TRANSFORMATIONS OF CURVES AND NETS OF CURVES IN THE PLANE

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TRANSFORMATIONS OF CURVES AND NETS OF CURVES IN THE PLANE

1. INTRODUCTION

It is the purpose of this thesis to study in some detail certain real point transformations of curves and nets of curves in the plane into curves and nets of curves* in the same plane. We first set up some useful formulae in the case in which the transformation is a general point transformation. We then specialize the transformations in various ways. For example, we consider certain transformations which we shall call E transformations. If we further specialize the transformations we find that the transformation is a transformation by reciprocal radii.

Let the non-homogeneous coordinates x_1, x_2 of a point P_x be given as analytic functions of two variables u, v. The locus of P_x is a net of curves in the plane.

Let the curves $u =$ const. and $v =$ const. be the lines respectively perpendicular to the $x-$ axis and parallel to the $x-$axis. The parametric equations of x may therefore be made to assume the simple form

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*V. G. Grove, Contributions to the theory of transformations of nets in a space , Transactions of the American Mathematical Society, Vol.35, No.3, pp.683-688 Hereafter referred to as Grove, Theory of transformations.

(1) $$x_1 = u, \; x_2 = v.$$

It follows therefore, that

(2) $$\begin{aligned} x_{1u} &= 1, & x_{1v} &= 0, \\ x_{2u} &= 0, & x_{2v} &= 1, \\ x_{uu} = 0, \; x_{uv} &= 0, & x_{vv} &= 0. \end{aligned}$$

Let P_y be a point in the given plane. It follows that its coordinates y_1, y_2 are defined by an expression of the form

(3) $$y = x + \theta x_u + \phi x_v.$$

wherein θ and ϕ are arbitrary functions of u and v.

The purpose of this thesis is to discuss the transformation of curves and nets of curves in the plane by the transformation (3).

2. THE TRANSFORMATION E

A transformation will be said to be an E transformation* if and only if the point of intersection of the tangent lines to the corresponding curves is equally distant from the corresponding points P_x and P_y.

Let the tangent to the curve $v = v(u)$ through the point P_x pass through the point z, whose coordinates

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*V. G. Grove, The transformation E of nets, Transactions of the American Mathematical Society, Vo. 33, No. 1, pp. 147-152.

are defined by an expression of the form

(4) $$z_1 = x + \mu_1 (x_u + \lambda x_v)$$

wherein

$$\frac{dv}{du} = \lambda, \quad \frac{dx}{du} = x_u + \lambda x_v .$$

Likewise let the tangent to the corresponding curve through the point P_y pass through the point z_2 whose coordinates are defined by an expression of the form

(5) $$z_2 = y + \mu_2 (y_u + \lambda y_v).$$

Computing the derivatives of y with respect to u and v, respectively, and making use of equations (2) we find that

(6) $$\begin{aligned} y_u &= (1 + \theta_u) x_u + \phi_u x_v, \\ y_v &= (1 + \phi_v) x_v + \theta_v x_u . \end{aligned}$$

By substituting in (5) the values for y_u and y_v given in (6) we obtain

(7) $$z_2 = x + \left\{\theta + \mu_2[(1+\theta_u) + \lambda\theta_v]\right\} x_u + \left\{\phi + \mu_2[\phi_u + \lambda(1+\phi_v)]\right\} x_v .$$

Let P_{z_1}, coincide with P_{z_2}. From (4) and (7) we obtain the following equations

(8) $$\begin{aligned} \mu_1 - \mu_2[(1+\theta_u) + \lambda\theta_v] &= \theta, \\ \lambda\mu_1 - \mu_2[\phi_u + \lambda(1+\phi_v)] &= \phi . \end{aligned}$$

Solving (8) for μ_1 and μ_2 we obtain

(9) $$\mu_1 = \frac{\phi[(1+\theta_u) + \lambda\theta_v] - \theta[\phi_u + \lambda(1+\phi_v)]}{(\phi_u + \lambda\phi_v) - \lambda(\theta_u + \lambda\theta_v)},$$

$$\mu_2 = \frac{\lambda\theta - \phi}{(\phi_u + \lambda\phi_v) - \lambda(\theta_u + \lambda\theta_v)} .$$

$$(\phi_u + \lambda\phi_v) - \lambda(\theta_u + \lambda\theta_v) \neq 0.$$

Let the distance from z_1 to x be d_1 and that from z_2 to y be d_2. It follows that

$$(10)\qquad \begin{aligned} d_1^2 &= \mu_1^2[\Sigma x_u^2 + 2\lambda\Sigma x_u x_v + \lambda^2\Sigma x_v^2], \\ d_2^2 &= \mu_2^2[\Sigma y_u^2 + 2\lambda\Sigma y_u y_v + \lambda^2\Sigma y_v^2]. \end{aligned}$$

If we define E, F, G and $\bar{E}, \bar{F}, \bar{G}$ respectively, by the formulae

$$(11)\qquad \begin{aligned} &E = \Sigma x_u^2,\ F = \Sigma x_u x_v,\ G = \Sigma x_v^2, \\ &\bar{E} = \Sigma y_u^2,\ \bar{F} = \Sigma y_u y_v,\ \bar{G} = \Sigma y_v^2, \end{aligned}$$

it follows from (2) that

$$(12)\qquad E = G = 1,\ F = 0;$$

and, likewise, from (6) it follows that

$$(13)\qquad \begin{aligned} \bar{E} &= (1+\theta_u)^2 + \phi_u^2, \\ \bar{F} &= \theta_v(1+\theta_u) + \phi_u(1+\phi_v), \\ \bar{G} &= (1+\phi_v)^2 + \theta_v^2. \end{aligned}$$

If our transformation is an E transformation it is necessary that

$$(14)\qquad d_1 = d_2$$

After some computation involving the use of equations (9) to (13), inclusive, this condition may be written

$$(15)\qquad \begin{aligned} &\left\{[\theta\phi_u - \phi(1+\theta_u)] + \lambda[\theta(1+\phi_v) - \theta_v\phi]\right\}^2(1+\lambda^2) \equiv \\ &(\lambda\theta - \phi)^2\left\{[(1+\theta_u)^2 + \phi_u^2] + 2\lambda[\theta_v(1+\theta_u) + \phi_u(1+\phi_v)] + \lambda^2[(1+\phi_v)^2 + \theta_v^2]\right\}. \end{aligned}$$

We may now consider (15) as an identity in λ and equate coefficients of corresponding terms. Hence

$$
\begin{aligned}
&[\theta\phi_u - \phi(1+\theta_u)]^2 = \phi^2[(1+\theta_u)^2 + \phi_u^2]\,,\\
&[\theta(1+\phi_v) - \theta_v\phi][\theta\phi_u - \phi(1+\theta_u)] =\\
&\qquad -\theta\phi[(1+\theta_u)^2+\phi_u^2] + \phi^2[\phi_u(1+\phi_v)+\theta_v(1+\theta_u)],\\
&[\theta(1+\phi_v)-\theta_v\phi]^2 + [\theta\phi_u - \phi(1+\theta_u)]^2 = \theta^2[(1+\theta_u)^2+\phi_u^2]\\
(16)\quad &-4\theta\phi[\phi_u(1+\phi_v)+\theta_v(1+\theta_u)] + \phi^2[(1+\phi_v)^2+\theta_v^2],\\
&[\theta(1+\phi_v)-\theta_v\phi][\theta\phi_u - \phi(1+\theta_u)] =\\
&\qquad \theta^2[\theta_v(1+\theta_u)+\phi_u(1+\phi_v)] - \theta\phi[(1+\phi_v)^2+\theta_v^2],\\
&[\theta(1+\phi_v)-\theta_v\phi]^2 = \theta^2[(1+\phi_v)^2+\theta_v^2]\,.
\end{aligned}
$$

The first and last equations in (16) may be written in the following simpler form, respectively,

$$
(17)\qquad
\begin{aligned}
&(\theta^2-\phi^2)\phi_u = 2\theta\phi(1+\theta_u),\\
&(\phi^2-\theta^2)\theta_v = 2\theta\phi(1+\phi_v).
\end{aligned}
$$

By combining the two equations of (17) we obtain

$$(18)\qquad (1+\theta_u)\theta_v + (1+\phi_v)\phi_u = 0.$$

Likewise by using (18) and the second, third, and fourth equations of (16) we obtain

$$(19)\qquad (1+\theta_u)^2+\phi_u^2 = (1+\phi_v)^2+\theta_v^2.$$

By combining (18) and (19) we obtain the equation

$$(\theta_v^2-\phi_u^2)[(1+\phi_v)^2+\theta_v^2] = 0.$$

Since we are restricting ourselves to real transformations we get from

$$(1+\phi_v)^2+\theta_v^2 = 0$$

$$1+\phi_v = 0,\quad \theta_v = 0$$

Therefore

$$\phi = -v + \mho_1(u),\quad \theta = \mho_2(u).$$

It follows that

$$y_1 = y_1(u), \; y_2 = y_2(u).$$

Hence the locus of P_y is not a net but a curve. We will therefore restrict ourselves to the case wherein

(20) $$\theta_v^2 = \phi_u^2.$$

From (18), the third equation of (16), and (19) we obtain

$$(\theta^2 - \phi^2)(\phi_u^2 - \theta_v^2) = 0.$$

The second factor in this equation leads to the same conclusion as (20). Furthermore

$$\theta^2 - \phi^2 \neq 0.$$

The existence of the above inequality may be demonstrated by assuming $\theta^2 - \phi^2 = 0$ and solving (17) for θ and ϕ. We obtain

$$\theta = -u + c_1, \; \phi = -v + c_2.$$

Substituting these solutions in (4) we find that

$$y_1 = c_1, \; y_2 = c_2.$$

Hence <u>if $\theta^2 - \phi^2 = 0$</u>, <u>the point P_y is a fixed point</u>. By using (17) and the fourth equation of (16) we find that (20) may be written in the form

(21) $$\theta_v = \phi_u$$

From (17) and (21) it follows that

(22) $$\theta_u + \phi_v + 2 = 0.$$

It is possible to show after some computation that the various equations in (16) to (22), inclusive, are all satisfied if θ and ϕ satisfy the following equations

$$
\begin{aligned}
&\phi_u = \theta_v , \\
(23) \qquad &\theta_u + \phi_v + 2 = 0, \\
&(\theta^2 - \phi^2)\theta_v = 2\theta\phi(1+\theta_u).
\end{aligned}
$$

Thus we arrive at the theorem:

<u>The transformation defined by equation (4) will be a transformation E if and only if the functions θ and ϕ satisfy the system of differential equations (23).</u>

3. THE RADIAL TRANSFORMATION R

We shall say that <u>a transformation of a plane into itself is a radial transformation if and only if the lines joining corresponding points in the plane pass through a fixed point.</u>

For example the coordinates of P_x are (u, v) and the coordinates of P_y are $(u+\theta, v+\phi)$. Hence the equation of the line $P_x P_y$ is

$$
\begin{vmatrix} x_1 & x_2 & 1 \\ u & v & 1 \\ u+\theta & v+\phi & 1 \end{vmatrix} = 0 .
$$

This line passes through the origin if and only if

(24) $$u\phi = v\theta.$$

Hence <u>a necessary and sufficient condition that our transformation be a radial transformation is that</u>

$$u\phi = v\theta.$$

4. A TRANSFORMATION WHICH IS BOTH E AND R.

If we impose the condition that our transformation be both an E and an R transformation it is necessary and sufficient that the following system of differential equations be satisfied

$$\begin{aligned} &u\phi = v\theta,\ \phi_u = \theta_v, \\ (25)\qquad &\theta_u + \phi_v + 2 = 0, \\ &(\theta^2 - \phi^2)\theta_v = 2\theta\phi(1+\theta_u). \end{aligned}$$

However the last of these may be disregarded since by computation we may show that it is satisfied if the other three are satisfied.

We shall now proceed to solve for θ and ϕ.

Differentiating the second and third equations in (25) with respect to u and v, respectively, we obtain

$$(26)\qquad \begin{aligned} &\phi_{uu} = \theta_{uv},\ \phi_{uv} = \theta_{vv}, \\ &\theta_{uu} + \phi_{uv} = 0,\ \theta_{uv} + \phi_{vv} = 0. \end{aligned}$$

combining the equations in (26) we obtain the Laplace differential equations

$$\theta_{vv} + \theta_{uu} = 0,$$

$$\phi_{uu} + \phi_{vv} = 0;$$

and, hence, know that a solution does exist. From the first and last equation of (25) we obtain

(27) $$(u^2 - v^2)\,\theta_v = 2uv(1 + \theta_u).$$

From the theory of differential equations we know that the solution of (27) may be found from the solutions of the following system

$$\frac{du}{2uv} = \frac{dv}{u^2 - v^2} = \frac{d\theta}{-2uv}.$$

If we solve the equation

$$\frac{du}{2uv} = \frac{d\theta}{-2uv}$$

we obtain

(28) $$u + \theta = c,$$

Also,

$$\frac{dv}{v^2 - u^2} = \frac{du}{2uv}.$$

This differential equation is homogeneous and may be solved by the usual methods. We obtain as its solution

(29) $$\frac{u^2 + v^2}{u} = c_2$$

Hence we know the general solution of (28) is of the form

(30) $$\theta + u = F(u\mu^2)$$

wherein

$$\mu^2 = \frac{1}{u^2 + v^2}$$

and F is an arbitrary function. Let $z = u\mu^2$. Differentiating the first equation of (25) with respect to u and substituting the result into the second, we obtain

$$v\theta_u - u\theta_v = \frac{v\theta}{u}.$$

Likewise, differentiating (30) with respect to u and v we obtain

(31) $$\theta_u = F'\frac{\partial z}{\partial u} - 1, \quad \theta_v = F'\frac{\partial z}{\partial v}.$$

We now differentiate z with respect to u and v and obtain

(32) $$\frac{\partial z}{\partial u} = 2u\mu\mu_u + \mu^2,$$
$$\frac{\partial z}{\partial v} = 2u\mu\mu_v.$$

By combining (31) and (32) we arrive at the conclusion that

$$F = K^2 u\mu^2$$

wherein K is an arbitrary non-zero constant. Hence

(33) $$\theta = (K^2\mu^2 - 1)u,$$
$$\phi = (K^2\mu^2 - 1)v.$$

Hence necessary and sufficient conditions that an E transformation be a radial transformation is that

$$\theta = (K^2\mu^2 - 1)u, \quad \phi = (K^2\mu^2 - 1)v.$$

5. THE TRANSFORMATION BY RECIPROCAL RADII.

Let there be given a circle whose center is O and radius $k \neq 0$. Let P_1 and P_2 be two points collinear with O such that

(34) $$\overline{OP_1} \times \overline{OP_2} = k^2.$$

The relation existing between points P_1 P_2 satisfying (34) is called the transformation by reciprocal radii.* The point O is called the center of inversion,** the given circle the circle of inversion, and its radius the radius of inversion.

Let the center of inversion be the origin. We now further consider the radial transformation*** E discussed in section 4.

If r_1 and r_2 denote the distances from the points P_x and P_y, respectively, to the origin we find readily that

$$r_1 = \sqrt{u^2+v^2}\ ,\quad r_2 = \sqrt{(u+\theta)^2+(v+\phi)^2}\ .$$

Hence

$$r_1\, r_2 = K^2 \tag{35}$$

Furthermore the coordinates of P_y are

$$y_1 = K^2 u\mu^2,\quad y_2 = K^2 v\mu^2. \tag{36}$$

They may be expressed in the form

$$y = \frac{K^2 x}{\Sigma x^2}$$

which is the formula of a transformation by reciprocal radii. We note that the transformation (36) may be written in the form $y = x + \theta x_u + \phi x_v$ wherein θ

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*Synder and Sisam, Analytic Geometry of space, New York, Henry Holt and Company, 1914, pp. 201-3.

**J. L. Coolidge, A treatise on the circle and the sphere, Oxford, At the Claredon Press, 1916, pp.21-2.

***Grove, Theory of transformations.

and ϕ are defined by (33).

Hence *if the points P_x of the plane are transformed by a transformation which is both E and radial into points P_y of the plane, the points P_y are the transformation of the points P_x by a transformation by reciprocal radii.*

6. THE TRANSFORMATION OF NETS IN THE PLANE

In this section we shall set up the system of partial differential equations by means of which a given net in the plane may be transformed into an arbitrary net in the plane.

We first make a transformation of coordinates from the original coordinates

$$x_1 = u, \quad x_2 = v$$

introduced in section 1 to coordinates ξ, η by means of the transformation

(37) $$\xi = \xi(u, v), \quad \eta = \eta(u, v).$$

Consider the differential equation of a general net

(38) $$(dv - A\,du)(du - B\,dv) = 0$$

wherein

(39) $$A = A(u, v), \quad B = B(u, v)$$

are arbitrary functions of the indicated arguments. Differentiation of (37) gives

$$
\begin{aligned}
d\xi &= \xi_u\,du + \xi_v\,dv, \\
d\eta &= \eta_u\,du + \eta_v\,dv.
\end{aligned} \tag{40}
$$

By a proper choice of ξ and η we may make the new parametric curves $\xi =$ const., $\eta =$ const. be the integral curves respectively of

$$du = B\,dv, \quad dv = A\,du.$$

It follows that

$$
\begin{aligned}
d\xi &= \rho(du - B\,dv) = 0, \\
d\eta &= \sigma(dv - A\,du) = 0.
\end{aligned} \tag{41}
$$

wherein ρ and σ are factors of proportionality. By comparison with (40) we find that (41) will be satisfied by choosing

$$\xi_v = -B\xi_u, \quad \eta_u = -A\eta_v. \tag{42}$$

We have thus made the general net parametric.

In addition, it is known that a parametric net in the plane* is both conjugate and asymptotic. Hence the coordinates of x satisfy the following system of differential equations

$$
\begin{aligned}
x_{uu} &= p x + \alpha x_u + \beta x_v, \\
x_{uv} &= c x + a x_u + b x_v, \\
x_{vv} &= q x + \gamma x_u + \delta x_v.
\end{aligned} \tag{43}
$$

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*E. P. Lane, Projective differential geometry of curves and surfaces, University of Chicago Press, Chicago, 1932, p. 133.

The case wherein $1-AB=0$ must be excluded, for if this were true, the net would degenerate into a one parameter family of curves

$$(du - B\,dv)^2 = 0.$$

Combining (2) and (43) we find that

$$(44) \qquad \begin{aligned} &p = q = c = 0, \\ &\alpha = \beta = a = b = \gamma = \delta = 0. \end{aligned}$$

If we define H and K by the formulae

$$(45) \qquad \begin{aligned} &H = c + ab - a_u, \\ &K = c + ab - b_v \end{aligned}$$

it follows that

$$H = K$$

Hence the <u>net</u> N_x <u>composed of the lines</u> $u =$ <u>const.</u>, $v =$ <u>const. has equal point invariants.</u>

Likewise the coordinates of y will satisfy a corresponding set of differential equations, namely

$$(46) \qquad \begin{aligned} &y_{uu} = \bar{p}\,y + \bar{\alpha}\,y_u + \bar{\beta}\,y_v, \\ &y_{uv} = \bar{c}\,y + \bar{a}\,y_u + \bar{b}\,y_v, \\ &y_{vv} = \bar{q}\,y + \bar{\gamma}\,y_u + \bar{\delta}\,y_v. \end{aligned}$$

Solving (6) for x_u and x_v we obtain

$$(47) \qquad \begin{aligned} &R x_u = (1+\phi_v)\,y_u - \phi_u\,y_v, \\ &R x_v = (1+\theta_u)\,y_v - \theta_v\,y_u. \end{aligned}$$

wherein

$$R = (1+\theta_u)(1+\phi_v) - \theta_v\phi_u \neq 0;$$

because if this were true, y would be a function of one variable which is contrary to our original hypothesis.

Calculation of the three second derivatives of y with respect to u and v and substitution of (47) gives

$$
(48)\qquad
\begin{aligned}
R y_{uu} &= [(1+\phi_v)\theta_{uu} - \theta_v \phi_{uu}] y_u \\
&\quad + [(1+\theta_u)\phi_{uu} - \phi\theta_{uu}] y_v, \\
R y_{uv} &= [(1+\phi_v)\theta_{uv} - \theta_v \phi_{uv}] y_u \\
&\quad + [(1+\theta_u)\phi_{uv} - \phi_u\theta_{uv}] y_v, \\
R y_{vv} &= [(1+\phi_v)\theta_{vv} - \theta_v \phi_{vv}] y_u \\
&\quad + [(1+\theta_u)\phi_{vv} - \phi_u\theta_{vv}] y_v.
\end{aligned}
$$

It follows that

$$
\bar{p} = \bar{c} = \bar{q} = 0,
$$

$$
(49)\qquad
\begin{aligned}
R\bar{\alpha} &= (1+\phi_v)\theta_{uu} - \theta_v\phi_{uu}, \\
R\bar{\beta} &= (1+\theta_u)\phi_{uu} - \phi_u\theta_{uu}, \\
R\bar{a} &= (1+\phi_v)\theta_{uv} - \theta_v\phi_{uv}, \\
R\bar{b} &= (1+\theta_u)\phi_{uv} - \phi_u\theta_{uv}, \\
R\bar{\gamma} &= (1+\phi_v)\theta_{vv} - \theta_v\phi_{vv}, \\
R\bar{\delta} &= (1+\theta_u)\phi_{vv} - \phi_u\theta_{vv}.
\end{aligned}
$$

Let us impose the condition that the transformation be an E transformation. By combining (13), (23), and (49) we find that

$$
\bar{\alpha} = \bar{b} = -\bar{\gamma} = P,
$$

$$
\bar{a} = -\bar{\beta} = \bar{\delta} = Q
$$

wherein

$$(50)\qquad P = \frac{1}{2}\frac{\partial}{\partial u}\log \bar{E}, \qquad Q = \frac{1}{2}\frac{\partial}{\partial v}\log \bar{E}.$$

It follows that

$$\bar{b}_v - \bar{a}_u = 0;$$

and

$$\bar{H} = \bar{K}.$$

Hence <u>a sufficient condition that the net</u> N_y <u>composed of lines respectively perpendicular and parallel to the</u> x- <u>axis have equal point invariants is that the transformation be an</u> E <u>transformation</u>.

Differentiating an arbitrary function X by the rules of the calculus we obtain

$$(51)\qquad X_u = X_\xi \xi_u + X_\eta \eta_u, \qquad X_v = X_\xi \xi_v + X_\eta \eta_v.$$

Substituting (42) into (51) we obtain

$$(52)\qquad X_u = \xi_u X_\xi - A\eta_v X_\eta, \qquad X_v = -B\xi_u X_\xi + \eta_v X_\eta.$$

Solving (52) for X_ξ and X_η we obtain

$$(53)\qquad X_\xi = \frac{X_u + A X_v}{\xi_u(1-AB)}, \qquad X_\eta = \frac{X_v + B X_u}{\eta_v(1-AB)}.$$

Calculation of the three second derivatives of $\mathbf{X}$ with respect to u and v gives

$$
\begin{aligned}
(54)\quad \mathbf{X}_{uu} &= \xi_u^2 \mathbf{X}_{\xi\xi} - 2A\xi_u\eta_v \mathbf{X}_{\xi\eta} + A^2\eta_v^2 \mathbf{X}_{\eta\eta} \\
&\qquad + \xi_{uu}\mathbf{X}_\xi - (A\eta_{uv} + A_u\eta_v)\mathbf{X}_\eta, \\
\mathbf{X}_{uv} &= -B\xi_u^2 \mathbf{X}_{\xi\xi} + (1+AB)\xi_u\eta_v \mathbf{X}_{\xi\eta} - A\eta_v^2 \mathbf{X}_{\eta\eta} \\
&\qquad -(B_u\xi_u + B\xi_{uu})\mathbf{X}_\xi + \eta_{uv}\mathbf{X}_\eta, \\
\mathbf{X}_{vv} &= B^2\xi_u^2 \mathbf{X}_{\xi\xi} - 2B\xi_u\eta_v \mathbf{X}_{\xi\eta} + \eta_v^2 \mathbf{X}_{\eta\eta} \\
&\qquad -B_v\xi_u + B\xi_{uv})\mathbf{X}_\xi + \eta_{vv}\mathbf{X}_\eta .
\end{aligned}
$$

Since $\mathbf{X}_{uu}$, $\mathbf{X}_{uv}$, and $\mathbf{X}_{vv}$ are linear combinations of $\mathbf{X}_u$ and $\mathbf{X}_v$ we may write the following equalities

$$
\begin{aligned}
\mathbf{X}_{uu} &= \alpha_1(\xi_u \mathbf{X}_\xi - A\eta_v \mathbf{X}_\eta) \\
&\qquad + \beta_1(-B\xi_u \mathbf{X}_\xi + \eta_v \mathbf{X}_\eta), \\
\mathbf{X}_{uv} &= a_1(\xi_u \mathbf{X}_\xi - A\eta_v \mathbf{X}_\eta) \\
(55)\qquad &\qquad + b_1(-B\xi_u \mathbf{X}_\xi + \eta_v \mathbf{X}_\eta), \\
\mathbf{X}_{vv} &= \gamma_1(\xi_u \mathbf{X}_\xi - A\eta_v \mathbf{X}_\eta) \\
&\qquad + \delta_1(-B\xi_u \mathbf{X}_\xi + \eta_v \mathbf{X}_\eta).
\end{aligned}
$$

By combining (54) and (55) and solving for $\mathbf{X}_{\xi\xi}$, $\mathbf{X}_{\xi\eta}$, and $\mathbf{X}_{\eta\eta}$ we obtain

$$
\begin{aligned}
&\xi_u^2(1-AB)^2\mathbf{X}_{\xi\xi} = \\
&\quad [(\alpha_1\xi_u - \beta_1 B\xi_u - \xi_{uu}) + 2A(B_u\xi_u + B\xi_{uu} + a_1\xi_u - b_1 B\xi_u) \\
&\quad + A^2(B_v\xi_u + B\xi_{uv} + \gamma_1\xi_u - \delta_1 B\xi_u)]\mathbf{X}_\xi \\
&\quad + [(A\eta_{uv} + A_u\eta_v - \alpha_1 A\eta_v + \beta_1\eta_v) + 2A(b_1\eta_v - a_1 A\eta_v - \eta_{uv}) \\
&\quad + A^2(\delta_1\eta_v - \gamma_1 A\eta_v - \eta_{vv})]\mathbf{X}_\eta ,
\end{aligned}
$$

$$\xi_u \eta_v (1-AB)^2 X_{\xi\eta} =$$

$$[B(\alpha_1\xi_u - \beta_1 B\xi_u - \xi_{uu}) + (1+AB)(B_u\xi_u + B\xi_{uu} + a_1\xi_u - b_1 B\xi_u)$$

$$+ A(B_v\xi_u + B\xi_{uu} + \gamma_1\xi_u - \delta_1 B\xi_u)]X_\xi$$

$$+[B(A\eta_{uv} + A_u\eta_v - \alpha_1 A\eta_v + \beta_1\eta_v) + (1+AB)(b_1\eta_v - a_1 A\eta_v - \eta_{uv})$$

$$+ A(\delta_1\eta_v - \gamma_1 A\eta_v - \eta_{vv})]X_\eta ,$$

(56)

$$\eta_v^2(1-AB)X_{\eta\eta} =$$

$$[B_v\xi_u + B\xi_{uv} + \gamma_1\xi_u - \delta_1 B\xi_u) + 2B(B_u\xi_u + B\xi_{uu} + a_1\xi_u - b_1 B\xi_u)$$

$$+ B^2(\alpha_1\xi_u - \beta_1 B\xi_u - \xi_{uu})]X_\xi$$

$$+[(\delta_1\eta_v - \gamma_1 A\eta_v - \eta_{vv}) + 2B(b_1\eta_v - a_1 A\eta_v - \eta_{uv})$$

$$+ B^2(A\eta_{uv} + A_u\eta_v - \alpha_1 A\eta_v + \beta\eta_v)]X_\eta .$$

It is evident that $X_{\xi\xi}$, $X_{\xi\eta}$, and $X_{\eta\eta}$ are linear combinations of X_ξ and X_η. We find the coefficients of X_ξ and X_η to be

$$\alpha_2 = \frac{(\alpha_1 - \beta_1 B) + 2A(a_1 - b_1 B + B_u) + A^2(B_v + \gamma_1 - \delta_1 B + BB_u)}{\xi_u(1-AB)^2} - \frac{\xi_{uu}}{\xi_u^2},$$

$$\beta_2 = \frac{[(A_u + AA_v - \alpha_1 A + \beta_1) + 2A(b_1 - a_1 A) + A^2(\delta_1 - \gamma_1 A)]\eta_v}{\xi_u^2(1-AB)^2},$$

(57) $$a_2 = \frac{B(\alpha_1 - \beta_1 B) + (1+AB)(a_1 - b_1 B) + A(B_v + \gamma_1 - \delta_1 B) + B_u}{\eta_v(1-AB)^2}$$

$$b_2 = \frac{B(A_u - \alpha_1 A + \beta_1) + AB(b_1 - a_1 A) + A(\delta_1 - \gamma_1 A) + (b_1 - a_1 A + A_v)}{\xi_u(1-AB)^2},$$

$$\gamma_2 = \frac{[(B_v + BB_u + \gamma_1 - \delta_1 B) + 2B(a_1 - b_1 B) + B^2(\alpha_1 - \beta_1 B)]\xi_u}{\eta_v^2(1-AB)^2},$$

$$\delta_2 = \frac{(\delta_1 - \gamma_1 A) + 2B(b_1 - a_1 A + A_v) + B^2(A_u - AA_v - \alpha_1 A + \beta_1)}{\;} - \frac{\eta_{vv}}{\eta_v^2}$$

By combining (44) and (57) we find that the original net N_x becomes, after the transformation (37), an integral net $N_{x'}$ of the system

$$
(58) \qquad
\begin{aligned}
x_{\xi\xi} &= \alpha' x_\xi + \beta' x_\eta, \\
x_{\xi\eta} &= a' x_\xi + b' x_\eta, \\
x_{\eta\eta} &= \gamma' x_\xi + \delta' x_\eta
\end{aligned}
$$

wherein

$$
\alpha' = \frac{2AB_u + A^2 B_v - A^2 B B_u}{\xi_u (1-AB)^2},
$$

$$
\beta' = \frac{(A_u + A A_v)\eta_v}{\xi_u^2 (1-AB)^2},
$$

$$
(59) \qquad a' = \frac{B_u + A B_v}{\eta_v (1-AB)^2}, \qquad b' = \frac{A_u B + A_v}{\xi_u (1-AB)^2},
$$

$$
\gamma' = \frac{(B_v + B B_u)\xi_u}{\eta_v^2 (1-AB)^2},
$$

$$
\delta' = \frac{2A_v B + B^2 A_u - B^2 A A_v}{\eta_v (1-AB)^2}.
$$

Likewise by combining (49) and (57) we find that the net N_y becomes, after the transformation (37), an integral net $N_{y'}$ of the system

$$
(60) \qquad
\begin{aligned}
y_{\xi\xi} &= \bar{\alpha}' y_\xi + \bar{\beta}' y_\eta, \\
y_{\xi\eta} &= \bar{a}' y_\xi + \bar{b}' y_\eta, \\
y_{\eta\eta} &= \bar{\gamma}' y_\xi + \bar{\delta}' y_\eta
\end{aligned}
$$

wherein

$$\bar{\alpha}' = \frac{(P+BQ)+2A(Q-BP+B_u)+A^2(B_v-P-BQ-BB_u)}{\xi_u(1-AB)^2} - \frac{\xi_{uu}}{\xi_u^2},$$

$$\bar{\beta}' = \frac{[(A_u+AA_v-AP-Q)+2A(P-AQ)+A^2(Q+AP)]\eta_v}{\xi_u^2(1-AB)^2},$$

(61)

$$\bar{a}' = \frac{(1+B^2)(Q-AP)}{\eta_v(1-AB)^2} + a',$$

$$\bar{b}' = \frac{(1+A^2)(P-BQ)}{\xi_u(1-AB)^2} + b',$$

$$\bar{\gamma}' = \frac{[(B_u+BB_v-P-BQ)+2B(Q-BP)+B^2(P+BQ)]\xi_u}{\eta_v^2(1-AB)^2},$$

$$\bar{\delta}' = \frac{(Q+AP)+2B(P-AQ+A_v)+B^2(A_u-AA_v-AP-Q)}{\eta_v(1-AB)^2} - \frac{\eta_{vv}}{\eta_v^2}.$$

wherein P and Q are defined by the formulas (50).

Suppose now that the net of curves ξ = const., η = const, is an orthogonal net. It follows that $A+B=0$. Under this condition the coefficients $\bar{a}'$, $\bar{b}'$ in (60) may be written in the form

(62)

$$\bar{a}' = \frac{Q+BP}{\eta_v(1+A^2)} + a' = \frac{1}{2}\frac{\partial}{\partial\eta}\log E + a',$$

$$\bar{b}' = \frac{P+AQ}{\xi_u(1+A^2)} + b' = \frac{1}{2}\frac{\partial}{\partial\xi}\log \bar{E} + b'.$$

We find readily that

$$\frac{\partial \bar{a}'}{\partial\xi} - \frac{\partial \bar{b}'}{\partial\eta} = \frac{\partial a'}{\partial\xi} - \frac{\partial b'}{\partial\eta}.$$

We may state our result in the theorem:

Any E transformation of an orthogonal net with equal point invariants is an orthogonal net with equal point invariants.

Impose the condition that the nets N_x and N_y be conformal, namely

$$\frac{E}{\bar{E}} = \frac{F}{\bar{F}} = \frac{G}{\bar{G}} .$$

From (12) and (13) it follows that

$$(1+\theta_u)^2 + \phi_u^2 = (1+\theta_v)^2 + \theta_v^2,$$

$$\theta_v(1+\theta_u) + \phi_u(1+\phi_v) = 0.$$

Hence if the transformation is an E transformation it is, also, a conformal transformation.

Computing E', F', and G' for the transformed net N_x' we obtain

$$E' = \frac{1+A^2}{\xi_u^2(1-AB^2} ,$$

(63) $$F' = \frac{A+B}{\xi_u \eta_v (1-AB)} ,$$

$$G' = \frac{1+B^2}{\eta_v^2(1-AB)^2} .$$

Likewise if we consider the net N_y', we obtain

$$\bar{E}' = \frac{\bar{E} + 2A\bar{F} + A^2\bar{G}}{\xi_u^2(1-AB)^2} ,$$

(64) $$\bar{F}' = \frac{B\bar{E} + (1+AB)\bar{F} + A\bar{G}}{\xi_u \eta_v (1-AB)^2} ,$$

$$\bar{G}' = \frac{B^2\bar{E} + 2B\bar{F} + \bar{G}}{\eta_v^2(1-AB)^2} .$$

Suppose now that the net N_x' is an orthogonal net. It follows that $A+B=0$. Let N_y' be an E transform of N_x'. Under these conditions we may write (63) and (64) in the form

$$
(65) \qquad
\begin{aligned}
E' &= \frac{1}{\xi_u^2(1+A^2)}, \\
F' &= 0, \\
G' &= \frac{1}{\eta_v^2(1+A^2)}.
\end{aligned}
$$

$$
(66) \qquad
\begin{aligned}
\bar{E}' &= \frac{\bar{E}}{\xi_u^2(1+A^2)}, \\
\bar{F}' &= 0, \\
\bar{G}' &= \frac{\bar{G}}{\eta_v^2(1+A^2)}.
\end{aligned}
$$

We may therefore state the theorem:

<u>Any E transform of an isothermally orthogonal net in the plane is an isothermally orthogonal net.</u>

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