DYNAMIC RESPONSE OF A MODEL CANTILEVER BRIDGE TO MOVING LOADS

Thesis for the Degree of M. S.

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Joseph D. Eisenberg

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THESIS

This is to certify that the

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DYNAMIC RESPONSE OF A MODEL CANTILEVER
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R. K. Wen
Major professor

Date Dec 12, 1961

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ABSTRACT

DYNAMIC RESPONSE OF A MODEL CANTILEVER

BRIDGE TO MOVING LOADS

by Joseph D. Eisenberg

A series of tests is performed in order to study the effects of the mass and the speed of moving loads upon the dynamic behavior of a model, three-span, cantilever bridge.

The model bridge is made up of three steel beams all of the same cross section. The moving loads are five steel balls of different masses. The different load speeds are obtained by releasing the balls from various heights on an inclined plank connected to the model structure.

Records of dynamic deflections are made at the centers of the three spans by means of differential transformers. Strain gages are also attached at these three points.

It is found that the dynamic effects in the bridge increase as the load progresses along the length of the structure. The maximum dynamic deflections measured were 17-36% larger than the maximum static deflection in the center span, and 25-58% larger than the maximum static deflections in the far span.

The maximum dynamic effects vary in an oscillating manner with speed. However, the general trend is an increase in dynamic

effects with an increase in speed. Also an increase in the mass of the load causes appreciably more than a proportionally greater increase in maximum deflection.

The experimental results are compared with "constant force" analytical results, i.e., the mass of the load assumed to be zero. The analytical results are in agreement with the trends of the effects of the variables as indicated by the experimental results.

DYNAMIC RESPONSE OF A MODEL CANTILEVER BRIDGE TO MOVING LOADS

By

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I INTRODUCTION

1.1 The General Problem

The problems caused by induced vibrations in a bridge have long been recognized. The particular area of the dynamic behavior of highway bridges under the influence of moving loads has been under study since the Second World War (1). These studies are very much needed since the present criteria for designing for the effect of dynamic loadings are without rational bases. For example, the A.A.S.H.O. impact formula is a function of bridge length alone. However, the number of physical variables actually involved in bridge vibration is large indeed, including, for example, such factors as vehicle suspension, axle spacing, roughness of roadway, and the masses of the bridge and the vehicle.

In 1957 a field study of highway bridges in the State of Michigan (2) found that among the different types of bridges tested, the cantilever bridge was the most susceptible to vibrations induced by moving loads. Since 1959 a fundamental study of the cantilever bridge has been in progress in the Department of Civil Engineering at Michigan State University. This thesis reports a phase of the experimental portion of this investigation.

THE

1.2 Object and Scope

The purpose of the work reported herein was to study the effects of the mass and the speed of moving loads upon the dynamic response of a model, three-span, symmetrical, cantilever bridge.

The results presented in this paper are those of a test program involving a single model bridge and several different moving masses. The quantities measured in the experiments were the deflections at the center of each of the three spans, and strains at the center of one of the spans. In the majority of the runs the strain at the center of the middle span was measured. Included in the results is a comparison between the experimental values of the deflections and theoretical values obtained by means of a constant force analytical solution (4).

It might be mentioned that with the exception of the studies of actual bridges in Ref. 2 no experimental work on the dynamic response of cantilever bridges has been reported in the technical literature.

1.3 Notation

Below is a list of the notation used in this paper.

A.F.	=	amplification factor defined in Eq. 2.5
L	=	length of bridge
m	=	subscript denoting model
m_b	=	total mass of bridge

m_l	=	mass of load
$\bar{m}_l$	=	mass parameter defined in Eq. 2.4
p	=	subscript denoting actual bridge
T	=	fundamental period
t	=	time in seconds
v	=	speed of load
y_s	=	maximum static deflection
y_d	=	maximum dynamic deflection
α	=	speed parameter defined in Eq. 2.1

II APPARATUS AND INSTRUMENTATION

2.1 Model Bridge and Moving Loads

The model bridge had been built prior to this series of tests, and the construction is completely discussed in Ref. 3. However, some of the information is repeated here to obviate the necessity of referring to the original report.

A schematic drawing of the model bridge is presented in Fig. 1. The model was made up of three steel beams, two 4 ft. 9-1/2 in. long and one 4 ft. long. The total length of the bridge was then 13.58 ft. The longer members made up the side spans and the overhangs. Each side span was pin supported at the outside end, and roller supported at a point four feet from the outside end. This left 9-1/2 in. cantilever arms. The four foot suspended span was pinned to the ends of the overhanging arms.

A groove 1/4 in. wide and 3/64 in. deep was cut into the top of each beam to serve as a guide for the moving loads.

The acceleration device was merely an inclined plank connected to a curved transition track which was in turn connected to a flat steel approach track. The approach track contained a groove identical with that of the bridge. A 1/16 in. space existed between the approach track and the bridge.

The loads used were five polished steel spheres. The balls weighed 0.501 lb., 1.181 lb., 1.979 lb., 3.063 lb., and 3.980 lb. For convenience in the report the balls are referred to by the normal weights of 1/2 lb., 1 lb., 2 lb., 3 lb., and 4 lb. respectively. These balls represent unsprung, smoothly rolling constant masses. In the series of dynamic tests the balls were manually released from different heights on the plank in order to obtain the several speeds desired.

Photographs and additional discussion of the model are found in Ref. 3. Table 1 contains a summary of basic information concerning the bridge model.

2.2 Measurements and Instrumentation

The measurements were recorded by means of a 4 channel Sanborn 150 Recording System. This system furnishes the power supply to the measuring devices in addition to acting as the recording system. The power is generated at 400 cps and thus can be used for differential transformers as well as strain gages.

In addition to the instruments used for the previous tests (3), which consisted of A-7, SR-4 strain gages at the centers of the three spans and a differential transformer at the center of the center span, differential transformers were installed for measuring the deflections of the centers of the two side spans. Due to the 400 cps exciting current it was found necessary to use shielded wire throughout the test set-up. This almost completely eliminated electrical disturbances effecting the output from the transducers.

Table 1. Bridge model characteristics

Length	13.58 ft.
Weight	18.1 lb.
Length of side span	2.00 ft.
Length of suspended span	2.00 ft.
Length of cantilever arm	0.79 ft.
Width of beam	1-1/8 in.
Depth of beam	3/8 in.
Moment of inertia, I_c	49×10^{-4} in.^{4}
Frequency first mode	10.00 cps
Frequency second mode	13.33 cps
Frequency third mode	16.70 cps

III TEST PROGRAM AND PROCEDURE

3.1 Parameters Studied

3.1.1 Speed Parameter

The speed parameter, α, is defined by the expression

$$\alpha = Tv/L \tag{2.1}$$

Where T is the fundamental period, L is the bridge length, and v is the load velocity. It may be seen that α is physically equal to the ratio of the fundamental period divided by the time required for the load to cross the bridge.

From the data presented in Ref. 2 the interval of 0.200 seconds and the length of 200 ft. may be taken respectively as the average fundamental period and the overall length of a typical 3 span, cantilever, highway bridge. Substituting these values in Eq. (2.1) and using the subscript p to denote full scale values, the following expression results

$$\alpha = v_p \times 10^{-3} \tag{2.2}$$

(It should be noted here that this definition of α differs slightly from that used in Ref. 3 and Ref. 4.)

Now setting the α value of Eq. 2.2 equal to the α for the model, and using the subscript m to denote the scale model values

$$v_p \times 10^{-3} = T_m v_m / L_m$$

or $$v_p \times 10^{-3} = \frac{0.100\ v_m}{13.58}$$

then $$v_m = 0.1358\ v_p \tag{2.3}$$

Thus a speed of 10 mph on the actual bridge is very nearly equivalent to a speed of 2 fps on the model bridge. In this study a maximum α of 0.0917, equivalent to a load speed of 62.3 mph, was attained.

3.1.2 Mass Parameter

The mass parameter, $\bar{m}_l$, is defined by the following expression,

$$\bar{m}_l = m_l/m_b \tag{2.4}$$

where m_l is the mass of the load and m_b is the total mass of the bridge. Again, from the data given in Ref. 2 the average weight of a 200 ft. long cantilever bridge may be taken as 1,240 kips. Then an A.A.S.H.O. H20-S16 vehicle, total weight 72 kips, on a 1,240 kip bridge could be closely approximated by a 1 lb. ball in the model study. The weights of the test balls chosen included and bracketed this load. The values of $\bar{m}_l$ corresponded to the five balls used in the test were 0.0277, 0.0652, 0.1092, 0.1692, and 0.2199.

3.2 Test Program

A total of 195 dynamic test runs was made. There were three runs for each of 13 velocities for each of the 5 loads. The speeds were increased in approximately 4 scale mph increments from 0 to 39 scale mph and in about 8 scale mph increments from 39 to 62 scale mph to make the total of 13 different velocities.

The dynamic response is measured in terms of "amplification factor," denoted by the symbol A.F. and defined by the following expression,

$$A.F. = y_d/y_s \tag{2.5}$$

where y_s is the maximum static deflection and y_d the maximum dynamic deflection. A similar relationship may be used in the case of the strain measurements.

Thus, in order to evaluate the dynamic data it was necessary to determine the maximum static response. To do this a static test at each measurement station and a crawl run were made for each load. The final 4 scale mph dynamic run for each load was also used as a crawl test. The average of the maximum response obtained in these three runs was used for y_s in all cases.

3.3 Test Procedure

As noted in Sec. 3.2, in each group 39 runs were made, three at each of the chosen speeds. The runs were made in a semi-randomized order to minimize extraneous effects during the experiments. For example, $\alpha = 0.035$ runs, (representing 24 scale mph) were the 7th, 27th, and34th.

The tests were run in five groups. One group was run for each mass. This method was found to be best due to limitations of the Sanborn Recording System. In order to get reliable readings, the attenuations had to be reduced for the lighter loads and increased for the heavier loads. After each change of the attenuation factor the recording system pre-amplifiers had to be adjusted. This was very time consuming. By running all tests for one mass together, frequent adjustments were unnecessary.

Prior to making any of the runs the elevation of each support was checked with a Dumpy level. The elevation of the near end, the end nearest the acceleration track, was used as the datum. At all other supports the track was set at the near end elevation. The bridge was also aligned by running a tight string along the length of the bridge. The string was fastened at each end of the bridge and the two intermediate supports were brought into alignment with the ends.

Following the check of alignment and elevation the static and crawl test were made. For the static tests the ball was placed on the bridge at each measurement station. A reading of the deflections and strains was made in each of the three cases, load on the near span, load on the center span, and load on the far span. Next a crawl test was run at about $\alpha = 0.004$. Again strain and deflection readings were made.

After completion of the static tests the dynamic tests were made. Marks had been placed on the acceleration track to denote the heights needed to obtain the 13 desired speeds. For a given run the ball was held manually at the proper mark. With the paper in the recorder running at 100 m.m. per second the load was released. The deflections of the three stations and the strain at one station were recorded as the load passed over the bridge.

IV RESULTS OF INVESTIGATION

4.1 General

Fig. 2 shows a typical record of a test run. This particular run was one with the 3 lb. ball at a speed of about 31 scale mph. The traces in this figure represent the history of dynamic response as a moving load passes over the bridge, and each trace may be called a history curve.

The top trace represents the history of the strain at the center of the center span. The vertical axis represents the strain, positive downward, and the horizontal axis represents time. It should be noted that actual numerical values have not been placed on the strain axis. Since it is only the ratio of the dynamic strain to the maximum static strain that is being studied, the actual values of the responses are not required.

The other three traces represent the deflections of the centers of the three spans. Going from top to bottom, the second trace on the sheet represents the near span, the third trace represents the far span, and the bottom trace represents the center span. As in the case of the strain curve and for the same reason mentioned the actual numerical values of deflection are not noted on the vertical axis.

The shape of the strain trace as the ball passes over the center span, t = 0.80 to t = 1.45, may be seen to be difficult to read. The high frequency response practically obliterates the predominant wave shape, and the exact total strain is questionable. Because of the poor quality of the strain traces, strain was measured in only one span, and there is no discussion of the strain amplification factors in this thesis.

The near span trace when the ball is on the near span and cantilever arm, t = 0.00 to t = 0.80, shows almost no dynamic response. Looking at the far and center span traces during this same time interval it may be seen that that dynamic response at all measurement points is very slight during this portion of the run.

In no test run was there observable dynamic response in the near span while the load was on that span. Therefore, the A.F. of the near span will not be considered in this paper.

Looking at the center and far span traces it may be seen that when the ball is on the respective spans (t = 1.46 to t = 2.23 for the far span and cantilever arm), a very definite wave form predominates. The period of the wave form in the center span is about 0.13 seconds, or the frequency is 7.69 cps. The period of the far span predominant wave form is about 0.09 seconds, the corresponding frequency being 11.1 cps. Now the unloaded fundamental frequency was found to be 10.0 cps, and the unloaded second and third frequencies were found to be 13.33 cps and 16.70 cps respectively (3). Thus the added mass of the load is seen to have significantly reduced the frequencies of the structure. The 7.69 cps is probably the modified first mode, and the 11.1 cps is probably the modified second mode.

It might be noted in passing that past t = 2.23, when the ball has left the bridge, all deflection traces show a very complicated vibration pattern. The center span seems to have a predominant 10 cps wave form with a wave form having about twice this frequency impressed upon it. These are probably first and third mode vibrations. The wave forms in the side spans are too complex to allow an estimate of the frequencies involved.

It may also be noted that an upward jump occurs in the near span when a load enters the bridge and the center span, and that a jump occurs in the far span as the load leaves the center span. These jumps, though hardly detectable at low speeds, get rather large at high α values. They are thought to be due in great part to the design of the connections.

The effect of this condition upon the maximum dynamic response of the bridge has not been ascertained.

4.2 Effects of Velocity Changes upon Amplification Factor

The test results are presented in the form of a series of spectrum curves. Essentially the spectrum curve is made up of a series of spectrum values, each value representing the maximum ordinate of a history curve. An examination of Fig. 3 will clarify the definition.

Each spectrum curve is associated with the response of one point of the bridge and one load. The abscissa represents the α value of a particular run. The ordinate represents the maximum A.F. of the point in question during that run. In this test series represented by Fig. 3, 3 runs were made at each of the desired α's with the exception of $\alpha = 0.058$ in which case only two runs were made. Every spectrum value is plotted in the figure. The average spectrum values are connected by straight lines to form the spectrum curve. It may be seen that the spread of spectrum values for a particular point in this figure is quite small in comparison to the differences between the several average points. Thus reasonable confidence may be placed in the resulting average values.

The general increase in A.F. with an increase in α is evident. True, there is an oscillating shape to the curve, but each maximum is larger than the preceding one. This oscillating but increasing characteristic of spectrum curves of A.F. vs. α has been noted previously in simple span bridges Ref. (5). In the constant force analytical study presented in Sec. 4.4 the same characteristic shape is found. In Fig. 3 the maximum average A.F. recorded is about 1.17 at $\alpha = 0.091$.

More spectrum curves are shown in Fig. 4 through Fig. 7 covering results of experiments with the 1/2, 2, 3, and 4 lb. loads. It may be noted that these curves are similar in character to that shown in Fig. 3. These spectrum curves are also for the center span. It may be seen that the spread of points in the 1/2 lb. curve, Fig. 4, is very large, and in the 4 lb. curve, Fig. 7, the spread is greater than in Fig. 3, Fig. 5, or Fig. 6. It seems that the spread tends to get larger for either a very small or a very large load.

The major differences between curves are the fact that the maximum A.F.'s occur at different α's for different loads, and the fact that the highest average A.F. in the range studied varies from curve to curve.

In Fig. 8 through Fig. 12 are presented the spectrum curves for the center point of the far span. Again the general increase of A.F. with an increase in α and the oscillatory nature of the curves are noted. The maximum A.F. values in the far span curves occur at very nearly the same α in all cases. These values, it may be seen, are different from the α values for maximums in the center span, however.

In this span, too, the highest average maximum A.F. varies from load to load. This change will be discussed in Sec. 4.3. It should be noted that in general the data scatter was greater in the far span spectrum curves than in those of the center span. The scatter of spectrum values in the 1/2 lb. spectrum curve is very large, and the average points must be viewed with considerable caution in this instance.

4.3 Effects of Changes in Mass of Load upon the Amplification Factor

Fig. 13 presents a plot of the highest average maximum A.F. within the α range studied against the parameter $\bar{m}_1$. It appears reasonably clear that A.F. tends to increase with an increase in the values of the mass parameter; or in this case of a fixed bridge, in the mass of the load.

The maximum A.F. for the center span which occurs with the 4 lb., $\bar{m}_1 = 0.2199$, load is about 1.350, and the maximum

A.F. for the far span, which also occurs for the 4 lb. load, is 1.580. These values are quite large. Even for the case of the 1 lb. load representing an H20-S16 vehicle the maximum A.F.'s are 1.17 and 1.34 for the center and far span respectively.

It would seem that the dynamic effects caused by a moving load as represented by the magnitudes of these amplification factors should be considered important in an actual bridge. Further, an increase in load apparently causes considerably more than than a proportionally greater increase in deflection.

4.4 Comparison with Constant Force Analytical Results

R. K. Wen and T. Toridis (3), (4) have developed an analytical solution for the dynamic response of a cantilever bridge to a constant moving force. This analysis has been programmed for solutions by use of the MISTIC computer at Michigan State University. Fig. 14 presents the analytical spectrum curve for the center span of the model bridge along with the 1 lb. and 4 lb. experimental spectrum curves. Fig. 15 presents the same information for the far span. For the sake of completeness, in Fig. 16 the analytical spectrum curve for the near span is shown.

Looking first at Fig. 14 it may be seen that the analytical curve has the same general characteristic as the experimental curves, an oscillating shape, but a general increase in A.F. with an increase in α, or velocity. The agreement in the phase of the "oscillations" between the analytical curve and the experimental $\bar{m}_1 = 0.0652$ curve is remarkable. However, the magnitude of the A.F. for the analytical curve is generally smaller than that of the experimental curves.

In Fig. 15 for the far span a similarity in shape between the analytical curve and the experimental curve $\bar{m}_1 = 0.0652$ may be seen also. It is again noted that the A.F. magnitudes are lower in the analytical curve. The smaller ordinates for the analytical curves may be explained by the fact that these curves are for a constant force solution, i.e., $\bar{m}_1 = 0$. Since, as pointed out previously (Fig. 13), A.F. tends to increase with $\bar{m}_1$, it seems reasonable that the $\bar{m}_1 = 0$ solutions yield smaller A.F.'s.

The maximum values of the A.F. obtained by the analytical solution for $\bar{m}_1 = 0$ are also shown in Fig. 13. It is seen that they certainly are in agreement with the trend indicated by the experimental results.

V CONCLUSION

5.1 Summary

The object of this study was to examine experimentally the effects of the speed and mass of moving loads upon the dynamic response of a three-span, symmetrical, cantilever bridge by means of tests on a model.

The dynamic response of the structure was gaged by the deflections at the centers of its three spans. The dynamic response in the near span was found to be negligible. The results in the center and far spans were presented in a series of ten spectrum curves, Fig. 3 through Fig. 12, in which A.F. was plotted against the speed parameter α. The range of velocities represented was from 0 to 62 mph in all cases.

All spectrum curves had the same characteristics, a general increase in A.F. with an increase in α, and an oscillating shape. From Fig. 13 it can be seen that an increase in the mass parameter $\bar{m}_1$, tends to cause an increase in the maximum A.F. Also from Fig. 13 it may be seen that the maximum A.F. values in the far span are higher than those in the center span in all cases.

Comparison of the experimental curves with constant force analytical curves showed that there was great similarity in shape, but that the analytical maximum A.F. values in the center and far span were lower than the maximum A.F. values determined experimentally. This is explained by the fact that the constant force solution corresponds to $\bar{m}_1 = 0$ and by the role of $\bar{m}_1$ mentioned above.

In both the center and far spans the A.F. would seem large enough to be of great significance in design considerations. The maximum A.F. with the 4 lb. load was 1.350 in the center span and 1.580 in the far span. A summary of important numerical results is presented in Table 2.

5.2 Suggestions for Future Study

Due to the physical characteristics of the model bridge no reliable strain readings were obtained. Since a study of the strain or bending moment is important from the standpoint of strength, such a study is suggested. It would seem that for that purpose considerable work has to be done in the modification of the model set-up.

The effects of jumps at the joints still remain in doubt. Therefore an examination of these, or preferably, of the general problem of road roughness would be of value.

Finally, the effect of a vibrating load, a sprung vehicle, on this type of bridge should also be examined.

Table 2--Highest average maximum amplification factors within range of speeds studied

center span

$\bar{m}_1$	(anal.) 0.0	0.0277	0.0652	0.1092	0.1692	0.2199
maximum A.F.	1.14	1.19	1.17	1.28	1.31	1.36
$\alpha_{(A.F.\ max.)}$	0.120	0.091	0.091	0.092	0.091	0.081

far span

$\bar{m}_1$	(anal.) 0.0	0.0277	0.0652	0.1092	0.1692	0.2199
maximum A.F.	1.24	1.25	1.34	1.46	1.58	1.58
$\alpha_{(A.F.\ max.)}$	0.120	0.069	0.070	0.080	0.080	0.080

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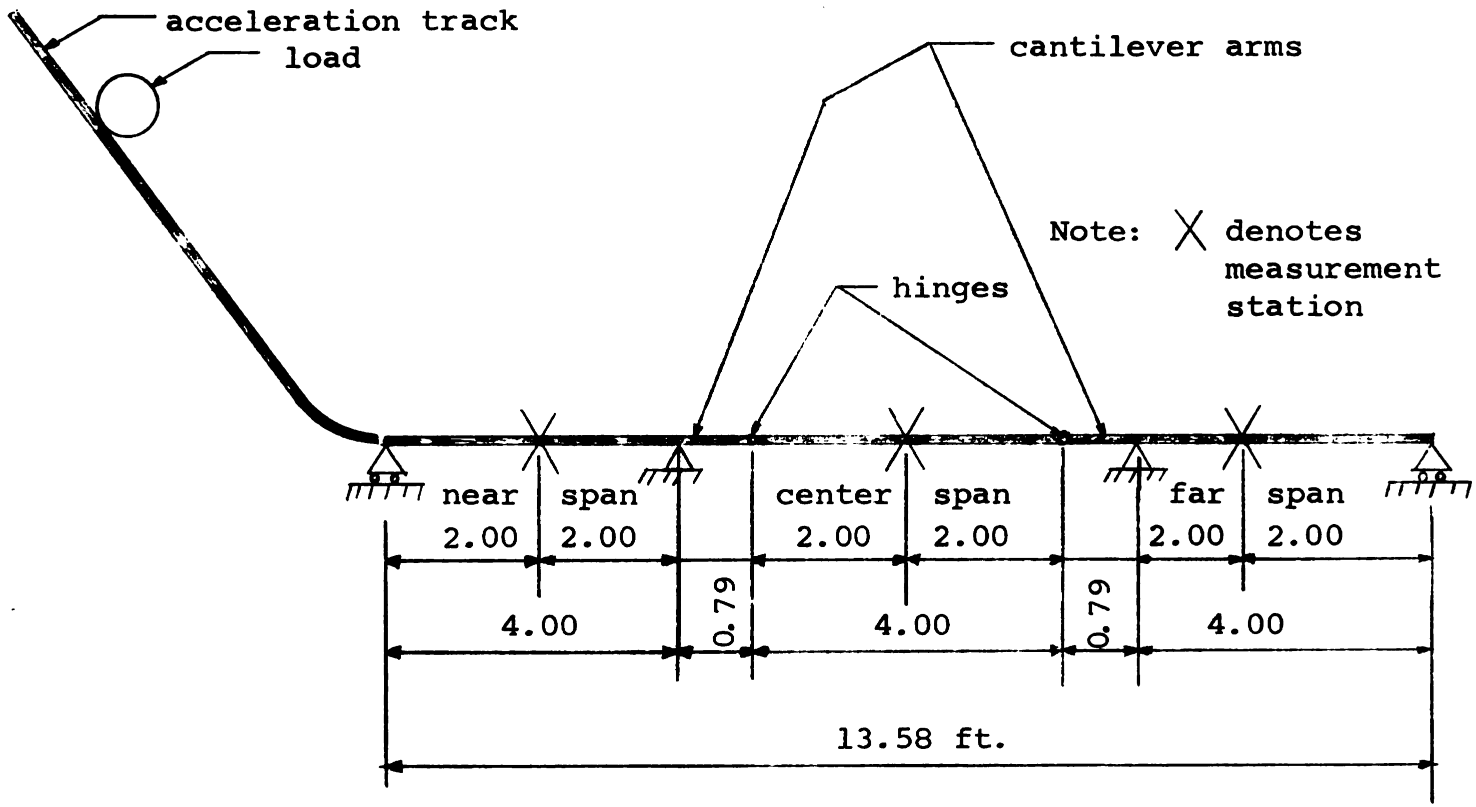

Fig. 1--Schematic drawing of bridge

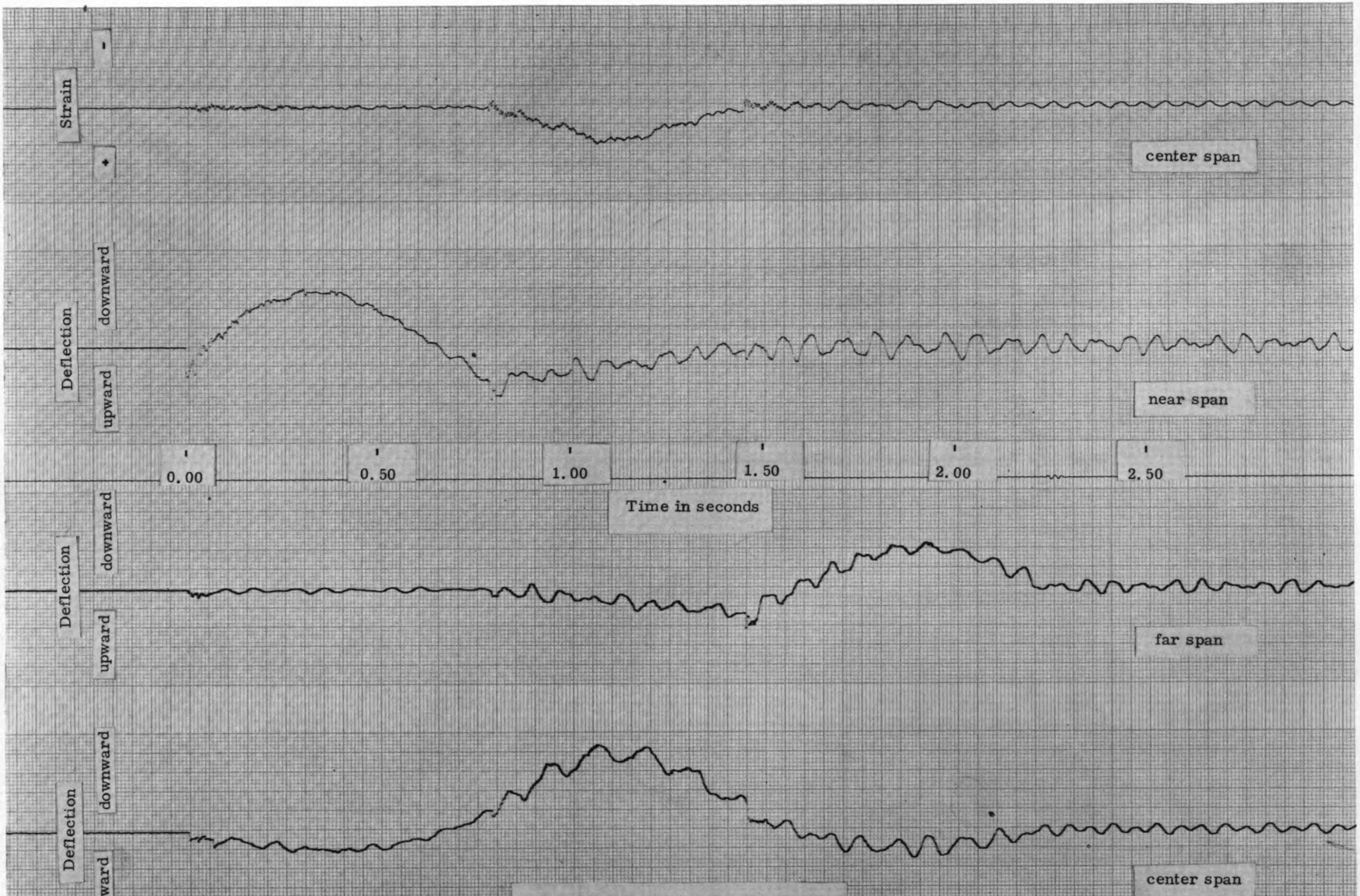

Fig . 2 - Test run with the 3 lb. load

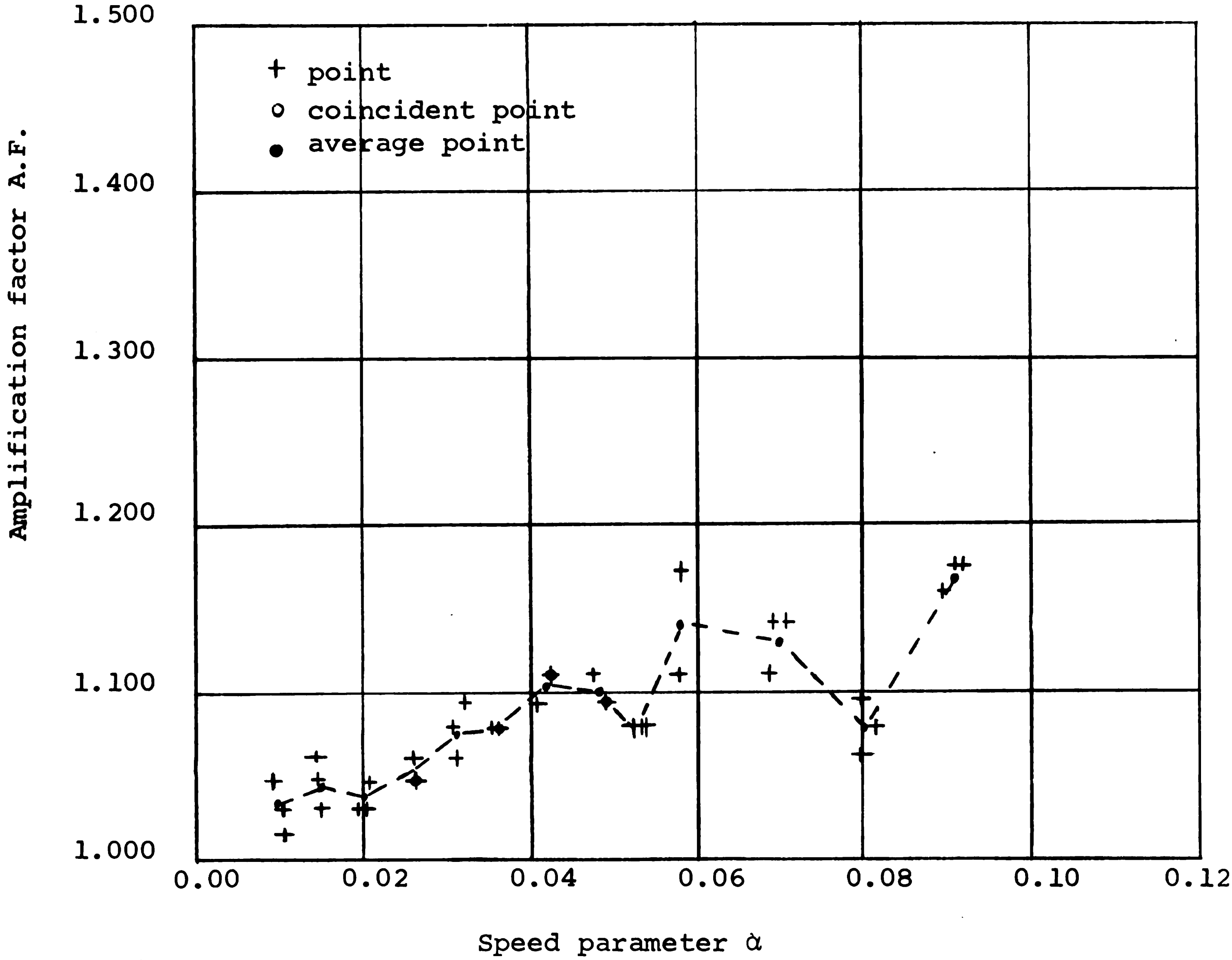

Fig. 3--Spectrum curve for center span

1 lb. load, $\bar{m}_1 = 0.0652$

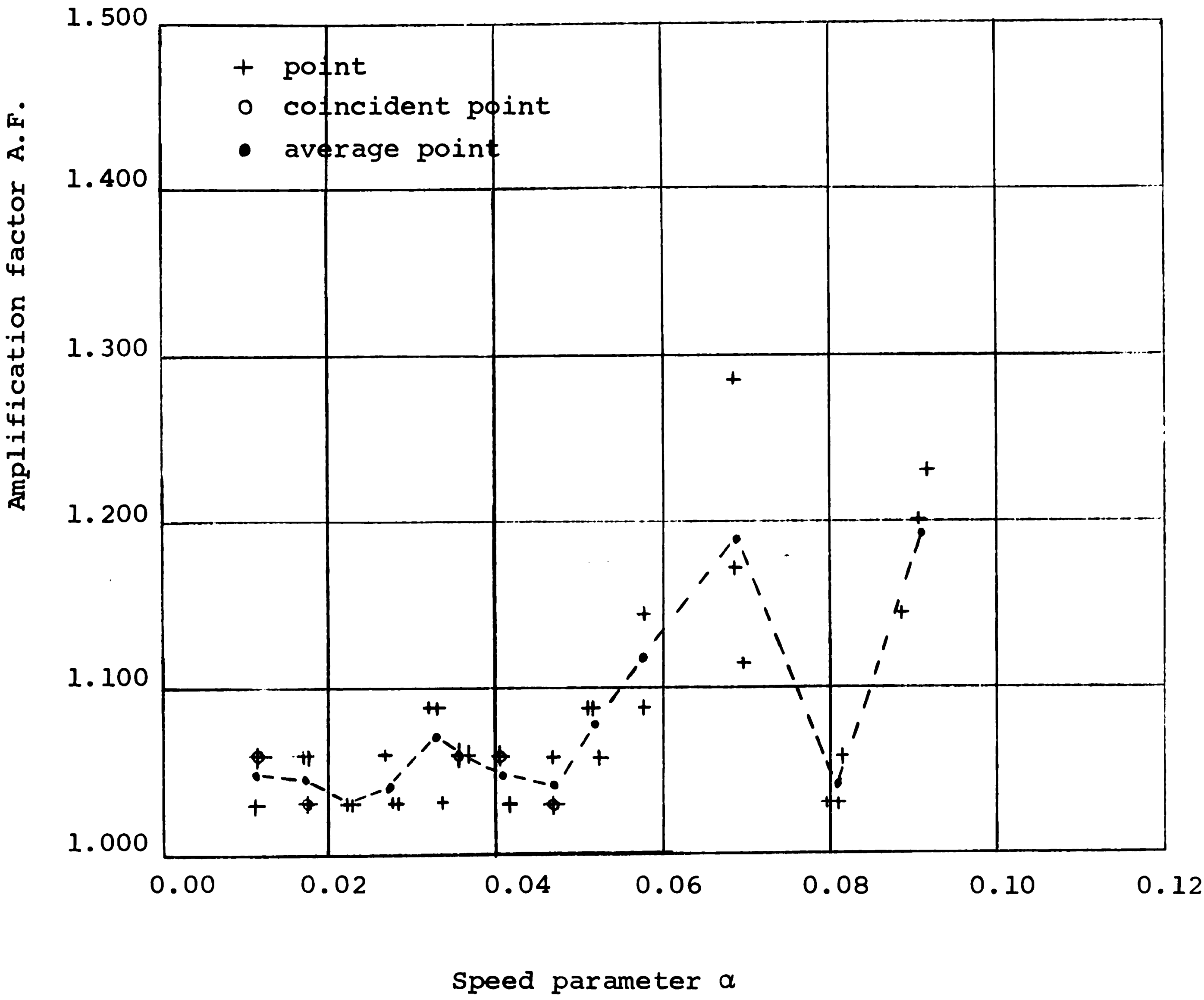

Fig. 4--Spectrum curve for center span

1/2 lb. load, $\bar{m}_1 = 0.0277$

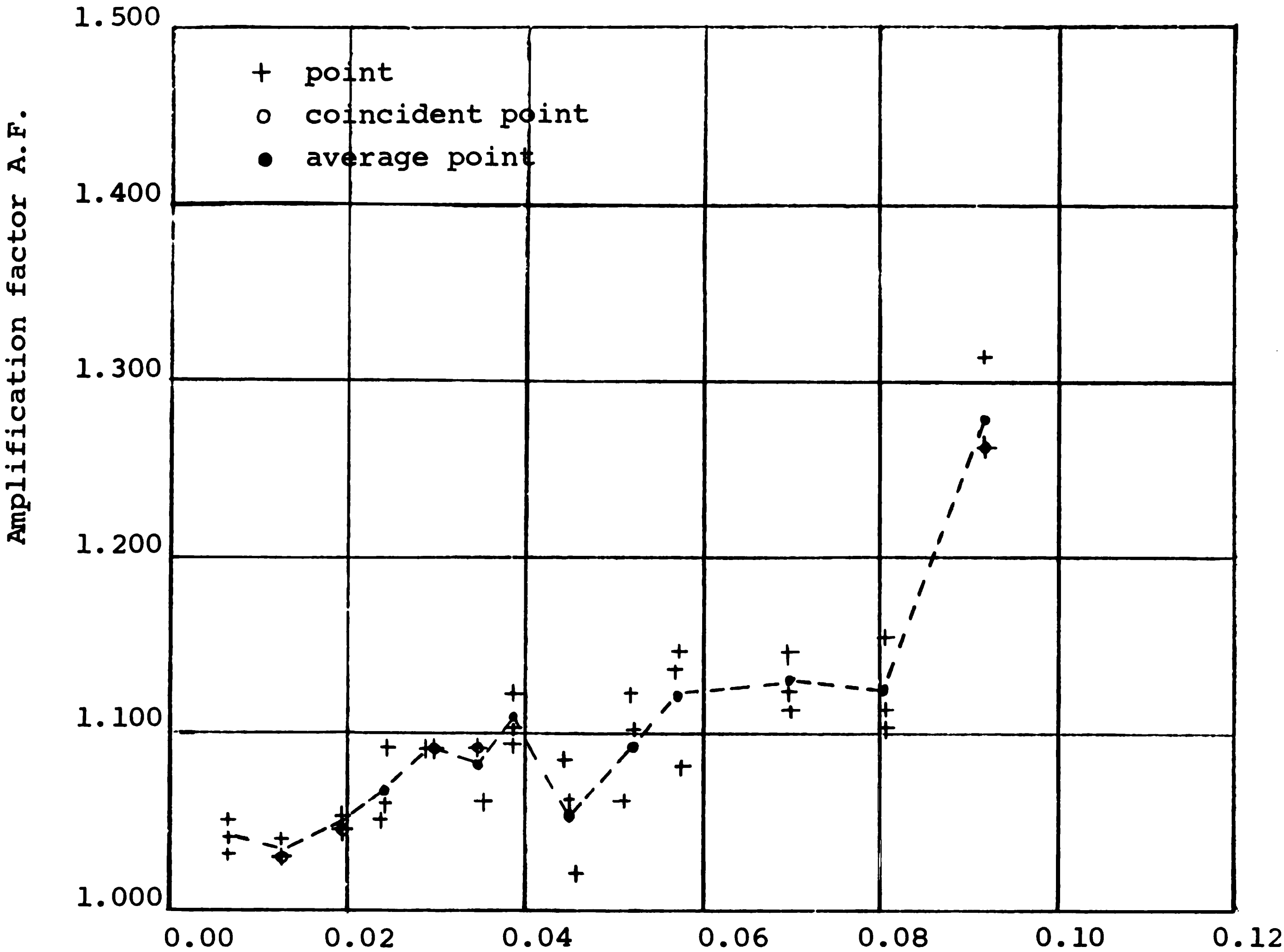

Fig. 5--Spectrum curve for center span

2 lb. load, $\bar{m}_1 = 0.1092$

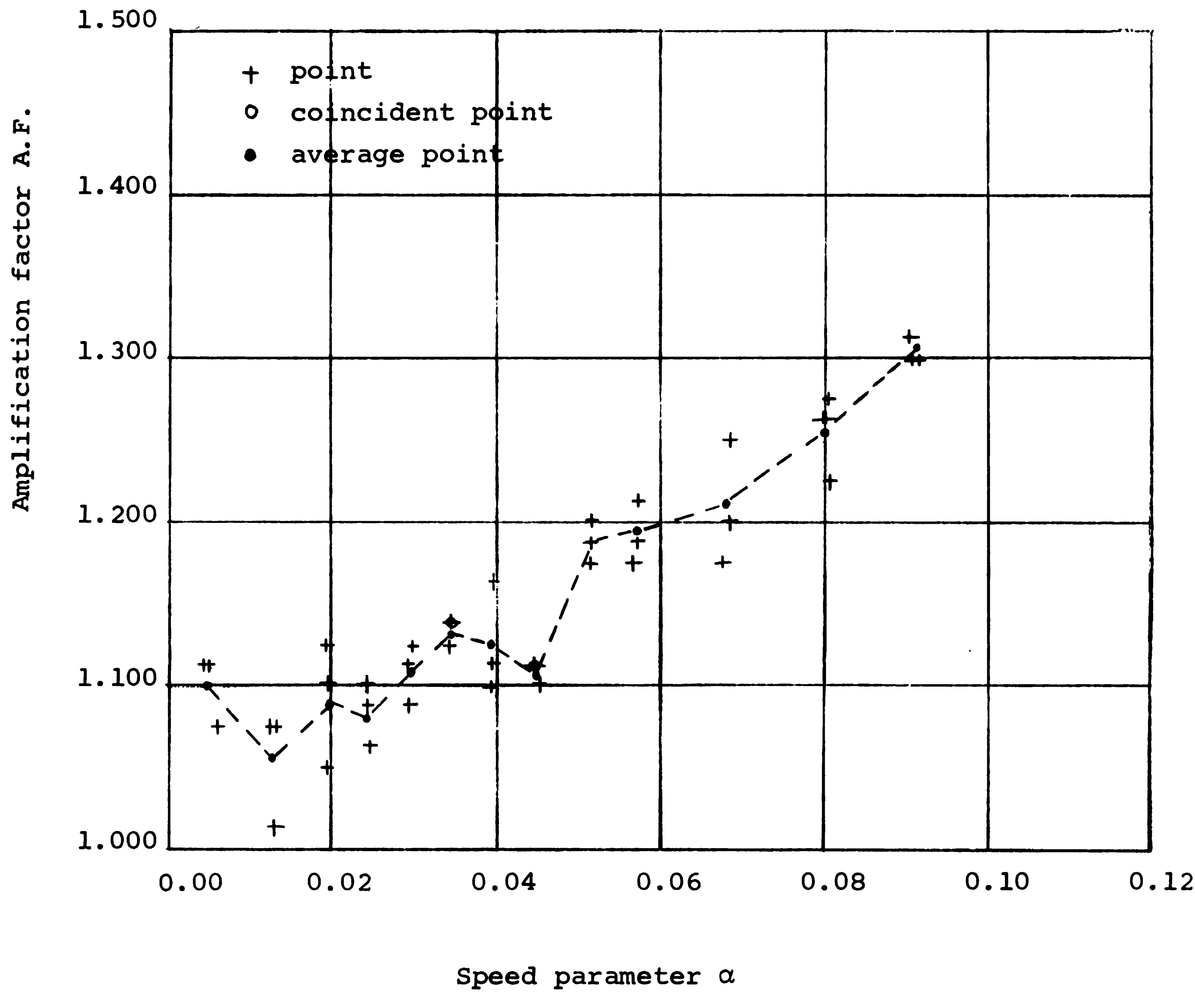

Fig. 6--Spectrum curve for center span

3 lb. load, $\bar{m}_1 = 0.1692$

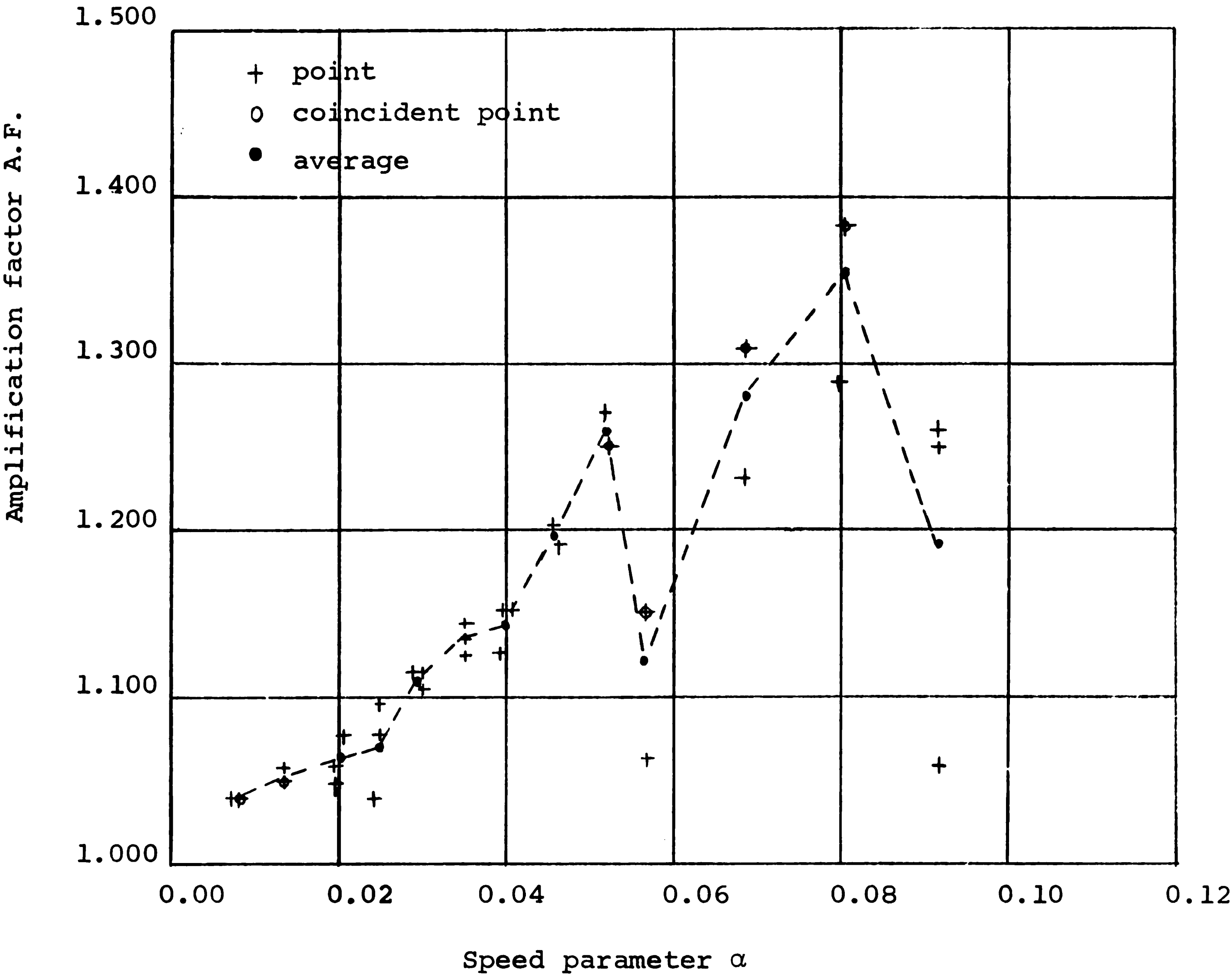

Fig. 7--Spectrum curve for center span

4 lb. load, $\bar{m}_1 = 0.2199$

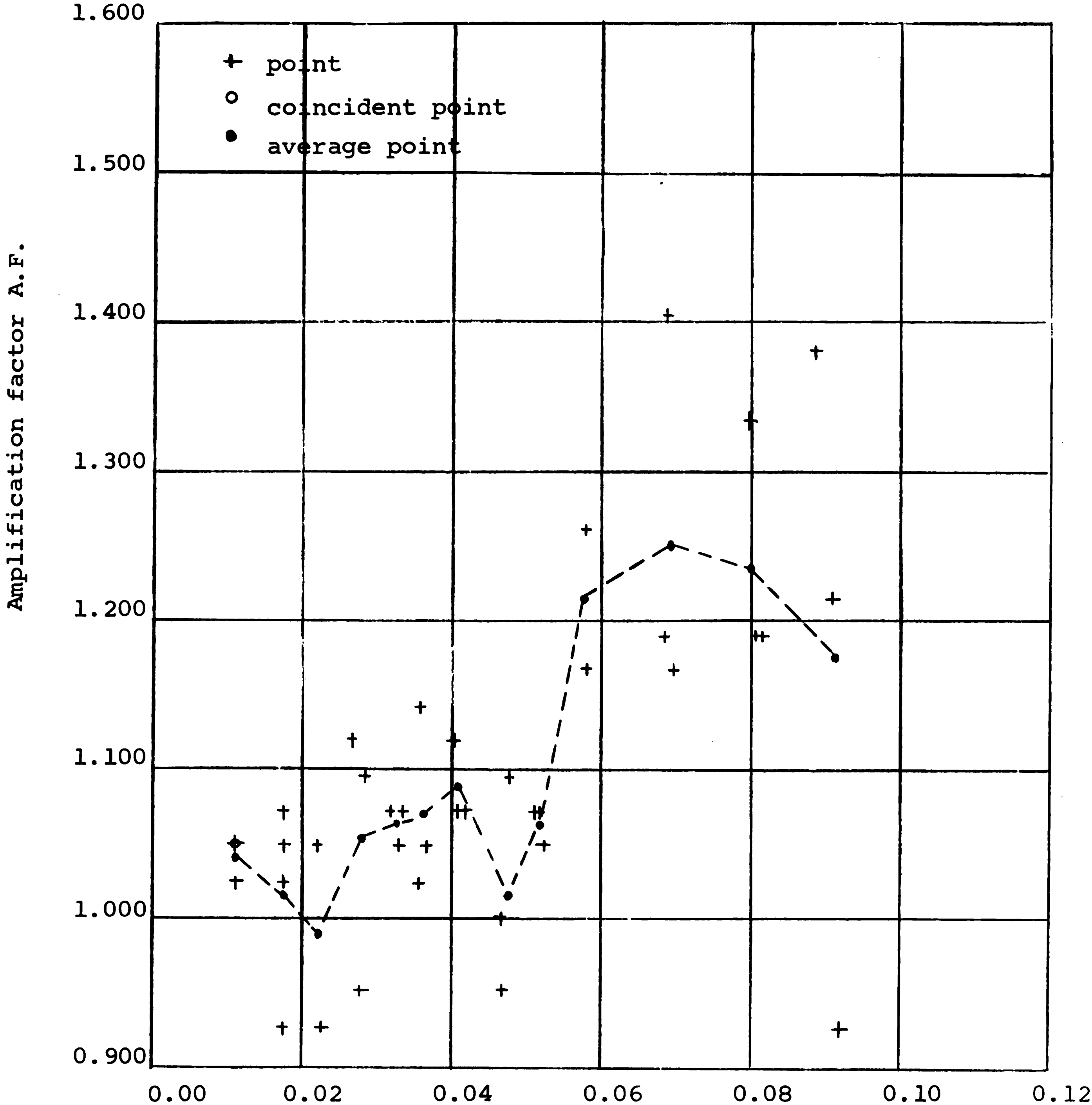

Fig. 8--Spectrum curve for far span

1/2 lb. load, $\bar{m}_1 = 0.0277$

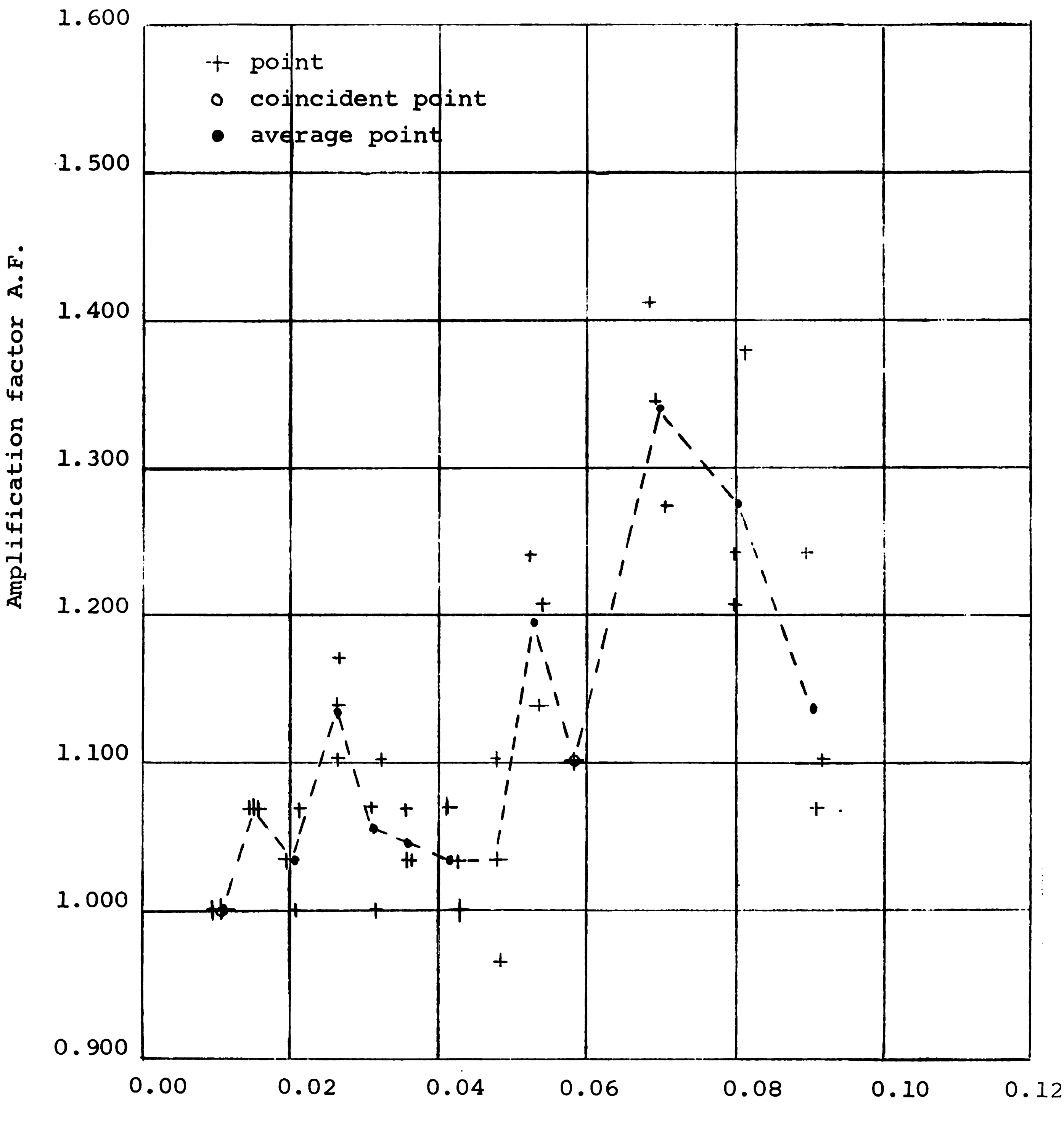

Fig. 9--Spectrum curve for far span

1 lb. load, $\bar{m}_1 = 0.0652$

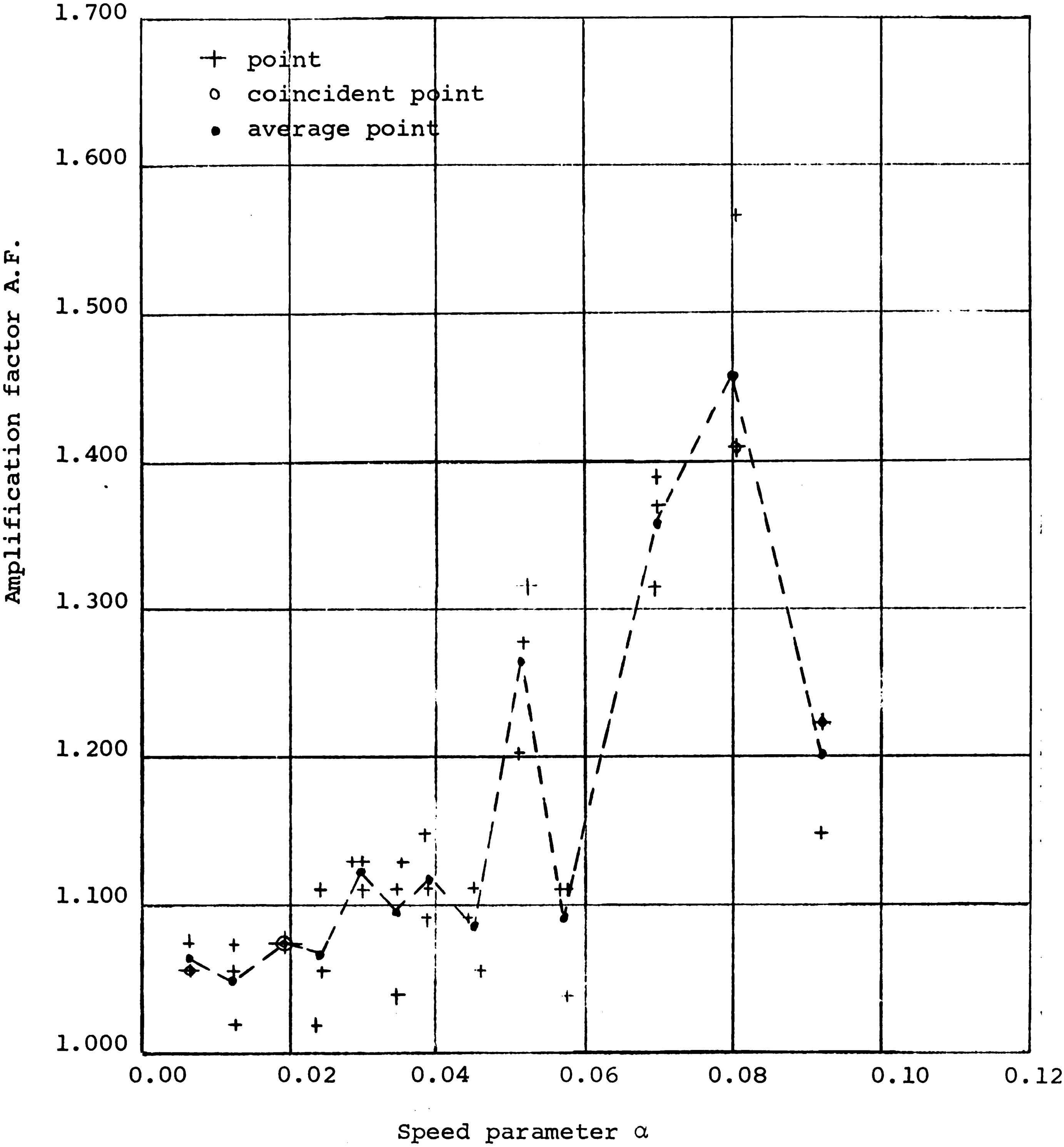

Fig. 10--Spectrum curve for far span

2 lb. load, $\bar{m}_1 = 0.1092$

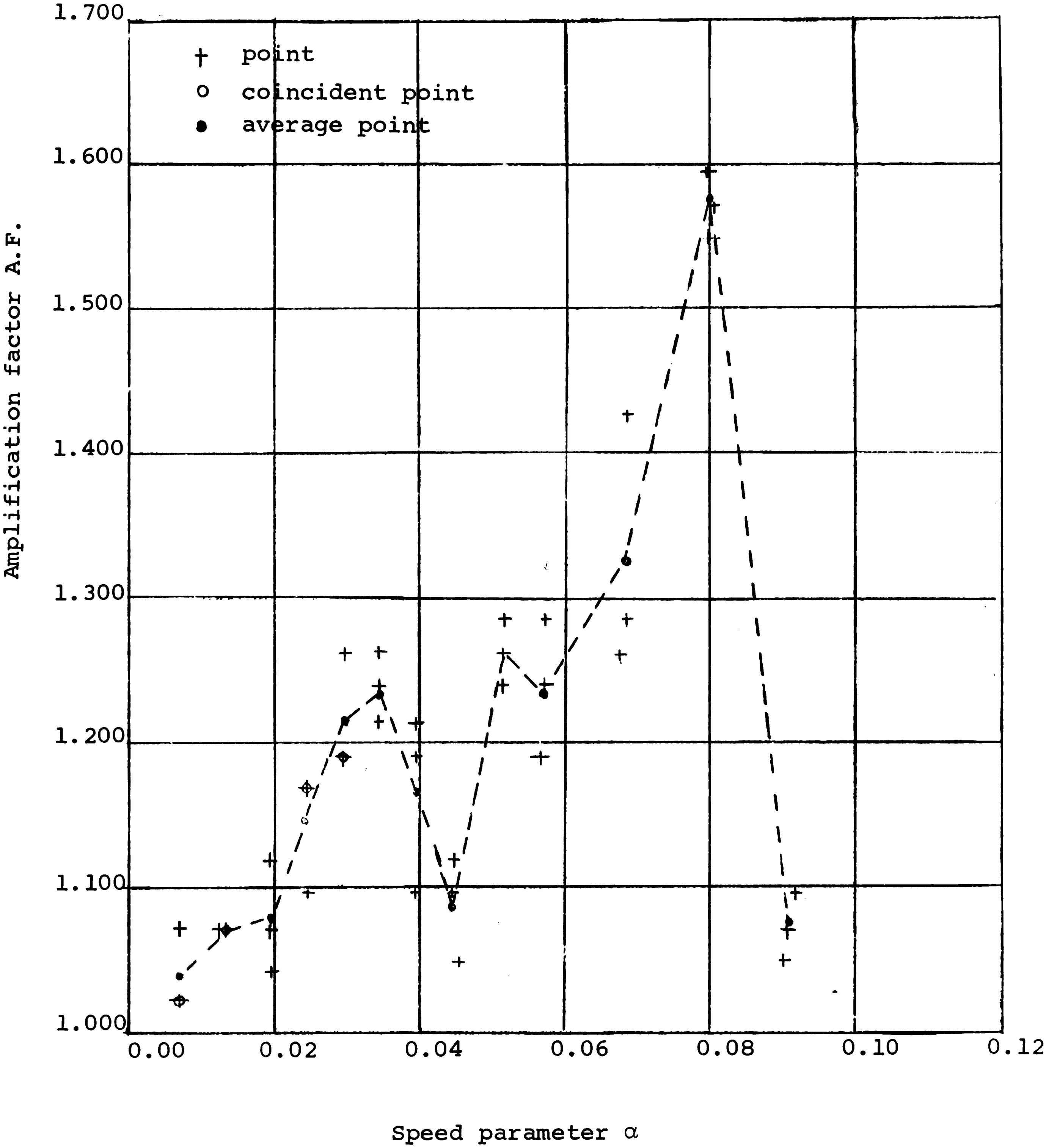

Fig. 11--Spectrum curve for far span

3 lb. load, $\bar{m}_1 = 0.1692$

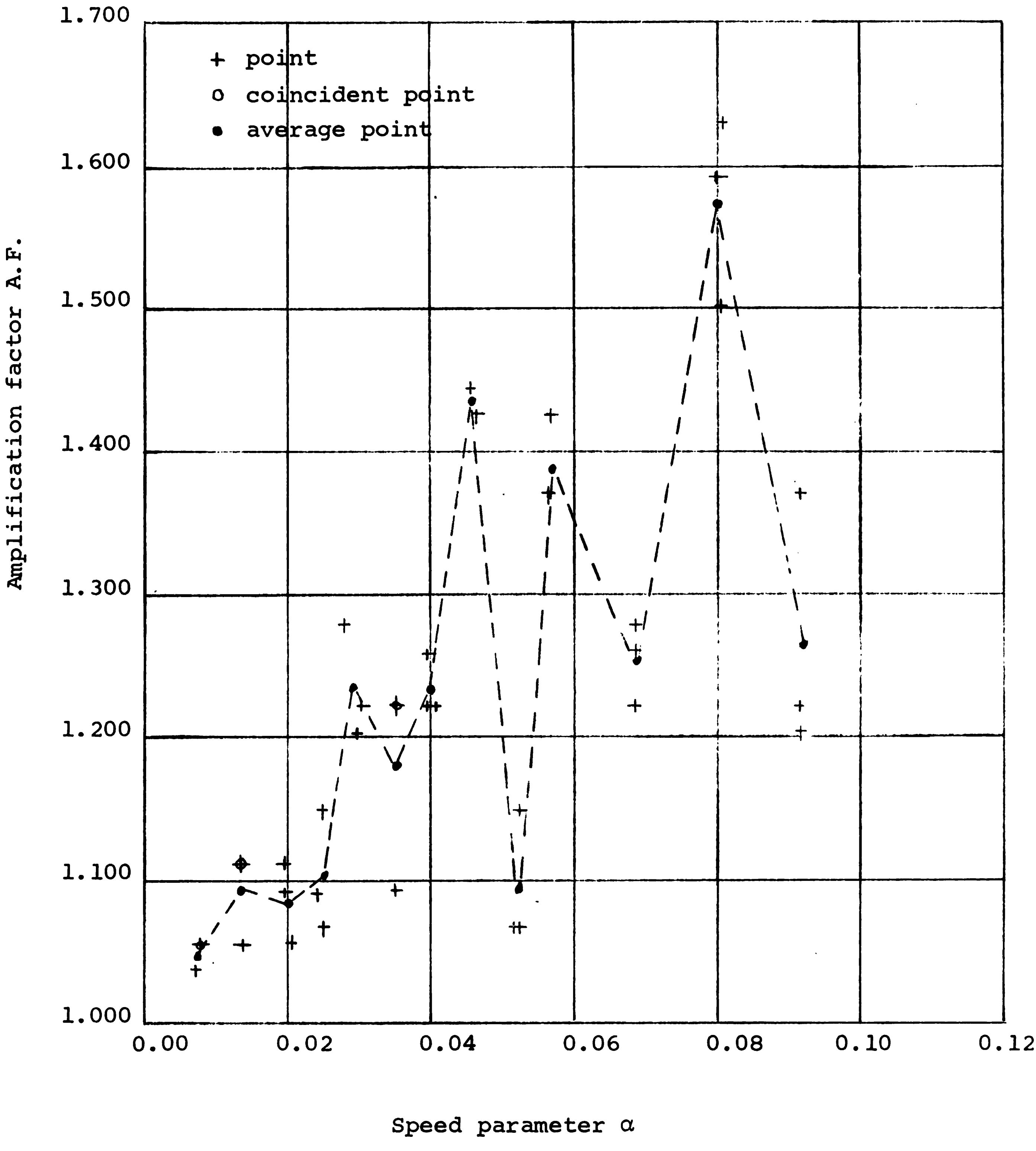

Fig. 12--Spectrum curve for far span

4 lb. load, $\bar{m}_1 = 0.2199$

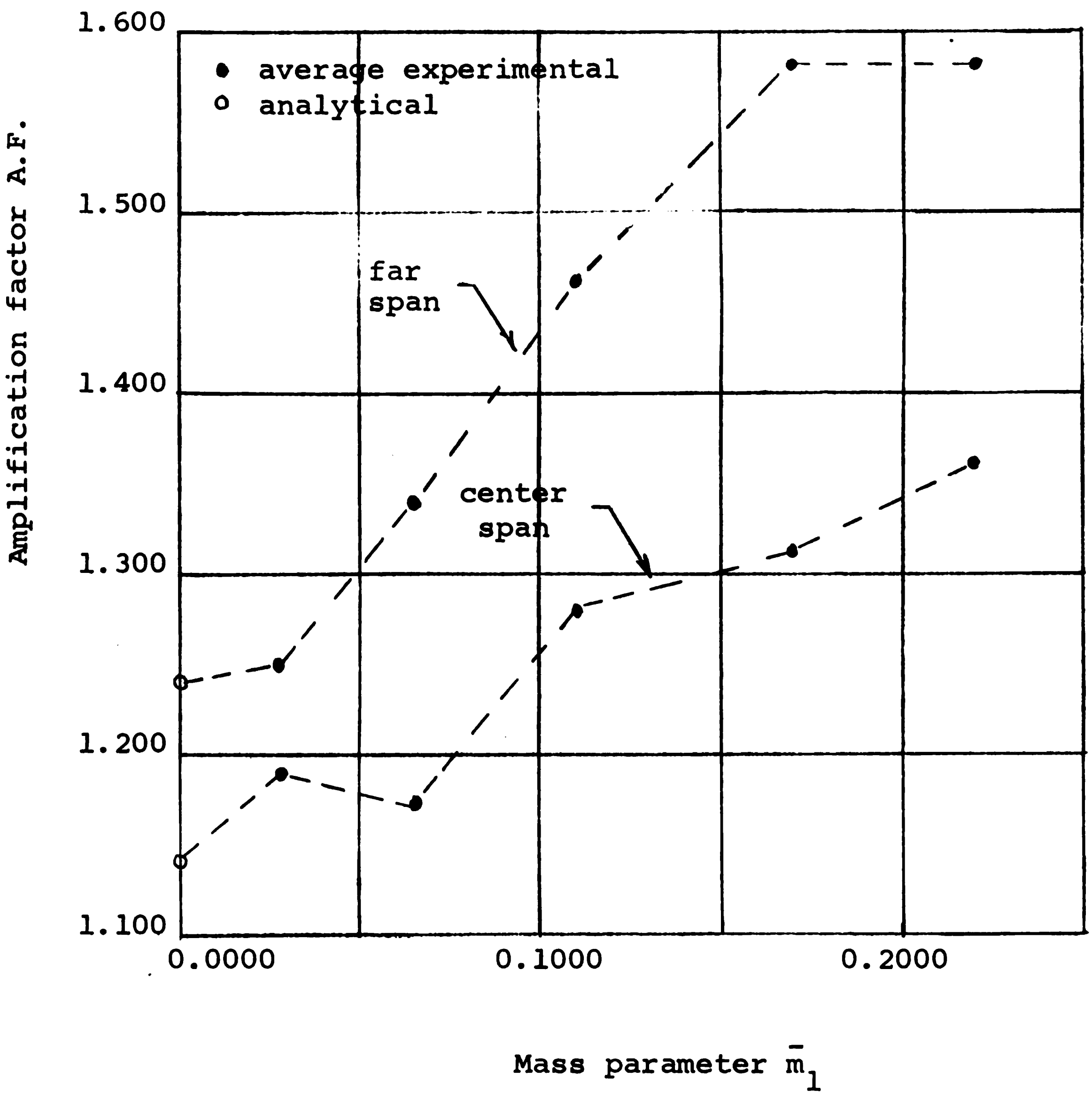

Fig. 13--Comparison of highest average maximum amplification factors

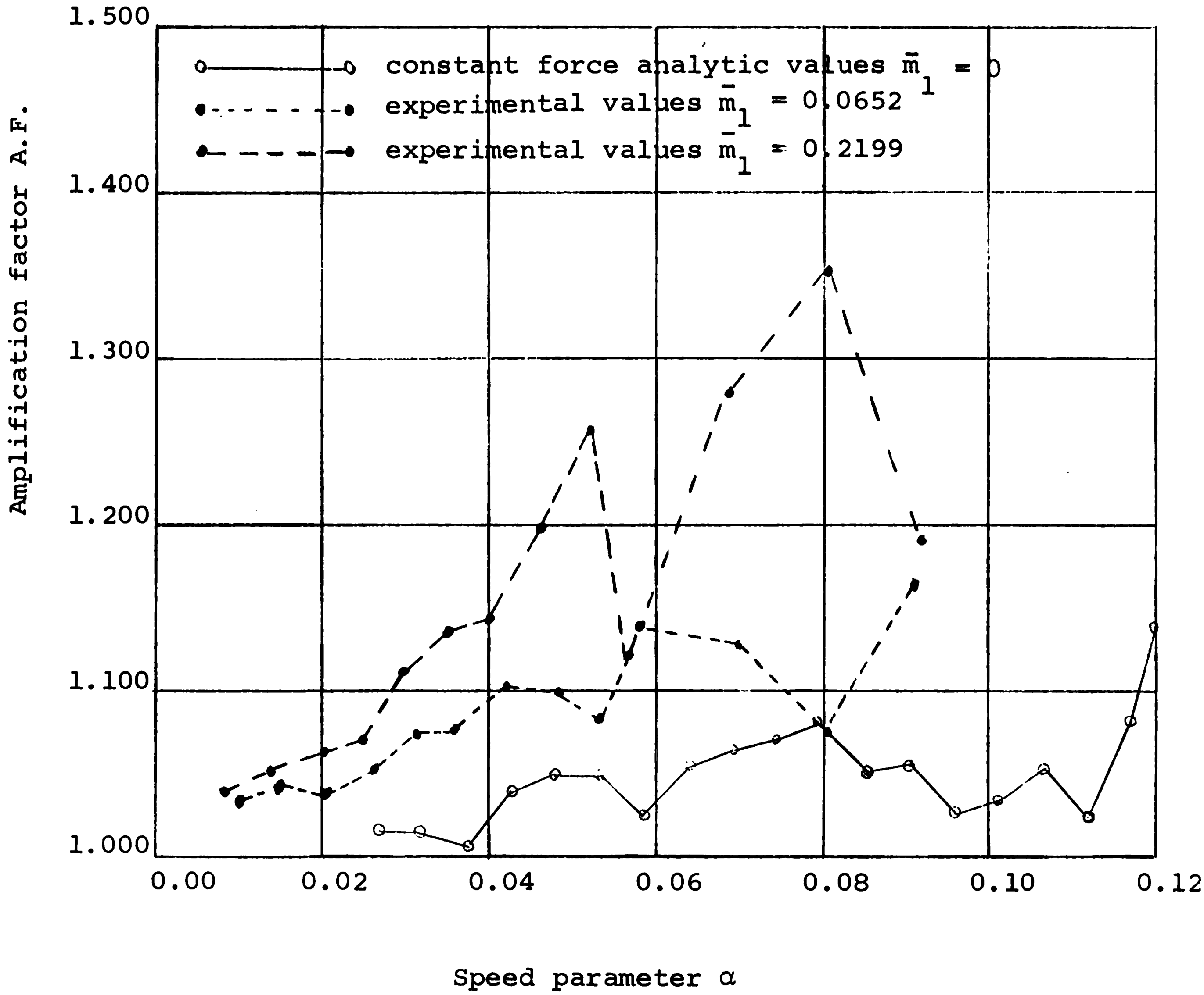

Fig. 14--Comparison of spectrum curves

for center span

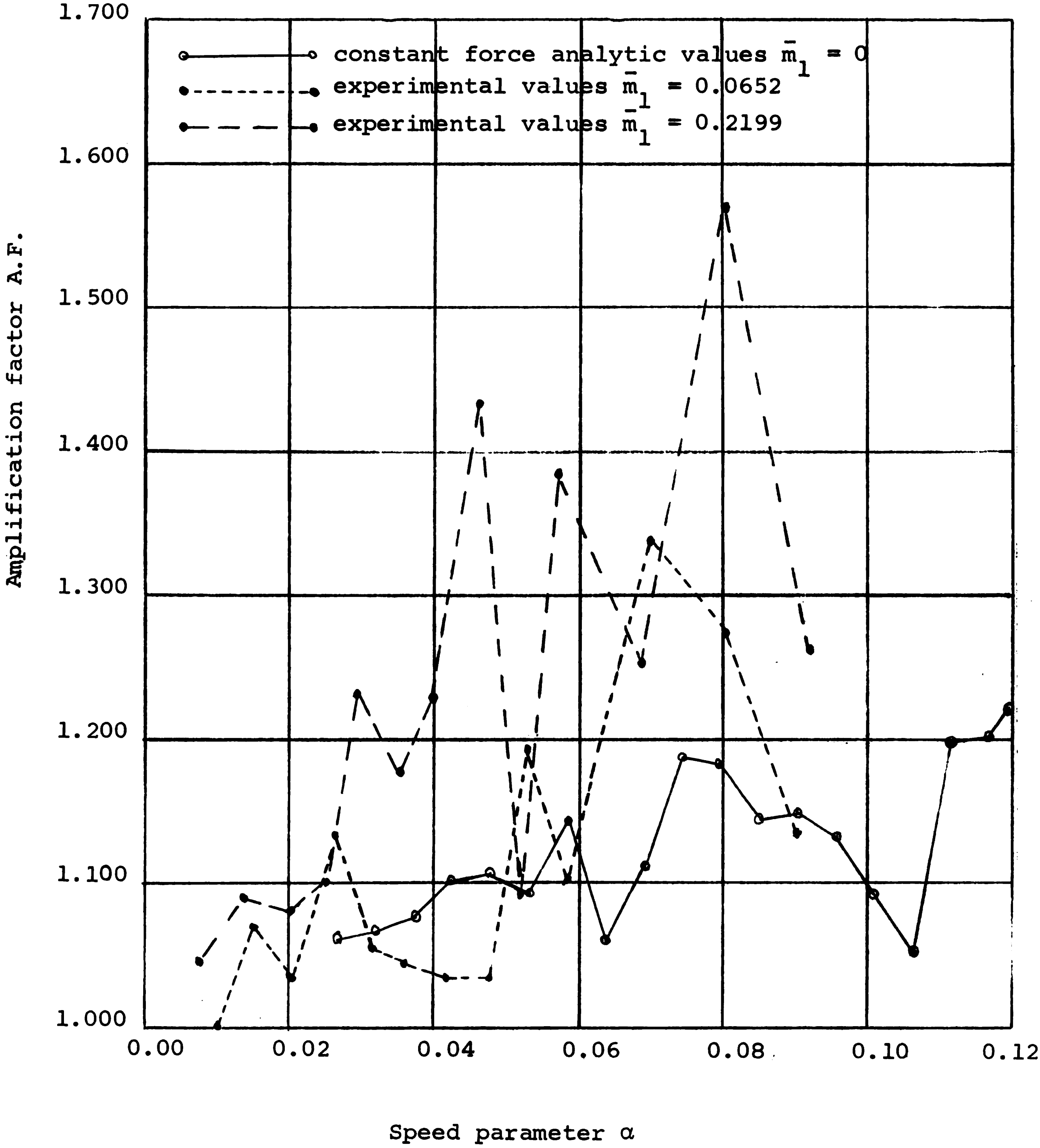

Fig. 15--Comparison of spectrum curves

for far span

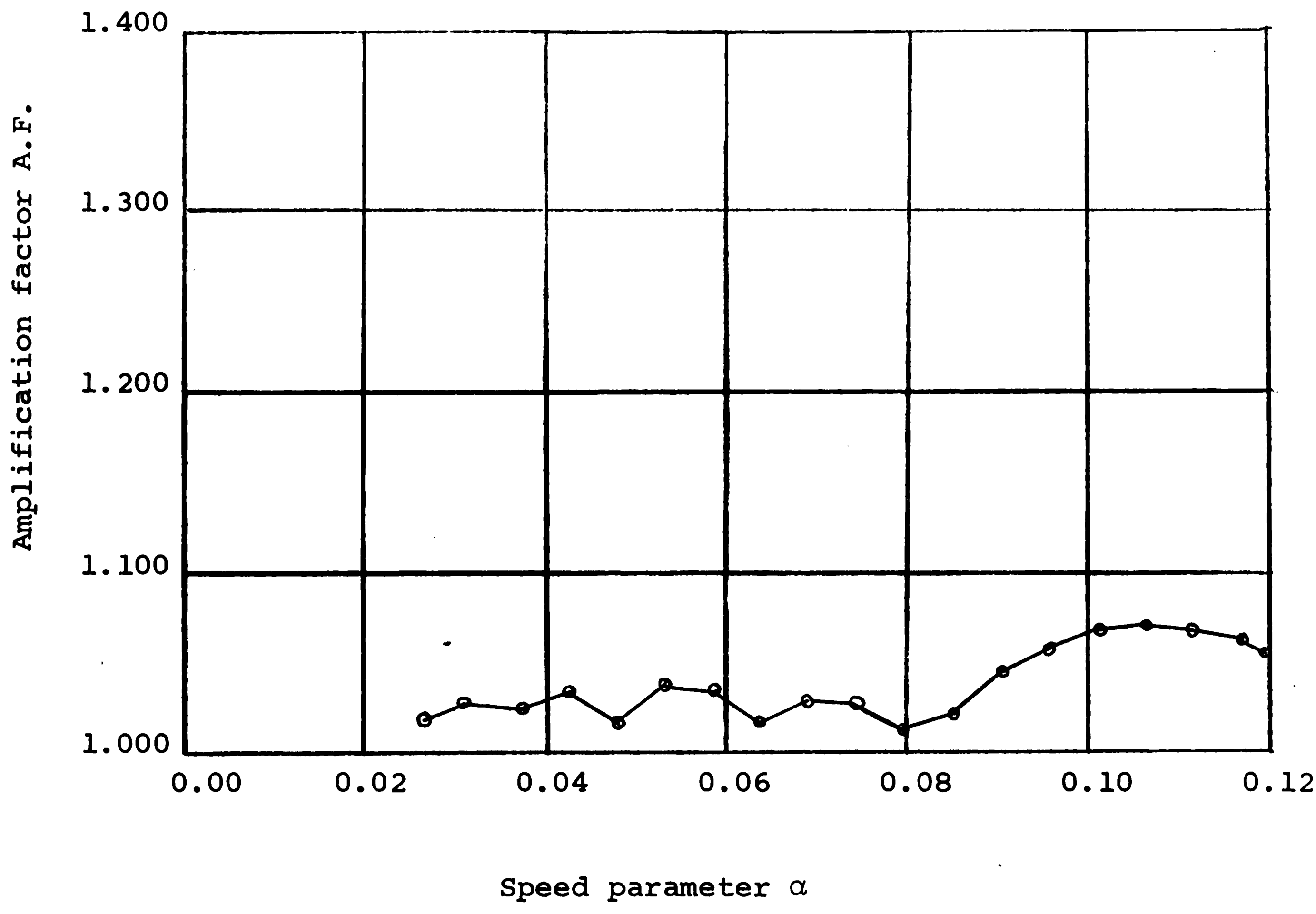

Fig. 16--Constant force analytic

spectrum curve for near span

$\bar{m}_1 = 0$