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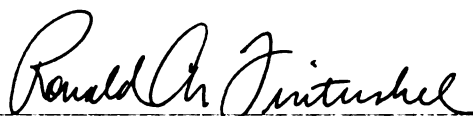
GROUP ACTIONS, COBORDISMS, AND OTHER
ASPECTS OF 4-MANIFOLD THEORY THROUGH THE
EYES OF FLOER HOMOLOGY

presented by

Nathan S. Sunukjian

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GROUP ACTIONS, COBORDISMS, AND OTHER ASPECTS OF 4-MANIFOLD
THEORY THROUGH THE EYLS OF FLOER HOMOLOGY

By

Nathan S. Sunukjian

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Submitted to
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ABSTRACT

GROUP ACTIONS, COBORDISMS, AND OTHER ASPECTS OF 4-MANIFOLD THEORY THROUGH THE EYES OF FLOER HOMOLOGY

By

Nathan S. Sunukjian

There are two main divisions of this dissertation, each dealing with a different aspect of smooth 4-manifold theory, and each employing a different variety of Floer homology as the central tool. In the first, we use monopole Floer homology to construct families of finite cyclic group actions that are equivariantly homeomorphic but not equivariantly diffeomorphic. In the second main division, we will use Heegaard-Floer homology to look at the relationship between a simple class of cobordisms and the Ozsvath-Szabo 4-manifold invariant. We will prove that the Ozsvath-Szabo invariant provides a lower bound on the complexity of certain cobordisms. To accomplish this, we will calculate the Heegaard-Floer homology of the plumbing of two spheres which have been plumbed zero times algebraically.

DEDICATION

To my parents and my friends at University Reformed Church, East Lansing.

ACKNOWLEDGMENT

Without the help of a number of people, I might still have written a dissertation, but it would not have been this one, and it would have been very, very bad. Danny Ruberman, Tom Parker, Ron Stern, and Ian Hambleton all offered ideas and encouragement when things seemed hopeless. Effie Kalfagianni lent her expertise about 3-manifolds, and Matt Hedden cheerfully allowed me to pester him with innumerable questions about Heegaard-Floer homology. Fellow students, Chris Hays, Cagri Karakurt, Jeff ‘the hipster’ Chapin, and Chris Cornwell all answered questions for me, and helped me brainstorm. Chris Hays was particularly indulgent in this respect. Adam Knapp and Inanc Baykur both shared their enthusiasm and answered questions. Tom Mark shared his enthusiasm and also bought me ice cream.

I am occasionally asked why I left my ancestral homeland “the golden paradise of California”. While there is not a single reason for this, most of the blame can be laid on my advisor, Ronald Fintushel. During my time here he has undeservedly treated me like a real mathematician. He answered questions with patience; was quick with encouragement, motivation, and advice; and endured many the Monday afternoon rant of a crazed lunatic (namely, me) with patient forbearance and a generosity for which I will be forever grateful.

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Chapter 1

Introduction

*My soul is an entangled knot,
Upon a liquid vortex wrought
By Intellect, in the Unseen residing.
And thine cloth like a convict sit,
With marlinspike untwisting it.
Only to find its knottiness abiding;
Since all the tools for its untying
In four-dimensioned space are lying...*

—James Clerk Maxwell

Ever since Donaldson's landmark work in the 1980's, gauge theory has played a central role in the study of 4-manifolds. However, the invariants arising from gauge theory are notoriously difficult to compute. Floer homology is an attempt to mediate this difficulty by, in a manner of speaking, breaking the problem up into pieces. Several versions of Floer homology have been defined, but they all have basically the same structure: For Y a 3-manifold, some group G_Y is defined; a 4-manifold X with

boundary Y has an associated relative invariant $\phi_X \in G_Y$; and to two four manifolds with a homeomorphic boundary, there is a pairing of their relative invariants, which ideally recovers some gauge theoretic invariant of a closed 4-manifold. Today there are three main sorts of Floer homology¹: Instanton Floer homology pioneered by Floer himself, which recovers Donaldson theory; the monopole Floer homology of Kronheimer and Mrowka, which is associated to Seiberg-Witten theory; and Heegaard Floer homology of Ozsvath and Szabo, which has an associated 4-manifold invariant, albeit one which lies outside the provenance of gauge theory proper. All three of these theories are conjectured to be equivalent, but to date the best and only real evidence for this is that it holds on all known examples.²

Various techniques have been developed for computing the Floer homology groups. In fact, the three varieties of Floer homology are formally similar enough that techniques for computing in one theory very often work in the other two. In particular, Floer's surgery exact triangle and the excision theorem have become mainstays.

In due course, some of the differences and relative advantages of the different varieties of Floer homology will become evident in this dissertation. Since the three theories are formally so similar, often the advantages of one theory over another will be manifest in the definitions themselves. In the first section, we will describe a simple situation involving monopole Floer homology. Our goal will be to show how monopole Floer homology can be used to construct 'exotic' group actions on 4-manifolds.

In the second section, we will turn our eyes to much broader questions about 4-manifolds. Specifically, we will define a particular surgery operation on 4-manifolds that is related to h-cobordisms. As a first step in investigating this surgery, we

¹For our purposes here we will ignore Floer homology theories such as Lagrangian-Floer homology and concentrate primarily on Floer homologies that give rise to 3-manifold invariants.

²While this manuscript was in preparation, Kutluhan, Lee, and Taubes announced a proof of the equivalence of Heegaard-Floer and monopole Floer homology in [23]

calculate the Heegaard-Floer homology of the 3-manifold on which this surgery is performed. Here, our calculation appeals directly to the definition of Heegaard-Floer homology. It is not clear how one would accomplish this computation in monopole Floer homology.

Chapter 2

Exotic group actions

The world of smooth 4-manifolds exhibits a beguiling array of exotic behavior.

1. *Exotic manifolds.* There exist 4-manifolds that are homeomorphic but not diffeomorphic.
2. *Exotic Surfaces.* There exist surfaces Σ and Σ' in a 4-manifold X , such that (X, Σ) is homeomorphic to (X, Σ') as pairs but not diffeomorphic.
3. *Exotic diffeomorphisms.* There exist homeomorphisms that are topologically isotopic but not smoothly isotopic.

In this chapter we are interested in investigating a more rigid version of the third item. That is, instead of considering a general diffeomorphism on a 4-manifold, we will look at diffeomorphisms that generate finite group actions. The following question arises: Do there exist smooth finite group actions on a 4-manifold that are equivariantly homeomorphic but not equivariantly diffeomorphic? In particular, are there such actions on irreducible manifolds? In this chapter we will answer this question in the affirmative by constructing such exotic group actions on 4-manifolds. In section 2.5 we will give a full statement of the circumstances to which our construction applies.

After briefly reviewing the history of exotic actions on 4-manifolds in section 2.1, we will survey a general strategy for producing exotic behavior in section 2.2. As an example, we'll review the technique of knot-surgery for producing exotic manifolds since our construction of exotic group actions is modeled on it. Once all the necessary machinery is in place, we will be able to construct exotic actions of finite cyclic groups on irreducible 4-manifolds. This result originally appeared in [13], and is joint work with Ronald Fintushel and Ronald Stern. The proof presented here is slightly different from the original: we remove all mention of 'twins' and S^1 actions. We shall end this chapter with various examples.

2.1 History

It has long been known that the fixed set and orbit data of a group action can tell us quite a bit about the action itself. In dimension 3, things are particularly rigid. The classical Smith conjecture from 1939 states that if a finite cyclic group acting S^3 has non-trivial fixed set, then that fixed set has to be the unknot. It was finally proved in 1978 using the combined work of Thurston, Meeks, Yau, Bass, and Gordon.

In dimension 4, the Smith conjecture is false. In 1966, Giffen constructed infinite families of finite group actions on S^4 with quotient S^4 and fixed set a knotted S^2 .

Whereas in dimension 3 we might say that group actions on S^3 are classified by their fixed set (i.e. there is only one such action with non-empty fixed set), finite group actions on S^4 which are a subaction of an S^1 action can also be classified. Fintushel showed that S^1 actions on S^4 are classified by their orbit data.

The classification of finite cyclic group actions on S^4 is far from complete, however. In 1976, Cappell-Shaneson constructed involutions on homotopy 4-spheres that were exotic in the sense that they were not equivariantly diffeomorphic to linear actions on S^4 , and Akbulut later showed that the homotopy 4-spheres constructed are in fact

S^4 , [6], [1]. Through different methods, Fintushel-Stern also constructed examples of exotic involutions on S^4 , [10].

The advent of Seiberg-Witten theory provided new opportunities for studying exotic group actions. For one thing, it provided obstructions to the existence of smooth actions. This is explained in more detail in Section 2.3.1. More constructive is the result of Ue [42] from 1998. Ue constructed free actions of finite groups on simply connected 4-manifolds that are equivariantly homeomorphic but not equivariantly diffeomorphic. The actions he constructed are distinguished by calculating the Seiberg-Witten invariant of the quotients. In Ue's construction, the 4-manifolds being acted upon can all be decomposed as a smooth connected sum with $S^2 \times S^2$, and it is this factor of $S^2 \times S^2$ that provides the flexibility to construct the exotic actions. We will offer an example modeled on Ue's actions in Section 2.2.1.

2.2 Exotic Constructions

Many constructions of exotic behavior on 4-manifolds follow the same general pattern:

1. Define some sort of surgery.
2. Check that the surgery doesn't change the topological type of whatever behavior you are studying.
3. Check using gauge theory that the surgery changes the smooth type.

As an example, we'll describe a neo-classical construction of exotic 4-manifolds due to Fintushel and Stern. To satisfy step 2, we'll use the following theorem of Freedman.

Theorem 1. [14] *If X and X' are smooth, simply connected 4-manifolds, then they are homeomorphic if and only if they have isomorphic cohomology rings.*

To satisfy step 3, we'll use the Seiberg-Witten invariant. This is described in detail in the next section. For now, it will suffice to know $SW_X \in \mathbb{Z}[H_2(X)]$ is an invariant of smooth 4-manifolds.

Knot surgery is a process whereby the neighborhood of a torus is replaced with something homologically equivalent, but “knotted”. Remarkably, this process does not change the homeomorphism type of a 4-manifold, and equally remarkable is the effect on the Seiberg-Witten invariant. Specifically, knot surgery is defined by replacing a copy of $D^2 \times T^2$ with $S^1 \times S^3 \setminus nb(K)$. If the surgered manifold is still simply connected, then one checks by Freedman's theorem that the homeomorphism type of the manifold is not changed.

Theorem 2. [12] *Suppose that T is an embedded torus in a 4-manifold X with $[T]^2 = 0$, and that K is a knot in S^3 . If X and $X \setminus T$ are simply connected, then X is homeomorphic to the knot surgered manifold $X_K := (X \setminus nb(T)) \cup_\phi (S^1 \times (S^3 \setminus nb(K)))$.*

Moreover, if ℓ_K is the longitude of K , and $\phi : \partial S^1 \times (S^3 \setminus nb(K)) \longrightarrow \partial X \setminus (D^2 \times T^2)$ identifies ℓ_K with ∂D^2 , then SW_{X_K} is obtained from SW_X via multiplication by the symmetrized Alexander polynomial of K :

$$SW_{X_K} = SW_X \cdot \Delta_K(2[T])$$

It is evident from this theorem that if X is a 4-manifold with $SW_X \neq 0$ and contains a suitable torus, then there exist an infinite number of manifolds that are homeomorphic but not diffeomorphic to X : apply knot surgery to X using an infinite collection of knots with distinct Alexander polynomials. There is an algebraic subtlety here, but it is minor enough that we have relegated it to the appendix.

2.2.1 A warm up: Exotic Involutions on $2X\#(S^2 \times S^2)$

Theorem 3. *Suppose that X is a 4-manifold to which the theorem 2 applies. Then there are an infinite number of exotic group actions on $2X\#S^2 \times S^2$.*

Proof. Let $\{X_{K_i}\}$ be a collection of non-diffeomorphic manifolds which all arise as knot surgery on X . If we take the 2-fold branched cover of $\{X_{K_i}\}$ over a trivially embedded torus we get $2X_{K_i}\#S^2 \times S^2$ (for this fact see [16] or [18]). It has been shown by Auckly [2], and independently by Akbulut [4], that $X_{K_i}\#S^2 \times S^2$ is diffeomorphic to $X\#S^2 \times S^2$ (see also [5] for a simplified proof). Hence, we have an infinite family of involutions on $2X\#S^2 \times S^2$ that are all the same topologically since they came from topologically equivalent branched covers, whereas these actions are smoothly distinct since their quotients are not diffeomorphic.

□

2.3 Seiberg-Witten theory basics

2.3.1 Seiberg-Witten on closed 4-manifolds

Let $\mathfrak{s}$ be a $Spin^c$ structure on a 4-manifold X , and let $\mathcal{B}(X, \mathfrak{s})$ be the set of gauge equivalence classes of pairs (A, ϕ) , where A is a $Spin^c$ connection and ϕ is a spinor field on X .

For a 4-manifold with a $Spin^c$ structure $\mathfrak{s}$, the Seiberg-Witten equations are:

$$D_A^+ \phi = 0 \tag{2.1}$$

$$1/2\rho(F_A^+ - \omega^+) - (\phi\phi^*)_0 = 0 \tag{2.2}$$

where ρ is the Clifford multiplication, $D_A^+ : \Gamma(S^+) \longrightarrow \Gamma(S^-)$ is the Dirac operator, and $(\phi\phi^*)_0$ is the trace free part of the endomorphism $\phi\phi^*$. The 2-form ω is an arbitrary perturbation.

The Simple Type Conjecture says that the moduli space of solutions $M(X, \mathfrak{s}) \subset \mathcal{B}(X, \mathfrak{s})$ to these equations is a zero dimensional manifold for all 4-manifolds with $b^+ \geq 2$ with a generic choice of ω ; we will assume this for the remainder of this paper. In this case, we define the *Seiberg-Witten invariant* $SW_X(\mathfrak{s})$, to be an algebraic count of the points in the moduli space (where signs are assigned via some choice of orientation). Witten shows that $SW_X(\mathfrak{s})$ depends only on the smooth structure of X , not on the choice of metric or perturbation. ([45], see [28] for a mathematically rigorous proof). A $Spin^c$ structure $\mathfrak{s}$ such that $SW_X(\mathfrak{s}) \neq 0$ is called a *basic class*. We encode the information given from this invariant as an element of $\mathbb{Z}[H^2(X)]$ by defining $SW_X := \sum SW(\mathfrak{s})c_1(\mathfrak{s})$ where the sum is taken over all $Spin^c$ structures on X . In the case that $H^2(X)$ has 2-torsion, we loose information when we pass to SW_X , but since we are primarily concerned with simply connected manifolds,

this is not a concern.

Note that if ϕ is a diffeomorphism of X , then $\phi_*(SW_X) = SW_X$. This provides a basic obstruction to the existence of certain smooth group actions. For example, Chen and Kwasik [7] use this idea to show how certain actions that exist on K3 cannot exist on exotic copies of K3.

A seminal result in Seiberg-Witten theory is the following theorem of Taubes:

Theorem 4. [41] *Suppose (X, ω) is a closed symplectic 4-manifold. Then $SW_X \neq 0$, and specifically $SW_X(\mathfrak{s}\omega) = 1$.*

2.3.2 Seiberg-Witten on 3-manifolds

Let $\mathfrak{s}$ be a $Spin^c$ structure on a 3-manifold Y . In [22], Kronheimer and Mrowka defined the ‘Monopole Floer homology group’ invariants, a collection of groups associated to $(Y, \mathfrak{s})$. For simplicity, we will restrict our attention to the circumstance where $\mathfrak{s}$ is torsion. In this case all of the groups defined by Kronheimer and Mrowka are equivalent. This invariant is called the *reduced monopole Floer homology* and we will denote it by $HM(Y, \mathfrak{s})$. It is constructed as follows: Let $\mathcal{B}(Y, \mathfrak{s})$ be gauge equivalence classes of pairs (A, Φ) where A is a $Spin^c$ connection and ϕ is a spinor field. Then the chain groups defining $HM(Y, \mathfrak{s})$ are generated by the elements of $\mathcal{B}(Y, \mathfrak{s})$ satisfying:

$$1/2\rho(F_{B^t}) - (\phi\phi^*)_0 = 0 \tag{2.3}$$

$$D_B\phi = 0$$

Here we fudge slightly: In fact, it is a suitably perturbed version of these equations that defines monopole Floer homology, and these perturbations are the source of much of the complexity in the theory. See [22] for the details.

We will primarily be concerned in this chapter with the monopole Floer homology of $S^1 \times \Sigma_g$ where Σ_g is a surface of genus g . In this case, we have no need to describe the differential.

Proposition 5. *Let $S^1 \times \Sigma_g$ be endowed with a product metric whose restriction to Σ_g has constant negative curvature, and let $\mathfrak{s}_{g-1}$ be the Spin^C structure characterized by $\langle c_1(\mathfrak{s}_{g-1}), \Sigma_g \rangle = 2g - 2$. Then the equations (2.3) have a unique solution, $[a_0] \in \mathcal{B}(S^1 \times \Sigma, \mathfrak{s}_{g-1})$, and consequently $HM(S^1 \times \Sigma_g, \mathfrak{s}_{g-1}) = \mathbb{Z}$.*

Proof. A detailed proof can be found in [8], [31], and [30]. Essentially the proof comes down to showing that (2.3) is invariant under the obvious S^1 action, and in this case these equations simplify to the abelian vortex equations on Σ_g . But the vortex equations can be solved explicitly. $\square$

2.3.3 Seiberg-Witten on 4-manifolds with boundary

The full story of Seiberg-Witten equations on 4-manifolds with boundary is a long one, told in its entirety in [22]. In general, if X is a 4-manifold with $\partial X = Y$, then associated to X we get an element, $\psi_X \in HM(Y, \mathfrak{s})$. We will restrict our attention to the simple case where $\partial X = S^1 \times \Sigma_g$.

Specifically, we'll consider the Seiberg-Witten equations on X with an infinite end, $X^* = X \cup S^1 \times \Sigma_g \mathbb{R}^+$. Let $\mathcal{B}(X^*, [a_0])$ be the subset of $\mathcal{B}(X^*)$ which limit to the element $[a_0]$ of Proposition 5 on the end of X^* . Then we can define $\psi_{X, \partial} \in HM(S^1 \times \Sigma_g, \mathfrak{s}_{g-1}) = \mathbb{Z}$ to be the count of isolated solutions to (2.1) in $\mathcal{B}(X^*, [a_0])$; the count of elements in the moduli space $M(X^*, [a_0])$, that is. Using suitable perturbations, $\psi_{X, \partial}$ is an invariant. What is more, we can decompose $\mathcal{B}(X^*, [a_0])$ along its path components into so called ' z -paths', $\mathcal{B}(X^*, [a_0]) = \coprod \mathcal{B}_z(X^*, [a_0])$, and similarly $M(X^*, [a_0]) = \coprod M_z(X^*, [a_0])$. Moreover, the set of path components of $\mathcal{B}(X^*, [a_0])$ is a principal homogeneous space for $H^2(X, S^1 \times \Sigma_g)$. Hence, if we make some

identification between $H^2(X, S^1 \times \Sigma_g)$ and the z -paths, we can define

$$SW_{X, \partial} = \sum_{h \in H^2(X, S^1 \times \Sigma)} (\#M_z(X^*, [a_0]))h \in \mathbb{Z}H^2(X, S^1 \times \Sigma)$$

which is invariant up to multiplication by an element of $H^2(X, S^1 \times \Sigma)$.

2.3.4 Seiberg-Witten invariants of pairs

We can define a smooth invariant of a pair (X, Σ) as follows.

Definition 6. Let X be a closed 4-manifold containing an embedded surface Σ , then we define $SW(X|\Sigma) = SW_{(X \setminus nb(\Sigma), \partial)}$ when $[\Sigma]^2 = 0$. When $[\Sigma]^2 = n > 0$, define $SW(X|\Sigma) = SW_{(X \setminus \#n \overline{CP^2} \setminus nb(\tilde{\Sigma}), \partial)}$ where $\tilde{\Sigma}$ is the total transform of Σ .

This notation is somewhat non-standard. Typically $SW(X|\Sigma)$ is only defined for the case that $[\Sigma]^2 = 0$, but the extension made here makes several theorems easier to state.

Theorem 7. *If X is a symplectic manifold and $\Sigma \subset X$ is a symplectic surface with $[\Sigma]^2 \geq 0$, then $SW_{X|\Sigma} \neq 0$*

The proof is standard, but we outline it here for the sake of completeness.

Proof. Assume $[\Sigma]^2 = 0$. Decompose X into $\overline{X} = X \setminus D^2 \times \Sigma$ and $D^2 \times \Sigma$ and let X_n define a family of metrics on X via

$$X_n = (\overline{X} \cup S^1 \times \Sigma \times [0, n]) \cup (D^2 \times \Sigma \cup S^1 \times \Sigma \times [0, n])$$

By Taubes's result above, $\mathfrak{s}_\omega$ is a basic class of X , and $\langle \mathfrak{s}_\omega, \Sigma \rangle = 2g - 2$ by the adjunction equality. Hence, the basic class $\mathfrak{s}_\omega$ restricts to $\mathfrak{s}_{g-1}$ on $S^1 \times \Sigma$. By a basic (but difficult) limiting argument, $M(X_n, \mathfrak{s}_{g-1})$ converges to $M(\overline{X}, [a_0]) \times M(D^2 \times \Sigma)$.

$\Sigma, [a_0])$ as n goes to infinity in some suitable compactification of $\bigcup_{n \in (0, \infty)} M((X_n, \mathfrak{s}_{g-1}))$.

But $M(X_n, \mathfrak{s})$ is (algebraically) non-trivial by Taubes's theorem above.

In the case that $[\Sigma]^2 = n > 0$, the same proof applies because the proper transform of Σ blown up n -times is still a symplectic surface. $\square$

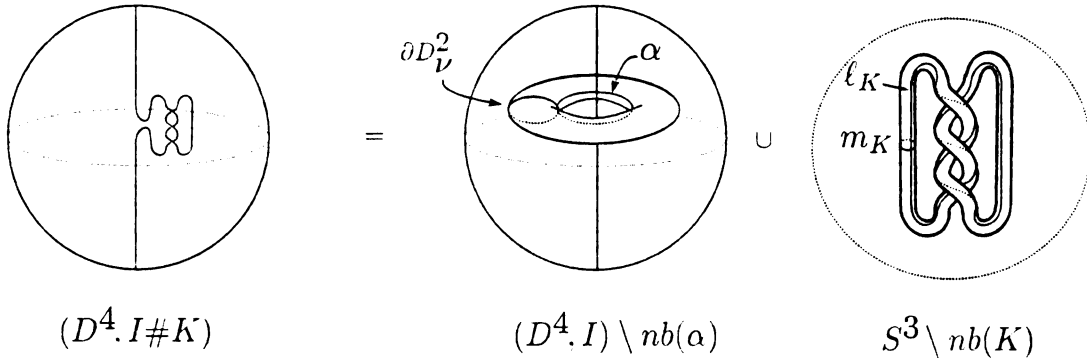


Figure 2.1: Decomposing (D^3, I) into pieces.

2.4 Knotted surfaces

We will examine two methods of knotting surfaces in 4-manifolds: twist spinning a knot to give a knotted S^2 in S^4 (originally defined by Zeeman, [46]), and the closely related technique of Fintushel-Stern of performing ‘rim surgery’ on a surface in an arbitrary 4-manifold. [11].

2.4.1 Twist spun knots

Heuristically, a spun knot is constructed by removing an annular neighborhood of the equator of a trivial S^2 in S^4 and replacing it with S^1 times a knotted arc. Let us spell this out in greater detail. Let $n, s \in S^2$ be the north and south pole respectively, and let $I \subset D^3$ be a straight segment from n to s in $D^3 \subset \mathbb{R}^4$.

Decompose S^4 into $S^1 \times D^3 \cup S^2 \times D^2$ by thinking of it as the boundary of $D^2 \times D^3$. The unknot can be seen in this decomposition as

$$(S^4, S^2) = (S^1 \times D^3, S^1 \times I) \cup (S^2 \times D^2, \{n, s\} \times D^2)$$

Now if K is a knot in S^3 , we can form the *spun knot* S_K in S^4 by replacing $S^1 \times I$ in the definition above with $I_K = S^1 \times I \# K$ (see Figure 2.1):

$$(S^1, S_K) = (S^1 \times D^3, I_K) \cup (S^2 \times D^2, \{n, s\} \times D^2) \quad (2.4)$$

Let us consider an alternate definition that will be easier to generalize. Let $\alpha \subset D^3$ be a meridian of $I \subset D^3$. Then $S^1 \times \alpha$ is a torus whose neighborhood we shall write as $S^1 \times \alpha \times D_\nu^2$.

Then

$$\begin{aligned} (S^1 \times D^3, I_K) &= (S^1 \times D^3, S^1 \times I) \setminus S^1 \times \alpha \times D_\nu^2 \\ &\quad \cup_\phi S^1 \times (S^3 \setminus nb(K)) \end{aligned}$$

The gluing map ϕ is characterized by

$$\begin{aligned} \phi_*([S^1]) &= [S^1] \\ \phi_*([\alpha]) &= [m_K] \\ \phi_*([\partial D_\nu^2]) &= [\ell_K] \end{aligned}$$

where m_K is the meridian to $K \subset S^3$, and ℓ_K is the longitude. See Figure 2.1.

A generalization of this construction is the *k-twist spun knot* $S_{K,k} \subset S^4$. Whereas we defined $S_K = I_K \cup \{n, s\} \times D^2$, we define $S_{K,k} = I_{K,k} \cup \{n, s\} \times D^2$ where $I_{K,k}$ is defined similarly to I_K above, except we use the gluing map ϕ characterized by

$$\begin{aligned} \phi_*([S^1]) &= [S^1] + k[m_K] \\ \phi_*([\alpha]) &= [m_K] \\ \phi_*([\partial D_\nu^2]) &= [\ell_K] \end{aligned}$$

Heuristically, $S_{K,k}$ spins K around k times as we go around the S^1 factor. Notice that by these definitions, both spun knots and twist spun knots can be constructed by performing knot surgery on the torus $S^1 \times \alpha$ in S^4 . This differs from knot surgery defined in Section 2.2 in an important respect however: In this case knot surgery does not change the ambient manifold S^4 ; it changes the embedding of a knotted sphere in S^4 .

2.4.2 Rim surgery.

Now we explore knotted surfaces in 4-manifolds that are more complicated than knotted spheres in S^4 .

Let Σ be a surface in an arbitrary 4-manifold X and let C be a simple closed curve in Σ that is homologically essential in Σ . Motivated by the definition of a twist spun knot:

$$(S^4, S_{K,k}) = (S^1 \times D^3, I_{K,k}) \cup (S^2 \times D^2, \{n, s\} \times D^2)$$

one can define *k-twist rim surgery* as:

$$(X, \Sigma_{K,k}) = (S^1 \times D^3, I_{K,k}) \cup (X \setminus nb(C), \Sigma \setminus nb(C))$$

This definition was originally made by Fintushel and Stern in [11], for the case of $k = 0$. The $k \neq 0$ case was explored by Kim and Ruberman in [19] and [20]. In contrast to twist spun knots in S^4 , twist rim surgery does not always change the topological type of the surface:

Theorem 8. *Say X is simply connected.*

- [11] *If $\pi_1(X \setminus \Sigma) = 1$, then (X, Σ) is homeomorphic to $(X, \Sigma_{K,0})$.*

- [20] If $\pi_1(X \setminus \Sigma) = \mathbb{Z}_d$ and $(d, k) = 1$, then (X, Σ) is homeomorphic to $(X, \Sigma_{K, k})$.

In (X, Σ) let $(C \times D^3, C \times I)$ be a tubular neighborhood of C . Define the *rim torus* to be $R = C \times \alpha \subset C \times D^3$ where α is a meridian of $I \subset D^3$. Note that this torus is homologically trivial in X , but homologically essential in $X \setminus nb(\Sigma)$. Since, we saw that twist spinning a knot was equivalent to doing knot surgery on just such a torus, the following theorem should be not too surprising.

Theorem 9. ([11], [9], see also [13]) *If $\Sigma \subset X$ has positive self intersection, then*

$$SW_{(X|\Sigma_{K, k})} = \Delta_{K(2R)} SW_{(X|\Sigma)}$$

where R is the rim torus corresponding to the curve C where the rim-surgery was performed.

Proof. Rim surgery is accomplished by replacing $(S^1 \times D^3, S^1 \times I)$ with $(S^1 \times D^3, I_{K, k})$. As with twist spun knots, this is equivalent to doing knot surgery on the torus which is S^1 times a meridian of I — the rim torus R , in this case. Recall $SW_{(X|\Sigma)}$ corresponds to finding solutions of the Seiberg-Witten equations on $X \setminus nb(\Sigma)$ (possibly blown up), and R is a homologically essential torus in this manifold. Fintushel and Stern's original proof of the knot surgery theorem [12] applied to closed manifolds, but the same proof works in this case once one recognizes that one should substitute z -paths where they originally spoke of $Spin^C$ structures.

□

Tom Mark has obtained an analogous result in Heegaard-Floer theory that applies to *any* symplectic surface in a symplectic manifold regardless of self-intersection, [26].

2.4.3 Examples

One can find an elliptic fibration structure on $K3$ such that a generic fiber is symplectic and has simply connected complement. This was the original example given in [11]. Complex algebraic curves also provide a large source of examples since they are automatically symplectic, and hence have non-trivial relative invariant by Theorem 7. So, for example, if V_d is a generic degree- d curve in CP^2 , we can apply rim-surgery to V_d as long as $\pi_1(CP^2 \setminus V_d)$ is finite cyclic. This is true by the Zariski Conjecture which says $\pi_1(CP^2 \setminus V_d) = \mathbb{Z}_d$. See [19] for this and other examples.

Similarly, in $S^2 \times S^2$ the curve U_d representing $d([S^2 \times pt] + [pt \times S^2])$ has $\pi_1(S^2 \times S^2 \setminus U_d) = \mathbb{Z}_d$ by the generalized Zariski conjecture [32]. Therefore we can also find an infinite family of exotic U_d 's.

2.5 Knotting group actions

We finally have all of the necessary machinery in place to construct the promised exotic actions. Before we do so, let us look at two model theorems. Our actions will arise as branched covers over rim surgered surfaces. First we'll consider the branched covers of twist spun knots in S^4 . In all that follows, denote the canonical d -fold branched cover of X over Σ as $(X, \Sigma)^d$ when $H_1(X \setminus \Sigma) = \mathbb{Z}$.

Theorem 10. *Let $S_{K,k} \subset S^4$ be a k -twist spun knot, and let $d \in \mathbb{Z}$ be relatively prime to k . Then $(S^4, S_{K,k})^d$ is diffeomorphic to S^4 .*

Proof. Giffen showed that such a branched cover is a homotopy 4-sphere, [15]. Gordon extended this, and showed it is a homotopy 4-sphere that admits an S^1 -action, [17]. Pao (using Fintushel's classification of S^1 actions on homotopy 4-spheres) showed that any homotopy 4-sphere admitting an S^1 -action is diffeomorphic to S^4 , [36].

□

Corollary 11. *Say U is a 2-handle attached to $S^1 \times D^3$ along $S^1 \times \{pt\}$. If we write $(S^1 \times D^3 \cup U, I_{K,k})^d$ as $(S^1 \times D^3, I_{K,k})^d \cup \{U_i\}_{i=1}^d$ where the U_i are the d disjoint lifts of U , then*

$$(S^1 \times D^3, I_{K,k})^d \cup U_1 = D^4$$

Proof. We can extend $I_{K,k} \subset S^1 \times D^3 \cup U$ to a twist spun knot in S^4 :

$$(S^4, S_{K,k}) = (S^1 \times D^3 \cup U, I_{K,k}) \cup (D^4, D^2 \cup D^2)$$

By Theorem 10, the d -fold branched cover is again just S^4 :

$$\begin{aligned}
S^4 &= (S^1 \times D^3 \cup U, I_{K,k})^d \cup (D^4, D^2 \cup D^2)^d \\
&= (S^1 \times D^3, I_{K,k})^d \cup (D^4 \cup U, D^2 \cup D^2)^d \\
&= (S^1 \times D^3, I_{K,k})^d \cup D^4 \cup U_1
\end{aligned}$$

This implies that $(S^1 \times D^3, I_{K,k})^d \cup U_1$ is diffeomorphic to D^4 .

□

Now we will construct actions on 4-manifolds that are locally just like those given in Corollary 11.

Theorem 12. [13] *Let Y be a simply connected 4-manifold with an embedded surface satisfying the following conditions:*

- Σ is of genus $g \geq 1$
- $[\Sigma]^2 \geq 0$
- $\pi_1(Y \setminus \Sigma) = \mathbb{Z}_d$
- the pair (Y, Σ) has non-trivial Seiberg-Witten invariant.
- Σ contains a non separating loop C which bounds an embedded 2-disk whose interior lies in $Y \setminus \Sigma$

*Let X be the d -fold branched cover of Y . Then X admits an infinite family of smooth $\mathbb{Z}_d$ actions that are topologically equivariant, but smoothly distinct.*¹

Proof. The first three conditions imply that Σ is a suitable surface for the rim-surgery construction, and the fourth provides a tool for indentifying the diffeomorphism types

¹The same proof works where X is the d' -fold branched cover of Y where d' divides d .

of the branched covers. Let k be an integer such that $(k, d) = 1$ and let $\{Y, \Sigma_{K_i, k}\}$ be a family of smoothly distinct pairs, where $\Sigma_{K_i, k}$ is obtained by k -twist rim surgery on C using some knot K_i . Let X_i be the d -fold branched cover over $\Sigma_{K_i, k}$, i.e. $X_i = (Y, \Sigma_{K_i, k})^d$. Then the induced $\mathbb{Z}_d$ actions on the X_i are all topologically equivariant, because they came from branched covers of topologically equivalent surfaces; and they are smoothly distinct because the images of their fixed sets are the surfaces $\Sigma_{K_i, k}$ which are smoothly distinct.

It only remains to show that X_i is diffeomorphic to X . Note that the branched covers only differ where the rim surgery was performed. Specifically, X_i is obtained from X by replacing $(C \times D^3, C \times I)^d$ with $(C \times D^3, I_{K, k})^d$. We'll look at a slightly larger region. Let U be a regular neighborhood of the disk bounded by C . Then we can obtain X_i from X by replacing $(C \times D^3, C \times I)^d \cup U_1$ with $(C \times D^3, I_{K, k})^d \cup U_1$ where U_1 is a lift of U . By the Corollary 11, $(C \times D^3, I_{K, k})^d \cup U_1 = D^4$. Therefore X_i is diffeomorphic to X .

□

2.5.1 Examples

In Section 2.4.3 we saw a number of surfaces to which the rim surgery construction applies. If we wish to use these examples to find exotic actions, it remains to check that the surfaces in these examples contain a suitable curve C that bounds an embedded disk.

For complex degree- d curves, $V_d \subset CP^2$, the curve V_d is the fiber of a pencil. Suppose $d > 2$. Then we can take C to be any loop in V_d bounding a vanishing cycle. The same is true for the curves $U_d \subset S^2 \times S^2$ described in Section 2.4.3 above.

By taking the branched cover over V_d , we get an infinite family of finite cyclic exotic group actions on, for example, $CP^2 \# 6\overline{CP}^2$ by looking at the 3-fold branched cover over V_3 . Actions on K3 can be constructed by looking at the 4-fold branched

cover over V_4 as well as the 2-fold branched cover over V_6 . The 3-fold branched cover of $U_3 \subset S^2 \times S^2$ is again K3. In fact, only $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$, can act on K3 in such a way that the fixed set is a connected surface – and we have constructed exotic actions in each of these cases. This can be shown by a bit of algebra using the following two formulas which relate the euler characteristic and signature of a manifold to those of its branched cover.

$$\chi(X) = d\chi(Y) - (d-1)\chi(\Sigma), \quad \sigma(X) = d\sigma(Y) - \frac{(d-1)(d+1)}{3d}[\Sigma]^2$$

Chapter 3

Complexity of cobordisms via Heegaard-Floer homology

We exhibited a variety of exotic behaviors in the last chapter. In this chapter, we will shift perspective slightly, and interpret exotic behavior on 4-manifolds through 5-dimensional techniques. In particular, we'll examine the following 2 questions from a 5-dimensional perspective:

1. How can you tell if homeomorphic 4-manifolds are diffeomorphic?
2. When is a self-homeomorphism of a 4-manifold isotopic to a self-diffeomorphism?

These are very difficult questions to answer in general. What the 5-dimensional perspective will give us is a way to quantify how far two manifolds are from being diffeomorphic, or how far a self-homeomorphism is from being a self-diffeomorphism. This is accomplished by measuring the complexity of a cobordism between two 4-manifolds as follows. Let M_1 and M_2 be homeomorphic 4-manifolds and let $\mathfrak{C}(M_1, M_2)$ be the set of all cobordisms from M_1 to M_2 that possess a decomposition with only 2- and 3-handles. Then we can define

$\Lambda(M_1, W, M_2) = \min\{\text{geometric intersection number of the belt sphere of the}$
 $2\text{-handles with the attaching spheres of the } 3\text{-handles}\}$

where the minimum is taken over all handle decompositions of W using only 2- and 3-handles. Define $\Lambda_1(M_1, W, M_2)$ similarly but take the minimum over handle decompositions of W with only one 2-handle and one 3-handle. Now define

$$\Lambda(M_1, M_2) = \min_{W \in \mathfrak{C}} \Lambda(M_1, W, M_2)$$

$$\Lambda_1(M_1, M_2) = \min_{W \in \mathfrak{C}} \Lambda_1(M_1, W, M_2)$$

$$\Lambda^h(M_1, M_2) = \min_{\substack{W \in \mathfrak{C} \\ W \text{ is an h-cob}}} \Lambda(M_1, W, M_2)$$

$$\Lambda_1^h(M_1, M_2) = \min_{\substack{W \in \mathfrak{C} \\ W \text{ is an h-cob}}} \Lambda_1(M_1, W, M_2)$$

The relation between Question 1 and the Λ invariants is transparent. It is not hard to show that $\Lambda^h(M_1, M_2) = 0$ if and only if M_1 and M_2 are diffeomorphic. And the greater Λ is, the greater the disparity between the smooth structures of M_1 and M_2 .

Λ and Λ^h have quite different behavior, however. Many families of exotic manifolds that have been constructed have $\Lambda = 2$. This is explained in [5]. On the other hand, Λ^h exhibits more interesting behavior, even for h-cobordisms of a manifold to itself (so called *inertial* h-cobordisms), e.g. the following theorem of Morgan and Szabo.

Theorem 13. [29] *For all $n \in \mathbb{Z}$ there exist 4-manifolds M_n and an inertial h-cobordism of M_n , say W_n , such that $\Lambda(M_n, W_n, M_n)$ is unbounded as n increases.*

These invariants also shed light on question 2 above.

In section 3.7 of this chapter we will construct an obstruction to a self homeomorphism of a 4-manifold being isotopic to a self diffeomorphism based on the Λ invariants. In particular, associated to ϕ , a self homeomorphism of a 4-manifold M , we will construct a cobordism (M, W^ϕ, M) such that $\Lambda(M, W^\phi, M) = 0$ if and only if ϕ is isotopic a diffeomorphism. The theorem of Morgan and Szabo gives examples of homeomorphisms that are not realized by diffeomorphisms. Contrast this with the previous chapter. In the previous chapter (Section 2.3.1) we saw how the Seiberg-Witten invariant provides such an obstruction. A partial motivation for our study here is to understand how these two obstructions are related.

Our goal in this chapter is to lay a framework for studying the Λ invariants. Ultimately, we would like to understand how, for instance, the Seiberg-Witten, or Ozsvath-Szabo 4-manifold invariants of cobordant 4-manifolds are related and to understand the Λ invariants. A full understanding is, at present, beyond our reach. As a first step in this direction, however, we will calculate the relevant Heegaard-Floer homology groups associated to *simple* cobordisms (defined in Section 3.1).

The main technical content of this chapter is a calculation of the Heegaard Floer homology for the plumbing of two spheres. At the end of this chapter we will use this calculation to derive a relationship between the 4-manifold invariants of M , and the Λ invariants for certain cobordisms.

3.1 Basic Definitions

The purpose of this section is to fix a consistent set of notation and terminology for basic handlebody theory.

A handle decomposition of a manifold is a thickened version of a cellular complex: An i -cell is defined to be a copy of D^i . We can “attach an n -cell to a space X ” using

a map $\partial D^i \rightarrow X$. That is, we attach cells by gluing their boundaries to a space. A *CW complex* is defined inductively by attaching cells of increasing dimension. We can construct n -manifolds in a similar way, but everything must be thickened: Define an n -dimensional *i-handle*, denoted h_i , to be $D^i \times D^{n-i}$, which is attached to an n -manifold M^n via a map $\alpha : (\partial D^i) \times D^{n-i} \rightarrow \partial M^n$ – we glue handles along the thickened region that we glued cells. We call $(\partial D^i) \times D^{n-i}$ the *attaching region* and we call α the *attaching map*. Additionally, we call $\partial D_i \times 0$ the *attaching sphere*, and $0 \times \partial D^{n-i}$ the *belt sphere*.

By elementary Morse theory, every n -manifold M^n has a handle decomposition where handles are attached in increasing index. We will denote this by:

$$M^n = \sum h_0 + \sum h_1 + \dots + \sum h_m$$

Also useful will be the *dual* of a handlebody decomposition whereby the roles of the belt spheres and attaching spheres are reversed: an i -handle h_i is an ‘upside down’ $(n-i)$ -handle $h_{n-i} = h_i^*$, the attaching region becomes $D^i \times \partial D^{n-i}$, etc.

Hence, we can also write:

$$\begin{aligned} M^n &= \sum h_0^* + \sum h_1^* + \dots + \sum h_n^* \\ &= \sum h_n + \sum h_{n-1} + \dots + \sum h_0 \end{aligned}$$

We can also define relative handlebodies which are built on an $(n-1)$ -manifold N :

$$M^n = I \times N + \sum h_1 + \dots + \sum h_n$$

In this case, we denote $\partial^- M = \{0\} \times N$ and $\partial^+ M = \partial M - \partial^- M$. For the subhandlebody composed of handles up to index i we will write $M^{(i)}$. By the *middle level* of a handle decomposition of odd dimension n , we will mean $\partial^+ M^{((n-1)/2)}$.

For odd dimensional manifolds, it will be convenient to convert the “bottom-up” handle decompositions described above into “middle-out” handle decompositions by dualizing the handles below the middle level. For example, we can convert handlebody decomposition of a 3-manifold ala:

$$\begin{aligned}
M^3 &= h_0 + \sum h_1 + \sum h_2 + h_3 \\
&= h_0^* + \sum h_1^* + (\Sigma \times I) + \sum h_2 + h_3 \\
&= h_3 + \sum h_2 + (\Sigma \times I) + \sum h_2 + h_3
\end{aligned}$$

That is, we attach two sets of 2-handles handles to the middle level, one going up, the other going down. For a 3-manifold we can actually draw a picture of a middle-out handlebody decomposition. Such a picture is called a Heegaard diagram and these are explored further in Section 3.4. Middle-out decompositions will also arise in our investigation of 5-dimensional cobordisms.

3.2 Simple cobordisms and surgery.

A *cobordism* between two smooth n -manifolds M_1 and M_2 is an $(n + 1)$ -dimensional manifold X with $\partial X = M_1 \cup M_2$. If the inclusion of M_1 (or equivalently M_2) is a homotopy equivalence, then X is called an *h-cobordism*. The *h-cobordism theorem* says that if two simply connected manifolds of dimension greater than 4 are h-cobordant, then they are actually diffeomorphic. It is the failure of this theorem in dimension 4 that is the source of the beguiling exotic behavior exhibited by smooth 4-manifolds.

Consider the simplest class of cobordisms between 4-manifolds: cobordisms that have a handlebody decomposition as a single 2 handle and 3 handle pair. We will refer to these as *simple cobordisms*. In this section we will characterize how a simple

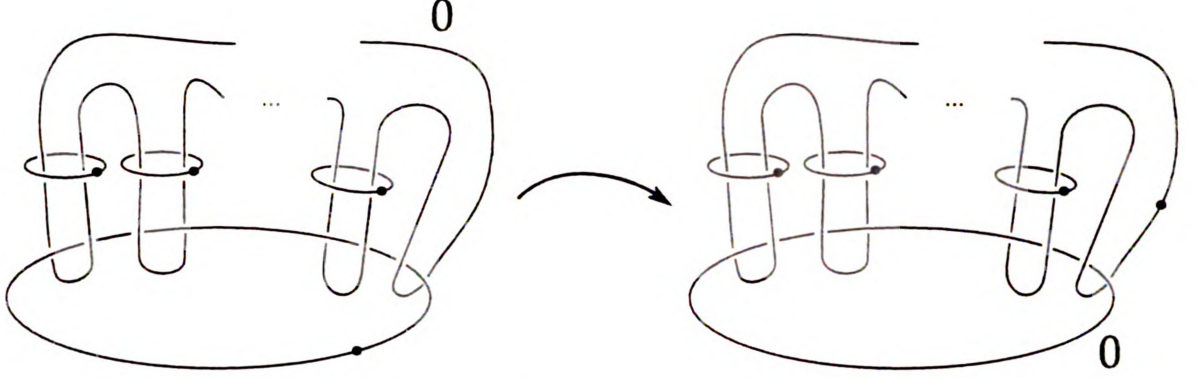


Figure 3.1: $T(n, m)$ surgery.

cobordism induces a surgery relationship between its boundary manifolds.

Proposition 14. *If M_1 and M_2 are simple cobordant 4-manifolds, then M_1 can be obtained from M_2 by a surgery of the type given in Figure 3.1. We will call this operation $T(n, m)$ surgery when the ‘outer’ 1- and 2-handles cross geometrically n times and algebraically m times.*

We shall refer to the 4-manifold given by this Kirby diagram as $D(n, m)$, and the three manifold which is its boundary as $T(n, m)$.

Notice that a degree-0 log transform is a $T(2, 0)$ surgery using this terminology.

Proof. A simple cobordism can be given a “middle-out” decomposition:

$$\begin{aligned} X &= M_1 \times I + h_2 + h_3 \\ &= h_3 + I \times X^{(2)} + h_3 \end{aligned}$$

Call the attaching regions for these two 3-handles S_α and S_β . In $X^{(2)}$, the union $S_\alpha \cup S_\beta$ is the neighborhood of two plumbed spheres, given in Kirby calculus by Figure 3.2. Then $M_2 = \partial^+(X^{(2)} + h_3)$, which is equal to surgery on S_β . This corresponds in Kirby calculus to changing the 0-framed 2-handle that corresponds to S_β into a dotted 1-handle. Similarly, M_1 is obtained by the same operation on the

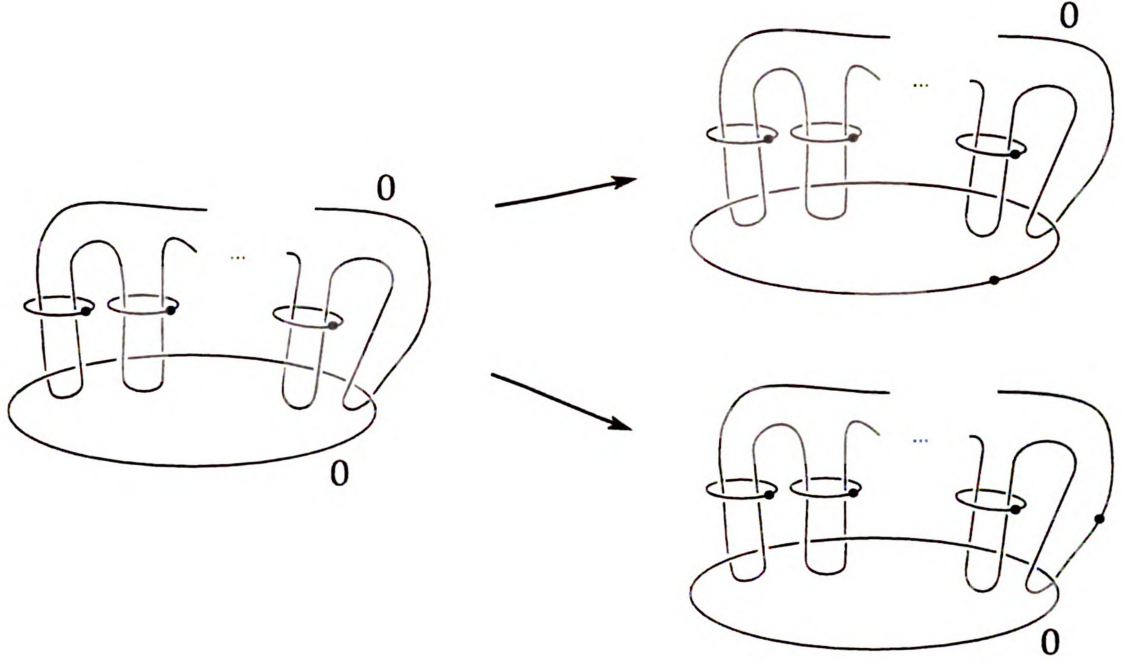


Figure 3.2: The figure on the left is the plumbing of two spheres S_α and S_β in $X^{(2)}$. The pictures on the right are the result of surgery on each of these spheres respectively.

other 2-handle of Figure 3.2.

□

Corollary 15. *If M_1 and M_2 are simple h-cobordant, then they are related by a $T(n, 1)$ surgery. Moreover, if M_1 (and M_2) have indefinite intersection forms, and $M_1 \# S^2 \times S^2$ is diffeomorphic to $M_2 \# S^2 \times S^2$, then M_1 is related to M_2 by both a $T(n, 1)$ surgery and a $T(n', 0)$ surgery.*

We remark that for non-spin manifolds, being simple h-cobordant is equivalent to $M_1 \# S^2 \times S^2$ being diffeomorphic to $M_2 \# S^2 \times S^2$. We do not know if this is true for manifolds which are spin.

Proof. The handles of a simple h-cobordism necessarily intersect algebraically once, proving the first part of the corollary. When $M_1 \# S^2 \times S^2$ is diffeomorphic to $M_2 \# S^2 \times S^2$ we can build a simple cobordism such that the handles intersect al-

gebraically once or not at all as follows: Let S_{1a}^2 and S_{1b}^2 be the obvious spheres in $M_1 \# S^2 \times S^2$ and similarly S_{2a}^2 and S_{2b}^2 in $M_2 \# S^2 \times S^2$

Let W_1 be the cobordism from $M_1 \# S^2 \times S^2$ to M_1 given by attaching a 2-handle to S_{1a}^2 . Similarly, define W_2 by attaching a 2-handle to S_{2a}^2 . We now form the cobordism $W = W_1 \cup_\phi W_2$ where ϕ is a diffeomorphism from $M_1 \# S^2 \times S^2$ to $M_2 \# S^2 \times S^2$. By a theorem of Wall, we may adjust our diffeomorphism such that $\phi_*([S_{1a}^2]) = [S_{2b}^2]$ or $[S_{2a}^2]$. Therefore, the two spheres to which we attach 3-handles to get M_1 and M_2 are $\phi([S_{1a}^2])$ and S_{2a}^2 or S_{2b}^2 which intersect algebraically once or not at all.

□

Remark 16. The $T(n, 1)$ surgeries coming from h-cobordisms are submanifolds of Akbulut corks (see, for example [21]), and $T(n, 0)$ surgeries correspond to plugs, [3].

3.3 Descriptions of the surgery 3-manifold $T(n, m)$

In the previous section, we showed how we could replace the problem of understanding simple cobordisms with the problem of understanding surgery along $T(n, m)$. A first step to understanding how 4-manifold invariants change under this surgery is to understand the Floer homology of $T(n, m)$. Before we can compute this, it will be useful to have as many descriptions of $T(n, m)$ as possible. Above we described it as the boundary of the plumbing of two S^2 's. They are plumbed n -times geometrically and algebraically m -times. Three additional descriptions of the manifold $T(n, m)$ manifold will be useful to us:

1. *Surgery description.* If we change the dotted 1-handles in Figure 3.1 into surgery curves with framing 0, then we have a surgery description of the 3-manifold $T(n, m)$.

2. *Decomposition into pieces.* Let A_n denote S^2 with n open disks removed, and denote the boundary components of A_n by $\bigsqcup_{i=0}^{n-1} S_i$. We can decompose the 3-manifold $T(n, m)$ as $S^1 \times A_n \cup_{\phi} S^1 \times A'_n$ where $\phi : S^1 \times \partial A_n \longrightarrow S^1 \times \partial A'_n$ is, restricted to each boundary torus, just one of the two orientation reversing maps exchanging the factors (use one of the maps on m of the boundary components, and the other on the remaining $(n - m)$ boundary components).

3. *Heegaard diagram.* This will be exhibited in Section 3.4

From description (2) we see a number of surfaces contained in $T(n, m)$: There are the n tori given by $S^1 \times S_i$ for $i = 0$ to $n - 1$. Call these tori T_i . In the case of $T(2n, 0)$ there are two other obvious surfaces of genus $n/2$, R_1 and R_2 : To form R_1 , we cap off $0 \times A_n \subset T(n, 0)$ by annuli in $S^1 \times A'_n$. More specifically, let γ_i be arcs in A'_n such that each component of $\partial A'_n$ contains exactly one endpoint of the γ_i . Then we can arrange ϕ such that $R_1 = 0 \times A_n \cup_{\phi} \cup_i S^1 \times \gamma_i$ is a closed, orientable surface. The obvious symmetry of $T(2n, 0)$ gives us a second such surface, R_2 . See Figure 3.3.

Lemma 17. • $H_2(T(2n, 0))$ is freely generated by the $2n - 1$ tori $\{T_i\}_{i=0}^{2n-2}$ and the surfaces R_1 and R_2 .

• $H_2(T(n, 1))$ is freely generated by the $n - 1$ tori $\{T_i\}_{i=0}^{n-2}$

Proof. This is a simple application of the Meyer-Vietoris sequence. □

3.4 Heegaard diagrams for 4-manifold theorists

A Heegaard diagram is just a representation of a ‘middle-out’ handlebody decomposition of a 3-manifold Y^3 . We will restrict our interest to decompositions with a single 3-handle on each side of the middle level. The middle level of such a decomposition is a surface which we will denote by Σ . Then $Y = h_3 + \sum h_2^{\alpha_i} + I \times \Sigma + \sum h_2^{\beta_i} + h_3$.

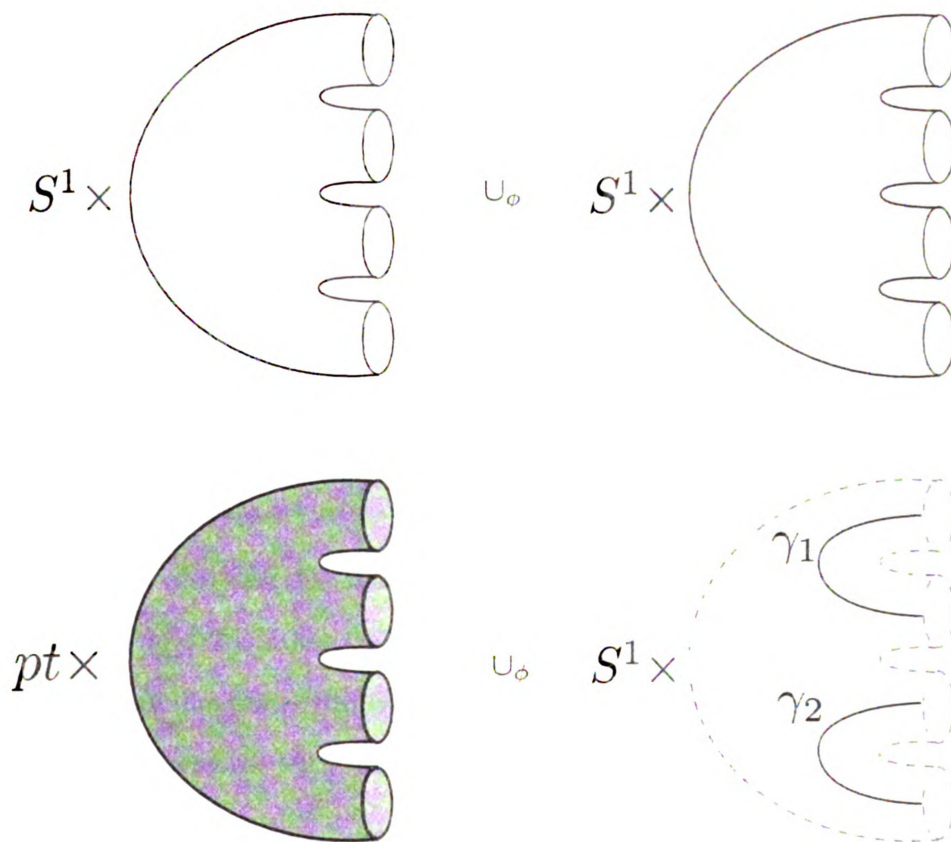


Figure 3.3: The top figure is a decomposition of $T(4,0)$ into two copies of $S^1 \times A_n$ and the bottom figure is the surface R_1 .

Since there are no framing issues to deal with in dimension 3, any such decomposition can be described by a triple (Σ, α, β) , where $\alpha = \cup_{i=1}^g \alpha_i$ — the set of homologically independent attaching curves for the 2-handles below the middle level — and similarly $\beta = \cup_{i=1}^g \beta_i$ the 2-handles above the middle. Such a picture is called a Heegaard diagram. Every closed, orientable 3-manifold admits a Heegaard diagram, and these diagrams are unique up to a certain set of ‘Heegaard moves’, [40].

3.4.1 ‘Bottom-up’ handlebody descriptions

Before we draw a Heegaard diagram of $T(n, m)$, we’ll first construct a standard bottom-up handle decomposition using 0, 1, 2 and 3 handles. A 0-handle is just a 3-ball; represent its boundary by the plane plus the point at infinity. It is not necessary to draw the 1-handles. It suffices to draw their attaching region in the plane: two disks. The 2-handles are also represented by their attaching regions, which are simple closed curves.

We can also describe relative handlebody decompositions this way by attaching 1-handles and 2-handles to a surface, or disjoint union of surfaces. Using relative handlebody diagrams will make it possible to glue handlebodies together, something that is cumbersome with ordinary Heegaard diagrams.

3.4.2 Induced handle structures on $S \times S^1$

Given a handlebody decomposition of a surface S (which may or may not be closed), we can induce a handlebody decomposition of $S \times S^1$ as follows.

The simplest case is when S is a disk D^2 , i.e just a 2-dimensional 0-handle. Then $S^1 \times D^2$ is a 0-handle and a 1-handle, both of dimension 3. Similarly, for a handlebody decomposition of a general surface S , any k -handle in S gives rise to a k -handle and a $k + 1$ handle in $S^1 \times S$. See Figure 3.4. Also, Figure 3.6b shows the this for $S = A_3$.

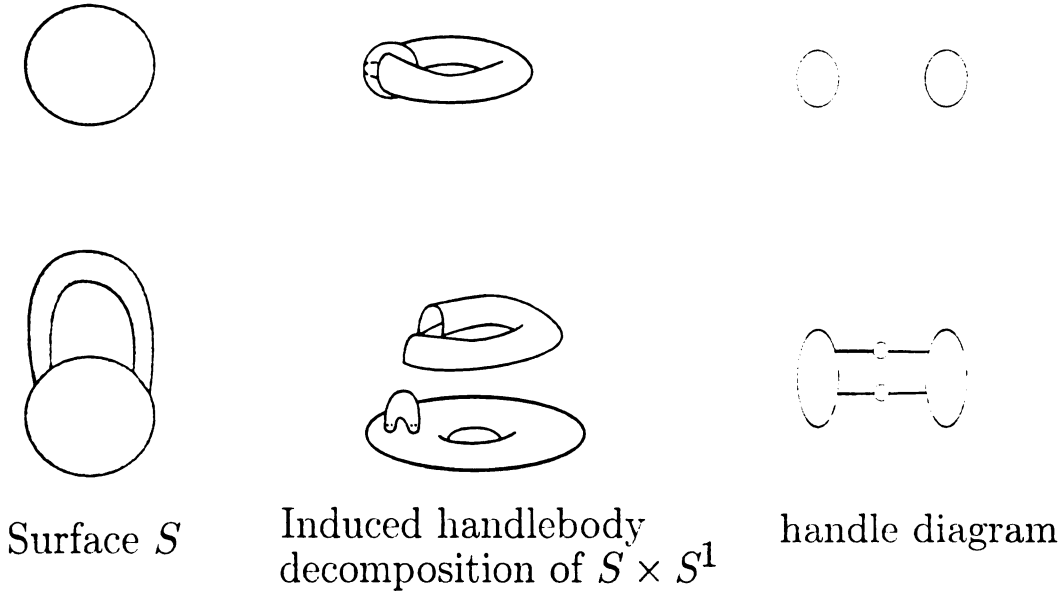


Figure 3.4: Crossing a surface with S^1

More generally, if we begin with a relative handlebody description of S , (i.e. one where $\partial^- S \neq \emptyset$), we see that $\partial^- S \times S^1$ is a collection of tori. On these tori, our procedure builds a relative Heegaard diagram by again attaching a k and a $k+1$ handle to these tori for each k -handle in S . In Figure 3.5 we use a relative handlebody description of the twice-punctured sphere to find a different picture that also describes $S^1 \times A_2$. See also Figure 3.6a for $S^1 \times A_3$.

3.4.3 Gluing handlebodies and constructing $T(n, m)$

Using this technique, we can find two descriptions of $S^1 \times A_3$, one with ∂^+ a collection of tori, and the other with ∂^- a collection of tori. We can now build, for example, $\Sigma_2 \times S^1$ by superimposing the tori from the second description on top of the boundary tori of the first. The boundary tori are shown in in Figure 3.6, and their superposition is given in Figure 3.7.

If, on the other hand, we superimpose the diagrams, but via different homeomorphisms of the boundary tori, then we can form, e.g. $T(3, 1)$, where we alternate

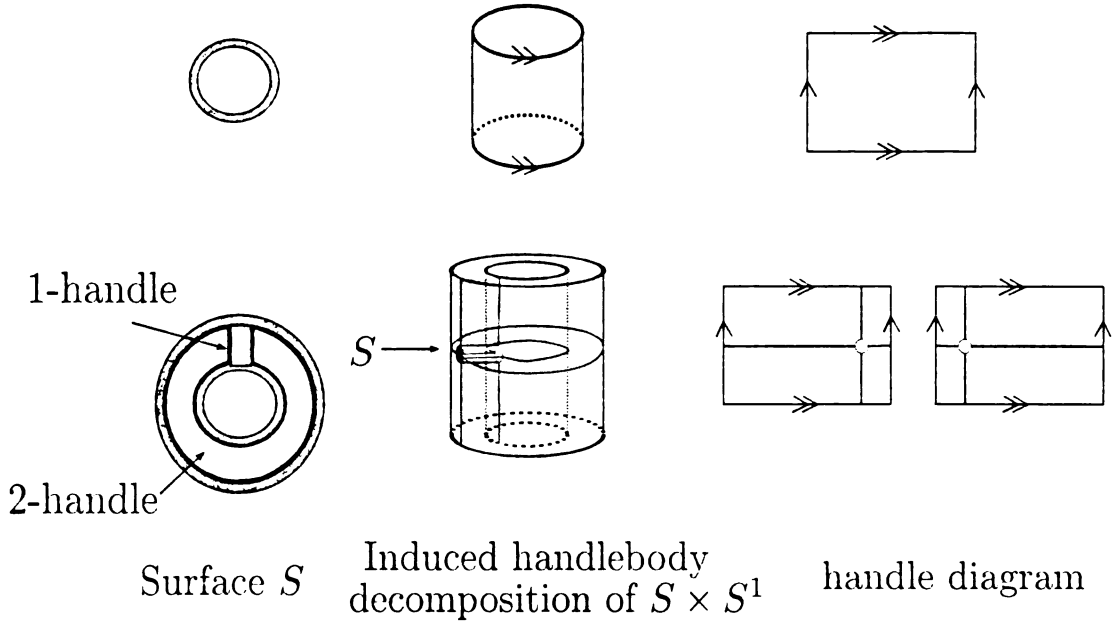


Figure 3.5: Two pictures of $A_2 \times S^1$. The top has $\partial^+ = T^2$ whereas we are more interested in the bottom picture, where $\partial^+ = \emptyset$

between right and left twists. It is straightforward to generalize this to get diagrams for $T(n, m)$, and more general pictures will be given in Section 3.6

3.4.4 Converting to Heegaard diagrams

To convert these standard handlebody diagrams to Heegaard diagram, we dualize the 1-handles to become β -curves on a surface. This is illustrated from two different perspectives in Figure 3.8, with the pictures on the bottom representing how we will normally draw our diagrams. The top pictures are to illustrate that, in fact, both diagrams actually represent a genus-1 handlebody.

3.4.5 Identifying the generators of $H_2(T(n, m))$

We will take this opportunity to find generators of $H_2(T(4, 0))$ in this Heegaard diagram, since we will need them later when we calculate Heegaard Floer homology. Recall by Lemma 17, $H_2(T(4, 0))$ is generated by three tori, plus R_1 and R_2 . The

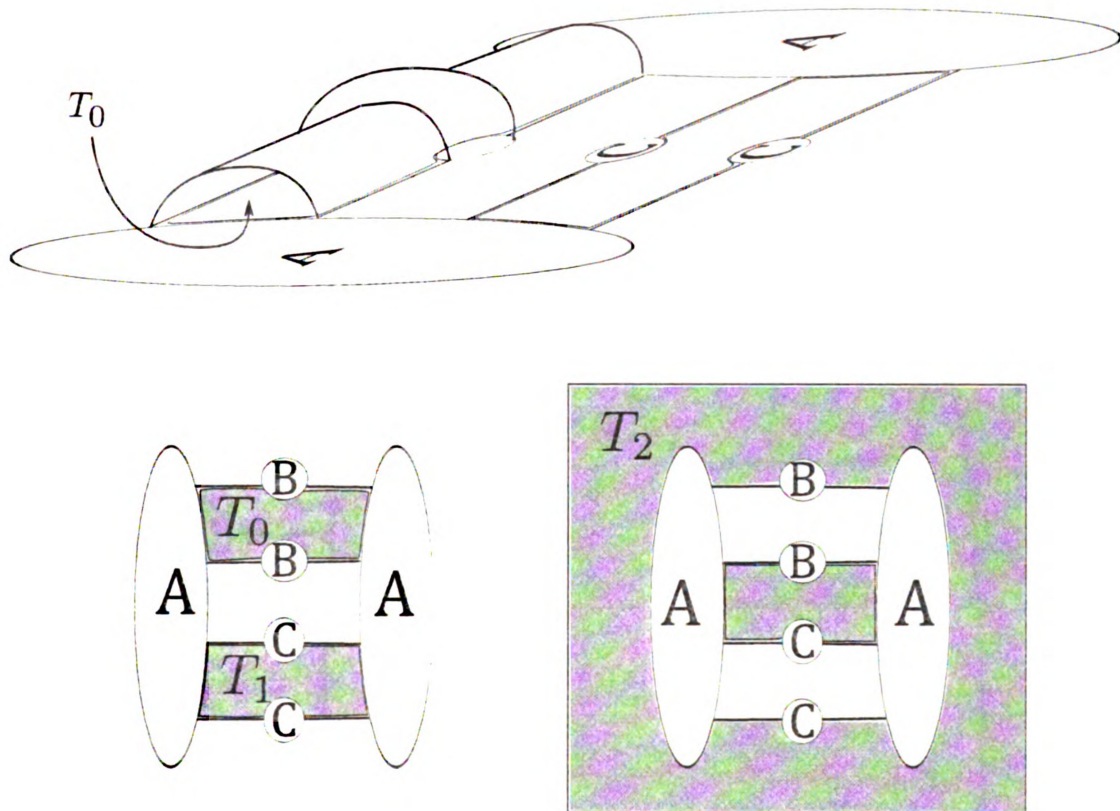


Figure 3.6: One of the tori is shown in the top picture where we have explicitly drawn a 1- and 2-handle. The other tori are represented more typically by the bottom pictures, where it is understood that part of each torus is contained on the 1- and 2-handles which are represented only by their attaching regions.

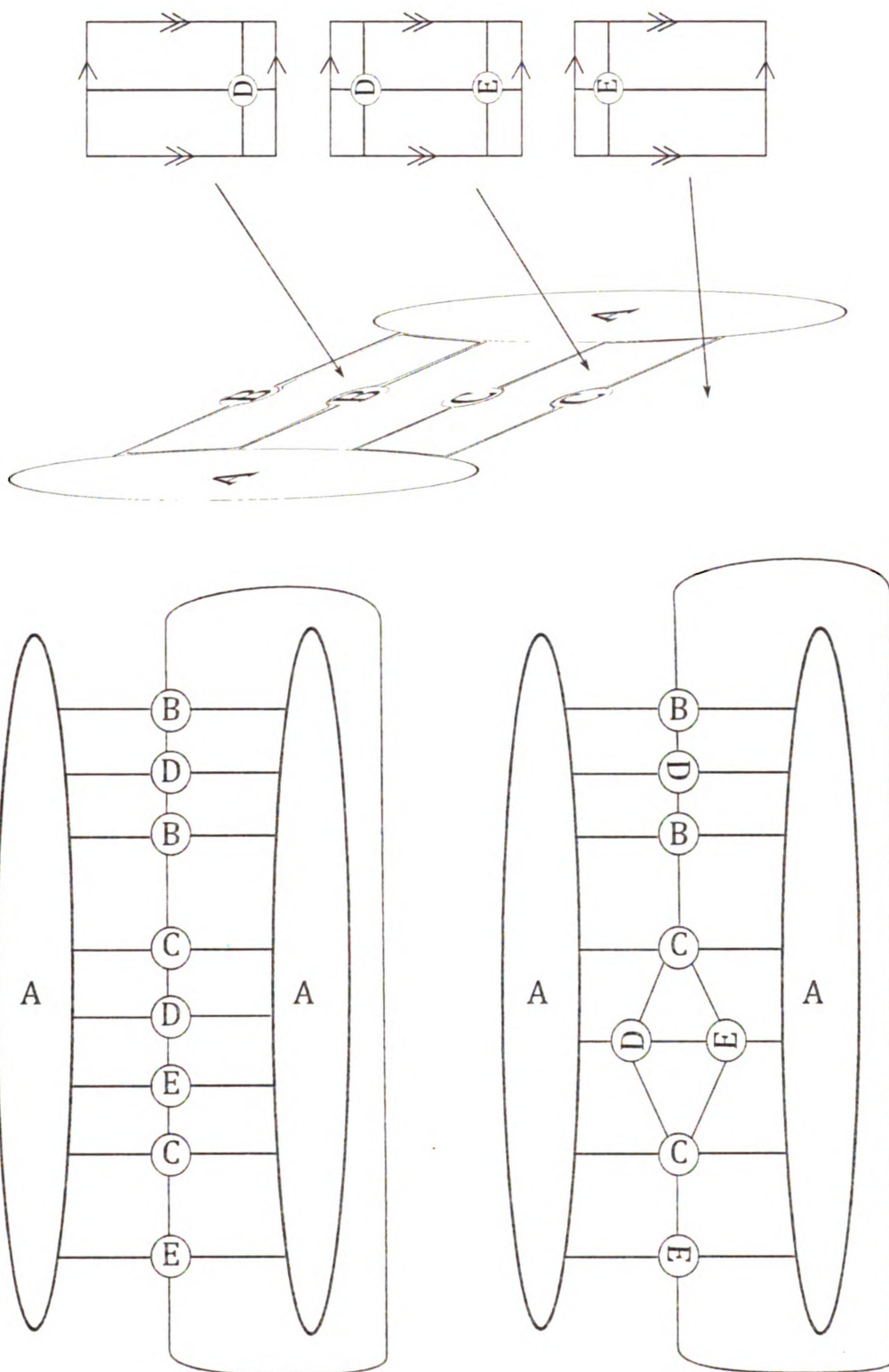


Figure 3.7: The top figure shows how to glue together two copies of $S^1 \times A_3$. The bottom left shows the standard gluing which gives $S^1 \times \Sigma_2$ while the alternate gluing on the right gives $T(3.1)$.

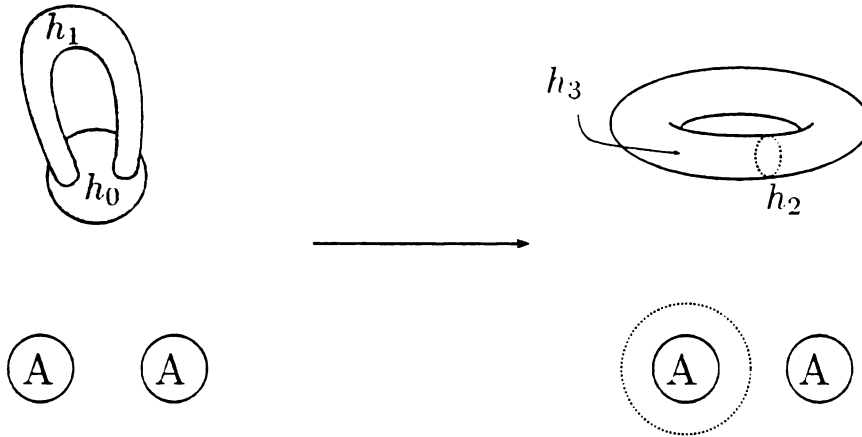


Figure 3.8: The pictures on the left represents a 1-handle attached to a 0-handle. To convert to a Heegaard diagram, dualize to get a 2-handle (plus an unpictured 3-handle) which we attach, in this case, to a torus. Note that we represent a 1-handle and the surface with the same notation: a labeled pair of circles.

tori are the tori that we glued along, as mentioned above (Figure 3.6). The other two generators are slightly harder to see. It is shown in Figures 3.9 and 3.10 how R_1 and R_2 are constructed from the pieces explained in Section 3.3: $pt \times A_4$ and two annuli from Figure 3.3. The annuli are in one copy of $S^1 \times A_4$ and $pt \times A_4$ is in the other. Pictures of the R_i in $T(2n, 0)$ for any integer n are given in Section 3.6

3.5 Background of Heegaard Floer homology

Given Y , a closed oriented 3-manifold, Ozsvath and Szabo define a collection of 3-manifolds invariants, $HF^+(Y)$, $HF^-(Y)$, $HF^\infty(Y)$, and $\widehat{HF}(Y)$. The original reference is [35]. Other surveys are [27] and [39].

These invariants are modeled on Lagrangian Floer homology: To a symplectic manifold M with two Lagrangian submanifolds L_1 and L_2 , a homology theory can be defined where the chain groups are freely generated by the points of $L_1 \cap L_2$, and the differentials are defined by counting J-holomorphic Whitney disks in M with boundary on $L_1 \cap L_2$.

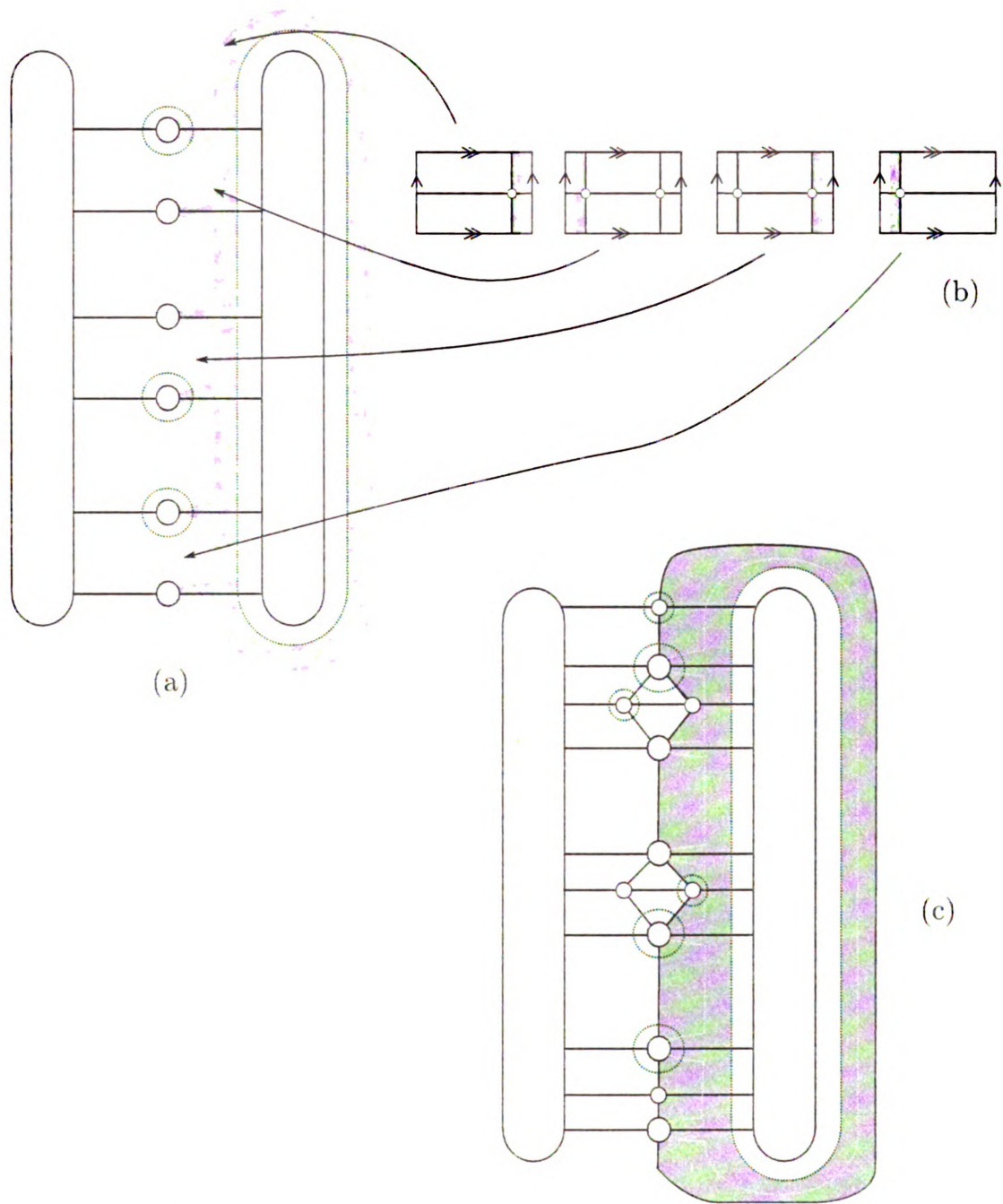


Figure 3.9: This is a picture of R_1 (shaded) in $T(4,0)$. (a) $A_4 \times pt$ is shaded. (b) The two annuli are shaded. (c) The composite, R_1 , is shaded with a perforation where A_4 is glued to the annuli.

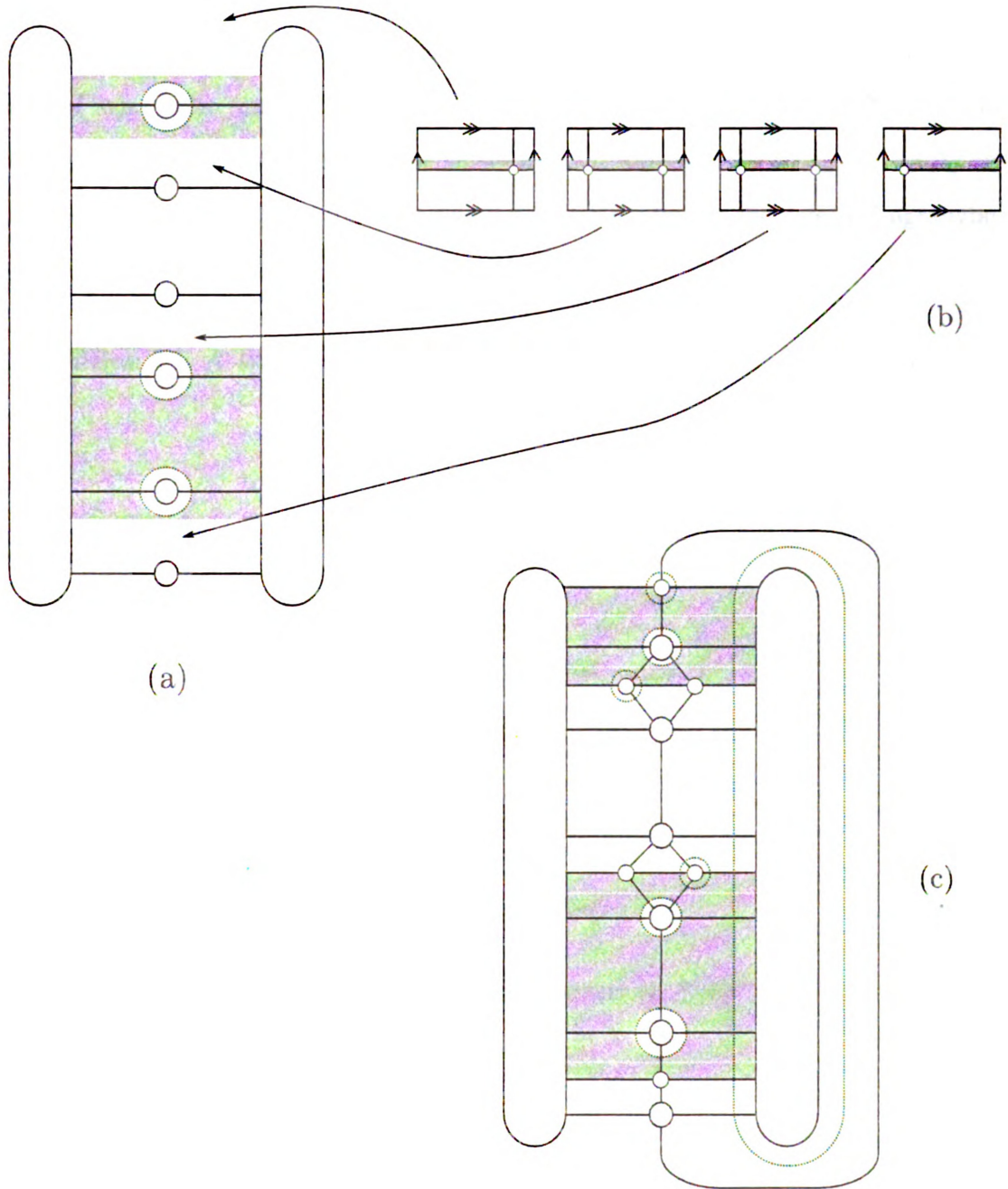


Figure 3.10: This is a picture of R_2 (shaded) inside $T(4,0)$. (a) The two annuli are shaded. (b) $pt \times A_4$ is shaded. (c) The composite is shaded with a perforation where A_4 is glued to the annuli.

The Heegaard Floer invariants fit into this structure as follows: Say a given three manifold Y has a genus g Heegaard diagram $(\Sigma, \alpha, \beta, z)$ where α and β are the sets of attaching curves, and z is a point in $\Sigma \setminus \alpha \cup \beta$. Then $Sym^g(\Sigma)$ will play the role of symplectic manifold in this Floer homology. If we think of $Sym^g(\Sigma)$ as the quotient of $\Sigma^{\times g}$, then the images of $T_\alpha := \alpha_1 \times \dots \times \alpha_g$ and $T_\beta := \beta_1 \times \dots \times \beta_g$ are transversely intersecting submanifolds of $Sym^g(\Sigma)$, playing the role of Lagrangian submanifolds.

Definition 18. A *Whitney disk* between points a and b in $T_\alpha \cap T_\beta$ is a map:

$$\phi : \{x \in \mathbb{C} | 0 \leq Re(x) \leq 1\} \longrightarrow Sym^g(\Sigma)$$

such that

$$\lim_{x \rightarrow \infty} \phi(x) = a$$

$$\lim_{x \rightarrow -\infty} \phi(x) = b$$

$$\phi(x) \in T_\alpha \quad \text{for } Re(x) = 0$$

$$\phi(x) \in T_\beta \quad \text{for } Re(x) = 1$$

If, additionally, J is an almost complex structure on $Sym^g(\Sigma)$, then ϕ is called a *J-holomorphic Whitney disk* when $\phi^* J = i$.

Heegaard Floer homology is only defined for suitably perturbed almost complex structures. Details can be found in [35] and we shall make no further mention of this technical point.

Definition 19. Let $\vec{x}$ and $\vec{y}$ be points in $T_\alpha \cap T_\beta$. Then $\pi_2(\vec{x}, \vec{y})$ is defined to be the set of homotopy classes of Whitney disks connecting $\vec{x}$ to $\vec{y}$ in $Sym^g(\Sigma)$

Definition 20. For any point w in Σ in the complement of the α and β curves, define

$$n_w : \pi_2(\vec{x}, \vec{y}) \longrightarrow \mathbb{Z}$$

by the algebraic intersection number

$$n_w(\phi) = \# \phi^{-1}(\{w\} \times \text{Sym}^{g-1}(\Sigma_g))$$

3.5.1 Definition of Heegaard-Floer homology

Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram for a three manifold Y . The chain groups of $\widehat{HF}$, HF^+ , HF^- , and HF^∞ associated to this diagram are defined respectively as:

$$\begin{aligned} \widehat{CF}(\alpha, \beta, z) &= \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{Z}[x] \\ CF^\infty(\alpha, \beta, z) &= \bigoplus_{i \in \mathbb{Z}} \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{Z}[x, i] \\ CF^-(\alpha, \beta, z) &= \bigoplus_{i < 0} \bigoplus_{x \in T_\alpha \cap T_\beta} \mathbb{Z}[x, i] \\ CF^+(\alpha, \beta, z) &= \frac{CF^\infty(\alpha, \beta, z)}{CF^-(\alpha, \beta, z)} \end{aligned}$$

The differentials are defined by:

$$\begin{aligned} \widehat{\partial} : \widehat{CF}(\alpha, \beta, z) &\longrightarrow \widehat{CF}(\alpha, \beta, z) \\ \vec{x} &\mapsto \sum_{\vec{y} \in T_\alpha \cup T_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi)=1}} \# \widehat{M}(\phi) \vec{y} \end{aligned}$$

and

$$\begin{aligned} \partial^\infty : CF^\infty(\alpha, \beta, z) &\longrightarrow CF^\infty(\alpha, \beta, z) \\ [\vec{x}, i] &\mapsto \sum_{y \in T_\alpha \cup T_\beta} \sum_{\substack{\phi \in \pi_2(\vec{x}, \vec{y}) \\ \mu(\phi)=1}} \widehat{M}(\phi)[\vec{y}, i - n_z(\phi)] \end{aligned}$$

where $\mu(\phi)$ is the Maslov index of ϕ , the expected dimension of the space of J-holomorphic Whitney disks, and $\#\widehat{M}(\phi)$ is the count of unparameterized holomorphic Whitney disks (i.e. mod out the set of J-holomorphic Whitney disks by the obvious $\mathbb{R}$ action) that are homotopic to ϕ . By various energy bounds, it can be shown that $\#\widehat{M}(\phi)$ is finite for a generic choice of almost complex structure on $Sym^g(\Sigma)$ (see [35], Section 3 for details).

Moreover, by Lemma 3.2 of [35], $\mu(\phi) \neq \emptyset$ only when $n_z(\phi) \geq 0$ (see also Proposition 29 below). Therefore CF^- and CF^+ are sub and quotient complexes respectively of CF^∞ , and hence the differential ∂^∞ defines all three homology groups $HF^+(Y)$, $HF^-(Y)$, and $HF^\infty(Y)$. That this notation makes no reference to the underlying Heegaard diagram is justified by the following theorem:

Theorem 21. [35] *If $(\Sigma, \alpha, \beta, z)$ is a weakly admissible Heegaard diagram, then $HF^+(Y)$ and $\widehat{HF}(Y)$ are invariants of Y ; and if it is strongly admissible, then $HF^-(Y)$ and HF^∞ are invariants of Y as well (that is, they do not depend on the particular Heegaard diagram chosen).*

What is meant by ‘weak and strong admissibility’ will be defined in Sec. 3.5.3 after a few more preliminaries.

As one further refinement, we remark that the chain complexes decompose according to $Spin^c$ structures on Y . So, for example, $CF^\infty(\alpha, \beta, z) = \sum_{\mathfrak{s} \in Spin^c(Y)} CF^\infty(\alpha, \beta, z, \mathfrak{s})$, which gives rise to the decomposition $HF^\infty(Y) = \sum_{\mathfrak{s} \in Spin^c(Y)} HF^\infty(Y, \mathfrak{s})$. There

are similar decompositions for the other Floer homology groups. An algorithm for determining this partitioning will be given in Proposition [?]

3.5.2 Calculating Heegaard Floer homology from a diagram

If we wish to do a ‘by hand’ calculation of these invariants, there are a number of things we need to find. First, we will divide up the generators by their $Spin^c$ structure. Second, we identify the homotopy classes of Whitney disks between all generators within a given $Spin^c$ structure. Then, we will calculate the Maslov index of each homotopy class of disks. And finally, if we are lucky, we will be able to count the number of J-holomorphic disks the homotopy classes with Maslov index 1. Techniques for accomplishing each of these steps are explained below. Proofs can be found in [35], unless otherwise indicated.

3.5.2.1 Domains

Insofar as Whitney disks in Sym^g are rather inconvenient to deal with, we will progressively simplify the information they contain, first by discussing an equivalence between Whitney disks and maps into Σ , and then by ‘discretizing’ such maps via a quantity called the domain.

Lemma 22. [35, Lemma 3.6] *There is a one-to-one correspondence between Whitney disks in $Sym^g(\Sigma)$, and maps of surfaces $\tilde{\phi} : F^2 \rightarrow \Sigma$ such that the following diagram commutes.*

$$\begin{array}{ccc} F^2 & \xrightarrow{\phi'} & \Sigma \times g \xrightarrow{p_1} \Sigma \\ f \downarrow & & \downarrow \\ D^2 & \xrightarrow{\phi} & Sym^g(\Sigma) \end{array}$$

where f is a branched covering map, p_1 is the projection to the first component, and $\tilde{\phi} = p_1 \circ \phi'$.

Such maps can be visualized in Σ by looking at a discretization called the domain which is the algebraic representation of such a map. A surprising amount of information about the homotopy class of a map $\phi \in \pi_2(x, y)$ is contained in this quantity.

Definition 23. Let $\Sigma = \cup \alpha_i \cup \beta_i = \bigsqcup D_i$, a disjoint union of regions. Then we define a *domain* to be a formal sum, $\sum a_i D_i$, where $a_i \in \mathbb{Z}$.

Moreover, choose a point in the interior of each region, $z_i \in D_i$. Now, to a Whitney disk $\phi \in \pi_2(x, y)$ we can associate a domain $\mathcal{D}(\phi) = \sum n_{z_i}(\phi) D_i$. Note this depends on the homotopy class of ϕ and not on the choice of the z_i 's.

3.5.2.2 Visualizing $\pi_2(x, y)$ using domains

Using the correspondence in the last section, we are now in a better position to understand $\pi_2(x, y)$. In particular, we will see how to construct the domains $D(\phi)$ for all $\phi \in \pi_2(x, y)$. We begin with the simpler case of $\pi_2(x, x)$.

Definition 24. A class $\phi \in \pi_2(x, x)$ is called a *periodic class* if $n_z(\phi) = 0$ and $D(\phi)$ is called a *periodic domain*. The set of periodic classes is denoted Π_x .

Note that the boundary of a periodic domain is a sum of α - and β -curves.

To a periodic class $\phi \in \Pi_x$, let $\tilde{\phi}$ be a surface $F \rightarrow \Sigma$ given by the correspondence in Lemma 22. Since the boundary of such a disk is a sum of α and β curves, we can define a closed surface in Y by adding the cores of the 2-handles defined by α and β . This gives us a map $\mathcal{H} : \Pi_x \rightarrow H_2(Y)$ which, because of the normalizing condition $n_z(\phi) = 0$, is actually an isomorphism.

When $g > 2$, it can be shown that $\pi_2(\text{Sym}^g(\Sigma_g)) = \mathbb{Z}$. Call a generator of this group $[S]$.

Proposition 25. [35] When $g > 2$, we have that $\pi_2(x, x)$ is isomorphic to $\mathbb{Z}_{<S>} \oplus H_2(Y)$.

When $\pi_2(x, y)$ is non-empty, it is a homogeneous space modeled on $\pi_2(x, x)$

It is also worth pointing out that the Maslov index is additive under this action. That is, $\mu(\phi_0 + \phi) = \mu(\phi_0) + \mu(\phi)$.

The group action is given by concatenation of Whitney disks. Proposition 25 tells us specifically that if we can find a representation of at least one Whitney disk in $\pi_2(x, y)$, we can find representations for all elements of $\pi_2(x, y)$ (and hopefully their corresponding domains) by adding periodic domains.

For example, if $\phi_0, \phi \in \pi_2(x, y)$ and $\{h_i\}$ is a basis for Π_x , then

$$D(\phi) = D(\phi_0) + \sum a_i D(h_i) + s[\Sigma] \quad \text{for some } a_i, s \in \mathbb{Z}$$

where by $[\Sigma]$ we mean simply a sum of all the regions of $\Sigma \setminus (T_\alpha \cap T_\beta)$

3.5.2.3 Calculations using domains.

Now we'll see how domains can help us conveniently partition the generators of our chain complex into $Spin^c$ structures as well as calculate the Maslov index of a Whitney disk.

Definition 26. Define the *euler measure* of a domain $D = \sum n_i D_i$ by

$$\hat{\chi}(D) = \sum n_i (\chi(D_i) - 1/4(\# \text{ of corner points of } D_i))$$

and define

$$\mu_{\vec{x}}(D) = \sum_{x_i \in \vec{x}} \frac{1}{4} (\text{sum of multiplicities of } D \text{ in the 4 regions bordering } x_i)$$

Finally, define the quantity

$$\mu(D, \vec{x}, \vec{y}) = \hat{\chi}(D) + \mu_{\vec{x}}(D) + \mu_{\vec{y}}(D)$$

It is worth pointing out that if D corresponds to an embedded surface F^2 whose boundary maps to a disjoint union of α - and β -curves, then $\hat{\chi}(D) = \chi(F^2)$. In this case, the above definition amounts to a convenient way of calculating the Euler characteristic of a surface by looking at the domain.¹

Proposition 27. [25] *The Maslov index of a Whitney disk $\phi \in \pi_2(x, y)$ can be computed via*

$$\mu(\phi) = \mu(D(\phi), x, y)$$

Proposition 28. [34, Theorem 4.9] *If $\phi \in \Pi_x$, then*

$$\langle c_1(\mathfrak{s}_x), \mathcal{H}(\phi) \rangle = \mu(D(\phi), x, x)$$

More generally, if $\phi \in \pi_2(x, x)$, then

$$\langle c_1(\mathfrak{s}_x), \mathcal{H}(\phi) \rangle = \mu(D(\phi), x, x) - 2n_z(\phi)$$

In other words, the $Spin^c$ structures and the Maslov index can be computed through completely combinatorial means from a Heegaard diagram.

Say $\mu(\phi) = 1$. In a few lucky cases we can determine $\#\widehat{M}(\phi)$ just based on the domain $D(\phi)$. This is the final step to finding the differential of our chain complexes.

Proposition 29. • *If $D(\phi)$ contains regions of negative multiplicity, then $\#\widehat{M}(\phi) = 0$.*

¹This simplifies finding, e.g. $\hat{\chi}(R_2)$ in Figure 3.15.

- If $D(\phi)$ is either a bigon or a square with multiplicity 1, then $\#\widehat{M}(\phi) = 1$.
- If $D(\phi)$ represents a disconnected region in Σ , then $\#\widehat{M}(\phi) = 0$.

The first two statements are standard. The third statement has a careful proof in [38] wherein a number of other cases are analyzed as well. In general, knowing the domain is insufficient to calculate $\#\widehat{M}(\phi)$ since this quantity depends on the almost complex structure on $Sym^g(\Sigma)$.

3.5.3 Admissible Heegaard diagrams

The Floer homology groups defined above are only invariants if we have what are called ‘admissible’ Heegaard diagrams. To understand this necessity, consider the following situation. Say $\phi_0 \in \pi_2(x, y)$ has $\mu(\phi_0) = 1$ and $\phi \in \Pi_x$ has $\mu(\phi) = 0$. Then $\phi_0 * n\phi \in \pi_2(x, y)$ has Maslov index 1 for all $n \in \mathbb{Z}$. Since the differentials in the Heegaard-Floer chain complex are defined by counting holomorphic Whitney disks in $\pi_2(x, y)$ with Maslov index 1, a priori our differentials might be infinite sums in this situation. However, if we begin with a Heegaard diagram which is properly ‘admissible’, we can guarantee that only finitely many homotopy classes of Whitney disks actually contain a holomorphic representative.

Definition 30. A Heegaard diagram is called *weakly admissibly* for a $Spin^c$ structure $\mathfrak{s}$ if every periodic domain D such that $\langle c_1(\mathfrak{s}), \mathcal{H}(D) \rangle = 0$ has regions of both positive and negative multiplicity.

A Heegaard diagram is called *strongly admissible* for a $Spin^c$ structure $\mathfrak{s}$ if every periodic domain D such that $\langle c_1(\mathfrak{s}), \mathcal{H}(D) \rangle = 2n \geq 0$ has a region of multiplicity greater than n .

Both of these conditions imply that for a given $n \in \mathbb{Z}$, only a finite number of $\phi \in \pi_2(x, y)$ such that $nz(\phi) = n$ will have $D(\phi)$ with only positive regions. Hence, by Proposition 29, only a finite number have holomorphic representatives.

3.6 The Heegaard Floer homology of multiply plumbed spheres

The Floer homology of plumblings is completely understood in a few basic situations. For example, the three manifold associated to a linear plumbing diagrams is just a lens space, whose Floer homology is considered in [34]. In [33], plumbing diagrams which are trees are considered. The manifolds considered here, $T(2n, 0)$, have plumbing diagrams that are not simply connected.

We will use the notation in Section 3.3 for representing generators of $H_2(T(2n, 0))$

Say $\mathfrak{s}_i \in Spin^c(T(2n, 0))$ is characterized by:

$$\langle c_1(\mathfrak{s}_i), [T_j] \rangle = 0 \quad \text{for all } j = 0, \dots, 2n - 1$$

$$\langle c_1(\mathfrak{s}_i), [R_1] \rangle = 0$$

$$\langle c_1(\mathfrak{s}_i), [R_2] \rangle = i$$

Theorem 31. *$HF^\infty(T(2n, 0), \mathfrak{s}_0)$ is isomorphic to $HF^\infty(T^3 \#_{(n-1)}(S^1 \times S^2), \mathfrak{s}_0)$ — a standard group— if we use coefficients in $\mathbb{Z}_2$. Similarly, $HF^\infty(T(n, 1), \mathfrak{s}_0)$ is isomorphic to $HF^\infty(\#_{(n-1)}(S^1 \times S^2))$ with $\mathbb{Z}_2$ coefficients.*

Proof. According to [24], if Y_1 and Y_2 are 3-manifolds, $\phi : H^1(Y_1) \longrightarrow H^1(Y_2)$ is an isomorphism that preserves the triple cup product, and $\mathfrak{s}$ is a torsion $Spin^c$ structure on Y_1 , then $HF^\infty(Y_1, \mathfrak{s})$ is isomorphic to $HF^\infty(Y_2, \phi(\mathfrak{s}))$ using $\mathbb{Z}_2$ coefficients. The triple cup product of $T(2n, 0)$ is characterized by $\langle PD(R_1) \cup PD(R_2) \cup PD(T_i), [T(n, m)] \rangle = 1$. This is evident from Figure 3.3. If the generators of $H^1(T^3 \#_{(n-1)}(S^1 \times S^2))$ are τ_1, τ_2, τ_3 and S_i for $i = 1$ to $n - 1$, then we have

an isomorphism

$$\phi : H^1(T(2n, 0)) \longrightarrow H^1(T^3 \# (n-1)(S^1 \times S^2))$$

$$PD(R_1) \mapsto \tau_1$$

$$PD(R_2) \mapsto \tau_2$$

$$PD(T_0) \mapsto \tau_3$$

$$PD(T_i) \mapsto \tau_3 + S_i \quad \text{for } i = 1, \dots, n-1$$

The proof for $HF^\infty(T(n, 1), \mathfrak{s}_0)$ is even easier. By Lemma 17, $H^1(T(n, 1))$ has vanishing triple cup product and $b_1 = n-1$, the same as $H^1(\#(n-1)S^1 \times S^2)$

□

Theorem 32. *Suppose $n \geq 2$. Then $\widehat{HF}(T(2n, 0), \mathfrak{s}_i) = \mathbb{Z}^{2\binom{2(n-1)}{n-1}}$ and $HF^+(T(2n, 0), \mathfrak{s}_i) = \mathbb{Z}^{\binom{2(n-1)}{n-1}}$ for $i = 2n-2$. Furthermore, both groups are trivial for $i > 2n-2$.*

Briefly, we will accomplish this calculation by:

- Identify the periodic domains.
- Check that we have an admissible Heegaard diagram.
- Find the points in the Heegaard diagram corresponding to $\mathfrak{s}_{2n-2}$.
- Find domains corresponding to all homotopy classes $\phi \in \pi_2(a, b)$ for all $a, b \in T_\alpha \cap T_\beta$ that correspond to $\mathfrak{s}_{2n-2}$.
- Compute $\mu(\phi)$ using the domain and compute $\widehat{\#M(\phi)}$ when $\mu(\phi) = 1$.

In Section 3.4 above, we demonstrated a method for constructing Heegaard diagrams for $T(n, m)$. In Figures 3.11 and 3.12 we exhibit a diagram for $T(2n, 0)$.

Following the process in Section 3.4, a diagram for $T(2n, 0)$ is constructed by stacking a piece of type I, then $(n - 2)$ pieces of type II, followed by a piece of type III. We shall refer to the β curve which is dual to a given 1-handle by, for example, referring to the dotted curve encircling the 1-handle ‘ a ’ as β_a . Notice that we have isotoped some of the β curves to ensure that our diagram will be weakly admissible. Curve β_Z has been wound $(l - 1)$ -times around for reasons that will become clear in the computation.

Lemma 33. *This Heegaard diagram is weakly admissible.*

Proof. If we can show that every nontrivial class $\phi_x \in \Pi_x$ such that $\mu(\phi_x) = 0$ has regions of both positive and negative multiplicity, then we have achieved weak admissibility. Equivalently, assume that for $\phi_x \in \Pi_x$ we have that $D(\phi_x)$ has only all positive or all negative regions. We will show that $D(\phi) = 0$. Our diagrams have a number of ‘test domains’ labeled that we will use to accomplish this.

A basis of the periodic domains is given by $R_1 - l[\Sigma]$, R_2 , and T_i for $i = 0, \dots, 2n - 2$. Therefore we can write $D(\phi_x) = r_1(R_1 - l[\Sigma]) + r_2R_2 + \sum_{i=0}^{2n-2} t_iT_i$. Then $\mu(\phi_x) = r_1\mu(R_1) + r_2\mu(R_2) + \sum \mu(T_i) = r_1(2n - 2)$. Since $n \geq 2$, this can be 0 only when $r_1 = 0$. It is also true that $r_2 = 0$ for the following reason: The multiplicity in $D(\phi)$ of region $D_{R_2}^b$ is r_2 while the multiplicity of region $D_{R_2}^a$ is $-r_2$ (see Figure 3.15). Now our assumption that $D(\phi)$ has only regions of positive multiplicity implies $r_2 = 0$.

The reason that the t_i vanish is slightly more intricate. Refer in the following to Figure 3.13. The multiplicity of region D_0 in $D(\phi)$ is t_0 , while the multiplicity of region $D_{\bar{0}}$ is $-t_0$. Hence $t_0 = 0$. The multiplicity of region D_j^a in $D(\phi)$ is $t_{j-1} - t_j$. Because the multiplicity of this region is non-negative, inductively $0 = t_0 \geq \dots t_{2n-1} \geq t_{2n-2}$. But the multiplicity of D_i^b is t_i which implies $t_i \geq 0$. Therefore $t_i = 0$. Therefore the only periodic domain with all non-negative regions

is the trivial domain.

□

We will see that on each β curve, there will be at most two intersection points which will be used in any generator corresponding to $\mathfrak{s}_{2n-2}$. For tidiness, these are the only intersections labeled on the figures. To refer to specific $\vec{x}$, we introduce the following shorthand. Order the elements in $\vec{x}$ in the order given by, e.g. $\vec{x} = \{x_Z^1, x_{B_1}^a, x_{C_1}^a, \dots, x_{B_{n-1}}^b, x_D^a\}$ which we shorthand as $x(1.a.a.\dots b.a)$.

Lemma 34. *The intersection points that correspond to $\mathfrak{s}_{2n-2}$ are of the form $x(i.\dots)$ where $i = 1$ or 2 and where there are $(n-1)$ b 's in the string.*

Proof. We claim that these are the intersection points that maximize $\langle c_1(\mathfrak{s}_x), R_1 \rangle$, and we will show this using Proposition 28 and Figures 3.11. Specifically, we will use a so-called ‘greedy’ algorithm: Choose a β curve and decide which intersections on this curve contribute the most to $\langle c_1(\mathfrak{s}_x), R_1 \rangle$. Move to the next β curve. If the intersection point that would contribute most to the sum is still available, then we will use it. Then repeat for the rest of the β curves each time choosing the intersections that will maximize the sum. If for some reason the maximally contributing intersection point is unavailable (that is, if we have previously used its α curve), then we need to check that if we reset our greedy algorithm, and began the entire process with this β curve, then we would not maximize $\langle c_1(\mathfrak{s}_x), R_1 \rangle$ – the best choice on this β curve does not maximize the sum. In this case, we choose the next best intersections available. This algorithm is carried out in Table 3.1 referencing Figures 3.11 and 3.12.

Conveniently, all such x 's actually correspond to points in $T_\alpha \cap T_\beta$ (a priori such a string might ‘double use’ an α curve). There are 2^{n+1} such generators.

Since $\chi(R_1) = 1 - 3n$, and $\mu(R_1, \vec{x}, \vec{x})$ is found by adding up the entries in Table 3.1, by Proposition 28, any such $\vec{x}$ will have $\langle R_1, \mathfrak{s}_x \rangle = 2n - 2$ if we place the basepoint z in region D_Z (Figure 3.11).

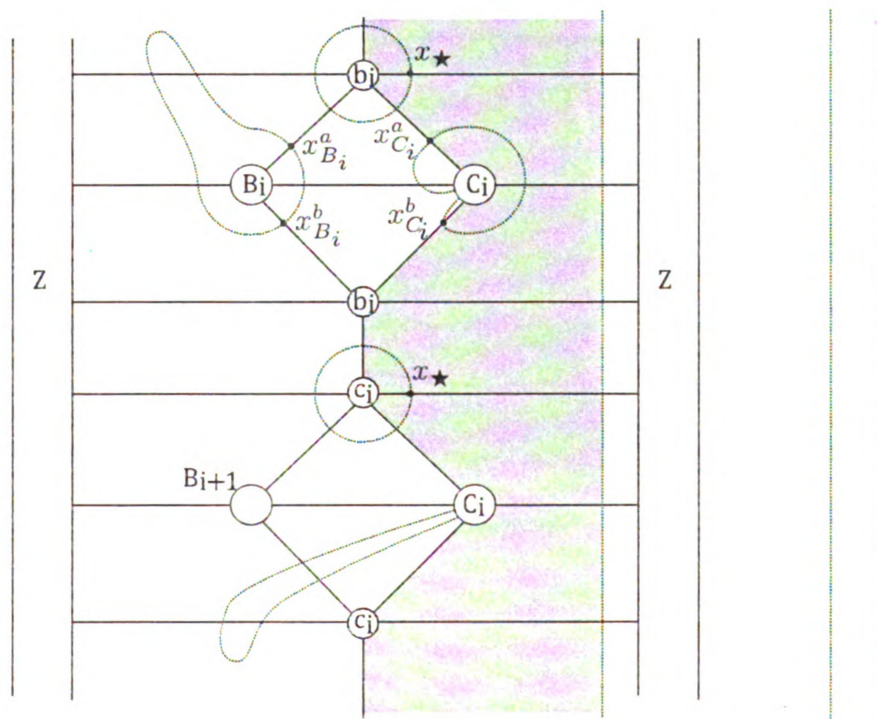
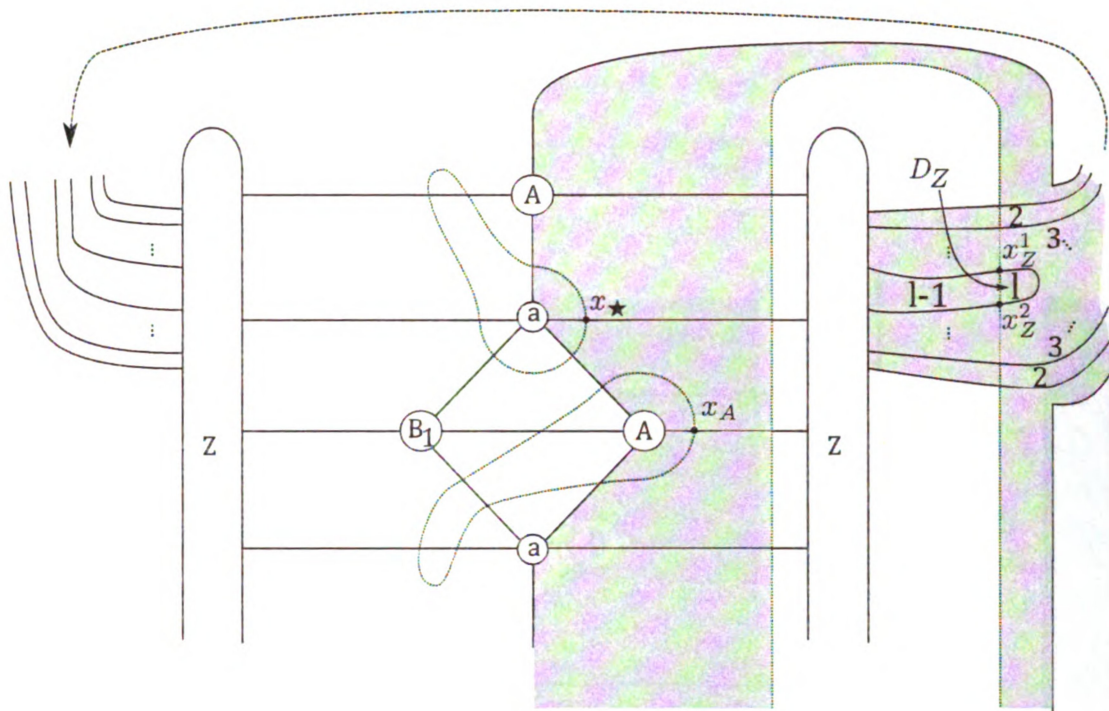


Figure 3.11: R_1 in pieces I and II

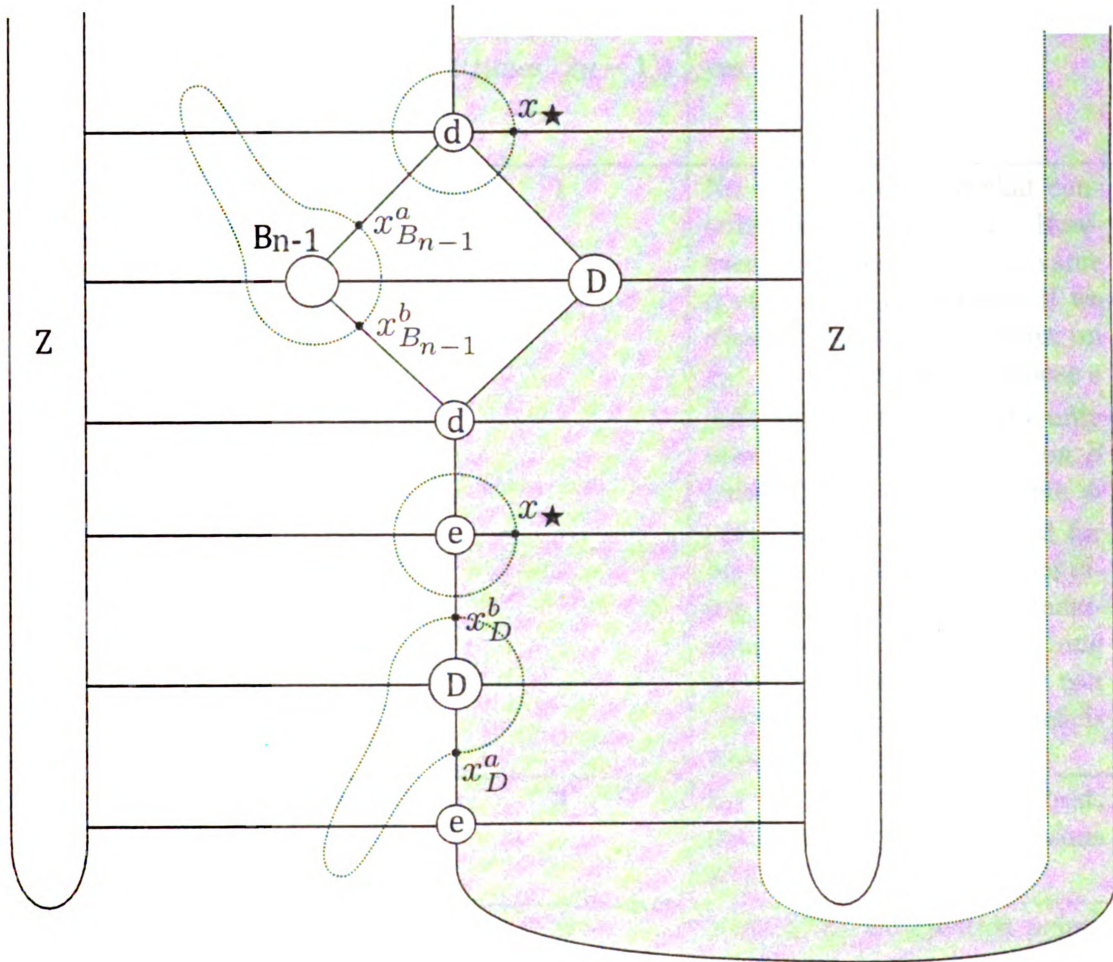


Figure 3.12: R_1 in piece III

Step	β - curve	possible intersections	contribution to $\mu(R_1, x, x)$	maximizes?
1	β_Z	x_Z^1 or x_Z^2	$4l - 4$	yes
2	β_A	x_A	2	yes
3	β_a β_{b_i} β_{c_i} β_d β_e	x_{star}	$2(\text{times } (2n - 1))$	yes
4	β_{C_i}	$x_{C_i}^a$ or $x_{C_i}^b$	1	No. There is a point that contributes 2 to $\mu(R_1, x, x)$. However, this point is on the same α curve as x_A , therefore if we chose the maximizing point on β_{C_i} , we would have to choose a different point on β_A , the only other intersection points on β either a) are on the same α curve as x_Z^1 and x_Z^2 , and for big enough l , any $\tilde{x}$ not including those points won't maximize; or b) contribute nothing to the sum, and again one can check that this creates a deficit which cannot be overcome.
5	β_D	x_D^a or x_D^b	1	No. But they are second best, and should be used, for reasons identical to the β_{C_i} case.
6	β_{B_i}	$x_{B_i}^a$ or $x_{B_i}^b$	0	None of the intersection points contribute to the sum. These are the only two intersections whose α curves have not already been used in previous steps.

Table 3.1: Algorithm for maximizing $\langle c_1(\mathfrak{s}_x), R_1 \rangle$

β - curve	intersections	contribution to $\mu(R_2, x, x)$
β_Z	x_Z^1, x_Z^2	0.0
β_A	x_A	1
$\beta_a, \beta_{b_i}, \beta_{c_i}, \beta_d, \beta_e$	x_{star}	$1 \times (2n - 1)$
β_{C_i}	$x_{C_i}^a, x_{C_i}^b$	0.2
β_D	$x_{D_i}^a, x_{D_i}^b$	0.2
β_{B_i}	$x_{B_i}^a, x_{B_i}^b$	0

Table 3.2: Data to calculate $\langle c_1(\mathfrak{s}_x), R_2 \rangle$

In Figures 3.13 and 3.14 we label the tori $T_0, \dots, T_{2n-2}$ (only some of the corresponding domains are shaded because, e.g. the domains T_0 and T_1 overlap). One checks, again using Proposition 28, that $\langle T_i, \mathfrak{s}_x \rangle = 0$. This is necessary, by the adjunction inequality, for the Floer groups to be non-trivial.

Now we will identify the subset of the $x(\dots)$ such that $\langle c_1(\mathfrak{s}_x), R_2 \rangle = 0$. The data compiled in Table 3.2 (calculated using Proposition 28 and Figures 3.15 and 3.16) shows that intersections labeled with superscript ‘b’ contribute 2 to μ_x while those with ‘a’ contribute nothing. If we say b is the number of intersections with superscript ‘b’ in a given intersection point, then since $\chi(R_2) = 2 - 4n$, we get that $\langle c_1(\mathfrak{s}_x), R_2 \rangle = 2 - 2n + 2b$. Therefore the $\vec{x} \in Spin^c(\mathfrak{s}_{2n-2})$ have exactly $n - 1$ intersections with superscript b . This implies that in $\widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s}_{2n-2})$ is generated by $2 \binom{2(n-1)}{n-1}$ elements.

□

Finally we are in position to calculate the Floer homology. We have identified $2 \binom{2(n-1)}{n-1}$ generators for $\widehat{CF}(\Sigma, \alpha, \beta, z, \mathfrak{s}_{2n-2})$ all of the form e.g. $x(2, a, b, a, \dots)$ where there are $n - 1$ b’s in this string. If we can show that $\hat{\partial} = 0$, then we are done. To accomplish this, we’ll find the general form of a domain $D(\phi)$ for *any* map $\phi \in \pi_2(x(-), x(\sim))$.

Consider the domain D_Z . We remarked in Proposition 29 that such a domain

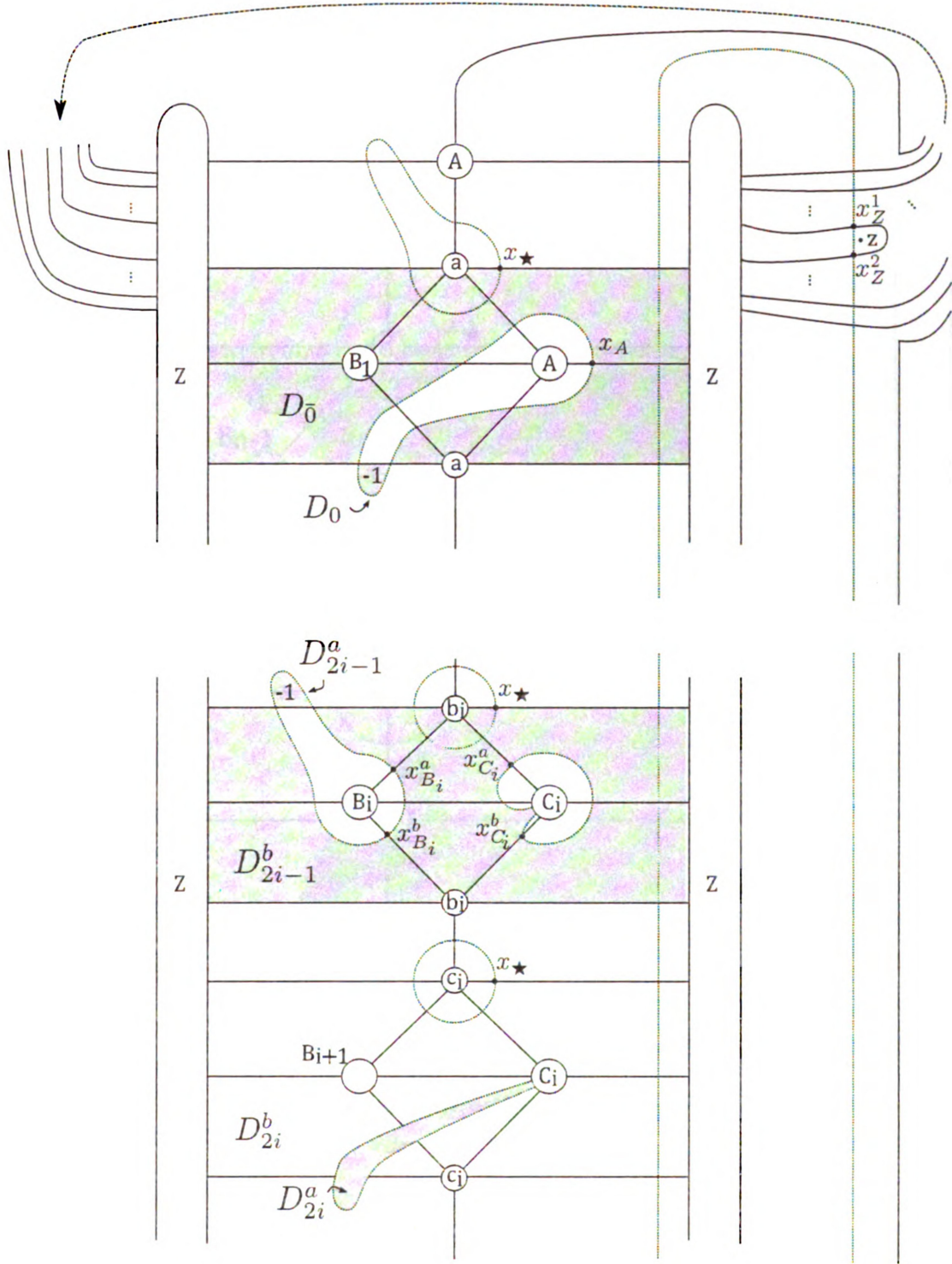


Figure 3.13: The top figure has the domain corresponding to T_0 shaded while the bottom picture has T_{2i-1} shaded. We have not shaded T_{2i} because it overlaps with T_{2i-1} .

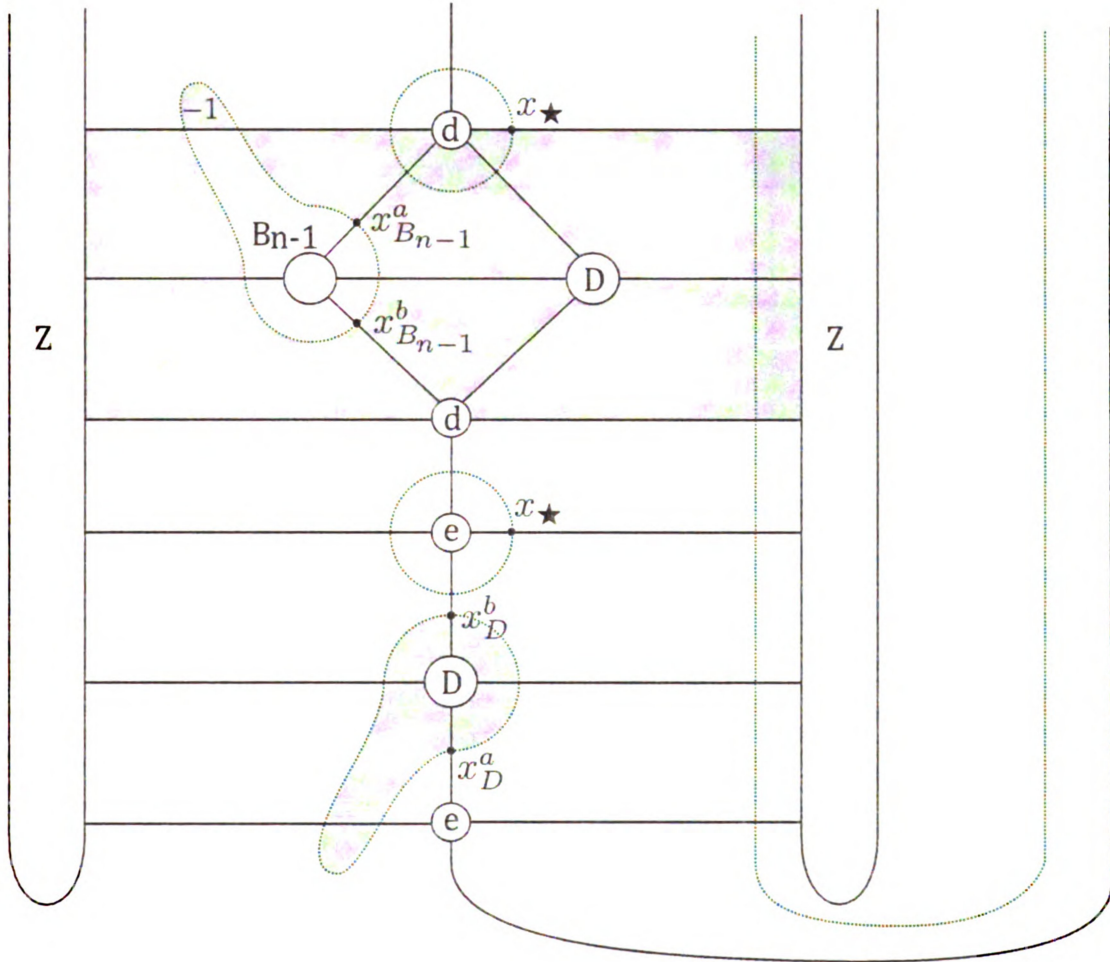


Figure 3.14: The domain corresponding to T_{2n-2} is shaded. We have not shaded T_{2n-1} because it overlaps with T_{2n-2} .

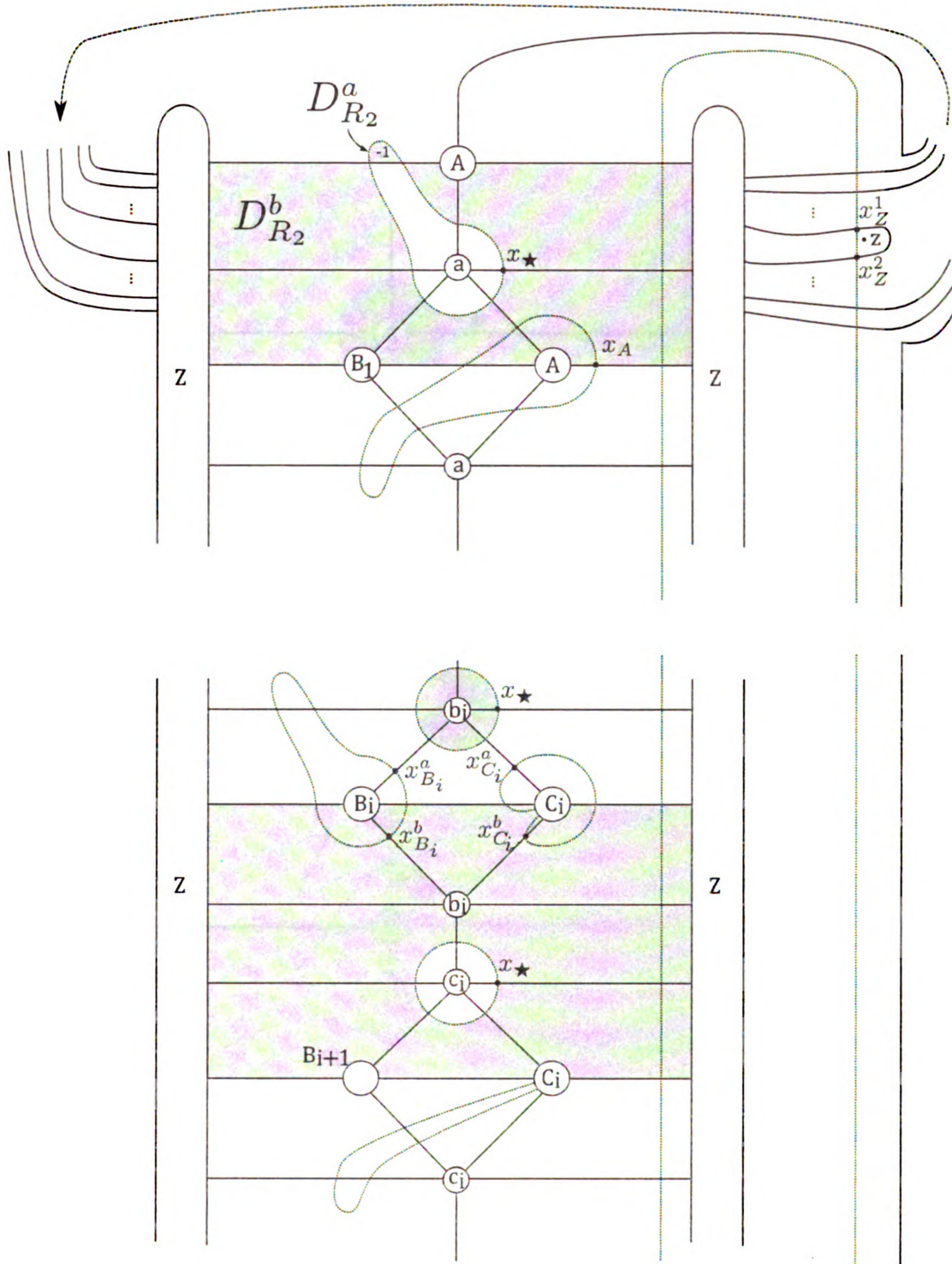


Figure 3.15: R_2 in pieces I and II of $T(2n, 0)$

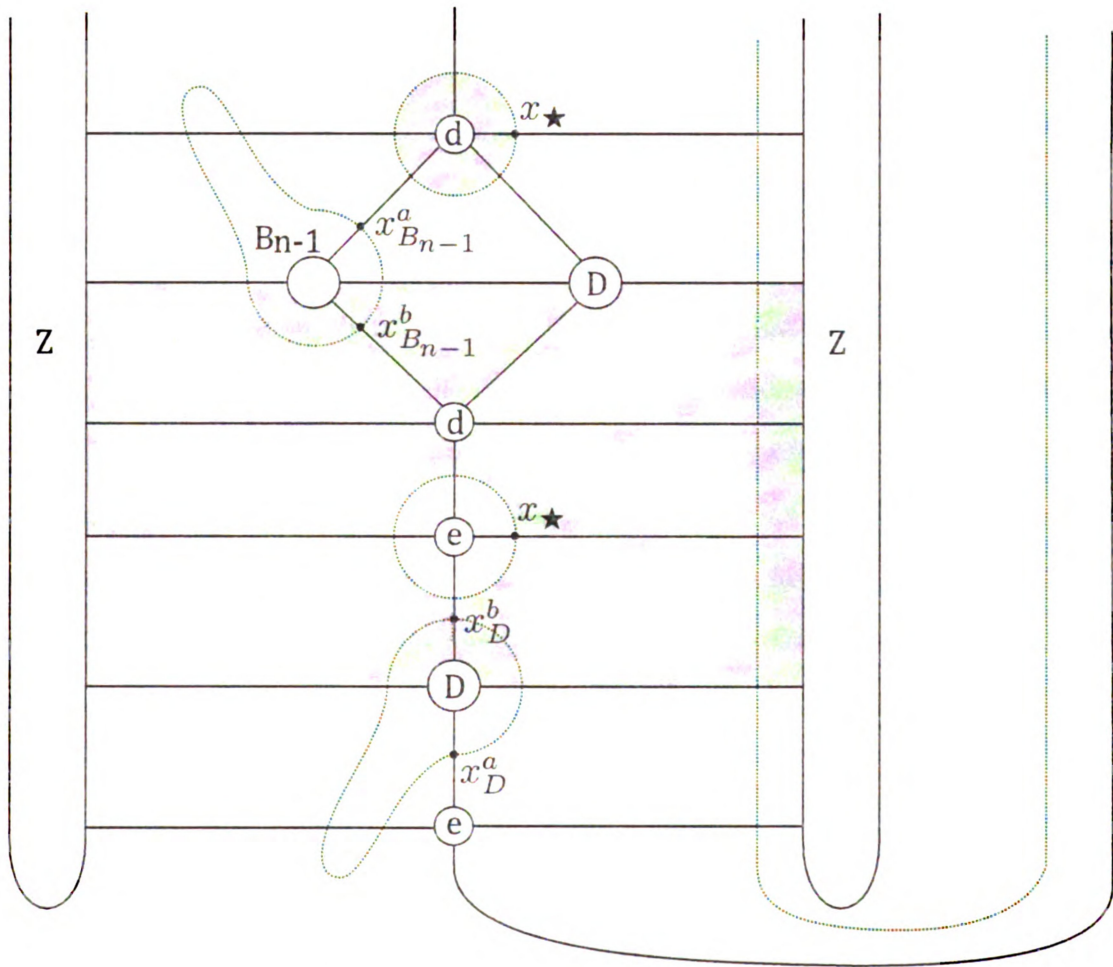


Figure 3.16: R_2 in piece III of $T(2n, 0)$

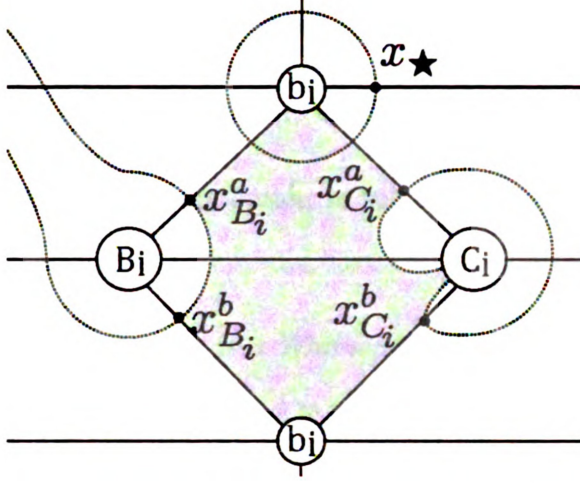


Figure 3.17: The domain Γ_i

corresponds to a holomorphic disk, and further we see that this Whitney disk is in $\pi_2(x(1, -), x(2, -))$. Similarly the domain $-D_Z$ corresponds to a Whitney disk in $\pi_2(x(2, -), x(1, -))$, however it has no holomorphic representative.

Furthermore, consider the annular domain Γ_i in Figure 3.17. This corresponds to a Whitney disk in $\pi_2(x(i, \dots, a, b, \dots), x(i, \dots, b, a, \dots))$ where $i = 1$ or 2 , a Whitney disk, that is, that leaves the net number of b 's constant. By summing the Whitney disks corresponding to the Γ_i , and to D_Z we can construct a Whitney disk between any two generators of $\widehat{CF}(\alpha, \beta, z, \mathfrak{s}_{2n-2})$.

Hence, for $\phi \in \pi_2(x(-), x(\sim))$, the domain will have the general form $D(\phi) = r_1 R_1 + r_2 R_2 + \sum t_i T_i + \sum \gamma_i \Gamma_i + \delta D_Z + s[\Sigma]$ where δ is either 1, -1, or 0.

Using the additivity of the Maslov index, and Proposition 27, we have that

$$\mu(\phi) = r_1(2l + 2n - 2) + \delta + 2s \quad (3.1)$$

Furthermore, observe that

$$n_z(\phi) = r_1 l + \delta + s. \quad (3.2)$$

We will use these two equations to glean information about $\widehat{\partial}$ and ∂^+ in the

following series of claims.

Claim 1. *If $\mu(\phi) = 1$, then $\delta = 1$ or -1 .*

Proof. The only option to rule out is the case $\delta = 0$. This cannot happen; if δ is 0, then by (3.1) we would have that $\mu(\phi)$ is even. $\square$

Claim 2. *If ϕ is a Whitney disk such that $\mu(\phi) = 1$, and ϕ admits a holomorphic representative, then ϕ is a member of either $\pi_2(x(1, -), x(2, \sim))$ or $\pi_2(x(2, -), x(1, \sim))$.*

Proof. This follows immediately from Claim 1. Notice in particular that if $\delta = 1$, then ϕ is in $\pi_2(x(1, -), x(2, \sim))$; whereas if $\delta = -1$, then ϕ is in $\pi_2(x(2, -), x(1, \sim))$ $\square$

Claim 3. *Suppose we have $\phi \in \pi_2(x(1, -), x(2, \sim))$ such that $\mu(\phi) = 1$ and $n_z(\phi) = 0$. Then ϕ cannot admit a holomorphic representative.*

Proof. Since ϕ is in $\pi_2(x(1, -), x(2, \sim))$, we have that $\delta = 1$. Additionally, $\mu(\phi) - 2n_z(\phi) = 1$ implies, by Equations (3.1) and (3.2), that $r_1(n - 1) = 1$. Since n is positive, this can only happen if $n = 2$ and $r_1 = 1$.

Suppose this is the case. Equation (3.2) now implies $1 + 1 + s = 0$, Which implies that s is negative. However, out of the regions under consideration, only $[\Sigma]$ contains the point at ∞ . Therefore $n_{\{\text{pt at } \infty\}}(\phi) = s$, which cannot be negative by Proposition 29 if ϕ is to admit a holomorphic representative. $\square$

Claim 4. *Suppose we have $\phi \in \pi_2(x(1, -), x(2, \sim))$ such that $\mu(\phi) = 1$ and $n_z(\phi) = 1$. Then $D(\phi) = D_Z$ and $\#\widehat{M}(\phi) = 1$ or -1 . That is, ϕ is in $\pi_2(x(1, -), x(2, -))$ and admits a unique holomorphic representative.*

Proof. Since ϕ is in $\pi_2(x(1, -), x(2, \sim))$, we have that $\delta = 1$. Additionally, $\mu(\phi) - 2n_z(\phi) = -1$ implies by Equations (3.1) and (3.2) that $r_1(n - 1) = 0$. Since $n > 0$, this implies that $r_1 = 0$, and by Equation (3.2), we can now say that $s = 0$ as well. Hence

$D(\phi)$ is of the form $r_2 R_2 + \sum t_i T_i + \sum \gamma_i \Gamma_i + \mathcal{D}_Z$. However, if there is a holomorphic Whitney disk corresponding to this domain, then according to Proposition 29 the domain $D(\phi)$ must be connected. D_Z is disconnected from R_1 , the Γ_i , and the T_i . Therefore r_2 , the t_i 's, and the γ_i 's must be 0 as well. Hence, $D(\phi) = D_Z$, and by Proposition 29 again, we have that $\# \widehat{M}(\phi) = 1$ or -1 since D_Z is a bigon. $\square$

Claim 5. *For all intersection points $x(2, -)$ and $x(1, \sim)$, we have that*

$$\sum_{\substack{\phi \in \pi_2(x(2, -), x(1, \sim)) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \widehat{M}(\phi) = 0$$

Proof. Here we will use the fact that $\partial^{+2} = 0$. Assume $i \gg 0$. Then by Claims 2, 3, and 4, we have that

$$\begin{aligned} \partial^+[x(1, -), i] &= \sum_{x(2, \sim)} \sum_{\substack{\phi \in \pi_2(x(1, -), x(2, \sim)) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \widehat{M}(\phi)[x(2, \sim), i] \\ &\quad + \sum_{x(2, \sim)} \sum_{\substack{\phi \in \pi_2(x(1, -), x(2, \sim)) \\ \mu(\phi)=1 \\ n_z(\phi)=1}} \# \widehat{M}(\phi)[x(2, \sim), i-1] \\ &\quad + \text{lower order terms} \\ &= 0 + [x(2, -), i-1] + \text{lower order terms} \end{aligned}$$

Therefore,

$$\begin{aligned}
\partial^{+2}[x(1, -), i] &= \partial^+([x(2, -), i - 1] + \text{lower order terms}) \\
&= \sum_{\substack{x(1, \sim) \phi \in \pi_2(x(2, -), x(1, \sim)) \\ \mu(\phi)=1 \\ n_z(\phi)=0}} \# \widehat{M}(\phi)[x(1, \sim), i - 1] + \text{lower order terms}
\end{aligned}$$

The claim now follows from the fact that $\partial^{+2} = 0$.

□

Now we are able to show that $\widehat{\partial} = 0$. Claim 2 implies that any holomorphic ϕ must be in either $\pi_2(x(1, -), x(2, \sim))$ or $\pi_2(x(2, -), x(1, \sim))$. For $\widehat{\partial}$, we are only interested in the Whitney disks where $n_z(\phi) = 0$, so by Claims 3 and 5 we have that $\# \widehat{M}(\phi) = 0$. Hence $\widehat{\partial} = 0$, and the statement about $\widehat{HF}$ follows.

Now we can turn our attention to HF^+ . Filter $CF^+(\alpha, \beta, z, \mathfrak{s}_{2n-2})$ by defining

$$\begin{aligned}
F_i &= \langle \{[x(1, -), j] | \mathfrak{s}_z(x(1, -)) = \mathfrak{s}_{2n-2} \ \& \ j < i\} \rangle \\
&\quad \bigoplus \langle \{[x(2, -), j] | \mathfrak{s}_z(x(1, -)) = \mathfrak{s}_{2n-2} \ \& \ j < i - 1\} \rangle
\end{aligned}$$

One can find the induced differential on the associated graded complex $\frac{F_i}{F_{i-1}}$, again by looking at domains.

By Claim 2, the only homotopy classes of maps that have Maslov index equal to 1 are in $\pi_2(x(1, -), x(2, \sim))$ and $\pi_2(x(2, -), x(1, \sim))$. However, in the associated graded complex this simplifies. Notice, in fact, that the only non-trivial differentials here will correspond to maps in $\pi_2(x(1, -), x(2, \sim))$ where $n_z(\phi) = 1$. By Claim 4, the only non-zero differential induced on $\frac{F_i}{F_{i-1}}$ is $[x(1, -), i] \mapsto [x(2, -), i - 1]$ or $[x(1, -), i] \mapsto -[x(2, -), i - 1]$ given by D_Z (the sign depends on how the moduli space is oriented, but either choice will give the same result). Consequently,

$$H\left(\frac{F_i}{F_{i-1}}\right) = \begin{cases} 0 & \text{when } i > 0 \\ \mathbb{Z} \binom{2(n-1)}{n-1} & \text{when } i = 0 \end{cases} \quad (*)$$

Associated to the short exact sequence

$$0 \rightarrow F_{i-1} \rightarrow F_i \rightarrow \frac{F_i}{F_{i-1}} \rightarrow 0$$

is a long exact sequence

$$\rightarrow H(F_{i-1}) \rightarrow H(F_i) \rightarrow H\left(\frac{F_i}{F_{i-1}}\right) \rightarrow$$

Using (*), this exact sequence, and the fact that $H(F_{-1}) = 0$, we arrive at

$$HF^+(T(2n, 0), \mathfrak{s}_{2n-2}) = \varinjlim H(F_i) = \mathbb{Z} \binom{2(n-1)}{n-1}.$$

3.7 Applications

We offer two calculations of the Λ invariants. The first is merely to offer some perspective on how the relationship between diffeomorphisms and h-cobordisms can be exploited. The second applies our Floer homology calculation.

Let ϕ^* be an automorphism of $H^*(M)$. Construct the h-cobordism W^ϕ as follows: Let W_1 and W_2 be cobordisms from M to $M \# S^2 \times S^2$ built out of a single 2-handle. Wall's theorem shows that there is a self-diffeomorphism ϕ of $M \# S^2 \times S^2$ that induces ϕ^* on $H^*(M)$ and is the identity on the cohomology of $S^2 \times S^2$. Then define $W^\phi = W_1 \cup_\phi W_2$. By a theorem of Quinn ([37]) and standard surgery theory, W^ϕ is determined up to *diffeomorphism* by ϕ^* .

Theorem 35. *An automorphism ϕ^* is induced from a diffeomorphism of M if and only if $\Lambda(M, W^\phi, M) = 0$*

Proof. If $\Lambda(M, W^\phi, M) = 0$, then W^ϕ is smoothly a product. That is, there is a diffeomorphism from $(M, M \times I, M)$ to (M, W^ϕ, M) which, when restricted to $M \times 1$, induces ϕ^* on $H^*(M)$.

Conversely, if ϕ^* is induced from a diffeomorphism of M , then we can arrange the diffeomorphism $\phi : M \# S^2 \times S^2 \longrightarrow M \# S^2 \times S^2$ such that it is just the identity on $S^2 \times S^2$. But by construction, this forces the handles of W^ϕ to cancel.

□

In the language of we have been using here, Morgan and Szabo prove the following

Theorem 36. [29] *Say $X_n = CP^2 \# m \overline{CP}^2$ where $m = (2n + 1)^2 + 1$. Then there exist automorphisms ϕ_n^* of $H^*(X_n)$ such that $\Lambda(X_n, W^{\phi_n}, X_n)$ is unbounded as n increases.*

From the perspective adopted here, this means that there exist homeomorphisms that are arbitrarily far from being diffeomorphisms.

As a second application of the Λ invariants, we derive the following elementary relationship between the 4-manifold invariant of Ozsvath-Szabo, and the complexity of certain cobordisms.

Theorem 37. *Say a simply connected spin 4-manifold M_1 has a Heegaard-Floer basic class $[h] \in H_2(X)$ with divisibility d , and self intersection 0.*

Say $M_1 \# S^2 \times S^2$ is diffeomorphic to $M_2 \# S^2 \times S^2$. Then there exists a cobordism W such that $\Lambda_1(M_1, W, M_2) \geq d + 2$

Proof. A theorem of Wall says that if two elements of H_2 have the same self intersection, divisibility, and are either both characteristic or regular (not characteristic, that is), then there is an automorphism of the cohomology ring taking one of these elements to the other. [43]. Wall also showed that this automorphism can be realized by a self diffeomorphism if the manifold splits as a smooth connect sum of $S^2 \times S^2$

and an indefinite manifold. [44]. Note that $(1/d)[h]$ is a regular homology class, because it has a dual Σ such that $\langle (1/d)[h], \Sigma \rangle = 1$, which cannot happen if $(1/d)[h]$ is characteristic, since X has an even intersection form.

Hence, there is a diffeomorphism $\phi : M_2 \# S^2 \times S^2 \longrightarrow M_1 \# S^2 \times S^2$ such that $\phi_*([S^2 \times pt]) = (1/d)[h]$. As in Proposition 15, we can construct a simple cobordism W using this diffeomorphism. Hence, if we take a handlebody decomposition of W where the belt sphere of the 2-handle intersects the attaching sphere of the 3-handle $2n$ times, then M_1 and M_2 are related by a $T(2n, 0)$ surgery in M_1 . Moreover, by construction we have $\langle (1/d)[h] \rangle = H_2(D(2n, 0)) \subset H_2(M_1)$ (recall that $D(n, m)$ is the 4-manifold corresponding to $T(n, m)$ surgery).

This implies that Floer homology maps that define the Ozsvath-Szabo 4-manifold invariant factor through $HF^+(T(2n, 0), \mathfrak{s}_d)$. For this to be non-trivial, the computation in the previous section requires $d \leq 2n - 2$.

□

Examples of manifolds satisfying the hypotheses of this theorem exist in abundance. Note also that this inequality has no dependence on the manifold M_2 , and that the cobordisms constructed here are never h-cobordisms.

Chapter 4

Appendix: Idiosyncrasies of the knot surgery formula

Recall the knot surgery formula, $SW_{X_K} = SW_X \cdot \Delta_K(2[T])$.

It is evident from this result that one can construct infinite families of exotic smooth manifolds. What is not *prima facie* evident is that knots with two different Alexander polynomials will *always* give non-equivalent knot surgeries. The purpose of this appendix is to clarify and resolve this issue.

Theorem 38. *If K_1 and K_2 are knots with different Alexander polynomials, then X_{K_1} and X_{K_2} cannot be diffeomorphic.*

This subtlety arises because of the somewhat imprecise way we have described $SW_X \in \mathbb{Z}[H_2(X)]$ as an invariant of X . It should really be thought of as an invariant *up to automorphisms of $\mathbb{Z}[H_2(X)]$* . Here is why: The Seiberg-Witten invariant is typically defined as a map $SW : Spin^c(X) \rightarrow \mathbb{Z}$. We encode this information as an element of $\mathbb{Z}[H_2(X)]$ by defining $SW_X := \sum SW(\mathfrak{s}) PD(c_1(\mathfrak{s}))$ where the sum is taken over all $Spin^c$ structures on X . When we do knot surgery on X to produce X_K , our new Seiberg-Witten invariant SW_{X_K} is an element of $\mathbb{Z}[H_2(X_K)]$. We can think of this as an element in $\mathbb{Z}[H_2(X)]$ — which is what we do implicitly in

the knot surgery formula because $H_2(X)$ is isomorphic to $H_2(X_K)$. In fact, this isomorphism is canonical, *but only with respect to the surgery*. Different knot surgeries, even surgeries that give diffeomorphic manifolds, will induce different isomorphisms of H_2 , and hence might manifest the resulting Seiberg-Witten invariants as different elements of $\mathbb{Z}[H_2(X)]$.

Consider the following illustrative example: Say X is a 4-manifold containing two tori T_1 and T_2 representing different homology classes such that there is a self-diffeomorphism of X taking T_1 to T_2 . For a single knot K , do knot surgery on T_1 and T_2 forming X_1 and X_2 . Clearly knot surgery can be performed in such a way that these manifolds are diffeomorphic, but note that their Seiberg-Witten invariants, as elements in $\mathbb{Z}[H_2(X)]$, will be different. According to the knot surgery formula, if $c_1(\mathfrak{s})$ is a basic class of X , then on X_1 we get new basic classes of the form $c_1(\mathfrak{s}) + n[T_1]$, whereas our new basic classes on X_2 are of the form $c_1(\mathfrak{s}) + n[T_2]$. However, the diffeomorphism of X_1 to X_2 induces an automorphism of $H_2(X)$ that takes $[T_1]$ to $[T_2]$ (and consequently takes SW_{X_1} to SW_{X_2}).

In the case at hand, where $\Delta_{K_1} \neq \Delta_{K_2}$ and we want to show X_{K_1} is not diffeomorphic to X_{K_2} , we will need to associate to each element of $\mathbb{Z}[H_2(X)]$ a quantity that is not affected by automorphisms of $H_2(X)$.

Definition 39. Suppose $H_2(X)$ is torsion free, α is an irreducible element of $\mathbb{Z}[H_2(X)]$, and ϕ is an automorphism of $\mathbb{Z}[H_2(X)]$. Define a map $\Gamma_{\alpha, \phi} : \mathbb{Z}[H_2(X)] \rightarrow \mathbb{Z}$ by

$$x \mapsto \sharp \text{ of elements of } \{\phi^n(\alpha) | n \in \mathbb{Z}\} \text{ that can be factored} \\ \text{out of } x \text{ counting multiplicity}$$

This map is well defined because $\mathbb{Z}[H_2(X)]$ is a UFD. Moreover, Γ has the following basic properties:

Proposition 40. (i) For $a, b \in \mathbb{Z}[H_2(X)]$, we have $\Gamma_{\alpha, \phi}(ab) = \Gamma_{\alpha, \phi}(a) + \Gamma_{\alpha, \phi}(b)$

$$(ii) \Gamma_{\alpha, \phi} \circ \phi = \Gamma_{\alpha, \phi}$$

$$(iii) \Gamma_{\alpha, \phi}(\Delta_K([T])) = \Gamma_{\alpha, id}(\Delta_K([T])) \text{ when } \alpha \in \mathbb{Z}[\langle [T] \rangle]$$

Proof. Only the third property deserves further comment. Suppose $\Gamma_{\alpha, \phi}(\Delta_k) > \Gamma_{\alpha, id}(\Delta_k)$. This means that $\phi^n(\alpha)$ can be factored out of Δ_k for some integer n such that $\phi^n(\alpha) \neq \alpha$. We will show there can be no such factor.

Since Δ_k can be factored into irreducibles that are in $\mathbb{Z}[\langle [T] \rangle]$, we have that $u\phi^n(\alpha) \in \mathbb{Z}[\langle [T] \rangle]$ for some unit $u \in \mathbb{Z}[H_2(X)]$. Since $\alpha \in \mathbb{Z}[\langle [T] \rangle]$, we can write $u\phi^n(\alpha) = u \sum a_i \phi^n([T]^i)$ for $a_i \in \mathbb{Z}$. This summation must have more than one term since otherwise $\phi^n(\alpha)$ would be a unit. Therefore, since $u\phi^n(\alpha) \in \mathbb{Z}[\langle [T] \rangle]$, we have that $u\phi^n([T]^i) = [T]^j$, and $u\phi^n([T]^{i'}) = [T]^{j'}$ for some $i \neq i'$ and $j \neq j'$. Therefore, $[T]^{-j'} = u^{-1}\phi([T]^{-i'})$, and this implies that $[T]^{j-j'} = \phi^n([T]^{i-i'})$. Since ϕ^n must preserve degree, we get that that $\phi([T]) = [T]$, and hence $\phi^n(\alpha) = \alpha$. $\square$

These properties are sufficient to prove theorem 38.

Proof. 38 Assume $\Delta_{K_1} \neq \Delta_{K_2}$ but that X_{K_1} is diffeomorphic to X_{K_2} . We will derive a contradiction. According to the knot surgery formula, the Seiberg-Witten invariants of X_{K_1} and X_{K_2} are $SW_X \cdot \Delta_{K_1}$ and $SW_X \cdot \Delta_{K_2}$ respectively. A diffeomorphism $\phi : X_{K_1} \rightarrow X_{K_2}$ induces an automorphism $\phi_* : \mathbb{Z}[H_2(X_{K_1})] \rightarrow \mathbb{Z}[H_2(X_{K_2})]$ where $\phi_*(SW_X \cdot \Delta_{K_1}) = SW_X \cdot \Delta_{K_2}$

Since $\Delta_{K_1} \neq \Delta_{K_2}$, we can choose α to be an irreducible element of $\mathbb{Z}[\langle [T] \rangle]$ that divides Δ_{K_1} with a greater multiplicity than it divides Δ_{K_2} . In other words $\Gamma_{\alpha, id}(\Delta_{K_1}) > \Gamma_{\alpha, id}(\Delta_{K_2})$.

In fact, via property (iii) we have $\Gamma_{\alpha, \phi_*}(\Delta_{K_1}) > \Gamma_{\alpha, \phi_*}(\Delta_{K_2})$.

To the equality $\phi_*(SW_X \cdot \Delta_{K_1}) = SW_X \cdot \Delta_{K_2}$ we apply Γ_{α, ϕ_*} (here short-handed as Γ) and use properties (i) and (ii) above:

$$\begin{aligned}\Gamma(\phi_*(SW_X \cdot \Delta_{K_1})) &= \Gamma(SW_X \cdot \Delta_{K_2}) \\ \Gamma(\phi_*(SW_X)) + \Gamma(\phi_*(\Delta_{K_1})) &= \Gamma(SW_X) + \Gamma(\Delta_{K_2}) \\ \Gamma(\Delta_{K_1}) &= \Gamma(\Delta_{K_2})\end{aligned}$$

This, however, contradicts our choice of α .

□

Remark 41. An essential hypothesis of this theorem was that $H_2(X)$ be torsion free: Otherwise $\mathbb{Z}[H_2(X)]$ is not a UFD and we cannot define Γ . Note, however, that in the case $H_2(X)$ has torsion, the same proof can be carried out as long as the image of SW_X in $\mathbb{Z}[H_2(X)/tor]$ is non-trivial. Simply replace every instance of $\mathbb{Z}[H_2(X)]$ above with $\mathbb{Z}[H_2(X)/tor]$.

Remark 42. The above proof can also be applied to rim surgery to show that any two knots with different Alexander polynomials will give rise to inequivalent rim-surgeries.

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