KINETICS OF SMALL SYSTEMS

Thesis for the Degree of M. S.
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ABSTRACT

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by Lowell E. Jacobs

Chemical kinetics is usually approached in a deterministic way, i.e., once the state of the system is known it can also be determined at any other time. However this approach does not allow for fluctuations in the system. The present work approaches chemical kinetics from a stochastic viewpoint which will allow fluctuations in the system to be recognized.

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INTRODUCTION

There has recently appeared a treatment of the classical thermodynamics of small systems, in which the well-known relations of classical thermodynamics have been extended to be applicable to small systems [1]. A similar extension is also possible in chemical kinetics and this thesis is part of a study in this direction.

The extension of kinetics to small systems (e.g., systems with one hundred particles or less) is effected by an application of stochastic processes to chemical kinetics. The classical approach to chemical kinetics can be called deterministic since once the state of the system is known at some time t_1, the state at any other time is automatically fixed, and no fluctuations or deviations about this value are recognized. However, in a stochastic model of kinetics the state of the system, for instance the concentration at time t, is represented by an integer-valued random variable and the fundamental solution to the problem is the probability density function of the random variable, from which various moments may be calculated. Fluctuations are then recognized through the second central moment, the variance.

Delbrück [2] seems to have the earliest representation of a chemical reaction by a stochastic model, in which he discussed fluctuations in an autocatalytic reaction. Singer [3] used this approach next in 1953 when he discussed apparent irreproducibilities in some chemical reactions and gave several examples of systems in which fluctuations, or irreproducibilities, have been reported. The reactions treated by Singer were a chain reaction without branching and one with branching. Renyi [4] next used this idea about a year later when he discussed the reaction $A + B \longrightarrow C$. Bartholomay [5, 6] has treated the irreversible unimolecular reaction $A \longrightarrow B$ and shown the connection between the two previous approaches and stationary Markov chains. Ishida [7] then used a multidimensional stochastic treatment to discuss the unimolecular-decomposition theory of Kassel. Bartholomay [8] also has discussed a stochastic model for the Michaelis-Menten formulation of an enzyme-substrate reaction.

The contribution that an idea such as the stochastic representation of chemical kinetics could make to the field of biology seems very useful. Several papers have already been published which use the stochastic approach to discuss reactions in cells and transitions of one type of cell to another. Berger [9], for example, discussed the transition of reticulocytes to red blood cells using a stochastic interpretation. Shea [10] has discussed histology from a statistical standpoint, and Fraser [11] has used this approach to study genetic systems.

The most recent work in applying stochastic models to chemical kinetics is that of McQuarrie [12], in which he discussed the systems, $A \longrightarrow B$, $A \rightleftarrows B$, and $A \xrightarrow{k_1} B$, $A \xrightarrow{k_2} C$. McQuarrie also has another paper [13] in which the stochastic model for the bimolecular reactions, $2A \longrightarrow B$, $A + B \longrightarrow C$, and $A \longrightarrow B$ autocatalytically are discussed.

The purpose of the present work was to apply a Monte Carlo process to a pure "death" reaction, similar to Kendall's procedure [14] for a simple "birth and death" reaction, and to calculate the individual fluctuations in the system in order that they might be compared graphically to the ensemble average fluctuations obtained from McQuarrie's formula. Kendall's procedure and the modifications applied to it will be discussed in detail later.

The general idea was simply to derive certain fundamental equations of probability concerning the system and then to set up a mathematically equivalent sampling process in order to solve the equations.

This is the general technique used in the Monte Carlo process. In this problem the sampling process and the calculations were carried out on the 160A computer at Michigan State University. The sampling process was simply a series of random digits obtained from The Rand Corporation [15]. The complete program appears on page 16. This amounts to a simulated kinetics experiment. The reactions under consideration are $A \longrightarrow B$ and $2A \longrightarrow B$.

Let us look at an example in order to see just what procedure was followed in this problem. Suppose we take the reaction $A \longrightarrow B$ and look at a system of five particles for five different runs. That means we have five particles in our system initially and we let them decay according to our probability equations. When this process is repeated five times we have a system of five particles for five runs. This is equivalent to an ensemble of five systems, each of which contains five particles initially. The systems that were constructed are listed on page 12 . From the results of computations on these systems the graphs of the mean and variance were drawn.

Examination of the System: $A \longrightarrow B$

Let us now consider the reaction $A \longrightarrow B$ [12] using a stochastic model. Let the integer-valued random variable, $X(t)$, be the number of A molecules in the system at time t. Then the stochastic model is completely defined by the following three assumptions:

(1) The probability of a transition $(x) \longrightarrow (x-1)$ in the interval $(t, t + \Delta t)$ is $kx\Delta t + 0(\Delta t)$, where k is a constant and $0(\Delta t)$ means that $0(\Delta t)/\Delta t \longrightarrow 0$ as $\Delta t \longrightarrow 0$.

(2) The probability of a transition $(x) \longrightarrow (x - j)$, where $j > 1$, in the interval $(t, t, + \Delta t)$ is $0(\Delta t)$.

(3) The reverse reaction occurs with probability zero.

Now let $P_x(t) = \text{Prob}\ \{X(t) = x\}$. That is, $P_x(t)$ is the probability that the number of A molecules is x at time t. Let $P_x(t + \Delta t) = \text{Prob}\ \{X(t + \Delta t) = x\}$. That is, $P_x(t + \Delta t)$ is the probability that the number of A molecules is x at time $t + \Delta t$.

The state $X(t + \Delta t) = x$ can be obtained in two ways. First, $X(t)$ may equal x and no transition occurs in time Δt; and second, $X(t)$ may equal $x + 1$ and a transition does occur in time Δt. $P_x(t + \Delta t)$ is thus a sum of two probabilities as follows:

$$P_x(t + \Delta t) = [k(x+1)\Delta t]\ P_{x+1}(t) + [1 - k\,x\,\Delta t]\ P_x(t) + 0(\Delta t) \qquad (1)$$

where $k(x + 1)\Delta t$ is the probability of a transition from the state $X(t) = x + 1$ to the state $X(t + \Delta t) = x$; $P_{x+1}(t)$ is the probability that $X(t) = x + 1$; $[1 - kx\,\Delta t]$ is the probability of no transition from the state $X(t) = x$; and $P_x(t)$ is the probability that $X(t) = x$. The term, $0(\Delta t)$, is added on in order to compensate for any higher order terms. Transpose $P_x(t)$ and divide through by Δt to get:

$$\frac{P_x(t + \Delta t) - P_x(t)}{\Delta t} = k(x + 1)P_{x+1}(t) - kxP_x(t) + 0(\Delta t)/\Delta t \qquad (2)$$

The $\lim\limits_{\Delta t \to 0}$ of the left hand side of this equation is by definition $\frac{dP_x}{dt}$.

Therefore:

$$\frac{dP_x}{dt} \doteq k(x + 1)\ P_{x+1}(t) - kxP_x(t) \qquad (3)$$

which is the differential-difference equation of the process. Now, by using the generating function of $P_x(t)$ [16], that is,

$$F\ (s, t) = \sum_{x=0}^{\infty} P_x(t)\ s^x\ , \qquad (4)$$

where $|\ s\ | < 1$ equation (3) may be transformed into a partial differential equation by multiplying through by s^x and summing over x from 0 to ∞.

$$\sum_{x=o}^{\infty} \frac{dP_x}{dt} s^x = k \sum_{x=o}^{\infty} (x+1) P_{x+1}(t) s^x - k \sum_{x=o}^{\infty} x P_x(t) s^x \qquad (5)$$

and so by noting equation (4), one obtains

$$\frac{\partial F}{\partial t} = k(1-s) \frac{\partial F}{\partial s} \qquad (6)$$

The solution of this equation using the initial condition $F(s, o) = s^{x_o}$, is

$$F(s,t) = [1 + (s-1)e^{-kt}]^{x_o} \qquad (7)$$

By using the following relationships:

$$E\left\{X(t)\right\} = \left(\frac{\partial F}{\partial s}\right)_{s=1} = \sum_{x=o}^{\infty} x P_x(t) \qquad \text{and}$$

$$D^2\left\{X(t)\right\} = \left(\frac{\partial^2 F}{\partial s^2}\right)_{s=1} + \left(\frac{\partial F}{\partial s}\right)_{s=1} - \left(\frac{\partial F}{\partial s}\right)^2_{s=1}$$

where $E\left\{X(t)\right\}$ is the expectation value or mean of X(t), and $D^2\left\{X(t)\right\}$ is the variance, the following equations result:

$$E\left\{X(t)\right\} = x_o e^{-kt} \qquad (8)$$

$$D^2\left\{X(t)\right\} = x_o e^{-kt}(1 - e^{-kt}) \qquad (9)$$

Note that the mean value of the stochastic representation is identical to the deterministic result, showing that the two representations are consistent in the mean. The stochastic model, however, also gives higher moments and so fluctuations can now be included in chemical kinetics.

Examination of the System 2A ⟶ B

Now consider the reaction 2A ⟶ B [13]. Here the procedure is similar to that for A ⟶ B, i.e., the bimolecular reaction is considered to be a pure "death" process with a continuous time parameter and transition probabilities. First consider a system of X_o molecules of reactant A at time $t = 0$. Let $X(t)$ equal the random variable, i.e., the number of A molecules is the system at time t. Then the stochastic model is completely defined by the following assumptions:

(1) The probability of the transition $x \longrightarrow x - 2$ in the time interval $(t, t + \Delta t)$ is $\frac{1}{2}k'x(x - 1)\Delta t + 0(\Delta t)$ where k' is a constant and $0(\Delta t)$ is defined so that $0(\Delta t)/\Delta t \longrightarrow 0$ as $\Delta t \longrightarrow 0$. The factor of $\frac{1}{2}$ eliminates counting collisions between the same two molecules twice. We shall let $\frac{1}{2}k' = k$ in subsequent equations.

(2) The probability of the transition $x \longrightarrow x - j$ where $j > 2$, in the time interval $(t, t + \Delta t)$ is $0(\Delta t)$.

(3) The probability of the transition $x \longrightarrow x + j$, where $j > 0$, in the time interval $(t, t + \Delta t)$ is zero.

Thus the following probability equation can be written,

$$P_x(t + \Delta t) = [k(x+2)(x+1)\Delta t]\, P_{x+2}(t) + [1 - kx(x-1)\Delta t]\, P_x(t) + 0(\Delta t) \qquad (10)$$

where the notation is the same as before. Now, by the standard procedure of transposing $P_x(t)$, dividing through by Δt, and taking the limit, the following differential-difference equation results;

$$\frac{dP_x}{dt} = k(x+2)(x+1)\, P_{x+2}(t) - kx(x - 1)\, P_x(t). \qquad (11)$$

This is transformed into a partial differential equation for the generating function, $F(s,t)$, viz:

$$\frac{\partial F}{\partial t} = (1 - s^2)\frac{\partial^2 F}{\partial s^2} \qquad (12)$$

whose solution for the initial condition $F(s, o) = s^{x_o}$ is:

$$F(s,t) = 1 + \sum_{j=2}^{x_0} A_j C_j^{-\frac{1}{2}}(s) \exp\left\{-\tfrac{1}{2} j(j-1)kt\right\} \qquad (13)$$

where

$$A_j = \left[\frac{2j-1}{2j}\right] \left[\frac{\Gamma(x_0+1) \quad \Gamma\left(\frac{x_0-j+1}{2}\right)}{\Gamma(x_0-j+1)\,\Gamma\left(\frac{x_0+j+1}{2}\right)}\right] \qquad (14)$$

and $C_j^{-\frac{1}{2}}(s)$ are Gegenbauer polynomials [17],

and $j = 2, 4, 6, ---, x_0$ (x_0 even).

It can be easily shown that:

$$\left\langle X \right\rangle = \sum_{j=2}^{x_0} A_j \exp\left\{-\tfrac{1}{2} j(j-1)kt\right\} \qquad (15)$$

$$\left\langle X(X-1) \right\rangle = \sum_{j=2}^{x_0} A_j \left[\frac{j(j-1)}{2}\right] \exp\left\{-\tfrac{1}{2} j(j-1)kt\right\} \qquad (16)$$

A study of second order reactions by the stochastic approach provides an excellent method of determining to what extent the deterministic rate equation is applicable to systems where the inherent statistical fluctuations are relatively large. McQuarrie [12] has stated "that when a deterministic rate equation such as $-\frac{dA}{dt} = kA^2$ is considered there is a tacit assumption that $\overline{A^2} = (\overline{A})^2$, which is true only for a delta function type density function, i.e., all central moments equal zero." Dole and Inokuti [18] have studied the first and second order kinetics for systems in which the number of reacting species was of the order of 10 or less. They considered the conditions under which the ratio $\overline{A^2} / (\overline{A})^2$ approaches unity. Their calculations seemed to indicate that the ratio approaches unity much more rapidly than one might expect.

Monte Carlo $A \longrightarrow B$

The procedure used in this problem was similar to that employed by Kendall [14]; the equations being modified to fit a pure "death" process.

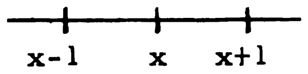
Kendall derives three differential-difference equations based on the probability equation for the birth and death process [16]:

$$P_x(t+\Delta t) = \mu\,(x+1)\Delta t\; P_{x+1}(t) + \lambda(x-1)\Delta t\; P_{x-1}(t) + [1-(\lambda+\mu)x\Delta t]P_x(t) \tag{17}$$

where λ is the expected birth rate and μ is the expected death rate. The standard procedure of transposing $P_x(t)$, dividing through by Δt, and taking the limit, leads to:

$$\frac{dP_x(t)}{dt} = \mu\,(x+1)\;P_{x+1}(t) + \lambda(x-1)\;P_{x-1}(t) - (\lambda+\mu)\;x\;P_x(t) \tag{18}$$

Imagine now the portion of the system represented by the following diagram:



.

By forbidding transitions out of the states $x \pm 1$ we can write three equations which give the probability of the system being in any one of the above three states at time $t + \Delta t$.

$$\begin{aligned} P_{x+1}(t+\Delta t) &= (\lambda\, x\, \Delta t)P_x(t) + P_{x+1}(t) \\ P_x(t+\Delta t) &= [1 - (\mu+\lambda)\; x\; \Delta t]P_x(t) \\ P_{x-1}(t+\Delta t) &= \mu\; x\; \Delta t P_x(t) + P_{x-1}(t) \end{aligned} \tag{19}$$

These equations lead to:

$$\begin{aligned} \frac{dP_{x+1}(t)}{dt} &= \lambda\; x\; P_x(t) \\ \frac{dP_x(t)}{dt} &= -\,(\mu+\lambda)\; x\; P_x(t) \\ \frac{dP_{x-1}(t)}{dt} &= \mu\; x\; P_x(t) \end{aligned} \tag{20}$$

Introduce a new variable u where $u = N\tau$; N is the number of particles present and τ is the length of time until the next "incident." Then, from the second equation in (20) it follows that the distribution of τ is: $e^{-(\lambda + \mu)N\tau} (\lambda + \mu)N d\tau \quad (o < \tau < \infty)$.

Kendall [14] has shown that this "birth-and-death" process is mathematically equivalent to a scheme of sequential sampling using random numbers. This process of deriving equations to explain a particular phenomenon and then employing a mathematically equivalent scheme to obtain a solution is called the "Monte Carlo" method. _This was exactly the procedure used in this problem, a pure "death" process (i.e., $\lambda = 0$). Here we are concerned with the second equation in (20). The solution of this equation is mathematically equivalent to the following scheme of sequential sampling: random variables are drawn independently from a series of random numbers. After each step in the sequential sampling process the following quantities are calculated:

$$\begin{aligned} N_r &= N_{r-1} - 1 \\ tr &= t_{r-1} + \frac{u_r}{N_{r-1}} \qquad (r \geq 1) \end{aligned} \tag{21}$$

The construction of the sample of u-variables is most easily effected by employing the transformation, $x = e^{-u}$ so that the x-variables must be uniformly distributed in $(0, 1)$.

Suppose we look at a typical set of calculations for a run with 10 particles.

r	x	u_r	N_{r-1}	τ_r	t_{r-1}
1	.32025	1.139	10	0.114	0.000
2	.27574	1.288	9	0.143	0.114
3	.79141	0.234	8	0.029	0.257
4	.33940	1.081	7	0.154	0.286
5	.53963	0.617	6	0.103	0.440
6	.39754	0.922	5	0.184	0.543
7	.18502	1.687	4	0.422	0.727
8	.65081	0.430	3	0.143	1.149
9	.41346	0.883	2	0.441	1.292
10	.18978	1.662	1	1.662	1.733
11	.66021	0.415	0	∞	3.395

Each interval is first assigned a random number X which is chosen independently. The u-variable is then obtained from the relation $x = e^{-u}$. The values of N are listed from 10 to 0 since the reaction is unimolecular and hence decays one particle at a time. Next τ is calculated from the equation $u = N\tau$. And finally t, the cumulative time, is obtained by adding the subsequent values of τ.

After we repeat this process for a series of runs we can tally the number of particles, N, in each run at various time intervals and thus obtain a mean for the system at each time interval. We can also obtain the variance at each time interval from the equation $s^2 = \frac{\Sigma N_i^2}{n} - \bar{N}^2$ where N_i is the number of particles present at time t_i.

Thus we can now graph the mean and the variance for each system and compare it with the theoretical curve obtained from McQuarrie's formula [12].

Monte Carlo $2A \longrightarrow B$

For the reaction $2A \longrightarrow B$ we can again write three equations which give the probability that the system is in one of the states, $x - 2$, x, or $x + 2$ at time $t + \Delta t$.

$$P_{x+2}(t + \Delta t) = [1 - k\,(x+2)(x+1)\Delta t]\,P_{x+2}(t) + k(x+4)(x+3)\,P_{x+4}(t)\Delta t$$

$$P_x(t + \Delta t) = k(x+2)(x+1)\Delta t P_{x+2}(t) + [1 - kx(x-1)\Delta t]P_x(t) \tag{22}$$

$$P_{x-2}(t + \Delta t) = k\,x\,(x-1)\Delta t\,P_x(t) + [1-k(x-2)(x-3)\Delta t]P_{x-2}(t)$$

By restricting transitions from x ± 2 we get:

$$\frac{dP_{x+2}(t)}{dt} = k\,(x+4)(x+3)P_{x+4}(t)$$

$$\frac{dP_x(t)}{dt} = -k\,x\,(x-1)\,P_x(t) \tag{23}$$

$$\frac{dP_{x-2}(t)}{dt} = k\,x\,(x-1)P_x(t)$$

Now it is clear that the distribution of τ is:

$$e^{-kN(N-1)\tau}\,kN(N-1)d\tau \tag{24}$$

This necessitates a change in equations (21) to:

$$\begin{aligned} N_r &= N_{r-1} - 2 \\ t_r &= t_{r-1} + \frac{u_r}{N_r(N_r-1)} \end{aligned} \tag{25}$$

Now the sampling process and the calculations can be carried out in an exactly analogous manner to that for the reaction $A \longrightarrow B$.

Discussion of Graphs

The graphs which resulted from this study are shown on the following pages. For the reaction $A \longrightarrow B$ the graphs shown are (1) mean, XBAR, vs time and (2) Variance vs time and for the reaction $2A \longrightarrow B$ the graphs shown are (1) $\frac{1}{XBAR}$ vs time and (2) Variance vs time. The graphs correspond to various systems which consisted of differing numbers of runs and particles as outlined in the table below. Inspection of the graphs shows how the individual fluctuations tend to follow the theoretical curves obtained from McQuarrie's formula.

Reaction: $A \longrightarrow B$

Systems

Particles	Runs
5	5
5	25
5	50
5	100
10	5
10	10
10	25
10	50
20	5
20	10
20	25
20	50
100	5
100	10

Reaction: $2A \longrightarrow B$

Systems

Particles	Runs
6	5
6	10
6	20
6	50
20	5
20	10
20	20
20	50
100	4
100	10

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APPENDIX

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L. E. JACOBS 21-1016T    A-B No. 4    8-1-63
        DIMENSION X(11),U(11),N(11,20),A(11),T(11),NMOL(42,20)
        READ 100, L, K
100     FORMAT (I3,3X,I3)
        PRINT 1
  1     FORMAT (1HO,23H10 PARTICLES FOR 5 RUNS)
        DO 230 I=1,L
        READ 101, (N(I,IRUN),IRUN=1,K)
101     FORMAT (26I3)
230     CONTINUE
        DO 220 IRUN=1,K
        PRINT 15, IRUN
 15     FORMAT (1HO,5HIRUN=I3
        PRINT 2
  2     FORMAT (1HO,49H     X        U        N        A        T)
        T(1)=0.000
        READ 102,(X(I),I=1,L)
102     FORMAT (6F8.8)
        DO 200 I=1,L
        U(I)=-LOGF(X(I))
        A(I)=U(I)/N(I,IRUN)
        PRINT 3.X(I),U(I),N(I,IRUN),A(I),T(I)
  3     FORMAT (1H2.F11.8,3X,F8.4,3X,I4,3X,F8.4,3X,F8.4)
200     T(I+1)=T(I) + A(I)
        TI=0.000
        NMOL(1,IRUN)=N(1,IRUN)
        I=2
        IY=1
        DO 210 IFORTY=1,40
        TI=TI + 0.1
        IY=IY + 1
        IF (TI-T(I)) 502,503,504
502     NMOL (IY,IRUN)=N(I-1,IRUN)
        GO TO 210
503     NMOL(IY,IRUN)=N(I,IRUN)
        GO TO 209
504     IF (TI-T(I+1)) 505,506,507
505     NMOL(IY,IRUN)=N(I,IRUN)
        GO TO 209
506     NMOL(IY,IRUN)=N(I+1,IRUN)
        GO TO 210
507     IF (TI-T(I+2)) 508,509,510
508     NMOL(IY,IRUN)=N(I+1,IRUN)
        GO TO 210
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```
509      NMOL(IY, IRUN)=N(I+2, IRUN)
         GO TO 209
510      IF (TI-T(I+3) 511, 512, 513
511      NMOL(IY, IRUN)=N(I+2, IRUN)
         GO TO 209
512      NMOL(IY, IRUN)=N(I+3, IRUN)
         GO TO 210
513      IF (TI-T(I+4)) 514, 515, 516
514      NMOL(IY, IRUN)=N(I+3, IRUN)
         GO TO 210
515      NMOL(IY, IRUN)=N(I+4, IRUN)
         GO TO 209
516      IF (TI-T(I+5)) 517, 518, 519
517      NMOL(IY, IRUN)=N(I+4, IRUN)
         GO TO 209
518      NMOL(IY, IRUN)=N(I+5; IRUN)
         GO TO 210
519      IF (TI-T(I+6)) 520, 521, 522
520      NMOL(IY, IRUN)=N(I+5, IRUN)
         GO TO 210
521      NMOL(IY, IRUN)=N(I+6, IRUN)
         GO TO 209
522      IF (TI-T(I+7)) 523, 524, 525
523      NMOL(IY, IRUN)=N(I+6, IRUN)
         GO YO 209
524      NMOL(IY, IRUN)=N(I+7, IRUN)
         GO TO 210
525      IF (TI-T(I+8)) 526, 527, 528
526      NMOL(IY, IRUN)=N(I+7, IRUN)
         GO TO 210
527      NMOL(IY, IRUN)=N(I+8, IRUN)
         GO TO 209
528      IF (TI-T(I+9)) 529, 530, 531
529      NMOL(IY, IRUN)=N(I+8, IRUN)
         GO TO 209
530      NMOL(IY, IRUN)=N(I+9, IRUN)
         GO TO 210
531      IF (TI-T(I+10)) 532, 533, 534
532      NMOL(IY, IRUN)=N(I+9, IRUN)
         GO TO 210
533      NMOL(IY, IRUN)=N(I+10, IRUN)
         GO TO 209
534      IF (TI-T(I+11)) 535, 536, 537
535      NMOL(IY, IRUN)=N(I+10, IRUN)
         GO TO 209
```

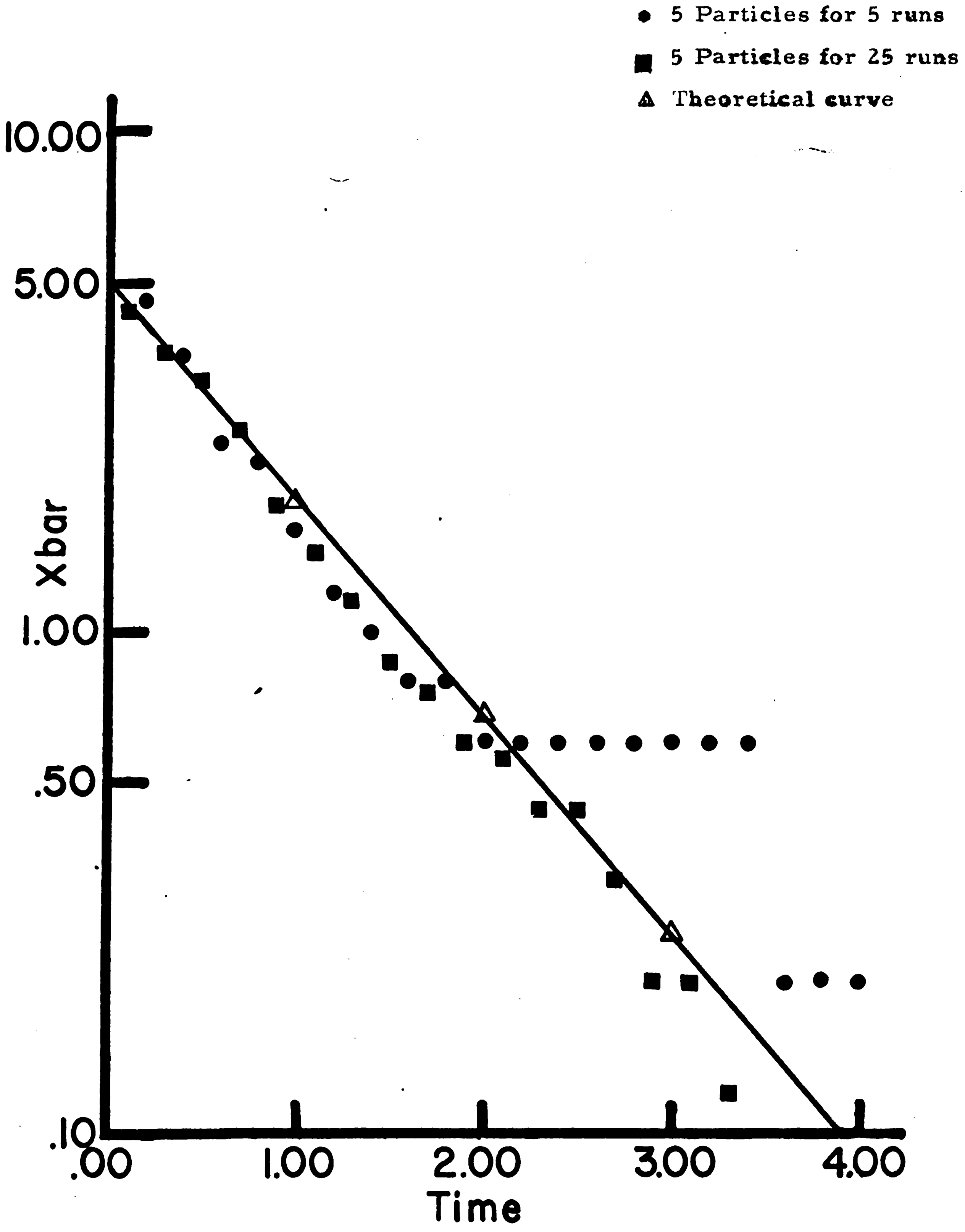
```
536     NMOL(IY,IRUN)=N(I+11,IRUN)
        GO TO 210
537     IF (TI-T(I+12)) 538,539,540
538     NMOL(IY,IRUN)=N(I+11,IRUN)
        GO TO 210
539     NMOL(IY,IRUN)=N(I+12,IRUN)
        GO TO 209
540     IF (TI-T(I+13)) 541,542,543
541     NMOL(IY,IRUN)=N(I+12,IRUN)
        GO TO 209
542     NMOL(IY,IRUN)=N(I+13,IRUN)
        GO TO 210
543     IF (TI-T(I+14)) 544, 545,545
544     NMOL(IY,IRUN)=N(I+13,IRUN)
        GO TO 210
545     NMOL(IY,IRUN)=N(I+14,IRUN)
209     I=I+1
210     CONTINUE
220     CONTINUE
        PRINT 8
  8     FORMAT(1H1,50HL.E.JACOBS, A-B NO.4 10 PARTICLES FOR
   5    RUNS 1-8)
        PRINT 6
  6     FORMAT (1HO,65H    TI    SUMXSQ    EXSQ   XBAR    XBARSQ
  1     VAR    STDEV)
        TI=0.000
        DO 203 IY=1,41
        SUMXSQ=000.00
        DO 201 IRUN=1, K
201     SUMXSQ=SUMXSQ + (NMOL(IY,IRUN)**2
        EXSQ=SUMXSQ/K
        SUM=000.00
        DO 202 IRUN=1,K
202     SUM=SUM + NMOL(IY,IRUN)
        XBAR=SUM/K
        XBARSQ=XBAR**2
        VAR=EXSQ-XBARSQ
        STDEV=SQRTF(VAR)
        PRINT 7, TI,SUMXSQ, EXSQ, XBAR, XBARSQ, VAR, STDEV
  7     FORMAT (1H2,F7.3,2X,F7.0,2X,F9.2,2X,F7.2,2X,F9.2,2X,F7.3,
      2X,F7.3)
203     TI=TI + 0.1
        STOP
        END
```

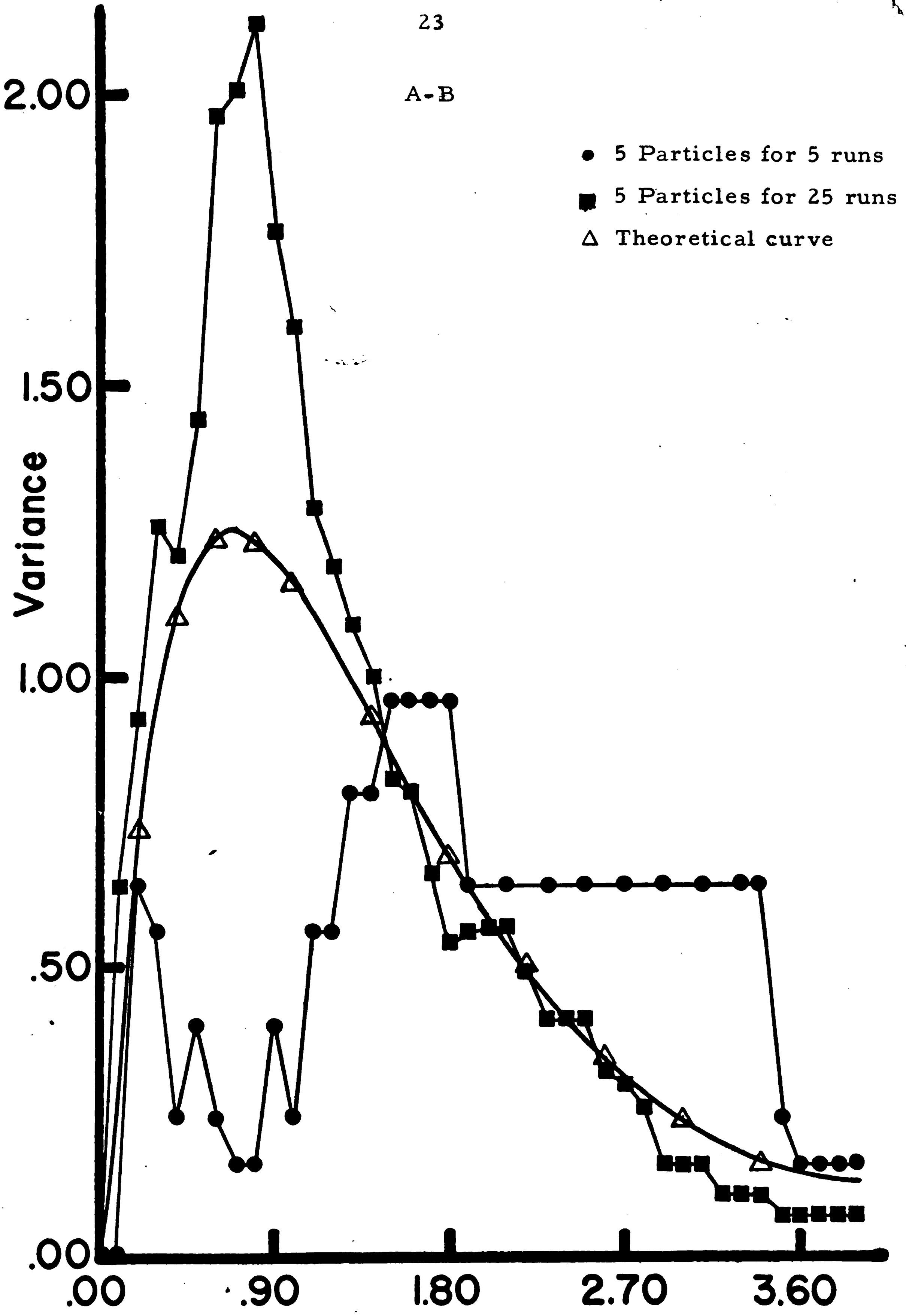
```
* L.E.JACOBS 21-1016T    2A-B No. 7   8-15-63
          DIMENSION X(11),U(11),N(11,10),A(11),T(11),NMOL(46,10)
          READ 100, L,K
100       FORMAT (I3, 3X,I3)
          PRINT 1
  1       FORMAT (1HO,24H20 PARTICLES FOR 10 RUNS)
          DO 230 I=1,L
          READ 101,(N(I,IRUN),IRUN=1,K)
101       FORMAT (26I3)
230       CONTINUE
          DO 220 IRUN=1,K
          PRINT 15,IRUN
 15       FORMAT (1HO,5HIRUN=I3)
          PRINT 2
  2       FORMAT (1H2,49H    X      U      N      A       T)
          T(1)=0.000
          READ 102,(X(I),I=1,L)
102       FORMAT (6F8.8)
          DO 200 I=1,L
          U(I)=-LOGF(X(I))
          A(I)=U(I)/(N(I,IRUN)*(N(I,IRUN)-1))
          PRINT 3,X(I),U(I),N(I,IRUN),A(I),T(I)
  3       FORMAT (IH2,F11.8,3X,F8.4,3X,I4,3X,F8.4,3X,F8.4)
200       T(I+1)=T(I) + A(I)
          TI=0.000
          NMOL(1,IRUN)=N(1,IRUN)
          I=2
          IY=1
          DO 210 IFORT=1,45
          TI=TI + 0.005
          IY=IY + 1
          IF (TI-T(I)) 502,503,504
502       NMOL (IY,IRUN)=N(I-1,IRUN)
          GO TO 210
503       NMOL(IY,IRUN)=N(I,IRUN)
          GO TO 209
504       (IF (TI-T(I+1)) 505,506,507
505       NMOL(IY,IRUN)=N(I,IRUN)
          GO TO 209
506       NMOL(IY,IRUN)=N(I+1,IRUN)
          GO TO 210
507       IF (TI-T(I+2)) 508,509,510
508       NMOL(IY,IRUN)=N(I+1,IRUN)
          GO TO 210
```

```
509     NMOL(IY, IRUN)=N(I+2, IRUN)
        GO TO 209
510     IF (TI-T(I+3)) 511, 512, 513
511     NMOL(IY, IRUN)=N(I+2, IRUN)
        GO TO 209
512     NMOL(IY, IRUN)=N(I+3, IRUN)
        GO TO 210
513     IF (TI-T(I+4)) 514, 515, 516
514     NMOL(IY, IRUN)+N(I+3, IRUN)
        GO TO 210
515     NMOL(IY, IRUN)=N(I+4, IRUN)
        GO TO 209
516     IF (TI-T(I+5) 517, 518, 519
517     NMOL(IY, IRUN)=N(I+4, IRUN)
        GO TO 209
518     NMOL(IY, IRUN)=N(I+5, IRUN)
        GO TO 210
519     IF (TI-T(I+6)) 520, 521, 522
520     NMOL(IY, IRUN)=N(I+5, IRUN)
        GO TO 210
521     NMOL(IY, IRUN)=N(I+6, IRUN)
        GO TO 209
522     IF (TI-T(I+7)) 523, 524, 525
523     NMOL(IY, IRUN)=N(I+6, IRUN)
        GO TO 209
524     NMOL(IY, IRUN)=N(I+7, IRUN)
        GO TO 210
525     IF (TI-T(I+8)) 526, 527, 528
526     NMOL(IY, IRUN)=N(I+7, IRUN)
        GO TO 210
527     NMOL(IY, IRUN)=N(I+8, IRUN)
        GO TO 209
528     IF (TI-T(I+9)) 529, 530, 531
529     NMOL(IY, IRUN)=N(I+8, IRUN)
        GO TO 209
530     NMOL(IY, IRUN)=N(I+9, IRUN)
        GO TO 210
531     IF (TI-T(I+10)) 532, 533, 534
532     NMOL(IY, IRUN)=N(I+9, IRUN)
        GO TO 210
533     NMOL(IY, IRUN)=N(I+10, IRUN)
        GO TO 209
534     IF (TI-T(I+11)) 535, 536, 537
535     NMOL(IY, IRUN)=N(I+10, IRUN)
        GO TO 209
```

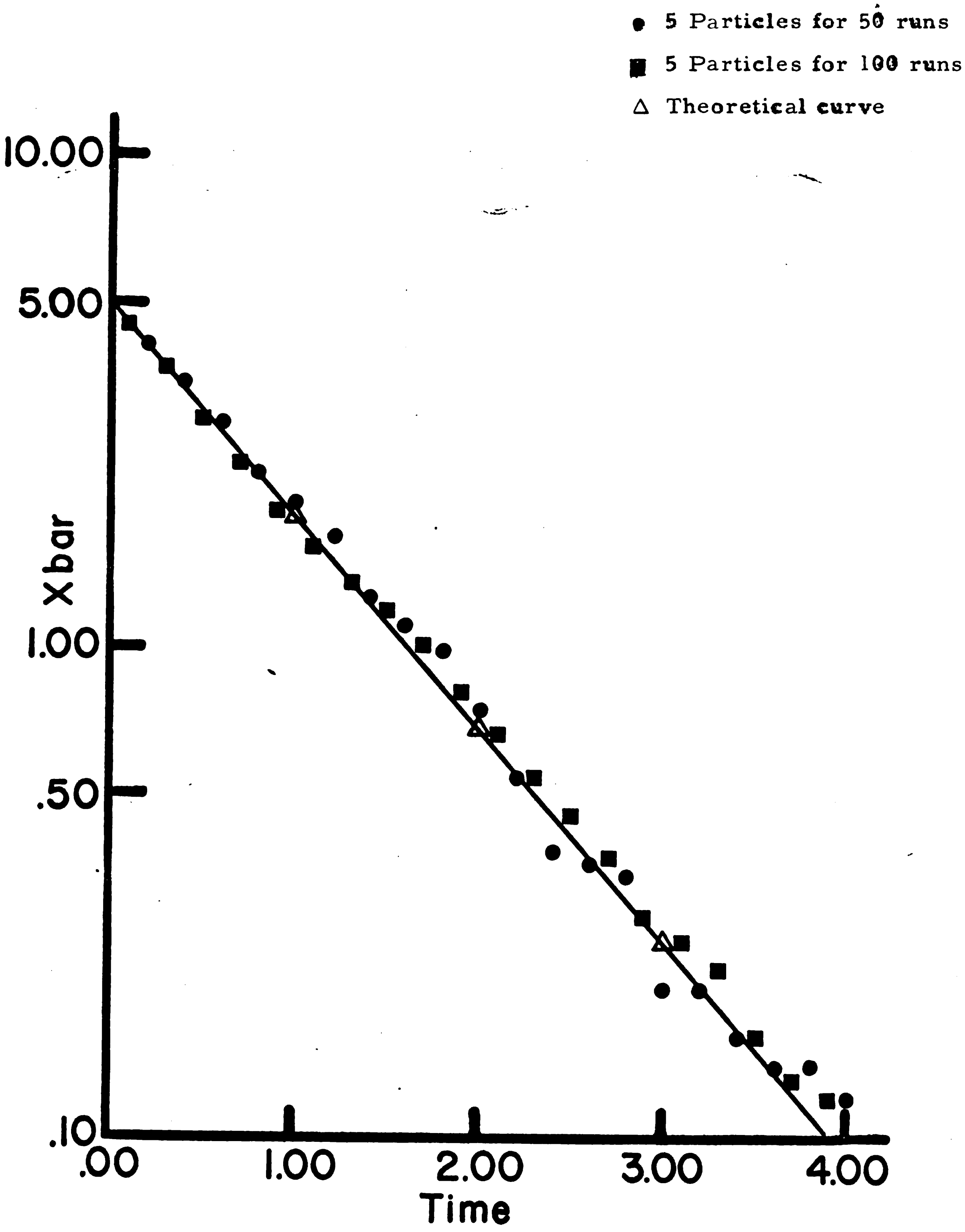
```
536       NMOL(IY, IRUN)=N(I+11, IRUN)
          GO TO 210
537       IF (TI-T(I+12)) 538, 539, 540
538       NMOL(IY, IRUN)=N(I+11, IRUN)
          GO TO 210
539       NMOL(IY,IRUN)=N(I+12, IRUN)
          GO TO 209
540       IF (TI-T(I+13)) 541, 542, 543
541       NMOL(IY, IRUN)=N(I+12, IRUN)
          GO TO 209
542       NMOL(IY, IRUN)=N(I+13, IRUN)
          GO TO 210
543       IF (TI-T(I+14)) 544, 545, 545
544       NMOL(IY, IRUN)=N(I+13, IRUN)
          GO TO 210
545       NMOL(IY, IRUN)=N(I+14, IRUN)
209       I=I+1
210       CONTINUE
220       CONTINUE
          PRINT 8
  8       FORMAT (1H1, 50HL. E. JACOBS 2A-B No. 7 20 PARTICLES FOR
     10   RUNS 8-15)
          PRINT 6
  6       FORMAT (1HO, 65H    TI    SUMXSQ    EXSQ    XBAR    XBARSQ
     1    VAR    STDEV)
          TI=0.000
          DO 203 IY=1, 46
          SUMXSQ=000.00
          DO 201 IRUN=1, K
201       SUMXSQ=SUMXSQ + (NMOL(IY, IRUN)**2
          EXSQ=SUMXSQ/K
          SUM=000.00
          DO 202 IRUN=1, K
202       SUM=SUM + NMOL(IY, IRUN)
          XBAR=SUM/K
          XBARSQ=XBAR**2
          VAR=EXSQ-XBARSQ
          STDEV=SQRTF(VAR)
          PRINT 7, TI, SUMXSQ, EXSQ, XBAR, XBARSQ, VAR, STDEV
  7       FORMAT (1H2, F7. 3, 2X, F7. 0, 2X, F9. 2, 2X, F7. 2, 2X, F9. 2, 2X
      F7. 3, 2X, F7. 3)
203       TI=TI+0. 005
          STOP
          END
```

A-B

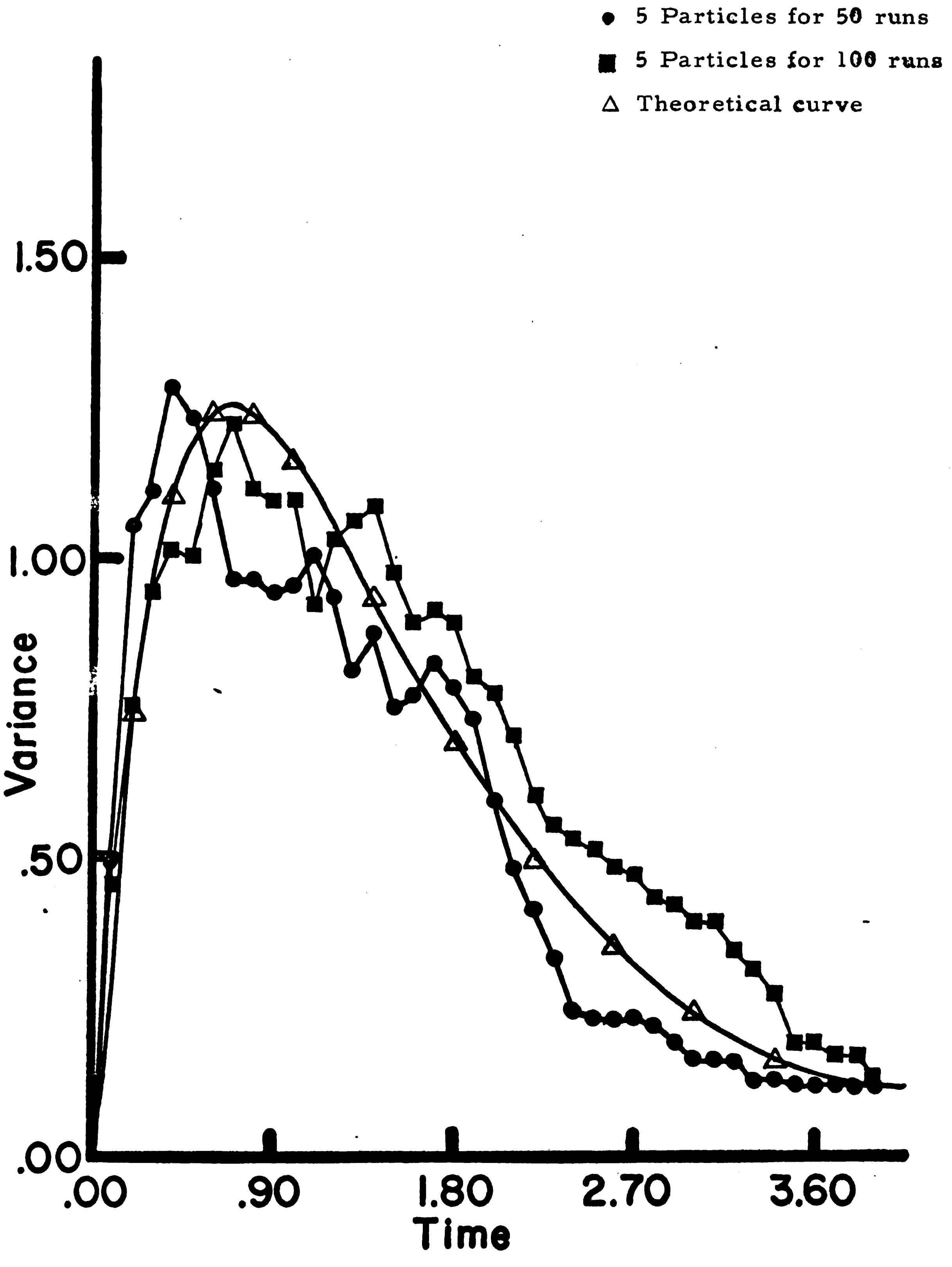

● 5 Particles for 5 runs
■ 5 Particles for 25 runs
△ Theoretical curve
10.00
5.00
1.00
.50
.10
Xbar
.00
1.00
2.00
3.00
4.00
Time


A-B
● 5 Particles for 5 runs
■ 5 Particles for 25 runs
△ Theoretical curve
Variance
2.00
1.50
1.00
.50
.00
.00
.90
1.80
2.70
3.60
Time

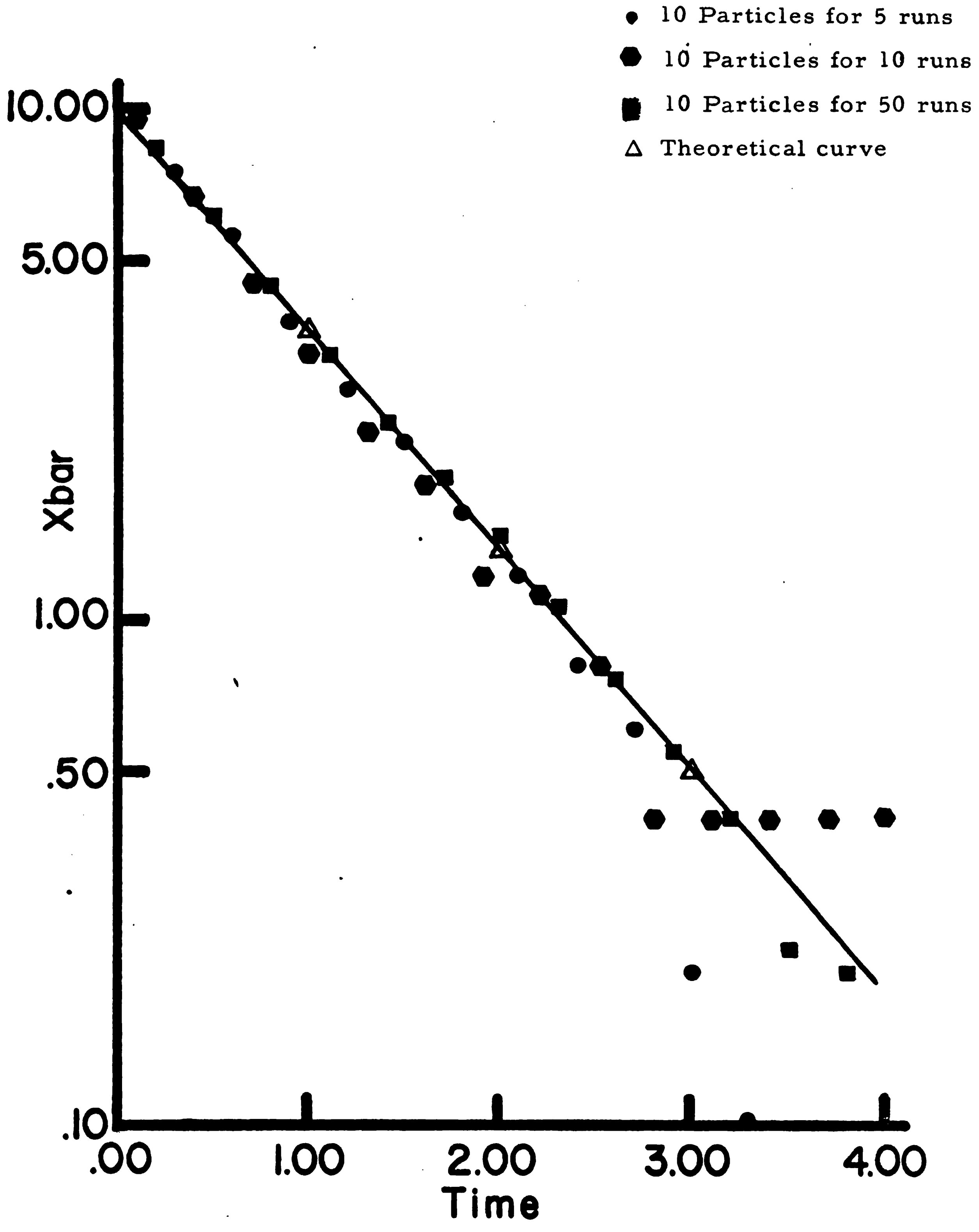
A→B


● 5 Particles for 50 runs
■ 5 Particles for 100 runs
△ Theoretical curve
10.00
5.00
1.00
.50
.10
Xbar
.00
1.00
2.00
3.00
4.00
Time

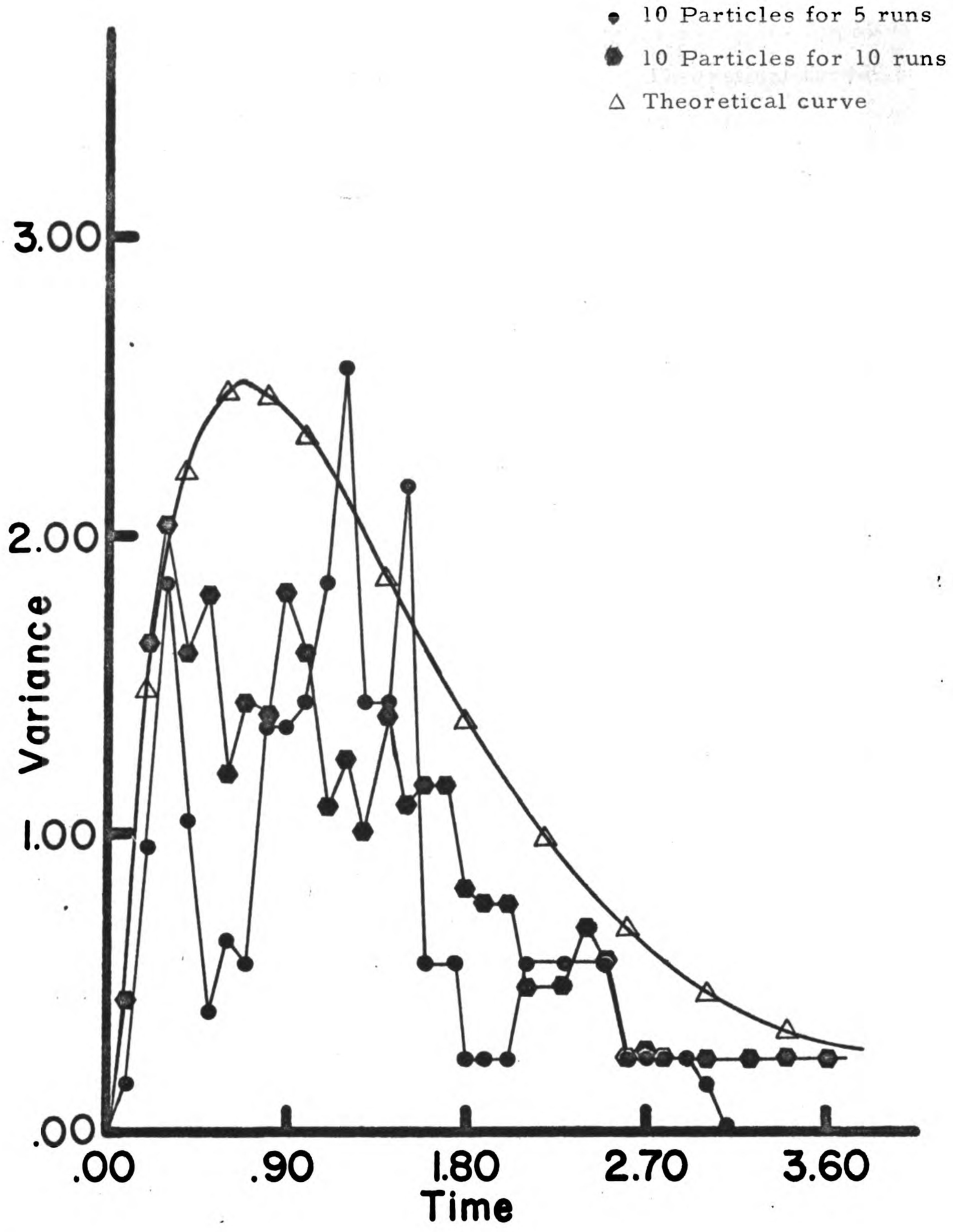
A-B


● 5 Particles for 50 runs
■ 5 Particles for 100 runs
△ Theoretical curve
1.50
1.00
.50
.00
Variance
.00
.90
1.80
2.70
3.60
Time

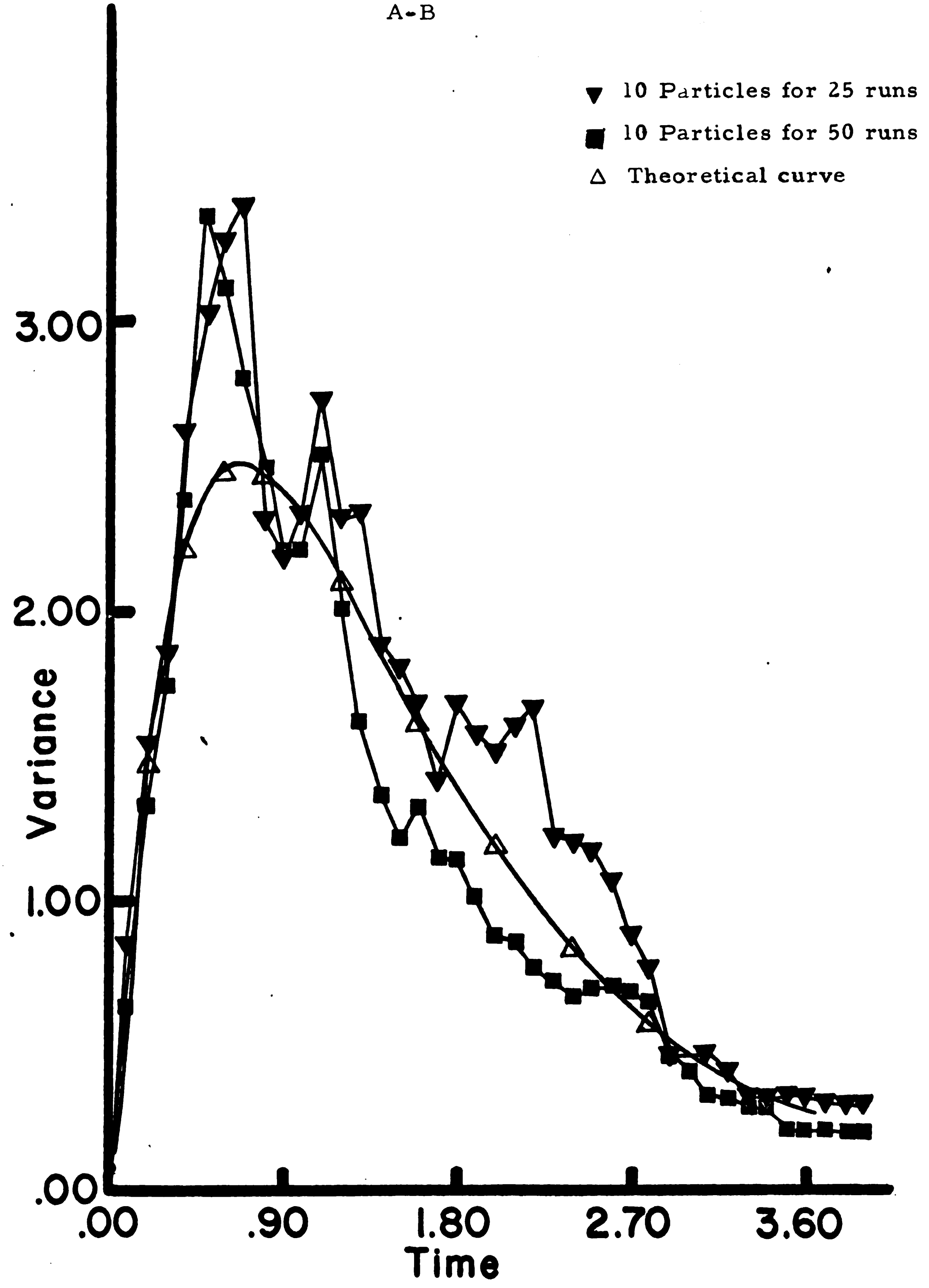
A-B

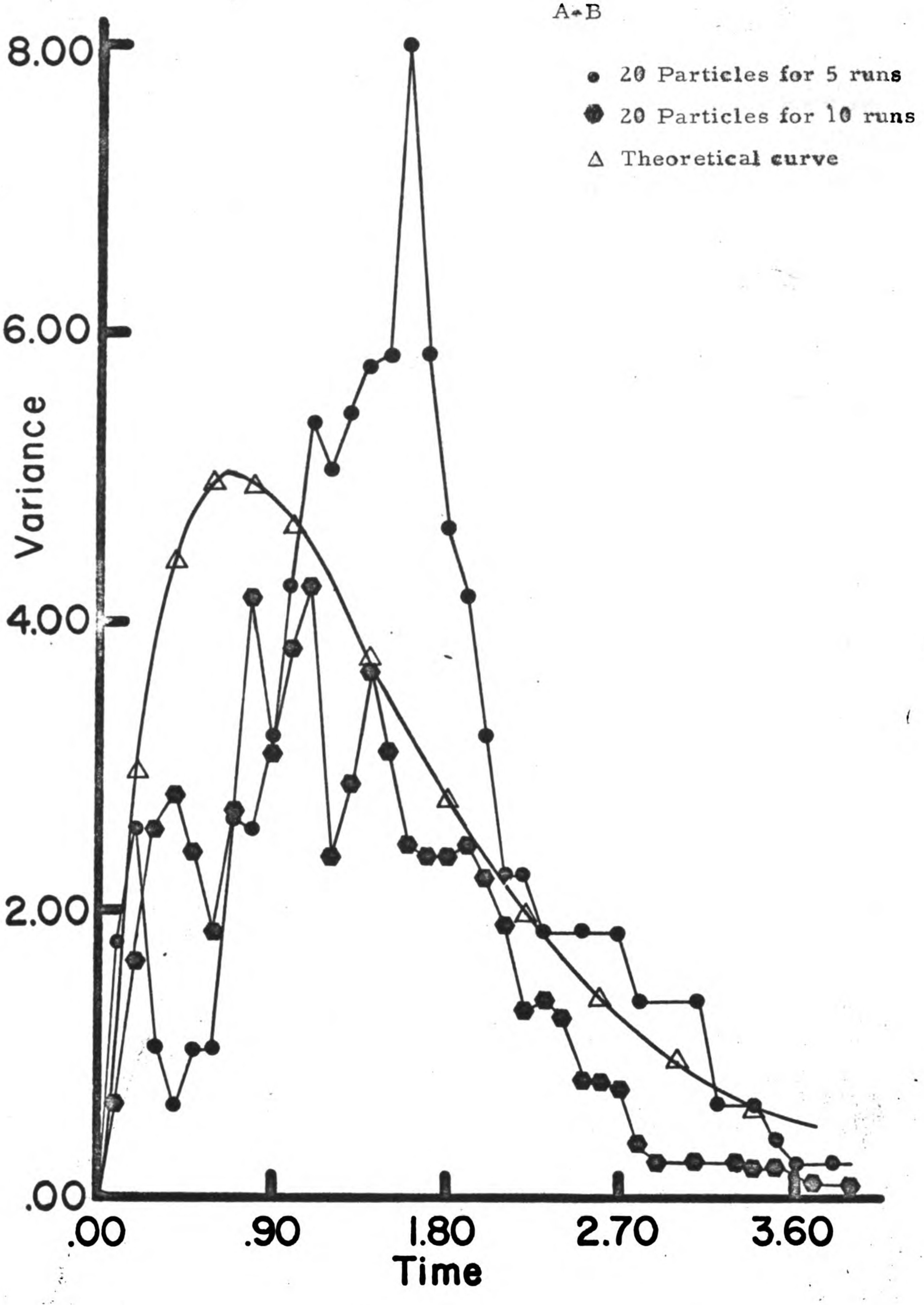

● 10 Particles for 5 runs
⬢ 10 Particles for 10 runs
■ 10 Particles for 50 runs
△ Theoretical curve
10.00
5.00
1.00
.50
.10
Xbar
.00
1.00
2.00
3.00
4.00
Time

A-B

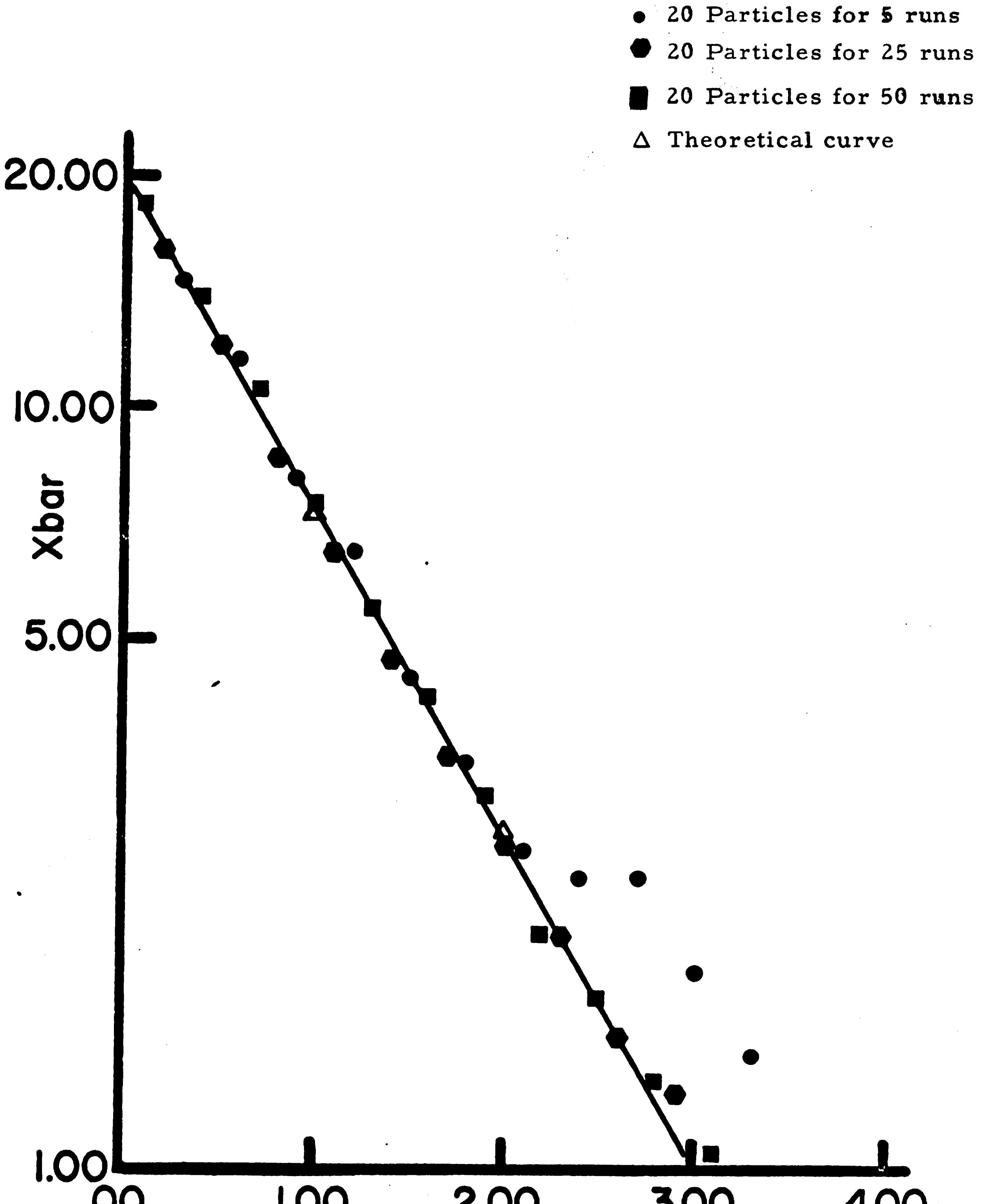

● 10 Particles for 5 runs
⬢ 10 Particles for 10 runs
△ Theoretical curve
3.00
2.00
1.00
.00
Variance
.00
.90
1.80
2.70
3.60
Time

A-B

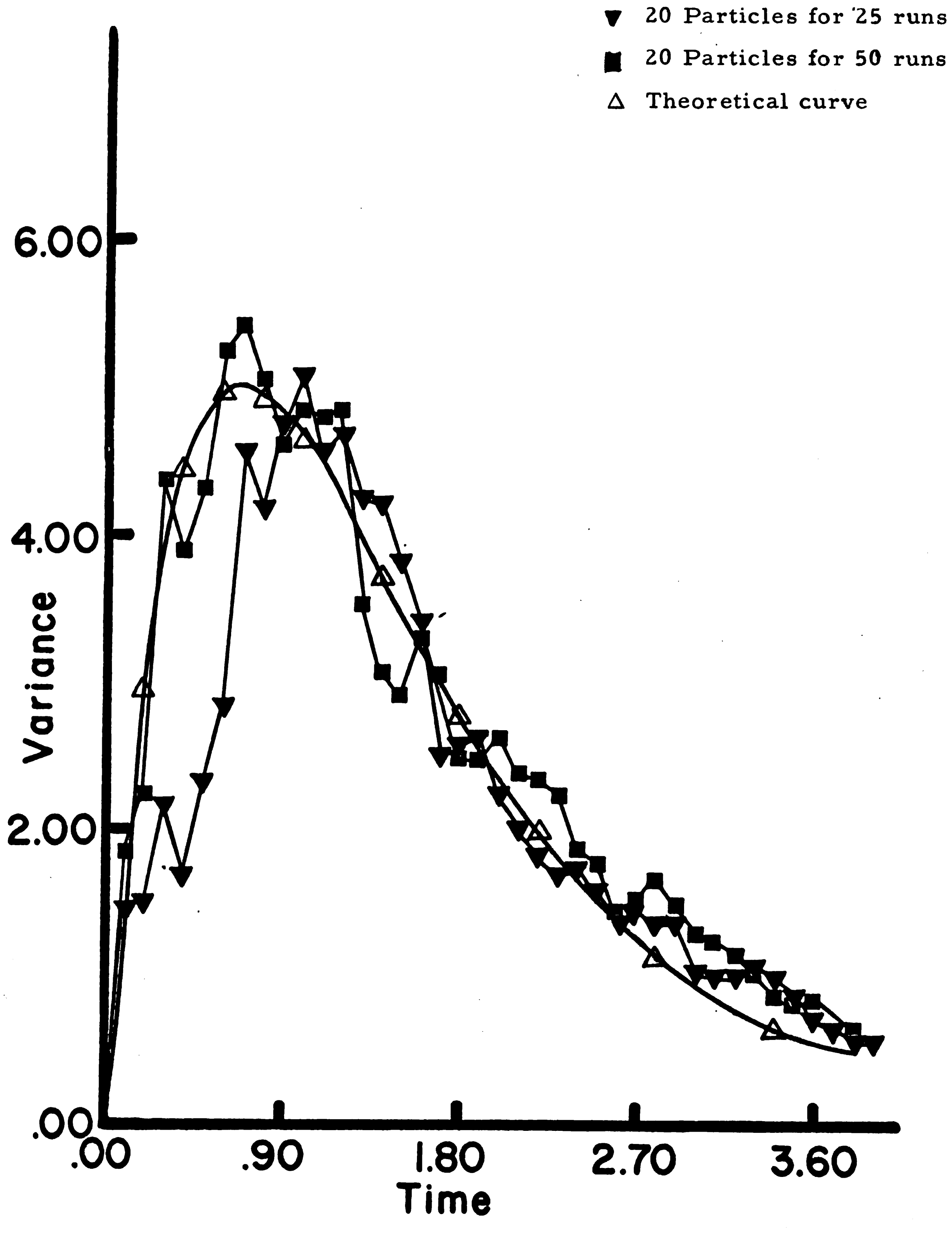

▼ 10 Particles for 25 runs
■ 10 Particles for 50 runs
△ Theoretical curve
3.00
2.00
1.00
.00
Variance
.00
.90
1.80
2.70
3.60
Time


A→B
● 20 Particles for 5 runs
⬢ 20 Particles for 10 runs
△ Theoretical curve
Variance
8.00
6.00
4.00
2.00
.00
.00
.90
1.80
2.70
3.60
Time

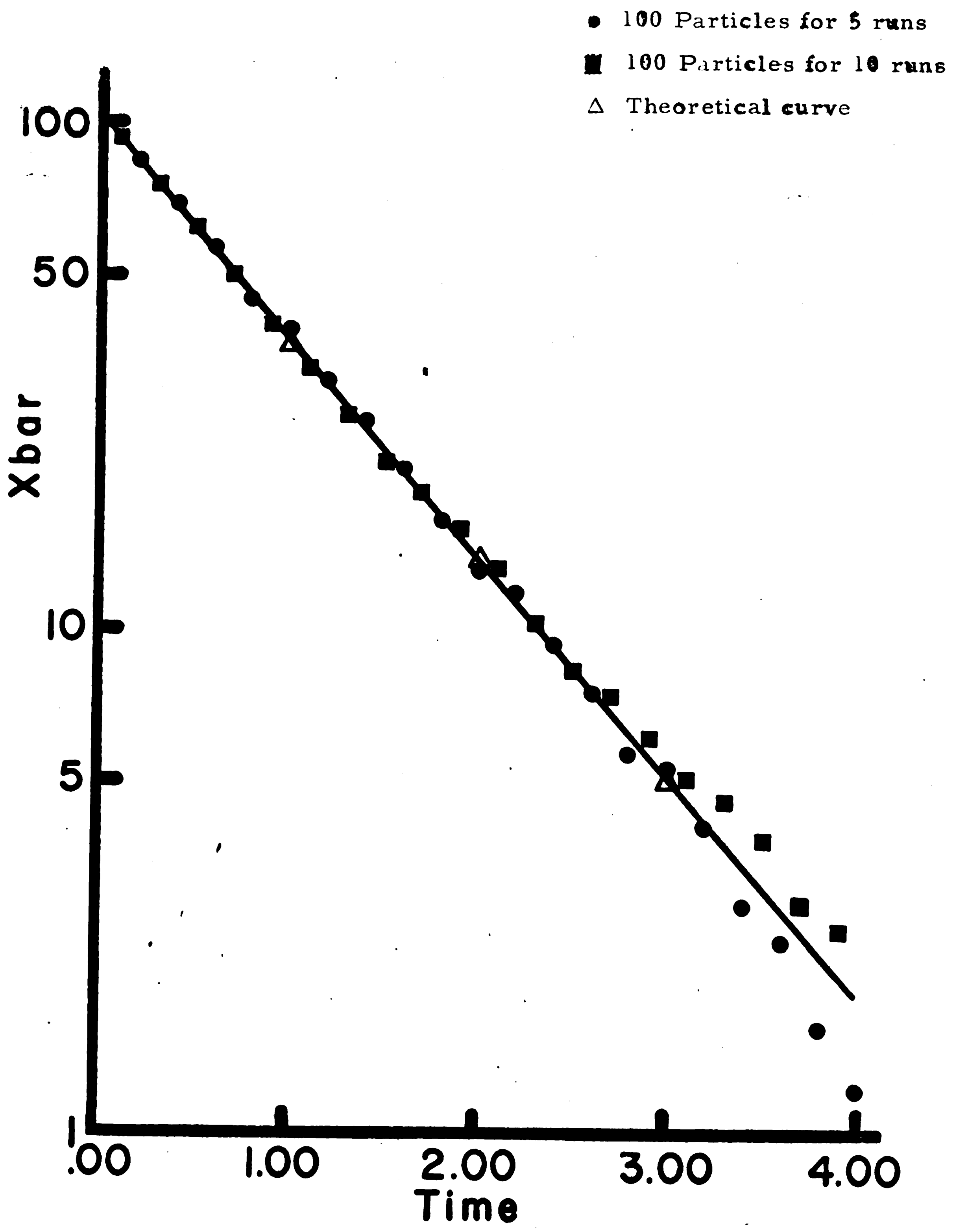
A-B

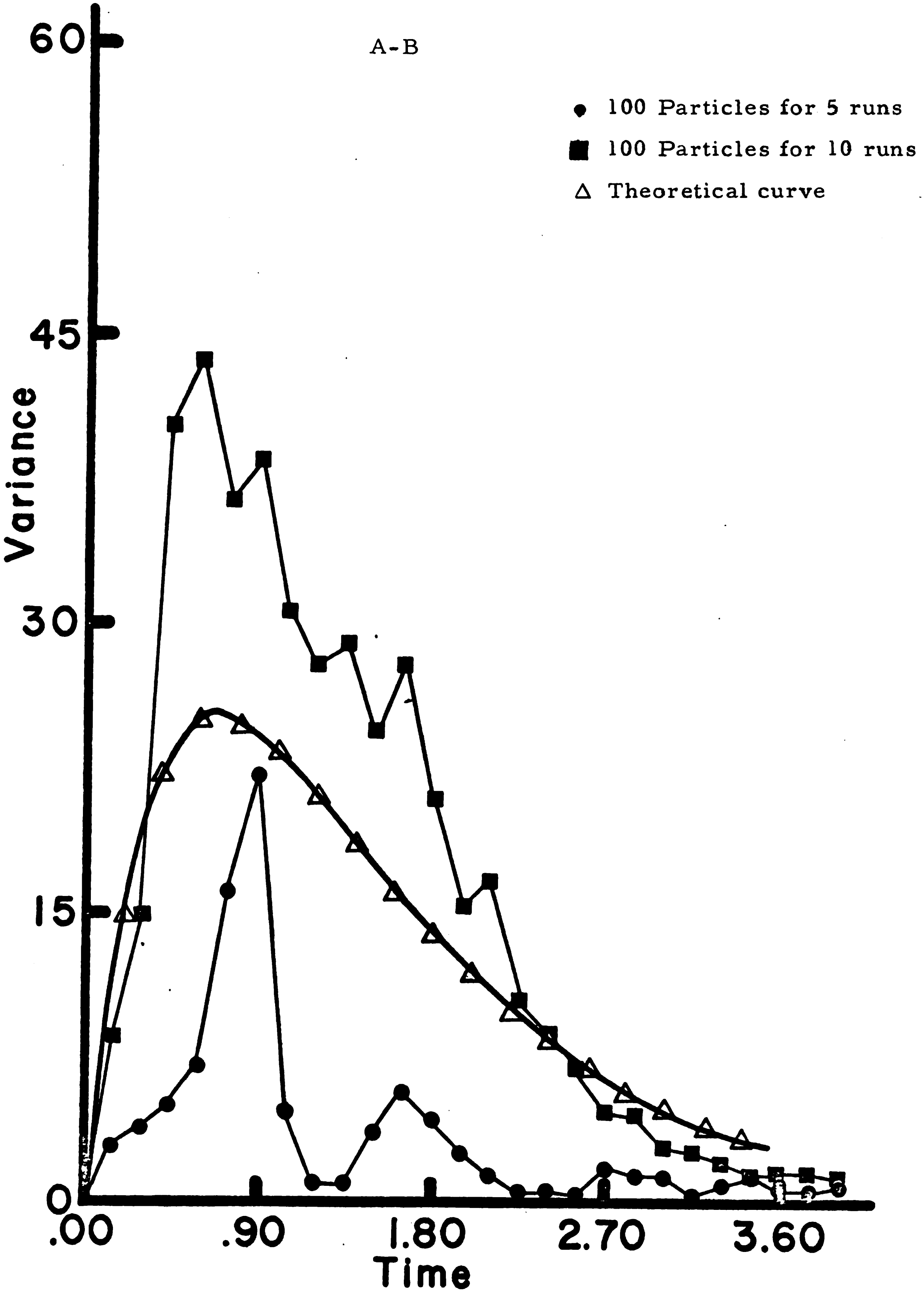

● 20 Particles for 5 runs
⬢ 20 Particles for 25 runs
■ 20 Particles for 50 runs
△ Theoretical curve
20.00
10.00
5.00
1.00
Xbar
.00
1.00
2.00
3.00
4.00
Time

A-B


▼ 20 Particles for 25 runs
■ 20 Particles for 50 runs
△ Theoretical curve
Variance
6.00
4.00
2.00
.00
.00
.90
1.80
2.70
3.60
Time

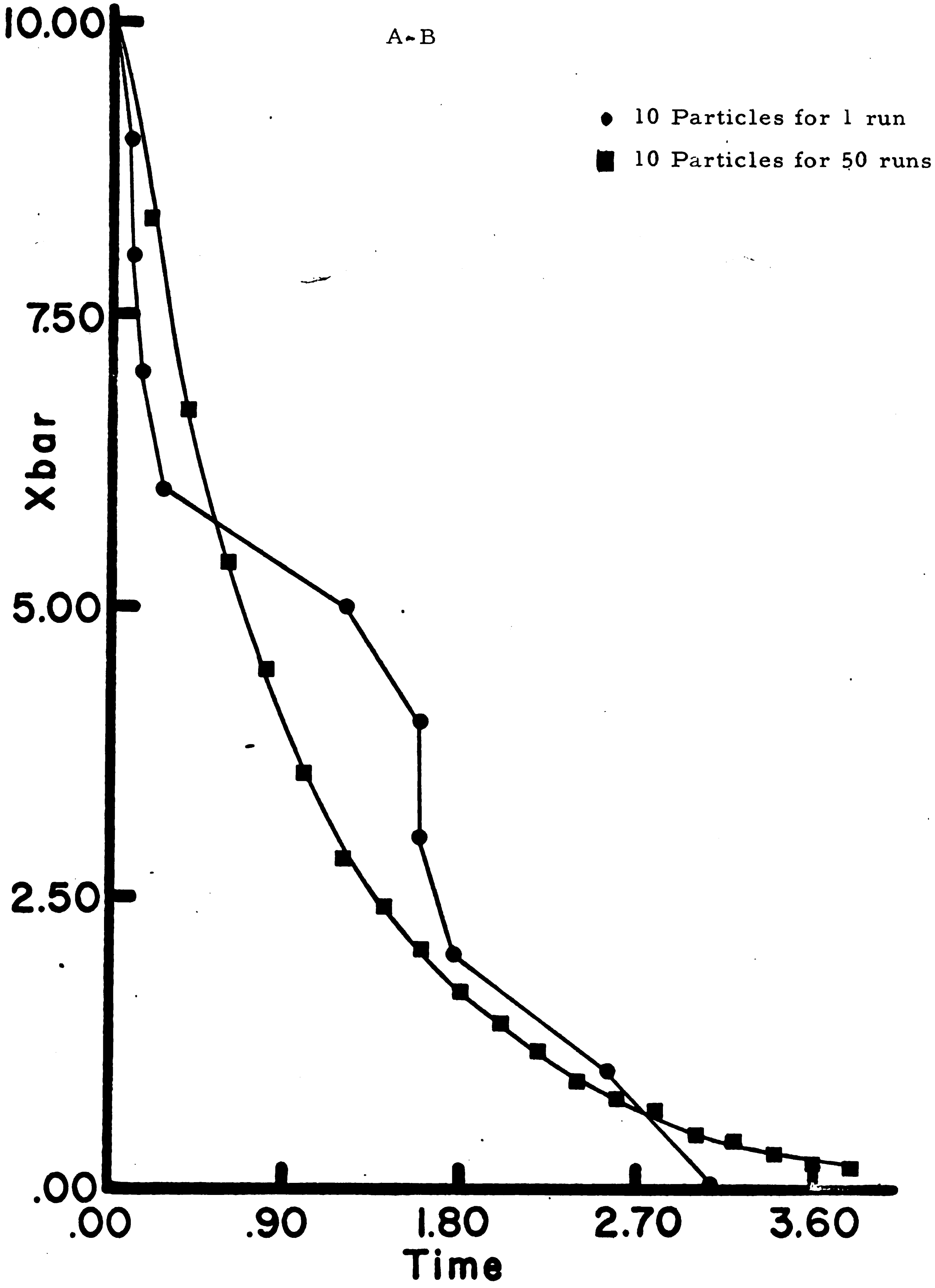
A-B


● 100 Particles for 5 runs
■ 100 Particles for 10 runs
△ Theoretical curve
100
50
10
5
1
Xbar
.00
1.00
2.00
3.00
4.00
Time



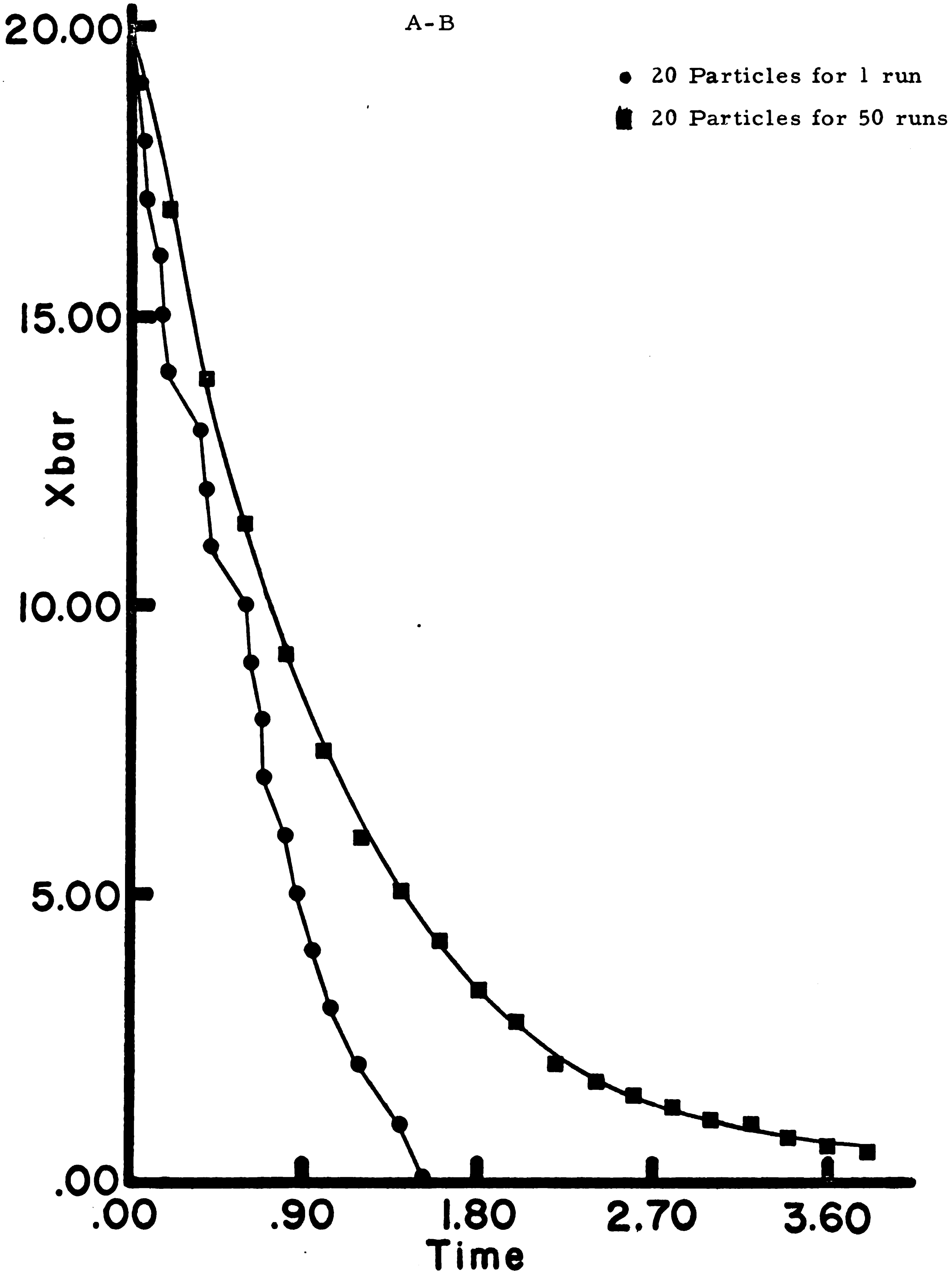
A-B

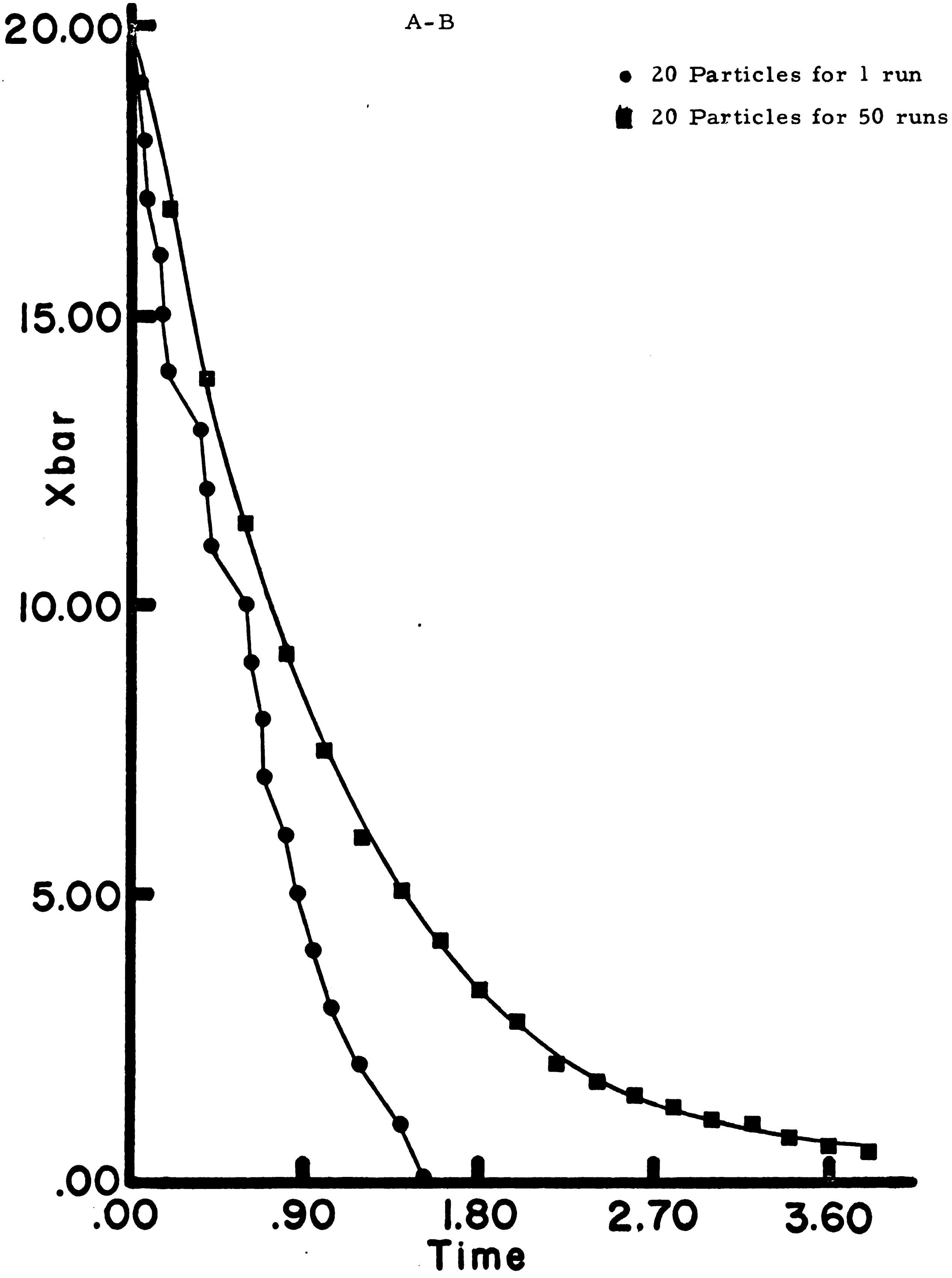
● 100 Particles for 5 runs
■ 100 Particles for 10 runs
△ Theoretical curve
Variance
60
45
30
15
0
.00
.90
1.80
2.70
3.60
Time



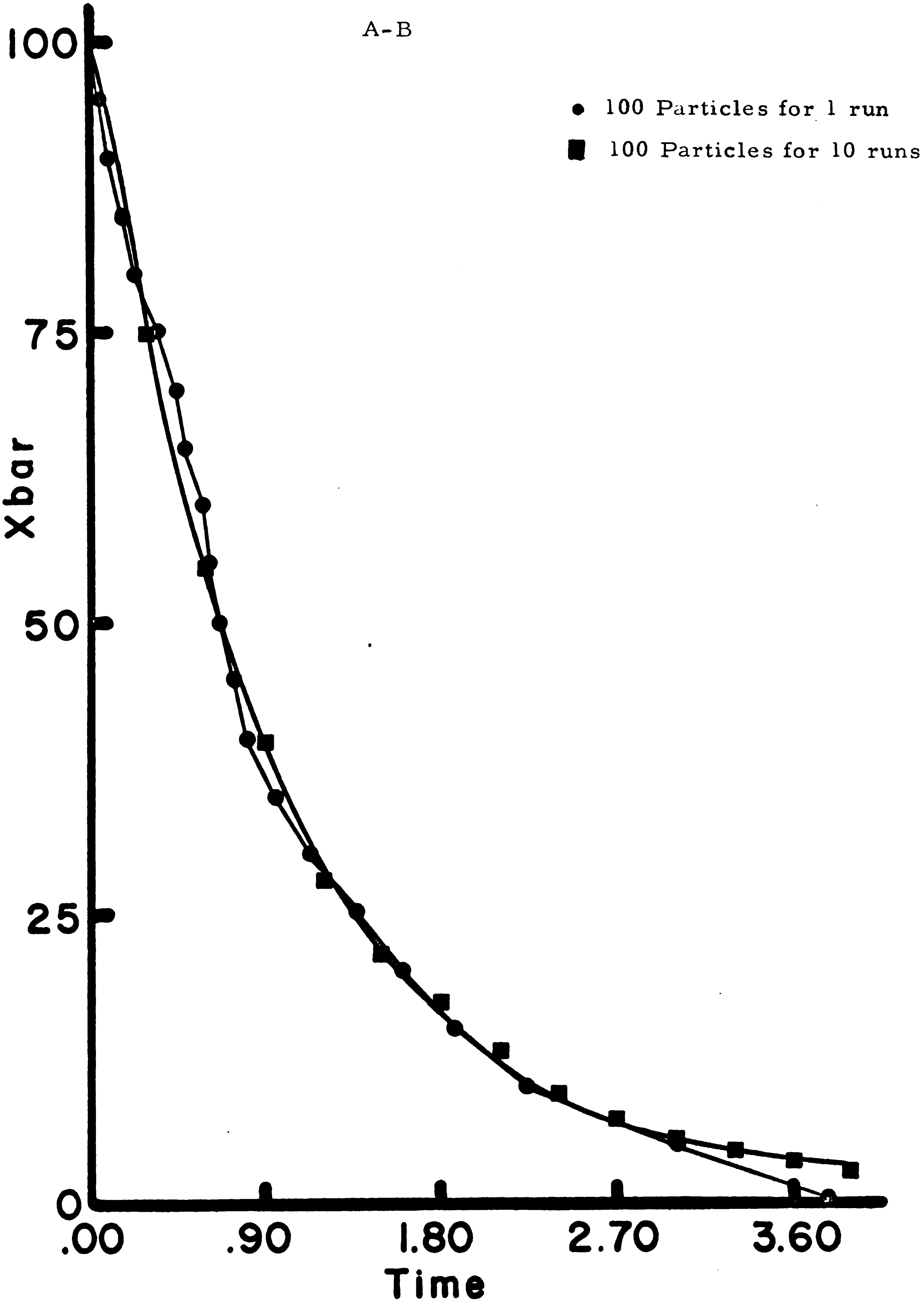
A-B

● 10 Particles for 1 run
■ 10 Particles for 50 runs
10.00
7.50
5.00
2.50
.00
Xbar
.00
.90
1.80
2.70
3.60
Time

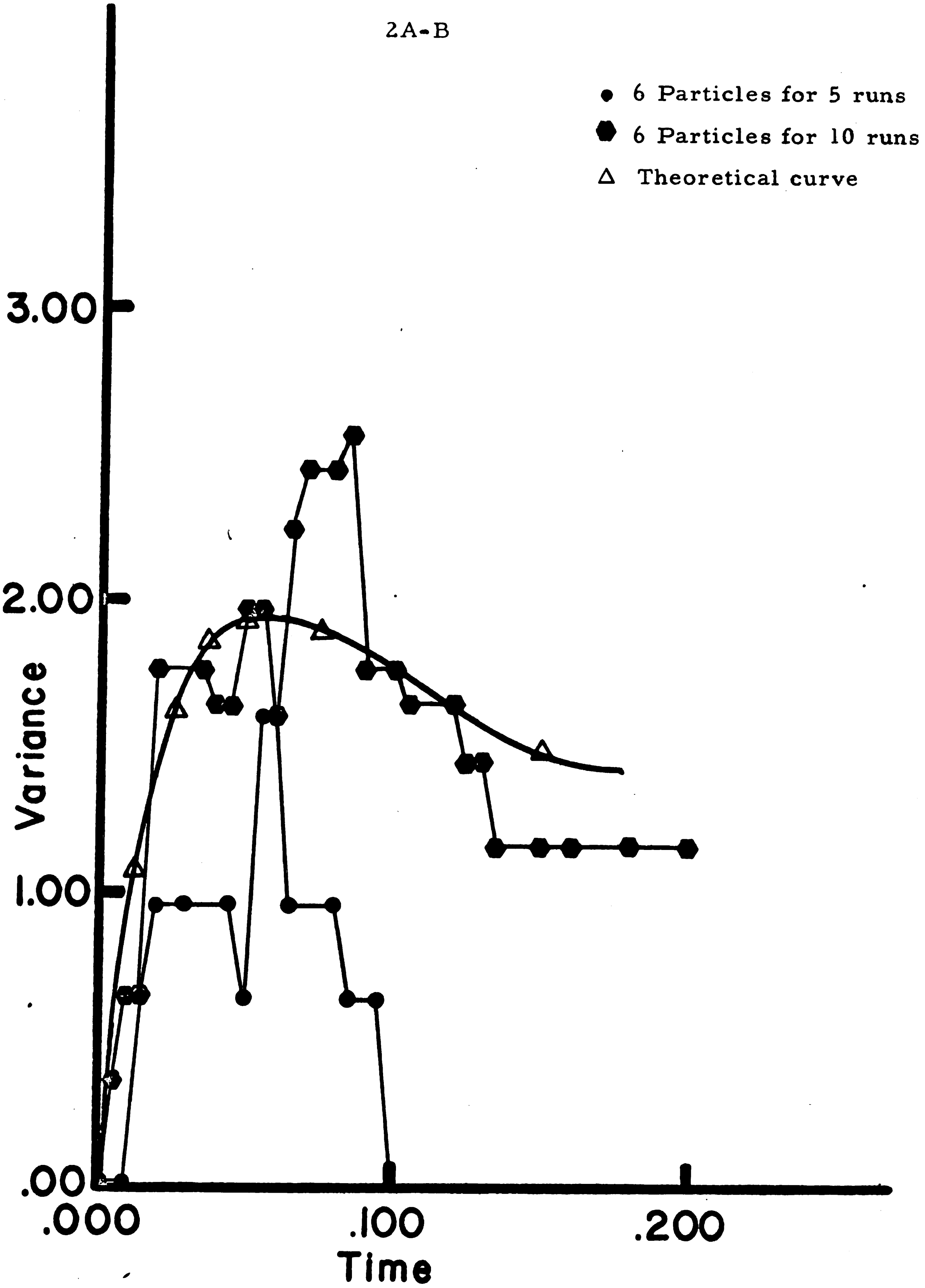

A-B
● 20 Particles for 1 run
■ 20 Particles for 50 runs
Xbar
20.00
15.00
10.00
5.00
.00
.00
.90
1.80
2.70
3.60
Time

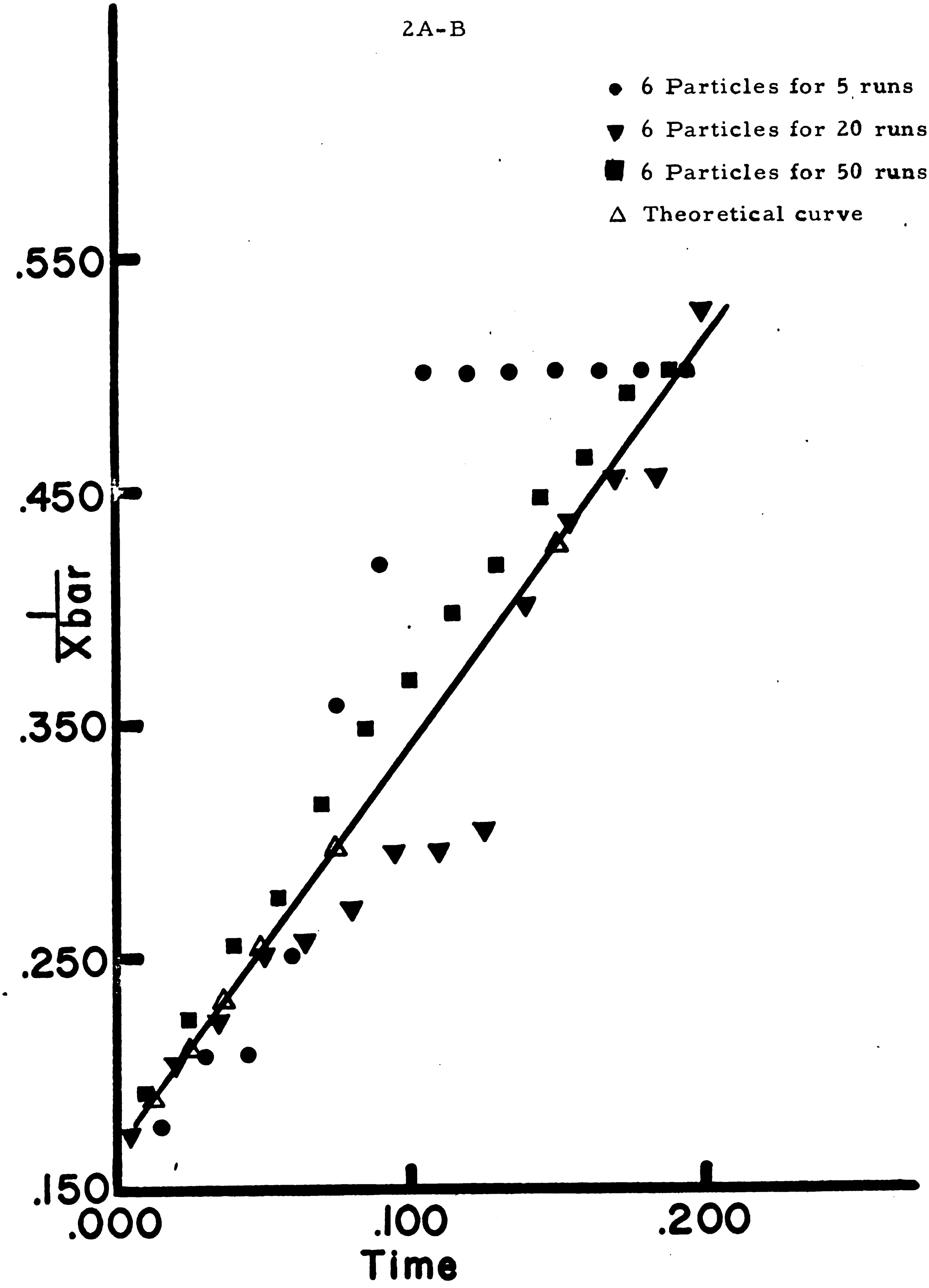

A-B
● 20 Particles for 1 run
■ 20 Particles for 50 runs
20.00
15.00
10.00
5.00
.00
Xbar
.00
.90
1.80
2.70
3.60
Time

A-B

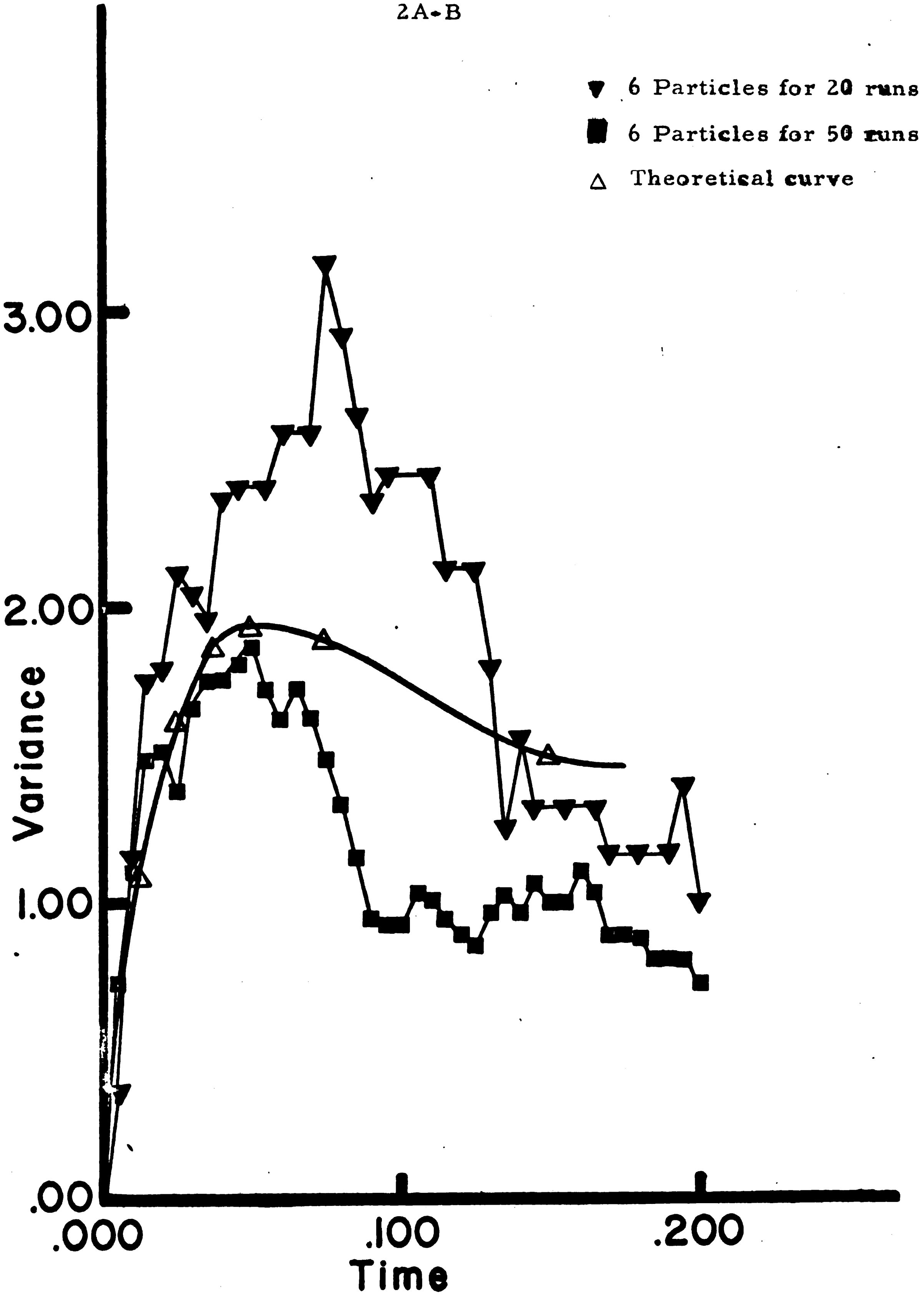

● 100 Particles for 1 run
■ 100 Particles for 10 runs
Xbar
100
75
50
25
0
.00
.90
1.80
2.70
3.60
Time

2A-B

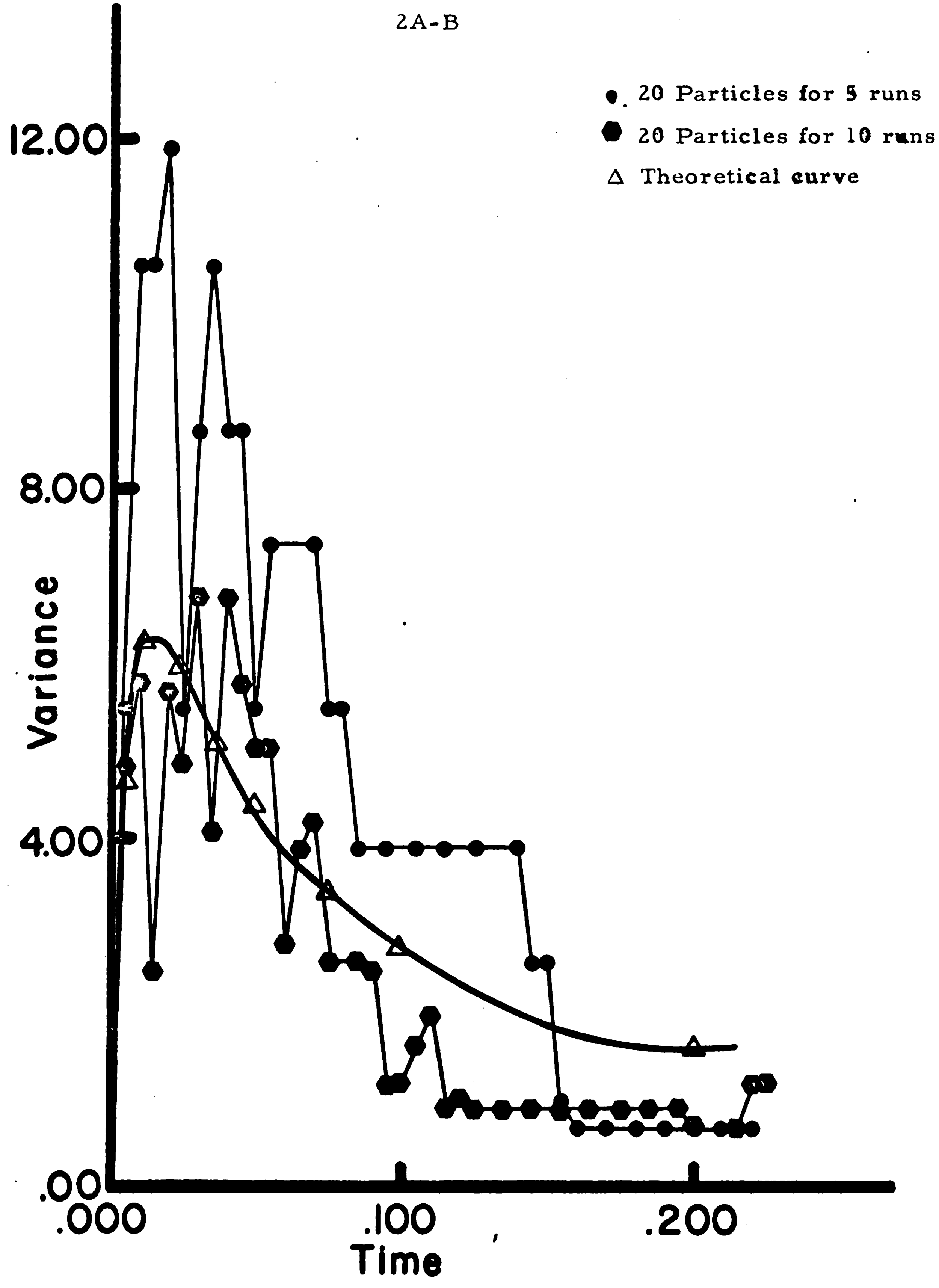

● 6 Particles for 5 runs
⬢ 6 Particles for 10 runs
△ Theoretical curve
3.00
2.00
1.00
.00
Variance
.000
.100
.200
Time


2A-B
● 6 Particles for 5 runs
▼ 6 Particles for 20 runs
■ 6 Particles for 50 runs
△ Theoretical curve
.550
.450
.350
.250
.150
$\frac{1}{\bar{X}}$
.000
.100
.200
Time

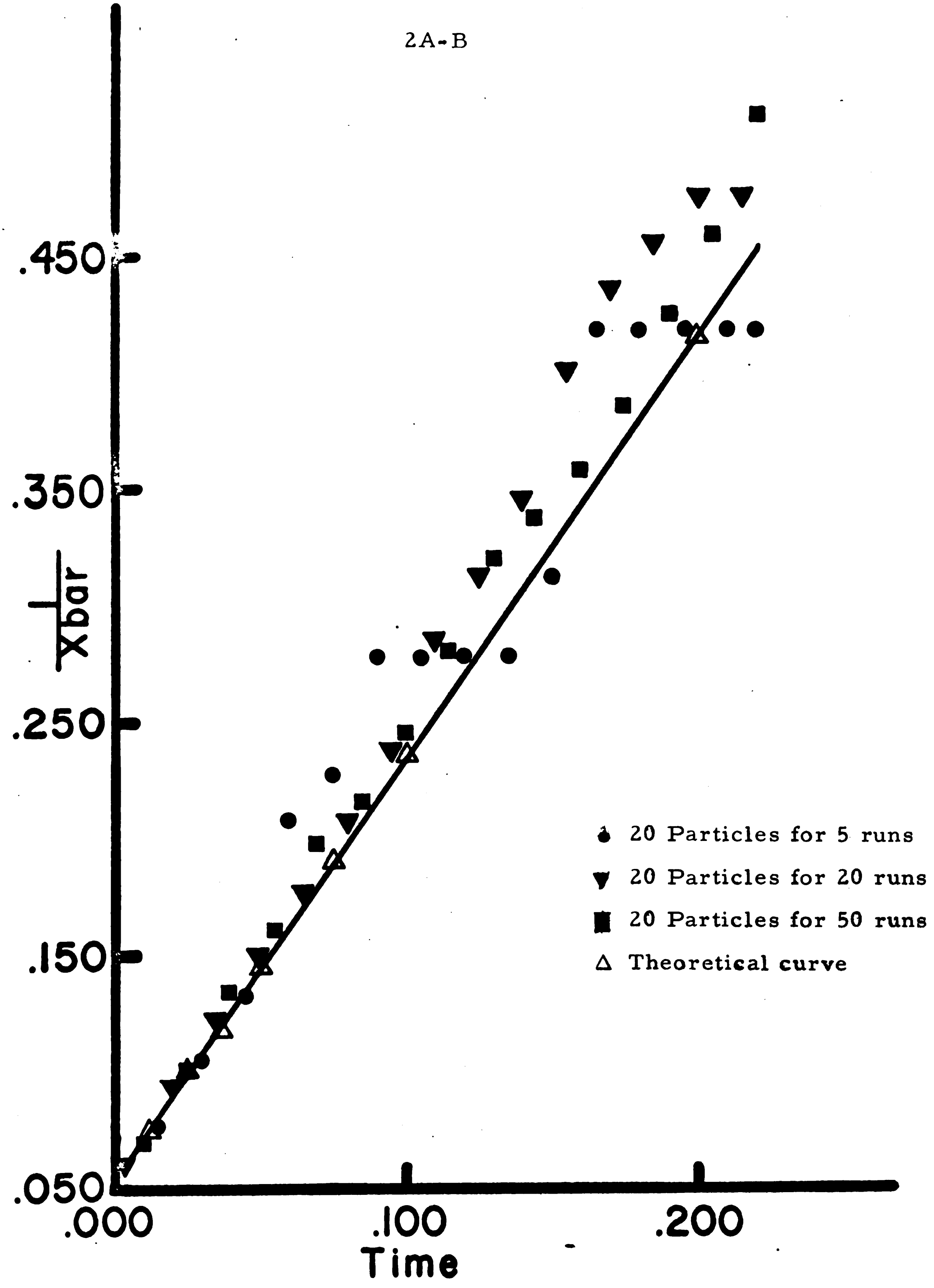
2A-B


▼ 6 Particles for 20 runs
■ 6 Particles for 50 runs
△ Theoretical curve
Variance
3.00
2.00
1.00
.00
.000
.100
.200
Time

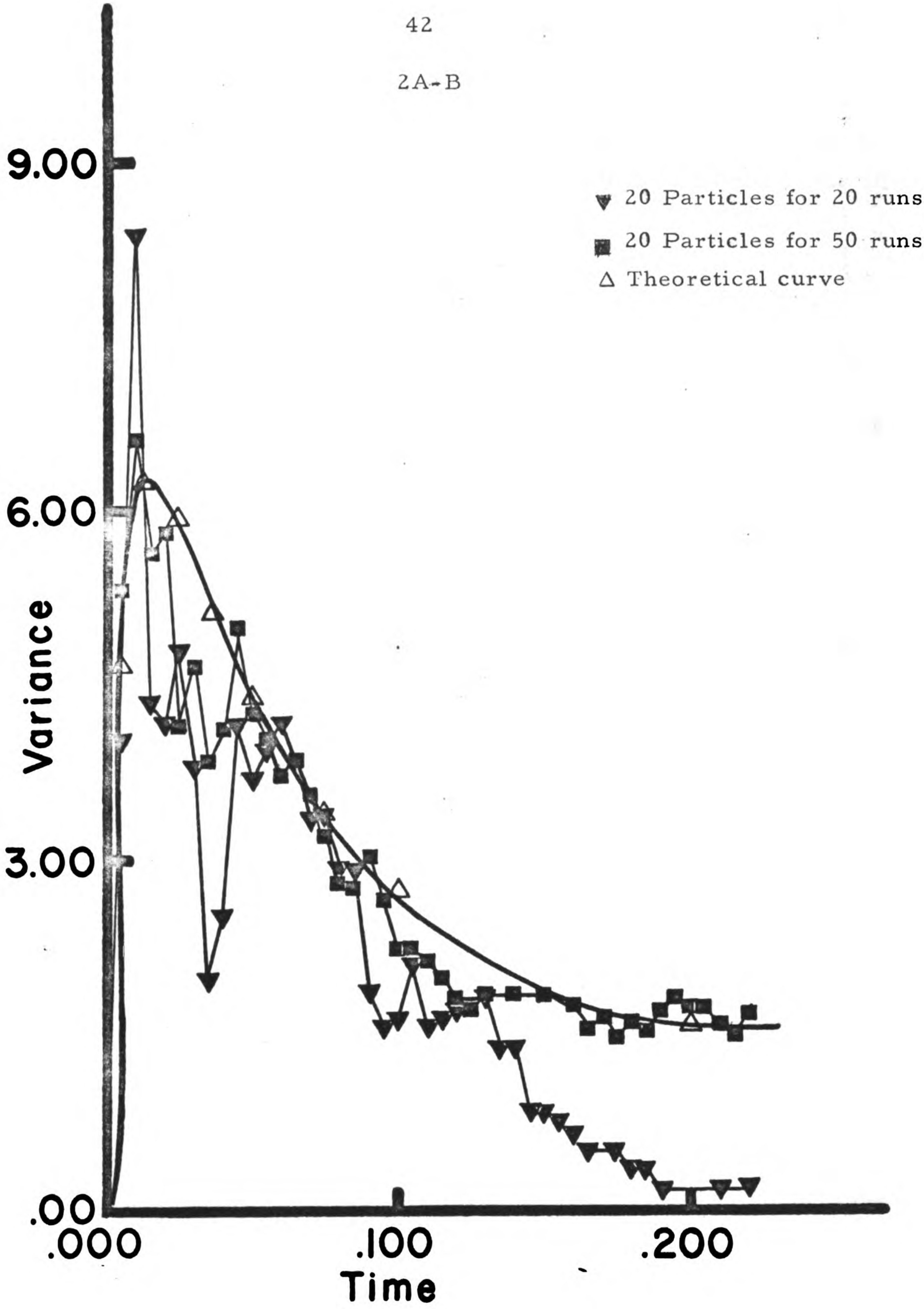
2A-B


● 20 Particles for 5 runs
⬢ 20 Particles for 10 runs
△ Theoretical curve
12.00
8.00
4.00
.00
Variance
.000
.100
.200
Time

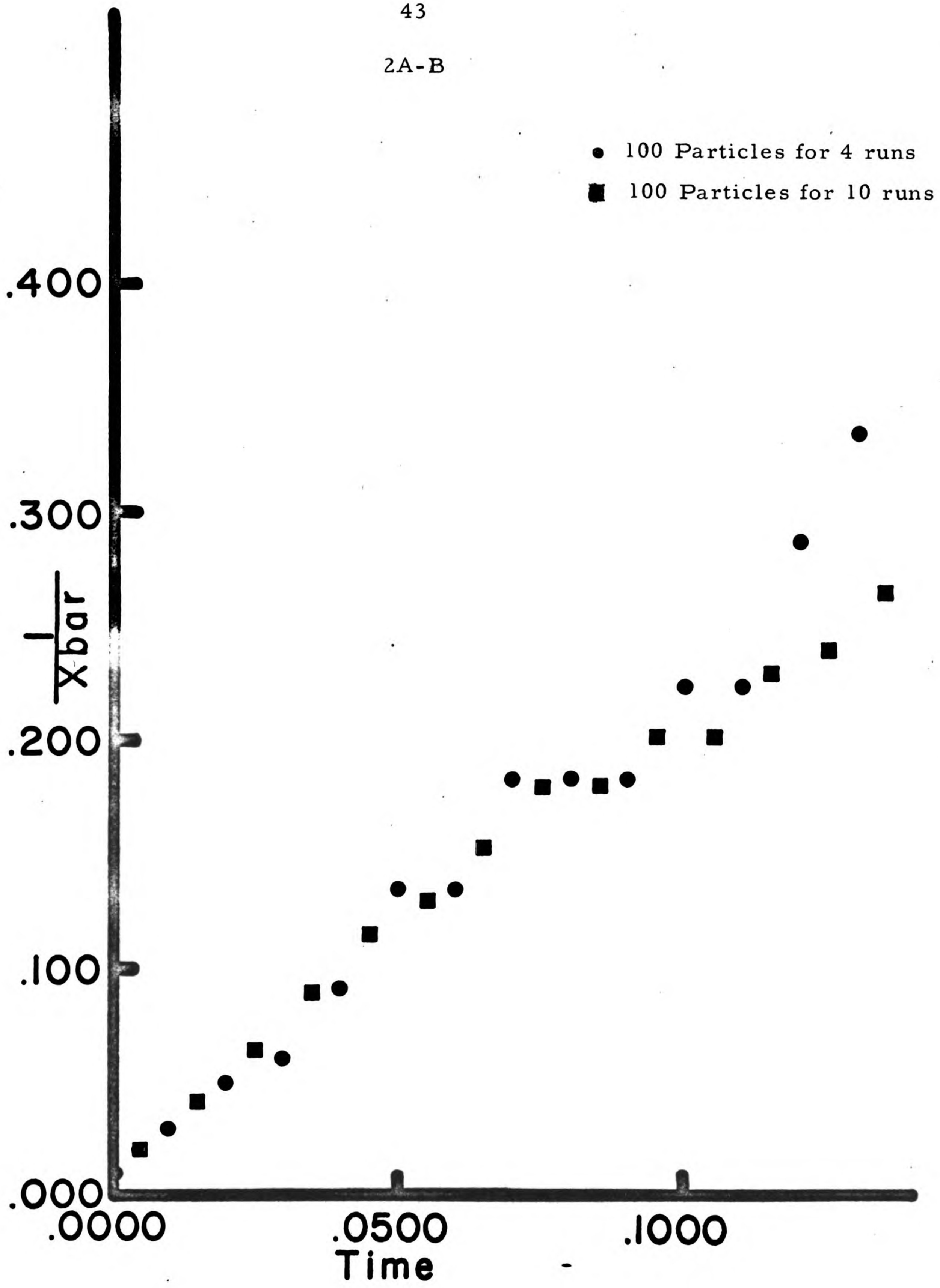
2A-B


.450
.350
.250
.150
.050
$\frac{1}{\text{Xbar}}$
.000
.100
.200
Time
● 20 Particles for 5 runs
▼ 20 Particles for 20 runs
■ 20 Particles for 50 runs
△ Theoretical curve

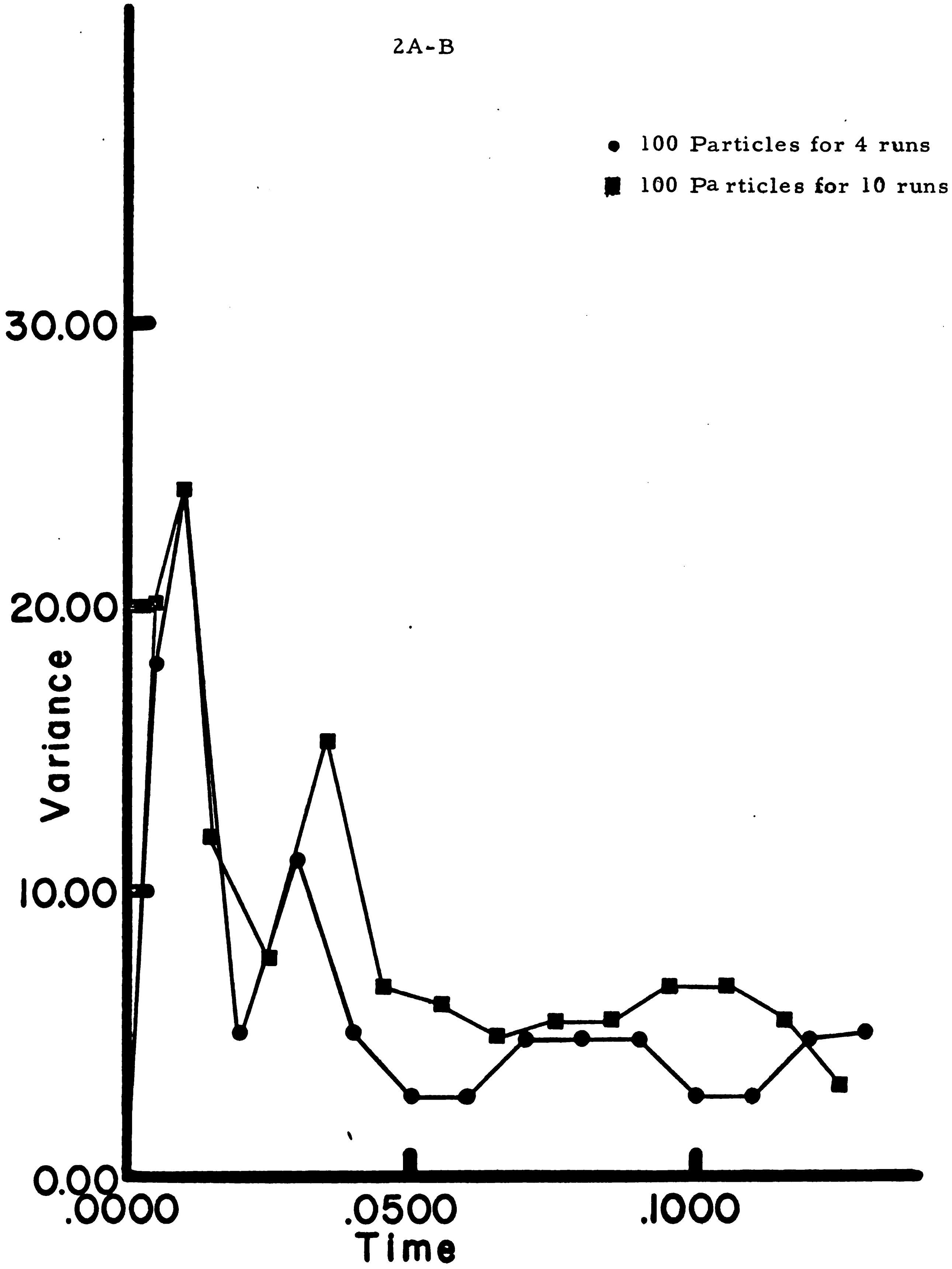
2A-B


9.00
6.00
3.00
.00
Variance
.000
.100
.200
Time
▼ 20 Particles for 20 runs
■ 20 Particles for 50 runs
△ Theoretical curve

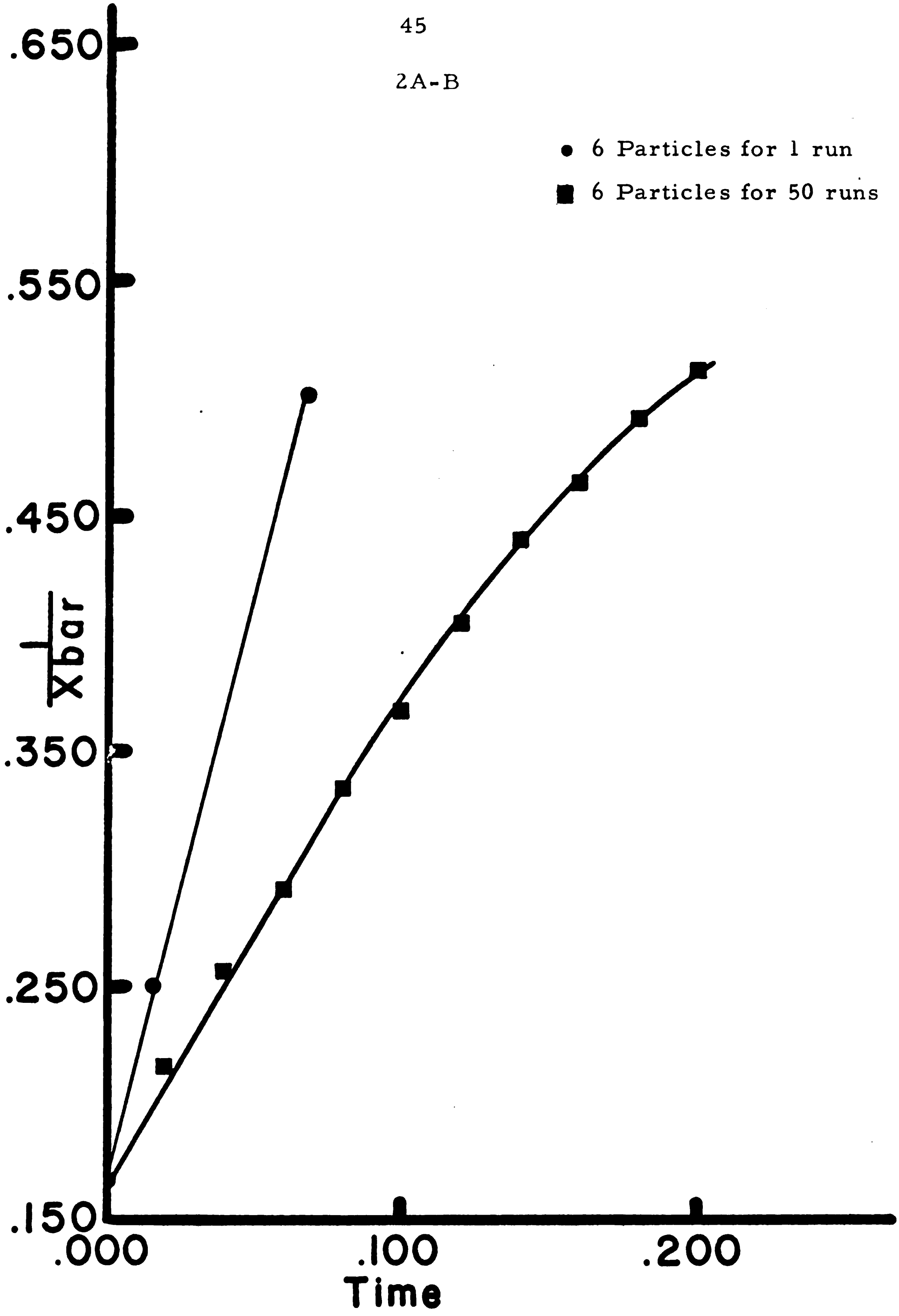
2A-B


● 100 Particles for 4 runs
■ 100 Particles for 10 runs
.400
.300
.200
.100
.000
$\frac{1}{\text{Xbar}}$
.0000
.0500
.1000
Time

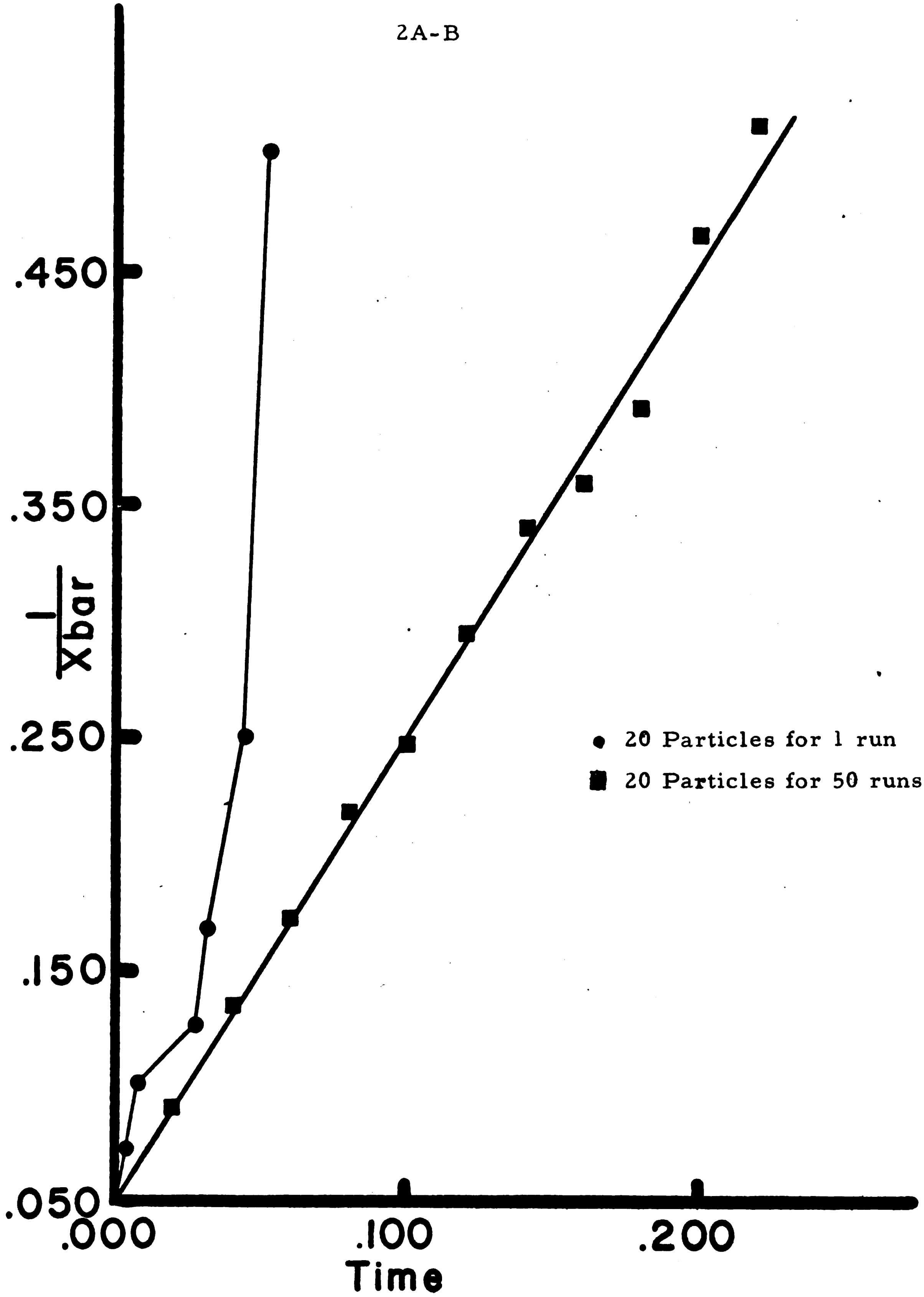
2A-B

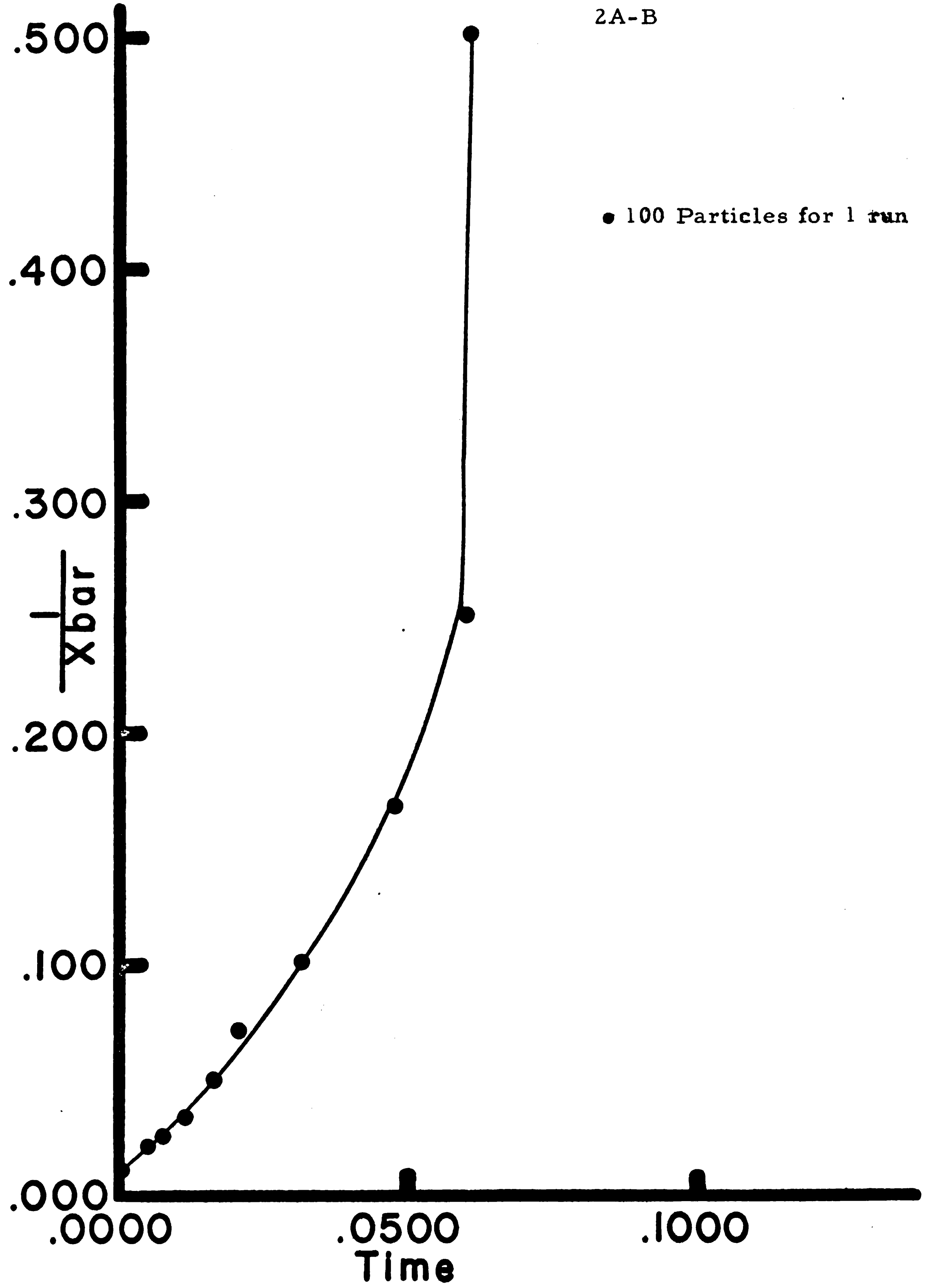

● 100 Particles for 4 runs
■ 100 Particles for 10 runs
30.00
20.00
10.00
0.00
Variance
.0000
.0500
.1000
Time

2A-B


● 6 Particles for 1 run
■ 6 Particles for 50 runs
.650
.550
.450
1/Xbar
.350
.250
.150
.000
.100
.200
Time

2A-B


.450
.350
.250
.150
.050
$\frac{1}{Xbar}$
.000
.100
.200
Time
● 20 Particles for 1 run
■ 20 Particles for 50 runs


2A-B
● 100 Particles for 1 run
.500
.400
.300
1/Xbar
.200
.100
.000
.0000
.0500
.1000
Time

[redacted]