine surface homeomorphisms. The upper semi-continuity of entropy was an open question for these maps. We prove that topological entropy for the Lozi maps can jump from zero to a value above 0.1203 as one crosses a particular parameter and hence it is not upper semi-continuous in general. Moreover, our results can be extended to a small neighborhood of this parameter and hence disprove a conjecture by Ishii and Sands which states that there are at most countable number of points of discontinuity of the entropy map.

We conclude with numerical results for the entropy of the Lozi maps for a large set of parameters which coincide with rigorous bounds given by Y. Ishii and D. Sands before.

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INTRODUCTION

In 1963, Edward N. Lorenz, a meteorologist and mathematician from MIT, published a paper[21] which included a set of three dimensional differential equations that had important implications for climate and weather predictions. He observed that the solution of the system exhibits complicated behavior that seemed to depend sensitively on initial conditions. In other words, small variations in initial conditions would cause large variations in the long term behavior of the system, a phenomenon now known as the "butterfly effect". Moreover, the solutions seemed to form a complicated picture which was later called a strange attractor. These discoveries planted the seeds for the chaos theory.

In 1976, Michel Henon, a French astronomer, introduced a simple two-dimensional map, called the Henon map, that exhibits chaotic behavior similar to that in the Lorenz system[11].

M. Henon studied:

$$H = H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 + y - ax^2 \\ bx \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0$$

A simple affine change of coordinates put this map in the form:

$$H = H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0$$

Originally, M. Henon used the parameters of $a = 1.4$ and $b = 0.3$. He observed that an initial point of the plane under the iterations of the map either diverges to infinity or approaches to a set now known as the Henon attractor. This set seemed to be locally homeomorphic to the product of an interval and a Cantor set(See Figure 1). Although the map is given by a very simple formula, the rigorous mathematical analysis of it turned out to be very difficult. The existence of an attractor was proved years later only in a small neighborhood of $b = 0$ [2].

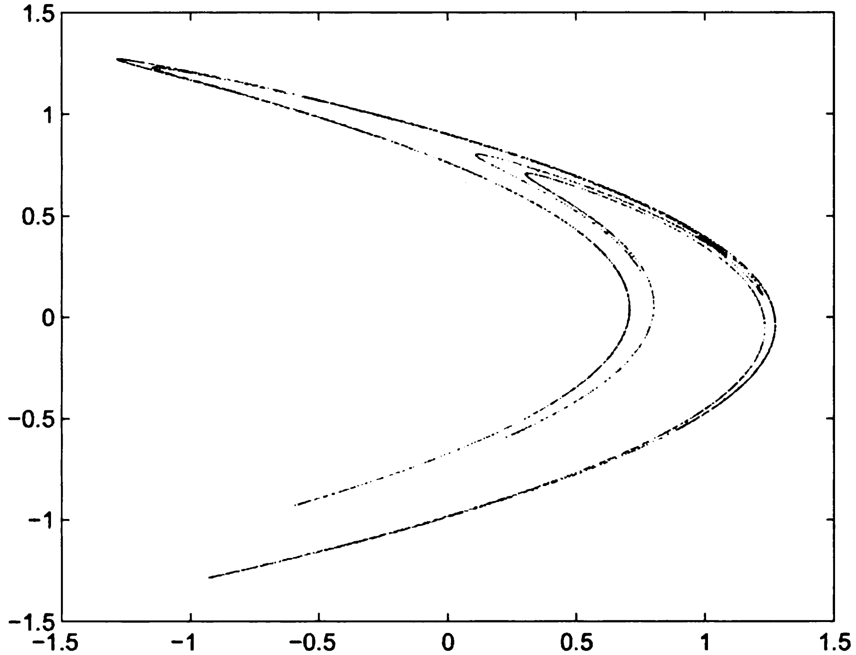
In 1978, Rene Lozi introduced another two dimensional family of maps which is very similar to the Henon family[22].
The Lozi family is given by:

$$\mathcal{L} = \mathcal{L}_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0.$$

Thus, the quadratic term ax^2 in the Henon family is replaced by the piecewise affine term $a|x|$. This results in a considerably simpler family of maps. For instance, in [26], Misiurewicz proved the existence of attractors for a large set of parameters. The triangular region in Figure 3 shows these parameters.

Past Work and Recent Progress

Symbolic dynamics has played an important role in the study of iterated maps. Milnor and Thurston have developed a kneading theory to study the topological dynamics of piecewise-monotone self maps of the interval[23]. In their study, the itinerary of the critical point, kneading sequence, is one of the most important ingredients. They proved that a continuous, piecewise monotone map of the interval with positive topological entropy is semi-conjugate to a continuous piecewise affine map with a constant slope and the same entropy. For unimodal maps (one critical point) with negative Schwarzian derivative and no periodic attractor, the kneading sequence gives the complete classification up to topological equivalence[17]. Also, the set of all admissible



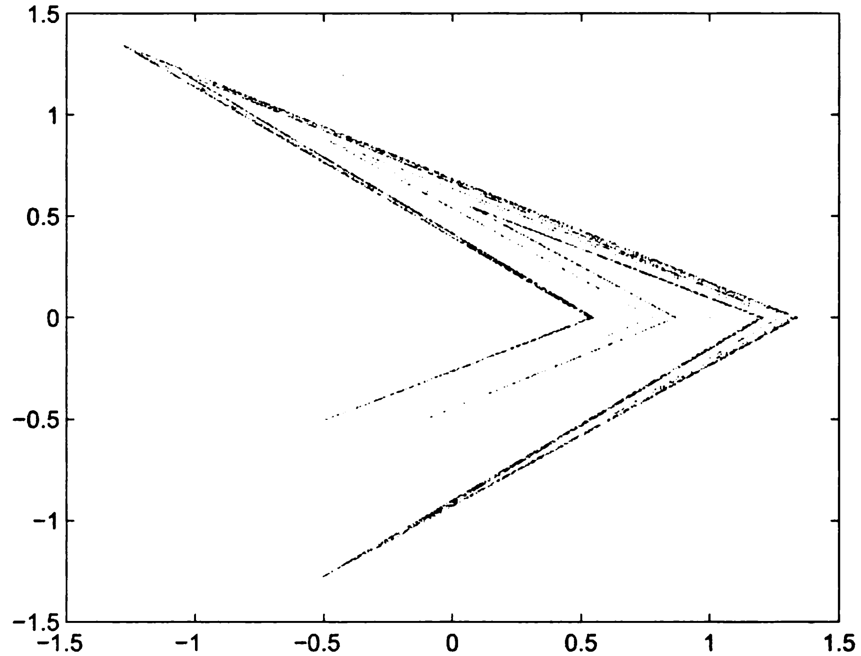


Figure 1: Henon attractor for $H_{1.4,0.3}$ and Lozi attractor for $\mathcal{L}_{1.7,0.5}$.

sequences can be obtained from the kneading sequence. Applications of the kneading theory lead to a proof of continuity of entropy and monotonicity of the kneading sequences for certain families of one dimensional maps.

On the other hand, there is no general symbolic theory for two dimensional maps. In [6], Cvitanovic, Gunaratne and Procaccia presented a two-piece partition of the plane which leads to symbolic dynamics of two symbols (-1 and 1) for the Henon map. If a symbolic sequence corresponds to an actual periodic orbit it was called admissible. They introduced subsets of the symbol space $\{-1,+1\}^{\mathbb{Z}}$ called "primary pruned region" and "pruning front" and conjectured that they specify all the periodic orbits (Pruning Front Conjecture(PFC)). In other words, if all the backward and forward iterations of a periodic sequence under the shift map stay away from the pruned region, then this periodic sequence corresponds to a periodic orbit in the phase space. This way they measured how far the given map is from a complete horseshoe in which case the pruned region is empty. They used this idea to obtain a numerical estimation for the topological entropy of the Henon map.

A Pruning Theory for surface homeomorphisms (in a more general setting) was given by A. Carvalho and Toby Hall[5].

Monotonicity of the Lozi family

Rigorous mathematical justification of the symbolic dynamics and an analog of Cvitanovic's Pruning Front Conjecture for the Lozi map was given by Y. Ishii, following suggestions of J. Milnor[12]. Ishii introduced a similar pruned region which distinguishes the sequences that correspond to a point in the non-wandering set(admissible sequences) from the sequences that do not(non-admissible). This characterization has powerful applications. For example, he gave the boundary of the region in the a-b

parameter space where the Lozi map has maximal entropy log 2(See Figure 3). Also, in a joint work with Sands, they proved the entropy increases monotonically with a for small fixed values of b:

Theorem 0.0.1 ([14]). *For every $a_* > 1$ there exists $b_* > 0$ such that, for any fixed b with $|b| < b_*$, the topological entropy of $\mathcal{L}_{a,b}$ is a non-decreasing function of $a > a_*$.*

The main step in the proof of this theorem is to show that when a increases, the primary pruned region decreases, and as a result entropy increases. It's natural to ask the following:

Question 1. *Is the entropy monotone in other directions?*

In this dissertation, we partly answered this question by proving the following theorem (for details see Theorem 2.1.3 below):

Let us define $\mathbb{R}^2_{>1+} = \{(a,b) \in \mathbb{R}^2 \mid a > 1 + |b|\}$.

Theorem 0.0.2. *For every $1 < a \leq 2$ there exist $N_a^1, N_a^2 \in \mathbb{R}^+$ and two lines $\gamma_{1,2} : (-\delta_{1,2}, \delta_{1,2}) \to \mathbb{R}^2_{>1+}$, $\delta_{1,2} > 0$, given by $\gamma_1(t) = (a+N_a^1 t, -t)$ and $\gamma_2(t) = (a+N_a^2 t, t)$ such that the topological entropy of $\mathcal{L}_{\gamma_1(t)}$ and $\mathcal{L}_{\gamma_2(t)}$ is a non-decreasing function of t.*

So we can get the monotonicity directions given in Figure 2.

Discontinuity of the topological entropy

A well known result in one dimensional dynamics is Misiurewicz and Szlenk's lap number entropy formula[27]. According to the formula, topological entropy of a piecewise monotone map of the interval, f, is given by: $h_{top}(f) = lim_{n\to\infty}\frac{1}{n}log\ell_n(f)$ where $\ell_n(f)$ is the number of monotone pieces of the nth iterate of f. In [16], Ishii and Sands give a similar lap number entropy formula for piecewise affine homeomorphisms of the plane. In [15], they use this formula to obtain some rigorous upper and lower

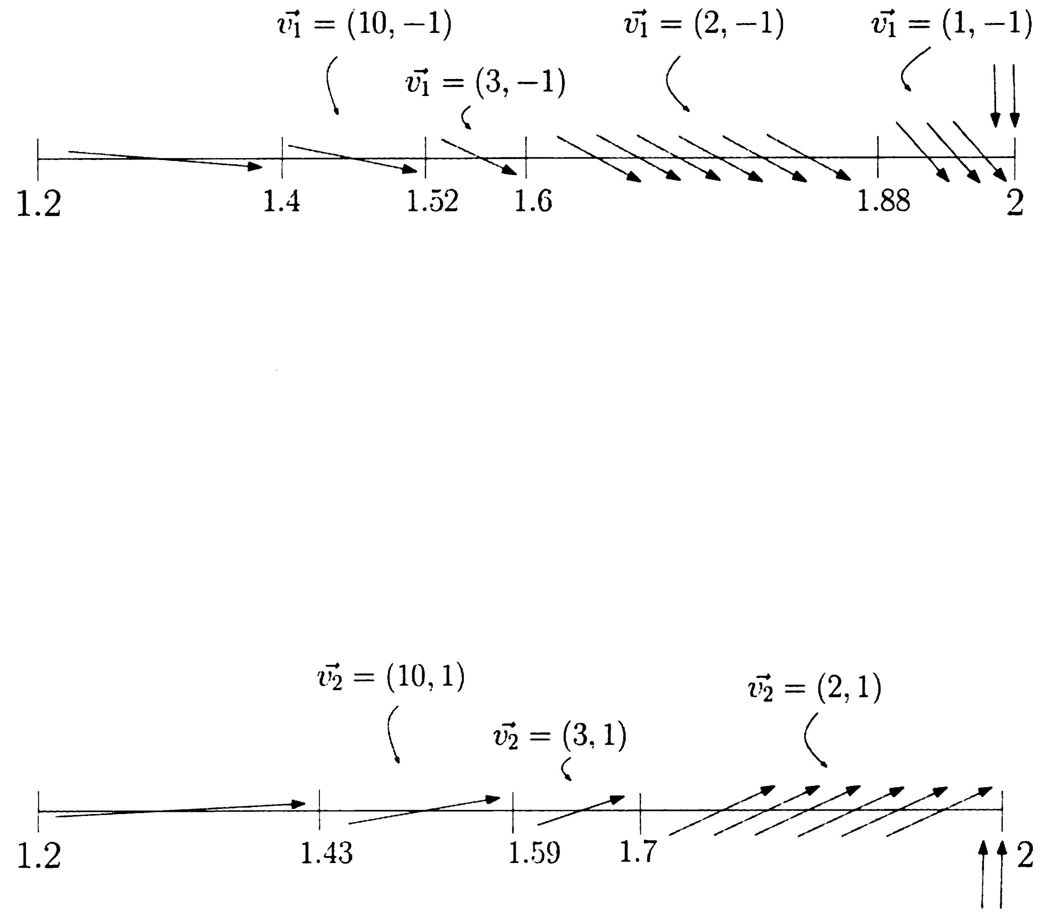


Figure 2: This figure shows the directions $\vec{v_1} = (N_a^1, -1)$ and $\vec{v_2} = (N_a^2, 1)$ along which entropy is non-decreasing. $1.2 < a \leq 2$ and b small.

bounds for the entropy of the Lozi family. Analyzing these results, they proposed the following conjecture for the Lozi family:

Conjecture 1 (Ishii and Sands [15]). *There are at most countable number of points of discontinuity of the entropy map* $(a, b) \to h(\mathcal{L}_{a,b})$.

Moreover, in [4], Buzzi proves a Katok-like theorem for piecewise affine surface homeomorphisms which shows the lower semi-continuity of the entropy map, $f \to h_{top}(f)$. The following question was asked by Buzzi:

Question 2. *Prove or disprove the upper semi-continuity of entropy for piecewise affine homeomorphisms of the plane.*

In this dissertation, we proved that the topological entropy of the Lozi maps is not continuous depending on the parameters showing that it can jump up from zero to a value above 0.1203 as one crosses the parameter $(a, b) = (1.4, 0.4)$ (see Theorem 3.0.5 below). Moreover, similar jumps occur in a small neighborhood of this point along the line $a = 1 + b$, disproving the above conjecture:

Theorem 0.0.3. *In general, the topological entropy of Lozi maps does not depend continuously on the parameters. For* $\epsilon_1 > 0$ *and small and* $|\epsilon_2|$ *small:*

(i) The topological entropy of the Lozi maps with $(a, b) = (1.4 + \epsilon_2, 0.4 + \epsilon_2)$, $h(\mathcal{L}_{1.4+\epsilon_2, 0.4+\epsilon_2})$, *is zero.*

(ii) The topological entropy of Lozi maps, $h(\mathcal{L}_{(1.4+\epsilon_1+\epsilon_2, 0.4+\epsilon_2)})$, *has a lower bound of* 0.1203.

Zero Entropy Parameters

In [15], Ishii and Sands give rigorous entropy computations for some rational values of a and b. Unfortunately, their algorithm gives poor results when entropy is close to zero. So, the precise shape of the set of zero entropy parameters is unknown. On the other hand, it is possible to use Brouwer's Translation Theorem to prove some sufficient conditions[15]. The light gray region in Figure 3 gives those parameters.

In this dissertation, we modified the use of Brouwer's Translation Theorem to prove the following (see Theorem 2.1.4):

Theorem 0.0.4. *In a neighborhood of the parameter* $(a, b) = (1, 0.5)$, $h_{top}(\mathcal{L}_{a,b}) = 0$.

The proof of this theorem can be extended to a larger set of parameters. But it gets more complicated especially as b gets close to 1. So, we give some numerical results where our proof seems to be working and obtain a picture (See the medium gray area in Figure 3) for zero entropy parameters.

Numerical Results

Besides the numerical studies of zero entropy parameters, we also introduced a numerical algorithm to approximate the entropy of Lozi maps. Our algorithm measures the growth rate of the number of intersections of the unstable manifold of the right fixed point with the y-axis. For most of the parameters, entropies we obtain are within ± 0.05 of the rigorous computations by Ishii and Sands. For example, for the original parameters studied by Lozi, $(a, b) = (1.7, 0.5)$, our entropy estimate is 0.5146 where as the rigorous upper bound obtained by them is 0.5087. Summary of our computations are given in the picture below. However, we are not able to yet obtain rigorous bounds for accuracy of our estimations. Another interesting point is that when our method is applied to the Henon family, one still gets very close approximations for the entropy. For example, for $(a, b) = (1.4, 0.3)$, we get 0.465 which is very close to a recent rigorous lower bound by Newhouse, Berz, Makino and Grote[29]. We will not go into any more details here.

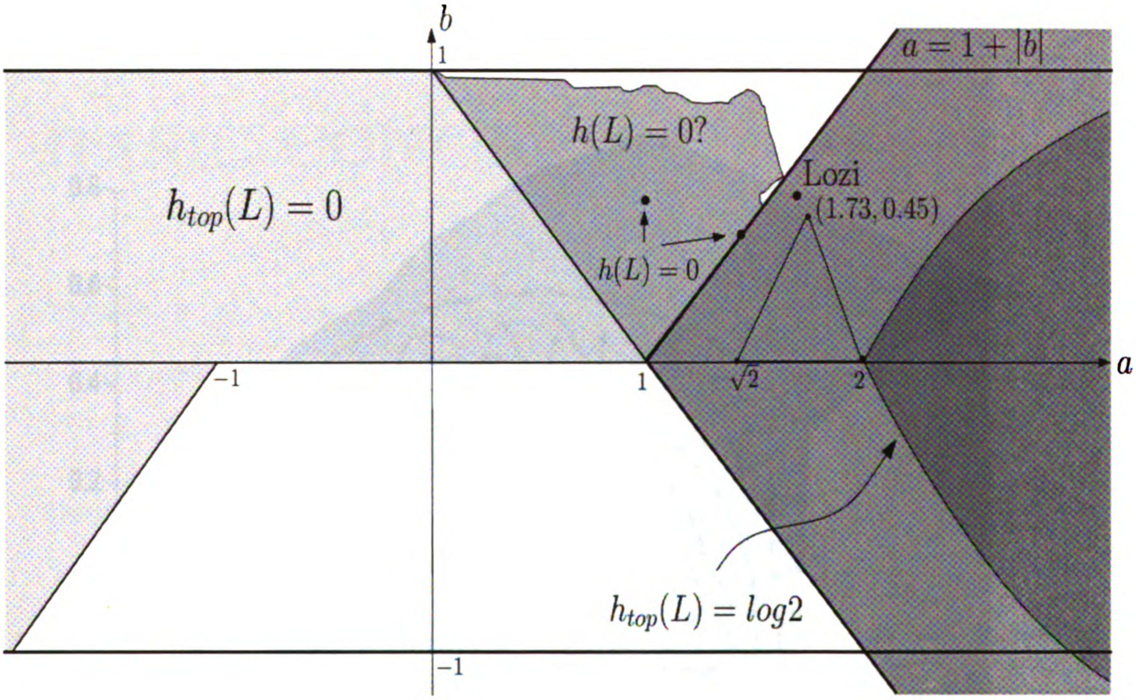


Figure 3: This picture gives a summary of recent results about the entropy of the Lozi family. When $a > 1 + |b|$, entropy is positive. The light gray area on the left gives the parameters for which entropy is zero. The darker gray area with complicated boundary represents our numerical results for zero entropy parameters. In the darkest gray region on the right, the entropy is log 2. In the triangular region, Misiurewicz proved the existence of strange attractors. Note that maps with $|b| > 1$, up to affine conjugacy, are inverses of maps with $|b| < 1$ and so not interesting. In the rest of the picture, nothing much is known.

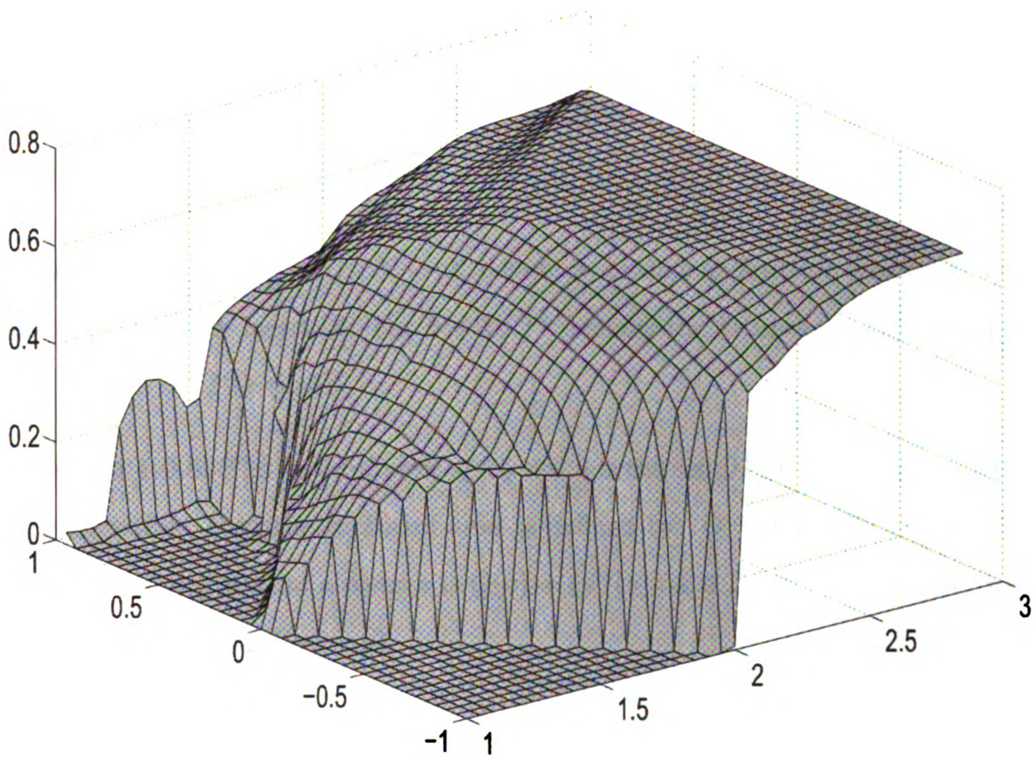


Figure 4: This is a summary of our numerical results for the entropy of Lozi family where $1 \le a \le 3$, $-1 \le b \le 1$ and the height is the entropy. Observe the consistency with Figure 3, especially zero entropy parameters when $b > 0$ and the maximal entropy parameters.

Chapter 1

BASIC CONCEPTS

Throughout this section let $f : X \to X$ denote a continuous function where (X, d) is a metric space with metric d. For $k \in \mathbb{Z}$, f^k denotes the k-times composition, i.e., $f^k = f \circ f \circ \cdots \circ f$ (k-times).

Definition 1.0.5. The *forward orbit of a point* a is the set $O^+(a) = \{f^k(a) : k \geq 0\}$. If f is invertible, then the *backward orbit* is defined by: $O^-(a) = \{f^k(a) : k \leq 0\}$. Then, the *(whole) orbit of a point* a is $O(a) = \{f^k(a) : k \in \mathbb{Z}\}$. If f is not invertible we take $f^{-1}(y) = \{x : f(x) = y\}$.

Definition 1.0.6. A point a is a *periodic point of period* n provided $f^n(a) = a$ and $f^j(a) \neq a$ for $0 < j < n$. If a has period one, then it is called a *fixed point*.

Now, let us give some fundamental definitions connected with convergence and stability of periodic points.

Definition 1.0.7. A point q is *forward asymptotic to* p provided $d(f^j(q), f^j(p))$ goes to zero as $j \to \infty$. If p is a periodic point with period n, then q is asymptotic to p if $d(f^{jn}(q), p)$ goes to zero as $j \to \infty$. The *stable set of* p is defined as:

$$W^s(p) = \{q : q \text{ is forward asymptotic to } p\}$$

If f is invertible, then a point q is said to be *backward asymptotic to* p provided $d(f^j(q), f^j(p))$ goes to zero as $j \to -\infty$. The *unstable set of* p is defined as:

$$W^u(p) = \{q : q \text{ is backward asymptotic to } p\}$$

The next notion has a great importance in dynamical systems theory and this document.

Definition 1.0.8 (Attractor). A compact region $N \subset X$ is called a *trapping region for* f provided $f(N) \subset \text{int}(N)$. A set Λ is called an *attracting set* provided there is a trapping region N such that $\Lambda = \cap_{k \geq 0} f^k(N)$. A set Λ is called an *attractor* if it is an attracting set and $f|\Lambda$ is topologically transitive, i.e., given any two open sets U and V in Λ, there is a non-negative number n such that $f^n(U) \cap V \neq \emptyset$.

Remark 1.0.9. The existence of attractors for the Lozi family was proven by Misiurewicz[26] for the parameters given in Figure 3. For these parameters, he also proved that the attractor is actually the closure of the unstable set(manifold) of the right fixed point.

1.1 Symbolic Dynamics

Following [20], let us define a finite alphabet set $\mathcal{N} = \{1, \ldots, n\}$. This set is a metric space with the metric $d(j, k) = 1 - \delta_{jk}$ where δ_{jk} is 0 if $j \neq k$ and 1 if $j = k$. The topology defined by this metric is a discrete topology and $\mathcal{N}$ is compact. Let us form two sequence spaces. The one-sided sequence space is $\Sigma_n^+ = \{1, \ldots, n\}^{\mathbb{N}}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and its points are in the form $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ where $\varepsilon_i \in \mathcal{N}$. The two sided sequence space is $\Sigma_n = \{1, \ldots, n\}^{\mathbb{Z}}$ where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and its points are in the form $(\ldots, \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ where $\varepsilon_i \in \mathcal{N}$. We put the product topology in each space. By Tychonoff's theorem for products of compact spaces, both spaces are compact.

A set of the form $[i_0, \ldots, i_\ell]_t = \{\varepsilon : \varepsilon_t = i_0, \ldots, \varepsilon_{t+\ell} = i_\ell\}$ is called a *cylinder set* or a *block*. The cylinder sets are both open and closed and they form a countable basis for the topology in each space. Every open set is a countable union of cylinder sets.

The product metric $d(\varepsilon, \varepsilon') = \sum_{i=0}^{\infty} \frac{1 - \delta_{\varepsilon_i \varepsilon'_i}}{2^i}$ on Σ_n^+ generates the topology. Similarly, the product metric $d(\varepsilon, \varepsilon') = \sum_{i=-\infty}^{\infty} \frac{1 - \delta_{\varepsilon_i \varepsilon'_i}}{2^i}$ on Σ_n generates the topology. Note that both Σ_n^+ and Σ_n are homeomorphic to the standard middle-third Cantor set. Next, let us define the shift map, σ, on both spaces as $\sigma(\varepsilon)_i = \varepsilon_{i+1}$. The shift map is a continuous, onto and n-to-1 on Σ_n^+ and it is a homeomorphism on Σ_n.

Definition 1.1.1. The dynamical system (Σ_n^+, σ) is called the *one-sided shift on n symbols* or *full one-sided n-shift.* Similarly, (Σ_n, σ) is called the *two-sided shift on n symbols* or *full n-shift.*

1.1.1 Subshift of Finite Type (SFT)

A *subshift* is a closed, shift-invariant subset of a full shift. Equivalently, let $\mathcal{D}$ be any set (finite or infinite) of cylinder sets. The set $S_{\mathcal{D}}$ of sequences that do not contain any element of $\mathcal{D}$ is a subshift, and any subshift can be expressed in this form.

Example 1.1.2. Even shift: Let $S_{\mathcal{D}}$ be the set of sequences consists of 0 and 1's so that between any two 1's there are even number of 0's. So, $\mathcal{D} = \{1(0)^{2k+1}1 : k \in \mathbb{N}\}$.

Definition 1.1.3. A *subshift of finite type (SFT)* is a shift space that can be described by a finite list of forbidden cylinder sets. Another classical definition is as follows:

Let A be a square $\{0, 1\}$ matrix with its rows and columns indexed by $\{1, \ldots, n\}$. One can define a closed, shift-invariant subset Σ_A of Σ_n^+ (or Σ_n) by selecting sequences

$\varepsilon \in \Sigma_A$ with the rule that $A_{\varepsilon_i \varepsilon_{i+1}} = 1$ for all $i \in \mathbb{N}$ (or $\mathbb{Z}$). The dynamical system (Σ_A, σ) consisting of this compact space Σ_A and the restriction of the shift map is the *one-sided (or two-sided) subshift of finite type defined by A* or *topological Markov shift defined by A*. The matrix A is called the *transition matrix.*

Example 1.1.4. The golden mean shift Σ_A is given by the transition matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Remark 1.1.5. Although subshifts of finite type have very nice properties which we will not discuss here, the Lozi maps which are discussed in this dissertation can not be represented by SFT's. In general, there are infinitely many forbidden cylinder sets. On the other hand, they can be approximated by SFT's (See [15]).

1.2 Topological Entropy

Topological entropy is a quantitative measurement of how complicated a map f is. A rough interpretation could be given as follows: Suppose one can distinguish two distinct points only if the distance between them is larger than a resolution ϵ. Then two orbits of length n obtained by taking n-iterations of these points under f can be distinguished provided that there is some iterate m between 0 and n for which their distance is greater than ϵ. Let $r(n, \epsilon, f)$ be the maximum number of such distinguishable orbits of length n. The entropy for a given ϵ, $h(\epsilon, f)$, is the growth rate of $r(n, \epsilon, f)$ as n goes to infinity. Then the entropy $h(f)$ is the limit of $h(\epsilon, f)$ as the resolution ϵ goes to zero.

Definition 1.2.1. Let $f : X \to X$ be a continuous map on a compact metric space (X, d) with a metric d. Two distinct points $x, y \in X$, $x \neq y$, are called (n, ϵ)*-separated* for a positive integer n and $\epsilon > 0$ if there is at least one m, $0 \leq m \leq n$, such that

$d(f^m(x), f^m(y)) > \epsilon$. A set $U \subset X$ is called an (n, ϵ)-*separated set* if every pair of distinct points $x, y \in U$, $x \neq y$, is (n, ϵ)-separated.

Let $r(n, \epsilon, f)$ be the maximum cardinality of an (n, ϵ)-separated set $U \subset X$. By compactness, this number is always finite.

Define $h(\epsilon, f) = \limsup_{n\to\infty} \frac{log(r(n, \epsilon, f))}{n}$. Then *topological entropy of* f, $h(f)$, is defined as:

$$h(f) = \lim_{\epsilon\to 0, \epsilon>0} h(\epsilon, f).$$

Note that this limit exists (can be infinite) because for $0 < \epsilon_2 < \epsilon_1$, $r(n, \epsilon_1, f) \leq r(n, \epsilon_2, f)$, so $h(\epsilon, f)$ is a monotone function of ϵ.

Remark 1.2.2. The original definition of topological entropy was given by Adler, Konheim and McAndrew, [1], using a different idea involving covers of open sets. The definition above was given by Bowen[30] and independently by Dinaburg[7].

Remark 1.2.3. Note that the Lozi map $\mathcal{L}$ is defined in $\mathbb{R}^2$ which is not compact. To be able to investigate the topological entropy of the Lozi maps, we take one-point compactification of $\mathbb{R}^2$ and extend the map continuously putting $\mathcal{L}(\infty) = \infty$. For more details about this continuous extension see [16].

Now, we would like to summarize some important theorems related to topological entropy. Most of the proofs can be found in [31].

Theorem 1.2.4. *Let X be compact, $f : X \to X$ be a continuous function and $k \geq 1$ be an integer. Then, the entropy of f^k is equal to k times the entropy of f, $h(f^k) = kh(f)$.*

Remark 1.2.5. If f is a homeomorphism, then theorem becomes $h(f^k) = |k|h(f)$ for any integer k.

Note that we make use of the above theorem when we prove our discontinuity result. In other words, we show that for some specific parameters $h(\mathcal{L}^4) > log\frac{1+\sqrt(5)}{2}$ and this implies $h(\mathcal{L}) = \frac{1}{4}h(\mathcal{L}^4) > \frac{1}{4}log\frac{1+\sqrt(5)}{2}$.

The next theorem says that the wandering orbits do not contribute to the entropy.

Theorem 1.2.6. *([30]) Let $f : X \to X$ be a continuous function on a compact metric space X. Let $\Omega \subset X$ be the nonwandering points of f Then, the entropy of f equals the entropy of f restricted to its nonwandering set, $h(f) = h(f|\Omega)$.*

Another related theorem states that any map whose nonwandering set is a finite set of points, has zero entropy:

Theorem 1.2.7. *Let $f : X \to X$ be a continuous function on a compact metric space X for which $\Omega(f)$ is a finite number of periodic points. (For example Morse-Smale diffeomorphism) Then the entropy of f is zero.*

We use the above theorem to prove the zero entropy results for some parameters in the Lozi map. In other words, we show for some specific parameters, the nonwandering set consists of periodic orbits only.

Now, let us give the definition for semi-conjugacy and conjugacy which will allow us to make comparisons between the dynamical properties(such as entropy) of two systems:

Definition 1.2.8. Let $f : X \to X$ and $g : Y \to Y$ be two maps. A map $k : X \to Y$ is called a *semi-conjugacy from f to g* if:

- k is continuous,
- k is onto and

- $k \circ f = g \circ k$.

The map k is called a *conjugacy* if it is a semi-conjugacy, one-to-one and has a continuous inverse. i.e. k is a homeomorphism.

The next theorem gives important relations about the entropy of semi-conjugate and conjugate systems:

Theorem 1.2.9. *Let X and Y be compact metric spaces. Let $f : X \to X$ and $g : Y \to Y$ be tow maps and $k : X \to Y$ be a continuous map with $k \circ f = g \circ k$*

(i) If k is onto (i.e.. a semi-conjugacy), then $h(f) \geq h(g)$.

(ii) If k is one-to-one (not necessarily onto), then $h(f) \leq h(g)$.

(iii) If k is onto and uniformly finite to one, then $h(f) = h(g)$. Note that k can be a conjugacy in this case.

Symbolic dynamics may be very helpful sometimes determining the entropy of a given system. If one can find a conjugacy or semi-conjugacy between a subshift and the given system, then we can use the above theorem to understand the entropy of the system. This is actually the idea behind the monotonicity results given in this document.

On the other hand, finding the entropy of an arbitrary subshift can also be difficult. The next theorem gives a useful tool to compute the entropy of subshifts:

Theorem 1.2.10. *(i) Let $\sigma : \Sigma_n \to \Sigma_n$ be the full shift on n symbols (one-sided or two-sided). Assume $\Sigma \subset \Sigma_n$ is a closed invariant subset. So, $(\Sigma, \sigma|\Sigma)$ is a subshift. Let ω_m be the number of words of length m in Σ, i.e.,*

$$\omega_m = \{(\varepsilon_0, \dots, \varepsilon_{m-1}) : \varepsilon_j = \varepsilon_j^{'} \text{ for } 0 \leq j < m \text{ for some } \varepsilon^{'} \in \Sigma\}$$

Then,

$$h(\sigma|\Sigma) = \limsup_{m \to \infty} \frac{log(\omega_m)}{m}$$

(ii) Let $A_{n\times n}$ be a transition matrix on n symbols. Let $\sigma_A : \Sigma_A \to \Sigma_A$ be the associated subshift of finite type (one-sided or two-sided). Then $h(\sigma_A) = log(\lambda_1)$ where λ_1 is the real eigenvalue of A such that $\lambda_1 \geq |\lambda_j|$ for all other eigenvalues λ_j of A.

Example 1.2.11. Using the second part of the above theorem, one can easily see that the entropy of the golden mean shift (defined before) equals to $log\frac{1+\sqrt(5)}{2}$.

Chapter 2

MONOTONICITY OF ENTROPY AND ZERO ENTROPY PARAMETERS

2.1 Preliminaries

Since its discovery in 1976, the Henon map[11] has been one of the most studied examples in dynamical systems. It was introduced by M. Henon as a simple model exhibiting chaotic motion. On the other hand, the Lozi map[22] which is a piecewise-affine analog of the Henon map has been also important since it has a simpler structure but similar chaotic behavior.

The Henon family is defined by:

$$H = H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - ax^2 + by \\ x \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0$$

while the Lozi family is defined by:

$$\mathcal{L} = \mathcal{L}_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0.$$

Thus, the quadratic term ax^2 in the Henon family is replaced by the piecewise affine term $a|x|$. This results in a considerably simpler family of maps. For instance, in [26] the existence of attractors is proved for a large set of parameters, while in the Henon family, this is only proven for very small $b \neq 0$(see [2]).

In this article we improve some of the entropy results obtained by Ishii and Sands in [14] and give some partial results about the parameters at which the topological entropy of the Lozi family is zero.

The following result about monotonicity was obtained in [14]:

Theorem 2.1.1. *For every $a_* > 1$ there exists $b_* > 0$ such that, for any fixed b with $|b| < b_*$, the topological entropy of $\mathcal{L}_{a,b}$ is a non-decreasing function of $a > a_*$.*

Our results can be summarized in the next three theorems:

Theorem 2.1.2. *For any fixed a^* in some neighborhood of $a = 2$, there exist $b_1^* > 0$ and $b_2^* < 0$ such that the topological entropy of $\mathcal{L}_{a,b}$ is a non-increasing function of b for $0 < b < b_1^*$ and a non-decreasing function of b for $b_2^* < b < 0$.*

Let us define $\mathbb{R}^2_{>1+} = \{(a, b) \in \mathbb{R}^2 \,|\, a > 1 + |b|\}$.

Theorem 2.1.3. *For every $1 < a \leq 2$ there exist $N_a^1, N_a^2 \in \mathbb{R}^+$ and two lines $\gamma_{1,2}$: $(-\delta_{1,2}, \delta_{1,2}) \to \mathbb{R}^2_{>1+}$, $\delta_{1,2} > 0$, given by $\gamma_1(t) = (a+N_a^1 t, -t)$ and $\gamma_2(t) = (a+N_a^2 t, t)$ such that the topological entropy of $\mathcal{L}_{\gamma_1(t)}$ and $\mathcal{L}_{\gamma_2(t)}$ is a non-decreasing function of t.*

Theorem 2.1.4. *In a small neighborhood of the parameters $a = 1$ and $b = 0.5$, topological entropy of $\mathcal{L}_{a,b}$, $h_{top}(\mathcal{L}_{a,b})$, is zero.*

Remark: The proof of this last result can be extended for other parameters as well. But it gets complicated especially when b is close to 1. So we give some numerical results for such parameters and obtain a picture (See Figure 2.6) for the zero entropy locus $H_0 = \{(a,b)|\, h_{top}(\mathcal{L}_{a,b}) = 0\}$ when $a > 0$ and $b > 0$.

Outline The remainder of this chapter is organized as follows. Section 2.2 gives an introduction to the Pruning Theory and some results by Ishii and Sands that we are going to use. Our monotonicity results are proved in Section 2.3. Then, Section 2.4 extends these results. Section 2.5 describes the results about the zero entropy locus.

2.2 Pruning Theory

The Pruning Theory was suggested by Cvitanović[6] as a way of obtaining symbolic dynamics for the Henon map. Certain conjectures were formulated which still remain unproved. Motivated by this, and following suggestions of J. Milnor, Ishii[12],[13] provided an analogous Pruning Theory for hyperbolic Lozi maps (ie. those satisfying $a > 1 + |b|$) and proved an appropriate "Pruning Conjecture" which yielded a good symbolic description of the bounded orbits of hyperbolic Lozi maps.

Let us recall the basic elements of this Pruning Theory:

Let Σ denote the symbol space $\{-1,+1\}^{\mathbb{Z}}$ with product topology. Define the shift map $\sigma : \Sigma \to \Sigma$ which is a continuous map given as $\sigma(\ldots\varepsilon_{-2},\varepsilon_{-1}\cdot\varepsilon_0,\varepsilon_1\ldots) = (\ldots\varepsilon_{-2},\varepsilon_{-1},\varepsilon_0\cdot\varepsilon_1\ldots)$. For any $\underline{\varepsilon}\in\Sigma$ we call $\underline{\varepsilon}^u = (\ldots\varepsilon_{-2},\varepsilon_{-1})$ the tail of $\underline{\varepsilon}$ and $\underline{\varepsilon}^s = (\varepsilon_0,\varepsilon_1\ldots)$ the head of $\underline{\varepsilon}$. Let C^u and C^s be the set of all tails and heads, respectively. So Σ may be identified with $C^u \times C^s$.

Define $p(\ldots,\varepsilon_{-2},\varepsilon_{-1})(a,b) = 1 - bs_{-2} + b^2 s_{-2}s_{-3} - b^3 s_{-2}s_{-3}s_{-4} + \ldots$

where s_n is defined as

$$s_n \equiv \cfrac{1}{-a\varepsilon_n + \cfrac{b}{-a\varepsilon_{n-1} + \cfrac{b}{-a\varepsilon_{n-2} + \cfrac{b}{\ddots}}}} \tag{2.1}$$

Similarly define $q(\varepsilon_0, \varepsilon_1 \ldots) = \hat{r}_0 - \hat{r}_0\hat{r}_1 + \hat{r}_0\hat{r}_1\hat{r}_2 - \ldots$

where $\hat{r}_n$ is defined as

$$\hat{r}_n \equiv \cfrac{1}{a\varepsilon_n + \cfrac{b}{a\varepsilon_{n+1} + \cfrac{b}{a\varepsilon_{n+2} + \cfrac{b}{\ddots}}}} \tag{2.2}$$

Note that $p(\underline{\varepsilon}^u)(a,b)$ and $q(\underline{\varepsilon}^s)(a,b)$ are defined on $C^u \times \mathbb{R}^2_{>1+}$ and $C^s \times \mathbb{R}^2_{>1+}$, respectively. In the rest of the paper, we identify p with $p \circ \tilde{\pi}_u$ and q with $q \circ \tilde{\pi}_s$ where $\tilde{\pi}_u : \Sigma \times \mathbb{R}^2_{>1+} \to C^u \times \mathbb{R}^2_{>1+}$ is the map $(\underline{\varepsilon})(a,b) \to (\underline{\varepsilon}^u)(a,b)$ and $\tilde{\pi}_s : \Sigma \times \mathbb{R}^2_{>1+} \to C^s \times \mathbb{R}^2_{>1+}$ is the map $(\underline{\varepsilon})(a,b) \to (\underline{\varepsilon}^s)(a,b)$. So, we consider p and q as functions $p, q : \Sigma \times \mathbb{R}^2_{>1+} \to \mathbb{R}$.

For the proof of the next lemma, see lemma 4.3 and 6.1 in [12].

Lemma 2.2.1. *For fixed $\underline{\varepsilon} \in \Sigma$, the functions $p(\underline{\varepsilon})$, $q(\underline{\varepsilon})$, $s_n(\underline{\varepsilon})$, $\hat{r}_n(\underline{\varepsilon}) : \mathbb{R}^2_{>1+} \to \mathbb{R}$ are real analytic in (a,b). Moreover, p, q, s_n, $\hat{r}_n$ and their partial derivatives with respect to the three variables $(a, b, \underline{\varepsilon})$ are continuous.*

Definition 2.2.2. We call

$$\mathcal{P}_{a,b} \equiv \{\underline{\varepsilon} \in \Sigma \,|\, (p-q)(\ldots \varepsilon_{-2}, \varepsilon_{-1} \cdot \varepsilon_0, \varepsilon_1 \ldots)(a,b) = 0\}$$

the *pruning front of* $\mathcal{L}_{a,b}$ and

$$\mathcal{D}_{a,b} \equiv \{\underline{\varepsilon} \in \Sigma \,|\, (p-q)(\ldots \varepsilon_{-2}, \varepsilon_{-1} \cdot \varepsilon_0, \varepsilon_1 \ldots)(a,b) < 0\}$$

the *primary pruned region of* $\mathcal{L}_{a,b}$. The pair $(\mathcal{P}_{a,b}, \mathcal{D}_{a,b})$ is known as the *pruning pair of* $\mathcal{L}_{a,b}$.

We call $\mathcal{A}_{a,b} \equiv \Sigma \setminus \bigcup_{n\in\mathbb{Z}} \sigma^n \mathcal{D}_{a,b} = \{\underline{\varepsilon} \in \Sigma \,|\, (p-q)(\sigma^n \underline{\varepsilon})(a,b) \geq 0 \,\forall n \in \mathbb{Z}\}$ the *admissible set.*

Definition 2.2.3. The set $\hat{\mathcal{P}}_{a,b} \equiv \mathcal{P}_{a,b} \cap \mathcal{A}_{a,b}$ is the *admissible pruning front.*

Let $K = K_{\mathcal{L}}$ denote the set of all points whose forward and backward orbits remain bounded. For a point $X \in K$ we put $\pi(X) = (\ldots \varepsilon_{-2}, \varepsilon_{-1} \cdot \varepsilon_0, \varepsilon_1 \ldots)$ where

$$\varepsilon_i \equiv \left\{ \begin{array}{ccc} +1 & \text{if} & \mathcal{L}^i(X)_x > 0 \\ * & \text{if} & \mathcal{L}^i(X)_x = 0 \\ -1 & \text{if} & \mathcal{L}^i(X)_x < 0 \end{array} \right\}$$

where $*$ can be both $+1$ and -1; and Y_x is the x-component of Y. An element of $\pi(X)$ is called an itinerary of X. So a point X can have more than one itinerary.

Now let us define the standard partial orders on $C^s \cup C^u$:

Definition 2.2.4.

1. Let $\underline{\varepsilon}^s$ and $\underline{\delta}^s$ be two distinct elements in C^s. Then there exists the smallest number $i \geq 0$ such that $\varepsilon_i \neq \delta_i$. We say $\underline{\varepsilon}^s <_s \underline{\delta}^s$ if one of the following is satisfied:

   (i) The number of $+1$'s in $\cdot\varepsilon_0 \ldots \varepsilon_{i-1}$ is even and $\varepsilon_i < \delta_i$,

(ii) The number of $+1$'s in $\cdot\varepsilon_0 \ldots \varepsilon_{i-1}$ is odd and $\varepsilon_i > \delta_i$, where order on the symbols is $-1 < +1$.

2. Let $\underline{\varepsilon}^u$ and $\underline{\delta}^u$ be two distinct elements in C^u. Then there exists the largest number $i < 0$ such that $\varepsilon_i \neq \delta_i$. When $b > 0$ (resp. $b < 0$), we say $\underline{\varepsilon}^u <_u \underline{\delta}^u$ if one of the following is satisfied:

   (i) The number of -1's (resp. $+1$'s) in $\varepsilon_{i-1} \ldots \varepsilon_0 \cdot$ is even and $\varepsilon_i < \delta_i$,

   (ii) The number of -1's (resp. $+1$'s) in $\varepsilon_{i-1} \ldots \varepsilon_0 \cdot$ is odd and $\varepsilon_i > \delta_i$, where order on the symbols is $-1 < +1$.

See Fig.2.1 for the case $b > 0$.

In [12], Ishii proves the following version of the Pruning Front Conjecture(PFC) which was motivated by Cvitanović *et al* [6].

Theorem 2.2.5 (the pruning front conjecture). *Suppose that $\mathcal{L}_{a,b}$ satisfies $a > |b|+1$ and let $\underline{\varepsilon} \in \{+1, -1\}^{\mathbb{Z}}$. Then there exists a point $X \in K_{\mathcal{L}}$ such that $\underline{\varepsilon} \in \pi(X)$ if and only if $\sigma^n \underline{\varepsilon}$ does not lie in $\mathcal{D}_{a,b}$ for all $n \in \mathbb{Z}$.*

Next, we will summarize the results of Ishii and Sands [14], which prove the monotonicity of the entropy in the positive a-direction.

Recall that the tent map $T_a : \mathbb{R} \to \mathbb{R}$ is given by $T_a(x) = 1 - a|x|$.

Definition 2.2.6. An *itinerary* of a point $x \in \mathbb{R}$ under the map T_a is an element of $i_a(x) \equiv \{\underline{\varepsilon}^s \in C^s \,|\, \varepsilon_i T_a^i(x) \geq 0 \;\; \forall i \geq 0\}$. We call $\kappa(a) \equiv i_a(1)$ the *kneading invariant* of T_a.

Proposition 2.2.7. *Suppose $1 < a \leq 2$. Then $\pi_s(\hat{\mathcal{P}}_{a,0}) = \kappa(a)$ where $\pi_s : \Sigma \to C^s$ is the map $\underline{\varepsilon} \to \underline{\varepsilon}^s$*

Lemma 2.2.8 (Stability of $\hat{\mathcal{P}}$). *Suppose $a > 1 + |b|$. Then for every neighborhood U of $\hat{\mathcal{P}}_{a,b}$ there exists a neighborhood V of (a, b) such that $\hat{\mathcal{P}}_{\hat{a},\hat{b}} \subset U$ for every $(\hat{a}, \hat{b}) \in V$.*

Definition 2.2.9. We say that $(\hat{\mathcal{P}}_{a,b}, \mathcal{A}_{a,b}) < (\hat{\mathcal{P}}'_{\hat{a},\hat{b}}, \mathcal{A}'_{\hat{a},\hat{b}})$ if $\mathcal{A}_{a,b} \subset \mathcal{A}'_{\hat{a},\hat{b}}$ and $\hat{\mathcal{P}}'_{\hat{a},\hat{b}} \cap \mathcal{A}_{a,b} = \emptyset$.

The main step in the proof of the monotonicity in [14] is the following theorem:

Theorem 2.2.10 (Local Monotonicity). *Suppose $f : (-\delta, \delta) \to \mathbb{R}^2_{>1+}$, $\delta > 0$, is C^1 and*

$$\left.\frac{d(p-q)(\underline{\varepsilon})f(t)}{dt}\right|_{t=0} > 0$$

for all $\underline{\varepsilon} \in \hat{\mathcal{P}}_{f(0)}$. Then there exists a C^1 neighborhood $\mathcal{F}$ of f and a neighborhood I of 0 such that for any C^1 curve $g \in \mathcal{F}$ the map $t \in I \to (\hat{\mathcal{P}}_{g(t)}, \mathcal{A}_{g(t)})$ is order preserving: if $t_1, t_2 \in I$ and $t_1 < t_2$ then $(\hat{\mathcal{P}}_{g(t_1)}, \mathcal{A}_{g(t_1)}) < (\hat{\mathcal{P}}_{g(t_2)}, \mathcal{A}_{g(t_2)})$.

It is also proven that if $(\hat{\mathcal{P}}_{a,b}, \mathcal{A}_{a,b}) < (\hat{\mathcal{P}}_{\hat{a},\hat{b}}, \mathcal{A}_{\hat{a},\hat{b}})$ then $h_{top}(\mathcal{L}_{a,b}) \leq h_{top}(\mathcal{L}_{\hat{a},\hat{b}})$.

In [14], Ishii and Sands show that $\dfrac{\partial(p-q)(\underline{\varepsilon})(a,0)}{\partial a} > 0$ for any $\underline{\varepsilon} \in \hat{\mathcal{P}}_{a,0}$. Then they use local monotonicity to prove the following:

Theorem 2.2.11. *For every $a_* > 1$ there exists $b_* > 0$ such that the map $a \in (a_*, \infty) \to (\hat{\mathcal{P}}_{a,b}, \mathcal{A}_{a,b})$ is order preserving for all $|b| < b_*$.*

So Theorem 2.1.1 follows from these facts.

2.3 Results about the monotonicity of the entropy

In [13], Ishii mentions that although we have monotonicity in the direction given above, we do not know anything about the monotonicity in b direction. We look for a solution to this question near the point $(a, b) = (2, 0)$.

Now we want to concentrate on the point $(a,b) = (2,0)$. We will first figure out the set $\hat{\mathcal{P}}_{2,0}$. Using the stability of $\hat{\mathcal{P}}$ this will give us some information about $\hat{\mathcal{P}}_{2,b}$ for $|b|$ small. After that we will use the local monotonicity by taking b-derivative of $(p-q)$ to show the monotonicity in b-direction around $(2,0)$.

Proposition 2.3.1. *Let* $\underline{\delta}^s = (+1,-1,-1,-1\ldots)$. *For* $(a,b) = (2,0)$ *we have* $\hat{\mathcal{P}}_{2,0} = \pi_s^{-1}(\underline{\delta}^s) = \{\underline{\delta}^u \cdot +1,-1,-1,-1\ldots \mid \underline{\delta}^u \in C^u\}$ *and* $\mathcal{D}_{2,0} = \emptyset$.

Proof. First note that by Proposition 2.2.7, $\pi_s(\hat{\mathcal{P}}_{2,0}) = \kappa(2) = (+1,-1,-1,-1\ldots)$. So, $\hat{\mathcal{P}}_{2,0} \subset \pi_s^{-1}(\underline{\delta}^s)$. To prove $\pi_s^{-1}(\underline{\delta}^s) \subset \hat{\mathcal{P}}_{2,0}$, we need to show that for any $\underline{\delta}^u \in C^u$ the sequence $\underline{\delta} = (\underline{\delta}^u \cdot +1,-1,-1,-1\ldots)$ is in $\hat{\mathcal{P}}_{2,0} = \mathcal{A}_{2,0} \cap \mathcal{P}_{2,0}$, i.e. $(p-q)(\sigma^n\underline{\delta})(2,0) \geq 0$ for $n \in \mathbb{Z}$ and $(p-q)(\underline{\delta})(2,0) = 0$. Note that for an arbitrary $\underline{\varepsilon} \in \Sigma$, $p(\underline{\varepsilon}^u)(2,0) = 1$ and $\hat{r}_n = \dfrac{1}{2\varepsilon_n}$ and $q(\underline{\varepsilon}^s)(2,0) = \dfrac{1}{2\varepsilon_0} - \dfrac{1}{2^2\varepsilon_0\varepsilon_1} + \dfrac{1}{2^3\varepsilon_0\varepsilon_1\varepsilon_2} - \ldots + (-1)^n\dfrac{1}{2^{n+1}\varepsilon_0\varepsilon_1\ldots\varepsilon_n} + \ldots$. So, $q(\underline{\varepsilon}^s)(2,0)$ is maximized at only $\underline{\delta}^s = (+1,-1,-1,-1\ldots)$ and its maximum value is $\sum_{i=1}^{i=\infty}\left(\dfrac{1}{2}\right)^i = 1$. This shows that for any $\underline{\delta} \in \pi_s^{-1}(\underline{\delta}^s)$, $(p-q)(\underline{\delta})(2,0) = 0$ and $(p-q)(\sigma^n\underline{\delta})(2,0) > 0$ for $n \neq 0$. This proves $\pi_s^{-1}(\underline{\delta}^s) \subset \hat{\mathcal{P}}_{2,0}$ and also $\mathcal{D}_{2,0} = \emptyset$. □

Lemma 2.3.2.

$$\left.\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b}\right|_{b=0} = \frac{1}{2\varepsilon_{-2}}$$

for $\underline{\varepsilon} \in \hat{\mathcal{P}}_{2,0}$.

Proof. Recall that

$$p(\ldots,\varepsilon_{-2},\varepsilon_{-1})(a,b) = 1 - bs_{-2} + b^2s_{-2}s_{-3} - b^3s_{-2}s_{-3}s_{-4} + \ldots$$

and

$$q(\varepsilon_0,\varepsilon_1\ldots)(a,b) = \hat{r}_0 - \hat{r}_0\hat{r}_1 + \hat{r}_0\hat{r}_1\hat{r}_2 - \ldots$$

where s_n and $\hat{r}_n$ are given by (2.1) and (2.2). Taking the partial derivative of p with respect b we get:

$$\frac{\partial p}{\partial b} = -s_{-2} - bs'_{-2} + 2bs_{-2}s_{-3} + b^2(s_{-2}s_{-3})' + \cdots$$

Since s_n are analytic $\forall n \leq -2$ we obtain:

$$\left.\frac{\partial p}{\partial b}\right|_{b=0} = -s_{-2}\Big|_{b=0} = \frac{1}{a\varepsilon_{-2}} = \frac{1}{2\varepsilon_{-2}}$$

Now for $\frac{\partial q}{\partial b}$; first note that for $\underline{\varepsilon}$ such that $\underline{\varepsilon}^s = (+1, -1, -1, -1\cdots)$ we have $q(\underline{\varepsilon})(a, b) = \dfrac{b}{(a+x)(b+x)}$ where $x = (a - \sqrt{a^2+4b})/2$ (See 3.2.2).
Since $\frac{\partial q}{\partial b}$ is continuous with respect to b; a calculation(See 3.2.3) shows that:

$$\left.\frac{\partial q}{\partial b}\right|_{b=0} = \lim_{b\to 0}\frac{\partial q}{\partial b} = \lim_{b\to 0}\frac{\partial}{\partial b}\left(\frac{b}{(a+x)(b+x)}\right) = \frac{1-\frac{2}{a}}{a(a-1)^2}.$$

So for $a = 2$ we have $\left.\frac{\partial q}{\partial b}\right|_{b=0} = 0.$ □

The previous lemma says that the sign of $\left.\dfrac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b}\right|_{b=0}$ depends on ε_{-2}.

Proof of the Theorem 2.1.2. First let's define:

$$\mathcal{X} \equiv \{\cdots\varepsilon_{-3}, +1, +1\cdot\varepsilon_0, \varepsilon_1, \varepsilon_2\cdots\}$$

$$\mathcal{Y} \equiv \{\cdots\varepsilon_{-3}, -1, \pm 1\cdot\varepsilon_0, \varepsilon_1, \varepsilon_2\cdots\}$$

$$\mathcal{Z} \equiv \{\cdots\varepsilon_{-3}, +1, -1\cdot\varepsilon_0, \varepsilon_1, \varepsilon_2\cdots\}$$

Also define the curve $f(t)$ by $t \in (-\delta, +\delta) \to (2, t) \in \mathbb{R}^2_{>1+}$ where $\delta > 0$.
Note that we have $\hat{\mathcal{P}}_{2,0} = \{\underline{\varepsilon}^u\cdot +1, -1, -1, -1\cdots \mid \underline{\varepsilon}^u \in C^u\}$ by Proposition 2.3.1 and $\mathcal{D}_{2,0}$ is empty.(See Figure 2.1).

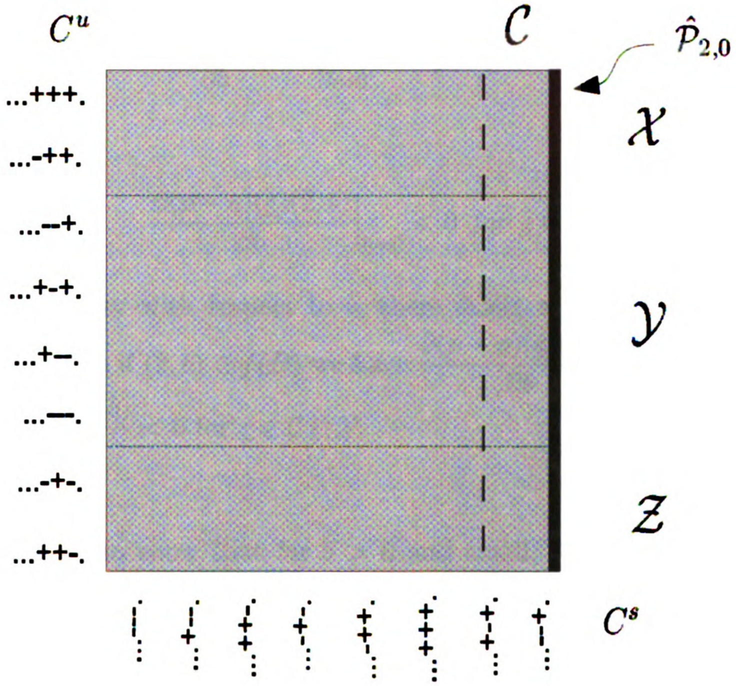


Figure 2.1: Symbol space ($b > 0$) and the sets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and $\mathcal{C}$

Then by Lemma 2.3.2, $\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b}$ is positive for $\underline{\varepsilon} \in \hat{\mathcal{P}}_{2,0} \cap (\mathcal{X} \cup \mathcal{Z})$ and negative for $\underline{\varepsilon} \in \hat{\mathcal{P}}_{2,0} \cap \mathcal{Y}$.

By continuity with respect to $\underline{\varepsilon}$ there exists a cylinder set $\mathcal{C}$ around $\hat{\mathcal{P}}_{2,0}$ such that

$$\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b}\Big|_{b=0} > 0 \ \ for\ \underline{\varepsilon} \in \mathcal{C} \cap (\mathcal{X} \cup \mathcal{Z})$$

and

$$\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b}\Big|_{b=0} < 0 \ \ for\ \underline{\varepsilon} \in \mathcal{C} \cap \mathcal{Y}.$$

Again by continuity with respect to b, there exists a neighborhood $B \subset (-\delta, +\delta)$ around 0 such that if $(2,b) \in f(B)$ we have $\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b} > 0$ for $\underline{\varepsilon} \in \mathcal{C} \cap (\mathcal{X} \cup \mathcal{Z})$ and $\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b} < 0$ for $\underline{\varepsilon} \in \mathcal{C} \cap \mathcal{Y}$.

Now we want to show that for $b > 0$ and small, $\hat{\mathcal{P}}_{2,b} \cap (\mathcal{X} \cup \mathcal{Z})$ is empty.(See Figure 2.3)

To do this, first observe that $C^u \times \mathcal{C}$ is a neighborhood of $\hat{\mathcal{P}}_{2,0}$. By stability of $\hat{\mathcal{P}}$(Lemma 2.2.8) there exists a neighborhood V of $(2,0)$ such that $\forall (a,b) \in V \ \hat{\mathcal{P}}_{a,b} \subset \mathcal{C}$.

We also know that $\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b} > 0$ for $\underline{\varepsilon} \in \mathcal{C} \cap (\mathcal{X} \cup \mathcal{Z})$. This means there exists a neighborhood $\tilde{B} \subset (-\delta, +\delta)$ around 0 where $(p-q)(\underline{\varepsilon})$ is increasing when b is increasing. This implies there exists $b_1^* > 0$ such that for every $(2,b)$ where $0 < b < b_1^*$ and for every $\underline{\varepsilon} \in \mathcal{C} \cap (\mathcal{X} \cup \mathcal{Z})$ we have

$$(p-q)(\underline{\varepsilon})(2,b) > (p-q)(\underline{\varepsilon})(2,0) \geq 0$$

In particular this tells us that all elements of $\hat{\mathcal{P}}_{2,b}$ are in $\mathcal{C} \cap \mathcal{Y}$. But then we know that for these elements $\frac{\partial(p-q)(\underline{\varepsilon})(2,b)}{\partial b} < 0$ and so using Theorem 2.2.10 the entropy is non-decreasing as b *decreases* to 0.

A similar argument applies for $b < 0$ and small where it can be shown that $\hat{\mathcal{P}}_{2,b} \subset \mathcal{C} \cap (\mathcal{X} \cup \mathcal{Z})$ and that the entropy is non-decreasing as b *increases* to 0.

□

2.4 Extension of the results to $1 < a \leq 2$

In this section we would like to prove some monotonicity properties for other a values as well. However, we are not able to prove the monotonicity in the vertical direction because it is not possible to use local monotonicity when we move away from $a = 2$. The reason behind this is the fact that for such a's and small b, $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}$ is positive for some $\underline{\varepsilon} \in \hat{\mathcal{P}}_{a,b}$ and negative for some other $\underline{\varepsilon} \in \hat{\mathcal{P}}_{a,b}$.

So we prove the next best thing: Monotonicity in the direction of lines which make some angle with the a-axis(See Figure 2.2). To prove this result we modify and use some of the computations done in [14].

Lemma 2.4.1. *(Lemma 11 in [14]) Suppose* $1 < a \leq 2$ *and* $\underline{\varepsilon}^s \in \kappa(a)$. *Then*

$$\frac{a^3 + 2a^2 - 6a + 2}{2a^2(a-1)} \leq \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} \leq \frac{a^3 + 2a^2 - 6a + 4}{2a^2(a-1)}$$

In particular; $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} \geq (\sqrt{2}-1)/2 > 0$ *if* $a \geq \sqrt{2}$.

Lemma 2.4.2. *(Corollary 13 in [14]) Suppose* $1 < a \leq 2$ *and* $\underline{\varepsilon}^s \in \kappa(a)$. *Then* $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} > 0$.

Lemma 2.4.3. *(Corollary 7 and Equation 3.11 in [14])*

Suppose $1 < a \leq 2$ *and* $\underline{\varepsilon}^s \in \kappa(a)$. *Then*

$$\sum_{i=0}^{\infty}(-1)^i \frac{\varepsilon_0 \dots \varepsilon_{i-1}}{a^i} = 0 \tag{2.3}$$

and

$$\sum_{j=0}^{\infty}(-1)^{i+j} \frac{\varepsilon_0 \dots \varepsilon_{i+j}}{a^{i+j+1}} = (-1)^i \frac{\varepsilon_0 \dots \varepsilon_{i-1}}{a^i} T_a^i(1) \tag{2.4}$$

where we define the empty product $\varepsilon_0 \dots \varepsilon_{-1}$ *to equal 1.*

Now, we use these results and similar techniques to prove the following:

Lemma 2.4.4. *Suppose* $1 < a \leq 2$ *and* $\underline{\varepsilon}^s \in \kappa(a)$. *Then*

$$\frac{1}{a\varepsilon_{-2}} - \frac{-2a^2 + 7a - 2}{2a^3(a-1)} \leq \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0} \leq \frac{1}{a\varepsilon_{-2}} - \frac{-2a^2 + 7a - 8}{2a^3(a-1)}$$

Proof. From the proof of Lemma 2.3.2 we know that $\frac{\partial p(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0} = \frac{1}{a\varepsilon_{-2}}$. So we need to find some upper and lower bound for $\frac{\partial q(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0}$.

Remember that $q(\varepsilon_0, \varepsilon_1 \dots) = \hat{r}_0 - \hat{r}_0\hat{r}_1 + \hat{r}_0\hat{r}_1\hat{r}_2 - \dots$.

Let's write $q(\varepsilon_0, \varepsilon_1 \dots) = T_0 - T_1 + T_2 - T_3 + \dots = \sum_{n=0}^{\infty}(-1)^n T_n$ where $T_n = \hat{r}_0\hat{r}_1 \dots \hat{r}_n$.

Now we have the following:

$$\hat{r}_n\Big|_{b=0} = \frac{1}{a\varepsilon_n + b\hat{r}_{n+1}}\Big|_{b=0} = \frac{\varepsilon_n}{a}$$

and

$$\hat{r}'_n\Big|_{b=0} = \frac{\partial(\hat{r}_n)(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0} = -\hat{r}_n^2(\hat{r}_{n+1} + b\hat{r}'_{n+1})\Big|_{b=0} = -\frac{\varepsilon_{n+1}}{a^3}$$

Taking term by term derivative of q, we get the following. Note that $\varepsilon_0 = +1$ and $\varepsilon_1 = -1$:

$$T'_0 = \hat{r}'_0 = -\frac{\varepsilon_1}{a^3} = \frac{1}{a^3}$$

$$-T_1' = -(\hat{r}_0'\hat{r}_1 + \hat{r}_0\hat{r}_1') = \frac{\varepsilon_1}{a^3}\frac{\varepsilon_1}{a} + \frac{\varepsilon_0}{a}\frac{\varepsilon_2}{a^3} = \frac{1}{a^4} + \frac{\varepsilon_0\varepsilon_2}{a^4}$$

$$T_2' = (\hat{r}_0\hat{r}_1)'\hat{r}_2 + (\hat{r}_0\hat{r}_1)\hat{r}_2' = -\frac{\varepsilon_2}{a^5} - \frac{\varepsilon_0}{a^5} - \frac{\varepsilon_0\varepsilon_1\varepsilon_3}{a^5}$$

$$-T_3' = -(\hat{r}_0\hat{r}_1\hat{r}_2)'\hat{r}_3 + (\hat{r}_0\hat{r}_1\hat{r}_2)\hat{r}_3' = \frac{\varepsilon_2\varepsilon_3}{a^6} + \frac{\varepsilon_0\varepsilon_3}{a^6} + \frac{\varepsilon_0\varepsilon_1}{a^6} + \frac{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_4}{a^6}$$

$$T_4' = (\hat{r}_0\hat{r}_1\hat{r}_2\hat{r}_3)'\hat{r}_4 + (\hat{r}_0\hat{r}_1\hat{r}_2\hat{r}_3)\hat{r}_4' = -\frac{\varepsilon_2\varepsilon_3\varepsilon_4}{a^7} - \frac{\varepsilon_0\varepsilon_3\varepsilon_4}{a^7} - \frac{\varepsilon_0\varepsilon_1\varepsilon_4}{a^7} - \frac{\varepsilon_0\varepsilon_1\varepsilon_2}{a^7} - \frac{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_5}{a^7}$$

$$\ddots = \ddots$$

Note that $\frac{\partial q(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0} = T_0' - T_1' + T_2' - \cdots + (-1)^n T_n' + \cdots$.

Claim: $T_n' = (\hat{r}_0\hat{r}_1\ldots\hat{r}_n)' = -\frac{1}{a^{n+3}}(\sum_{i=0}^{n-1} \varepsilon_0\ldots\overline{\varepsilon_i\varepsilon_{i+1}}\ldots\varepsilon_n + \varepsilon_0\varepsilon_1\ldots\overline{\varepsilon_n}\varepsilon_{n+1})$, $n \geq 2$ where $\varepsilon_0\ldots\varepsilon_{i-1}\overline{\varepsilon_i}\varepsilon_{i+1}\ldots\varepsilon_n$ means ε_i is missing in the term.

Proof of the Claim: Note that $T_n' = (T_{n-1}\hat{r}_n)' = T_{n-1}'\frac{\varepsilon_n}{a} + (\hat{r}_0\hat{r}_1\ldots\hat{r}_{n-1})\hat{r}_n' = T_{n-1}'\frac{\varepsilon_n}{a} - \frac{\varepsilon_0\varepsilon_1\ldots\overline{\varepsilon_n}\varepsilon_{n+1}}{a^{n+3}}$. So the claim follows by induction.

$$
\left[\begin{array}{c|ccccccc}
-T_1' & & & & & \frac{1}{a^4} & \frac{\varepsilon_0\varepsilon_2}{a^4} \\
T_2' & -\frac{\varepsilon_2}{a^5} & & & & -\frac{\varepsilon_0}{a^5} & -\frac{\varepsilon_0\varepsilon_1\varepsilon_3}{a^5} \\
-T_3' & \frac{\varepsilon_2\varepsilon_3}{a^6} & \frac{\varepsilon_0\varepsilon_3}{a^6} & & & \frac{\varepsilon_0\varepsilon_1}{a^6} & \frac{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_4}{a^6} \\
T_4' & -\frac{\varepsilon_2\varepsilon_3\varepsilon_4}{a^7} & -\frac{\varepsilon_0\varepsilon_3\varepsilon_4}{a^7} & -\frac{\varepsilon_0\varepsilon_1\varepsilon_4}{a^7} & & -\frac{\varepsilon_0\varepsilon_1\varepsilon_2}{a^7} & -\frac{\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_5}{a^7} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^n T_n' & -\frac{(-1)^n \overline{\varepsilon_0\varepsilon_1}\varepsilon_2\ldots\varepsilon_n}{a^{n+3}} & -\frac{(-1)^n \varepsilon_0\overline{\varepsilon_1\varepsilon_2}\ldots\varepsilon_n}{a^{n+3}} & \cdots & \cdots & -\frac{(-1)^n \varepsilon_0\ldots\overline{\varepsilon_{n-1}\varepsilon_n}}{a^{n+3}} & -\frac{(-1)^n \varepsilon_0\ldots\overline{\varepsilon_n}\varepsilon_{n+1}}{a^{n+3}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
\sum_{n=1}^{\infty}(-1)^n T_n' & \star_1 & \star_2 & \star_3\cdots & \star_n\cdots & S & \frac{\varepsilon_0\varepsilon_2}{a^4} + R
\end{array}\right]
$$

Let us organize the terms in the following matrix form. The terms in the first column($(-1)^n T_n'$) equal to the sum of the terms in the corresponding row. Also note that the last row denotes the sum of the terms in the corresponding column.

Since the series which give the derivative of q is absolutely convergent, regrouping the suitable terms together, we can write:

$$\frac{\partial q(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{b=0} = \frac{1}{a^3} + \frac{\varepsilon_0\varepsilon_2}{a^4} + \sum_{n=1}^{\infty} \star_n + S + R$$

where

$$\star_1 = -\frac{\overline{\varepsilon_0\varepsilon_1}\varepsilon_2}{a^5} + \frac{\overline{\varepsilon_0\varepsilon_1}\varepsilon_2\varepsilon_3}{a^6} - \frac{\overline{\varepsilon_0\varepsilon_1}\varepsilon_2\varepsilon_3\varepsilon_4}{a^7} + \dots$$

$$\star_2 = \frac{\varepsilon_0\overline{\varepsilon_1\varepsilon_2}\varepsilon_3}{a^6} - \frac{\varepsilon_0\overline{\varepsilon_1\varepsilon_2}\varepsilon_3\varepsilon_4}{a^7} + \frac{\varepsilon_0\overline{\varepsilon_1\varepsilon_2}\varepsilon_3\varepsilon_4\varepsilon_5}{a^8} - \dots$$

$$\star_n = (-1)^n\frac{\varepsilon_0\cdots\overline{\varepsilon_{n-1}\varepsilon_n}\varepsilon_{n+1}}{a^{n+4}} + (-1)^{n+1}\frac{\varepsilon_0\cdots\overline{\varepsilon_{n-1}\varepsilon_n}\varepsilon_{n+1}\varepsilon_{n+2}}{a^{n+5}} + \dots$$

and

$$S = \frac{1}{a^4} - \frac{\varepsilon_0}{a^5} + \frac{\varepsilon_0\varepsilon_1}{a^6} - \dots + (-1)^{i+1}\frac{\varepsilon_0\cdots\varepsilon_i}{a^{i+5}} + \dots \qquad i \geq -1$$

and

$$R = -\frac{\varepsilon_0\varepsilon_1\overline{\varepsilon_2}\varepsilon_3}{a^5} + \frac{\varepsilon_0\varepsilon_1\varepsilon_2\overline{\varepsilon_3}\varepsilon_4}{a^6} - \dots + (-1)^{j+1}\frac{\varepsilon_0\cdots\varepsilon_{j-1}\overline{\varepsilon_j}\varepsilon_{j+1}}{a^{j+3}} \qquad j \geq 2$$

First let's start with observing that by the equation (2.3):

$$S = \frac{1}{a^4}\left(1 - \frac{\varepsilon_0}{a} + \frac{\varepsilon_0\varepsilon_1}{a^2} - \frac{\varepsilon_0\varepsilon_1\varepsilon_2}{a^3} + \dots\right) = 0 \tag{2.5}$$

Secondly,

$$|R| \leq \frac{1}{a^5}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \dots\right) = \frac{1}{a^4(a-1)} \tag{2.6}$$

For $\sum_{n=1}^{\infty} \star_n$, by the equation (2.4) we have:

$$\star_1(-a^2\varepsilon_0\varepsilon_1) = (-1)^2 \frac{\varepsilon_0\varepsilon_1}{a^2} T_a^2(1)$$

$$\star_2(-a^2\varepsilon_1\varepsilon_2) = (-1)^3 \frac{\varepsilon_0\varepsilon_1\varepsilon_2}{a^3} T_a^3(1)$$

and

$$\star_n(-a^2\varepsilon_{n-1}\varepsilon_n) = (-1)^{n+1} \frac{\varepsilon_0\varepsilon_1\dots\varepsilon_n}{a^{n+1}} T_a^{n+1}(1)$$

So

$$\sum_{n=1}^{\infty} \star_n = -\frac{1}{a^4}\sum_{i=0}^{\infty}(-1)^i \frac{\varepsilon_0\varepsilon_1\dots\varepsilon_{i-1}}{a^i} T_a^{i+2}(1)$$

Again by (2.3) we also have

$$\sum_{n=1}^{\infty} \star_n = -\frac{1}{a^4}\sum_{i=0}^{\infty}(-1)^i \frac{\varepsilon_0\varepsilon_1\dots\varepsilon_{i-1}}{a^i} (T_a^{i+2}(1) - \alpha)$$

for any $\alpha \in \mathbb{R}$. Let $\alpha = \dfrac{1+T_a(1)}{2}$ and $\delta = \dfrac{1-T_a(1)}{2} = \dfrac{a}{2}$. Since $T_a^i(1) \in [T_a(1), 1]$ for every $i \geq 0$ we have $-\delta \leq T_a^i(1) - \alpha \leq \delta$ for every $i \geq 0$. Note that by direct calculation $T_a^2(1) - \alpha = \delta(2a-3)$. This gives us:

$$\sum_{n=1}^{\infty} \star_n = -\frac{1}{a^4}(T_a^2(1) - \alpha - \frac{\varepsilon_0}{a}(T_a^3(1) - \alpha) + \frac{\varepsilon_0\varepsilon_1}{a^2}(T_a^4(1) - \alpha) - \dots)$$

$$\leq -\frac{1}{a^4}(\delta(2a-3) - \delta(\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \dots)) = -\frac{2a^2-5a+2}{2a^3(a-1)} \qquad (2.7)$$

Similar calculations show that

$$\sum_{n=1}^{\infty} \star_n \geq -\frac{2a^2-5a+4}{2a^3(a-1)} \qquad (2.8)$$

Now, combining (2.5),(2.6),(2.7) and (2.8) we get the desired result. □

Proof of the Theorem 2.1.3. By lemma 2.4.2 we know that for any $1 < a \leq 2$, $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} > 0$. Also by the previous lemma for any such a, $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{(a,0)}$ has an upper and lower bound. So there exist $N_a^1 \in \mathbb{R}^+$ such that

$$N_a^1 \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} - \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{(a,0)} > 0$$

and $N_a^2 \in \mathbb{R}^+$ such that

$$N_a^2 \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}\Big|_{(a,0)} + \frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}\Big|_{(a,0)} > 0$$

This means that the directional derivatives of $(p-q)(\underline{\varepsilon})(a,b)$ in the direction $\vec{v_1} = (N_a^1, -1)$ and $\vec{v_2} = (N_a^2, 1)$ are both positive. So by local monotonicity theorem, result follows. □

Since we have explicit upper and lower bounds for both $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial a}$ and $\frac{\partial(p-q)(\underline{\varepsilon})(a,b)}{\partial b}$ we can compute the directions in which the entropy is non-decreasing (See Figure 2.2).

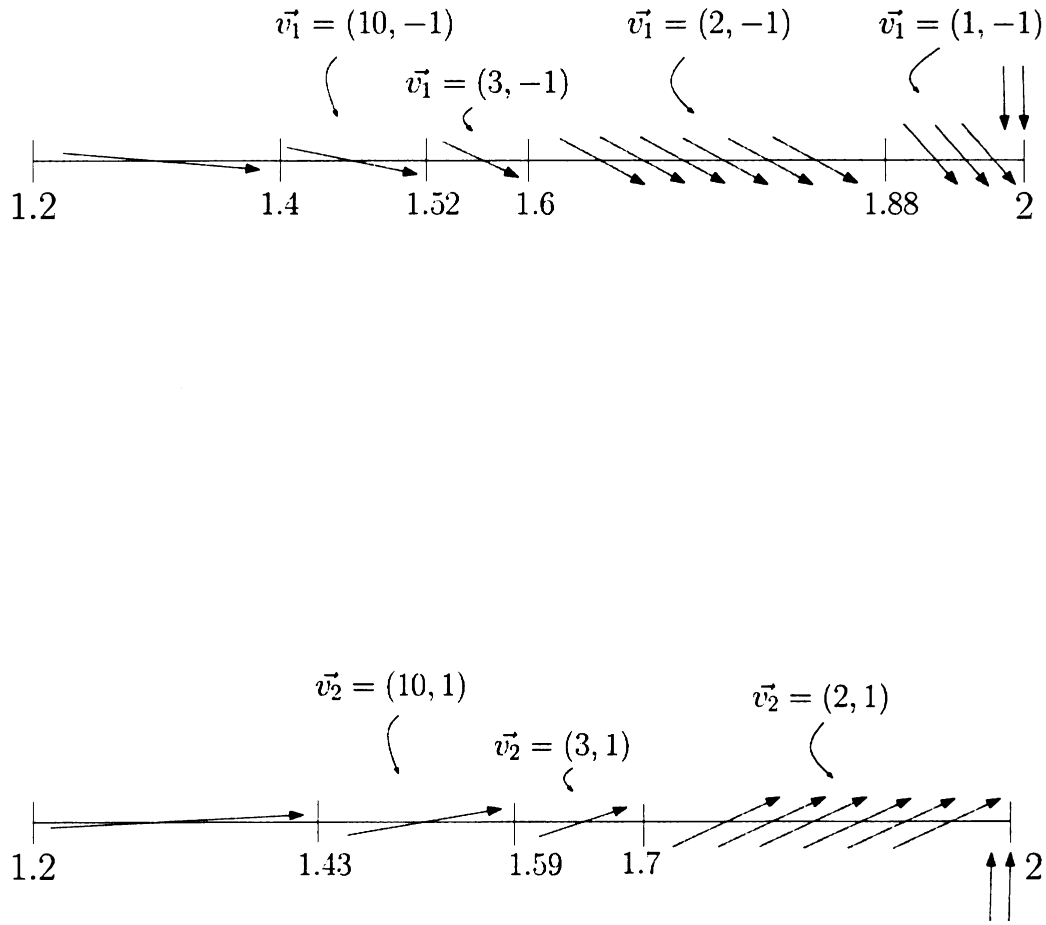
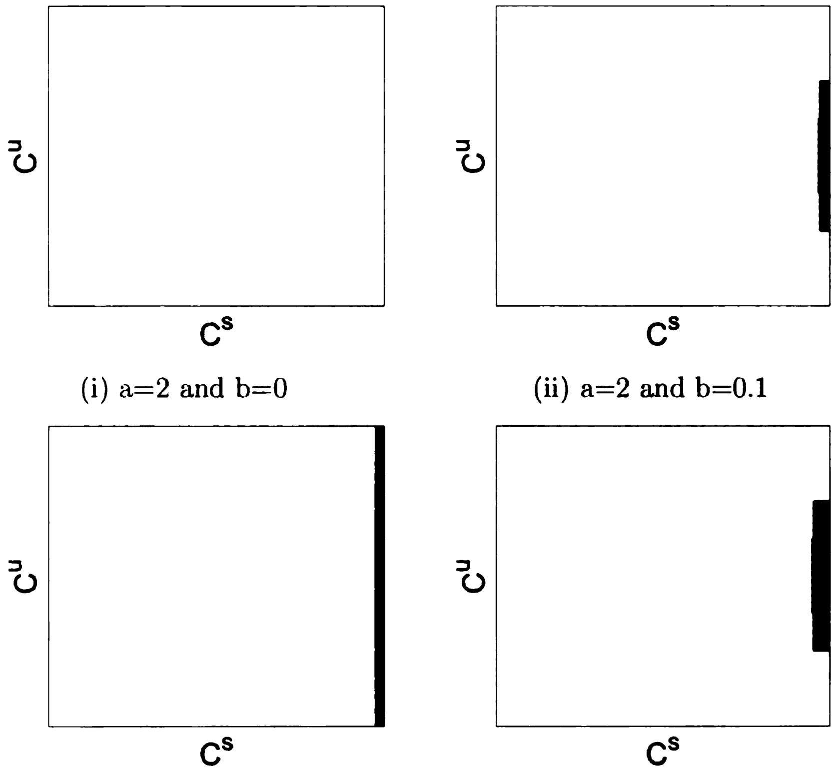


Figure 2.2: This figure shows the approximate monotonicity results for different N_a^1 and N_a^2 values where $1.2 < a \leq 2$. The topological entropy is non-decreasing in the direction of arrows.



(i) a=2 and b=0

(ii) a=2 and b=0.1

(iii) a=1.95 and b=0

(iv) a=1.95 and b=0.01

Figure 2.3: This figure shows the primary pruned regions, $\mathcal{D}_{a,b}$, of maps for given parameters. The x-axis represents C^s and the y-axis represents C^u. One can expect to find some elements of $\hat{\mathcal{P}}_{a,b}$ at the boundary of $\mathcal{D}_{a,b}$.

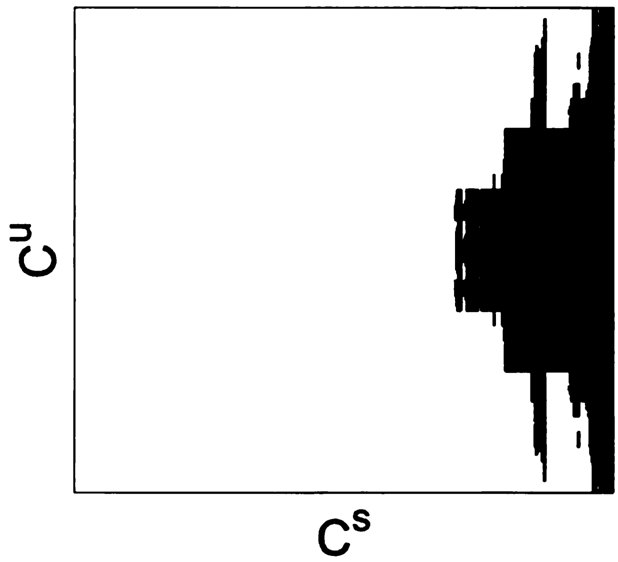


Figure 2.4: Primary pruned region, $\mathcal{D}_{a,b}$, for original parameters studied by Lozi: a=1.7 and b=0.5

2.5 Results about the zero entropy locus

In this section we turn our attention to the parameters for which $h_{top}(\mathcal{L}_{a,b}) = 0$. Note that it is enough to consider the maps with $|b| \leq 1$ since the maps with $|b| > 1$ are, up to affine conjugacy, inverses of the maps with $|b| < 1$.

Let us first review the following theorem:

Theorem 2.5.1 ([15]). *If the Lozi map $\mathcal{L}_{a,b}$ satisfies either (i) $-1 < b < 0$ and $a \leq b - 1$, (ii) $0 < b \leq 1$ and $a \leq -b + 1$, then $h_{top}(\mathcal{L}_{a,b}) = 0$.*

Proof. If $a \leq b - 1 \leq -a$ then $\mathcal{L}_{a,b}$ has no fixed points. When $b < 0$, $\mathcal{L}_{a,b}$ is orientation preserving, so by Brouwer's translation theorem[3] it has an empty non-wandering set and therefore zero entropy, proving (i). When $0 < b \leq 1$ and $b - 1 \leq a \leq 1 - b$, there exists a unique saddle fixed point $p = (1/(1 + a - b), 1/(1 + a - b))$ in the first quadrant. Also note that there is no other period-two points. Now $v^s = (\lambda, 1)$ where $\lambda = (-a + \sqrt{a^2 + 4b})/2$ is a stable direction at p and $W^s_+(p) = \{p + v^s t \in \mathbb{R}^2 | t > 0\}$

is invariant under $\mathcal{L}_{a,b}$. Also $\mathbb{R}^2 \setminus (W^s_+(p) \cup \{p\})$ is homeomorphic to $\mathbb{R}^2$ and $\mathcal{L}^2_{a,b}$ has no fixed points there. Since $\mathcal{L}^2_{a,b}$ is orientation preserving when $b > 0$, $h_{top}(\mathcal{L}_{a,b}) = h_{top}(\mathcal{L}^2_{a,b})/2 = 0$. □

Now let us start stating our results by the following theorem:

Theorem 2.5.2. *For $a = 1$ and $b = 0.5$, $h_{top}(\mathcal{L}_{a,b}) = 0$.*

Proof. First note that when $0 < b < 1$ and $1 - b < a < b + 1$, $\mathcal{L}_{a,b}$ has two saddle fixed points: $p_1 = (1/(1 + a - b), 1/(1 + a - b))$ in the first quadrant and $p_2 = (1/(1 - a - b), 1/(1 - a - b))$ in the third quadrant. Also there are two attracting period-two points: $n_1 = (N, (1 - aN)/(1 - b))$ in the fourth quadrant and $n_2 = ((1-aN)/(1-b), N)$ in the second quadrant where $N = (1+a-b)/[(b-1)^2+a^2]$. By a direct calculation of $\mathcal{L}^4_{a,b}$, one can check that there are no other period-four points.

Now $v^s_1 = (\lambda^s_1, 1)$ where $\lambda^s_1 = (-a + \sqrt{a^2 + 4b})/2$ is a stable direction at p_1 and $W^s_+(p_1) = \{p_1 + v^s_1 t \in \mathbb{R}^2 | \, t > 0\}$ is invariant under $\mathcal{L}_{a,b}$. Similarly, $v^u_2 = (-\lambda^u_2, -1)$ where $\lambda^u_2 = (a + \sqrt{a^2 + 4b})/2$ is an unstable direction at p_2 and $W^u_+(p_2) = \{p_2 + v^u_2 t \in \mathbb{R}^2 | \, t > 0\}$ is invariant under $\mathcal{L}_{a,b}$.

The more challenging part is to show that the right and left parts of the unstable manifold of p_1 are attracted by n_1 and n_2, respectively. We will show this happens when we consider $\mathcal{L}^4$. Now, let Z be the intersection of the line $\ell_1 = \{p_1 + v^u_1 t \in \mathbb{R}^2 | \, t > 0\}$ and the x-axis where $v^u_1 = (-\lambda^u_1, -1)$ and $\lambda^u_1 = (-a - \sqrt{a^2 + 4b})/2$. See Figure 2.5.

Claim: For $a = 1$ and $b = 0.5$, $\mathcal{L}^{4m}_{a,b}(Z) \to n_1$ as $m \to \infty$.

Proof of the claim: Let us use $\mathcal{L}_{1,0.5} = \mathcal{L}$. Let P be the polygon whose corners are given by Z, $\mathcal{L}^2(Z)$, $\mathcal{L}^4(Z)$ and $\mathcal{L}^6(Z)$. Since $\mathcal{L}^8(Z) \approx (1.223, -0.375)$ is in P, $\mathcal{L}^2(P) \subset P$, i.e., P is invariant under $\mathcal{L}^2$. Now consider the Lyapunov function $V(x, y) = (x - \pi_1(n_1))^2 + (y - \pi_2(n_1))^2$ where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the projections to the x-coordinate and y-coordinate, respectively. It is not hard to see (with the help of a computer if necessary) that $V(\mathcal{L}^4(x, y)) - V(x, y) < 0 \ , \forall (x, y) \in P \setminus \{n_1\}$. This implies that,(see for ex. [8]), Z (actually every $(x, y) \in P \setminus \{n_1\}$) is asymptotically stable to n_1 under $\mathcal{L}^4$.

Similarly it can be shown that $\mathcal{L}(Z)$ is asymptotically stable to n_2 under iterations of $\mathcal{L}^4$. Now let $W_r(p_1)$ be the forward iterations (under $\mathcal{L}^4$) of the line segment connecting p_1 and Z. Similarly let $W_\ell(p_1)$ be the forward iterations (under $\mathcal{L}^4$) of the line segment connecting p_1 and $\mathcal{L}(Z)$. To complete the proof of the theorem, we apply Brouwer's translation theorem to $\mathcal{L}^4$. Note that $\mathbb{R}^2 \backslash (W^s_+(p_1) \cup \{p_1\} \cup W^u_+(p_2) \cup \{p_2\} \cup W_r(p_1) \cup \{n_1\} \cup W_\ell(p_1) \cup \{n_2\})$ is homeomorphic to $\mathbb{R}^2$ and $\mathcal{L}^4$ has no fixed points there. Since $\mathcal{L}^4$ is orientation preserving $h_{top}(\mathcal{L}) = h_{top}(\mathcal{L}^4)/4 = 0$. □

Proof of the Theorem 2.1.4: The proof of the above theorem, using similar Lyapunov functions, works for the parameters in a small neighborhood of $(a, b) = (1, 0.5)$ as well.

Remark: When we move away from a neighborhood of $(a, b) = (1, 0.5)$, it is sometimes the case that the unstable manifold of the right fixed point intersects with the stable manifold of the same fixed point causing a homoclinic point and positive entropy. The parameters for which $\mathcal{L}_{a,b}$ is numerically observed to have zero entropy is given in Figure 2.6. For more details see [32]. Note that since positive entropy occurs as a result of a homoclinic intersection of the stable and unstable manifolds of a fixed point (which are piecewise linear), the boundary of the zero entropy locus is expected to be piecewise algebraic. But writing the equations explicitly requires more work.

The case a=1+b: When $a = 1 + b$ and $b > 0$, it can be shown that the portion of the line $\ell : y = -x + (1 - b^2)/(a(1 + b^2))$ that stays in the region given by $1 + ax + by \geq 0$, $1 - a(1 + ax + by) + bx \leq 0$, $x \leq 0$ and image of that portion of the line ℓ under $\mathcal{L}_{a,b}$ give all the period-four points except the fixed points of $\mathcal{L}_{a,b}$. In other words there are infinitely many period-four points that lie on two line segments. But it can be again observed numerically that as long as there are no homoclinic points, the unstable manifold of the right fixed point is attracted by these two line segments causing the entropy to be zero. Note that when $a > 1 + b$, the period-two points become saddles, so we can expect that some portion of the line $a = 1 + b$, $b > 0$ is a part of the boundary of the zero entropy locus. See Figure 2.6.

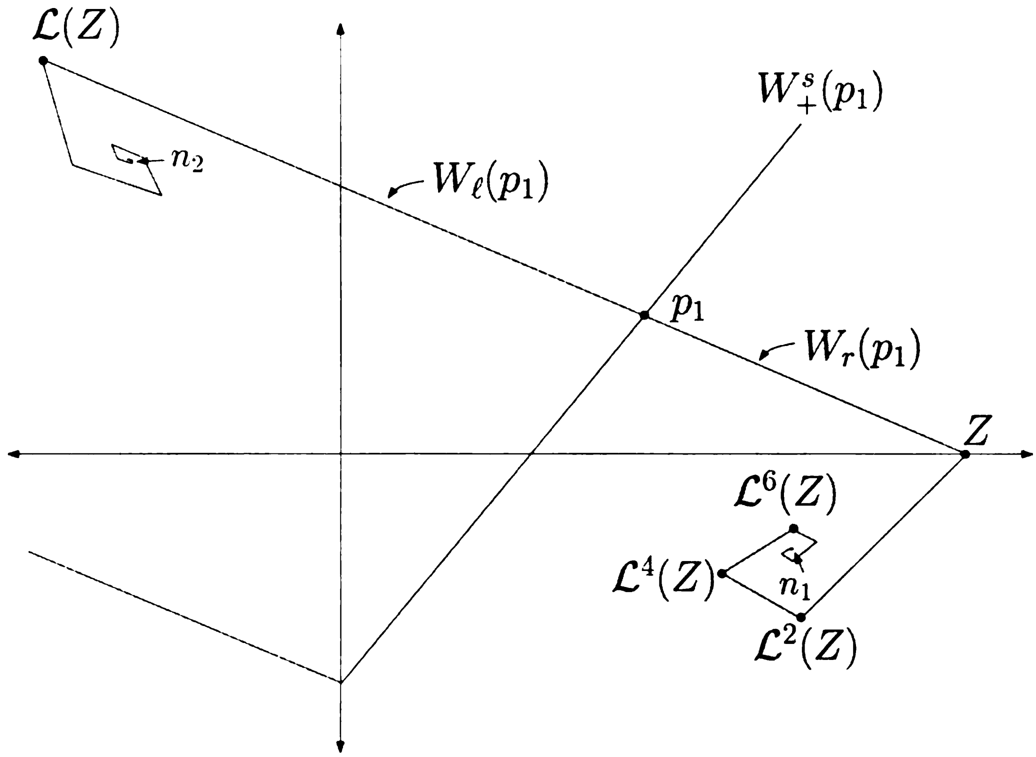


Figure 2.5: The picture shows the unstable and stable manifolds of the right fixed point of $\mathcal{L}_{1,0.5}$.

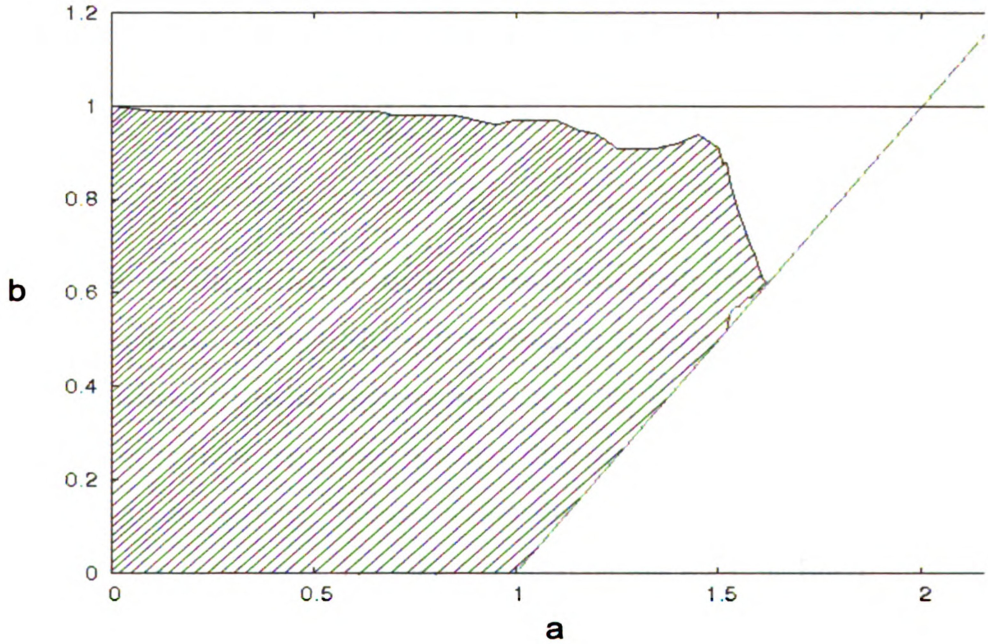


Figure 2.6: The shaded region gives the parameters for which $h_{top}(\mathcal{L}_{a,b}) = 0$.

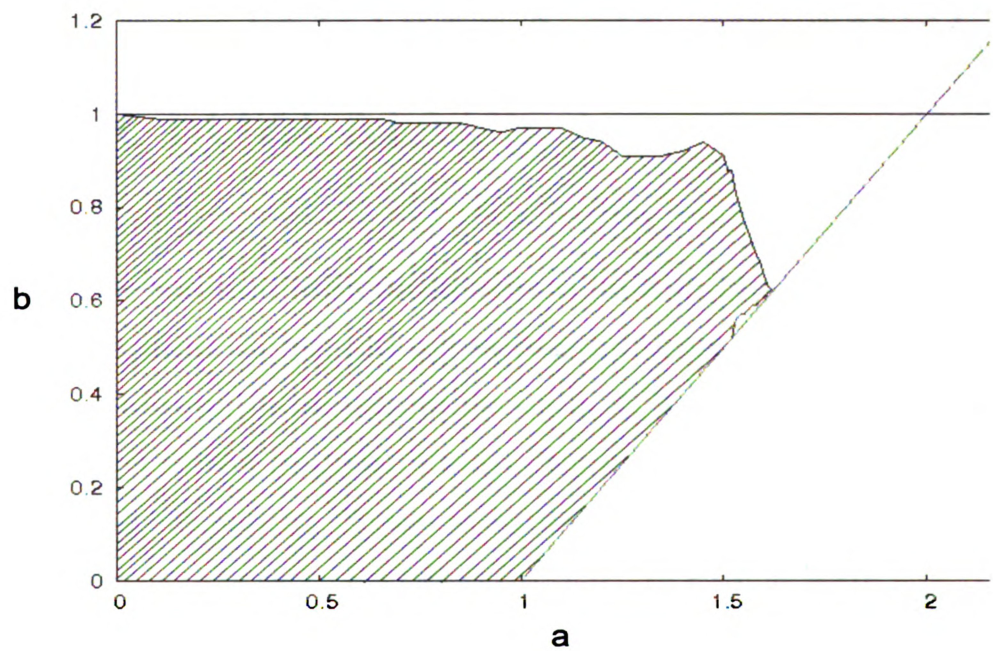


Figure 2.6: The shaded region gives the parameters for which $h_{top}(\mathcal{L}_{a,b}) = 0$.

Chapter 3

DISCONTINUITY OF ENTROPY FOR LOZI MAPS

There have been some recent developments in the study of piecewise affine surface homeomorphisms. In [16], Ishii and Sands give a lap number entropy formula for piecewise affine surface homeomorphisms and in [4], Buzzi proves that under the assumption of positive topological entropy, there are finitely many ergodic measures maximizing the entropy. He also shows that topological entropy is lower semi-continuous for these maps. The following question was asked by Buzzi:

Question 3. *Prove or disprove the upper semi-continuity of entropy for piecewise affine homeomorphisms of the plane.*

Also, Ishii and Sands, motivated by their rigorous entropy computations for the Lozi family, made the following conjecture:

Conjecture 2 (Ishii and Sands [15]). *There are at most countable number of points of discontinuity of the entropy map* $(a, b) \to h(\mathcal{L}_{a,b})$.

Our goal is to answer Buzzi's above question by showing that topological entropy of the Lozi map is not upper semi-continuous at a given parameter. Moreover, our results can be extended to disprove the above conjecture by Ishii and Sands.

Let us start with a review of the subject:

Piecewise affine maps: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function where $n \in \mathbb{Z}^+$. An *affine subdivision* of f is a finite collection $\mathcal{U} = \{U_1, \dots, U_N\}$ of pairwise disjoint non-empty open subsets of $\mathbb{R}^n$ such that their union is dense in $\mathbb{R}^n$ and $f|_{U_i} = A_i|_{U_i}$ for each $i = 1, \dots, N$ where $A_i : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine map. A *piecewise affine map* is a continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ for which there exists an affine subdivision.

Example 3.0.3. Lozi maps are piecewise affine homeomorphisms of the plane given by:

$$\mathcal{L} = \mathcal{L}_{a.b} : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}, \quad a,b \in \mathbb{R},\ b \neq 0.$$

Note that $\mathcal{U} = \{U_1, U_2\}$ where $U_1 = \{(x, y) \in \mathbb{R} \mid x > 0\}$ and $U_2 = \{(x, y) \in \mathbb{R} \mid x < 0\}$.

Let us first review some of the related results in different dimensions. Throughout this paper, we will denote the topological entropy of a map f by $h(f)$.

In one dimension, one can work with piecewise monotone functions. Let I denote a compact interval of $\mathbb{R}$. A map $T : I \to I$ is called a *piecewise monotone* function if there exists a partition of I into finitely many subintervals on each of which the restriction of T is continuous and strictly monotone. Two piecewise monotone maps T_1 and T_2 are said to be ε-close, if they have the same number of intervals of monotonicity and the graph of T_2 is contained in an ε-neighborhood of the graph of T_1 considered as subsets of $\mathbb{R}^2$. It was proved by Misiurewicz and Szlenk[27] that the entropy map $f \to h(f)$ is lower semi-continuous for piecewise monotone continuous maps. They also gave upper bounds for the jumps up of the entropy. For unimodal maps(two-piece continuous monotone maps) entropy is continuous for all maps for which it is positive.

In higher dimensions, let $C^r(M^n)$ denote the set of C^r self maps of an n-dimensional compact manifold. It is a classical result of Katok[19] that the entropy map is lower semi-continuous for $C^{1+\alpha}$ diffeomorphisms on compact surfaces. Yomdin[33] and Newhouse[28] proved that entropy is upper semi-continuous in $C^\infty(M^n)$ for $n \geq 1$. Combining these two results, one can get the continuity of entropy in $C^\infty(M^2)$. This result does not hold for homeomorphisms on surfaces. Also, Misiurewicz[24] constructed examples showing that entropy is not continuous in $C^\infty(M^n)$ for $n \geq 4$ as well as examples[25] showing that entropy is not upper semi-continuous in $C^r(M^n)$ where $r < \infty$ and $n \geq 2$.

For piecewise affine surface homeomorphisms, the following Katok-like theorem(see [18]) is given by Buzzi[4]:

Theorem 3.0.4. *Let $f : M \to M$ be a piecewise affine homeomorphism of a compact affine surface. Let S be the singularity locus of M, that is, the set of points x which have no neighborhood on which the restriction of f is affine. For any $\varepsilon > 0$, there is a compact invariant set $K \subset M \setminus S$ such that $h(f|K) > h(f) - \varepsilon$. Moreover $f : K \to K$ is topologically conjugate to a subshift of finite type.*

The lower semi-continuity of the entropy follows from the above theorem. The goal of this paper is to disprove the upper semi-continuity by showing a jump up of the entropy in Lozi maps. Our results can be summarized as follows:

Theorem 3.0.5. *In general, the topological entropy of Lozi maps does not depend continuously on the parameters: For $\epsilon_1 > 0$ and small and $|\epsilon_2|$ small.*

(i) The topological entropy of Lozi maps with $(a, b) = (1.4 + \epsilon_2, 0.4 + \epsilon_2)$, $h(\mathcal{L}_{1.4+\epsilon_2, 0.4+\epsilon_2})$, is zero.

(ii) The topological entropy of Lozi maps, $h(\mathcal{L}_{(1.4+\epsilon_1+\epsilon_2, 0.4+\epsilon_2)})$, has a lower bound of 0.1203.

3.1 Lower Bound Techniques

There are some computer assisted techniques to give rigorous lower bounds for the topological entropy of maps like Henon and Ikeda. They were first introduced by Zygliczyński [34] and developed in [10] and [9]. There are also more recent methods by Newhouse, Berz, Makino and Grote[29] which gives better lower bounds for the Henon map.

Let us review the following ideas which were used in [9].

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map and $N_1, N_2, \dots N_p$ be p pairwise disjoint quadrilaterals. Note that we can parametrize each N_i with the unit square $I^2 = [0,1] \times [0,1]$ by choosing a homeomorphism $h_i : I^2 \to N_i$. We call the edges $h_i(\{0\} \times [0,1])$ and $h_i(\{1\} \times [0,1])$ "vertical" and the edges $h_i([0,1] \times \{0\})$ and $h_i([0,1] \times \{1\})$ "horizontal". We define a covering relation between two quadrilaterals in the following way: (See Figure 3.4)

Definition 3.1.1. We say N_i f-covers N_j and write $N_i \implies N_j$ if:

(i) For each $\rho \in [0,1]$, $f(h_i(\{0\} \times \{\rho\}))$ and $f(h_i(\{1\} \times \{\rho\}))$ are located geometrically on the opposite sides of N_j.

(ii) For each $\rho \in [0,1]$, there are two numbers $t_\rho^1, t_\rho^2 \in (0,1)$ such that $f(h_i(\{t_\rho^1\} \times \{\rho\}))$ lies in one of the vertical edges of N_j and $f(h_i(\{t_\rho^2\} \times \{\rho\}))$ lies in the other vertical edge of N_j and $\forall\, t_\rho^1 < t < t_\rho^2$, $f(h_i(\{t\} \times \{\rho\})) \in N_j$.

(iii) For $0 \leq t < t_\rho^1$ and $t_\rho^2 < t \leq 1$, $f(h_i(\{t\} \times \{\rho\})) \cap N_j$ is empty.

If one can show the existence of these quadrilaterals and associated cover relations, they can be used to give rigorous lower bounds for the topological entropy of f:

Theorem 3.1.2. *([9]) Let $N_1, N_2, \dots N_p$ be pairwise disjoint quadrilaterals and f :*

$\mathbb{R}^2 \to \mathbb{R}^2$ *be continuous. Let* $A = (a_{ij})$ *be a square matrix where* $1 \leq i, j \leq p$ *and*

$$a_{ij} = \begin{cases} 1 & if \quad N_i \implies N_j \\ 0 & otherwise \end{cases}$$

Then f *is semi-conjugate to the subshift of finite type with transition matrix* A. *In particular,* $h(f) \geq log(\lambda_1)$ *where* λ_1 *is the largest magnitude eigenvalue (*$\lambda_1 \geq |\lambda_j|$ *for all eigenvalues of* A*).*

Note that there is no easy way to detect these quadrilaterals. They are usually found by trial and error. In [9], Galias introduces 29 disjoint sets around the non-wandering set of the Hénon map and covering relations between these sets. The transition matrix obtained gives a lower bound of 0.43 for the topological entropy of the Hénon map. Note that these bounds also hold in a small neighborhood of the studied parameter. Later, this bound is improved in [29] using different techniques.

3.2 Discontinuity of entropy for Lozi maps

Since Lozi maps are piecewise affine surface homeomorphisms, topological entropy of these maps are lower semi-continuous[4]. In other words, if parameters are slightly changed, entropy of the map can not jump down. There are also some monotonicity results(see [14] and Theorem 2.1.2 and Theorem 2.1.3 above) about the entropy of these maps around the parameter $b = 0$. It is also known that the topological entropy is continuous for all $\mathcal{L}_{a,b}$ where $a > 1$ and $b = 0$.

We first prove that the entropy jumps from zero to a positive value if parameters are slightly changed from $(a, b) = (1.4, 0.4)$ to $(a, b) = (1.4 + \epsilon, 0.4)$ where $\epsilon > 0$ and small.

Theorem 3.2.1. *For $\epsilon > 0$ and small:*

(i) The topological entropy of Lozi maps with $(a, b) = (1.4, 0.4)$, $h(\mathcal{L}_{1.4,0.4})$, is zero.

(ii) The topological entropy of Lozi maps, $h(\mathcal{L}_{(1.4+\epsilon,0.4)})$, has a lower bound of 0.1203.

Proof of the Theorem 3.2.1 (i).

Let's denote $\mathcal{L}_{1.4,0.4} = \mathcal{L}$. We will prove that $h(\mathcal{L}^4) = 0$. Note that $\mathcal{L}^4$ has the following fixed points: (i) $p_1 = (1/2, 1/2)$ and $p_2 = (-5/4, -5/4)$, (ii) the closed line segment ℓ_1 which connects $(0, 15/29)$ to $(-20/29, 35/29)$ and (iii) $\mathcal{L}(\ell_1)$.

Note that p_1 is a saddle fixed point and $v_1^s = (\lambda_1^s, 1)$ where $\lambda_1^s = (-7 + \sqrt{89})/10$ is a stable direction at p_1 and $W_+^s(p_1) = \{p_1 + v_1^s t \in \mathbb{R}^2 \,|\, t > 0\}$ is invariant under $\mathcal{L}$(or $\mathcal{L}^4$). Similarly, p_2 is a saddle point and $v_2^u = (-\lambda_2^u, -1)$ where $\lambda_2^u = (7 + \sqrt{89})/10$ is an unstable direction at p_2 and $W_+^u(p_2) = \{p_2 + v_2^u t \in \mathbb{R}^2 \,|\, t > 0\}$ is invariant under $\mathcal{L}^4$.

Let's call the left and right parts of the unstable manifold at p_1; $W_\ell(p_1)$ and $W_r(p_1)$, respectively. If we can show that $W_\ell(p_1)$ is attracted by ℓ_1 and $W_r(p_1)$ is attracted by $\mathcal{L}(\ell_1)$ then we can use the Brouwer's translation theorem in $U = \mathbb{R}^2 \setminus (W_+^s(p_1) \cup \{p_1\} \cup W_+^u(p_2) \cup \{p_2\} \cup W_r(p_1) \cup \ell_1 \cup W_\ell(p_1) \cup \mathcal{L}(\ell_1))$ which is homeomorphic to $\mathbb{R}^2$. Since $\mathcal{L}^4$ has no fixed points in U and it is orientation preserving, $h(\mathcal{L}^4) = 4h(\mathcal{L}) = 0$.

$W_\ell(p_1)$ is attracted to ℓ_1: Now, let Z be the intersection of the line $m = \{p_1 + v_1^u t \in \mathbb{R}^2 \,|\, t > 0\}$ and the x-axis where $v_1^u = (-\lambda_1^u, -1)$ and $\lambda_1^u = (-7 - \sqrt{89})/10$. In other words, Z is the first intersection point of $W_r(p_1)$ with the x-axis. Note that $W_\ell(p_1) = \bigcup_{n=0}^{\infty} \mathcal{L}^{4n}(\{p_1 - v_1^u t \,|\, 0.1 > t > 0\})$, i.e. forward iterations of a small piece in the unstable direction. Let the portion of $W_\ell(p_1)$ which connects $\mathcal{L}(Z)$ and $\mathcal{L}^5(Z)$ be called W. It is not hard to see that $W_\ell(p_1) = \bigcup_{n=-\infty}^{\infty} \mathcal{L}^{4n}(W)$. We want to show that every $x \in W$(so every $x \in W_\ell(p_1)$) is attracted to ℓ_1.

Trapping Region: We introduce a trapping region R around ℓ_1 such that any point $x \in R$ is attracted to a point in ℓ_1. Let:

$$R_1 = (-20/29, 35/29 + 0.2)$$

$$R_2 = (-20/29 + 0.1, 35/29 - 0.25)$$

$$R_3 = (0, 15/29 - 0.25)$$

$$R_4 = (-0.2, 15/29 + 0.5)$$

Let's call the left and right end points of ℓ_1; F_1 and F_2, respectively. Let R be the hexagon with vertices R_1,F_1,R_2,R_3,F_2 and R_4. The sides F_1R_2 and F_2R_4 are parallel to each other with slope $-5/2$ and they are stable directions at F_1 and F_2, respectively. Since R_1 is in the stable manifold of a point in ℓ_1, it is attracted to ℓ_1 under iterations of $\mathcal{L}^4$. Similarly, R_4 is attracted to F_2 since it is in the stable manifold of F_2. So, the quadrilateral with vertices R_1,F_1,F_2 and R_4 is mapped to thinner and thinner quadrilaterals for which one of the sides is always $\ell_1 = F_1F_2$. Similarly, the quadrilateral with vertices F_1,R_2,R_3 and F_2 is mapped towards ℓ_1(See Figure 3.2). So, R is a trapping region.

We want to show that more and more portions of W is mapped into R under forward iterations of $\mathcal{L}^4$. Let's start with the part of W which connects $\mathcal{L}(Z)$ and $\mathcal{L}^3(Z)$. The image of this line segment(under $\mathcal{L}^4$) is the portion of $W_\ell(p_1)$ which connects $\mathcal{L}^5(Z)$ and $\mathcal{L}^7(Z)$(See Figure 3.1). Let's call this portion $\overline{W}$. $\mathcal{L}^5(Z)$ and $\mathcal{L}^7(Z)$ are both in R but there is a part of $\overline{W}$ which is still outside of R which we denote by $\overline{\overline{W}}$, ie. $\overline{\overline{W}}$ is the closure of $\overline{W} \setminus R$. Note that $\ell_c : y = 1 - 1.4(1 + 1.4x + 0.4y) + 0.4x$ is a critical line for $\mathcal{L}^4$ around F_1, ie. images of lines which transversally intersect ℓ_c are

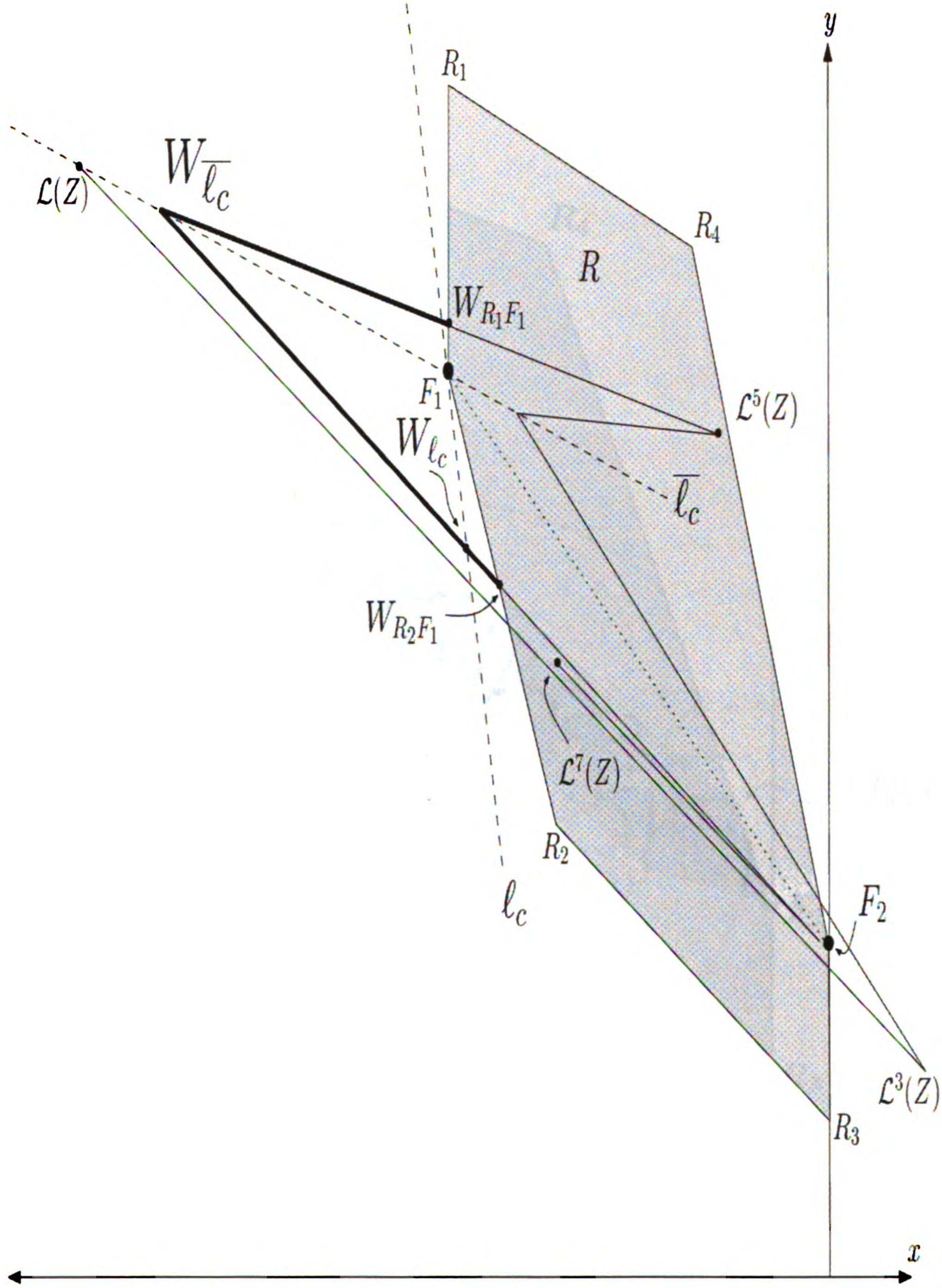


Figure 3.1: This figure shows a portion of the left unstable manifold of the fixed point p_1. Note that all the points on the line segment connecting F_1 to F_2 are period-4 points of $\mathcal{L}$

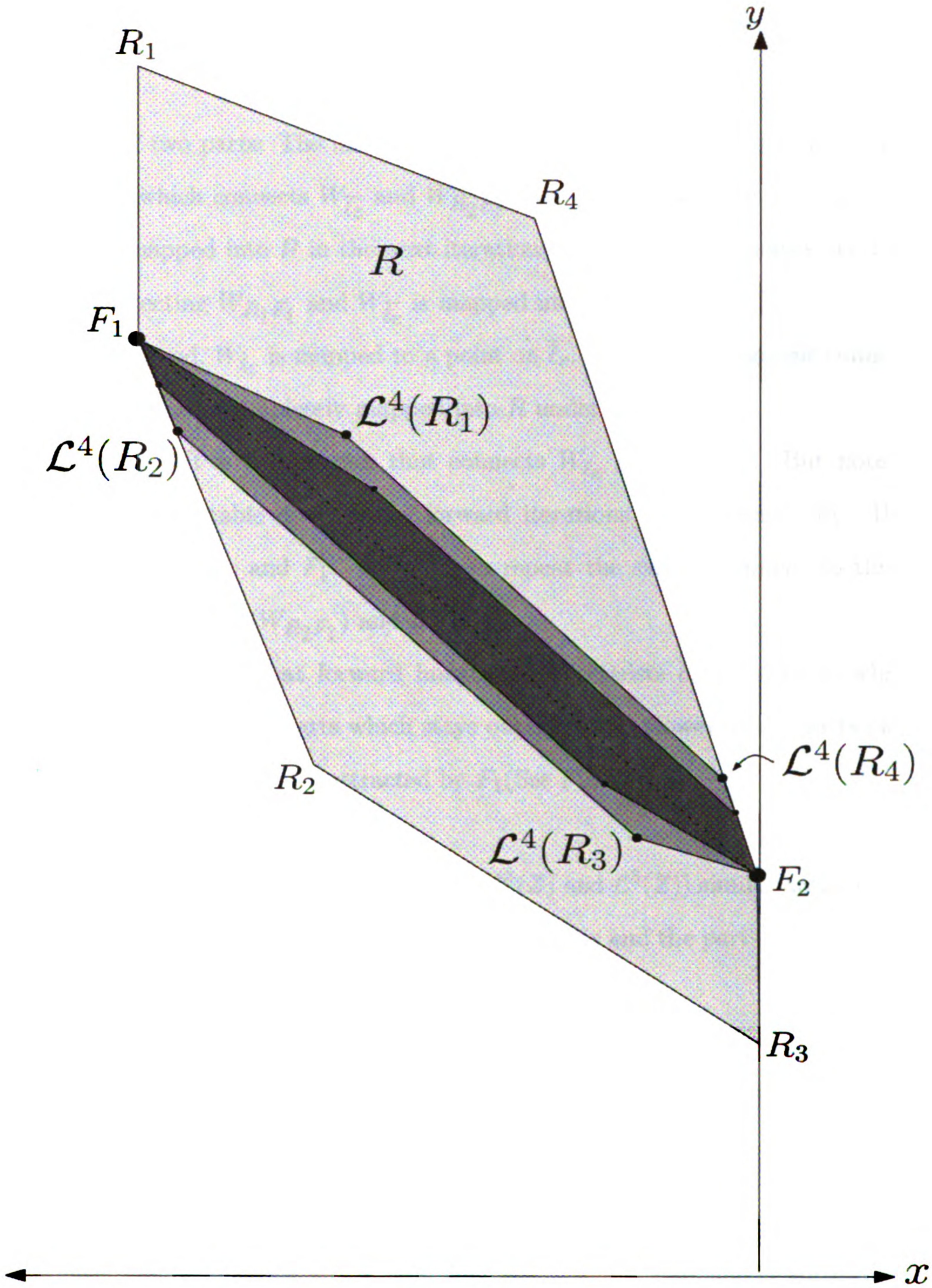


Figure 3.2: Trapping region R(gray) and images $\mathcal{L}^4(R)$(darker) and $\mathcal{L}^8(R)$(darkest).

broken lines. Let $\overline{\ell_c} = \mathcal{L}^4(\ell_c)$. Also, let $W \cap R_1F_1 = W_{R_1F_1}$, $W \cap R_2F_1 = W_{R_2F_1}$, $W \cap \overline{\ell_c} = W_{\overline{\ell_c}}$ and the intersection point of W and ℓ_c which stays below $\overline{\ell_c}$ be W_{ℓ_c}.

$\overline{\overline{W}}$ consists of two parts: The line segment which connects $W_{R_1F_1}$ and $W_{\overline{\ell_c}}$ and the line segment which connects $W_{\overline{\ell_c}}$ and $W_{R_2F_1}$.(See Figure 3.3). It is not hard to see that $W_{\overline{\ell_c}}$ is mapped into R in the next iteration(under $\mathcal{L}^4$) so all points on the line segment connecting $W_{R_1F_1}$ and $W_{\overline{\ell_c}}$ is mapped into R, too.
On the other hand, W_{ℓ_c} is mapped to a point on $\overline{\ell_c}$. So, the line segment connecting W_{ℓ_c} and $W_{\overline{\ell_c}}$ is also completely mapped into R under $\mathcal{L}^8$.
The only part left is the portion that connects W_{ℓ_c} and $W_{R_2F_1}$. But note that $W_{R_2F_1}$ is on the stable direction so forward iterations move towards F_1. W_{ℓ_c} is mapped between $W_{\overline{\ell_c}}$ and F_1. So, one can repeat the same argument to this line segment connecting $\mathcal{L}^4(W_{R_2F_1})$ and $\mathcal{L}^4(W_{\ell_c})$.
This analysis explains that forward images of $\overline{\overline{W}}$ consists of some parts which is mapped into R and some parts which stays outside of R. However, the parts outside of R gets shorter and shorter attracted by F_1(See Figure 3.3).

Now, for the other portion of W(connecting $\mathcal{L}^3(Z)$ and $\mathcal{L}^5(Z)$) similar arguments can be done while this time the critical line ℓ_c is the y-axis and the parts outside of R are either mapped into R or attracted by F_2.

Also, note that $W_\ell(p_1)$ is attracted to ℓ_1 implies that $W_r(p_1) = \mathcal{L}(W_\ell(p_1))$ is attracted to $\mathcal{L}(\ell_1)$. □

Proof of the Theorem 3.2.1 (ii).

We want to show that for any $\epsilon > 0$ and small, there are various subsets which factor onto symbolic systems and so give lower bounds for the map $\mathcal{L}_{(1.4+\epsilon,0.4)}$ by Thm. 3.1.2.

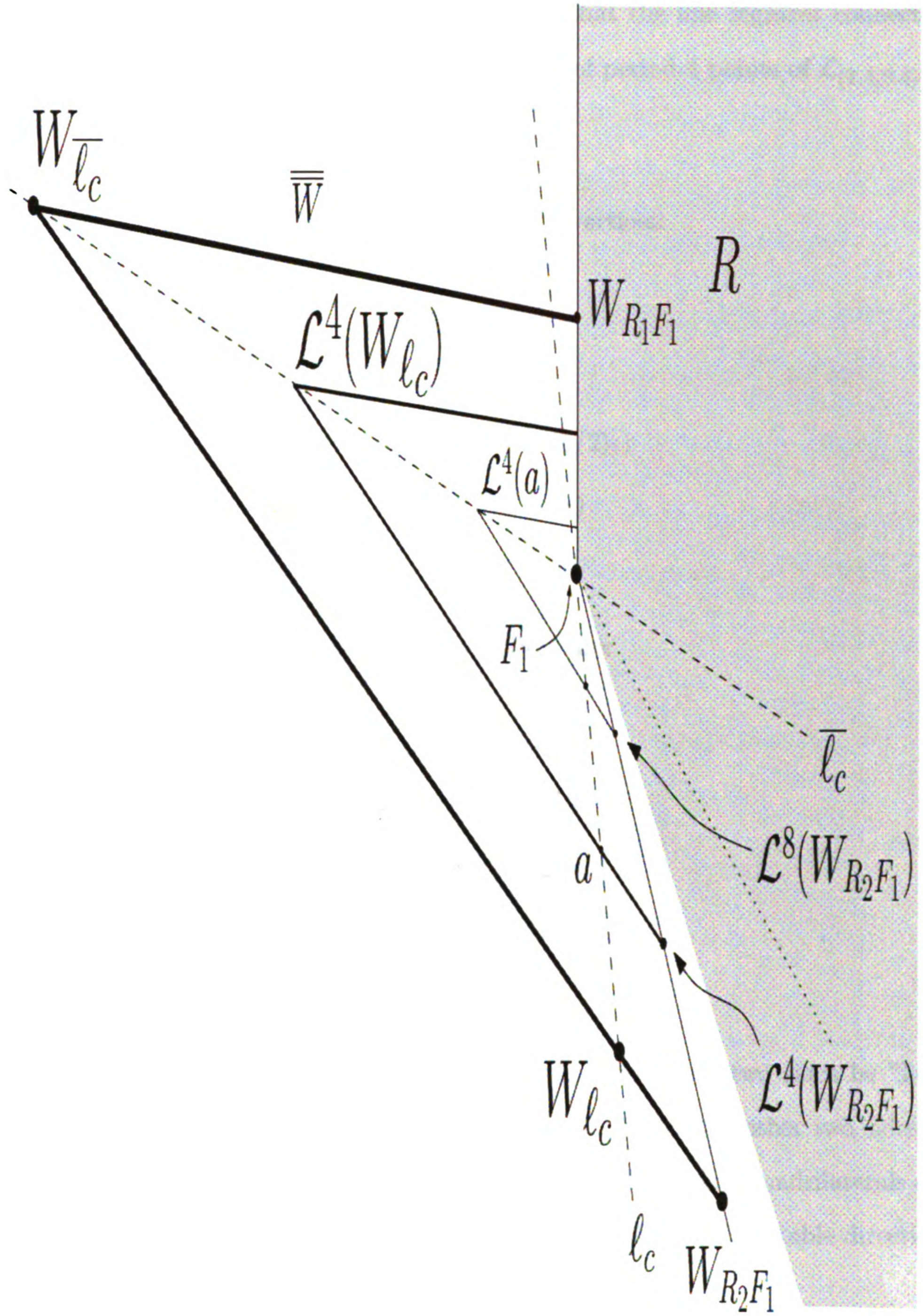


Figure 3.3: The set $\overline{\overline{W}}$(thickest solid broken line) and the part of the images $\mathcal{L}^4(\overline{\overline{W}})$(thinner) and $\mathcal{L}^8(\overline{\overline{W}})$(thinnest) which stay outside of R. Note that everything above $\overline{\ell_c}$ is mapped into R under $\mathcal{L}^4$.

Fix an $\epsilon > 0$ and denote $\mathcal{L}_\epsilon = \mathcal{L}_{(1.4+\epsilon,0.4)}$. Note that the line segment connecting $F_1 = (-20/29, 35/29)$ and $F_2 = (0, 15/29)$ consists of period-4 points of $\mathcal{L}_{(1.4,0.4)}$.

Now, let N_1 be the quadrilateral given by the four vertices:

$$A = (0, 15/29 - \epsilon)$$

$$B = (\epsilon, 15/29 + (7/2)\epsilon)$$

$$C = ((5/2)\epsilon, 15/29 + (5/2)\epsilon)$$

$$D = ((3/2)\epsilon, 15/29 - 2\epsilon)$$

Also let N_2 be the quadrilateral whose vertices are:

$$E = (-3\epsilon, 15/29 + (7/2)\epsilon)$$

$$F = (-2\epsilon, 15/29 + (5/6)\epsilon)$$

$$G = (0, 15/29 - (1/2)\epsilon)$$

$$H = (-\epsilon, 15/29 + (13/6)\epsilon)$$

For N_1, let the sides AB and CD be "vertical" and the other two sides be "horizontal". Similarly for N_2, let EF and GH be "vertical" and the other two sides be "horizontal". Note that the images of N_1 and N_2 under $\mathcal{L}_\epsilon^4$ are also quadrilaterals and vertical edges are contracted since they are chosen very close to the stable directions around $(0, 15/29)$ and $(-20/29, 35/29)$.

By direct calculation, it can be shown that the images of the vertices under the map

$\mathcal{L}_\epsilon^4$ are given by(See Figure 3.4):

$$\mathcal{L}_\epsilon^4(A) = (\frac{30476}{18125}\epsilon + O(\epsilon^2), \frac{15}{29} - \frac{6363}{3625}\epsilon + O(\epsilon^2)) \approx (1.68\epsilon, \frac{15}{29} - 1.75\epsilon)$$
$$\mathcal{L}_\epsilon^4(B) = (\frac{6188}{3625}\epsilon + O(\epsilon^2), \frac{15}{29} - \frac{1319}{725}\epsilon + O(\epsilon^2)) \approx (1.70\epsilon, \frac{15}{29} - 1.81\epsilon)$$
$$\mathcal{L}_\epsilon^4(C) = (-\frac{4769}{1450}\epsilon + O(\epsilon^2), \frac{15}{29} + \frac{847}{290}\epsilon + O(\epsilon^2)) \approx (-3.28\epsilon, \frac{15}{29} + 2.92\epsilon)$$
$$\mathcal{L}_\epsilon^4(D) = (-\frac{120153}{36250}\epsilon + O(\epsilon^2), \frac{15}{29} + \frac{21639}{7250}\epsilon + O(\epsilon^2)) \approx (-3.31\epsilon, \frac{15}{29} + 2.98\epsilon)$$
$$\mathcal{L}_\epsilon^4(E) = (-\frac{9283}{18125}\epsilon + O(\epsilon^2), \frac{15}{29} + \frac{1554}{3625}\epsilon + O(\epsilon^2)) \approx (-0.51\epsilon, \frac{15}{29} + 0.42\epsilon)$$
$$\mathcal{L}_\epsilon^4(F) = (-\frac{23209}{54375}\epsilon + O(\epsilon^2), \frac{15}{29} + \frac{3792}{10875}\epsilon + O(\epsilon^2)) \approx (-0.42\epsilon, \frac{15}{29} + 0.34\epsilon)$$
$$\mathcal{L}_\epsilon^4(G) = (\frac{36363}{18125}\epsilon + O(\epsilon^2), \frac{15}{29} - \frac{7494}{3625}\epsilon + O(\epsilon^2)) \approx (2.00\epsilon, \frac{15}{29} - 2.06\epsilon)$$
$$\mathcal{L}_\epsilon^4(H) = (\frac{113584}{54375}\epsilon + O(\epsilon^2), \frac{15}{29} - \frac{22917}{10875}\epsilon + O(\epsilon^2)) \approx (2.08\epsilon, \frac{15}{29} - 2.10\epsilon)$$

It is not hard to see that we have the following covering relations: $N_1 \implies N_1$, $N_1 \implies N_2$ and $N_2 \implies N_1$. So the transition matrix is given by:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

where the largest magnitude eigenvalue is $\frac{\sqrt{5}+1}{2}$. Since we are using $\mathcal{L}_\epsilon^4$ during the process $h(\mathcal{L}_\epsilon)= \frac{1}{4}h(\mathcal{L}_\epsilon^4) \geq \frac{1}{4}log\frac{\sqrt{5}+1}{2} > 0.1203$ by Thm. 3.1.2. □

Now, we can extend our results from $(a, b) = (1.4, 0.4)$ to $(a, b) = (1.4+\epsilon_2, 0.4+\epsilon_2)$ where $|\epsilon_2|$ is small:

Proof of the Theorem 3.0.5 .

Let $\mathcal{L}$ denote $\mathcal{L}_{(1.4+\epsilon_2, 0.4+\epsilon_2)}$.

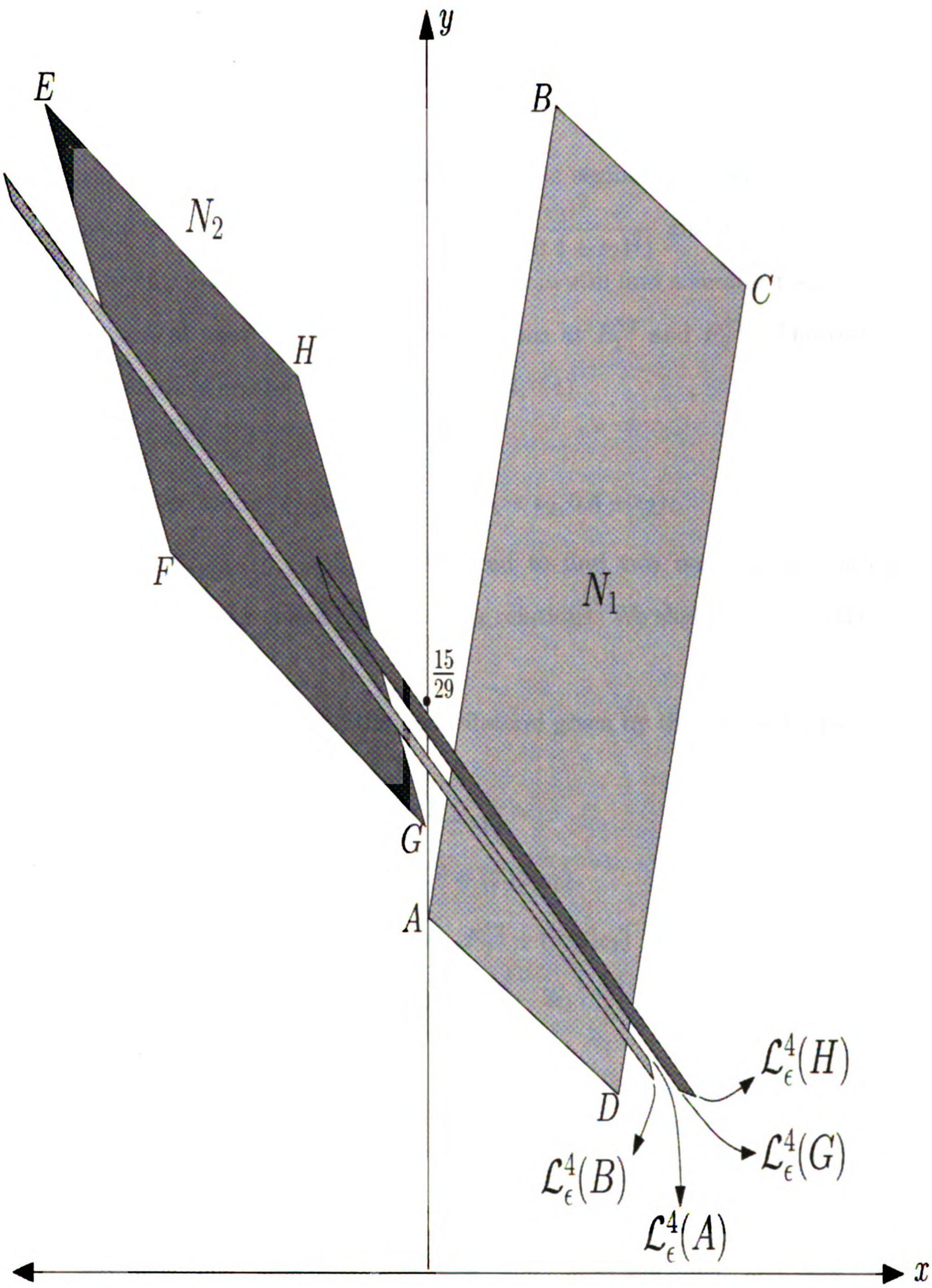


Figure 3.4: This figure shows the quadrangles N_1 and N_2 and their images(thinner boxes). Notice the covering relations: $N_1 \Longrightarrow N_1$, $N_1 \Longrightarrow N_2$ and $N_2 \Longrightarrow N_1$

(i) The entropy is zero for $\mathcal{L}$:

For $|\epsilon_2|$ small and fixed, we still have two line segments of period-4 points: the line segment connecting $F_2^{\epsilon 2} = \dfrac{1-(0.4+\epsilon_2)^2}{(1.4+\epsilon_2)(1+(0.4+\epsilon_2)^2)}$ and $F_1^{\epsilon 2} = \mathcal{L}^2(F_2^{\epsilon 2})$ and the image of this line segment under $\mathcal{L}$. So, we can still find a similar trapping region using the vertical lines and the stable directions at $F_1^{\epsilon 2}$ and $F_2^{\epsilon 2}$. The rest of the proof is the same as in the case of $(a, b) = (1.4, 0.4)$.

(ii) The lower bound for $(a, b) = (1.4 + \epsilon_1 + \epsilon_2, 0.4 + \epsilon_2)$:

Let $\mathcal{L}_{\epsilon_1} = \mathcal{L}_{(1.4+\epsilon_1+\epsilon_2, 0.4+\epsilon_2)}$. We need to find two boxes as in the case of $(a, b) = (1.4, 0.4)$ which give us the covering relations. We slightly modify the points we used before:

For $\epsilon_1 > 0$ and small, let N_1 be the quadrilateral given by the four vertices:

$$\tilde{A} = (0, F_2^{\epsilon 2} - \epsilon_1)$$
$$\tilde{B} = (\epsilon_1, F_2^{\epsilon 2} + (7/2)\epsilon_1)$$
$$\tilde{C} = ((5/2)\epsilon_1, F_2^{\epsilon 2} + (5/2)\epsilon_1)$$
$$\tilde{D} = ((3/2)\epsilon_1, F_2^{\epsilon 2} - 2\epsilon_1)$$

Also let N_2 be the quadrilateral whose vertices are:

$$\tilde{E} = (-3\epsilon_1, F_2^{\epsilon 2} + (7/2)\epsilon_1)$$
$$\tilde{F} = (-2\epsilon_1, F_2^{\epsilon 2} + (5/6)\epsilon_1)$$
$$\tilde{G} = (0, F_2^{\epsilon 2} - (1/2)\epsilon_1)$$
$$\tilde{H} = (-\epsilon_1, F_2^{\epsilon 2} + (13/6)\epsilon_1)$$

In other words, ϵ is replaced with ϵ_1 and 15/29 is replaced with $F_2^{\epsilon_2}$. Although finding the images of these points under $\mathcal{L}^4_{\epsilon_1}$ looks difficult, it is not hard to see the differences between this case and the case $(a, b) = (1.4, 0.4)$. For example, $\mathcal{L}^4_{\epsilon_1}(\tilde{A})$ consists of terms including ϵ_1 and some others not including ϵ_1. Observe that if ϵ_1 equals zero then $F_2^{\epsilon_2}$ is a period-4 point, so the terms *not including* ϵ_1 in $\mathcal{L}^4_{\epsilon_1}(\tilde{A})$ add up to $F_2^{\epsilon_2}$. On the other hand, the terms including ϵ_1 can be made arbitrarily close to the terms including ϵ in the $(a, b) = (1.4, 0.4)$ case by choosing small enough ϵ_2 values and letting $\epsilon_1 = \epsilon$. So, our new boxes also satisfy the previous covering relations giving the same lower bound (0.1203) for the entropy.

□

APPENDIX

Proposition 3.2.2. *For $\underline{\varepsilon}$ with $\underline{\varepsilon}^s = (+1, -1, -1, -1\dots)$ we have*

$$q(\underline{\varepsilon})(a,b) = \frac{b}{(a+x)(b+x)} \quad \textit{where } x = (a - \sqrt{a^2+4b})/2.$$

Proof. First note that since $\underline{\varepsilon}^s = (+1, -1, -1, -1\dots)$,

$$\hat{r}_0 = \cfrac{1}{a + \cfrac{b}{-a + \cfrac{b}{-a + \cfrac{b}{\ddots}}}} = \frac{1}{a+x} \text{ where } x = \cfrac{b}{-a + \cfrac{b}{-a + \cfrac{b}{\ddots}}}.$$

Note that the continued fraction for x can be written as $x = \dfrac{b}{-a+x}$ and this equation gives two solutions. We choose $x = (a - \sqrt{a^2+4b})/2$ as in [12].

Also note that $\hat{r}_n = \cfrac{1}{-a + \cfrac{b}{-a + \cfrac{b}{-a + \cfrac{b}{\ddots}}}}$ for all $n \geq 1$. So $\hat{r}_n = \dfrac{x}{b}$ for $n \geq 1$.

Now we have $q(\underline{\varepsilon})(a,b) = \hat{r}_0 - \hat{r}_0\hat{r}_1 + \hat{r}_0\hat{r}_1\hat{r}_2 - \dots = \dfrac{1}{a+x}(1 - \dfrac{x}{b} + \dfrac{x^2}{b^2} - \dots) =$

$$\frac{1}{a+x}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{b}\right)^n\right) = \frac{b}{(a+x)(b+x)}.$$ □

Lemma 3.2.3. *For $1 < a \leq 2$ and $\underline{\varepsilon} \in \hat{\mathcal{P}}_{2,0}$, $\frac{\partial q}{\partial b}\Big|_{b=0} = \lim_{b\to 0}\frac{\partial q}{\partial b} =$*

$$\lim_{b\to 0}\frac{\partial}{\partial b}\left(\frac{b}{(a+x)(b+x)}\right) = \frac{\left(1-\frac{2}{a}\right)}{a(a-1)^2}$$

For $a = 2$, we have $\lim_{b\to 0}\frac{\partial q}{\partial b} = 0$.

Proof. Note that by the above proposition, we have

$$\frac{\partial q}{\partial b}\Big|_{b=0} = \lim_{b\to 0}\frac{\partial q}{\partial b} = \lim_{b\to 0}\frac{\partial}{\partial b}\left(\frac{b}{(a+x)(b+x)}\right)$$

Note that $\frac{\partial q}{\partial b} = \frac{(a+x)(b+x) - b[x'(b+x) + (a+x)(1+x')]}{(a+x)^2(b+x)^2}$. To find $\lim_{b\to 0}\frac{\partial q}{\partial b}$, we need to apply L'Hospital's Rule twice. Applying L'Hospital's Rule the first time, $\lim_{b\to 0}\frac{\partial q}{\partial b}$, after cancelation, becomes:

$$\lim_{b\to 0}\frac{-b[x''(b+x) + x'(1+x') + x'(1+x') + (a+x)x'']}{2(a+x)x'(b+x)^2 + (a+x)^2 2(b+x)(1+x')}$$

which equals:

$$\lim_{b\to 0}\frac{-b[x''(a+b+2x) + 2x'(1+x')]}{2(a+x)(b+x)[(x'(b+x) + (a+x)(1+x')]}$$

Applying the L'Hospitals' Rule again, the b-derivative of the numerator becomes:

$$-[x''(a+b+2x) + 2x'(1+x')] - b[x'''(a+b+2x) + x''(1+2x') + 2x''(1+x') + 2x'x'']$$

and the b-derivative of the denominator becomes:

$$2[(x'(b+x)+(a+x)(1+x'))(x'(b+x)+(a+x)(1+x'))+(a+x)(b+x)(x''(b+x)+$$

$$2x'(1+x')+(a+x)x'')]$$

Now, taking the limit of the numerator and denominator as b goes to 0 gives the result. Note that $\lim_{b\to 0} x = \lim_{b\to 0} \frac{a-\sqrt{a^2+4b}}{2} = 0$ and $\frac{\partial x}{\partial b}\Big|_{b=0} = x'\big|_{b=0} = -\frac{1}{a}$ and $\frac{\partial^2 x}{\partial b^2}\Big|_{b=0} = x''\big|_{b=0} = \frac{2}{a^3}$:

$$\lim_{b\to 0} \frac{\partial q}{\partial b} = \frac{\frac{2}{a}\left(1-\frac{2}{a}\right)}{2a^2\left(1-\frac{1}{a}\right)^2} = \frac{\left(1-\frac{2}{a}\right)}{a(a-1)^2}$$

□

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