FUNCTIONS OF TWO COMPLEX VARIABLES

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FUNCTIONS OF TWO COMPLEX VARIABLES

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CONTENTS

INTRODUCTION

The theory of functions of several complex variables has been the subject of study by mathematicians for approximately the last fifty years. Publications in the field began to make an appearance at about the beginning of this century. However, most of the work in functions of several complex variables has been done since 1925, making it one of the newest fields in mathematics. Although this subject has never enjoyed the tremendous popularity attained by some of the other modern trends in mathematics, much has been accomplished in the field, mainly by men in Germany and Italy. Among those mathematicians who have made noteworthy contributions to the study of functions of several complex variables, mention might be made of the Germans, H. Behnke, and his pupil, P. Thullen, Stefan Bergmann, F. Hartogs, H. Kneser, and P.J.Myerberg, and the Italians, E. E. Levi, and F. Severi, and a Frenchman, H. Cartan.

In general the investigations into the field of functions of several complex variables have been made along the same lines which were followed in developing the theory for functions of one complex variable. That is to say, in the former as in the latter, studies have been made of analyticity, continuity, the Cauchy integral, power series expansions, singularities, zeros, and the like. However, the results have been varied. In some cases, theorems are transferable almost word for word from one field to the other, while in other cases the differences are so marked as to be rather astonishing.

The purpose of the following discussion is quite simple. We shall limit our investigations to functions of two complex variables. This will simplify the statement of theorems without too great a loss in generality, since almost without exception, theorems which are proved for functions of two complex variables can be generalized to the case of *n* complex variables. Our purpose will be to contrast or compare some of the differences and similarities in theorems as they are stated for functions of one complex variable with the corresponding theorems for the case of two complex variables. In a few instances, a proof will be given; more frequently examples will be used. However, the main object of our discussions will be to point out the differences or similarities between corresponding theorems for the two cases.

In conclusion it might be stated that this work uses few theorems that have not been found in published form. An attempt will be made to give proper references. Mention might also be made of the material to be found on this subject. For a very complete bibliography of material published prior to 1933, the reader is referred to Behnke, H. and Thullen, P., *Theorie der Funktionen Mehrerer Komplexer Veranderlichen*, pp. 109-113. A few more recent references are listed at the end of this paper.

CHAPTER I

FUNCTIONS. GEOMETRICAL REPRESENTATION. REGIONS.

Functions. Fundamental to a discussion of functions in some particular field is the definition of a function. If for each pair of values (z_1, z_2) of two complex variables, $z_1 = x_1 + iy_1$, $z_2 = x_2 = iy_2$, where (z_1, z_2) is a point of a region S of a 4-space of points (x_1, x_2, y_1, y_2), there is determined a value or set of values for a third complex variable w, then w is called a function of the two complex variables z_1 and z_2 $[w = f(z_1, z_2)]$ for the region S. As an example, $w = 3z_1 + 2z_2$ is a function of the two complex variables z_1 and z_2, since for any pair of finite values given to z_1 and z_2, a value for w is determined. Here S could be considered as consisting of all finite values of z_1 and z_2, respectively.

Geometrical Representation. Our definition of a function of two complex variables has introduced the idea of a region. In order to clarify the concepts of the regions we shall use, we must first see what type of geometrical representation can be used for functions of two complex variables.

As in the theory for one complex variable, where we conveniently adopted the idea of two complex planes, one for the independent variable and one for the functional value, so in the case of two complex variables we might use three planes, a z_1-plane, a z_2-plane, and a w-plane. Now, given any function of z_1 and z_2, we have for each pair of values (z_1, z_2) a corresponding value for the function. Each such pair of values is called a point. For example, the function $w = 3z_1 + 2z_2$ at the point $(1+i, -i)$ has for its w-value, $w = 3+i$. Hence, the corresponding values of z_1, z_2, and w can be mapped on their respective planes, as shown in figure 1.1.

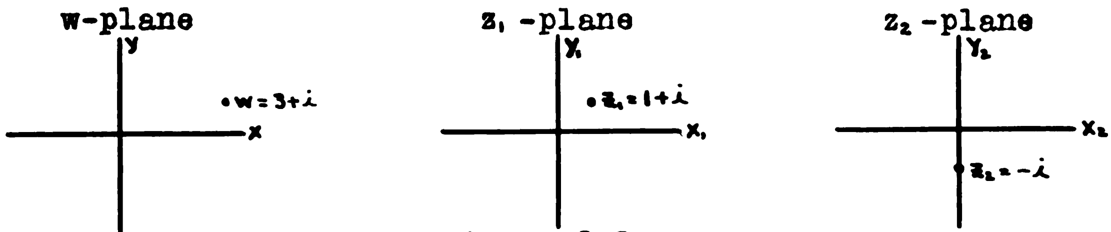

Figure 1.1.

Regions. It is obvious that we might wish to confine the discussion of a given function to sets of values of z_1 and z_2 other then their whole complex planes. There are several types of regions which have been defined in the theory of functions of two complex variables, such as the Reinhardt field, the Hartogs field, circular fields, and a few others.* However, we shall use only three special types of regions in our discussions.** These are the generalized dicylinder, the dicylinder, and the hypersphere.

Generalized Dicylinder. The point (z_1, z_2) is said to be contained in a generalized dicylinder $\mathfrak{S}$ if its z_1-coordinate belongs to a simply-connected region S_1 in the

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* Behnke, H. and Thullen, P., Theorie der Funktionen Mehrerer Komplexer Veranderlichen, pp. 1-20. (Hereafter referred to as Behnke.)

**Bochner, S., Functions of Several Complex Variables, Part III, p. 161. (Hereafter referred to as Bochner.)

z_1-plane and its z_2-coordinate belongs to the simply-connected region S_2 in the z_2-plane. Pictorially, S

Figure 1.2. A Generalized Dicylinder.

might appear as shown in figure 1.2.

<u>Dicylinder</u>. A dicylinder about the point (a_1, a_2) consists of all points (z_1, z_2) such that

$$|z_1 - a_1| < d_1, \quad |z_2 - a_2| < d_2.$$

In figure 1.3 we have illustrated such a region. The

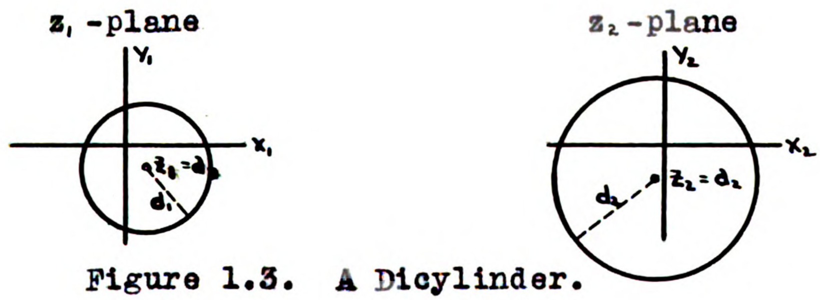

Figure 1.3. A Dicylinder.

dicylinder, we see, is a special case of the generalized dicylinder where S_1 and S_2 are now circles of radius d_1, d_2 about $z_1 = a_1$ and $z_2 = a_2$, respectively.

<u>Hypersphere</u>. A hypersphere about the point (a_1, a_2) consists of all points (z_1, z_2) such that

$$|z_1 - a_1| + |z_2 - a_2| < d.$$

The value that z_1 may take to give a point in the hypersphere depends on the value given to z_2, or vice-versa. Of these three regions defined, we shall use the first two almost exclusively.

In the theory of functions of one complex variable we often wish to consider the idea of a closed region, that is, a region in which the boundary points are included. In the case of a dicylinder or a generalized dicylinder a boundary point will be a point (z_1, z_2) such that,

a) z_1 lies on the boundary of S_1 and z_2 is anywhere in S_2,

b) z_2 lies on the boundary of S_2 and z_1 is anywhere in S_1,

c) z_1 lies on the boundary of S_1, z_2 on the boundary of S_2.

We recall from the theory for one complex variable that when we consider $z = x+iy$, where x and y are real, independent variables, we may then express

$$f(z) = u(x,y) + iv(x,y)$$

where u and v are both real functions of x and y. Similarly, in the theory of functions of two complex variables, if we consider $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$, where x_1, x_2, y_1, y_2, are four independent, real variables, we may then express

$$f(z_1, z_2) = u(x_1, x_2, y_1, y_2) + iv(x_1, x_2, y_1, y_2)$$

where u and v are real functions. From this we see that a function of two complex variables is in reality a function of four independent real variables. Therefore, in discussing the theory geometrically, we must think in terms of a four-space. This, at times, causes some difficulty in visualizing our procedures geometrically.

Also, by considering $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, as mentioned in the preceeding paragraph, we may write equivalent definitions for the dicylinder and the hypersphere. Let $a_1 = \alpha_1 + i\beta_1$, and $a_2 = \alpha_2 + i\beta_2$. Then the dicylinder about (a_1, a_2) is given by

$$(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < d_1^2, \quad (x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < d_2^2.$$

and the hypersphere about (a_1, a_2) is given by

$$(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 + (x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < d^2.$$

CHAPTER II

SOME THEOREMS ON CONTINUITY.

In this chapter we shall take up some theorems on the continuity of functions of two complex variables, first, because they are of interest in themselves, and secondly, because they will be of use in proving further theorems in the following chapters.

To define a continuous function of two complex variables we require, as for functions of one complex variable, the idea of a limit. We say that a function $f(z_1,z_2)$ has a limit P at the point (a_1,a_2) $\left[\lim\limits_{\substack{z_1 \to a_1 \\ z_2 \to a_2}} f(z_1,z_2) = P\right]$, if for every positive, real there exists a δ, such that

$$\left|f(z_1,z_2) - P\right| < \epsilon$$

for all z_1 and z_2 such that

$$|z_1 - a_1| < \delta, \quad |z_2 - a_2| < \delta.$$

Further, we say that a function of two complex variables, $f(z_1,z_2)$, is continuous at the point (a_1,a_2) if

$$\lim_{\substack{z_1 \to a_1 \\ z_2 \to a_2}} f(z_1,z_2) = f(a_1,a_2) .$$

We have a theorem concerning limits in the theory for functions of one complex variable,*

Theorem 2.1a.**

H_1). Given $z = x + iy$, $a = \alpha + i\beta$, $P = A + iB$,

and $f(z) = u(x,y) + iv(x,y)$.

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* Townsend, E. J., Functions of a Complex Variable, p. 27. (Hereafter referred to as Townsend.)

** We shall use this form for the statement of our theorems in order to bring out the differences or similarities between the theory for one variable and that for two variables.

C_1). <u>The necessary and sufficient condition that</u>

$$\lim_{z \to a} f(z) = P$$

<u>is that</u>

$$\lim_{\substack{x \to \alpha \\ y \to \beta}} u(x,y) = A \quad \underline{\text{and}} \quad \lim_{\substack{x \to \alpha \\ y \to \beta}} v(x,y) = B.$$

We shall prove the corresponding theorem for functions of two complex variables.

<u>Theorem 2.1b.</u>

H_1). <u>Given</u> $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $a_1 = \alpha_1 + i\beta_1$, $a_2 = \alpha_2 + i\beta_2$, $P = A + iB$, <u>and</u> $f(z_1, z_2) = u(x_1, x_2, y_1, y_2) + iv(x_1, x_2, y_1, y_2)$.

C_1). <u>The necessary and sufficient condition that</u>

$$\lim_{\substack{z_1 \to a_1 \\ z_2 \to a_2}} f(z_1, z_2) = P$$

<u>is that</u>

$$\lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} u(x_1, x_2, y_1, y_2) = A \quad \underline{\text{and}} \quad \lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} v(x_1, x_2, y_1, y_2) = B.$$

We prove the necessity of the condition first. We have given then that

$$\lim_{\substack{z_1 \to a_1 \\ z_2 \to a_2}} f(z_1, z_2) = P.$$

By the definition of a limit this means that given $\frac{\epsilon}{\sqrt{2}} > 0$, there exists a δ_ϵ (δ dependent on ϵ) such that for

$$|z_1 - a_1| < \delta \quad \text{and} \quad |z_2 - a_2| < \delta,$$

it follows that

$$\left| f(z_1, z_2) - P \right| < \frac{\epsilon}{\sqrt{2}}.$$

This in turn means that

$$(1) \qquad \left| u + iv - A - iB \right| < \frac{\epsilon}{\sqrt{2}}$$

for

$$(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < \delta^2 \quad \text{and} \quad (x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < \delta^2.$$

By a theorem for functions of one complex variable*

(2) $$\left|\frac{u + iv - A - iB}{\sqrt{2}}\right| \leqslant |u - A| + |v - B| \quad .$$

Substituting this in (1) we get

(3) $$|u - A| + |v - B| < \epsilon$$

for

$$(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < \delta^2 \quad \text{and} \quad (x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < \delta^2 .$$

Now, for the same conditions on x_1, x_2, y_1, y_2,

(4) $$|u - A| < \epsilon \quad \text{and} \quad |v - B| < \epsilon \; .$$

From the definition of a limit, (4) states that

$$\lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} u = A \quad \text{and} \quad \lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} v = B.$$

For the sufficiency part of the proof, we have given that

$$\lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} u = A \quad \text{and} \quad \lim_{\substack{x_1 \to \alpha_1 \\ x_2 \to \alpha_2 \\ y_1 \to \beta_1 \\ y_2 \to \beta_2}} v = B.$$

By the definition of a limit these mean that given $\frac{\epsilon}{2} > 0$, there exist $\delta_1(u)$, $\delta_2(u)$, $\delta_1(v)$, $\delta_2(v)$, such that

$$|u - A| < \frac{\epsilon}{2}$$

for $(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < [\delta_1(u)]^2$ and $(x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < [\delta_2(u)]^2$.

and

$$|v - B| < \frac{\epsilon}{2}$$

for $(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < [\delta_1(v)]^2$ and $(x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < [\delta_2(v)]^2$.

Adding together $|u - A|$ and $|v - B|$, we obtain

(5) $$|u - A| + |v - B| < \epsilon$$

for $(x_1 - \alpha_1)^2 + (y_1 - \beta_1)^2 < \delta^2$ and $(x_2 - \alpha_2)^2 + (y_2 - \beta_2)^2 < \delta^2$

where δ is the minimum of $\delta_1(u)$, $\delta_2(u)$, $\delta_1(v)$, $\delta_2(v)$.

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* Townsend, p.10.

We know that

$$\left|f(z_1,z_2) - P\right| = \left|(u+iv)-(A+iB)\right| = \left|(u-A)+i(v-B)\right| \leqslant \left|u-A\right| + \left|v-B\right| .$$

Substituting in (5) we get

$$\left|f(z_1,z_2) - P\right| < \epsilon$$

for

$$(x_1-\alpha_1)^2 + (y_1-\beta_1)^2 < \delta^2 \quad \text{and} \quad (x_2-\alpha_2)^2 + (y_2-\beta_2)^2 < \delta^2.$$

This is nothing but the definition of the limit

$$\lim_{\substack{z_1 \to a_1 \\ z_2 \to a_2}} f(z_1,z_2) = P.$$

Our main purpose in proving the above theorem was to make possible a theorem on the continuity of a function at a point. The theorem for functions of a single complex variable is *

Theorem 2.2a.

H_1). Given $z = x+iy$ and $f(z) = u(x,y)+iv(x,y)$.

C_1). The necessary and sufficient condition that $f(z)$ be continuous at the point $z = a$ is that $u(x,y)$ and $v(x,y)$ are both continuous at a.

The corresponding theorem for functions of two complex variables is almost identically stated.

Theorem 2.2b.

H_1). Given $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, and $f(z_1,z_2) = u(x_1,x_2,y_1,y_2) + iv(x_1,x_2,y_1,y_2)$.

C_1). The necessary and sufficient condition that $f(z_1,z_2)$ be continuous at the point (a_1,a_2) is that $u(x_1,x_2,y_1,y_2)$ and $v(x_1,x_2,y_1,y_2)$ ane both continuous at $(\alpha_1,\alpha_2,\beta_1,\beta_2)$.

The proof follows immediately from theorem 2.1b.

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* Townsend, p. 35.

The next theorem on the continuity of functions of two complex variables has no counterpart in the theory for one variable. Instead it corresponds to a theorem of two real variables.

Theorem 2.3.

H_1). In a given region S, $f(z_1, z_2)$ is continuous in z_1 uniformly with respect to z_2.

H_2). $f(z_1, z_2)$ is continuous in z_2.

C_1). $f(z_1, z_2)$ is continuous in both variables together.

The final theorem of this chapter concerns sequences of continuous functions. For functions of one complex variable we have *

Theorem 2.4a.

H_1). $f_0(z), f_1(z), f_2(z), \ldots$ is a sequence of functions converging uniformly to a limiting function $f(z)$.

H_2). Each $f_i(z)$ is continuous in a region S.

C_1). $f(z)$ is continuous is S.

For functions of two complex variables we can prove a similar theorem.

Theorem 2.4b.

H_1). $f_0(z_1, z_2), f_1(z_1, z_2), f_2(z_1, z_2), \ldots$ is a sequence of functions converging uniformly with respect to both z_1 and z_2 to a limiting function $f(z_1, z_2)$.

H_2). Each $f_i(z_1, z_2)$ is continuous in both variables in a region S.

C_1). $f(z_1, z_2)$ is continuous in both variables in S.

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* Copson, E. T., Theory of Functions of a Complex Variable, p. 94. (Hereafter referred to as Copson).

To prove the theorem, suppose we are given an ϵ. Then consider

$$\left| f(z_1, z_2) - f(z_1+p, z_2+q) \right| .$$

This can be written

$$\left| f(z_1, z_2) - f(z_1+p, z_2+q) \right| \tag{8}$$

$$= \left| f(z_1, z_2) - f_n(z_1, z_2) + f_n(z_1, z_2) - f_n(z_1+p, z_2+q) + f_n(z_1+p, z_2+q) - f(z_1+p, z_2+q) \right| .$$

$$\leqslant \left| f(z_1, z_2) - f_n(z_1, z_2) \right| + \left| f_n(z_1, z_2) - f_n(z_1+p, z_2+q) \right| + \left| f_n(z_1+p, z_2+q) - f(z_1+p, z_2+q) \right| . \tag{8a}$$

Consider the terms of (8a) separately. By H1 of the theorem there exists an N such that

$$\left| f(z_1, z_2) - f_n(z_1, z_2) \right| < \frac{\epsilon}{3} \tag{9}$$

and

$$\left| f_n(z_1+p, z_2+q) - f(z_1+p, z_2+q) \right| < \frac{\epsilon}{3} \tag{10}$$

for $n > N$. Also, since each $f_n(z_1, z_2)$ is continuous in S, then by the definition of continuity, given $\frac{\epsilon}{3}$ there exists a δ (dependent on $\frac{\epsilon}{3}$ and also on both z_1 and z_2) such that

$$\left| f_n(z_1, z_2) - f_n(z_1+p, z_2+q) \right| < \frac{\epsilon}{3} \tag{11}$$

for

$$|p| < \delta \quad \text{and} \quad |q| < \delta .$$

Hence for each z_1 and z_2 in S and for any positive ϵ there exists a δ which is dependent on ϵ and also on z_1 and z_2, such that by substituting (9), (10), and (11) in (8a) we have

$$\left| f(z_1, z_2) - f(z_1+p, z_2+q) \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for

$$|p| < \delta \quad \text{and} \quad |q| < \delta .$$

This is merely the definition of the continuity of $f(z_1, z_2)$ in both variables, and hence our theorem is proved.

CHAPTER III

FUNCTIONS REGULAR IN A REGION.
OSGOOD'S THEOREM. HARTOGS' THEOREM.

Functions regular in a region. For functions of two complex variables, we define a function to be regular in a region S if the derivatives of all orders, iterated and mixed, exist and are continuous and bounded in every region interior to S.* A function which has a region in which it is regular is often referred to as an analytic function.

It should be noted that when we refer to a function of two complex variables as being regular or analytic, we mean that the function is analytic in both variables together. Osgood's theorem and Hartogs' theorem express conditions on a function for such regularity.

Osgood's theorem. In order to prove Osgood's theorem we require several preliminary lemmas.

Lemma 1.**

H_1). f(z) is regular in a region S.

H_2). f(z) is bounded in S with an upper bound M.

H_3). A is a region whose closure is interior to S.

H_4). α is the distance from $\bar{A}$ to the boundary of S. ℓ is the length of the boundary of S.

C_1). For z and z+h interior to A it follows that:

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* Bochner, p. 164.
** Bochner, p. 162.

1). $\left|f(z+h) - f(z)\right| \leqslant \dfrac{M\cdot|h|\cdot\ell}{2\pi\alpha^2}$.

2). $\left|f'(z)\right| \leqslant \dfrac{M\cdot\ell}{2\pi\alpha^2}$.

3). $\left|\dfrac{f(z+h) - f(z)}{h} - f'(z)\right| \leqslant \dfrac{M\cdot|h|\cdot\ell}{2\pi\alpha^3}$.

These are readily proved by the use of Cauchy's formula.

Lemma 2 deals with functions of two complex variables.

Lemma 2.

H_1). <u>$f(z_1,z_2)$ is defined in a dicylinder</u> $P(a_i,r)$; <u>that is, the dicylinder;</u> $|z_1-a_1|<r$, $|z_2-a_2|<r$.

H_2). $|f(z_1,z_2)| \leqslant M$ <u>for</u> (z_1,z_2) <u>in</u> $P(a_i,r)$.

H_3). <u>For every point</u> (z_1',z_2') <u>in</u> $P(a_i,r)$, $f(z_1,z_2')$ <u>is analytic in</u> z_1, <u>for</u> (z_1,z_2') <u>in</u> $P(a_i,r)$ <u>and</u> $f(z_1',z_2)$ <u>is analytic in</u> z_2 <u>for</u> (z_1',z_2) <u>in</u> $P(a_i,r)$.

C_1). <u>In any dicylinder</u> $P(a_i,r_1)$, <u>where</u> $r_1< r$, $\dfrac{\partial f(z_1,z_2)}{\partial z_1}$ <u>and</u> $\dfrac{\partial f(z_1,z_2)}{\partial z_2}$ <u>exist and are analytic in each variable separately</u>.

It will be sufficient to carry through the proof for $\dfrac{\partial f}{\partial z_1}$.

To prove that $\dfrac{\partial f}{\partial z_1}$ is analytic in z_1, we recall that by the definition of a partial derivative at a point, we need consider $f(z_1,z_2)$ as a function of z_1, only. Since by hypothesis we know that $f(z_1,z_2)$ is analytic in z_1 for (z_1,z_2) in $P(a_i,r)$, then by the theorem on derivatives of functions of one complex variable, $\dfrac{\partial f}{\partial z_1}$ exists and is analytic in z_1 for the same dicylinder $P(a_i,r)$.*

To prove that $\dfrac{\partial f}{\partial z_1}$ is analytic in z_2 we make use of the following device. Let h_n be a sequence of complex

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* Townsend, p. 77.

quantities approaching zero as $n \to \infty$. Then consider the sequence of functions

$$F_n(z_1, z_2) = \frac{f(z_1 + h_n, z_2) - f(z_1, z_2)}{h_n}$$

Considering $f(z_1, z_2)$ as a function of z_1 alone, from (3) of lemma 1 we get

$$\left| F_n - \frac{\partial f}{\partial z_1} \right| \leq \frac{M \cdot |h_n| \cdot \ell}{2\pi \alpha^3}$$

in a dicylinder $P(a_i, r_1)$, where $r_1 < r$. This means that $F_n(z_1, z_2)$ converges uniformly in both variables to $\frac{\partial f}{\partial z_1}$, since $\frac{M \cdot |h_n| \cdot \ell}{2\pi \alpha^3}$ is independent of the values given to z_1 and z_2. Now, considering $f(z_1, z_2)$ as a function of z_2 alone, we know by hypothesis that $f(z_1, z_2)$ is analytic in z_2. Therefore, $F_n(z_1, z_2)$, considered now as a function of z_2 alone, is also analytic in z_2. By the Weierstrass theorem for functions of a single complex variable, since, in the dicylinder $P(a_i, r_1)$, $F_n(z_1, z_2)$ converges uniformly to $\frac{\partial f}{\partial z_1}$ in z_2 and each $F_n(z_1, z_2)$ is analytic in z_2, then the limiting function $\frac{\partial f}{\partial z_1}$ is analytic in z_2 for (z_1, z_2) in $P(a_i, r_1)$.*

We may now proceed with the statement and proof of Osgood's theorem.**

<u>Theorem 3.1.</u>

H_1). <u>$f(z_1, z_2)$ is defined in the dicylinder $P(a_i, r)$.</u>

H_2). <u>$|f(z_1, z_2)| \leq M$ for (z_1, z_2) in $P(a_i, r)$.</u>

H_3). <u>For every point (z_1', z_2') in $P(a_i, r)$, $f(z_1, z_2')$ is analytic in z_1 for (z_1, z_2') in $P(a_i, r)$ and $f(z_1', z_2)$ is analytic in z_2 for (z_1', z_2) in $P(a_i, r)$.</u>

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* Copson, p. 95.
** Bochner, p. 163.

C_1). <u>Then for $r_d < r$ the derivatives of all orders, iterated and mixed, exist and are continuous and bounded in the dicylinder $P(a_i, r_d)$.</u> [<u>In</u> other words, $f(z_1, z_2)$ is analytic in both variables together.]

First we shall show that the iterated derivatives of all orders exist in $P(a_i, r_d)$. Considering $\frac{\partial^n f}{\partial z_1^n}$ for example, by the definition of a partial derivative at a point, we need consider $f(z_1, z_2)$ as a function of the single variable z_1, only. Applying the theorem on derivatives of functions of one complex variable, we know that all the derivatives with respect to z_1 exist at any point in $P(a_i, r)$, and hence at any point in the interior dicylinder $P(a_i, r_d)$.* Corresponding reasoning would prove the existence of the iterated derivatives with respect to z_2.

To show that the iterated derivatives are bounded, consider $f(z_1, z_2)$ as a function of z_1 alone. From (2) of lemma 1 we have then

$$\left| \frac{\partial f}{\partial z_1} \right| \leq \frac{M \cdot \ell}{2\pi \alpha^2}$$

for (z_1, z_2) in $P(a_i, r_1)$. This bound is independent of the value of z_1 and also of the fixed value assigned to z_2. Hence $\frac{\partial f}{\partial z_1}$ is uniformly bounded for (z_1, z_2) in $P(a_i, r_1)$. In a dicylinder $P(a_i, r_1)$ we know now by lemma 2 and from what we have just proved that $\frac{\partial f}{\partial z_1}$ satisfies the hypotheses of both lemmas. Considering $\frac{\partial f}{\partial z_1}$ as our function, now, we can apply (2) of lemma 1 to show that $\frac{\partial^2 f}{\partial z_1^2}$ is uniformly

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* Townsend, p. 77.

bounded in a dicylinder $P(a_i, r_2)$, where $r_2 < r_1$. Using this and also applying lemma 2 on the function $\frac{\partial f}{\partial z_1}$, we see that $\frac{\partial^2 f}{\partial z_1^2}$ now satisfies the hypotheses for both lemmas. This enables us to carry on the same discussion for $\frac{\partial^3 f}{\partial z_1^3}$. Hence, by continuing in a similar manner and using a sequence of dicylinders, each one of which is contained in the one preceeding, we can show that any iterated derivative $\frac{\partial^n f}{\partial z_1^n}$ is uniformly bounded in a dicylinder $P(a_i, r_n)$. Moreover, we can show that any iterated derivative satisfies the hypotheses of both lemmas in $P(a_i, r_n)$, a fact which we shall use in further parts of this proof. We have confined our proof to the iterated derivatives with respect to z_1 since the proof for the iterated derivatives with respect to z_2 follows through in exactly the same way.

To conclude our discussion of the iterated derivatives, we must show that they are continuous in both variables together.* Since $\frac{\partial^n f}{\partial z_1^n}$ satisfies the hypotheses of lemma 1 in $P(a_i, r_n)$ as shown in the preceeding paragraph, then considering $\frac{\partial^n f}{\partial z_1^n}$ as a function of z_1 alone and applying (1) of lemma 1, we have

$$(4) \qquad \left| \frac{\partial^n f(z_1 + h, z_2)}{\partial z_1^n} - \frac{\partial^n f(z_1, z_2)}{\partial z_1^n} \right| \leq K \cdot |h|$$

where K is a constant depending on the sequence of dicylinders used. From (4) we have that $\frac{\partial^n f}{\partial z_1^n}$ is continuous in z_1 uniformly with respect to z_2, since the right hand side of the inequality is independent of the fixed value assigned to z_2. Next, since we can show that any iterated derivative is

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* The continuity in both variables could also be shown by using theorem 2.4b.

analytic in z_2 by the use of lemma 2, then $\frac{\partial^n f}{\partial z_1^n}$ is analytic in z_2. The continuity of $\frac{\partial^n f}{\partial z_1^n}$ in z_2 follows from its analyticity. $\frac{\partial^n f}{\partial z_1^n}$ now satisfies the hypotheses of theorem 2.3 and therefore is continuous in both variables in a dicylinder $P(a_i, r_{n+1})$. A similar discussion would verify the continuity of the iterated derivatives with respect to z_2.

We shall next discuss the mixed derivatives. As we have already shown, any iterated derivative $\frac{\partial^n f}{\partial z_1^n}$ satisfies the hypotheses of both lemmas in a dicylinder $P(a_i, r_n)$. Applying lemma 2 to the function $\frac{\partial^n f}{\partial z_1^n}$, we have the existence of $\frac{\partial^{n+1} f}{\partial z_1^n \partial z_2}$ and moreover, its analyticity in each variable. Also for the same function, from (2) of lemma 1, we have that $\frac{\partial^{n+1} f}{\partial z_1^n \partial z_2}$ is uniformly bounded in $P(a_i, r_{n+1})$. We now have $\frac{\partial^{n+1} f}{\partial z_1^n \partial z_2}$ satisfying the hypotheses of both lemmas in $P(a_i, r_{n+1})$. Consequently, we can apply these lemmas to $\frac{\partial^{n+1} f}{\partial z_1^n \partial z_2}$ considered as the function and show that $\frac{\partial^{n+2} f}{\partial z_1^n \partial z_2^2}$ exists, is analytic in each variable, and is uniformly bounded in a dicylinder $P(a_i, r_{n+2})$. By repeated application of both lemmas we can show that any mixed derivative $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ is bounded in $P(a_i, r_{n+m})$. Furthermore, we can show that $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ satisfies the hypotheses of both lemmas.

Finally, we must show the continuity of $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$. As mentioned in the last paragraph $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ can be shown to satisfy the hypotheses of both lemmas in $P(a_i, r_{n+m})$. Hence, considering $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ as a function of z_1 alone, from (1) of lemma 1, we get

$$(5) \quad \left| \frac{\partial^{n+m} f(z_1+h, z_2)}{\partial z_1^n \partial z_2^m} - \frac{\partial^{n+m} f(z_1, z_2)}{\partial z_1^n \partial z_2^m} \right| \leq K_1 \cdot |h|$$

where K_1 is a constant depending on the sequence of dicylinders used. Hence, from (5) we have that $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ is continuous in z_1 uniformly with respect to z_2. Also since we can show $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ to be analytic in z_2, it is necessarily continuous in z_2. Using these two facts and applying theorem 2.3 we have the continuity of $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$ in both variables, in a dicylinder $P(a_i, r_{n+m+1})$.

Now we see that we have shown that any iterated or mixed derivative, all of which can be represented by $\frac{\partial^{n+m} f}{\partial z_1^n \partial z_2^m}$, exists and is uniformly bounded and continuous in both variables in a dicylinder $P(a_i, r_{n+m+1})$. The sequence of dicylinders used was such that each one was necessarily contained in the preceeding one. We can therefore select any dicylinder $P(a_i, r_d)$ interior to $P(a_i, r)$ and show that Osgood's theorem holds in $P(a_i, r_d)$ by making the decrease in radius from one dicylinder to the next $\frac{(r-r_d)}{2^{n+m+1}}$.

Hartogs' theorem is less restrictive than Osgood's theorem in that it does not require the property of boundedness. We hereby state Hartogs' theorem.*

<u>Theorem 3.2.</u>

H_1). <u>$f(z_1, z_2)$ is defined in a region</u> S.

H_2). <u>For every point (z_1', z_2') of</u> S, <u>$f(z_1, z_2')$ is analytic for (z_1, z_2') in</u> S <u>and $f(z_1', z_2)$ is analytic for (z_1', z_2) in</u> S.

C_1). <u>In</u> S, <u>$f(z_1, z_2)$ is analytic in both variables.</u>

The proof of this theorem is too long and involved to be presented here. We shall use Hartogs' Theorem frequently to determine the analyticity of various functions.

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* Boohner, pp. 164-172.

CHAPTER IV

THE CAUCHY-RIEMANN EQUATIONS. THE THEOREM OF THE MAXIMUM.

The Cauchy-Riemann Equations. The extension of the Cauchy-Riemann equations to the case of two complex variables leads to several interesting results.

The theorem dealing with the Cauchy-Riemann equations for functions of a single complex variable is *

Theorem 4.1a.

H_1). In a given finite region S, $u(x,y)$ and $v(x,y)$ are two real, single-valued functions of the real variables, x and y.

C_1). The necessary and sufficient condition that the complex function $w = u(x,y) + iv(x,y)$ be regular in S is that the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, exist and are continuous in S and satisfy the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Immediately, using Hartogs' theorem, we can state a corresponding theorem for functions of two complex variables.

Theorem 4.1b.

H_1). In a given finite region S, $u(x_1, x_2, y_1, y_2)$ and $v(x_1, x_2, y_1, y_2)$ are two real, single-valued functions of the real variables x_1, x_2, y_1, y_2.

C_1). The necessary and sufficient condition that the

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* Townsend, p. 83.

complex function $w = u(x_1, x_2, y_1, y_2) + iv(x_1, x_2, y_1, y_2)$ be regular in both variables in S is that the partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial u}{\partial y_i}$, $\frac{\partial v}{\partial x_i}$, $\frac{\partial v}{\partial y_i}$, $(i=1,2)$ exist and are continuous in S and satisfy the partial differential equations,

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i} \qquad (i = 1,2).$$

The proof of this theorem is made quite obvious when we recall that in Hartogs' theorem the analyticity of a function in each complex variable separately is a necessary and sufficient condition that the function be analytic in both variables together.

We have a further interesting theorem dealing with the application of the Cauchy-Riemann equations to functions of two complex variables.*

Theorem 4.2.

H_1). Given a real, single-valued function $v(x_1, x_2, y_1, y_2)$ which is continuous in a region S.

H_2). In S, the second-order mixed derivatives of v exist, are continuous, and satisfy the relations,

(a) $$\frac{\partial^2 v}{\partial x_i^2} + \frac{\partial^2 v}{\partial y_i^2} = 0 \qquad (i = 1,2)$$

(b) $$\frac{\partial^2 v}{\partial x_1 \partial y_2} = \frac{\partial^2 v}{\partial x_2 \partial y_1}$$

(c) $$\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial y_1 \partial y_2} = 0 \quad .$$

C_2). Then there exists a real function $u(x_1, x_2, y_1, y_2)$ such that the complex function $w = u + iv$ is regular in S.

* Bochner, pp. 176-177.

To prove this theorem, suppose we have $w = u + iv$, a function which is regular in a region S. Then applying theorem 4.1b we see that necessarily,

(1) $$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial y_1}$$

(2) $$\frac{\partial u}{\partial y_1} = -\frac{\partial v}{\partial x_1}$$

(3) $$\frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial y_2}$$

(4) $$\frac{\partial u}{\partial y_2} = -\frac{\partial v}{\partial x_2} \quad .$$

Therefore, if we are given a real function, $v(x_1, x_2, y_1, y_2)$, in order for some other real function, $u(x_1, x_2, y_1, y_2)$, to form with v an analytic function $u + iv$, u must satisfy the four partial differential equations, (1), (2), (3), (4). The necessary and sufficient condition that these possess a solution for u is that the second-order mixed derivatives are independent of their order of differentiation. The second-order mixed derivatives, independent of their order of differentiation, are,

$$\frac{\partial^2 u}{\partial x_1 \partial x_2}, \quad \frac{\partial^2 u}{\partial y_1 \partial y_2}, \quad \frac{\partial^2 u}{\partial y_2 \partial x_1}, \quad \frac{\partial^2 u}{\partial y_1 \partial x_2}, \quad \frac{\partial^2 u}{\partial y_1 \partial x_1}, \quad \frac{\partial^2 u}{\partial y_2 \partial x_2}.$$

Our procedure will be to take each one of these mixed derivatives, obtain it in as many ways as possible by differentiating (1), (2), (3), (4) above, equate the proper second-order mixed derivatives of v, and see what conditions we get on the function v.

Differentiating equations (1) and (3) with respect to x_2 and x_1, respectively, we get

(1a) $$\frac{\partial^2 u}{\partial x_2 \partial x_1} = \frac{\partial^2 v}{\partial x_2 \partial y_1} ,$$

(3a) $$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial^2 v}{\partial x_1 \partial y_2} .$$

Equating the right hand sides of (1a) and (3a), we obtain

$$(5)\qquad \frac{\partial^2 v}{\partial x_2 \partial y_1} = \frac{\partial^2 v}{\partial x_1 \partial y_2} .$$

Differentiating (2) and (4) with respect to y_2 and y_1, respectively, we get through a similar procedure,

$$\frac{\partial^2 v}{\partial y_2 \partial x_1} = \frac{\partial^2 v}{\partial y_1 \partial x_2}$$

which is merely a restatement of (5).

Differentiating (1)with respect to y_2 and (4) with respect to x_1, we get,

$$\frac{\partial^2 v}{\partial y_2 \partial y_1} = -\frac{\partial^2 v}{\partial x_1 \partial x_2},$$

or

$$(6)\qquad \frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial y_1 \partial y_2} = 0 .$$

Differentiating (3) with respect to y_1 and (2) with respect to x_2, we get,

$$\frac{\partial^2 v}{\partial y_1 \partial y_2} = -\frac{\partial^2 v}{\partial x_2 \partial x_1},$$

or

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial y_1 \partial y_2} = 0,$$

which is a restatement of (6).

Differentiating (1) with respect to y_1 and (2) with respect to x_1, we obtain,

$$\frac{\partial^2 v}{\partial y_1^2} = -\frac{\partial^2 v}{\partial x_1^2},$$

or

$$(7)\qquad \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial y_1^2} = 0 .$$

Differentiating (3) with respect to y_2 and (4) with respect to x_2, we get,

$$\frac{\partial^2 v}{\partial y_2^2} = -\frac{\partial^2 v}{\partial x_2^2},$$

or

$$(8)\qquad \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 v}{\partial y_2^2} = 0.$$

Thus we have the conditions (5), (6), (7), and (8) for v in order that a function u exist to make $w = u + iv$ regular in S. These conditions are stated in the theorem as (a), (b), and (c).

Let us apply this theorem to several examples.

Example 1. Consider

$$v = 2x_1 y_1 + 2x_2 y_2 + x_1 y_2 + x_2 y_1 \quad .$$

This function v is real, single-valued, and continuous for all finite values of z_1 and z_2. Hence the first hypothesis of theorem 4.2 is satisfied. Computing our mixed derivatives, we find that

$$\frac{\partial^2 v}{\partial x_1^2} = 0, \frac{\partial^2 v}{\partial y_1^2} = 0, \frac{\partial^2 v}{\partial x_2^2} = 0, \frac{\partial^2 v}{\partial y_2^2} = 0, \frac{\partial^2 v}{\partial x_1 \partial y_2} = 1, \frac{\partial^2 v}{\partial x_2 \partial y_1} = 1, \frac{\partial^2 v}{\partial x_1 \partial x_2} = 0, \frac{\partial^2 v}{\partial y_1 \partial y_2} = 0.$$

These derivatives exist and are continuous for all values of z_1 and z_2. Substituting these values in the equations (a), (b), (c), we see that H_2 is satisfied. Hence there must exist a function u, such that $u + iv$ is regular for all finite z_1 and z_2. Such a u is

$$u = x_1^2 - y_1^2 + x_1 x_2 - y_1 y_2 + x_2^2 + y_2^2 \; ,$$

coming from the function

$$\begin{aligned} w &= z_1^2 + z_1 z_2 + z_2^2 \\ &= (x_1 + iy_1)^2 + (x_1 + iy_1)(x_2 + iy_2) + (x_2 + iy_2)^2 \\ &= (x_1^2 - y_1^2 + x_1 x_2 - y_1 y_2 + x_2^2 + y_2^2) + i\,(2x_1 y_1 + 2x_2 y_2 + x_1 y_2 + x_2 y_1) \\ &= u + iv \; . \end{aligned}$$

This function, $w = z_1^2 + z_1 z_2 + z_2^2$, is regular for all finite values, as predicted by our application of the theorem.

Example 2. Consider

$$v = \frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2} \quad .$$

Immediately we see that v is not defined for $x_2 = 0$, $y_2 = 0$, that is, for the point $z_2 = 0$. Hence, any region S, which we consider must exclude $z_2 = 0$. Our second-order mixed derivatives are

$$\frac{\partial^2 v}{\partial x_1^2} = 0\,, \qquad \frac{\partial^2 v}{\partial x_2^2} = \frac{2y_2^3 x_1 - 6y_1 y_2^2 x_2 - 6y_2 x_1 x_2^2 + 2y_1 x_2^3}{(x_2^2 + y_2^2)^3}\,,$$

$$\frac{\partial^2 v}{\partial y_1^2} = 0\,, \qquad \frac{\partial^2 v}{\partial y_2^2} = \frac{-2y_1 x_2^3 + 6y_2 x_1 x_2^2 + 6y_1 y_2^2 x_2 - 2y_2^3 x_1}{(x_2^2 + y_2^2)^3}\,,$$

$$\frac{\partial^2 v}{\partial x_1 \partial y_2} = \frac{y_2^2 - x_2^2}{(x_2^2+y_2^2)^2}, \quad \frac{\partial^2 v}{\partial x_2 \partial y_1} = \frac{y_2^2 - x_2^2}{(x_2^2+y_2^2)^2}, \quad \frac{\partial^2 v}{\partial x_1 \partial x_2} = \frac{2x_2 y_2}{(x_2^2+y_2^2)^2}, \quad \frac{\partial^2 v}{\partial y_1 \partial y_2} = \frac{-2x_2 y_2}{(x_2^2+y_2^2)^2}.$$

Substituting the proper derivatives in the equations (a), (b), (c) of the theorem, we find that all are satisfied. Thus, by the conclusion of the theorem there must exist a function u such that $u + iv$ will be regular for all finite values of z_1 and z_2 except for $z_2 = 0$. Such a u is

$$u = \frac{x_1 x_2 + y_1 y_2}{(x_2^2 + y_2^2)}$$

coming from the function,

$$w = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2} = u + iv\,.$$

As predicted, the function $w = \frac{z_1}{z_2}$ is regular in any finite region which does not include points with their z_2-coordinate equal to 0.

It might be interesting to investigate an example which does not satisfy the hypotheses of the theorem. Almost any v chosen at random would fail to satisfy the conditions of the theorem. For example, $v = x_1^2 + x_2 y_2$ would give $2 = 0$ for equation (a) of the theorem.

<u>The Theorem of the Maximum</u>. For functions of one complex variable the theorem of the maximum is *

<u>Theorem 4.3a.</u>

H_1). $f(z)$ <u>is regular in a region</u> S <u>and continuous</u>

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* Copson, p. 162.

on the boundary of S.

C_1). There exists no point, $z = a$, interior to S such that

$$|f(z)| \leq |f(a)|$$

for all z within S, unless $f(z) \equiv f(a)$.

The corresponding theorem for functions of two complex variables follows almost word for word.*

Theorem 4.3b.

H_1). $f(z_1, z_2)$ is regular in a region S and continuous on the boundary of S.

C_1). There exists no point (a_1, a_2), interior to S, such that

$$|f(z_1, z_2)| \leq |f(a_1, a_2)|$$

for all (z_1, z_2) in S unless $f(z_1, z_2) \equiv f(a_1, a_2)$.

Expressed more simply, both these theorems state that a function which is regular in an open region and continuous on the boundary of that region attains no maximum value in that open region unless the function is a constant. The extreme similarity between the theorems for the two cases is due to the fact that the theorem for functions of two complex variables is proved by considering each variable separately and applying the theorem for functions of one complex variable.

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* Bochner, p.175.

CHAPTER V

THE CAUCHY THEOREM. CAUCHY'S INTEGRAL FORMULA. TAYLOR'S EXPANSIONS. ASSOCIATED RADII OF CONVERGENCE.

In the theory of functions of one complex variable, the Cauchy theorem and the Cauchy integral formula are extremely important in that they enable us to obtain power series expansions for the functions being considered. In the case of functions of two complex variables we use some similar theorems for the same purpose.

The Cauchy Theorem. The Cauchy theorem for functions of a single complex variable is *

Theorem 5.1a.

H_1). A region S is bounded by an ordinary closed curve C.**

H_2). $f(z)$ is regular in S.

H_3). $f(z)$ is continuous on C.

C_1). $\int_C f(\zeta)\,d\zeta = 0$. [$\zeta$ indicates values of z on C.]

For functions of two complex variables we have a corresponding theorem. ***

Theorem 5.1b.

H_1). F is a two-dimensional, closed, two-sided surface.

H_2). F possesses a real, analytic representation

$$\varphi_i(x_1, x_2, y_1, y_2) = 0. \qquad i = 1, 2.$$

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* Copson, p. 61.

** By an ordinary curve is meant a curve that may be broken up into a finite number of divisions, each of which is either a rectilinear segment parallel to one of the coordinate axes or else has the property that it is determined by a function, y (x), where (x) and its inverse function, x (y), are single-valued and have first derivatives that are continuous except at most at the end points.

*** Behnke, p. 41.

H_3). F can contract in a region S to a point or a curve.

H_4). $f(z_1, z_2)$ is regular in S.

H_5). $f(z_1, z_2)$ is continuous on F.

C_1). $\int_F f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 = 0$. [$\zeta_1, \zeta_2$ indicate values of z_1, z_2 on F.]

We should note that the surface F is a boundary surface of a region in which $f(z_1, z_2)$ is regular. Also, it is the surface over which we integrate to obtain the conclusion for our theorem.

The converse of the Cauchy theorem, known as Morera's theorem, is also of interest. For functions of one complex variable it is *

Theorem 5.2a.

H_1). $f(z)$ is continuous in a region S.

H_2). $\int_C f(\zeta) d\zeta = 0$ for any closed curve C in S, where C incloses only points of S.

C_1). $f(z)$ is regular in S.

For functions of two complex variables. Morera's theorem reads,**

Theorem 5.2b.

H_1). $f(z_1, z_2)$ is continuous in a region S.

H_2). $\int_F f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2 = 0$ taken over the boundary surface, F, of any dicylinder lying completely in S.

C_1). $f(z_1, z_2)$ is regular in S.

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* Townsend, p. 80.
** Behnke, p. 41.

The interesting fact about Morera's theorem for functions of two complex variables is that a special surface of integration, namely the boundary surface of a dicylinder, may be used.

The Cauchy Integral Formula. For functions of one complex variable this important theorem is usually stated as *

Theorem 5.3a.

H_1). S is a finite, closed region with a boundary C which consists of a finite number of ordinary curves.

H_2). $f(z)$ is regular within S and continuous on C.

C_1). For any inner point z of S

$$f(z) = \left(\frac{1}{2\pi i}\right) \int_C \frac{f(\zeta)}{(\zeta - z)}\, d\zeta .$$

In the case of functions of two complex variables, we do not have a complete generalization.** Whereas in Cauchy's theorem we were able to use a surface of integration, for this theorem the integration is made over two curves in succession.

Theorem 5.3b.

H_1). S is a generalized dicylinder with boundaries C_1 in the z_1-plane and C_2 in the z_2-plane.

H_2). $f(z_1, z_2)$ is regular inside S and continuous on the boundary of S.

C_1). For any point (z_1, z_2) of S

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_2\, d\zeta_1 .$$

[ζ_1 and ζ_2 indicate values of z_1 on C_1 and z_2 on C_2 respectively.]

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* Copson, p. 66.
** Behnke, p. 40.

As we shall show, this theorem is proved by applying twice the Cauchy integral formula for functions of one variable.

We have given a region S, a generalized dicylinder, which consists of a region S_1 in the z_1-plane and a region S_2 in the z_2-plane. Now if we take z_2 fixed at some point z_2' in S_2, we then have a function of one variable which is regular in S_1. Applying theorem 5.3a, the Cauchy integral formula for the case of one variable, and integrating about C_1, we get

$$(1) \qquad f(z_1, z_2') = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta_1, z_2')}{(\zeta_1 - z_1)} d\zeta_1 ,$$

for any value of z_1 in S_1.

Next take the function $f(\zeta_1, z_2')$ from (1) and consider ζ_1 as fixed on C_1, with z_2' now varying. Applying theorem 5.3a once more we have

$$f(\zeta_1, z_2') = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_2 - z_2')} d\zeta_2 .$$

Substituting this for $f(\zeta_1, z_2')$ in (1) and dropping the prime from z_2', we have that for any point (z_1, z_2) in S

$$(2) \qquad f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1}\int_{C_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 d\zeta_1 .$$

It is interesting to note that in proving the Cauchy intergral formula for functions of two complex variables we have not been forced to use all the conditions given in the hypotheses. We might restate the theorem, giving only such hypotheses as were used in the proof of the formula (2).

Theorem 5.4.

H1). There exists a z_2', where $|z_2'| < d_2$, such that $f(z_1, z_2)$ is regular at all points (z_1, z_2') where $|z_1| \leq d_1$.

H2). For each ζ_1 where $|\zeta_1| = d_1$, $f(\zeta_1, z_2)$ is analytic in z_2 for $|z_2| < d_2$ and continuous for $|z_2| = d_2$.

H3). $f(\zeta_1, \zeta_2)$ is continuous on $|\zeta_1| = d_1$, $|\zeta_2| = d_2$.

C1). For any point (z_1, z_2) in S

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{C_1} \int_{C_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \, d\zeta_2 \, d\zeta_1 \, .$$

A further interesting observation can be made concerning functions which have the same boundary values. For the theory of functions of two variables, as in the theory for functions of one variable, we can see that if two functions, say $f(z_1, z_2)$ and $g(z_1, z_2)$, have equal values at all points of the boundary of a region of the type we have been considering, then the two functions are identical throughout the region.

Taylor's Series Expansions. Once we have discussed the Cauchy integral theorem, the next logical step is to investigate the possibility of using the theorem, as we do for functions of one complex variable, in developing some kind of a Taylor's series expansion for a function we know to be regular in a given region.

We now state the theorem concerning Taylor's series for functions of one complex variable. *

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* Copson, p. 73.

Theorem 5.5a.

H1). f(z) is regular in the neighborhood $|z - a| < R$ of the point $z = a$.

H2). f(z) can be represented in that neighborhood as a convergent power series of the form

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots$$

We might recall that this series is uniformly convergent when

$$|z - a| \leq R_1, \quad \text{where} \quad R_1 < R.$$

We state the corresponding theorem for functions of two complex variables. *

Theorem 5.5b.

H1). $f(z_1, z_2)$ is regular in a dicylinder S:

$$|z_1 - a_1| < d_1, \quad |z_2 - a_2| < d_2.$$

H2). $f(z_1, z_2)$ is continuous on the boundary of S.

C1). $f(z_1, z_2)$ can be represented in S as a convergent double power series of the form,

$$f(z_1, z_2) = \sum_{m,\ell=0}^{\infty} a_{m,\ell}(z_1 - a_1)^m (z_2 - a_2)^\ell ,$$

where

$$a_{m,\ell} = \frac{1}{m!\,\ell!} \frac{\partial^{m+\ell} f(a_1, a_2)}{\partial z_1^m \, \partial z_2^\ell} .$$

To prove this theorem, let us suppose that we have, as stated in the hypotheses, a function $f(z_1, z_2)$ which is regular in a dicylinder S, and continuous on the boundary of that dicylinder.

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* Behnke, p. 40.

First, let us fix z_2 at z_2', some value such that $|z_2' - a_2| < d_2$. We then have $f(z_1, z_2')$, a function of z_1 alone, which is regular for $|z_1 - a_1| < d_1$. Now, using Taylor's theorem for functions of one complex variable, theorem 5.5a, we can represent $f(z_1, z_2')$ by

$$
\begin{aligned}
f(z_1, z_2') = {} & f(a_1, z_2') + \frac{\partial f(a_1, z_2')}{\partial z_1} \cdot \frac{(z_1 - a_1)}{1!} + \cdots \\
& \cdots + \frac{\partial^m f(a_1, z_2')}{\partial z_1^m} \cdot \frac{(z_1 - a_1)^m}{m!} + \cdots .
\end{aligned} \tag{3}
$$

Consider the general term

$$
\frac{\partial^m f(a_1, z_2')}{\partial z_1^m} \cdot \frac{(z_1 - a_1)^m}{m!} \tag{4}
$$

of the series. Take $f(a_1, z_2')$ and let z_2' vary. We again have a function of one complex variable. Applying theorem 5.5a once more we get an expansion

$$
\begin{aligned}
f(a_1, z_2') = {} & f(a_1, a_2) + \frac{\partial f(a_1, a_2)}{\partial z_2'} \cdot \frac{(z_2' - a_2)}{1!} + \cdots \\
& \cdots + \frac{\partial^\ell f(a_1, a_2)}{\partial z_2^\ell} \cdot \frac{(z_2' - a_2)^\ell}{\ell!} + \cdots .
\end{aligned} \tag{5}
$$

By a well known theorem of complex variable this series may be differentiated term by term.* Substituting (5) in (4) and performing the differentiation, we obtain a general term of (4), and thereby a general term of (3) also,

$$
\frac{1}{m!\,\ell!} \cdot \frac{\partial^{m+\ell} f(a_1, a_2)}{\partial z_1^m \, \partial z_2^\ell} (z_1 - a_1)^m (z_2 - a_2)^\ell . \tag{6}
$$

The z_2' has been replaced by z_2, since it was adopted as a means of notation, only. We see that in (6) we have obtained the general term $a_{m,\ell}(z_1 - a_1)^m (z_2 - a_2)^\ell$ as given in

* Townsend, p. 237.

the statement of the theorem.

We can obtain an equivalent, although less useful form for the coefficient $a_{m,\ell}$. Applying the theorem concerning derivatives of functions of one complex variable on the general term, (4), of the series (3) we obtain for (4)*

$$(7) \qquad \left(\frac{\cancel{m!}}{2\pi i}\right)\int_{C_1}\frac{f(\zeta_1,z_2)}{(\zeta_1-a_1)^{m+1}}\,d\zeta_1\cdot\frac{(z_1-a_1)^m}{\cancel{m!}}\;.$$

Fixing ζ_1 on C_1 and applying the same theorem on the general term of the Taylor's expansion of $f(\zeta_1,z_2)$ we obtain

$$(8)\; f(\zeta_1,z_2) = \cdots + \left(\frac{\cancel{\ell!}}{2\pi i}\right)\int_{C_2}\frac{f(\zeta_1,\zeta_2)}{(\zeta_2-a_2)^{\ell+1}}\,d\zeta_2\cdot\frac{(z_2-a_2)^{\ell}}{\cancel{\ell!}} + \cdots$$

Substituting (8) in (7) we find for the coefficient of the general term of the expansion for $f(z_1,z_2)$

$$a_{m,\ell} = \left(\frac{1}{2\pi i}\right)^2\int_{C_1}\int_{C_2}\frac{f(\zeta_1,\zeta_2)}{(\zeta_1-a_1)^{m+1}(\zeta_2-a_2)^{\ell+1}}\,d\zeta_2\,d\zeta_1\;.$$

This is an equivalent form for the coefficient in (6).

We shall now discuss several examples of Taylor's series expansions for functions of two complex variables.

Example 1. Expand $f(z_1,z_2) = 2z_1 + 3z_2^2$ about $(1+i,2)$. This function is regular for all finite values; hence, the power series expansion will be convergent for any dicylinder about the point $(1+i,2)$. Finding the derivatives and substituting values as required in the conclusion of theorem 5.5b, we obtain a finite power series,

$$f(z_1,z_2) = (14+2i) + 2(z_1-1-i) + 12(z_2-2) + 3(z_2-2)^2\,.$$

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* Townsend, p.77.

<u>Example 2.</u> Expand $f(z_1, z_2) = \dfrac{z_1}{z_2 - i}$ about the point $(1,1)$. This function has a singular point at $z_2 = i$. Hence our expansion will be convergent only for dicylinders such that $|z_2 - 1| < \sqrt{2}$. By taking derivatives and using an induction we find that

$$\frac{\partial f}{\partial z_1} = \frac{z_2}{z_1 - i}, \quad \frac{\partial^m f}{\partial z_1^m} = 0, (m>1), \quad \frac{\partial^\ell f}{\partial z_2^\ell} = \frac{(-1)^\ell \ell!\, z_1}{(z_2 - i)^{\ell+1}},$$

$$\frac{\partial^{1+\ell} f}{\partial z_1 \partial z_2^\ell} = \frac{(-1)^\ell \ell!}{(z_2 - i)^{\ell+1}}, \text{ and } \frac{\partial^{m+\ell} f}{\partial z_1^m \partial z_2^\ell} = 0 \quad \text{for } (m>1).$$

Substituting these in the expressions given in the conclusion of theorem 5.5b, we obtain for the expansion about the point $(1,1)$,

$$f(z_1, z_2) = \frac{1}{1-i} + \frac{(z_1 - 1)}{1-i} + \sum_{\ell=1}^{\infty} \frac{(-1)^\ell (z_2 - 1)^\ell}{(1-i)^{\ell+1}} + \frac{(z_1 - 1)}{1} \sum_{\ell=1}^{\infty} \frac{(-1)^\ell (z_2 - 1)^\ell}{(1-i)^{\ell+1}}$$

<u>Associated Radii of Convergence.</u> In dealing with power series expansions of functions of one complex variable we define the radius of convergence of the power series. If we describe a circle having a radius r about the point z a such that the power series being considered converges for all values of z within the circle and diverges for all values of z outside the circle, then we define r to be the <u>radius of convergence</u> of the power series*. Moreover, the power series converges uniformly and absolutely for all values of z such that $|z - a| < r_1$, where $r_1 < r$, and ordinary convergence may occur only on the circle $|z - a| = r$.

In investigating the possibility of radii of con-

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* Townsend, p. 230.

vergence when considering functions of two complex variables, mathematicians have introduced the idea of associated radii of convergence.** A pair of positive numbers, r_1', r_2', are called a pair of <u>associated radii of convergence</u> of the power series expanded about the point (a_1, a_2) if this series converges for

$$|z_1 - a_1| < r_1' \quad , \quad |z_2 - a_2| < r_2'$$

and diverges for

$$|z_1 - a_1| > r_1' \quad , \quad |z_2 - a_2| > r_2' \quad .$$

It can be seen that the values of r_1' and r_2' depend on each other; that is, the power series might still converge if r_1' were larger and r_2' were smaller, and vice-versa. Hence, if we set

$$|z_1 - a_1| = r_1 \quad \text{and} \quad |z_2 - a_2| = r_2,$$

then the pair of associated radii of convergence of the power series $p(z_1 - a_1, z_2 - a_2)$ describe a curve $\varphi(r_1, r_2) = 0$ in an $r_1 r_2$-plane, and this in turn would mean a three-dimensional manifold in the $z_1 z_2$, four-space. This three-dimensional manifold is the boundary of the region of convergence of the power series. We shall designate this region of convergence as a region K.

Analogous to the theory for one complex variable, the power series converges absolutely and uniformly inside K, and ordinary convergence can occur only on the boundary of K. However, ordinary convergence may also occur on the planes $z_1 = 0$, $z_2 = 0$, protruding out from K. Such protruberances from the region K are referred to as <u>spines</u> of the convergent space.

** Behnke, pp. 36-39.

__Example.__ Consider the series $\sum z_1 z_2^n$. This series possesses a region of convergence $|z_2| < 1$. It also possesses the spine

$$z_1 = 0 \quad , \quad |z_2| \geqslant 1 \quad .$$

There are several interesting properties of the associated radii of convergence.

__Property 1.__ Monotone property. If $r_1' < r_1''$ and r_2' corresponds to r_1' while r_2'' corresponds to r_1'', then $r_2' \geqslant r_2''$. That is, if the radius in the z_1-plane increases, the radius in the z_2-plane either remains the same or decreases.

__Property 2.__ If r_1', r_2' is a pair of associated radii of convergence of the power series $p(z_1 - a_1, z_2 - a_2)$, then there exists at least one point (z_1', z_2') where

$$|z_1'| = r_1' \quad , \quad |z_2'| = r_2'$$

at which the regular function represented by $p(z_1 - a_1, z_2 - a_2)$ becomes singular. This corresponds to the statement in the theory of functions of one complex variable that the power series expansion of an analytic function holds out to the nearest singular point.

CHAPTER VI

SINGULARITIES.

As in the theory of functions of one complex variable, we say that a function of two complex variables, $f(z_1, z_2)$, has a <u>singular point</u> at (a_1, a_2) if $f(z_1, z_2)$ is not regular at (a_1, a_2) but has points in any neighborhood of (a_1, a_2) at which it is regular.

As in the case of one complex variable, these singular points may be classified as non-essential singular points and essential singular points. However, as we shall see, in the theory of functions of two complex variables the non-essential singular points themselves are of two types, poles or points of indeterminacy.

We say that a function $f(z_1, z_2)$ has a <u>non-essential singularity</u> at the point (a_1, a_2) if there exists a neighborhood $U(a_1, a_2)$ of the point and two functions $g(z_1, z_2)$ and $h(z_1, z_2)$, which have no common factors and are regular in $U(a_1, a_2)$, such that

$$f(z_1, z_2) = \frac{g(z_1, z_2)}{h(z_1, z_2)}$$

in $U(a_1, a_2)$ and $h(a_1, a_2) = 0$. If $g(a_1, a_2) \neq 0$, then (a_1, a_2) is called a <u>pole</u>, or <u>non-essential singularity of the first kind</u>. If $g(a_1, a_2) = 0$, then (a_1, a_2) is called a <u>point of indeterminacy</u>, or a <u>non-essential singularity</u> of the second kind. All singularities which are not non-essential singularities we define as <u>essential singularities</u>. Thus, for example, the function, $f(z_1, z_2) = \dfrac{z_1}{z_2 - a}$ has poles at all points of the type (z_1, a), and a point of indeterminacy at the point $(0,a)$, while the function $f(z_1, z_2) = e^{\frac{z_1}{z_2 - a}}$ has essential singularities at all points of the type (z_1, a).

In discussing functions of two complex variables, reference is often made to functions which are meromorphic in a region. Essentially, we say that a function is <u>meromorphic</u> in a region if it possesses only regular points and poles in that region. More accurately, we define a function $f(z_1, z_2)$ to be <u>meromorphic</u> in a region S of the four-space if *

(a) there is an exceptional point set E not decomposing S; [that is, S - E is connected],

(b) $f(z_1, z_2)$ is regular in S - E,

(c) corresponding to any point of E there exists a neighborhood and two functions $g(z_1, z_2)$ and $h(z_1, z_2)$, which are regular in that neighborhood, such that $f = \dfrac{g}{h}$ [the fraction being in its lowest terms] in the points of S - E lying in the neighborhood, and

(d) the set E must be the minimum set having this property.

In studying the singularities of functions of two complex variables, much of our work will be dependent on a theorem that is generally referred to as the continuity theorem. We shall here state the theorem without proof.**

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* Bochner, p. 198.

** Behnke, p. 49. A proof for this theorem may be found in Bochner, pp. 199-201.

Theorem 6.1.

H1). $f(z_1, z_2)$ is regular on a circle:

$$|z_1 - a_1| = r, \qquad z_2 = a_2 .$$

This circle lies on a two-dimensional plane:

$$z_1 \text{ varying}, \qquad z_2 = a_2 .$$

We shall designate this plane as $\mathcal{E}_0$.

H2). There exists a sequence of two-dimensional planes, designated $\{\mathcal{E}_\nu\}$, which converges to $\mathcal{E}_0$. That is

$$z_2 = \epsilon^{(\nu)} \quad \text{such that} \quad \lim_{\nu \to \infty} \epsilon^{(\nu)} = a_2 .$$

H3). $f(z_1, z_2)$ is regular on these planes for $|z_1 - a_1| \leq r$.

C1). $f(z_1, z_2)$ is regular on $\mathcal{E}_0$ for $|z_1 - a_1| \leq r$.

If we wish to consider single-valued functions only, the continuity theorem may be stated in somewhat more useful form.*

Theorem 6.2.

H1). $f(z_1, z_2)$ is regular on a circle:

$$|z_1 - a_1| = r, \qquad z_2 = a_2 .$$

H2). $f(z_1, z_2)$ is singular at (a_1, a_2).

C1). There always exists a $d > 0$, such that on each plane, $z_2 = \epsilon_2$, where $|a_2 - \epsilon_2| < d$, there is at least one singular point for values of z_1 inside of $|z_1 - a_1| < r$.

This theorem may be proved directly from theorem 6.1.

We shall illustrate theorem 6.2 with examples of several different functions.

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* Behnke, p. 49.

__Example 1.__ Consider $f(z_1, z_2)$ $e^{\frac{z_2}{z_1 - a_1}}$. This function has essential singularities at the points where $z_1 = a_1$. The hypotheses of theorem 6.2 are satisfied since the function is regular on the circle: $|z_1 - a_1| = r$, $z_2 = a_2$, and since the function is singular at (a_1, a_2). Hence, as we would expect, the conclusion of the theorem holds for this function. Moreover, d may have any positive, real value, since no matter what complex value we take for , we have a singular point (a_1, ϵ_2) on the plane $z_2 = \epsilon_2$. This singular point (a_1, ϵ_2) completes the requirements of the theorem by having its z_1-value inside $|z_1 - a_1| < r$.

Note that this function has a two-dimensional manifold, or more specifically a plane, of singularities; that is, for $z_1 = a_1$, any value in the z_2-plane gives an essential singular point.

__Example 2.__ Consider $f(z_1, z_2) = \frac{1}{z_1 - 2z_2}$. This function, we see, has poles at all points (z_1, z_2) for which $z_1 = 2z_2$. Now, applying theorem 6.2, we find that all the hypotheses are satisfied, since the function is regular for values on the circle: $|z_1 - 2a_1| = r$, $z_2 = a_1$, and since the function is singular at $(2a_1, a_1)$. Our next step is to determine in what way the conclusion of the theorem applies. On any plane $z_2 = \epsilon_2$ we have a singular point when $z_1 = 2\epsilon_2$. Conceivably, our choice of the plane $z_2 = \epsilon_2$ might be such that the value $z_1 = 2\epsilon_2$ might lie outside the circle $|z_1 - 2a_1| = r$. Hence, for a singular point $(2\epsilon_2, \epsilon_2)$ satisfying the conditions of the theorem, the z_1-value,

$2\epsilon_2$, must be such that $|2\epsilon_2 - 2a_1| < r$. Simplified, this requires that $|\epsilon_2 - a_1| < \frac{r}{2}$, meaning that the choice $z_2 = \epsilon_2$ must be made within a distance $\frac{r}{2}$ of a_1. In short, for this function, $d = \frac{r}{2}$.

Observe that in this example, also, the function has a two-dimensional manifold of singularities, since for each value of z_2 there is a corresponding value of z_1 giving a pole.

Example 3. Consider $f(z_1, z_2) = \frac{(z_1 - 1)^2}{(z_1 - z_2)}$. It is interesting to note that this function has poles at all points (a, a) except at the point $(1,1)$ where it has a point of indeterminacy since the function takes on a value $\frac{0}{0}$ there.

Seeking to apply theorem 6.2, we again find the hypotheses satisfied, the function being regular on the circle: $|z_1 - 1| = r$, $z_2 = 1$ and having a singularity at the point $(1,1)$. The conclusion of the theorem naturally follows. On any plane $z_2 = \epsilon_2$ we have a singularity when $z_1 = \epsilon_2$. In order for the value of z_1 giving a singular point to be inside the circle $|z_1 - 1| = r$, we see that our choice of ϵ_2 must be such that $|\epsilon_2 - 1| < r$. This means that the value $z_2 = \epsilon_2$ must be chosen inside a circle of radius r about the point $z_2 = 1$ in the z_2-plane. In other words, for this example, $d = r$.

There are several interesting consequences of the continuity theorem which we shall now proceed to state and discuss.

<u>Consequence 1.</u>

H_1). <u>$f(z_1, z_2)$ is regular in a region S of the four-space except for a one-dimensional manifold at the most.</u>

C_1). <u>Then $f(z_1, z_2)$ is regular in the whole interior of S.</u>

Before proving this consequence, let us investigate its meaning. In simpler form, this consequence states that for functions of two complex variables, if there exist any singularities of the function in a given four-dimensional region, then there must be at least a two-dimensional manifold, or a double infinity, so to speak, of singularities of the function in that region. For example, a function cannot have one point of singularity, such as (a_1, a_2). Neither can a function have a one-dimensional manifold of singularities. Referring back to the examples already discussed in this chapter, we see that in each case, the singularities formed a two-dimensional manifold.

This is quite in contrast with functions of one complex variable. For functions of one complex variable, singular points are for the most part isolated. This is always true in the case of poles.* An essential singular point is in most cases isolated. However, an essential singular point may be the limit point of an infinite

* Townsend, p. 270.

sequence of poles or of an infinite sequence of essential singular points.*

We shall now indicate a proof of consequence 1 for the case where S is a generalized dicylinder. By hypothesis our function $f(z_1, z_2)$ is known to be regular everywhere in S except for a one-dimensional manifold. Suppose we consider the case where z_1 is fixed and z_2 describes a curve in the z_2-plane.

By hypothesis we know $f(z_1, z_2)$ to be regular

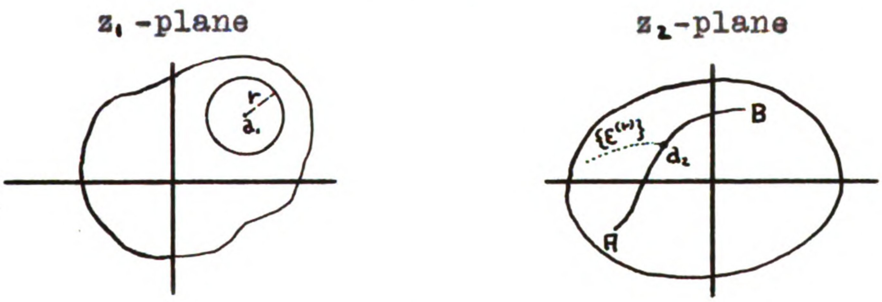

Figure 6.1.

in S except on the one-dimensional manifold: $z_1 = a_1$, z_2 on arc AB. We will take any point a_2 on AB and show that the function is regular at (a_1, a_2). In S on the z_2-plane take a sequence of points, $\{\mathcal{E}^{(\nu)}\}$, none of which lie on AB, with a_2 as the limit point. Construct a circle in S about the point $z_1 = a_1$. Let r designate the radius of the circle. These constructions will be possible for any a_2 chosen in S. Our next step is to apply theorem 6.1, the continuity theorem. Our $f(z_1, z_2)$ is regular on the circle: $|z_1 - a_1| = r$, $z_2 = a_2$, and it is regular for $|z_1 - a_1| \leqslant r$ on the sequence of

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* Townsend, p. 271.

planes $z_2 = \mathcal{E}^{(\nu)}$. Hence, by the theorem, $f(z_1, z_2)$ is regular for $|z_1 - a_1| \leqslant r$, $z_2 = a_2$, and so the point (a_1, a_2) is a regular point. Since this is true for any choice of a_2, there cannot be any such manifold of singularities. A proof for the general case would be similar.

Consequence 2.

H_1). Given a dicylinder S:

$$|z_1 - a_1| < d_1, \qquad |z_2 - a_2| < d_2.$$

H_2). On each two-dimensional plane $z_2 = b_2$, where b_2 is a value in $|z_2 - a_2| < d_2$, there is at most one singular point of the regular function, $f(z_1, z_2)$.

H_3). $f(z_1, z_2)$ has at least one singular point, P, in S.

C_1). There exists exactly one two-dimensional surface, F, passing through P such that all singularities of $f(z_1, z_2)$ which lie in S lie on F, and each point of F is a singular point of $f(z_1, z_2)$.

C_3). F satisfies an equation $z_1 - g(z_2)$ for $|z_2 - a_2| < d_2$.

Several examples will serve to illustrate this consequence.

Example 1. Consider $f(z_1, z_2) = \dfrac{z_2}{z_1 - b_1}$ in a dicylinder: $|z_1 - a_1| < d_1$, $|z_2 - a_2| < d_2$. Take d_1 such that $d_1 > |a_1 - b_1|$, that is, such that b_1 is in the dicylinder. The hypotheses of consequence 2 are all satisfied since on each two-dimensional plane $z_2 = b_2$, where b_2 is a value in $|z_2 - a_2| < d_2$, we have only one singular point (b_1, b_2), and since the function has at least one singularity in the dicylinder; for example, the point $(b_1,\ a_2+\frac{d_2}{2})$. Now, the question is, can we find the surface F of the type described in the conclusions of consequence 2. Consider the plane $z_1 = b_1$, z_2 varying. This plane passes through $(b_1, a_2+\frac{d_2}{2})$. All the singular points of $f(z_1,z_2)$ lie on this plane, for in order that a point be a singularity, it must have a z_1-coordinate, $z_1 = b_1$. Furthermore, each point of this plane is a singular point of $f(z_1,z_2)$ for the same reason. Finally, this plane has an analytic representation, $z_1 = b_1$.

Example 2. Consider $f(z_1,z_2) = \frac{1}{(z_1-z_2)(z_1+z_2)}$ in a dicylinder about the origin: $|z_1| < d_1$, $|z_2| < d_2$. Immediately we see that the second hypothesis of consequence 2 is not satisfied, since for any plane $z_2 = a_2$, where a_2 is a value in $|z_2| < d_2$, there exist two singular points (a_2,a_2) and $(-a_2,a_2)$. Hence it is impossible to find any two-dimensional surface F on which all the singular points of $f(z_1,z_2)$ lie.

Consequence 3.

H_1). The point set F lies in a region S and is cut by any two-dimensional plane $z_2 = b_2$ once at the most.

H_2). $f(z_1,z_2)$ is regular at every point of S which does not belong to F.

H_3). $f(z_1,z_2)$ is singular in at least one point of F.

C_1). $f(z_1,z_2)$ is singular at every point of F.

The meaning of this consequence may be somewhat clarified by the discussion of an example.

<u>Example 1</u>. Consider $f(z_1, z_2) = \sin\left(\frac{1}{z_1 - z_2}\right)$. Let S be a generalized dicylinder about the origin, and let F be the set of points (z_1, z_2) lying in S such that $z_1 = z_2$. We see that F is cut by any plane $z_2 = b_2$ in one point at the most, namely, the point (b_2, b_2). Also $f(z_1, z_2)$ is regular at every point (z_1, z_2) of S where $z_1 \neq z_2$; and $f(z_1, z_2)$ possesses an essential singularity at the point $(1,1)$. Thus, all the hypotheses of consequence 3 are satisfied, and $f(z_1, z_2)$ should be singular at every point of F. Upon examining our function, we see that at every point (z_1, z_2) for which $z_1 = z_2$, $f(z_1, z_2)$ possesses an essential singularity. These are the points of F.

<u>Consequence 4</u>.

H_1). $f(z_1, z_2)$ <u>is regular at all boundary points of a region</u> S.

H_2). S <u>is a closed, finite region with a connected boundary</u>.

C_1). $f(z_1, z_2)$ <u>can be analytically continued to each inner point of</u> S.

This is the most important of the consequences we shall discuss. To better show its importance we might restate it more simply. Consequence 4 states that <u>if a function is singular anywhere in the interior of a closed, finite region with a connected boundary, then that function is necessarily</u>

singular in at least one boundary point of the region.

Herein the theory of functions of two complex variables differs greatly from the theory of functions of one complex variable. In the case of one variable a function may be regular on the boundary of a region, yet still have singularities inside that region. For example, the function $f(z) = \dfrac{1}{z-a}$ has a singular point $z = a$, yet on the boundary of any region enclosing $z = a$ as an inner point, $f(z)$ is regular.

We might investigate a few examples of functions of two complex variables as consequence 4 applies to them.

Example 1. Consider $f(z_1, z_2) = \dfrac{z_1}{z_2}$. We shall use three different regions to illustrate three different applications.

(a). Let S be the dicylinder: $|z_1| < d_1$, $|z_2 - a_2| < d_2$, where $|a_2| > d_2$, as shown in figure 6.2. The function

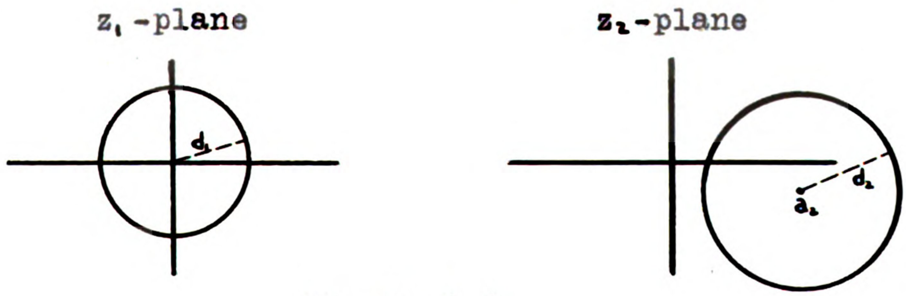

Figure 6.2.

$f(z_1, z_2) = \dfrac{z_1}{z_2}$ is regular at all points of the boundary, since the value $z_2 = 0$ has been excluded from the region. By the conclusion of consequence 4, there can be no singular points inside the region. We see that this is true, since all the singularities of the function lie on the plane $z \quad 0$.

(b). Let S be the dicylinder: $|z_1| < d_1$, $|z_2| < d_2$. The function $f(z_1, z_2) = \dfrac{z_1}{z_2}$ has a pole at the point $(\frac{d_1}{2}, 0)$.

Since this point is interior to the dicylinder, by the restatement of consequence 4 there must be at least one singular point on the boundary of the dicylinder. The point $(d_1, 0)$ is such a point. In fact, every point $(z_1, 0)$, where $|z_1| = d_1$, is a singular point on the boundary of S.

(c). It should be noticed that a function may have a singular point on the boundary of the region which is being considered without being singular at any point interior to the region. This does not contradict the statements of the theorem. If we consider the same function, $f(z_1, z_2) = \frac{z_1}{z_2}$, in a dicylinder: $|z_1| < d_1$, $|z_2 - a_2| < |a_2|$, we have such a situation. For this

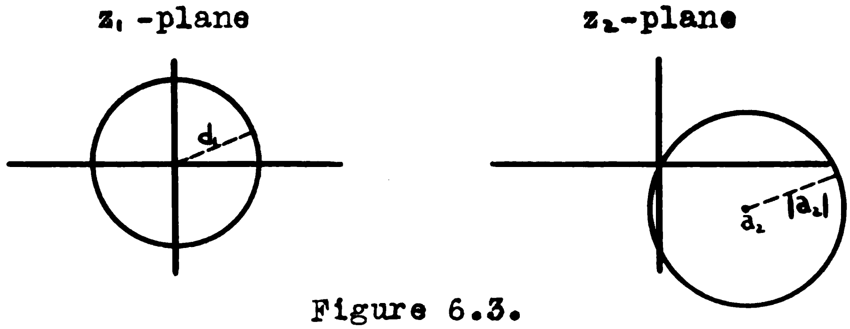

Figure 6.3.

dicylinder the function has poles at all the boundary points $(z_1, 0)$ such that $|z_1| = d_1$. Yet there are no singularities inside the region, since in order for a point to be a singular point, its z_2-coordinate must be zero, and only boundary points possess such a value for their z_2-coordinate.

CHAPTER VII

ZEROS. NON-ESSENTIAL SINGULAR POINTS. MITTAG-LEFFLER'S THEOREM. THE WEIERSTRASS PRODUCT THEOREM.

Zeros. For functions of two complex variables we can define zeros in the same way that they are defined for functions of one complex variable. If $f(z_1, z_2)$ vanishes at the point (a_1, a_2) and is regular in a neighborhood of (a_1, a_2), we say that (a_1, a_2) is a zero of $f(z_1, z_2)$.

From our discussion of singularities, where those singularities might be confined to poles, we would expect that if there exist any zeros of a function in some region there will be at least a two-dimensional manifold of such zeros. Suppose we wish to consider a function $f(z_1, z_2)$ for its zeros in a region S. We will accomplish the same end by considering the function $\frac{1}{f(z_1, z_2)}$ for its poles in S. By consequence 1 of theorem 6.1, if $\frac{1}{f(z_1, z_2)}$ has any poles, it will possess at least one two-dimensional manifold of poles. Hence, $f(z_1, z_2)$, if it has any zeros in S, will possess at least one two-dimensional manifold of zeros in S. In a similar way the other consequences of the continuity theorem may be interpreted in terms of zeros by using the function $\frac{1}{f(z_1, z_2)}$.

In order to determine these zero manifolds as we shall call them, the preparation theorem of Weierstrass is used.*

Theorem 7.1.

H_1). $f(z_1, z_2)$ is regular at the point (a_1, a_2).

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* Behnke, p. 57.

H_2). $f(a_1, a_2) = 0$, but $f(z_1, z_2) \not\equiv 0$.

C_1). There exists a neighborhood $U(a_1, a_2)$ of the point (a_1, a_2) such that there exist:

a). a function $\frac{1}{Q(z_1, z_2)}$ which is regular in U and does not vanish in U,

b). a whole number $\mu \geqslant 0$,

c). a function $\psi(z_1, z_2)$ which is identically equal to 1, or of the form

(1)
$$\psi(z_1, z_2) = (z_1 - a_1)^m + A_1(z_2)(z_1 - a_1)^{m-1} + \cdots + A_m(z_2).$$

where the $A_i(z_2)$ are regular in U and vanish at $z_2 = a_2$,

such that $f(z_1, z_2)$ can be written in the form,

(2)
$$f(z_1, z_2) = (z_2 - a_2)^{\mu} \frac{1}{Q(z_1, z_2)} \psi(z_1, z_2).$$

C_2). If $f(z_1, a_2) \equiv 0$, then the factor $(z_2 - a_2)^{\mu}$ is omitted from representation (2).

A function of type (1) is called a distinguished polynomial. Thus we define a distinguished polynomial with respect to $z_2 = a_2$ to be a function of the form

$$(z_1 - a_1)^m + A_1(z_2)(z_1 - a_1)^{m-1} + \cdots + A_m(z_2),$$

where the $A_i(z_2)$ are regular in the neighborhood of a point $z_2 = a_2$ and vanish at $z_2 = a_2$.

Observing (2) we note that we can now find, in the neighborhood $U(a_1, a_2)$, the manifolds upon which $f(z_1, z_2)$ equals zero. Naturally, one such manifold will be the plane $z_2 = a_2$ if the factor $(z_2 - a_2)^{\mu}$ occurs in the

representation (2). The factor $\frac{1}{Q(z_1,z_2)}$ does not vanish in $U(a_1,a_2)$, by definition; hence, this factor will give rise to no zeros of the function $f(z_1,z_2)$ in $U(a_1,a_2)$. Obviously, we can obtain all other zero manifolds of $f(z_1,z_2)$ by setting the factor $\Psi(z_1,z_2)$ equal to zero and solving the resulting m'th degree equation. This gives us m roots,

$$z_1 = g_1(z_2),\quad z_1 = g_2(z_2),\quad \ldots\ ,\quad z_1 = g_m(z_2).$$

These are the m manifolds upon which $\Psi(z_1,z_2)$ equals zero, and therefore they are the m manifolds upon which $f(z_1,z_2)$ equals zero in the neighborhood $U(a_1,a_2)$.

Bochner gives a method for finding this distinguished polynomial $\Psi(z_1,z_2)$ for a given function, $f(z_1,z_2)$.* We shall outline the method and illustrate with an example. This method uses the origin, (0,0), as the point (a_1,a_2), in a neighborhood of which we desire to investigate the zero manifolds of $f(z_1,z_2)$. This choice does not limit the generality of the method, since our given function can always be transformed by use of substitutions $z_1' = (z_1 - a_1)$ and $z_2' = (z_2 - a_2)$, enabling us to consider the transformed function at (0,0), rather than the given function at (a_1,a_2). Also, we shall consider functions such that $f(z_1,a_2) \not\equiv 0$, permitting us by conclusion 2 of the theorem to omit the factor $(z_2 - a_2)^{\mu}$ from representation (2).

Thus we are considering $f(z_1,z_2)$ about the point (0,0), and by the preparation theorem, this function has a representation,

$$(3)\qquad f(z_1,z_2) = \frac{1}{Q(z_1,z_2)}\Psi(z_1,z_2) \quad .$$

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* Bochner, pp. 183-184.

Our task is to find $Q(z_1, z_2)$ and $\Psi(z_1, z_2)$. Let

$$\Psi(z_1, z_2) = B(z_1, z_2) + H(z_1, z_2).$$

This enables us to choose $B(z_1, z_2)$ arbitrarily. Then we must find an $H(z_1, z_2)$ which added to $B(z_1, z_2)$ will give a distinguished polynomial, $\Psi(z_1, z_2)$. Substituting $B(z_1, z_2) + H(z_1, z_2)$ for $\Psi(z_1, z_2)$ in (3) and solving for $H(z_1, z_2)$, we get

(4) $$H(z_1, z_2) = f(z_1, z_2)Q(z_1, z_2) - B(z_1, z_2).$$

Therefore, to find $H(z_1, z_2)$ we must find $Q(z_1, z_2)$, given $f(z_1, z_2)$ and having chosen $B(z_1, z_2)$ arbitrarily. Since $\dfrac{1}{Q(z_1, z_2)}$ must not vanish in $U(a_1, a_2)$, $Q(z_1, z_2)$ must be regular in $U(a_1, a_2)$, and hence may be expanded in a power series

(5) $$Q(z_1, z_2) = \sum_{m,n=0}^{\infty} q_{m,n} z_1^m z_2^n .$$

The coefficients $q_{m,n}$ may be found by the use of a recursion formula which involves the coefficients $b_{m,n}$ and $f_{m,n}$ from the expansions

(6) $$B(z_1, z_2) = \sum_{m,n=0}^{\infty} b_{m,n} z_1^m z_2^n ,$$

(7) $$f(z_1, z_2) = \sum_{m,n=0}^{\infty} f_{m,n} z_1^m z_2^n$$

for $B(z_1, z_2)$ and $f(z_1, z_2)$. This recursion formula is *

(8) $$q_{m,n} = b_{m,n} - \sum_{\mu=0}^{m+k} \sum_{\nu=0}^{n-1} q_{\mu,\nu} f_{m+k-\mu,n-\nu} - \sum_{\mu=0}^{m-1} q_{\mu,n} f_{m+k-\mu,0}$$

where k is the power of the first term in the series

(9) $$f(z_1, z_2) = \sum_{m=0}^{\infty} f_m(z_2)\, z_1^m$$

for which $f_m(0) \neq 0$.

__Example.__ Consider

$$f(z_1, z_2) = z_2^2 + 3z_2 z_1 + z_1^3 + z_1^2 - 3z_2 z_1^2$$

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* Bochner, p. 184.

for its zero manifolds in a neighborhood $U(0,0)$. In most cases the function being investigated for its zero manifolds would be an infinite series. The zero manifolds of the function we are investigating would be most easily found by setting the function, as it stands, equal to zero and solving the resulting equation for its roots. However, for purposes of illustration, we shall carry through the work for this function in the manner described on pages 53 and 54.

Expressing $f(z_1, z_2)$ as a series in the form of (9), we obtain

(10) $$f(z_1, z_2) = z_2^2 + (3z_2)z_1 + (1 - 3z_2)z_1^2 + z_1^3 .$$

Since $(1-3z_2) \neq 0$ when $z_2 = 0$, then $k = 2$.

We then choose $B(z_1, z_2) = z_1^2$. Considered as a power series in the form (6), this means that all $b_{m,n} = 0$, except $b_{2,0} = 1$. Considering $f(z_1, z_2)$ as a power series in the form (7), we get for the coefficients,

$$p_{0,0} = 0, \quad p_{0,1} = 0, \quad p_{0,2} = 1, \ldots$$
$$p_{1,0} = 0, \quad p_{1,1} = 3, \quad p_{1,2} = 0, \ldots$$
$$p_{2,0} = 1, \quad p_{2,1} = -3, \ldots$$
$$p_{3,0} = 1, \ldots \text{ and all other } p_{m,n} = 0.$$

Next, using our recursion formula, (8), we find

$q_{0,0} = 1$, $q_{0,1} = 6$, $q_{0,2} = 53$, $q_{0,3} = 567$, $q_{0,4} = 6744$, . . .

$q_{1,0} = -1$, $q_{1,1} = -12$, $q_{1,2} = -142$, $q_{1,3} = -1770$, $q_{1,4} = -23101$, . . .

$q_{2,0} = 1$, $q_{2,1} = 18$, $q_{2,2} = 267$, $q_{2,3} = 3825$, $q_{2,4} = 54795$, . . .

$q_{3,0} = -1$, $q_{3,1} = -24$, $q_{3,2} = -428$, $q_{3,3} = -6948$, $q_{3,4} = -108846$, . . .

$q_{4,0} = 1$, $q_{4,1} = 30$, $q_{4,2} = 625$, $q_{4,3} = 11355$, $q_{4,4} = 193570$, . . .

Substituting these coefficients in the expansion (5) for $Q(z_1, z_2)$ we obtain that expansion up to and including $m=4$, $n=4$. Multiplying this by $f(z_1, z_2)$, we get

$$\begin{aligned} \Psi(z_1, z_2) &= B(z_1, z_2) + H(z_1, z_2) \\ &= z_1^2 + (3z_2 + 17z_2^2 + 147z_2^3 + 1559z_2^4 + \ldots)z_1 \qquad (11) \\ &\quad + (z_2^2 + 6z_2^3 + 53z_2^4 + \ldots). \end{aligned}$$

This, we see, is a distinguished polynomial since the coefficient of the highest power of z_1 is equal to 1 and all the other coefficients of powers of z_1 are regular in the neighborhood of the origin and vanish at the origin. To find the zero manifolds in the neighborhood of $(0,0)$, setting (11) equal to zero and solving for the two roots, we obtain the zero manifolds

$$z_1 = g_1(z_2), \text{ and } z_1 = g_2(z_2),$$

for the given function in the neighborhood of $(0,0)$.

Our discussion of zeros now leads us to *

Theorem 7.2.

H_1). $f(z_1, z_2)$ is regular at (a_1, a_2).

H_2). $f(a_1, a_2) = 0$, but $f(z_1, z_2) \not\equiv 0$.

C_1). In a sufficiently small neighborhood $U(a_1, a_2)$ of the point (a_1, a_2) all zeros of $f(z_1, z_2)$ lie on a finite number of two-dimensional analytic pieces of surface.

C_2). All the points of these analytic pieces of surface, if they lie in $U(a_1, a_2)$, are zeros of $f(z_1, z_2)$.

* Behnke, p. 59.

Thus we see that the zeros of functions of two complex variables differ from the zeros of functions of one complex variable in much the same way as do singularities for the two cases. For functions of two complex variables, if there are any zeros they will form at least one two-dimensional manifold. In contrast to this, functions of one complex variable can have a zero at a point. The function $f(z) = z - a$, for example, has a zero at the point $z = a$.

In the theory of functions of one complex variable we have a theorem dealing with the zeros of a function which is sometimes referred to as the unicity theorem.*

Theorem 7.3a.

H_1). $f(z)$ is regular in a region S.

H_2). $f(z)$ is equal to zero at an infinite sequence of points, z', z'', z''', . . . , which have a point interior to S as a limit point.

C_1). $f(z) \equiv 0$ in S.

The unicity theorem for functions of two complex variables is **

Theorem 7.3b.

H_1). $f(z_1, z_2)$ is regular in a region S.

H_2). $f(z_1, z_2)$ is equal to zero everywhere in S', where S' is a region interior to S.

C_1) $f(z_1, z_2) \equiv 0$ in S.

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* Copson, p. 74.
** Bochner, p. 174.

We shall omit the proof of this theorem.

It is interesting to note that in the case of one complex variable we need know that the function is zero at a sequence of points, only, while in the case of two variables we must know that the function equals zero everywhere in a subregion of the region in which the function is regular. The reason for this difference can be pointed out if we recall that functions of two complex variables which are not identically zero can still have two-dimensional manifolds upon which they are equal to zero. Hence, when considering a function of two complex variables, we could get a sequence of points, lying on the zero manifolds of the function, at which the function takes on the value zero, without having our function identically equal to zero. However, if we know the function to be zero at an infinite sequence of points which do not lie on these zero manifolds, then it necessarily follows that the function is identically equal to zero in the region being considered.

Closely related to the unicity theorem for functions of one complex variable we have the following theorem.*

Theorem 7.4a.

H_1). $f(z)$ and $g(z)$ are two functions regular in a region S.

H_2). $f(z)=g(z)$ at an infinite sequence of points, $z', z'', z''', \ldots$, which have a point interior to S as a limit point.

C_1). $f(z)\equiv g(z)$ in S.

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* Townsend, p. 248.

We can state a corresponding theorem for functions of two complex variables.

Theorem 7.4b.

H_1). $f(z_1, z_2)$ and $g(z_1, z_2)$ are regular in a region S.

H_2). $f(z_1, z_2) = g(z\ , z\ )$ for all points (z_1, z_2) in a region S' interior to S.

C_1). $f(z_1, z_2) \equiv g(z\ , z\ )$ in S.

Non-essential Singular Points. We have defined non-essential singular points in chapter VI. A function $f(z_1, z_2)$ is said to have a non-essential singularity at the point (a_1, a_2) if there exist a neighborhood $U(a_1, a_2)$ of the point and two functions $f(z_1, z_2)$ and $g(z_1, z_2)$, which have no common factors and are regular in $U(a_1, a_2)$, such that $f(z_1, z_2) = \dfrac{g(z_1, z_2)}{h(z_1, z_2)}$ in $U(a_1, a_2)$ and $h(a_1, a_2) = 0$.

We also mentioned that there are two types of non-essential singular points. A non-essential singular point is called a pole or a non-essential singularity of the first kind if $g(a_1, a_2) \neq 0$. In this case $f(z_1, z_2)$ tends toward infinity as (z_1, z_2) approaches the point (a_1, a_2). We say that $f(z_1, z_2)$ has the value ∞ at the point (a_1, a_2). A non-essential singular point is called a point of indeterminacy or a non-essential singularity of the second kind if $g(a_1, a_2) = 0$.

It is possible to show that if (a_1, a_2) is a point of indeterminacy of the function $f(z_1, z_2)$ and if α is any preassigned value, that $f(z_1, z_2)$ assumes the value α somewhere in any neighborhood of (a_1, a_2).* Also, for functions of two complex variables, points of in-

* Bochner, p. 199.

determinacy are always isolated. This follows from the fact that $g(z_1, z_2) = 0$, $h(z_1, z_2) = 0$ give four equations in the four variables x_1, x_2, y_1, y_2 when we substitute $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ and equate real and imaginary parts. Solving these four equations, we will obtain a finite number of solutions, that is, points where $g(z_1, z_2) = 0$ and $h(z_1, z_2) = 0$. Also, for functions of two complex variables the points of indeterminacy are always limit points of non-essential singularities of the first kind. These statements are clearly illustrated by example 3 of theorem 6.2.

In conclusion we might state a theorem for non-essential singular points corresponding to theorem 7.2 on zeros.*

Theorem 7.5.

H1). $f(z_1, z_2)$ has a non-essential singular point at (a_1, a_2).

C1). All the singularities of $f(z_1, z_2)$ in a sufficiently small neighborhood $U(a_1, a_2)$ of the point (a_1, a_2) lie on a finite number of analytic pieces of surface.

C2). All the points on these pieces of surface, so far as they lie in $U(a_1, a_2)$ are non-essential singular points of $f(z_1, z_2)$.

Mittag-Leffler's Theorem and the Weierstrass Product Theorem. In the theory of functions of one complex variable these two theorems concern the possibilities of forming a function which has poles or zeros at an infinite number of

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* Behnke, p. 61.

previously assigned points.

Mittag-Leffler's theorem for functions of a single complex variable is *

Theorem 7.6a.

H_1). Given an infinite set of points, $z_1, z_2, z_3, \ldots, z_k, \ldots$ such that

$$0 < |z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots \leqslant |z_k| \leqslant \cdots$$

H_2). $\lim\limits_{k \to \infty} z_k = \infty$.

H_3). Corresponding to each z_k there is given an arbitrarily chosen integral function of $\dfrac{1}{z - z_k}$, namely, $G_k\left(\dfrac{1}{z - z_k}\right)$.

C_1). There exists a single-valued, analytic function which is regular for all finite values of z except $z = z_k$.

C_2). This function has $G_k\left(\dfrac{1}{z - z_k}\right)$ as the principal part of its expansion in the neighborhood of z_k.

The Weierstrass product theorem, on the other hand, concerns zeros.**

Theorem 7.7a.

H_1). Given an infinite set of points, $z_1, z_2, z_3, \ldots, z_k, \ldots$ not including the origin, such that

$$|z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots \leqslant |z_k| \leqslant \cdots$$

H_2). $\lim\limits_{k \to \infty} z_k = \infty$.

H_3). $G(z)$ is an integral function.

C_1). There exists a transcendental integral function

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* Townsend, p. 304.
** Townsend, p. 313.

<u>of the form</u>

$$\varphi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right) e^{\frac{z}{z_k} + \frac{1}{2}\left(\frac{z}{z_k}\right)^2 + \cdots + \frac{1}{m_k - 1}\left(\frac{z}{z_k}\right)^{m_k - 1}}$$

<u>having the points</u> z_k <u>and no others as zero points.</u>

C_2). <u>The function</u>

$$F(z) = e^{G(z)} \cdot \varphi(z)$$

<u>is the most general function having this property.</u>

The difficulty in generalizing these theorems to the case of two complex variables lies in the fact that for functions of two complex variables, as we have already seen, the zeros and poles are not isolated but lie on two-dimensional manifolds. However, theorems have been developed which, although not complete generalizations, may be called the corresponding theorems for functions of two complex variables.

First, we require two definitions. Two functions, $f(z_1, z_2)$ and $h(z_1, z_2)$, which are meromorphic in the neighborhood of a point P, are called <u>equivalent with respect to subtraction at P</u> in case the difference, $f(z_1, z_2) - h(z_1, z_2)$, is regular at P. Two functions, $f(z_1, z_2)$ and $h(z_1, z_2)$, which are regular in the neighborhood of a point P, are called <u>equivalent with respect to division at P</u> in case the quotient, $\dfrac{f(z_1, z_2)}{h(z_1, z_2)}$, is regular and different from zero at P.

The theorem concerning poles, corresponding to Mittag-Leffler's theorem is *

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* Behnke, p. 64.

Theorem 7.6b.

H_1). To each point P of a dicylinder $\mathfrak{Z}$ let there be associated a neighborhood U(P) and a function $f_P(z_1, z_2)$ which is meromorphic there, such that for any point Q chosen from U(P), where $f_Q(z_1, z_2)$ is the function associated with Q, the functions $f_P(z_1, z_2)$ and $f_Q(z_1, z_2)$ are equivalent with respect to subtraction.

C_1). There exists a function, $F(z_1, z_2)$, meromorphic everywhere in $\mathfrak{Z}$, which is at each point P equivalent with respect to subtraction with the associated function $f_P(z_1, z_2)$.

The theorem for functions of two complex variables corresponding to the Weierstrass product theorem is *

Theorem 7.7b.

H_1). To each point P of a dicylinder $\mathfrak{Z}$ let there be associated a neighborhood U(P) and a function $f_P(z_1, z_2)$ which is regular there, such that for any point Q chosen from U(P), where $f_Q(z_1, z_2)$ is the function associated with Q, $f_P(z_1, z_2)$ and $f_Q(z_1, z_2)$ are equivalent with respect to division.

C_1). There exists a function $G(z_1, z_2)$, regular in $\mathfrak{Z}$, which is at each point P equivalent with respect to division with the associated function $f_P(z_1, z_2)$.

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* Behnke, p. 65.

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