A TEST OF REVEALED PREFERENCE THEOREMS

Thesis for the Degree of M. S.

MICHIGAN STATE UNIVERSITY

Georg Hasenkamp

1970

ABSTRACT

A TEST OF REVEALED PREFERENCE THEOREMS

By

Georg Hasenkamp

The consistency of assumptions made in an economic theory of consumer behavior can be tested by the revealed preference theorems. This consistency is frequently assumed in demand estimation, though very little research attention has been directed toward its verification.

Revealed preference theory uses actual market observations to determine a preference ordering among different combinations of commodities. For example, if the x^i denote commodity vectors purchases in period i, and p^i the corresponding vector of prices paid, then a condition of $p^o x^o \geq p^o x'$ implies that x^o is revealed to be preferred over x', since x' could have been bought but was not. A consistent behavior stipulated by revealed preference theory requires that $p'x' < p'x^o$ must be true in order to avoid a circular preference pattern. This consistency of behavior was tested in the study. The method of testing used concepts of directed graph theory. An appendix on graph theory has been added to explain the necessary concepts of this mathematical tool.

The test of the revealed preference theorems was performed with data provided by the Michigan State University Consumer Panel. Families participating in this panel made weekly detailed reports on prices paid and quantities of commodities purchased. Since only data on food purchases were available, the demand independence of food from non-food

commodities had to be assumed in order to perform a valid test. Also the assumption of a four-week demand period was made.

The results of the study showed extensive violations to the revealed preference theorems. This indicates possible deficiencies in the assumptions of the economic theory of consumer behavior. Some modifications are suggested; use of a dynamic rather than a static approach, and the introduction of the perception threshold. The perception threshold is introduced because individuals might not perceive small variations in prices and quantities of commodities.

A TEST OF REVEALED PREFERENCE THEOREMS

By

Georg Hasenkamp

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Agricultural Economics

1970

Meinen lieben Eltern gewidmet

ACKNOWLEDGEMENTS

A number of individuals have been of considerable assistance during the development of this thesis. I am especially indebted to Dr. Anthony Y.C. Koo who acted as my research supervisor, and who was largely responsible for the development of my interest in revealed preference theory. His encouragement and support is very much appriciated.

It is a special pleasure to express my gratitude to Dr. Lester V. Manderscheid, my academic advisor, for his devoted help and most valuable comments and suggestions. I am thankful for his interest in my graduate student education.

My particular thanks are due to Professor Wilhelm Krelle of "der Universität Bonn" and Dr. James Shaffer for most helpful discussions with regard to the introduction of perception thresholds into the theory of consumer behavior.

For programming assistance I am particularly indebted to Peggy Thomsen and Laura Robinsin of the Computer Lab for the Department of Agricultural Economics.

Many people read earlier drafts of this thesis and offered valuable advice and criticism. I wish to thank especially the comments of my academic advisor Dr. Lester V. Manderscheid, my research supervisor Dr. Anthony Y.C. Koo, my teachers Dr. Jan Kmenta and Dr. James B. Ramsey, and my colleague Lloyd Teigen.

The research for this thesis was supported by a National Science Foundation grant received by Dr. Anthony Y.C. Koo, and by the Department of Agricultural Economics. During my years as a graduate student at Michigan State University I received financial support in form of a research assistantship from the Agricultural Experiment Station. I hope that this thesis is at least a partial reward for the very much appriciated financial support.

And finally, thanks are due to my wife Barbara. She had to tolerate my grumpy nature when things did not run as smoothly as they should have during the study project, yet she receives little comfort from the results obtained. She anticipated these results, - by introspection of her own shopping habits.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER I

INTRODUCTION

This thesis contains the results of a test of the revealed preference theorems. The purpose of the study was to test whether actual market behavior is consistent with the underlying basic assumptions of an economic theory of consumer behavior as developed in Chapter II. This consistency is frequently assumed in demand estimation, for example Barten (3), (4), Goldberger (10), Frisch (8), and Theil (36) demonstrate this in a rigorous manner. Yet, even though the assumption of consistency between the theoretical model and actual market behavior is so crucial, very little research attention has been directed toward its verification.

The test was performed on the data provided by the Michigan State University Consumer Food Purchase Panel of 1955 to 1958, (29). The panel consisted of a sample of about 250 families in Lansing, Michigan. The participating families made weekly detailed reports on their food purchases in terms of quantities bought and prices paid. The commodities were coded into 129 different groups, but no account of specific brands was made. Only families participating the full 52 weeks of at least one year were included in the study. Particular attention will be paid to those 64 families who participated all four years in the panel.

Revealed preference theory is based upon market observations. These observations are decisions made by the consumer unit

under a set of data, given a particular choice function. [1]
Each choice function has an underlying preference relation over the commodity space. Uzawa (37), Yokoyama (38), and Debreu (7) illustrated this in a brilliant manner; their influence will be noticed in Chapter II.

It seems logical to consider first preference relations over the commodity space, and to examine the restrictions it places on the choice function. Once this hypothetical framework is constructed, one may use revealed preference theory in turn to test the consistency of the underlying axioms and primitives. This will be illustrated in Chapter II.

The concept of the utility function, as it is so customarily used in micro-economic textbooks, will not be used in this thesis. The traditional approach to the economic theory of consumer behavior assumes the existence of a "well behaved" utility function defined on the commodity space. Subject to side constraints, the consumer unit is assumed to act as if it had to maximize the hypothetical utility function.

In contrast, the modern theory postulates certain axioms defined for the pertinent problem, and then derives specific properties in

---

1) The choice function might also be called demand function. But one has to be careful not to confuse this with the demand function in the Marshallian sense, where demand for a single commodity is a function of its own price, *ceteris paribus* all other prices, income, and assumptions. In the choice function the vector of all commodities is a function of all prices, income, and *ceteris paribus* other assumptions. This reflects the idea that decisions are made simultaneously, and are a general rather than partial equilibrium solution.

forms of theorems. Besides being intellectually more satisfying, the modern approach achieves the same results in the economic theory of consumer behavior with less stringent assumptions. It is even possible for the consumer unit to behave consistently under a specific preference relation, yet no order preserving utility function need exist. An example is shown by Debreu (6), who observed that a consumer unit with a lexicographic preference ordering on the commodity space cannot possess an order preserving utility function. With a lexicographic preference ordering a bundle $(x_1'', \text{---}, x_n'')$ is preferred over $(x_1', \text{---}, x_n')$ if (i) $x_1'' > x_1'$, regardless of the values of the other elements, or (ii) if $x_1'' = x_1'$, then $x_2'' > x_2'$, and so on. 1/

In Chapter II it will be indicated that at least the same basic assumptions used in the axiomatic treatment are also needed for the utility function approach. Therefore, if it is found that the assumptions for the axiomatic treatment are in contradiction to actual market behavior, one has also shown the implausibility of the utility function approach.

The theory of consumer behavior hypothesizes, just as any other scientific theory, an abstract picture of a real world phenomenon, and as such it can never be completly realistic. The purpose of proposed theories is to predict and describe in a "sufficiently" accurate and simple manner a real world phenomenon.

---

1) A lexicographic preference ordering is perhaps used by consumer units for choosing between the same commodity (or at least similar commodities) of different brands, colors, or other non-price related characteristics.

The stimulus to research and progress in science arises because one can easily disagree as to whether or not a theory meets the requirement stipulated by the word "sufficient". That is, the desire exists (or at least it should) to improve the accuracy of the predictive power which a theory might have, or to test the consistency of the underlying assumptions of the theory with empirical data.

The empirical test performed is described and the results are presented in Chapter III. The negative results of this test established that the theory as developed in Chapter II must contain assumptions which are inconsistent with each other. In Chapter IV some possible modifications of the pure theory of consumer behavior are suggested.

CHAPTER II

A PURE THEORY OF CONSUMER BEHAVIOR

Go, wondr'ous creature
Mount where Science guides.
Go, measure earth, weigh air,
    and state the tides,
Instruct the Planets in what orbs to run,
Correct old Time, and regulate the Sun;
Go, teach Eternal Wisdom how to rule - - -
Then drop into thyself, and be a fool.

A. Pope

In this chapter a theory of consumer behavior will be developed in an axiomatic manner. The theory will be defined for a consumer unit. This consumer unit can be a single person, or a group of individuals. What distinguishes consumer units is the communal decision making in a market setting; - i.e. a consumer unit acts as one economic agent regardless of the number of individuals involved. No attention will be paid to the problem of how the decision process within a consumer unit functions; this would be a completely different problem with its own flavor of interest.

Each consumer unit has a particular preference relation P defined over the commodity space Q. This preference relation P is assumed to be unchanging over the time periods considered. The dimension of the commodity space can vary from one consumer unit to the other. The scope and perception which the consumer unit has of the market setting will determine the dimension of the commodity space. Thus, the commodity space Q for a given consumer unit will be the set of all conceivable commodity vectors and is denoted as

$$Q = \left\{ x = (x_1, \text{---} , x_n) : x \geq 0 \right\}$$

where x_i , $i = 1, --- , n$, denote commodities, and n is the number of conceivable commodities. For all practical purposes only non-negative vectors have to be considered. Q is a subset of the real number space R^n.

For convenience it is assumed that the consumer unit can conceive of every possible convex combination of commodity vectors in Q, - i.e. if $x, y \in Q$ are conceivable, then so is $(1 - \mu) + \mu y \in Q$, for some scalar $0 < \mu < 1$. Of course, this is not a realistic assumption since commodities are actually conceived in their natural units of measurement. However, discarding this assumption in favor of making Q an integer space only complicates proofs and would not add any significant insights into the theory.

To make the problem an economic and interesting one, the consumer unit is assumed to act in a market setting; that is, the consumer unit conceives simultaneously for the set Q a price vector defined as

$$p = (p_1, --- , p_n), \quad p > 0.$$

Corresponding to every commodity x_i there exists a positive price p_i. The case of commodities with negative prices will not be considered.[1] The price vector p is contained in the positive subset of the real number space R^n.

In addition, the consumer unit receives a flow of income over time. For a given price vector, the flow of income can be converted

---

1) The case where a resale price has to be considered, in particular for durable goods, is also not treated here. This is a challenging subject of its own interest. Some of the problems encountered by considering durable goods are studied by L.V. Manderscheid (26).

into a flow of commodities without any external constraints. Because income is a flow over time, the flow of commodities has to be measured in a unit of a period of time. From the outset, this period of time is assumed to be some standard period and is then implicitly understood. Income is denoted by the real scalar Y such that $0 \leq Y < \infty$, given the period of time involved.

To summarize, the Primitives of the theory are:

Primitive I: The consumer unit consists of k individuals, $k \in \{1, \text{---}, m < \infty\}$, and acts as one economic agent.

Primitive II: The consumer unit possesses a preference relation P defined on the commodity space.

Primitive III: The consumer unit conceives without error of n commodities and their corresponding prices to form the commodity space

$$Q = \left\{ x = (x_1, \text{---}, x_n) : x \geq 0 \right\}$$

and the price vector

$$p = (p_1, \text{---}, p_n), \quad p > 0.$$

Primitive IV: The consumer unit can conceive of every convex combination of conceivable vectors in Q.

Primitive V: The consumer unit receives an income Y, $0 \leq Y < \infty$, in a specific period of time. This flow of income can be converted into a flow of commodities at the given prices.

These primitives alone do not allow one to predict a behavior pattern. In addition a number of Axioms have to be postulated. These are:

Axiom 1: The preference relation P, read as "preferred to", is a binary relation defined on Q, such that for every $x, y \in Q$ only one is true, either xPy or not(xPy).

Axiom 2: P is irreflexive; - i.e. for every $x \in Q$, not(xPx) is true.

Axiom 3: P is transitive; - i.e. for $x, y, z \in Q$, xPy and yPz implies xPz.

Axiom 4: P is monotone; - i.e. for $x, y \in Q$, $x \geq y$, at least one inequality in the vector holding, xPy is true. This simply means that more is preferred over less.

Axiom 5: P is continuous on Q; - i.e. for some $x \in Q$, the sets $\{y: y \in Q, yPx\}$ and $\{z: z \in Q, xPz\}$ are open and disjoint.

Axiom 6: For a given $x \in Q$, the set $\{y: y \in Q, \text{not}(xPy)\}$ is convex. If P is not a lexicographic preference ordering then the set defined is strictly convex, and for a vector y in the set, $(1-\mu)y + \mu x \; Px$ is true for a scalar $0 < \mu < 1$.

Axioms 5 and 6 are mathematically the most restrictive axioms, but they are needed only to prove Theorem 4 below.

If it is desired to obtain an order preserving utility function, then some modifications of the axioms are necessary as Debreu (6) has shown. One would have to define a new preference relation as follows: for every $x,y \in Q$, $yP^{*}x$ if and only if $\text{not}(xPy)$. The relation P^{*} is a complete quasi-ordering, and it reads "preferred or indifferent to". In particular, the indifference relation T can then be defined as: given an $x \in Q$, the indifference set to it is $\{y: y \in Q, xTy\} = \{y: y \in Q, yP^{*}x - yPx\}$, where $yP^{*}x - yPx$ means y is P^{*} but not P related to x. Another acceptable definition for the indifference set to $x \in Q$ is: $\{y: y \in Q, xTy\} = \{y: y \in Q, yP^{*}x\} (\cdot) \{y: y \in Q, xP^{*}y\}$ where $(\cdot)$ indicates the logical intersection of the two sets. For a lexicographic preference ordering the set $\{y: y \in Q, xTy\}$, for a given $x \in Q$, contains only one element, namely x itself.

However, as will be demonstrated below, it is possible to derive meaningful theorems without the assumption of an indifference set.

<u>Lemma 1</u>: For $x,y \in Q$, xPy implies $\text{not}(yPx)$.

Proof: (by contradiction) Suppose xPy and yPx are both true. By Axiom 3 this implies xPx, which contradicts Axiom 2.

Simply requesting the consumer unit to order the commodity space by the relation P does not create an economic problem. The problem becomes an economic one if Primitive V is used. This is why one more axiom is needed. But before stating it, a definition is called for.

The budget set for a given price vector p and a given income Y is defined to be $X(p,Y) = \{x: x \in Q, px \leq Y\}$, and the budget hyperplane is the set $\{x: x \in Q, px = Y\}$. Thus, the budget set is the set of all conceivable commodity vectors in Q which can be converted from the income Y at the given price vector. Since Y is a flow over a period of time, so must be $X(p,Y)$ respectively.

Lemma 2: $X(p,Y)$ is a compact (closed and bounded) subset of Q.

Proof: By definition $X(p,Y)$ is a subset of Q; it remains to show that $X(p,Y)$ is compact. By Primitive III all prices are positive, and by Primitive V income Y is bounded. In addition to the origin, the extreme points of $X(p,Y)$ are given by $x_i^* = Y/p_i$, $i = 1, \text{---} , n$. Since $X(p,Y)$ is the set of all convex combinations of these extreme points, it follows that $X(p,Y)$ is compact. (See Lancaster (22), page 267, the Krein-Milman Theorem.)

Axiom 7: The consumer unit always acts rationally, which is defined as follows: If the consumer unit makes a choice on the budget set $X(p,Y)$, it will choose a commodity vector x^o such that $x^o \in X(p,Y)$, and x^oPx for every other commodity vector $x \in X(p,Y)$. x^o is called the optimal vector for the set $X(p,Y)$. From the definition of x^o, and from Axiom 5 it follows that x^o is the only vector for which $X(p,Y)$ intersects with the set $\{y: y \in Q, \text{not}(x^oPy)\}$.

Axiom 7 needs some further interpretation. Since x^o is chosen for the same period of time for which Y is defined, this would imply that the decision on the budget set is made with the same period in mind as the period for which the flow of income is defined. This is merely a convenience, strictly speaking this need not be so.

The budget period for which purchasing decisions are defined, and the income period need not coincide. But if budget decisions over longer than one period are ignored, one can redefine the income period so it coincides with the budget period. This is done implicitly in the rest of this chapter.

These five primitives and seven axioms are sufficient to derive some meaningful theorems on the economic behavior of the consumer unit.

Theorem 1: The optimal commodity vector is unique.

Proof: (by contradiction) Suppose on the contrary that there are two optimal commodity vectors, x^o and x^{oo}. By Axiom 7 x^oPx^{oo} and $x^{oo}Px^o$ both must hold. This is in violation to Lemma 1.

Theorem 2: Let x^o be the optimal vector for the budget set $X(p,Y)$, then $px^o = Y$; - i.e. the consumer unit converts all his income into commodities.

Proof: (by contradiction) Suppose $px^o < Y$. The difference is $(Y - px^o) > 0$. Take any one of the commodities, say the i^{th}. With the income not spent, the consumer unit could purchase additional $0 < d_i = (Y - px^o)/p_i$ units. Alternatively, this can be denoted by the vector v_i whose i^{th} element equals d_i, all other elements equal to zero. Then $p(x^o + v_i) = px^o + pv_i = px^o + (Y - px^o) = Y$. Thus $(x^o + v_i) \in X(p,Y)$. Since d_i was positive, by Axiom 4 $(x^o + v_i)Px^o$, contrary to the definition of x^o being the optimal vector.

Corollary 1: The optimal vector x^o must lie on the boundary of the budget set $X(p,Y)$.

Proof: (by contradiction) Suppose x^o is not on the boundary of the set $X(p,Y)$. Then at least one component of x^o could be increased. This leads to a contradiction as outlined in the proof of Theorem 2.

<u>Corollary 2</u>: If x^o is the optimal vector for the budget set $X(p,Y)$, then for every $z \in Q$, such that $\text{not}(x^o P z)$ and $x^o \neq z$ is true, $pz > px^o = Y$; - i.e. $z \notin X(p,Y)$.

Proof: (by contradiction) Suppose $\text{not}(x^o P z)$ is true and $pz \leq px^o$. This implies $z \in X(p,Y)$. Therefore, x^o could not have been the optimal vector.

<u>Theorem 3</u>: (Homogeneity theorem) The optimal vector x^o is invariant to scalar changes in all prices and in Y. That is, if x^o is optimal for $X(p,Y)$, then it is also optimal for $X(\lambda p, \lambda Y)$, given a scalar $0 < \lambda < \infty$.

Proof: By Theorem 1 the optimal vector for a given budget set is unique. It remains to show that the budget sets $X(p,Y)$ and $X(\lambda p, \lambda Y)$ are the same subset of Q. In addition to the origin which is common to both budget sets, the extreme points for $X(p,Y)$ are all $x_i^* = Y/p_i$, for $i = 1, \text{---}, n$; and for $X(\lambda p, \lambda Y)$, $x_i^{**} = \lambda Y / \lambda p_i$. Since the λ's cancel, it follows that $x_i^* = x_i^{**}$ for all i. Hence, $X(p,Y) = X(\lambda p, \lambda Y)$ by lemma 2.

Theorem 3 needs some further comments. Implicit to it is the assumption that the preference relation P on the commodity space is independent of the price vector p and income Y. If this were not true, Theorem 3 could not hold.

<u>Theorem 4</u>: Given P is not a lexicographic preference ordering, then any bounded commodity vector x' can conceivably be optimal for a suitable price vector p' and income Y'.

Proof: p' and Y' have to be chosen such that the following conditions hold: $x'Px$ for every $x \in X(p',Y')$, $x' \in X(p',Y')$, and $p'x' = Y'$. By Axioms 5 and 6 the set $\{y: y \in Q, \text{not}(x'Px)\}$ is closed and contains the open subset $\{y: y \in Q, yPx'\}$. Axiom 7 requires that the intersection of $X(p',Y')$ and $\{y: y \in Q, yPx'\}$ is empty. Thus x' is a vector on the boundary of the convex, closed set $\{y: y \in Q, \text{not}(x'Py)\}$ (see Corollary 1). By the Minkowski Theorem (see Lancaster (22), page 264) there exists a supporting hyperplane through x' such that $p'x' = Y'$.

If the boundary of the set defined in the proof for Theorem 4 is free of any "kinks", then the price vector p' and income Y' are unique. This is the case in the traditional indifference hyperplane approach, excluding the case of some infinite prices. However, from Theorem 4 one cannot conclude the uniqueness of p' and Y' as a general case.

<u>Theorem 5</u>: Let x^o be the optimal vector for the budget set $X(p,Y)$, and x' for $X(p',Y')$; $p' \neq p$, $Y' \neq Y$, $x^o \neq x'$. If $\text{not}(x^oPx')$ and $\text{not}(x'Px^o)$ are both true, then $(p - p')(x^o - x') < 0$.

Proof: Since $\text{not}(x^oPx')$ is true, by Axiom 7 $x' \notin X(p,Y)$. Similarly $x^o \notin X(p',Y')$. By Corollary 2 this implies $Y = px^o < px'$, and $Y' = p'x' < p'x^o$; or $(px^o - px') < 0$ and $(p'x' - p'x^o) < 0$. Adding these two inequalities gives, after factoring, $(p - p')(x^o - x') < 0$.

<u>Corollary 3</u>: Using the same premises as for Theorem 5, if the price of only the j^{th} commodity differs, then for $x_j^o > 0$, $x_j' > 0$, $(p_j - p_j')(x^o - x_j') < 0$.

Proof: Same as for Theorem 5. The results follow since in $(p - p')$ only the j^{th} element differs from zero. 1/

Since $Y \neq Y'$, Theorem 5 and in particular Corollary 3 does not provide a proof for a downward sloping Marshallian demand curve. The clumsy notation of $\text{not}(x^o P x')$ and $\text{not}(x' P x^o)$ can be simplified if the existence of the indifference set, defined on page 9, is assumed. In that case $\text{not}(x^o P x')$ and $\text{not}(x' P x^o)$ implies $x^o T x'$.

<u>Theorem 6</u>: Let $Y' > Y$, and let x^o be the optimal vector for the budget set $X(p,Y)$, and x' for $X(p,Y')$, then $x' P x^o$.

Proof: Changes in income simply shift the budget hyperplane in a parallel fashion. Therefore, $X(p,Y)$ is contained in $X(p,Y')$. But this implies $x^o \in X(p,Y')$, hence $x' P x^o$ by Axiom 7 and by Corollary 1.

By now it should be obvious that the postulated axiomatic treatment of the economic behavior of consumer units leads to some powerful conclusions. There is a one-to-one correspondence between the set of all possible budget sets $X(p,Y)$, and the set of the optimal vector x^o.

---

1) Note that it is necessary to assume $x_j^o > 0$. If $x_j^o = 0$, then $p' x^o = p x^o$. Similarly, $x_j' > 0$ must hold.

Since this relation between the optimal vector and the budget set played such an important role in the theorems, it is useful to reformulate this correspondence: The choice (or demand) function is defined to be the correspondence between the set of all the budget sets, and the associated set of optimal commodity vectors. This choice function is denoted as $x = d(p,Y)$. Since the budget set is defined for a period of time, so must be the choice function. If Theorem 4 holds, then the set of optimal commodity vectors is the set of all bounded vectors in Q.

One of the more interesting questions asked in economics is the following: Suppose in a certain period p' and Y' are given and give rise to the demand vector $x' = d(p',Y')$. At the beginning of the next period, before the decision to purchase commodities is made, the price of the j^{th} commodity changes, say it increases from p'_j to p''_j, everything else retaining their previous values. [1] Let $p'' = (p'_1, \text{---}, p''_j, \text{---}, p'_n)$ denote the new price vector. What can be said about the changes (if any ?) in the composition of the demand vector from x' to $x'' = d(p'',Y')$?

Since
$$\begin{aligned} p''x' &= p'_1x'_1 + \text{---} + p''_jx'_j + \text{---} + p'_nx'_n \\ &= p'x' + (p''_j - p'_j)x'_j \\ &= Y' + (p''_j - p'_j)x'_j \end{aligned}$$
By hypothesis, the j^{th} price increased, hence $(p''_j - p'_j) > 0$. Now suppose the special case where $x'_j > 0$; - i.e. the j^{th} commodity was

---

1) The case for a decrease is analogous.

actually purchased. Then $(p''_j - p'_j)x'_j > 0$, thus $p''x' > p'x' = Y'$. This means that the original vector cannot be purchased in the new situation, or in mathematical notation $x' \notin X(p'',Y')$.

<u>Lemma 3</u>: Given the above premises, then $x' \neq x''$.

Proof: Since $x' \notin X(p'',Y')$ and $x'' = d(p'',Y')$, the conclusion follows.

By Lemma 3 we know that the consumer unit will react to the increase in p_j by buying a different commodity vector, given that $x'_j > 0$.

<u>Lemma 4</u>: $x'' \in X(p',Y')$; - i.e. the demand vector x'' is contained in the budget set of the previous period.

Proof: The budget set $X(p',Y')$ is made up of all the convex combinations of the extreme points (see Lemma 2). Since only the j^{th} extreme point will change to $x^{\#}_j = Y'/p''_j < Y'/p'_j$, for $p''_j > p'_j$. Therefore, the budget set $X(p'',Y')$ is contained in the set $X(p',Y')$, <u>a fortiori</u> so is x''.

<u>Theorem 7</u>: $x'Px''$; - i.e. the consumer unit is worse off with the rise in p_j, given $x'_j > 0$.

Proof: By definition $x' = d(p',Y')$. Using Lemma 4, $x'' \in X(p',Y')$, and by Lemma 3 $x' \neq x''$. Therefore, by Axiom 7 $x'Px''$.

To explain the reaction to the price increase in p_j in more detail one can reason as follows: Since $x' \notin X(p'',Y')$, at least some of the change in the composition of the demand vector from x' to x'' can be explained by the fact that the budget set decreased (see Lemma 4). The rest of the change is assumed to result because x_j is now relatively more expensive. Of course, it is desirable to determine at least the direction of change attributable to each of these causes.

To answer this, a little trick has to be used. The difficulty arises because there is no direct measure for the change in the budget set, and the trick is to parameterize changes in the budget set by the variable Y. Using Y as a parameter simply shifts the budget hyperplane in a parallel fashion.

Let the hypothetical income level $Y*$ be defined as follows: $Y* = p''x' = Y' + (p''_j - p'_j)x'_j$; for $x'_j > 0$, $Y* > Y'$. Now the hypothetical budget set is formed as $X(p'',Y*)$. Since $x' \in X(p'',Y*)$, it would be possible to buy the original demand vector in this hypothetical situation. Also, $Y* > Y'$ implies that $X(p'',Y')$ is contained in $X(p'',Y*)$. Let the hypothetical situation give rise to a hypothetical demand vector $x* = d(p'',Y*)$.

Lemma 5: $x'' \in X(p'',Y*)$; - i.e. the new demand vector is contained in the hypothetical budget set.

Proof: Since $p''x'' = Y' < Y*$, the claim follows.

Lemma 6: $x*Px''$; - i.e. the hypothetical demand vector $x*$ is preferred over the new vector x''.

Proof: By Theorem 6 directly, since Y was increased.

<u>Lemma 7</u>: If $x^* \neq x'$, then x^*Px' ; - i.e. if x^* and x' are not identical, then the hypothetical demand vector is preferred over the original.

Proof: Since $p''x' = Y^*$, $x' \in X(p'',Y^*)$. By hypothesis $x^* \neq x'$, thus x^*Px' by Axiom 7. One has to assume $x^* \neq x'$ since Theorem 4 does not guarantee the uniqueness of the price vector and income level involved.

Now each component of the demand vector will be considered separately. The fundamental equation (or identity) is defined as follows: for $i,j = 1, --- , n$;

$$\frac{x_i'' - x_i'}{p_j'' - p_j'} = \frac{x_i'' - x_i^*}{p_j'' - p_j'} + \frac{x_i^* - x_i'}{p_j'' - p_j'}$$

Since $(p_j'' - p_j') > 0$, the sign of $(x_i'' - x_i')$ will indicate the direction of change in x_i due to the new situation created by the price increase in p_j. To simplify notation, let $(p_j'' - p_j') = \Delta p_j$, and $(x_i'' - x_i') = \Delta x_i$. Then the fundamental equation reads as:

$$\frac{\Delta x_i}{\Delta p_j} = \frac{(x_i'' - x_i^*)}{\Delta p_j} + \frac{(x_i^* - x_i')}{\Delta p_j}$$

Because $x' = d(p',Y')$, $x'' = d(p'',Y')$, and $x^* = d(p'',Y^*)$, the change from x^* to x'' is a pure effect of the parametric change in the budget set, measured by the decrease of Y^* to Y'. Hence, the first term on the right hand side of the fundamental equation indicates that part of Δx_i which can be attributed to the change in the budget set. The second term must therefore be attributed to the fact that the relative price relation changed between the commodities.

Suppose $x'_j > 0$, by definition $p''x' = Y* = Y' + \Delta p_j x'_j$, or $-(Y' - Y*) = \Delta p_j x'_j$. Solving for Δp_j gives:

$$\Delta p_j = \Delta Y / x'_j \qquad \text{for} \quad \Delta Y = -(Y' - Y*).$$

(Note that the sign of ΔY is determined by the sign of Δp_j.) Thus, a measure for the parametric change in Y is available, and substituting in the first term on the right hand side of the fundamental equation gives: for $i,j = 1, --- , n$;

$$\frac{\Delta x_i}{\Delta p_j} = \frac{x'_j(x''_i - x^*_i)}{\Delta Y} + \frac{(x^*_i - x'_i)}{\Delta p_j}$$

<u>Theorem 8</u>: If $x* \neq x'$, and if $\Delta p_j > 0$, then $(x^*_j - x'_j) < 0$, given $x'_j > 0$.

Proof: From above $p''x' - p'x' = \Delta p_j x'_j$. By construction, $Y* = p''x' = p''x*$, and by Lemma 7 $x*Px'$. This in turn implies by Corollary 2 $x* \notin X(p',Y')$, or $p'x* > p'x' = Y'$. Multiplying the last inequality by -1 to get $-p'x* < -p'x'$ and adding $Y*$ to both sides gives, after factoring, $(p'' - p')x* < (p'' - p')x'$. Since only the j^{th} element of $(p'' - p')$ differs from zero, this gives $(x^*_j - x'_j)\Delta p_j < 0$. Dividing by $\Delta p_j > 0$ does not change the sign of the inequality and proves the claim.

In conventional literature the fundamental equation is known in a different form as the Slutsky equation. 1/ If Δx_i tends to zero faster

---

1) See Allen (1) and Samuelson (31) on the Slutsky equation, and Yokoyama (38) on the fundamental equation.

than Δp_j does, so that $\Delta x_i / \Delta p_j$ has a finite limit, not always equal to zero, then, as Δp_j tends to zero, the above defined fundamental equation will become the Slutsky equation. The term $x'_j(x''_i - x^*_i)/\Delta Y$ translated into the Slutsky equation is always referred to as the "income effect"; however, in this thesis it will be called the budget set effect. To measure the change in the budget set, use was made of the (already existing) variable Y. There is no essential need to use Y; other parameters could be used instead. 1/ The second term $(x^*_i - x'_i)/\Delta p_j$ is called the substitution effect. It is the change due to the fact that x_j becomes relatively more expensive as p_j increases.

Now a more definite answer can be given to the question of the direction of change in $\Delta x_i / \Delta p_j$. In particular, for $\Delta p_j > 0$ and $x'_j > 0$ initially, is it possible to ascertain $\Delta x_j / \Delta p_j < 0$? By Theorem 8 it follows for $x^* \neq x'$ (perhaps the most common case) that $(x^*_j - x'_j) < 0$. The theory of this chapter (and for that matter all existing theories) cannot specify the sign of $(x''_j - x^*_j)$, and therefore of $x'_j(x''_j - x^*_j)/\Delta Y$. This sign has to be assumed or specified from an outside source. A similar reasoning holds for all other terms of the $\Delta x_i / \Delta p_j$.

In discussing the fundamental equation it was necessary to assume $x'_j > 0$. 2/ Clearly, in reality this is not the case for most

---

1) The condition for the parameter is that its variation should include x' in the hypothetical budget set, and at least two commodities, including x_j must vary.

2) Remember that the commodity space consists of all conceivable commodities, not commodities actually purchased.

commodities. Many elements of the demand vector will be equal to zero. Thus, suppose $x'_j = 0$, what will be the implications ? The fundamental equation reduces to $\Delta x_i / \Delta p_j = (x^*_i - x'_i) / \Delta p_j$. The term expressing the change in the budget set drops out. That this is correct can also be shown as follows: First suppose an increase in p_j . If $x'_j = 0$, then $p''x' = p'x' = Y'$, or $x' \in X(p'',Y')$. Since $X(p'',Y')$ is a subset of $X(p',Y')$ this implies $x'' = x'$, and $x'Px$ for every $x \in X(p',Y')$, __a fortiori__ $X(p'',Y')$ as well.

This is a strong, and obvious result; it implies $\Delta x_i = 0$ for $i = 1, --- , n$. The increase in p_j will have no effect on the demand vector actually purchased. 1/ Also, since $p''x' = Y^* = Y'$, and therefore $X(p'',Y') = X(p'',Y^*)$ it follows that $x^* = x'$. Theorem 8 can be modified for the case $x'_j = 0$ as follows:

__Theorem 8':__ If $\Delta p_j > 0$, and $x'_j = 0$, then (i) $(x^*_i - x'_i) = 0$ for every $i = 1, --- , n$; and (ii) $x' = x''$; $x'' = x^*$.

Proof: See arguments provided above.

Secondly, what if $\Delta p_j < 0$, and $x'_j = 0$; - i.e. the price of the commodity x_j decreases ? Again, since $p''x' = Y'$ we have $x' \in X(p'',Y') = X(p'',Y^*)$. Because $p''_j < p'_j$ and by Lemma 2, $X(p',Y')$ is a subset of $X(p'',Y')$. Unfortunately, nothing more definite can be said than simply $\Delta x_i / \Delta p_j \lesseqgtr 0$. In particular, it is impossible to

---

1) The initial condition was $x'_j = 0$ by assumption.

say by how much p_j must decrease before a change in the demand vector will be observed, unless additional knowledge is available from an outside source.

As indicated in Chapter I, a set of theorems is only meaningful if the theorems can be tested empirically for their validity. A necessary test condition for the validity of the above theorems is provided by the revealed preference theorems. 1/ Revealed preference theory utilizes actual market observations of at least two demand periods to test whether the observed demand vector is in violation to the theoretical restrictions imposed on it. This is the most direct and simplest way of testing the consistency of the underlying axioms and primitives, since it is practically impossible to compute the actual budget set for each observed demand vector.

To start with the simplest case: Suppose two observations on demand vectors are available, $x' = d(p',Y')$ and $x'' = d(p'',Y'')$. By computing $p''x'$ and $p'x''$ one can determine which observed demand vector is contained in the other budget set. For example, suppose $p'x'' \leq Y' = p'x'$. This implies $x'' \in X(p',Y')$.

<u>Theorem 9</u>: If $x'' \neq x'$ and $x'' \in X(p',Y')$, then $x'Px''$.

Proof: Follows immediately by Axiom 7.

---

1) Revealed preference theory is by no means the only test that can be performed on the theory of consumer behavior.

The demand vector x' is, under these circumstances, defined to be revealed preferred to x''. This can be denoted by the relation R, - i.e. $x'Rx''$ if the following conditions hold: (i) $x' = d(p',Y')$, $x'' = d(p'',Y'')$, and (ii) $p'x'' \leq p'x'$. From Theorem 9 it follows that $x'Rx''$ implies $x'Px''$ if $x' \neq x''$ is known.

<u>Theorem 10</u>: If the consumer units acts in accord with the proposed theory, and if $x'Rx''$, then $\text{not}(x''Rx')$ is true for $x'' \neq x'$.

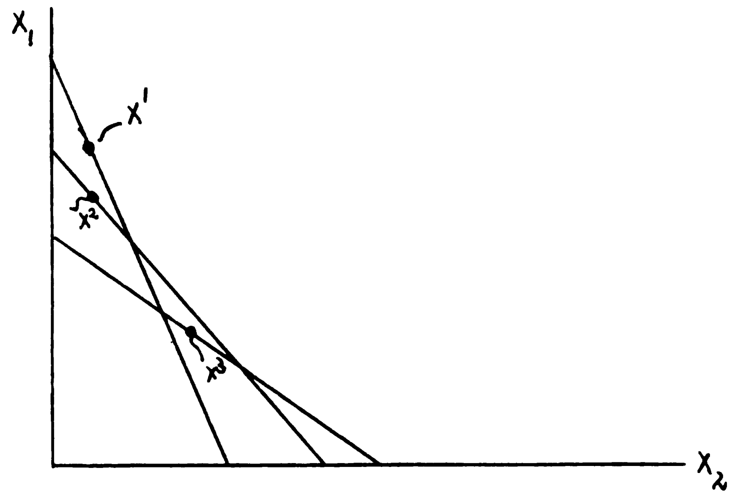
Proof: (by contradiction) Suppose $x'Rx''$ and $x''Rx'$ were both true. By definition of the relation R and by Theorem 9 this implies $x'Px''$ and $x''Px'$; this violates Lemma 1.

In words, Theorem 10 means if in situation "one" the consumer unit could have purchased x'' but did not in favor of x', then in situation "two" the vector x' may not be contained in the budget set of situation "two". This analysis can be extended to several observations.

<u>Theorem 11</u>: If all the k relations x^1Rx^2; x^2Rx^3; --- , and x^kRx^{k+1} hold, the x^i being observed, non-identical demand vectors, and if the consumer unit behaves in accord with the outlined theory, then $p^{k+1}x^1 > p^{k+1}x^{k+1}$.

Proof: (by contradiction) Suppose $p^{k+1}x^1 \leq p^{k+1}x^{k+1}$, so that $x' \in X(p^{k+1},Y^{k+1})$. Since x^iRx^j , $x^i \neq x^j$, implies x^iPx^j, using Axiom 3 this implies x^1Px^{k+1}. But $p^{k+1}x^1 \leq Y^{k+1}$ implies $x^{k+1}Px^1$. Thus, a contradiction to Lemma 1 is established.

As indicated in the proof of Theorem 11, x^1Rx^2; --- ; x^kRx^{k+1} does imply x^1Px^{k+1} for non-identical demand vectors; however, it does not imply x^1Rx^{k+1}. A little counter example will show why the relation R is not transitive: Consider the two-commodity case, and denote the situations by x^1, x^2, and x^3.



As illustrated, x^1Rx^2, x^2Rx^3, but $p^1x^3 > Y^1$ (and $p^3x^1 > Y^3$); thus, it is not true that x^1Rx^3.

A final remark is needed. Suppose a test along the lines of Theorems 10 and 11 were conducted, and the consumer unit did not violate these theorems. This would not prove that the postulated theory of consumer behavior is correct. Satisfying Theorems 10 and 11 is a necessary, but it is not a sufficient condition. One observation that violates the test criteria of Theorems 10 and 11 provides evidence that the specific consumer unit does not behave in accord with the postulated theory of this chapter.

Conclusions:

Chapter II contains a testable theory of consumer behavior which was developed from some basic primitives and axioms. The same primitives and axioms are necessary for the traditional utility function and indifference curve approach; but meaningful theorems can be derived with less stringent assumptions than required by the traditional approach.

CHAPTER III

THE REVEALED PREFERENCE TEST PERFORMED

To test the revealed preference theorems of Chapter II only data provided on food purchases were available. This seems to make the test questionable in its validity, since the pure theory of Chapter II was specified for all conceivable commodities. Indeed, special assumptions have to be introduced in order to make the test performed a valid one.

The first assumption is the demand independence of food from all non-food commodities. The necessary condition for demand independence of commodities, say x_i and x_j, $i \neq j$, is defined by a commodity grouping according to the requirement that the substitution effect in the fundamental equation is identical to zero for either changes in p_i or p_j. This is a characteristic of the preference relation P, and is commonly defined as want-independence. 1/ Thus, for want-independent commodities the fundamental equation reads as follows:

$$\Delta x_i / \Delta p_j = x'_j (x''_i - x^*_i) / \Delta Y \qquad \text{for } i \neq j$$

It is assumed that this condition of want-independence holds between all food and non-food commodities treated as groups; - i.e. the price change of any food commodity has no substitution effect on any non-food commodity and vice versa.

---

1) Strotz (34),(35), Frisch (8), Houthakker (16), Gorman (11), and Clarkson (5) provide good references on this point.

Since even for want-independent commodities the budget set effect $x'_j(x''_i - x^*_i)/\Delta Y$ of the fundamental equation will not be identical to zero, a sufficient condition for demand independence has to be specified. This condition is provided by the following assumption: Before decisions on the purchase of a demand vector are made, the (anticipated) income flow for the coming demand period is divided into a food and non-food budget. 1/ Any price change of non-food commodities during the period will have no effect on the (prior) food budget allocation, and vice versa for changes in food prices. 2/

Thus, given this assumption of demand independence, the demand vector for a specific period can be partitioned into:

$$\begin{pmatrix} x_F \\ x_{NF} \end{pmatrix} = \begin{pmatrix} d(p_F, Y_F) \\ d(p_{NF}, Y_{NF}) \end{pmatrix}$$

where F indicates food, NF the non-food subvectors; and Y_F the food budget and Y_{NF} the non-food budget are determined at the beginning of the demand period. For a specific period one can simply focus attention on the food demand subvector without regard to the omitted variables in the general demand vector.

---

1) There might exist a further subgrouping among the non-food commodities.

2) Of course, if prices do change then this will have an effect on the budget allocation for the next demand period. This also means that demand independence can be defined only within a single demand period, but not for several periods.

The second assumption needed to make the test procedure valid deals with the time period for which a food demand vector is defined. The data used were recorded per week; this prevents making the demand period a fraction of weeks. On the other hand, if all the information provided in the 52 weeks of a year is to be used, a factoring of 52 will show that the demand period assumed has to be either one, two, four, thirteen, or twenty-six weeks long. The choice was made to partition the year into 13 four-week periods. This assumption will certainly not hold for many food purchases, especially if the consumer unit follows the practice of canning and deep-freezing food items. In that case the demand period would be a year or so, rather than four-weeks long.

There is no doubt that the assumption made on the length of the food demand period is fairly arbitrary and is vulnerable to criticism.

The apparent contradiction of combining four succesive weeks, each having its own action of purchases, into one demand period where all the separate actions of purchases are treated as one decision, must be resolved by assuming that the decisions for all purchases are made at the beginning of the demand period. This, then, separates the act of deciding to buy a demand vector from the several acts of actually purchasing the commodities. It implicitly supposes that the consumer unit must consider without error the (anticipated) food prices for the coming demand period in the decision process.

Theorem 2 of Chapter II permits one to treat the total expenditures of a demand period as the particular food budget. Also, no specific information on the budget is needed to perform revealed preference tests.

The weekly reported food purchases were not added up into one 129-dimensional vector. Rather, the observed demand vector should be considered as consisting of four 129-dimensional subvectors, each of them corresponding to one of the four weeks composing a demand period. Besides being computationally simpler to handle, this approach also avoids the computation of "average" prices for a demand period and permits the use of correct prices for each week instead.

For a specific consumer unit, let q^i denote the 516-dimensional vector containing the necessary information of a demand vector for period $i = 1, --- , 13$ in a given year, and $x^{i,j}$, $j = 1, --- , 4$, the 129-dimensional subvector of q^i for the j^{th} week of period i. The 129 lelements of the subvectors are coded according to the aggregation scheme for food commodities.

Analogously, the price vector for the i^{th} period, $i = 1, --- , 13$ is denoted by $r^i = (p^{i,1}, p^{i,2}, p^{i,3}, p^{i,4})$, where the $p^{i,j}$ are reported 129-dimensional vectors of prices paid during week j of period i.

The computation of $q^i r^i = x^{i,1}p^{i,1} + x^{i,2}p^{i,2} + x^{i,3}p^{i,3} + x^{i,4}p^{i,4}$ is straight forward, it is the sum of the inner products for each weekly vector. In computing $q^i r^j$, $i \neq j$, of a specific consumer unit in one of the four years, an additional rule for computing the inner product was used: Consider the term $x^{i,2}p^{j,2}$ for example, and suppose the quite possible case that the k^{th} element of $x^{i,2}$ is positive; - i.e. the consumer unit purchased the k^{th} commodity in the second week of period i, in contrast, the k^{th} element of $p^{j,2}$ equals zero because commodity k was not purchased during this week. If this happened, then first $p^{j,1}$ was searched for a price information for commodity k, then $p^{j,3}$, and if still no information was found, then $p^{j,4}$. We believe that this search procedure did provide a price information in virtually all cases, since none of the 129 commodities appeared to be of seasonal nature. 1/ This modification of the computation of the inner products for $q^i r^j$, $i \neq j$, is implicitly assumed in the rest of the chapter.

Using the available data of q^i and r^i, $i = 1, ---, 13$, a price-quantity matrix

$$PQ_{k,t} = \begin{pmatrix} q^1 r^1 & q^2 r^1 & --- & q^{13} r^1 \\ q^1 r^2 & q^2 r^2 & --- & q^{13} r^2 \\ & & & \\ q^1 r^{13} & q^2 r^{13} & --- & q^{13} r^{13} \end{pmatrix}$$

was computed, where k denotes the consumer unit and t the year.

A. Y. C. Koo (19) applied an ingeneous way of representing preference relations over a finite number of commodity vectors by Boolean matrices. Any binary relation between two objects can be represented by either 1, if the relation is true, or by 0, if the relation is not true. The corresponding Boolean matrix B reflects this information of a binary relation over a finite number of objects in the following manner: the typical element of B,

$$b_{ij} = \begin{cases} 1 & \text{if the relation is true between object } i \text{ and object } j. \\ 0 & \text{if the relation is not true.} \end{cases}$$

---

1) If these searches were in vain, then the price was set to zero. We should have, but did not count how often the search was in vain.

Thus, the columns and rows of B are indexed in the same order as the objects involved; which also implies that the order of B is the same as the number of objects.

For example, if m commodity vectors are the objects involved, and the preference relation P is applied, then the resulting Boolean matrix B will have the following properties:

(i) B is of order m.

(ii) The diagonal elements of B are equal to zero by Axiom 2.

(iii) If $b_{ij} = 1$, then $b_{ji} = 0$ by Lemma 1.

(iv) $B^m = 0$; - i.e. B raised to the m^{th} power is a zero matrix. By Axiom 3 and by Lemma 1 the diagonal elements of B^i, $i = 1, --- , m$, are always equal to zero. Applying Theorems 1 and 2 of Appendix II, B^m must be a zero matrix. 1/

By applying the revealed preference relation R on the price - quantity matrix $PQ_{k,t}$ a Boolean matrix $V_{k,t}$ of order 13x13 was generated, where the typical elements of $V_{k,t}$ are:

$$v^{ij} = \begin{cases} 1 & \text{if } q^i r^i \geq q^j r^i , \; i \neq j, \\ & \quad i,j = 1, --- , 13 \\ 0 & \text{otherwise.} \end{cases}$$

The Boolean matrix $V_{k,t}$ is analogous to the concept of the adjacency matrix of graph theory. Indeed, matrix operations from the theory of directed graphs will be applied to $V_{k,t}$.

---

1) All necessary concepts of graph theory are provided in Appendix II.

Theorem 12: If the consumer unit whose revealed preference structure is represented by the matrix $V_{k,t}$ acts in accord with the proposed theory, and if $v^{ij} = 1$, then $v^{ji} = 0$.

Proof: Since $v^{ij} = 1$ implies $q^i R q^j$, then, by Theorem 10, not$(q^j R q^i)$ must be true, that is $v^{ji} = 0$.

The preference relation P can be used to extend the revealed preference matrix $V_{k,t}$ to an indirectly revealed preference matrix $S_{k,t}$, analogous to the reachable matrix in graph theory. The computation of $S_{k,t}$ uses Theorem 3b of Appendix II, that is,

$$S_{k,t} = \left[(I + V_{k,t})^{13} - I\right] \#$$

where I is a 13x13 identity matrix. The transitive property of P enables this approach.

Theorem 13: If the consumer unit with an indirectly revealed preference matrix $S_{k,t}$ behaves in accord with the proposed consumer theory, then the diagonal elements of $S_{k,t}$ are equal to zero.

Proof: (by contradiction) Suppose s^{ii}, the i^{th} diagonal element of $S_{k,t}$ is equal to one. By Theorem 1 of Appendix II and by Theorem 11 there exists a sequence of non-identical commodity vectors such that $q^i P q^w$, --- , $q^n P q^i$ is true; hence, by Axiom 3, $q^i P q^i$ results, which violates Axiom 2.

The empirical test performed was essentially to compute the $V_{k,t}$ matrices for all consumer units participating in the panel for the year t, and then to search whether Theorems 12 or 13 were violated. Any violation is indicated by the existence of cycles in $V_{k,t}$. [1] In addition to testing whether every matrix $V_{k,t}$ contained cycles, we also tried to measure the extent of violations to the revealed preference theorems. In particular, the following procedure was applied to each matrix $V_{k,t}$: The matrix was raised to an integer power until some diagonal elements were positive (if any), thus indicating a violation to Theorem 12 or 13. The commodity vector revealing the most inconsistencies [2,3] (i.e. having the largest diagonal element) was then outpermuted by row and column interchanges to the bottom right of the matrix and ignored from then on.

The same procedure was repeated to the decremented matrix until a submatrix of $V_{k,t}$ was established, whose first n columns (and rows) representing n commodity vectors were free of any inconsistency (i.e. were acyclic) within each other; and whose bottom right 13 - n columns and rows represent the outpermuted commodity vectors. For each outpermuted commodity vector the number of violations to Theorem 12 or 13 were counted and added up to a total number of inconsistencies shown by the consumer unit.

Since the first outpermuted commodity vector could not show more than 12 inconsistencies, the second vector not more than 11 etc.,

---

1) For a definition of a cycle see Appendix II.

2) In case of a tie the vector with the lower column index was taken.

3) Inconsistency means here a violation to Theorem 11 (or 10 equivalently) or to Theorem 13 (or 12)

the maximum number of inconsistencies a consumer unit could have is $78 = (12 + 11 + \text{---} + 1)$. The minimum is 0. This provides a scale of measurement to be used in comparing consumer units in their extent of violations to Theorems 12 and 13.

Another useful measure is the power to which $V_{k,t}$, or a submatrix of it, has to be raised in order to detect an inconsistency. By Theorem 1 of Appendix II the power indicates how many commodity vectors are involved in the sequence $q^i R q^j$; --- ; $q^d R q^f$ that leads to inconsistencies. The tables below contain the results of the test performed.

These results are striking, the majority of consumer units violated the revealed preference theorems rather extensively. Those few consumer units who showed no violations, - i.e. for whom all 13 commodity vectors compose an acyclic matrix $V_{k,t}$, provided in most cases no information at all; their matrix $V_{k,t}$ consisted mostly of zero's.

Remembering that the number of violations to the revealed preference theorems is measured on the scale between 0 and 78, Table 2 underscores the extensive nature of these violations.

Table 3 contains the frequencies, over all consumer units, of the commodity vectors q^i, $i = 1, \text{---}, 13$, being outpermuted in the process of obtaining an acyclic submatrix of $V_{k,t}$.

Table 1:

Grouping of consumer units according to the size of the non-inconsistent (acyclic) submatrix.

consumer units

number of vectors in the acyclic submatrix	1958 number	1958 % of total	1958 % cumulative	1957 number	1957 % of total	1957 % cumulative	1956 number	1956 % of total	1956 % cumulative	1955 number	1955 % of total	1955 % cumulative
13	9	4.3	4.3	4	2.5	2.5	6	4.3	4.3	5	3.6	3.6
12	4	1.9	6.2	3	1.9	4.4	6	4.3	8.6	3	2.1	5.7
11	6	2.8	9.0	6	3.8	8.2	4	2.8	11.4	8	5.7	11.4
10	8	3.8	12.8	3	1.9	10.1	4	2.8	14.2	7	5.0	16.4
9	13	6.2	19.0	7	4.4	14.5	9	6.4	20.6	10	7.1	23.5
8	15	7.1	26.1	8	5.1	19.6	6	4.3	24.9	17	12.1	35.6
7	14	6.6	32.7	11	7.0	26.6	8	5.7	30.6	14	10.0	45.6
6	17	8.0	40.7	14	8.9	35.5	18	12.9	43.5	12	8.6	54.2
5	26	12.3	53.0	18	11.4	46.9	20	14.3	57.8	20	14.3	68.5
4	25	11.8	64.8	31	19.6	66.5	27	19.3	77.1	17	12.1	80.6
3	38	18.0	82.8	32	20.2	86.7	20	14.3	91.4	13	9.3	89.9
2	31	14.7	97.5	20	12.6	99.3	10	7.1	98.5	13	9.3	99.2
1	5	2.4		1	0.6		2	1.4		1	0.7	

total 211 total 158 total 140 total 140

Table 2:

Grouping of consumer units according to number of violations to the revealed preference theorems.

total number of inconsistencies per $V_{k,t}$ matrix	number of consumer units in 1958	1957	1956	1955
0 - 9	26	14	17	20
10 - 19	26	19	14	28
20 - 29	18	10	18	18
30 - 39	25	21	18	20
40 - 49	28	26	24	15
50 - 59	32	29	24	16
60 - 69	31	25	19	13
70 - 78	25	14	6	10

Table 3:

Frequencies of the commodity vectors outpermuted.

year	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}
1958	146	134	116	112	97	103	117	99	98	87	126	138	151
1957	109	99	106	90	72	69	125	84	73	80	91	111	110
1956	95	97	79	77	64	61	65	65	52	57	73	100	94
1955	105	83	73	60	58	58	72	50	48	64	69	86	93

A chi-aquare test is appropriate to test the hypothesis that the probability is the same (i.e. 1/13) for each commodity vector to be outpermuted. Thus, for each year the statistics

$$\chi^2 = \sum (f_i - nw)^2 / nw$$

was calculated, where f_i = actual frequency of q^i

$w = 1/13$

$n = 1525$ in 1958 (the total number of outpermuted vectors)

$= 1219$ in 1957

$= 979$ in 1956

$= 919$ in 1955

The hypothesis had to be rejected at the 1 % level, thus, perhaps indicating a seasonal influence on the preference structures.

Indeed, partitioning a year into three 4-period "seasons" 1/ increased the number of consumer units who showed no violations to the revealed preference theorems. This is illustrated in Table 4.

Table 4:

Improvement of test results by forming seasons.

	number of consumer units with no violations in			
	1958	1957	1956	1955
tested with 13 periods	9	4	6	5
tested with "seasons"	20	13	19	17

1) The commodity vector q^{13} was ignored in forming these "seasons". This had to be done since 13 is a prime number.

The improvement in performance by forming "seasons" is at most modest; in particular, one could hardly argue that a "seasonal" difference of preference pattern provides an explanation for the negative results obtained at first.

Another surprising result was that in most cases Theorem 12 (or 10) would have been sufficient to detect inconsistencies. This is evident from Table 5.

Table 5:

Grouping of the total number of inconsistencies according to violations to Theorem 12 (or 10) and to Theorem 13 (or 11).

number of commodity vectors in the sequence leading to an inconsistency	number of outpermuted commodity vectors for all consumer units in 1958	1957	1956	1955
2 (i.e. Theorem 12)	1522	1217	978	917
3 (i.e. Theorem 13)	3	2	1	2
more than 3 (i.e. also Theorem 13)	none	none	none	none

The quest to find explanations for the widespread inconsistent behavior suggested focusing on those 64 consumer units who participated all four years in the panel. Table 6 contains the most important test results of these 64 consumer units.

Table 6:

Test results for the 64 consumer units participating all four years in the panel.

consumer unit code;	year	number of inconsistencies	size of the acyclic submatrix	consumer unit code;	year	number of inconsistencies	size of the acyclic submatrix
	55	25	7		55	21	9
62	56	19	6	161	56	29	6
	57	49	4		57	30	7
	58	41	6		58	6	11
	55	53	3		55	21	8
163	56	65	3	172	56	31	5
	57	41	5		57	30	4
	58	12	9		58	32	5
	55	0	13		55	12	11
188	56	0	13	252	56	0	13
	57	4	10		57	16	8
	58	0	13		58	18	6
	55	58	4		55	52	5
258	56	45	5	280	56	44	6
	57	62	2		57	60	3
	58	54	4		58	48	5
	55	17	7		55	0	13
308	56	25	7	362	56	0	13
	57	24	7		57	0	13
	58	43	5		58	2	11
	55	3	11		55	41	4
397	56	13	9	453	56	50	4
	57	4	11		57	49	4
	58	9	10		58	45	5
	55	52	4		55	16	7
545	56	58	3	554	56	24	7
	57	48	5		57	37	5
	58	50	4		58	29	7
	55	67	2		55	31	9
586	56	63	4	600	56	16	9
	57	62	3		57	49	3
	58	66	3		58	43	5

Table 6: continued

consumer unit code;	year	number of inconsistencies	size of the acyclic submatrix	consumer unit code;	year	number of inconsistencies	size of the acyclic submatrix
616	55	0	13	634	55	42	5
	56	0	13		56	46	5
	57	0	13		57	49	4
	58	0	13		58	68	2
763	55	73	2	851	55	53	4
	56	67	2		56	42	4
	57	75	2		57	66	3
	58	78	2		58	66	3
956	55	50	4	1011	55	7	10
	56	64	3		56	2	12
	57	59	3		57	16	8
	58	55	4		58	10	10
1192	55	3	10	1257	55	35	5
	56	14	9		56	18	10
	57	31	7		57	34	4
	58	11	8		58	39	5
1259	55	37	4	1289	55	44	4
	56	42	4		56	59	4
	57	37	5		57	66	3
	58	39	6		58	53	4
1383	55	40	6	1448	55	40	6
	56	31	6		56	35	5
	57	29	7		57	54	4
	58	13	9		58	35	8
1460	55	62	3	1480	55	51	5
	56	57	3		56	56	3
	57	68	3		57	33	6
	58	73	2		58	23	8
1485	55	6	10	1536	55	60	3
	56	1	12		56	54	3
	57	21	8		57	65	3
	58	11	10		58	68	3

Table 6: continued

consumer unit code;	year	number of in-consistencies	size of the acyclic sub-matrix	consumer unit code;	year	number of in-consistencies	size of the acyclic sub-matrix
1566	55	41	5	1650	55	18	7
	56	62	3		56	5	10
	57	56	4		57	11	8
	58	64	3		58	31	7
1653	55	31	7	1656	55	34	5
	56	23	6		56	46	5
	57	18	9		57	53	4
	58	8	9		58	43	5
1673	55	37	6	1693	55	30	6
	56	58	5		56	45	5
	57	43	4		57	50	4
	58	38	4		58	56	3
1726	55	34	6	1736	55	74	2
	56	27	7		56	74	2
	57	47	3		57	74	2
	58	40	6		58	71	2
1896	55	33	5	1900	55	13	9
	56	26	6		56	7	10
	57	59	3		57	31	6
	58	50	3		58	42	5
1951	55	23	8	2000	55	17	8
	56	11	8		56	30	9
	57	17	9		57	39	6
	58	43	5		58	10	9
2156	55	14	8	2218	55	59	3
	56	35	6		56	42	5
	57	45	5		57	47	4
	58	36	5		58	58	3
2222	55	25	6	2226	55	23	7
	56	27	9		56	32	7
	57	26	6		57	55	4
	58	13	8		58	31	6

Table 6: continued

consumer unit code;	year	number of in-consistencies	size of the acyclic sub-matrix	consumer unit code;	year	number of in-consistencies	size of the acyclic sub-matrix
	55	45	5		55	9	9
2240	56	43	4	2255	56	1	12
	57	63	2		57	4	11
	58	51	4		58	11	10
	55	29	8		55	2	11
3122	56	21	7	3125	56	0	13
	57	30	6		57	3	11
	58	20	8		58	1	12
	55	31	6		55	10	10
3128	56	51	5	3139	56	18	7
	57	25	7		57	34	7
	58	2	12		58	23	9
	55	39	5		55	57	3
3140	56	54	4	3146	56	68	3
	57	57	4		57	63	3
	58	51	3		58	63	2
	55	40	4		55	4	11
4004	56	58	5	4019	56	60	3
	57	54	4		57	41	5
	58	40	6		58	38	4
	55	20	8		55	30	6
4020	56	23	7	5207	56	51	3
	57	51	3		57	62	3
	58	27	6		58	55	4
	55	42	4		55	55	4
5396	56	49	5	5513	56	33	6
	57	48	4		57	67	3
	58	27	7		58	27	5
	55	64	3		55	13	8
5624	56	67	3	6014	56	0	13
	57	68	2		57	7	11
	58	52	5		58	0	13

A striking feature of Table 6 is the relative constancy of the test results for each consumer unit over the four years. To find out whether, over the years, the consumer units also revealed a similar preference structure within the acyclic submatrix, a computer program was written that permuted the acyclic submatrix into an upper-triangular form. The order of the index for the columns and rows according to commodity vectors q^i indicates the order of preference between the commodity vectors (given some elements of the indirectly revealed preference submatrix are equal to one).

The results were disappointing; the acyclic preference pattern for all consumer units differed radically over the years, even for consumer units with relatively few violations to the revealed preference theorems. This can be explained by the fact that matrices $V_{k,t}$ revealing few inconsistencies were mostly composed of zero's; - i.e. they contained many non-comparable commodity vectors. On the other hand, matrices $V_{k,t}$ with many inconsistencies must lead to non-representative acyclic submatrices; - i.e. nothing conclusive can be said about the preference structure between the periods of a year.

Table 7 is similar to Table 3. It contains the frequencies of outpermuted commodity vectors for the 64 consumer units. For these 64 consumer units it was desired to test the hypothesis that the grouping scheme of 13 periods and the four years is independent.

Table 7:

Frequencies of outpermuted commodity vectors for the 64 consumer units participating all four years.

year	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	row total
1958	41	38	36	27	31	24	36	33	25	20	34	39	45	429
1957	47	39	45	33	24	27	52	33	30	31	36	46	45	488
1956	40	40	33	37	23	27	33	27	21	26	29	39	46	421
1955	44	35	36	20	27	29	33	22	22	30	29	39	47	413
column total	172	152	150	117	105	107	154	115	98	107	128	163	183	1751

The test statistics of this contingency test has a chi-square distribution with 36 d.f.

$$\chi^2 = \sum \left[n_i - E(n_i)\right]^2 / E(n_i)$$

where n_i = actual cell frequency in Table 7,

n = 48 (number of cells),

$E(n_i)$ = expected cell frequency.

The hypothesis of independent grouping cannot be rejected at the 1 % level, indicating that the frequencies of outpermuted commodity vectors did not differ significantly among the four years.

Available qualitative data on the 64 consumer units were used in the following regression equation to test their relevance in explaining the number of inconsistencies. The hypothesis tested is that all coefficients equal zero.

Let $y_{i,t}$ = number of inconsistencies for consumer unit i in year t; $i = 1, \text{---}, 64$; $t = 1955, \text{--}, 1958$
$0 \leq y_{i,t} \leq 78$

$$x^1_{i,t} = \begin{cases} 1 & \text{if consumer unit } i \text{ had children in year } t \\ 0 & \text{otherwise} \end{cases}$$

$$x^2_{i,t} = \begin{cases} 1 & \text{if head of consumer unit } i \text{ had at least a high-school education} \\ 0 & \text{otherwise} \end{cases}$$

$$x^3_{i,t} = \begin{cases} 1 & \text{if head of consumer unit } i \text{ had at least a junior college education} \\ 0 & \text{otherwise} \end{cases}$$

$$x^4_{i,t} = \begin{cases} 1 & \text{if wife (for a married head) of consumer unit } i \text{ worked in year } t \\ 0 & \text{otherwise} \end{cases}$$

$x^5_{i,t}$ = age of head of consumer unit i in year t; measured in decades

$x^6_{i,t}$ = number of individuals composing the consumer unit i in year t; actual number

$x^7_{i,t}$ = yearly income of consumer unit i in year t; measured in 1000 - Dollars.

The estimated equation (with standard errors in parenthesis below the estimated coefficients) reads:

$$y_{i,t} = \underset{(10.061)}{45.506} - \underset{(3.683)}{5.795}\, x^1_{i,t} - \underset{(3.181)}{1.284}\, x^2_{i,t} + \underset{(3.623)}{9.110}\, x^3_{i,t} + \underset{(4.337)}{4.732}\, x^4_{i,t}$$

$$+ \underset{(1.367)}{.949}\, x^5_{i,t} - \underset{(1.126)}{3.562}\, x^6_{i,t} - \underset{(.623)}{.278}\, x^7_{i,t}$$

Number of observations: 254; $R^2 = .189$; $F = 8.23$ with 7;246 d.f. Thus, the hypothesis has to be rejected at the 1 % level. However, one can hardly argue that the qualitative variables have a strong explanatory power.

It is interesting to note that consumer units having children tend to show fewer inconsistencies. Perhaps, a lower per capita income requires more care in planning the food purchases. Since the wife does most of the purchasing, in general, it was natural to expect a tendency to more inconsistencies if the wife worked; a working wife will have less time for a careful planning.

It is hard to explain why consumer units with the head having at least a junior college education tend to have more inconsistencies. The relative insignificance of income was also surprising, since one would expect a positive and relatively larger coefficient for $x^7_{i,t}$, because wealthier consumer units can more easily "afford" to be inconsistent. But, as so often is the case, intuition was misleading.

Conclusions:

The negative results of the test performed indicate that the theory of consumer behavior, as it was developed in Chapter II and the beginning of Chapter III, does lead to contradictions if it is confronted with empirical data. At least some of the underlying axioms and primitives of the theory, or the additional assumptions of Chapter III, cannot be realistic and the theory has to be revised in a way that it can meet the challange of empirical tests.

CHAPTER IV

REVISIONS ON THE PURE THEORY OF CONSUMER BEHAVIOR

> O happy he, who still renews
> The hope, from Error's deeps to rise forever!
> That which one does not know, one needs to use;
> And what one knows, one uses never.
>
> J.W. von Goethe (Faust)

It would have been illusionary to expect a result of the study in which the majority of consumer units did not reveal any violations to the revealed preference theorems. For this to happen, ideal test conditions have to exist; but clearly, they almost never do.

There are always deficiencies in data collecting, however small, and changes in the environmental setting of the consumer units do happen over time. It was hoped that these factors would not influence the test significantly, and perhaps they did not. In any case, the results are in such an extensive contradiction to the predicted behavior that one should search for potential deficiencies in the pure theory.

Before examining the basic assumptions made in Chapter II in a new light, it is perhaps more constructive to consider first the assumptions made at the beginning of Chapter III. These had to be made in addition to those of Chapter II in order to perform a valid test.

Use was made of weekly reported data, grouped into demand periods. As mentioned in Chapter III, specifying a demand period of exactly 4-weeks long is fairly arbitrary. The difficulty arises because it was assumed implicitly that the demand period, for which the decision to consume a commodity vector is defined, is identical to the period in which the acts of purchasing took place. There need not be the slightest relation between these two periods, especially if a commodity is durable. Except for canning and freezing it was hoped that these two periods are fairly similar for food commodities. Violations to this assumption probably account for a large share of the negative results.

The assumption of demand independence of food from non-food commodities within one demand period still seems plausible, (the independence over several periods was not claimed, nor is it needed).
Of course, this claim has to be verified, but the data did not permit this.

Now attention will be paid to the 5 primitives and 7 axioms of Chapter II. In essence, a violation to the revealed preference theorems implies that the consumer unit did not act rationally as defined in Axiom 7. Does this also imply that the consumer unit does not want to act rationally ? I do not think so. One has to distinguish the intent to act rationally from the realisation whether the action taken was indeed the rational one. As humans we request the freedom to commit mistakes, yet this freedom was denied to the hypothetical consumer unit.

The reason for this denial was the static nature of the theory. If the freedom is granted to make mistakes in purchasing what is considered to be the optimal commodity vector, and the possibility exists to realize and avoid these mistakes at subsequent decisions, then a dynamic theory is needed.

The most likely source for committing mistakes will lie with a non-fulfillment of Primitive III. Perhaps, it is too much to require that the consumer unit is able to conceive of very small variations in the commodity vectors and the price vectors without error.

W. Krelle (20), (21), gives a very stimulating discussion on the existence of a "perception threshold". 1/ The perception threshold applies to the commodity space Q and the price vector p.

On the commodity space Q the perception threshold is defined as follows: For each initially conceived commodity vector $x \in Q$, there exists a level of variation about each element of x, such that the consumer unit perceives no change of it at first. Only as time goes by will the consumer unit become aware of the occured variations.

Let $g_i(x_i,t)$ denote the threshold function for the i^{th} commodity. According to Weber's Law (21), (32), $g_i(x_i,t) \geq 0$ is a non-decreasing function of x_i, because the same absolute variation of x_i is easier conceived at a lower initial level than at a higher. This threshold function is defined in the following context:

any $x_i' \in \left\{ x_i : x_i \geq 0, \ |x_i - x_i^o| \leq g_i(x_i,t) \right\}$ will

appear to be the same quantity as the initial level x_i^o at time t after a very small change occured. As time increases, the threshold value will vanish, - i.e. $\lim\limits_{t \to m < \infty} g_i(x_i,t) = 0$. This means that the

---

1) Professor Krelle uses the term "Schwellenwert der Fühlbarkeit", which translates to "threshold of sensitivity". However, here the term perception threshold is used as in Professor Shaffer's lecture notes (32).

consumer unit can conceive of new levels of the x_i without error as time goes on. The range of the perception threshold, and the speed of a complete realization for small variations in the level of commodities is a characteristic of the consumer unit.

This can be generalized to the n-dimensional commodity space Q: given an initial vector $z^o \in Q$ subject to a small change of at least one component, every $y \in S(z^o,t) = \{z: z_i \geq 0, |z_i - z_i^o| \leq g_i(z_i^o,t)$ for $i = 1, --- , n\}$ will appear to be the same as z^o itself at time t after the change occured. As time increases, $S(z^o,t)$ will eventually contain only z^o itself. The set $S(z,t)$ so defined is called the threshold set to a commodity vector $z \in Q$.

<u>Amendment to Primitive III, part a</u>: The consumer unit can conceive of distinct vectors in Q only up to the perception threshold.

<u>Axiom 8</u>: The preference ordering P orders Q up to the perception threshold; - i.e. given an initial $x \in Q$, for every $y \in S(x,t)$ not(xPy) and not(yPx) are both true.

The original Axioms 1, 2, and 3 can be adopted unchanged. However, the following revisions are necessary:

<u>Amendment to Axiom 4</u>: P is monotone up to the perception threshold; - i.e. for $x \geq y$, $x,y \in Q$, at least one inequality holding, and for y the initial vector, then xPy is true only if $x \notin S(y,t)$.

Now Axiom 4 states that the consumer unit prefers more over less only if the increase in the components of the vectorexceeds the perception threshold.

Lemma 8: For a given time period there exists commodity vectors $x, y, z \in Q$, such that not(xPy) and not(yPz) are true, yet xPz is also true.

Proof: Choose $z \leq y \leq x$, at least some inequalities holding, but $z \in S(y,t)$, $y \in S(x,t)$, and $z \notin S(x,t)$ are true. Then by Axiom 4 not(yPz), not(xPy), as well as xPz follow.

Lemma 8 provides a rationale for not assuming the existence of an indifference set (as was shown not necessary in Chapter II). From the premises to Lemma 8 it also follows that not(zPy) and not(yPx) are true; thus, if the indifference relation T were assumed one would have zTy and yTx, yet xPz is also true. The indifference relation is not transitive if a perception threshold exists, and is therefore of little interest.

Axioms 5 and 6 would have to be rejected, at least for changes of the commodity vector not fully conceived. Only if enough time after each change in a commodity vector is permitted so that all perception thresholds are ineffective, then Axioms 5 and 6 can still be valid. These two Axioms were used only in the proof of Theorem 4, where it was claimed that any bounded commodity vector can conceivably be bought for a suitable price vector and income. Since the premise of Theorem 4 provides an initial, fully conceived vector, only the following amendments are needed:

Amendment to Axioms 5 and 6: The vector $x \in Q$ and all vectors in the sets defined in Axioms 5 and 6 must be fully conceived.

The perception threshold on the price vector is similarly defined. For a given price level p_i', given a small price change for commodity x_i, the consumer unit conceives all

$$p_i'' \in \left\{ p_i : p_i > 0, \left|p_i' - p_i\right| \leq h_i(p_i', t')\right\}$$

and p_i' as equivalent at time t' after the change took place. The price threshold function so defined is non-decreasing in p_i and decreasing in time t for all $h_i(p_i, t)$, $i = 1, ---, n$.

Amendment to Primitive III, part b: The consumer unit conceives of different price vectors only up to the perception threshold; - i.e. if small changes occur from an initial price level p', then all

$$p'' \in \left\{ p : p_i > 0, \left| p_i' - p_i \right| \leq h_i(p_i', t'), \; i = 1, ---, n \right\}$$

will appear to be equivalent to p' itself at time t'.

Incorporating the perception threshold into the theory, it is now possible to give a formal restatement of Axiom 7:

Amendment to Axiom 7: The consumer unit attempts to purchase the optimal vector as defined in Axiom 7. The sequence of actions can be described as follows: Starting from an initial actual budget set $X(p^o, Y^o)$, the consumer unit conceives a price vector in the set

$$\left\{ p : p_i > 0, \left| p_i - p_i^o \right| \leq h_i(p_i^o, t), \; i = 1, ---, n \right\}, \text{ say } p'',$$

and makes the choice on the conceived budget set $X(p'', Y^o)$. The actual choice made by the consumer unit will be a commodity vector $x''' \in S(x'',t)$, where x'' is the optimal vector for the set $X(p'',Y^o)$. Given $X(p^o,Y^o)$ does not change for a sufficiently long time interval, all $g_i(x_i,t)$ and $h_i(p_i,t)$ will tend to zero, - i.e. the optimal vector for the actual budget set will be attained. 1/

Since the definition of the optimal vector remains the same, the theorems of Chapter II do not have to be revised. Only their significance is reduced, since the optimal vector is a commodity combination to which the consumer unit will tend, but not necessarely buy. While income or the food budget is not likely to change very often, prices do change more frequently, and then only in small amounts. Under these circumstances will the optimal vector practically never be attained, only a close approximation to it. Furthermore, the hope to observe values of the choice (or demand) function has to be abandoned, since the observed correspondence between the conceived budget sets and the purchased commodities is governed by a different, more complicated relation.

This also implies that, in general, revealed preference theory does not provide a test anymore on the consistency of the underlying assumptions made in the theory of consumer behavior. Only for completly adjusted situations, - i.e. where all the perception thresholds vanished, can revealed preference theory still provide a test.

---

1) It is possible that there exists also a perception threshold for income. This case is not considered here. But the treatment of a perception threshold for income is along the same lines as for prices.

The question arises, is the theory of consumer behavior that incorporates concepts of perception thresholds still testable by using market observations ? It is, under the condition that all threshold functions $g_i(x_i,t)$ and $h_i(p_i,t)$ are measurable. Once these functions are measured, one can set up an exact bound by how much the inequalities of the original revealed preference theorems may be violated before an inconsistency is evident.

For example, given the observed inconsistent situation of

$$p^o x^o \geq p^o x' \quad \text{and} \quad p'x' \geq p'x^o \qquad x^o \neq x'$$

which violates Theorem 10, then there exists a scalar λ , determined by the threshold functions, $0 < \lambda < 1$, and multiplying both $p^o x^o$ and $p'x'$ by λ reverses at least one inequality. Let the λ so defined be denoted as the threshold parameter. In the above inequalities $p^o x^o$ and $p'x'$ are multiplied by a λ, $0 < \lambda < 1$, rather than $p^o x'$ and $p'x^o$ by a parameter greater than 1, because it is assumed that the actual purchases contain the (measured) thresholds.

This threshold parameter λ will be closer to 1, the smaller the perception thresholds are. Similarly, for violations to Theorem 11 and Theorem 13 there will exist a scalar λ which disrupts the sequence of observed preference relations which leads to an inconsistency.

Assuming that the above hypothesis is is correct, one can estimate bounds for the threshold parameter as follows: Given $p^o x^o \geq p^o x'$ and $p'x' \geq p'x^o$, then λ_i , $0 < \lambda_i < 1$, $i = 1,2$ have to be calculated such that the two equalities $\lambda_1 p^o x^o = p^o x'$ and $\lambda_2 p'x' = p'x^o$ are attained. Then, choosing as the threshold parameter any $\lambda^{**} < \max(\lambda_1, \lambda_2)$

will reverse at least one inequality, and any $\lambda^* < \min(\lambda_1, \lambda_2)$ will reverse both inequalities of the revealed inconsistent situation. 1,2/

The values of $\min(\lambda_1, \lambda_2)$ and $\max(\lambda_1, \lambda_2)$ were calculated for the 64 consumer units who participated all four years in the panel. To eliminate observations which contained possible data deficiencies, only those consumer units with less than 50 violations to Theorem 10 (or 12) were considered for the results presented below. 3/ The year was subdivided into 3 four-period "seasons", again, the observed demand vector for period 13 had to be ignored. Then $\min(\lambda_1, \lambda_2)$ and $\max(\lambda_1, \lambda_2)$ were computed for every two observed demand vectors of a "season" which violated Theorem 10 (or 12). Table 8 gives an indication of the results obtained. The values presented are for all consumer units considered in computing the bounds of the threshold parameter.

Table 8:

Values of the estimated bounds of the threshold parameter.

	minimum	maximum	mean	standard diviation
$\min(\lambda_1, \lambda_2)$	.59	.99	.91	.06
$\max(\lambda_1, \lambda_2)$	.81	.99	.95	.04

1) Of course, any value between $\min(\lambda_1, \lambda_2)$ and $\max(\lambda_1, \lambda_2)$ will also reverse at least one inequality.

2) If the requirement for the threshold parameter is that both inequalities have to be reversed, then $\min(\lambda_1, \lambda_2)$ is an upper bound estimate of the threshold parameter.

3) Since practically all violations to the revealed preference theorems were in relation to Theorem 10 (or 12), $\min(\lambda_1, \lambda_2)$ and $\max(\lambda_1, \lambda_2)$ values were computed only for two commodity vectors.

Of those consumer units disregarded because of too many inconsistencies, some showed a $\min(\lambda_1, \lambda_2)$ value of as low as .47. However, the results presented in Table 8 seem to be close to anticipated values.

In an attempt to explain variations in $\min(\lambda_1, \lambda_2)$, some of the available qualitative data were used in the following two regressions to test the significance of the variables involved. 1/

Let $y_{i,t}$ = the average of observed values for $\min(\lambda_1, \lambda_2)$ of consumer unit i in season t of a year

$x^1_{i,t}$ = number of individuals in consumer unit i, season t

$$x^2_{i,t} = \begin{cases} 1 & \text{if head of consumer unit } i \text{ had at least a high-school education} \\ 0 & \text{otherwise} \end{cases}$$

$$x^3_{i,t} = \begin{cases} 1 & \text{if head of consumer unit } i \text{ had at least a junior college education} \\ 0 & \text{otherwise} \end{cases}$$

$x^4_{i,t}$ = age of head of consumer unit i; measured in decades

$x^5_{i,t}$ = income of consumer unit i in season t; measured in 1000 Dollars.

Regression (i):
$$y_{i,t} = \underset{(.049)}{.857} - \underset{(.004)}{.001}\, x^1_{i,t} + \underset{(.017)}{.034}\, x^2_{i,t} + \underset{(.023)}{.017}\, x^3_{i,t} + \underset{(.006)}{.006}\, x^4_{i,t} + \underset{(.012)}{.007}\, x^5_{i,t}$$

$R^2 = .074$; 71 observations used; $F = 1.0529$ with 5;65 d.f.

---

1) Only $\min(\lambda_1, \lambda_2)$ was used in the regressions, since its interpretation can be more restrictive. See footnote 2 on page 55.

The hypothesis that all coefficients to $x^1_{i,t}$, --- , $x^5_{i,t}$ are equal to zero cannot be rejected at the 1 % level.

Regression (ii): $y_{i,t} = \underset{(.014)}{.905} + \underset{(.025)}{.011} \; x^5_{i,t} / x^1_{i,t}$

$R^2 = .002$; 71 observations used.

The hypothesis that the coefficient to $x^5_{i,t} / x^1_{i,t}$, - i.e. income per capita in consumer unit i, is equal to zero cannot be rejected at the 1 % level.

Clearly, the results of the regression equations show the insignificance of the variables involved. Particularly disappointing is the insignificance of income and per capita income. Since income shifts the budget hyperplane in a parametric fashion, one would expect $\min(\lambda_1, \lambda_2)$ to be lower for higher income levels, - i.e. a negative coefficient to income, because a higher level of consumption in each commodity increases the levels of the $g_i(x_i,t)$ function. The results obtained instead can be explained, assuming Weber's Law still holds, that a higher income level does not lead to a proportionally higher level of consumption of the same commodities. This effect is fairly common for food commodities. 1/

One cannot deny that the above approach of quantifying concepts of perception thresholds is fairly basic, at least it does no justice to some of the fine lines written on this subject in Krelle (21).

---

1) Nevertheless, $\min(\lambda_1, \lambda_2)$ should be inversely related to income if also a perception threshold for income exists.

However, the approach presented above permitted the use of already generated data (for testing the revealed preference theorems) with little additional computational effort and computer cost, and for this reason might find some merit in its simplicity.

There are many other violations to the maintained assumptions of the tested theory that could explain part of the negative results of the study. The assumption of a preference relation P which is constant over time certainly has to be considered with suspicion. Discarding the assumtion of a time constant preference relation would, however, invalidate all of the theory derived in Chapter II. One can argue that the preference relation changes over time because consumer units develop a liking for new products only through experimental buying. Also, the dimension of the commodity space can be changing over time because of new products offered on the market, or because advertising makes consumer units aware of already existing products. However, these problems should be more relevant for non-food rather than for food commodities.

Conclusions:

The incorporation of perception thresholds into the economic theory of consumer behavior is only one possible modification. There are many other reasonable approaches to consider the implication of the test results.

Of particular interest is the implication which the study might have on the demand estimation approaches of Barten (3), (4), Theil (36), Frisch (8), and Goldberger (10) which basicly assume a consistent behavior of the consumer units, although at a more aggregated level than the unit as it was defined in this thesis. An answer to this problem is not attempted here, but it is a challanging subject and deserves effort and research from serious economists.

Another implication of the results obtained is the implausibility of obtaining meaningful index numbers for micro-economic units, as the unit was defined in the study. 1/ Each violation of the revealed preference theorems also gives inconsistent Paasche and Laspeyre index numbers for the consumer unit.

The extent to which the revealed inconsistent behavior of consumer units will also have implications on index numbers and demand equations at a more aggregated level is an interesting problem of aggregation. This problem has its own flavor of interest that should merit more research attention.

These, and perhaps many more questions remain unanswered in this thesis. The purpose of the study presented in this thesis was merely to test the plausibility of the assumption for a theory of consumer behavior as outlined in Chapter II. However, it is hoped that this thesis will provide a stimulus and an incentive for more effort and research on the questions left unanswered.

---

1) Samuelson (31) discusses the Paasche and Laspeyre index numbers.

APPENDICES

APPENDIX I

THE MAGNETIC DATA TAPES USED IN THE STUDY

A comprehensive description of the qualitative data, and of the weekly reported prices and purchased commodities, can be found in the mimeograph: "Description of Data Stored on Magnetic Tape from M.S.U. Consumer Panel Survey 1952 - 58", Miscellaneous Series No. 1969-2 of the Department of Agricultural Economics at Michigan State University.

Four new tapes containing the codes for the consumer units, the price quantity matrices $PQ_{k,t}$, and the corresponding Boolean matrices $V_{k,t}$ were generated for the study. Their numbers are:

For 1955	5=(KØØ PQBØLMAT 55),RW,MT(287)
1956	5=(KØØ PQBØLMAT 56),RW,MT(88)
1957	5=(KØØ PQBØLMAT 57),RW,MT(143)
1958	5=(KØØ PQ BØL MAT,1,1,999),RW,MT(165)

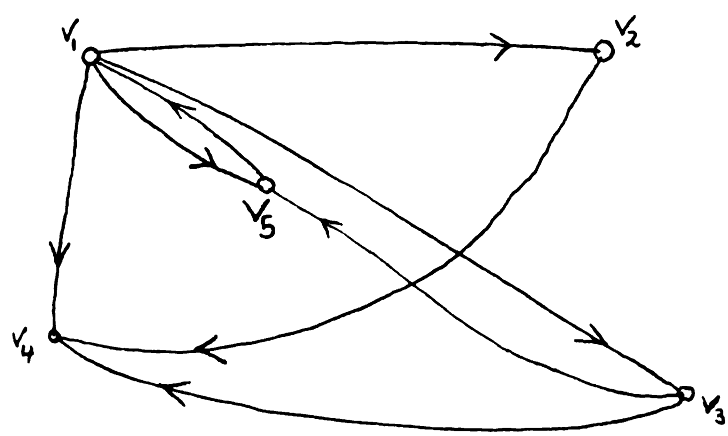
APPENDIX II

SOME BASIC CONCEPTS OF DIRECTED GRAPHS [1]

The theory of directed graphs is founded on the primitive concepts of points and lines. The set of points is finite, and may not be empty; whereas the finite set of lines connecting points may be empty. The lines in a graph are directed in the sense that each originates at one point and terminates at another. No line may originate and terminate at the same point. The existence of a unique directed line between any two points is determined by an irreflexive, binary relation defined on the set of points.

To show an example, suppose a graph D consists of the set of points $P = (v_1, v_2, v_3, v_4, v_5)$, and the set of lines $L = (v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_3v_4, v_3v_5, v_5v_1)$, where v_iv_j indicates a line, determined by a binary relation on P, which originates at v_i and terminates at v_j (in the figure below this is indicated by an arrow head). Then D can be illustrated by a figure of the form:

Figure 1:



---

1) For books on this subject, Harary (14) is recommended. This appendix is by no means a complete treatment of the theory of directed graphs.

It is of no significance that the lines cross, also, it is immaterial how the points are located in the figure, since the concepts of directed graphs is free of a geometric orientation.

Such a graph, as symbolized by Figure 1, can also be represented in matrix form. The (square) adjacency matrix A of a directed graph has one row and one column for each point in P. A is then defined to be:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is in } L \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for the above example:

$$A = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 0 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 1 & 0 & 0 & 0 & 0 \end{array}$$

While the representation of a directed graph by a figure has more intuitive appeal, the matrix form suits itself better for computations. Since for the adjacency matrix A the order of row and column subscripts is important, it follows: if columns i and j are interchanged, rows i and j have to be interchanged to obtain an equivalent adjacency matrix for the directed graph.

The above introductory remarks on the theory of directed graphs will hopefully ease the interpretation of the following definitions and theorems:

Let a graph D have the set of points $P = (v_1, \text{---}, v_n)$ and the set of lines $L = (v_i v_j, \text{---}, v_t v_p)$ for $i \neq j$, $t \neq p$, and $1 \leq i,j,t,p \leq n$, where n is a finite integer.

<u>Definition 1</u>: A sequence is an alternating arrangement of points and directed lines. The sequence is said to be open if the initial and terminal points differ; the sequence is closed if these points are the same.

<u>Definition 2</u>: A path from v_i to v_j is a collection of distinct points $v_s, v_r, \text{---}, v_t$, together with lines $v_i v_s$, $v_s v_r$, --- , $v_t v_j$. Such a path is trivial if it consists of a single point.

From the definitions it follows that every path is a sequence, but not every sequence is a path, since the points in the sequence need not be distinct.

<u>Definition 3</u>: The length of a sequence from v_i to v_j is the number of lines in it.

Since it is possible that there are several paths of various length from v_i to v_j, another definition is needed.

Definition 4: The geodesic from v_i to v_j is a path of minimum length.

Definition 5: The distance from v_i to v_j is the length of the geodesic from v_i to v_j. If there is no path from v_i to v_j, then the distance is infinite.

Definition 6: A cycle is a path of length greater than one originating and ending at the same point.

Let A be the adjacency matrix for a graph D, and $A^2 = A \cdot A$ be a matrix multiplication of A with itself. Then the typical element a^2_{ij} of A^2 indicates the number of sequences of length 2 from v_i to v_j. To show that this is true, simply multiply out:

$$a^2_{ij} = a_{i1}a_{1j} + a_{i2}a_{2j} + \text{---} + a_{in}a_{nj}$$

Since a_{ik} and a_{kj} are either equal to 1 or to 0, depending on the existence of a sequence of length 1 from v_i to v_k, and v_k to v_j respectively, it follows: if the product $a_{ik}a_{kj} = 1$, then there must exist the sequence with points v_i, v_k, v_j and the lines $v_i v_j$, $v_k v_j$. There are 2 lines, hence the length of the sequence is 2. This result can be generalized:

Theorem 1: For A, the adjacency matrix of a directed graph D, the entry a^n_{ij} of A^n is the number of sequences from v_i to v_j of length n. The closed sequences can be read off the main diagonal.

<u>Corollary</u>: The number of sequences a^n_{ij} in A^n will indicate the number of paths from v_i to v_j if n is the lowest power to which A has to be raised in order to make a^n_{ij} positive for the first time. In that case, n is also the distance between v_i and v_j.

Clearly, for n points in the graph, no path with distinct initial and terminal points can have a length greater than $n-1$, and no cycle can have a length greater than n. The proof of Theorem 1 follows by induction along the pattern as indicated for A^2.

<u>Theorem 2</u>: If a directed graph D with m points contains no cycles, - i.e. is acyclic, then $A^m = 0$.

Proof: Since D contains no cycles, the longest sequence it can contain is of length $m-1$; hence, by Theorem 1, $A^m = 0$.

Suppose it is desired to determine the existence of a sequence between any two points of length less than of equal to n in a graph D having n points. Let this be indicated by an $n x n$ reachable matrix S, whose element $s_{ij} = 1$ if there exists at least one sequence from v_i to v_j of length $\leq n$, otherwise $s_{ij} = 0$. The diagonal elements s_{ii} of S will indicate the existence of any cycles about the point v_i.

To compute the reachable matrix S, Boolean algebra is needed. Let the operator $\#$ be defined only for integers $c \geq 0$, then

$$c \# = \begin{cases} 1 & \text{if } c \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3 a: For A the adjacency matrix of a directed graph D with n points, $S = (A + A^2 + \text{---} + A^n)\ \#$

Proof: The elements a^t_{ij} of A^t indicate the number of sequences of length t from v_i to v_j. If v_i can reach v_j along lines in L, then there exists a path from v_i to v_j whose length is $\leq n$. The operator $\#$ is used because we are interested only in the existence of at least one sequence.

This way of finding S has an intuitive appeal, however, it is cumbersome to compute and can be simplified to:

Theorem 3 b: $S = \left[(I + A)^n - I\right]\ \#$, where I is the identity matrix.

Proof: By definition of the operator $\#$, $cA\ \# = A$ for every integer $c \geq 1$. Hence $(I + A)^2 = (I + 2A + A^2)$, and applying $\#$ on both sides gives the same result as $(I + A + A^2)\ \#$. In general, $(I + A)^n\ \# = (I + A + \text{---} + A^n)\ \#$. Thus, take $\left[(I + A)^n - I\right]\ \# = (A + \text{---} + A^n)\ \# = S$.

Now revealed preference structures symbolized by Boolean matrices and directed graph theory can be synthesized. The set of points in the theory of directed graphs becomes the set of observed demand vectors, and the set of lines indicating a preference is determined by a revealed preference relation on the commodity space. The adjacency matrix will result for the relation R, and the reachable matrix is equivalent to the preference matrix $S_{k,t}$. In the terminology of graph theory, a cycle is equivalent to a violation to the revealed preference theorems.

BIBLIOGRAPHY

BIBLIOGRAPHY

1) Allen, R.G.D.: "Professor Slutsky's Theory of Consumers' Choice," Review of Economic Studies, 3, 1936, pp. 120-129.

2) Arrow, K.J.: "Rational Choice Functions and Orderings," Economica, 26, 1959, pp. 158-160.

3) Barten, A.P.: "Consumer Demand Equations under Conditions of Almost Additive Preferences," Econometrica, 32, 1964, pp. 1-38.

4) Barten, A.P.: "Estimating Demand Equations," Econometrica, 36, 1968, pp. 213-251.

5) Clarkson, G.P.E.: The Theory of Consumer Demand: A Critical Appraisal, Englewood Cliffs, N.J.: Prentice-Hall, 1963.

6) Debreu, G.: "Representation of a Preference Ordering by a Numerical Function," Chapter XI in Thrall, R.M., C.H. Coombs, and R.L. Davis (eds.), Decision Processes, New York: John Wiley & Son, 1954, pp. 159-160.

7) Debreu, G.: Theory of Value: An Axiomatic Analysis of Economic Equilibrium, New York: John Wiley & Son, 1959, Cowles Monograph 17.

8) Frisch, R.: "A Complete Scheme for Computing All Direct and Cross Demand Elasticities in a Model with Many Sectors," Econometrica, 27, 1959, pp. 177-196.

9) Georgescu-Roegen, N.: "Threshold in Choice and the Theory of Demand," Econometrica, 26, 1958, pp. 157-168.

10) Goldberger, A.S.: "Functional Form and Utility: A Review of Consumer Demand Theory," Systems Formulation, Methodology, and Policy Workshop Paper 6703, Social Systems Research Institute, The University of Wisconsin, Madison, 1967.

11) Gorman, W.M.: "Separable Utility and Aggregation." Econometrica, 27, 1959, pp. 469-481.

12) Green, H.A.J.: "Some Logical Relations in Revealed Preference Theory," Economica, 24, 1957, pp. 315-323.

13) Harary, F.: Graph Theory, Reading: Addison Wesley, 1969.

14) Harary, F., R.Z. Norman, and D. Cartwright: Structural Models: An Introduction to the Theory of Directed Graphs, New York: John Wiley & Son, 1965.

15) Hicks, J.R.: A Revision of Demand Theory, New York: Oxford University Press, 1956.

16) Houthakker, H.S.: "Additive Preferences," Econometrica, 28, 1960, pp. 244-257.

17) Houthakker, H.S.: "Revealed Preference and Utility Function," Economica, 17, 1950, pp. 159-174.

18) Kamke, E.: Theory of Sets, New York: Dover, 1950.

19) Koo, A.Y.C.: "An Empirical Test of Revealed Preference Theory," Econometrica, 31, 1963, pp. 646-664.

20) Krelle, W.: Präferenz- und Entscheidungstheorie, Tübingen: J.C.B. Mohr (Paul Siebeck), 1968.

21) Krelle, W.: Preistheorie, Tübingen: J.C.B. Mohr (Paul Siebeck), 1961.

22) Lancaster, K.: Mathematical Economics, New York: The Macmillan Co., 1968.

23) Lemmon, E.J.: Beginning Logic, London: Thomas Nelson and Sons, 1965.

24) Lindgren, B.W.: Statistical Theory, New York: The Macmillan Co., 1968.

25) Malinvaud, E.: Statistical Methods of Econometrics, Amsterdam: North-Holland Publishing Co., 1966.

26) Manderscheid, L.V.: "Incorporating Durable Goods into Consumption Theory," mimeographed paper, presented at the meeting of the Econometric Society in New York on December 28, 1965, Department of Agricultural Economics, Michigan State University, 1965.

27) Newman, P.: "A Further Note on Revealed Preference," Economica, 26, 1959, pp. 158-160.

28) Newman, P.: "The Foundations of Revealed Preference Theory," Oxford Economic Papers, N.S. 7, 1955, pp. 151-168.

29) Quackenbush, G.G., and J.D. Shaffer: Collecting Food Purchases Data by Consumer Panel, Michigan State University, Agricultural Experiment Station Technical Bulletin 279, E. Lansing, 1960.

30) Richter, M.K.: "Revealed Preference Theory," Econometrica, 34, 1966, pp. 635-645.

31) Samuelson, P.A.: Foundations of Economic Analysis, Cambridge: Harvard University Press, 1947.

32) Shaffer, J.D.: "Notes for a Theory of Personality for a Theory of Consumer Behavior," mimeographed lecture notes, Department of Agricultural Economics, Michigan State University, 1969.

33) Stoll, R.R.: Sets, Logic and Axiomatic Theories, San Francisco: W.H. Freeman, 1961.

34) Strotz, R.H.: "The Empirical Implications of a Utility Tree," Econometrica, 25, 1957, pp. 269-280.

35) Strotz, R.H.: "The Utility Tree - A Correction and Further Appraisal," Econometrica, 27, 1959, pp. 482-488.

36) Theil, H.: "The Information Approach to Demand Analysis," Econometrica, 33, 1965, pp. 67-87.

37) Uzawa, H.: "Preference and Rational Choice in the Theory of Consumption," Chapter 9 in Arrow, K.J, S. Karlin, and P. Suppes (eds.), Mathematical Methods in the Social Sciences, 1959, (Stanford 1960), pp. 129-148.

38) Yokoyama, T.: "A Logical Foundation of the Theory of Consumer's Demand," Osaka Economic Papers, 2, 1953, pp. 71-79.