ALLOWABLE BEAM DEFLECTION AS LIMITED BY PLASTER STRAIN

Thesis for the Degree of M. S.

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Florence Helen Dyer

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This is to certify that the

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FLORENCE DYER

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Charles O. Harris
Major professor

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ALLOWABLE BEAM DEFLECTION AS LIMITED BY PLASTER STRAIN

By

Florence Helen Dyer

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INTRODUCTION

The object of this thesis is to determine a criterion for the calculation of the maximum value of beam deflection due to live load which will not cause the plaster strain in tension, in an associated ceiling or wall panel, to exceed a maximum value beyond which plaster failure would result.

In the fifth edition of the Steel Construction Manual of AISC (3) is the existing specification:

> "Beams and girders supporting plastered ceilings shall if practicable be so proportioned that the maximum live load deflection will not exceed 1/360 of the span."

It seems that this blanket rule of thumb might be refined for designs of greater precision if the value for the limiting strain were determined for various plasters; and if the relation between strain and deflection were stated mathematically for representative conditions of loading.

As far as could be determined by reference to the Industrial Arts Index and the Engineering Index no articles are available on plaster strain caused by beam deflection.

Plaster

The plasters used were those packaged by the U. S. Gypsum Company. Several types and mixtures were tested; basically they all contain calcined gypsum. Plaster of paris is the pure form ($2CaSO_4$ plus H_2O). Keenes cement, a hard finish plaster, is made by mixing alum with the calcined gypsum and recalcining.

The finishing lime plaster is made with hydrated lime, $Ca(OH)_2$, slaked and mixed with guaging plaster. In this case the development of strength is a progressive process depending on the formation of $CaCO_3$ by CO_2 in the air.

THEORY

Plastered Beam

The maximum deflection (3) in a beam of length 'l', under uniform loading is:

$$(1) \qquad \delta = \frac{5}{384}\,\frac{wl^4}{EI}$$

where 'δ' is the deflection; 'w' is the load per unit of length; 'E' is the modulus of elasticity, a constant for the material of the beam within the proportional limit; and 'I' is the moment of inertia about the horizontal axis of symmetry of the section called the neutral axis. The moment of inertia (4) in Figure 1 is:

$$(2) \qquad I = \frac{bd^3}{12}$$

The maximum stress (4) in a homogeneous beam, as in Figure 1, is:

$$(3) \qquad \sigma = \frac{Mc}{I} \quad , \quad \text{the flexure formula,}$$

where 'σ' is the unit stress; 'M' is the maximum moment; and 'c' is the distance from the neutral axis to the extreme fiber.

The amount of stress in any fiber is proportional to the distance from the neutral axis as shown by the arrows in Figure 2. Therefore the maximum stress occurs in the extreme fiber. By Hookes Law (4), strain is directly proportional to stress:

$$(4) \qquad E = \frac{\sigma}{\epsilon} \quad \text{or} \quad \epsilon = \frac{\sigma}{E}$$

or combined with equation (3)

$$(5) \qquad \epsilon_{max} = \frac{M_{max}c}{EI}$$

Thus the maximum strain denoted by 'ϵ' also occurs in the extreme fiber. In the case of a plastered beam as shown in Figure 3, it is necessary to consider the two materials acting together as an equivalent homogeneous section (Figure 4), where the ratio of the widths b_1/b is directly proportional to the ratio of the modulus of elasticity of plaster to the modulus of elasticity of the beam; the thickness 't' of the plaster remains unchanged.

So, $$\frac{b_1}{b} = \frac{E_p}{E_B} \qquad \text{or} \qquad b_1 = \frac{E_p\,b}{E_B} .$$

Let $$g = \frac{E_p}{E_B} \qquad \text{then} \qquad b_1 = g\,b .$$

It will be shown that the value of 'I' of the transformed section in Figure 4, differs from the value of 'I' of the beam section in Figure 1, but that the difference is very small and may be disregarded. Also, an exact value for the distance from the neutral axis to the extreme fiber '$\bar{y}$' will be found. However, the value of '$\bar{y}$' differs so slightly from the value of $d/2 + t$ (the 'c' distance in Figure 3.) that this difference may also be disregarded.

The distance from the neutral axis to the extreme fiber of the transformed section in Figure 4, is found by balancing the moments of the areas about the neutral axis:

$$bd\left(\frac{d}{2} + t - \bar{y}\right) = b_1 t\left(\bar{y} - \frac{t}{2}\right)$$

$$\frac{bd^2}{2} + bdt - bd\bar{y} = b_1 t\bar{y} - \frac{b_1 t^2}{2}$$

$$(b_1 t + bd)\,\bar{y} = \frac{bd^2}{2} + bdt + \frac{b_1 t^2}{2}$$

$$\bar{y} = \frac{\frac{bd^2}{2} + bdt + \frac{b_1 t^2}{2}}{b_1 t + bd} .$$

substitute: $b_1 = bg$, where $g = \frac{E_p}{E_B}$

$t = dk$, where $k = t/d$

$$\bar{y} = \frac{\frac{bd^2}{2} + bkd^2 + \frac{bgk^2 d^2}{2}}{bgkd + bd}$$

$$\bar{y} = \frac{\frac{d}{2}\,(1 + 2k + gk^2)}{1 + gk} .$$

The term gk^2 in the numerator may be neglected since it is very small compared to the other terms, so:

$$\bar{y} = \frac{d/2\ (1 + 2t/d + \cdots}{1 + \frac{E_p}{E_B} \times \frac{t}{d}} ;$$

This is the distance from the neutral axis of the transformed section to the extreme fiber, that is:

$$c = \frac{(d/2 + t)}{1 + t/d\ \frac{E_p}{E_B}} ,$$

and neglecting the term $t/d\ \frac{E_p}{E_B}$ in the demonimator

(6) $\quad c = \frac{d}{z} + t$

The value of $t/d \frac{Ep}{E_B}$ will be very small. The modulus of elasticity of plaster will nearly always be much smaller than the modulus of elasticity of the beam material. $\frac{1}{10}$ would be a large value for t/d occurring when 1/2" of plaster is applied to a 5" beam.

The moment of inertia of the transformed section about its centroidal axis is found by using the parallel axis theorem (4).

$$\bar{I} = \frac{bd^3}{12} + bd\left[d/2 + kd - \bar{y}\right]^2 + \underbrace{\frac{bgk^3d^3}{12}}_{\text{very small}} + bgkd\left[\bar{y} - \frac{kd}{2}\right]^2$$

$$= \frac{bd^3}{12} + bd\left[\left(\frac{d}{2} + kd\right)\left(1 - \frac{1}{1-gk}\right)\right]^2 + \ldots + bd.gk\left[\frac{d/2 + kd}{1 + gk} - \frac{kd}{2}\right]^2$$

$$= \frac{bd^3}{12} + \frac{bd^3}{12} \cdot 3\,\frac{(1-2k)(gk)^2}{(1+gk)^2} + \frac{bd^3}{12} \cdot 3gk\,\frac{(1+k - \underbrace{gk^2}_{\text{very small}})^2}{(1+gk)^2}$$

$$= \frac{bd^3}{12} + \frac{bd^3}{12} \cdot \underbrace{\frac{3\,g^2k^2(1 + \ldots)}{1 + 2\,gk + \ldots}}_{\text{very small}} + \frac{bd^3}{12} \cdot \frac{3gk\,(1 + 2gk^2 + \ldots}{1 + 2gk + \ldots}$$

$$= \frac{bd^3}{12}\left[1 + \frac{3gk}{1 + 2gk}\right].$$

Thus, the moment of inertia of the transformed section is

(7) $$I = \frac{bd^3}{12}\left[1 + \frac{3t/d\,\frac{Ep}{E_B}}{1 + 2t/d\,\frac{Ep}{E_B}}\right],$$

and disregarding the effect of plaster

$$I = \frac{bd^3}{12}$$

When the value of 'Ep' approaches the value of 'E_B', the value of 'I' in equation (7) should be used. Since $\frac{Ep}{E_B}$ is small, it is assumed in the following computations that the moment of inertia of the transformed section is equal to the moment of inertia of the beam.

The tensile strain for the transformed section is found by combining equations (1),(5) and (6).

$$\frac{\epsilon_{max}}{\delta_{max}} = \frac{\frac{M_{max}\, C}{EI}}{5/384 \cdot \frac{Wl^4}{EI}} = \frac{384}{5} \; \frac{M_{max}\,(d/2 + t)}{Wl^4}$$

The maximum value of moment (4) for a beam with uniform loading is:

(8) $$M_{max} = \frac{Wl^2}{8}$$

$$\text{So} \quad \epsilon_{max} = \delta \cdot \frac{384 \times Wl^2\,(d/2 + t)}{5 \times 8\; Wl^4}$$

$$= \frac{\delta}{l^2} \cdot \frac{384}{40} \cdot \frac{d}{2} \; (1 + 2\; t/d)$$

(9) $$\epsilon_{max} = \frac{\delta}{l} \cdot \frac{d}{l} \cdot \frac{384}{80} \; (1 + 2t/d) = \frac{\delta}{l} \cdot \frac{d}{l} \cdot \frac{24}{5} \, (1 + 2\; t/d)$$

Similarly for a beam under third point loading (4), (see Figure 11).

$$M_{max} = \frac{Pl}{3}$$

(10) $$\delta_{max} = \frac{23Pl^3}{648\; EI}$$

(11) $$\epsilon_{max} = \frac{\delta}{l} \cdot \frac{d}{l} \cdot \frac{108}{23} \quad (1+2t/d)$$

The family of curves in Figure 5 shows the manner in which δ/l varies for the different values of d/l and t/d when the maximum strain is taken as 0.0005.

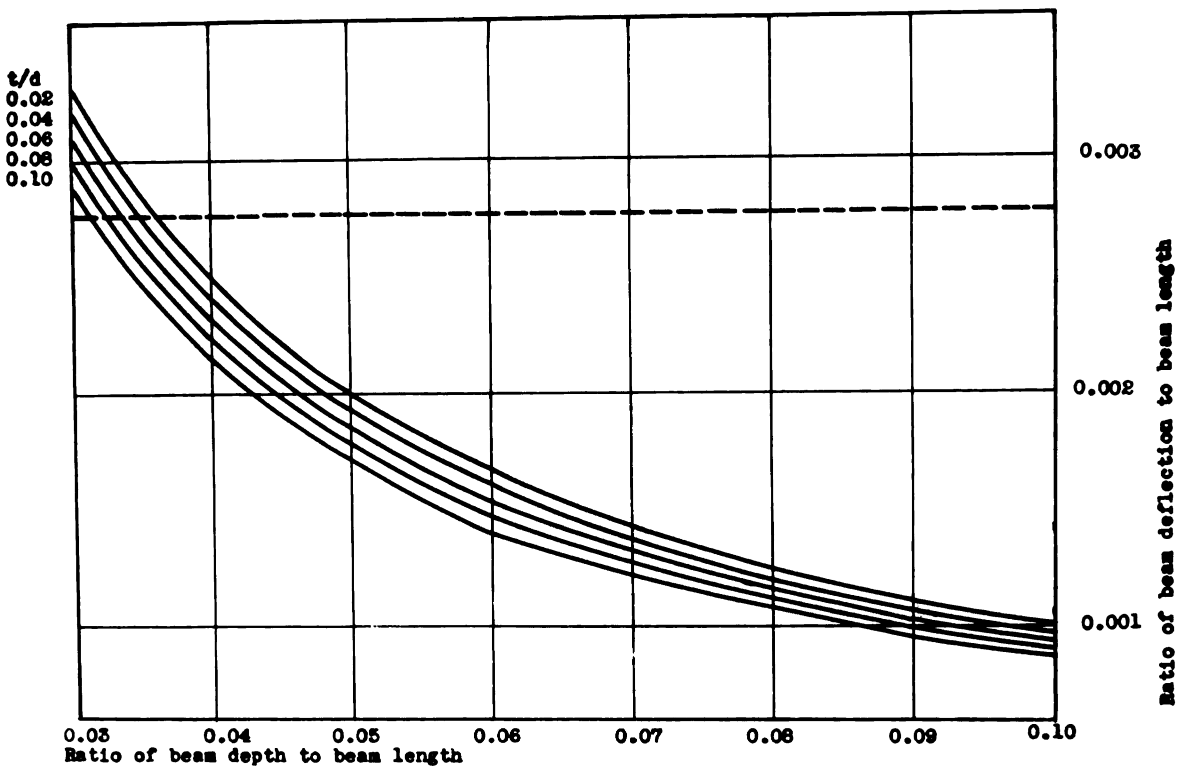

Figure 5 Relation of deflection-length to depth-length ratio for a range of t/d values. Plaster strain taken as: 0.0005

Plastered Wall Panel

When a plastered wall panel is supported as in Figure 6, the deflection of the supporting beam at 'A' results in distortion of the plastered wall panel. It is assumed that the support at 'B' in Figure 6 does not deflect. The relation between the deflection of the beam and the unit strain in the plaster is found as follows:

The unit strain, γ, in the plaster is

$$(12) \qquad \gamma = \frac{\delta}{L}$$

Where 'δ' is the deflection in the beam of 'A' in Figure 6 and 'L' is the length of the plaster panel. The shearing modulus of elasticity, 'G', is the ratio of unit shearing stress 'S_s' to unit strain, γ.

$$(13) \qquad G = \frac{S_s}{\gamma}, \text{ or}$$

$$G\gamma = S_s = \frac{G\delta}{L}$$

The panel is in a state of pure shear so the principle stresses are:

$$(14) \qquad S_c = S_t = S_s$$

as shown by the stresses on elemental areas in Figure 7 and Figure 8, or the maximum unit strain is, from Hookes law,

$$(15) \qquad \epsilon = \frac{S_t + \mu S_c}{E} = \frac{S_s\,(1 + \eta)}{E} = \frac{G\delta}{L}\frac{(1+\eta)}{E}$$

Where 'E' is the modulus of elasticity in tension and 'η' is Poissons ratio. Since

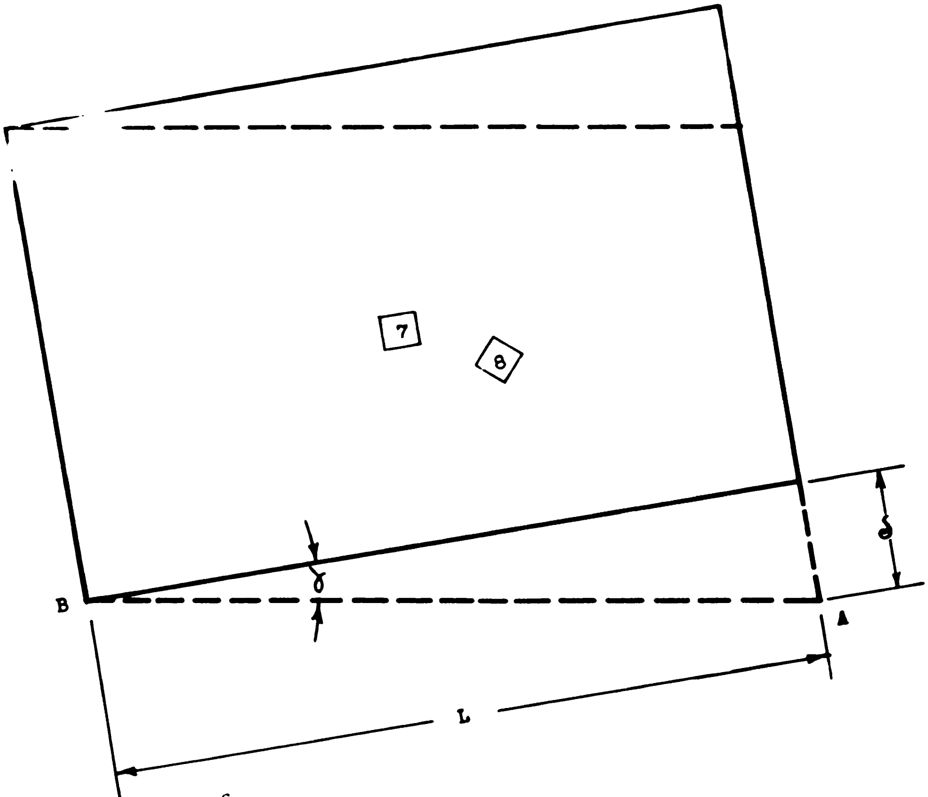

Figure 6

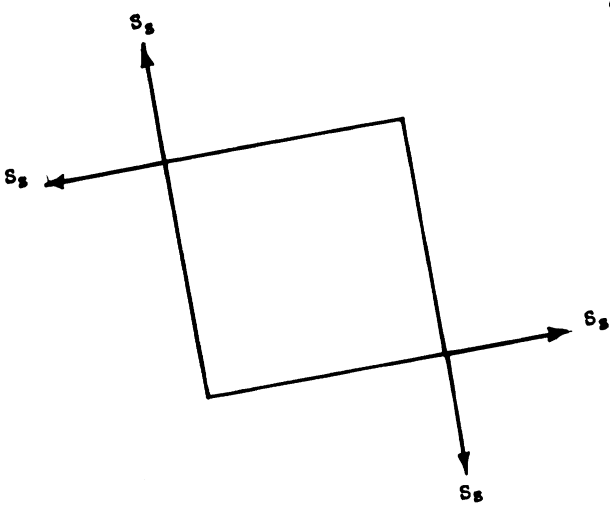

Figure 7

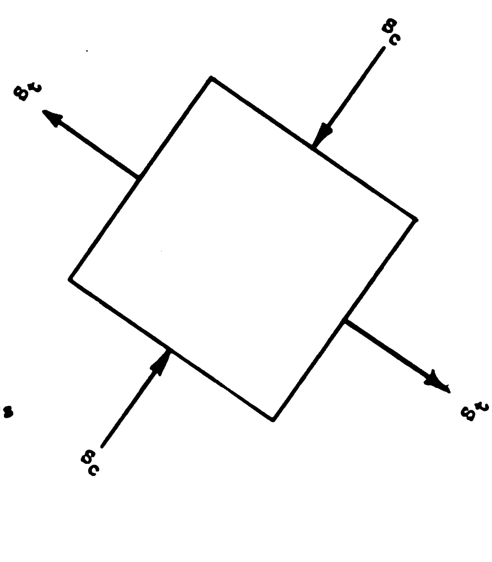

Figure 8

(16) $$E = 2(1 + \mu)\,G,$$

it follows that,

(17) $$\epsilon = \delta/2L$$

or $$\frac{\delta}{L} = 2\epsilon .$$

EXPERIMENT

The plaster test bars were made up in a form as shown in Figure 9. The nominal size of each bar was 1" x 3" x 24". Three sets of forms were used so that three similar bars could be made at one time.

Nine different mixtures of three bars each were made. The proportioning of the mixtures was that recommended by the Gypsum Company (5).

As soon as the plaster had set enough to be sufficiently firm the bars were removed from the forms and stored so that air could circulate freely on all sides for uniform drying.

Small holes on the surface of the bars were patched with plaster of paris to reduce the concentration of stress at those points.

When the bars were thoroughly dry, usually about one week, after forming, they were submitted to a bending test on the apparatus shown in Figure 10. The supports at 'A' and "A'" were placed 8' apart, or approximately at the third points of the bar. The load applied at 'B' was divided evenly between 'C' and 'C''. With this arrangement the bar was subjected to a constant moment between the two supports as shown in the diagram in Figure 11. It is believed (1) that the breaking stress in a test of this kind on brittle material is less than center loading due to the break always occurring at a weaker section. The weight of the bar was so small in comparison with the applied loads that the weight was neglected in the calculations of stress and strain.

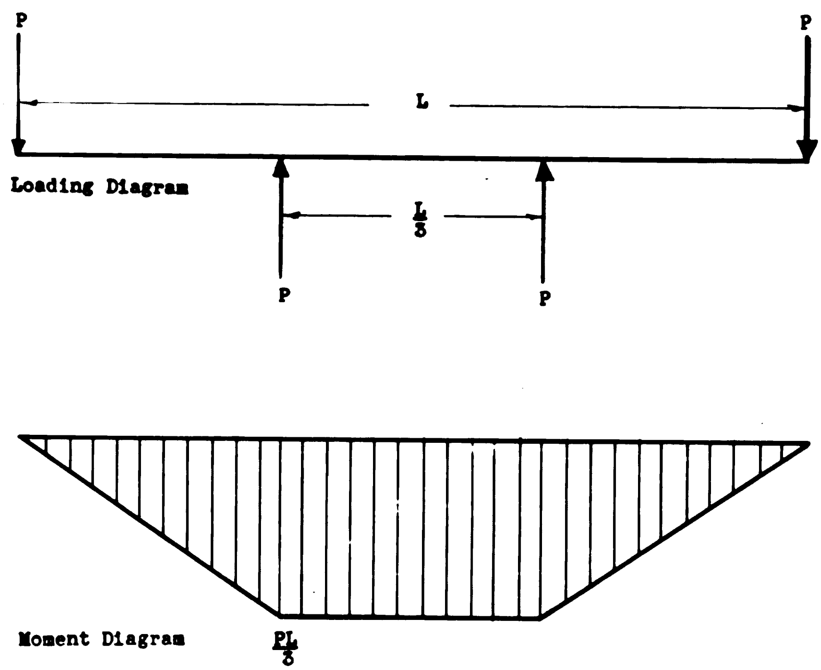

Figure 11

In the testing machine the loads were applied gradually and readings of deflection were made by means of dial guages, 'D' and 'D'' in Figure 10. From these values and the careful measurement of dimensions of each bar, values were calculated for the modulus of elasticity, maximum tensile stress, and maximum strain in each bar. Average values of these quantities are tabulated in Table I, and the stress strain curves are shown in Figure 12.

The modulus of elasticity was calculated as follows: The expression for maximum deflection under third point loading is;

$$\delta_{max} = \frac{23\ Pl^3}{648\ EI} \quad \text{or} \quad \frac{23Wl^3}{1296EI}$$

Where P is the load at one end and W is the total load applied at 'B' in Figure 10. From that equation:

$$E = \frac{23Wl^3}{1296\ \delta_{max}\ I}$$

The value of the length, 'l', was 23.8". δ_{max} was the sum of the two guage readings for any one load.

The moment of inertia, 'I', was calculated for each bar using:

$$I = \frac{bd^3}{12}$$

where 'b' was the width of the bar, nominally 3", and 'd' is the height of the bar, nominally 1". There were minor differences between the dimensions of different bars due to variable shrinkage in the plaster and perhaps some swelling of the wood in the forms.

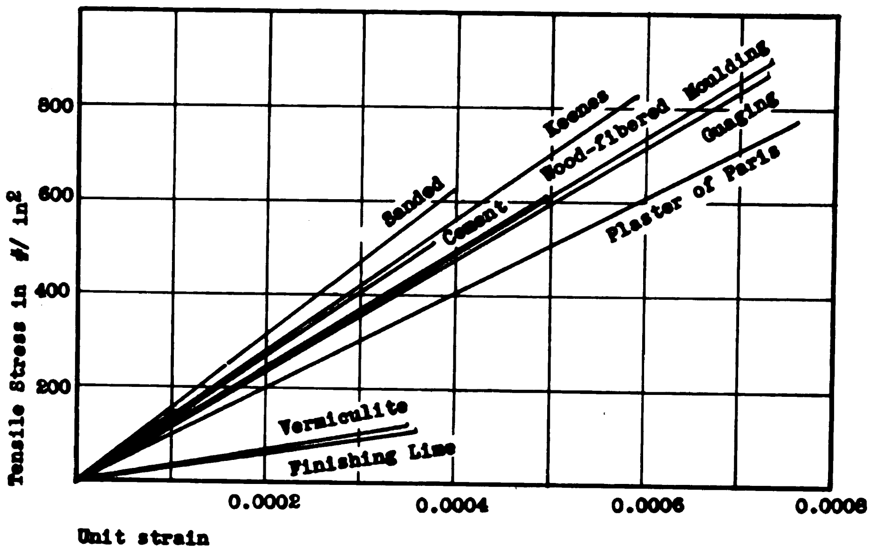

Figure 12 Average stress-strain relationships from bending tests on plaster bars

TABLE I

Mix	Average specific Wt.#/in^3	Average E x 10^b	Average Max. (tension)	Average Max. in	Average Max.
Cement plaster neat	0.0542	1.37	513	0.0435	0.000375
Cement plaster sand 1:2 by wt	0.0653	1.58	625	0.0446	0.000395
Finishing lime plaster	0.0373	0.33	118	0.0417	0.00359
Cement plaster vermiculite 1:1 by vol.	0.0325	0.37	129	0.0711	0.00350
Average for graph Figure 13					0.0037
Wood fibered plaster neat	0.0511	1.25	612	0.0542	0.000498
Keenes cement hard	0.0580	1.40	827	0.0637	0.000590
Average for graph Figure 13					0.000544
Plaster of paris	0.0475	1.02	777	0.0887	0.000760
Guaging plaster	0.0528	1.19	885	0.0868	0.000728
Moulding plaster	0.0479	1.24	905	0.0830	0.000731
Average for graph Figure 13					0.00074

Tensile stress was calculated from the flexure formula, $\sigma = \frac{Mc}{I}$, where 'M' was the maximum moment due to the given load, 'W':

$$M = \frac{W}{2} \times 7.9 \quad \text{or} \quad M = W \cdot 3.95$$

$$c = d/2$$

and 'I' is the same value as before a constant for each bar. The value of strain was the ratio of the stress to the modulus of elasticity.

$$\epsilon = \frac{\sigma}{E}$$

DISCUSSION

The family of curves in Figure 5 represents the variation in the limiting allowable ratio of beam deflection to beam length, δ/l, as the depth to length ratio of the beam, d/l, changes when the plaster strain is assumed to be 0.0005. The third variable is the ratio of plaster thickness to the beam depth, t/d. Each of the five curves represents a different value of that ratio. The dashed line is, $\delta/l = \frac{1}{360}$, or the specified maximum value of δ/l.

The graph in Figure 5 shows that the depth to length ratio is equally as important as the deflection to length ratio, because as the 'd/l' ratio increases the limiting allowable 'δ/l' decreases.

From the bending tests conducted on the plaster bars the average maximum values of strain were calculated (see Table I).

Using these values the three curves in Figure 13 were plotted. The t/d ratio is taken as 0.06 because that is the average t/d ratio used previously and the variation between the different values of t/d was small (Figure 5). The values of ϵ max are those found experimentally. Of the nine types of plaster tested there seemed to be three groupings for ϵ, shown averaged in Table I. These values were low compared with a value of 0.0013 for plaster of paris in <u>Properties of Engineering Material</u>, (2).

It is evident from these two graphs Figure 5 and Figure 13, that the limiting value of 'δ/l' is dependent upon the value of d/l and the value of maximum plaster strain; and is affected, though in a lesser degree, by the thickness of the plaster.

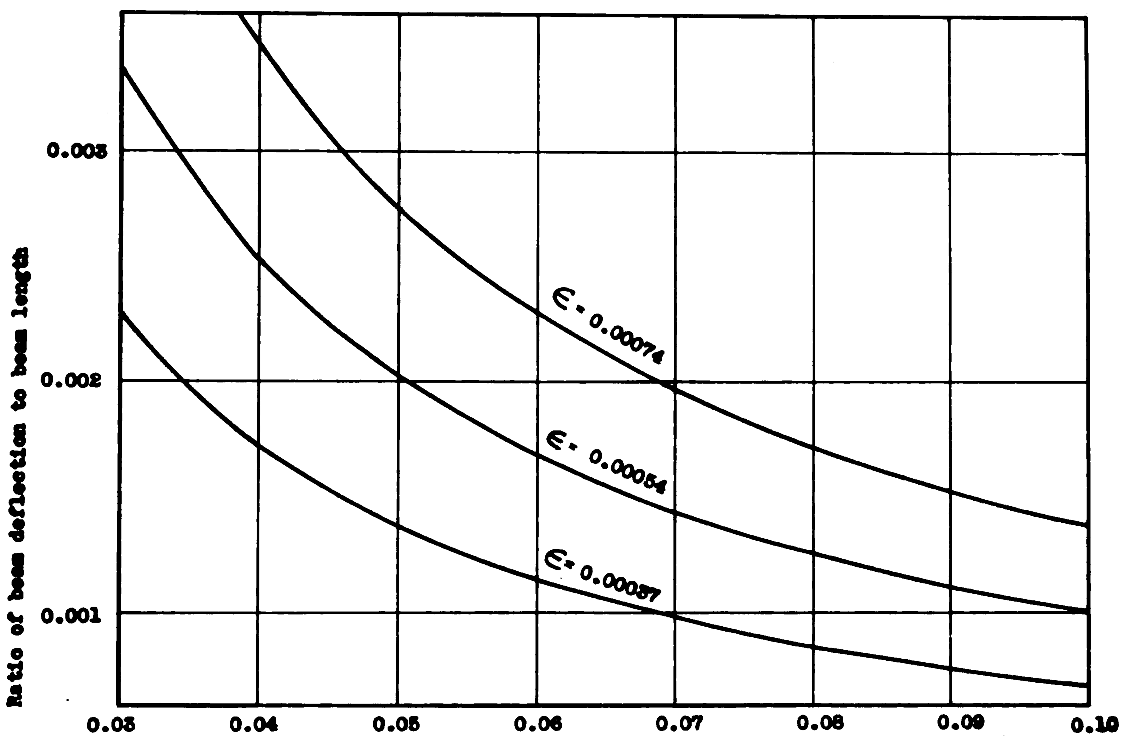

Figure 13 Relation of deflection-length to depth-length ratio for experimental values of plaster strain. t/d taken as: 0.06.

In equation (9) where

$$\epsilon_{max} = \frac{\delta_{max}}{l} \frac{d}{l} \times \frac{24}{5} (1 + 2\ t/d)$$

it is seen that the maximum allowable deflection is inversely proportional to the maximum value of plaster strain and to the depth of the beam and directly proportional to the square of the length; thus,

$$\delta_{max} = k \frac{l^2}{\epsilon_{max}\ d}$$

where $$k = \frac{5}{24(1 + 2\ t/d)}$$

The maximum allowable deflection to length ratio, δ/l, may be found by using the graph in Figure 14, if the value of maximum plaster strain is known and the depth to length ratio, d/l, of the beam may be calculated. The value for 'd/l' is found on the abscissa and its ordinate is followed until it meets the curve of the desired plaster strain. The vertical coordinate of this point is the maximum allowable deflection to length ratio for that beam.

Comparing the δ/l values found by using equation (9) with the allowable maximum ratio of $\frac{1}{360}$ for average values of plaster strain it is seen that ' $\frac{\delta}{l} = \frac{1}{360}$ ' exceeds the allowable maximum of equation (9) as values of d/l increase; though for the lower values of d/l the value of δ/l might be considerably greater than ' $\frac{\delta}{l} = \frac{1}{360}$ '

Though a δ/l of $\frac{1}{360}$ has been considered "safe", actually the maximum live load deflection for which the beam is designed may

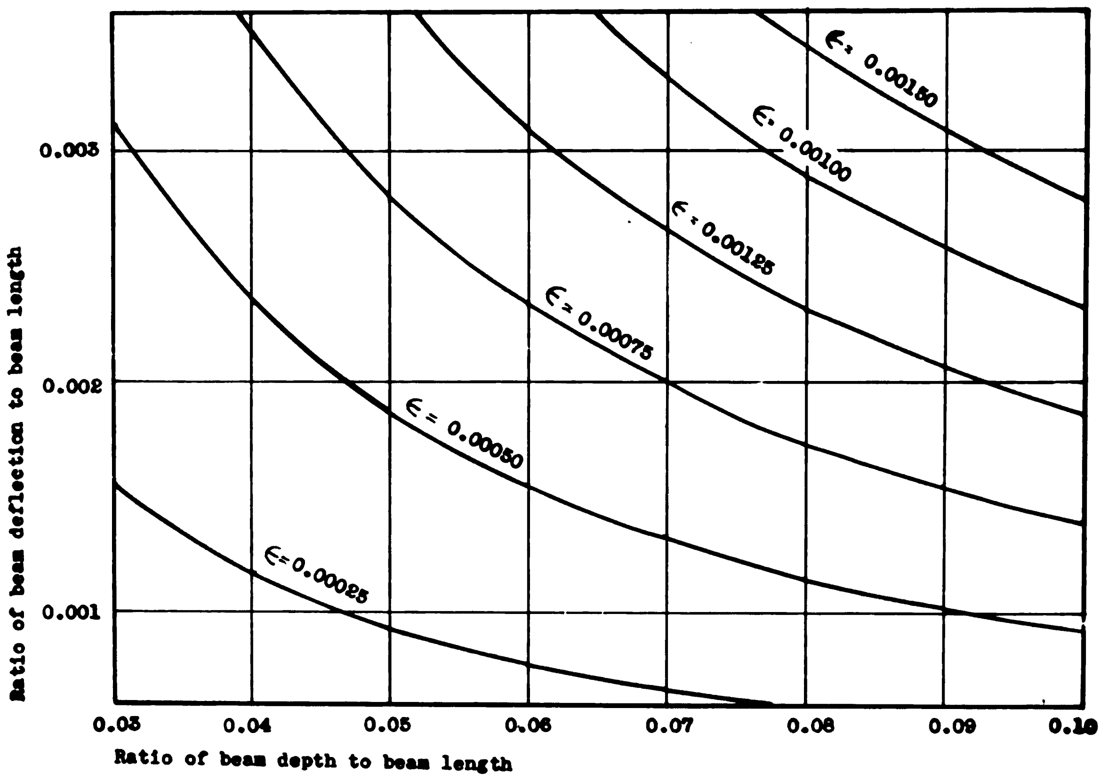

Figure 14 Relation of deflection-length to depth-length ratio for a range of plaster strain values. t/d taken as: 0.06.

never be attained; also the action of the lath, especially metal lath may strengthen the beam and so reduce deflection; and in the plastered ceiling, failure in the plaster may produce a fine crack or series of cracks which does not constitute failure of the ceiling because they are so small.

Equation (17), $\delta_{max} = 2\epsilon_{max} L$ shows that in the case of a plaster panel the actual amount of the deflection of the supporting beam is critical when the ratio of the deflection to the length of the plastered panel equals twice the maximum allowable strain in the plaster.

CONCLUSIONS

The foregoing tests and discussion seem to support the following conclusions:

(1) The limiting value of deflection in a beam under a uniform live load which supports a plastered ceiling is:

$$\delta_{max} = k \frac{l^2}{\epsilon_{max}\, d}$$

Where 'l' is the length of the beam; 'd' is the depth of the beam, ϵ_{max} is the maximum unit strain in the plastered ceiling; and 'k' $= \frac{5}{24(1 + 2\, t/d)}$ or $\approx$.186 when t/d is taken as .06.

(2) The limiting value of deflection in beam which supports on end of a plaster panel at its midpoint when the other support of the panel is fixed is:

$$\delta_{max} = 2\, \epsilon_{max}\, L$$

Where 'L' is the length of the panel, 'ϵ_{max}' is the maximum unit strain in the plastered panel.

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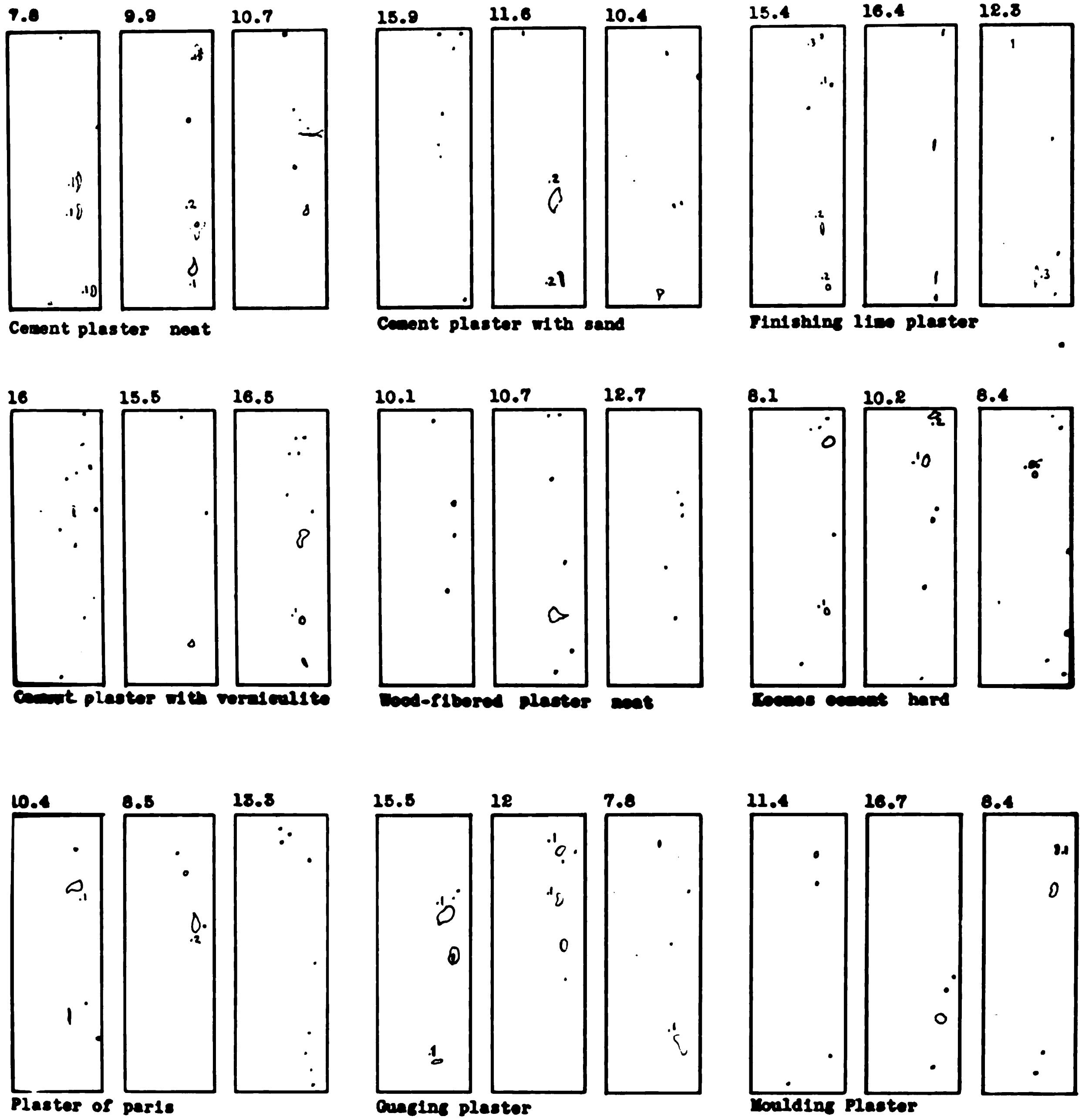

Right side of bar is tensile side.

Number above each cross-section is the distance of the break from the end of the bar in inches (distance from right-hand end facing bar in test).

Numbers on the faces of the cross-sections represent the aproximate values of the depths of the stress-raisers outlined nearby.

Stress-raisers in the compression side were not outlined.

APPENDIX Figure 15