A NUMERICAL METHOD FOR DETERMINING CHARACTERISTIC VALUES OF THE WAVE EQUATION

Thesis for the Degree of M. S.

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AN ABSTRACT

Submitted to the College of Engineering
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ABSTRACT

Methods of solving the two dimensional wave equation assuming wave guide boundary conditions are discussed. In particular, numerical methods are mainly stressed. Difference equations, relaxation methods, and iteration methods receive the most emphasis. The objective of the thesis is to find a method for determining characteristic values.

The result is an iteration program which will determine characteristic values for a uniform cross section wave guide. The method approximates the wave equation by difference equations. The eigenvalues are then computed by an iteration process. Alternative methods of solving the wave equation and methods of improving the existing program are given. The results of two applications are given as a check on the method. Finally, more difficult problems to which the same method might be applied are discussed.

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CHAPTER I

INTRODUCTION

A. Guided Waves

As is obvious from the name itself, the purpose of a wave guide is to transfer electromagnetic radiation or energy from one place to another by means of some sort of guiding structure. The design of the wave guide may be as varied as the applications to which it is applied. In many cases the wave guide is a length of hollow pipe which has as its physical characteristics the critical dimensions of its cross section, the conductivity of the metal structure, and the magnetic permeability, dielectric constant, and conductivity of the dielectric material in its interior.

This type of wave guide is not the only type of interest, but it is this general type of wave guide which will be considered in this thesis. In other words, the wave guides considered in this thesis are wave guides for which: (1) the cross section is uniform, (2) the dielectric material in the wave guide has constant dielectric constant ϵ, constant magnetic permeability μ, and zero conductivity, (3) the metal of which it is made

is assumed to be a perfect conductor. This last stipulation is of no great practical concern since most wave guides are constructed out of metal with high enough conductivity so that the metal can be assumed to be a perfect conductor for all practical purposes.

A wave guide has many modes of operation and, in fact, infinitely many. However, it is usually only the lower order modes which are of interest.

Propagation of electromagnetic energy in the wave guide is governed by a homogeneous partial differential equation called the wave equation. This equation has solutions only for certain discrete values of one of its parameters. These discrete values are the characteristic values or eigenvalues of the wave equation. Each characteristic value corresponds to its own mode as was mentioned in the preceeding paragraph.

Once the characteristic values are known, it is then a somewhat simpler matter to solve for the eigenvectors of eigenfunctions which represent the field in the wave guide.

As it turns out for a uniform cross section wave guide, it is only necessary to solve a two dimensional wave equation in order to determine the characteristic values. The dimensions involved are of course the X and Y dimensions of the cross section. Solution of the two dimensional wave equation then becomes a matter of

finding the characteristic values for the modes of interest.

The wave equation is not unique in form but is found in problems involving many different branches of science. It so happens, that the wave equation is identically the same as the equation governing the vibrations of thin elastic membranes of arbitrary cross section, which are rigidly clamped on some or all of their edges. The vibrating membrane is brought up here because it has certain analogies which help understand the wave guide. The analogies will be explained after a few brief remarks on the wave guide.

From the characteristic value of a mode of propagation in a wave guide, it is possible to calculate from classical formulas (1) the associated characteristics of the wave guide such as the cut-off frequency, the attenuation of the guide, the power handling capacity, and the mode separation. The latter is naturally dependent on more than one eigenvalue.

Now for the analogies. The characteristic values associated with the vibrating membrane are related to the extreme positions in which it is possible for the membrane to vibrate. The lowest possible frequency at which the membrane will vibrate is analogous to the lowest cut-off frequency or cut-off frequency of the dominant mode of the wave guide.

The lowest frequency at which the membrane will vibrate in a more complex nature corresponds, clearly, to the cut-off frequencies of the higher order modes. The cut-off frequency of the wave guide, is the frequency below which propagation of energy in the wave guide is not possible for that mode. The rate at which the oscillations of the membrane are damped is analogous to the attenuation of the fields in the wave guide. The power handling capacity, which is determined by the maximum fields allowable before the dielectric material breaks down, is analogous to the extreme amplitude in which the membrane can vibrate without damage. Obviously, mode separation is its own analogy.

It is these properties which are of final interest, however, it is not the intent of this thesis to discuss these properties of the wave guide but rather the characteristic values or eigenvalues, from which these properties can be calculated.

In order to analyze a wave guide of particular shape, it is first necessary to determine the eigenvalues. This is a very difficult job in all but the very simplest cases. The purpose of this thesis can now be stated.

The object of this thesis is to develop a general program for a digital computer, in particular the MISTIC, which will, when given the necessary details about the boundary conditions, compute the desired characteristic

values.

B. The Wave Equation

In the introduction the wave equation was mentioned as a homogeneous partial differential equation which governs the electromagnetic fields in a wave guide. The wave equation in general form is

$$\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{or} \quad \nabla^2 H = \mu\epsilon \frac{\partial^2 H}{\partial t^2} \tag{1-1}$$

These equations are easily derived from Maxwell's equations. They are based on the assumption that the dielectric material has zero conductivity, is homogeneous, linear, and isotropic. They are identical for the electric or magnetic field and must be satisfied by all components of the electric and magnetic field.

If it is now assumed that the fields vary with time as $e^{j\omega t}$, the wave equations become

$$\nabla^2 E = -\omega^2 \mu\epsilon E, \quad \nabla^2 H = -\omega^2 \mu\epsilon H \tag{1-2}$$

If it is further assumed that the propagation in the Z direction (the direction of propagation) varies as $e^{-\gamma z}$, where γ is known as the propagation constant, the equations then become:

$$\nabla_{xy}^2 E = -(\gamma^2 + \omega^2 \mu\epsilon) E, \quad \nabla_{xy}^2 H = -(\gamma^2 + \omega^2 \mu\epsilon) H \tag{1-3}$$

In these equations $\nabla^2_{xy}E$ is now the two dimensional Laplacian in the transverse plane; the third partial derivative having been set equal to $\gamma^2 E$ because of the assumption that E_z varies as $e^{-\gamma z}$. Finally, it is convenient to replace $\gamma^2 + \omega^2\mu\epsilon$ by K_c^2 and the equations are then, simply

$$\nabla^2_{xy} E = -K_c^2 E \quad , \quad \nabla^2_{xy} H = -K_c^2 H \tag{1-4}$$

where $K_c^2 = \gamma^2 + \omega^2\mu\epsilon$

These equations have solutions only for discrete values of K_c^2. It is the K_c's with which the thesis is concerned. These numbers are such that $K_{c_1}^2 \leqq K_{c_2}^2 \leqq K_{c_3}^2 \leqq$ . . . . etc., where K_{c_1} is the eigenvalue of the dominant mode and K_{c_2} is the characteristic value of the next highest order mode, etc. Once the K_c is known the cut-off frequency f_c, for example, is easily found by letting $\gamma = 0$ so that:

$$f_c = \frac{K_c}{2\pi\sqrt{\mu\epsilon}} \tag{5}$$

It is common to classify solutions to the wave equation inside a wave guide into three general types. These modes of operation are called transverse electromagnetic, transverse electric, and transverse magnetic; TEM, TE, and TM respectively. The TEM mode has no E or H component in the Z direction (i.e. direction of propagation) and may in many cases not even be a possible mode

of operation. Its propagation usually requires at least two separate conductors. The TE and TM modes are the usual modes of propagation in the wave guide. The TE modes are called transverse electric modes because they have no E component in the axial Z direction, and likewise, the TM has no H component in the Z direction.

As was mentioned before, the wave equation must be satisfied by all of the individual components of the electric and magnetic field present in the problem being analyzed. For this reason, it is not necessary to solve the wave equation for the entire electric or magnetic field if the main interest is merely the characteristic values. One is at liberty to pick the component of E or H field which is most convenient in applying the boundary conditions. What can be done in practice is to decide what mode is to be analyzed and pick for solution the component of the electric or magnetic field, for example, E_x, E_y, E_z, H_x, H_y, or H_z, which allows the easiest application of boundary conditions.

Thus far the wave equation has been considered only in a rectangular coordinate system. This is because the detailed method given in the following chapters uses the rectangular coordinate system. However, it is just as logical to write the wave equation in a cylindrical or spherical coordinate system or any

other coordinate system.

A common method of solution of the two dimensional or the three dimensional wave equation in any coordinate system is known as separation of variables (1, p. 145) or as the product solution method. The solution is assumed to be a product of functions, each of which is a function of only one variable of the coordinate system. The assumed product solution is substituted into the differential equation, and it is then possible to separate it into ordinary differential equations which can be solved individually. Of course, there still remains the application of the boundary conditions. When this is done the individual solutions are multiplied together giving the final product solution.

This is easily done when the geometry of the wave guide matches nicely the coordinate system. For instance, the solution to the wave equation for a retangular wave guide can be found quite easily by writing the wave equation in the rectangular coordinate system. The solution turns out to be trigonometric functions which are easily matched to the rectangular geometry of the wave guide. In the case of the cylindrical wave guide, Bessel functions result for the radial variation in the solution, having written the wave equation in cylindrical coordinates. Again, the Bessel functions are easily matched to the circular boundary. In a like manner it

is even possible to solve a parabolic wave guide (2) using a parabolic coordinate system. However, when the cross section of the wave guide does not fit the coordinate system, applying the boundary conditions becomes quite difficult. As a result, solution of the wave equation is very difficult except for these few special cases above.

A great deal of work has been done during the past decade or so on analyzing the ridge wave guide. S. B. Cohn has shown (3) the ridge wave guide to have a lower cut-off frequency and higher mode separation than the corresponding rectangular wave guide. This particular feature has lead to its commercial use in airline weather penetration radar.

The only difference between the rectangular wave guide and the ridge wave guide is the superposition of a rectangular indentation on one of the sides of the rectangular cross section. Yet, this simple change is enough to make the problem extremely difficult to solve.

More recently, work has been done on a semicircular ridge wave guide. The solution of this problem has shown (4) the semicircular ridge wave guide to be superior in power handling capicity as compared with a rectangular ridge guide of the same mode separation factor.

There are other means of solving the wave equation

analytically besides a product solution, but each seems to require a great deal of ingenuity and specialization to a particular problem. It would be valuable to have a general method of solution to the wave equation. A convenient subterfuge is a numerical solution.

Many types of numerical solutions are possible. Some of these will be discussed broadly so as to give a wider outlook on the problem, and then the particular method used will be discussed in detail.

One numerical method is to substitute difference equations for the differential equation. It is then possible to write a difference equation for each node of a grid system imposed on the cross section of the wave guide. The next step is to effect a simultaneous solution of all the difference equations. Since the number of equations necessarily must be great in order to obtain accuracy, the simultaneous solution of these finite difference equations is not always easy.

The following chapters will discuss the difference equations and methods of solving the difference equations. Finally, Chapter IV will discuss a program for MISTIC, embodying these ideas, which will solve the wave equation given any arbitrary cross section of wave guide.

CHAPTER II

DIFFERENCE EQUATIONS

A. First Order Difference Equations

Writing difference equations for a differential equation involves two procedures. First, it is necessary to impose a grid system on the domain of the problem so that a difference equation can be written for each node. The second is to find a suitable approximation or difference equation for the original differential equation. Thus, instead of having to solve one differential equation, the problem reduces to a matter of finding a simultaneous solution for the numerous difference equations.

The wave equation for which one would like to find a difference equation is

$$\nabla^2 E = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} = -k_c^2 E \qquad (2\text{-}1)$$

Actually, once one finds a difference expression for $\nabla^2 E$ the entire equation (2-1) is easily written in difference equation form. Therefore, the main objective of this chapter will be to discuss difference expressions for $\nabla^2 E$.

To begin with one imposes on the cross section of

the wave guide a grid system. A portion of this grid system is shown in Figure 2.1.

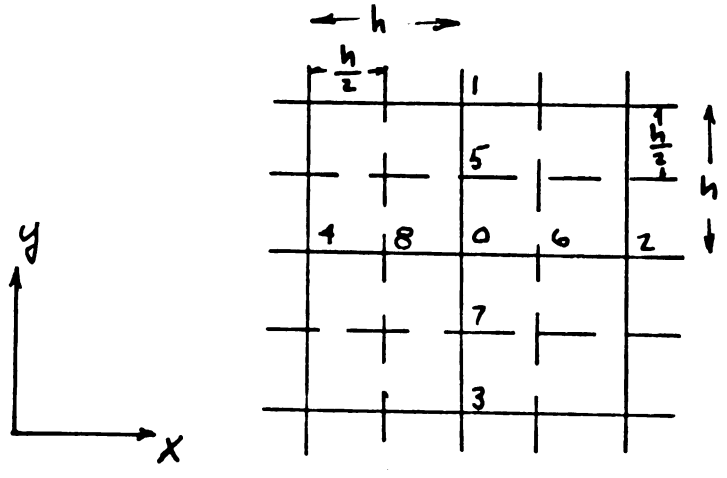

Figure 2.1

The solid lines show the actual grid lines; while the dotted lines are only for the purpose of deriving the difference equation. A derivation of the first order difference equation in rectangular coordinates (5, p. 51) will now be given.

Since E is a function of two variables, the definition of $\frac{\partial E}{\partial X}$ is

$$\left.\frac{\partial E}{\partial x}\right|_0 = \lim_{h\to 0}\left[\frac{E(x_0+\frac{h}{2}, y_0) - E(x_0-\frac{h}{2}, y_0)}{h}\right] \qquad (2\text{-}2)$$

where $\left.\frac{\partial E}{\partial X}\right|_0$ means the partial derivative $\frac{\partial E}{\partial X}$ at the point 0, etc. If h is taken small enough,

$$\left.\frac{\partial E}{\partial x}\right|_0 \approx \frac{E_6 - E_8}{h} \qquad (2\text{-}3)$$

and similarly,

$$\left.\frac{\partial E}{\partial X}\right|_6 \approx \frac{E_2 - E_0}{h} \quad \text{and} \quad \left.\frac{\partial E}{\partial X}\right|_8 \approx \frac{E_0 - E_4}{h} \tag{2-4}$$

Now, since

$$\frac{\partial^2 E}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial E}{\partial X}\right)$$

it follows that

$$\left.\frac{\partial^2 E}{\partial X^2}\right|_0 \approx \frac{\left.\frac{\partial E}{\partial X}\right|_6 - \left.\frac{\partial E}{\partial X}\right|_8}{h} \tag{2-5}$$

By substituting equations (2-4) into equation (2-5) one obtains

$$\left.\frac{\partial^2 E}{\partial X^2}\right|_0 \approx \frac{E_2 + E_4 - 2E_0}{h^2} \tag{2-6}$$

This is the finite difference equation for $\frac{\partial^2 E}{\partial X^2}$.

In a similar manner $\frac{\partial^2 E}{\partial X^2}$ becomes:

$$\left.\frac{\partial^2 E}{\partial y^2}\right|_0 \approx \frac{E_1 + E_3 - 2E_0}{h^2} \tag{2-7}$$

Combining equations (2-6) and (2-7) one obtains

$$\nabla^2 E_0 = \frac{E_1 + E_2 + E_3 + E_4 - 4E_0}{h^2} \tag{2-8}$$

In equation (2-8) the expression $\frac{E_1+E_2+E_3+E_4-4E_0}{h^2}$ is the finite difference approximation for the Laplacian, $\nabla^2 E$.

As a check on the error (5, p. 52) of this approximation, assume the function E(x,y) can be expanded in a Taylors series. First, check the error in the

approximation for $\frac{\partial^2 E}{\partial X^2}\Big|_o$. Assume E(x,y) to be a function of X with Y constant; one may write

$$E(X_i) = a_o + a_1(X_o - X_i) + a_2(X_o - X_i)^2 + a_3(X_o - X_i)^3 + a_4(X_o - X_i)^4 + \ldots \tag{2-9}$$

Where i = 2,4,. . . .

With $X = X_o$ as the origin, one then obtains

$$a_o = E_o,\ a_1 = \frac{\partial E}{\partial X}\Big|_o,\ 2!a_2 = \frac{\partial^2 E}{\partial X^2}\Big|_o,\ 3!a_3 = \frac{\partial^3 E}{\partial X^3}\Big|_o, \ldots$$

Now letting i = 2 and 4, one obtains:

$$E_2 = a_o + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 + \ldots \tag{2-10}$$

$$E_4 = a_o - a_1 h + a_2 h^2 - a_3 h^3 + a_4 h^4 + \ldots \tag{2-11}$$

Adding equations (2-10) and (2-11),

$$E_2 + E_4 = 2a_o + 2a_2 h^2 + 2a_4 h^4 + \ldots \tag{2-12}$$

Now solving equation (2-12) for a_2, one obtains

$$2!\,a_2 = \frac{\partial^2 E}{\partial X^2}\Big|_o = \frac{E_2 + E_4 - 2E_o}{h^2} - \frac{h^2}{12}\frac{\partial^4 E}{\partial X^4}\Big|_o - \ldots \tag{2-13}$$

Therefore, the error in the finite difference equation (2-6) is given by the algebraic sum of the terms $-\frac{h^2}{12}\frac{\partial^4 E}{\partial X^4}\Big|_o$. . . etc.

By an analogous procedure, one can find an analogous error associated with $\frac{\partial^2 E}{\partial Y^2}\Big|_o$. Hence, the total error in approximating $\nabla^2_{xy} E$ is just twice that assumed

by $\frac{\partial^2 E}{\partial X^2}\Big|_0$ alone.

B. Higher Order Difference Equations

Thus far only first order difference equations have been mentioned. A great deal more accuracy can be accomplished by using higher order difference equations. For example, in an auxiliary method of solving the wave equation for a rectangular wave guide with a grid system of ten pivotal points, the error in determining the lowest characteristic value was 2.26% for the first order difference equation. However, applying the second order difference equation in the same method produced an error of only 0.08%; while a third order difference equation produced a low error of 0.0105%.

The second order difference equations can easily be found by assuming a power series expansion. Given the points of a grid system as shown in Figure 2.2, first, compute $\frac{\partial^2 E}{\partial X^2}$.

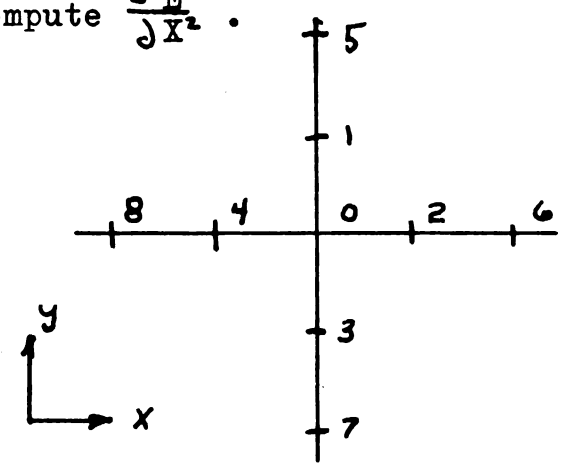

Figure 2.2

Assume that the E field existing at these points is given

as a power series expanded about the center point. i.e.

$$E(X_i) = a_0 + a_1(X_0 - X_i) + a_2(X_0 - X_i)^2 + a_3(X_0 - X_i)^3 + a_4(X_0 - X_i)^2 + \cdots \quad (2\text{-}14)$$

Where i = 2, 4, 6, and 8.

It then follows that

$$E_2 = E_0 + a_1 h + a_2 h^2 + a_3 h^3 + a_4 h^4 \quad (2\text{-}15)$$

$$E_4 = E_0 - a_1 h + a_2 h^2 - a_3 h^3 + a_4 h^4 \quad (2\text{-}16)$$

$$E_6 = E_0 + 2a_1 h + 4a_2 h^2 + 8a_3 h^3 + 16 a_4 h^4 \quad (2\text{-}17)$$

$$E_8 = E_0 - 2a_1 h + 4a_2 h^2 - 8 a_3 h^3 + 16 a_4 h^4 \quad (2\text{-}18)$$

Adding equations (2-15) and (2-16) and adding equations (2-17) and (2-18) it follows that

$$E_2 + E_4 = 2E_0 + 2a_2 h^2 + 2a_4 h^4 \quad (2\text{-}19)$$

$$E_6 + E_8 = 2E_0 + 8a_2 h^2 + 32 a_4 h^4 \quad (2\text{-}20)$$

By multiplying equation (2-19) by sixteen and then subtracting equation (2-20) from equation (2-19), a_4 is eliminated, and the resulting equation can be solved for $2a_2$. Finally one has

$$2a_2 = \frac{\partial^2 E}{\partial X^2}\Big|_0 = \frac{16(E_2 + E_4) - (E_6 + E_8) - 30 E_0}{12 h^2} \quad (2\text{-}21)$$

Equation (2-21) is the second order difference equation for $\frac{\partial^2 E}{\partial X^2}$.

Similarly the second order partial derivative in

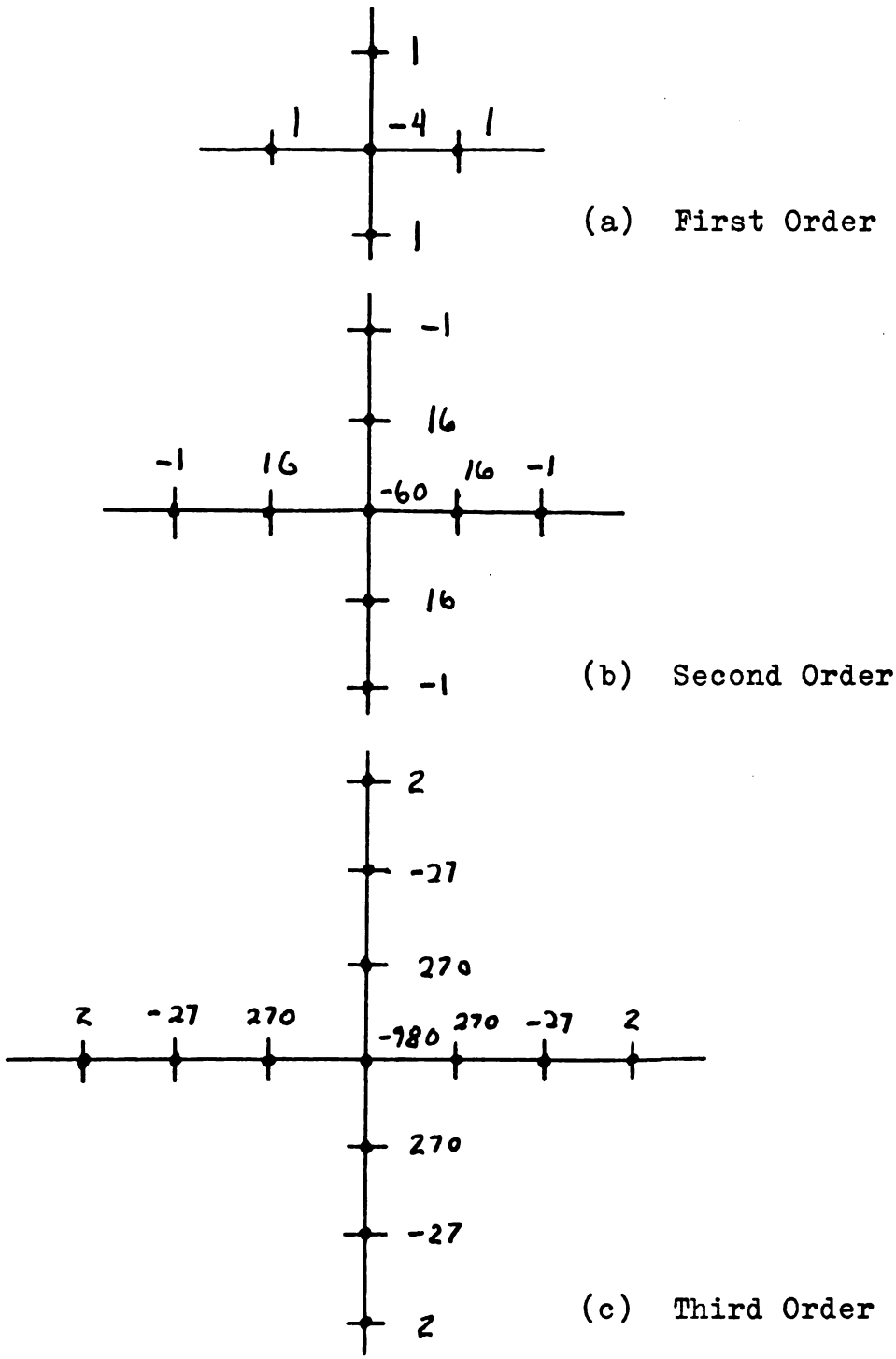
1
1 -4 1
1
(a) First Order
-1
16
-1 16 -60 16 -1
16
-1
(b) Second Order
2
-27
270
2 -27 270 -180 270 -27 2
270
-27
2
(c) Third Order

Figure 2.3

the other direction is

$$\left.\frac{\partial^2 E}{\partial y^2}\right|_o = \frac{16(E_1+E_3)-(E_5+E_7)-30E_o}{12h^2} \qquad (2\text{-}22)$$

Combining equation (2-22) and (2-21) one obtains

$$12\nabla^2 E = \frac{16(E_1+E_2+E_3+E_4)-(E_5+E_6+E_7+E_8)-60E_o}{h^2} \qquad (2\text{-}23)$$

The ∇^2 operator in difference equation form has been derived for the first and second order cases. ∇^2 operators which consider higher orders, points on the diagional, or even three dimensions can be derived in a perfectly analogous manner. Some of these operators (6, p. 170) are shown in Figure 2.3.

C. Special Type Difference Equations

Besides difference equations of higher orders there are many other ways in which a difference expression can be written for the Laplacian. For one thing, it is not absolutely necessary to use a grid system with constant h. One might use a rectangular grid with h as the distance between nodes in the X direction and k as the distance between nodes in the y direction. It would be nice if h could be entirely variable. This would allow fitting a very close grid in regions of high irregularity and allow a very coarse grid in regions where there is little variation of the fields.

Many types of coordinate systems may be used,

for example, a skew rectangular coordinate system or a triangular coordinate system. One other system which does hold a great deal of interest is the cylindrical coordinate system.

The two dimensional Laplacian, to which one is restricted here, in the cylindrical coordinate system is

$$\nabla^2 F = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \phi^2} \qquad (2\text{-}24)$$

The grid system for this equation is shown in Figure 2.4. The difference expressions for the derivatives of equation (2-24) can be derived in a similar manner to those in the rectangular coordinate system (6, p. 224)

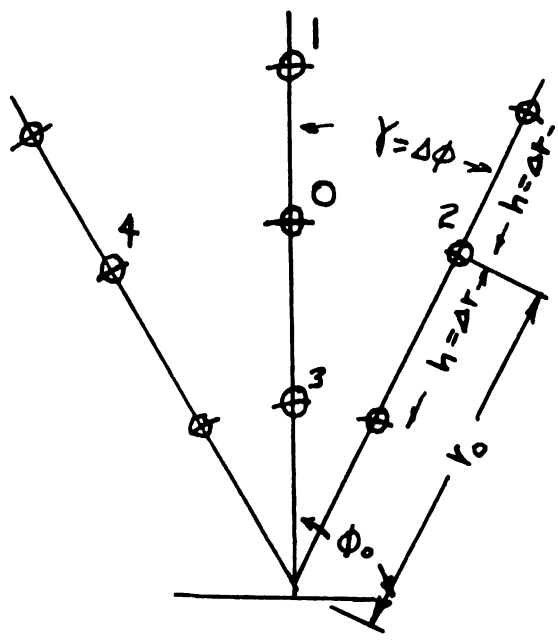

Figure 2.4

and are

$$\frac{\partial^2 E}{\partial r^2} = \frac{E_1 + E_3 - 2E_0}{h^2}$$
$$\frac{\partial E}{\partial r} = \frac{E_1 - E_3}{2h} \qquad (2\text{-}25)$$
$$\frac{\partial^2 E}{\partial \phi^2} = \frac{E_2 + E_4 - 2E_0}{\gamma^2}$$

where $h = \Delta r$, $= \Delta \phi$

Substituting equation (2-25) into equation (2-24) one obtains in cylindrical coordinates.

$$\nabla^2 E_0 = \left(1 + \frac{h}{2r_0}\right)E_1 + \left(\frac{h}{r_0\gamma}\right)^2 E_2 + \left(1 - \frac{h}{r_0\gamma}\right)E_3 + \left(\frac{h}{r_0\gamma}\right)^2 E_4 - 2\left[1 + \left(\frac{h}{r_0\gamma}\right)^2\right]E_0 \qquad (2\text{-}26)$$

where r_0 is the distance from the origin to E_0.

Equation (2-26) is the first order difference equation for the Laplacian $\nabla^2 E$ in a cylindrical coordinate system.

It is entirely possible to write higher order difference equations in cylindrical coordinates or with a variable h or γ. However, the equations become very complex with these increasing variations.

CHAPTER III

METHODS FOR SOLVING THE DIFFERENCE EQUATIONS

A. Matrix Methods

Many methods of solving difference equations are possible, but just a few will be discussed here. The first to be discussed are two matrix methods which were investigated by the author. They were not found too helpful because of the lack of digital computer programs to handle large enough matrices or determinants. But they will be discussed since the only missing link is a computer with sufficient memory. By using a computer with larger memory, solution by these methods may be feasible.

The wave equation, equation (2-1), can be written in matrix form as:

$$\begin{bmatrix} \quad \nabla^2 \quad \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} = -K_c^2 \begin{bmatrix} \quad \mathcal{U} \quad \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix} \qquad (3\text{-}1)$$

In equation (3-1) $\left[\nabla^2\right]$ represents the ∇^2 operator matrix.

Its typical entries can be filled in directly from the difference equation (2-8) and the grid system of the particular cross section of wave guide for which the wave equation is to be solved. Each row of the ∇^2 matrix corresponds to a difference equation for a single node, and the columns correspond to the coefficients of the E values. The $[u]$ is, of course, the unit matrix. The column matrices are n dimensional space vectors representing the E field in the wave guide.

By multiplying K_c^2 times the unit matrix and subtracting the right hand side of the matrix equation from both sides of equation (3-1); one obtains

$$\begin{bmatrix} a_{11}-k_c^2 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-k_c^2 & \cdots & & \vdots \\ a_{31} & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn}-k_c^2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ \vdots \\ \vdots \\ E_n \end{bmatrix} = 0 \qquad (3\text{-}2)$$

This matrix equation has non zero solutions only when the determinant of the coefficient matrix is zero. Obtaining an approximate solution to the wave equation is, then, just a matter of finding the eigenvalues and eigenvectors of the coefficient matrix. Programs for the MISTIC are available to compute eigenvalues and eigenvectors of symmetric matrices up to forty by forty. Once the wave equation is written in this form its

solution follows simply by using these programs.

This method was used by the author to determine the characteristic values of a rectangular wave guide, of aspect ratio two to one, using a grid mesh of ten points. Very good accuracy for the dominant mode was obtained by using second and third order difference equations, as was mentioned in Chapter II.

However, there is an inherent problem. A grid system of more than just a few points, in this case ten, can not be used because the boundary conditions make the matrix asymmetrical even if the boundary of the guide is symmetrical. This is because boundary conditions enter the difference equations for nodes on the boundary of the mesh but not for those equations in the interior of the mesh.

A program to compute the determinant of up to forty by forty asymmetrical matrices is available for MISTIC. This routine could be used by first making an approximation to the value of an eigenvalue. Using this value one could compute the determinant of the matrix. If the approximation were correct, which is highly unlikely, the determinant would be zero. Most likely the determinant would not be zero, so another approximation would be made. Again the determinant would be computed. If this approximation is not correct either, some method would be used to pick a better value

on the basis of the previous two. This process could be continued until a sufficiently accurate answer is obtained.

A program of this nature is conceivable. However, one would still be limited to a grid system of forty points. This is certainly better than ten points, but it is desirable to have as many nodes as possible so as to be able to handle irregular shapes with more accuracy.

B. Relaxation

Since 1938 Sir Richard Southwell and a small group of his followers have been busy developing their numerical method known as relaxation. Relaxation (7) is a method for solving a number of simultaneous equations by successive approximations. It is a very convenient way of solving the difference equations representing the wave equation.

Rather than discuss relaxation for the general difference equation, the discussion here will be limited to the form of difference equation which represents the wave equation. The form of the difference equations for the wave equation is easily obtained. The finite difference expression for the two dimensional Laplacian was given in equation (2-8). Substituting this into the two dimensional wave equation (2-1), one obtains

$$E_1 + E_2 + E_3 + E_4 - 4E_o = -h^2 K_c^2 E_o \qquad (3\text{-}3)$$

Equation (3-3) is the basic difference equation form of the wave equation and will be used later.

The subject of the wave equation will now be left for a moment. The following is strictly about relaxation. For the sake of simplicity in understanding relaxation assume equation (3-3) is instead

$$E_1 + E_2 + E_3 + E_4 - 4E_o = -10 \qquad (3\text{-}4)$$

In equation (3-4) E_o is the value of the field at the central point of a grid system and E_1, E_2, E_3, and E_4 are the values of the fields at nodes about the central point. A simple example using difference equations like equation (3-4) will suffice to show the principle of relaxation.

Assume there are only two nodes in the grid system, as shown in Figure 3.1, and hence only two simultaneous equations. Call the value of the E's at the two points,

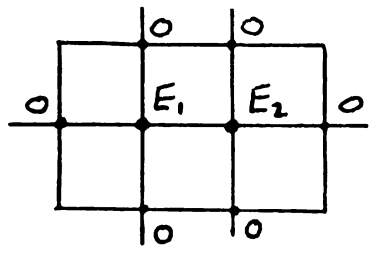

Figure 3.1

E_1 and E_2. If the boundary points are all zero as shown, the two simultaneous equations would be, from equation (3-4)

$$E_2 + 0 + 0 + 0 - 4E_1 = -10$$
$$0 + 0 + E_1 + 0 - 4E_2 = -10 \qquad (3\text{-}5)$$

As the first step in relaxation, rewrite the equations in residual form as

$$E_2 - 4E_1 + 10 = R_1 \qquad (3\text{-}6)$$
$$E_1 - 4E_2 + 10 = R_2 \qquad (3\text{-}7)$$

If one arbitrarily picks values of E_1 and E_2, the left hand side of equation (3-6) determines a value of R_1, which is a measure of the error introduced by these arbitrary values, E_1 and E_2. In the same manner one obtains a value of R_2. R_1 and R_2 are called residuals and are a function of the values E_1 and E_2. Obviously if E_1 and E_2 are both correct, R_1 and R_2 will both be zero. The object of the relaxation method is to reduce the residuals to zero, since the equations are satisfied if the residuals are zero.

This is usually done by changing the node value about which the difference equation is written, so as to decrease the residual to zero. It can be noticed, in larger examples than the one given here, that reducing a residual to zero may not reduce the total

residuals, as they may just spread to adjacent points. This is the case for both points in the example. So the process becomes one of pushing the residuals over the boundaries.

If special methods are not used, convergence is obtained as quickly as is possible by looking for the highest residuals and "relaxing" them before points with lower residuals. Numerous special methods are available for speeding up the process of convergence. But most of them depend upon the judgement and experience of the person applying the relaxation process.

C. Iteration and Eigenvalue Methods

Imagine a great number of equations to be solved by a relaxation process. If the number of equations is large enough, it is conceivable that just finding the equation with the highest residual may pose some problem, at least require a considerable amount of calculation. The iteration process is basically the same as relaxation, except that the iteration process does not search first for the point with the highest residual. It just relaxes each point in order, in a systematic manner. Convergence is still obtained since when a point next to the boundary is relaxed the total residuals are reduced. However, convergence is usually slower.

If equation (3-3) is solved for E_o, one obtains

$$E_o = \frac{E_1 + E_2 + E_3 + E_4}{4 - h^2 K_c^2} \qquad (3\text{-}8)$$

Let $4-h^2K_c^2 = \delta$, equation (3-8) then becomes

$$E_o = \frac{E_1 + E_2 + E_3 + E_4}{\delta} \qquad (3\text{-}9)$$

Equation (3-9) is a simple formula for reducing the residual at any node to zero. This can be seen from equation (3-10) which is equation (3-9) written in

$$R = E_1 + E_2 + E_3 + E_4 - \delta E_o \qquad (3\text{-}10)$$

residual form. To find E_o at any node one merely adds up the surrounding points, E_1, E_2, E_3, and E_4 and divides by δ. This automatically gives a new value of E_o for which the residual is zero.

The iteration process is just a matter of relaxing all points in the grid system one by one in this manner. It can be clearly seen from equation (3-10) that changing a node point by one unit changes the residual of that point by δ while only changing the residual of the surrounding four points by one. Thus, the residuals are either reduced or are spread toward the boundaries where they are eventually reduced.

So far the problem has been simplified. A value of δ has been assumed when actually it is δ which one wishes to find, or rather K_c^2 which differs from δ only by the constant 4. What this means is that the residuals

can not be reduced completely in the above iteration process unless the value of δ is correct.

The complete iteration method will now be described. A grid system is first drawn on the cross section of the wave guide to be solved. The nodes are numbered from one to n. A set of E values are assumed at the nodes and an approximate average for δ is found by the formula (3-11).

$$\delta = \frac{\sum_{k=1}^{n} (E_1 + E_2 + E_3 + E_4)}{\sum_{k=1}^{n} E_0} \qquad (3\text{-}11)$$

where E_0 is the value of E at a node and E_1, E_2, E_3, and E_4 are the points surrounding E_0. This value of δ is then used to compute a whole new set of E's by use of formula (3-9). This should reduce the residuals somewhat. However, further relaxation may not tend to reduce residuals much more. So, after one iteration through all points of the mesh, equation (3-10) is again used to produce a more accurate δ. Then the residuals are reduced again by computing a new set of E's. This process is continued until the value of δ becomes constant within the degree desired.

A proof of the convergence of this method is not available. All that can be said is that it has provided convergence in several problems that have been investigated, and the answers obtained by this method agree

theoretically with other methods.

A heuristic discussion of this convergence is as follows. Suppose one is given a set of node values, and from these values a δ is computed which comes closest to fitting all of the node values. By fitting the node values, is meant that by adding up any four nodes around a central node and dividing them by δ, one obtains a value for the central node as close to the existing value as is possible, for all nodes and that value of δ. This value of δ should be closely approximated by equation (3-11). Now, if the node values are all relaxed by equation (3-9), using this new value of δ, the total residuals will be reduced. Therefore these node values should be closer to the true values. Since these node values are better than the first set it follows that a new δ computed from these node values will be better than the first δ, etc.

CHAPTER IV

DIGITAL COMPUTER PROGRAMS

A. The Iteration Program

This chapter deals specifically with a program, written for MISTIC, to solve the wave equation. The actual program is written as a closed subroutine and is given in Appendix I. The general method used in this program will be given here briefly, and then it will be gone over again in slightly more detail.

The first step in writing the program was to decide on a difference equation. The program uses a first order difference equation written in rectangular coordinates, in particular equation (3-3).

One equation of this form is assumed at each node of a rectangular grid system imposed on the cross section of the wave guide. These equations form a set of simultaneous equations and are solved in this program by the iteration method discussed in Chapter III.

When the eigenvalue has converged to the desired accuracy, it can be printed out along with the values of the field at the nodes. These node values are called the eigenvectors. The number of iterations can also be

printed out if desired.

The program stores the eigenvectors in memory locations 17 through 17+n-1. Locations 17+n through 17+2n-1 are reserved for the interconnection information consecutively, for example, location 17+n contains the information as to how node 1 is connected. The information for the 1's interconnection in Figure 4.1 is punched on tape as 020 04K 000 1S2. Each group is a

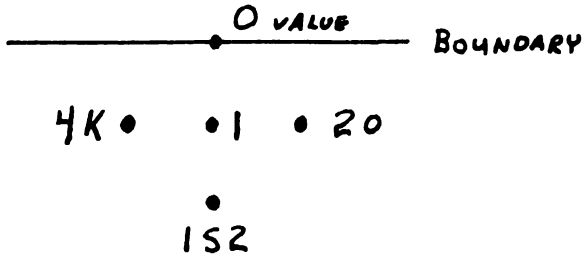

Figure 4.1

three digit sexadecimal number. The order in which the four groups are punched on the tape is irrelevant. Node 0 corresponds to a zero boundary condition. If there were a normal derivative boundary condition rather than a zero boundary condition, the tape would appear as 020 04K 001 1S2 indicating node 1 as its own neighbor.

This information is read in and assembled into a MISTIC word with nine binary bits per node number and stored consecutively starting at location 17+n. This is done by the packing routine and must be done before the main routine is entered. When the packing routine is through, each node has stored in some memory location

the addresses of its surrounding four points. This information is used by the program time and time again in computing both new node values and new values of δ. The packing routine is given in Appendix II.

When the number of nodes is great, writing the interconnections can be quite tedious, especially since they must be written in sexadecimal. Therefore, a program was written to write the interconnection tape automatically.

This program requires four bits of information about the grid system. It requires the number of points in the grid system and the number of horizontal rows of points. It also must be given the number of points in each horizontal row. Finally, it also requires the number one must add to a point in a row to obtain the point below it in the next lower horizontal row. This number is the same for each individual row, but one must be given for each horizontal row. The interconnections are then written assuming zero boundary conditions. With a slight modification the program can be used with other types of boundary conditions. This program is given in Appendix III.

When the iteration program is entered, the first thing it does is to compute a value of δ, using equation (3-11), the initial E values, and the interconnection information. Now that the program has a value of δ to

use it starts one complete iteration, changing the E values, using equation (3-9) and the interconnection information. While it computes new E values it also computes the numerator and denominator of equation (3-11). Then when it has completed an iteration, it has only to divide the sum corresponding to the numerator, by the sum corresponding to the denominator of equation (3-11) in order to obtain the trial eigenvalue for the next iteration. The iteration process can now begin all over again.

After each iteration the absolute value of the new δ is compared with the absolute value of the old δ. When the absolute difference of these two values is less than a predetermined amount called the accuracy constant, the program quits and transfers control back to a master program. Before it transfers control it puts the final eigenvalue in location 16. Location 17 and up contain the correct E values. Thereafter, the print out of this information is easily available.

B. Application of the Program

The values at the nodes may represent any rectangular component of the electric or magnetic field one wishes. However, the boundary conditions are entirely dependent upon which component is used. These boundary conditions then show up in the interconnections. If a

TM mode is to be investigated, the most probable choice of component would be E_z since everywhere on the boundary E_z must be zero no matter what the shape of the cross section. The most probable choice for a TE mode would be H_z. In this case, the normal derivative of H_z must be zero at the boundary. This implies that a node point next to the boundary must have the same value as the point on the boundary.

If the cross section of wave guide has any physical symmetry it is usually possible to solve only a partial region, the other points being filled in by symmetry. The boundary conditions on the new boundary between the regions of symmetry have boundary conditions of the same nature as the actual boundaries. If the field has even symmetry with respect to this boundary, the normal derivative of the field must be zero. If the field has odd symmetry with respect to the boundary, the field itself must be zero along the boundary. By using symmetry it is possible to get considerably more accuracy because more points are concentrated in a smaller area of the total cross section.

Besides the interconnection tape, the program also requires initial values. Of course the closer the initial values are to a normal mode, the faster the convergence. Convergence will be to the mode which comes closest to the initial distribution of E values.

If the initial values are assumed to be identical, convergence is not necessarily to the dominant mode. Consequently, one must use as initial values, values which are thought to approximate the desired mode. Since the lower modes are usually the ones of interest, one usually does have a general idea as to what the fields will look like. If necessary the desired cross section could be slowly changed from some known shape, each time using for the initial E values on a newly perturbed cross section, the final E values for the unperturbed shape.

C. Program Improvements

Most likely the greatest improvement with the least amount of change to the program could be accomplished by using second order or higher difference equations. This would, nevertheless, mean a considerable cut in the maximum number of nodes which could be used. For the second order difference equation eight interconnection addresses would be needed for each node rather than four. It would probably reduce the maximum number of nodes available from about four hundred at present to less than three hundred.

If the cross section to be analyzed was more similar to a circle than to a rectangle, it follows that a similar program written using a cylindrical coordinate

system difference equation would give more accurate results than the present program. This would be due to the fact that the cylindrical grid system would more nearly fit the boundary. In order to get an idea of this type of error, the present program was used to compute the characteristic value of the TM_{01} mode for a circular wave guide. These results are given at the end of this chapter.

More accurate results, using this program could be obtained by using extrapolation. A number of characteristic values could easily be obtained, each from a smaller mesh size. From these values a more accurate characteristic value could be extrapolated.

A more classical method of approximating K_c^2 after each iteration was found after the program had been written. It is called the Rayleigh quotient (7, p. 173) and is supposed to give a good approximation to K_c^2. It is possible that use of the Rayleigh quotient would give faster convergence. It is a weighted average rather than just a simple average. The Rayleigh quotient is shown in equation (4-1).

$$K_c^2 = \frac{-\iint E \nabla^2 E \, dx\, dy}{\iint E^2 \, dx\, dy} \qquad (4\text{-}1)$$

As a summation equation (4-1) becomes equation (4-2).

Obviously, whenever the grid system does not

exactly coincide with the boundary some error will result. This error can be made smaller by using Fox's correction

$$K_c^2 = \frac{-\Sigma E_0 \nabla^2 E_0}{\Sigma E_0^2} \qquad (4\text{-}2)$$

formula (7, p. 65) for points next to a boundary. Fox's formula would be used rather than equation (3-9) and is given in equation (4-3). A typical boundary on which it might be used is shown in Figure 4.2.

$$E_0 = \frac{\frac{2E_1'}{\eta(1+\eta)} + \frac{2E_2'}{\epsilon(1+\epsilon)} + \frac{2E_3}{1+\eta} + \frac{2E_4}{1+\epsilon}}{\frac{2}{\eta} + \frac{2}{\epsilon} - h^2 K_c^2} \qquad (4\text{-}3)$$

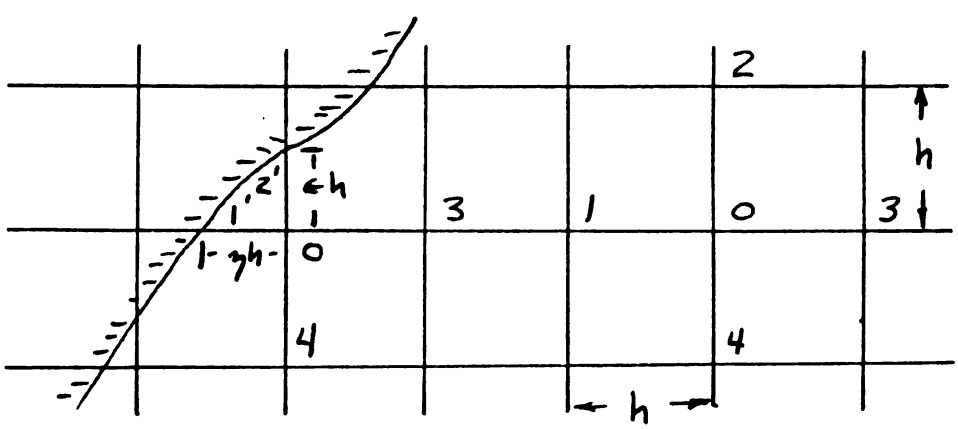

Figure 4.2

The more complicated procedures like this one were not employed because it was hoped to keep the program as simple as possible to allow for the maximum number of nodes.

D. Results

Two results will be discussed. The first shows the accuracy of this method versus other methods and the second shows what type of accuracy can be expected when the boundary is such that it does not fit the grid system very well.

In the first example a simple rectangular wave guide was solved. The dominant mode was investigated by using the E_y component. E_y must be zero on the vertical sides, and the normal derivative of E_y must be zero on the upper and lower boundaries. A grid system of ten points was used as shown in Figure 4.3.

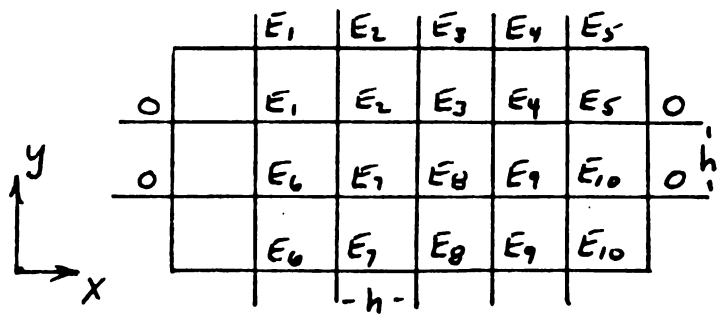

Figure 4.3

The dielectric material is assumed to be free space. The aspect ratio is two to one. The initial values used agreed with the known solution of the TE_{10} mode only in the first digit.

This very same problem was solved using the matrix method of Chapter II. If h is given as one meter, which

is convenient for comparison purposes, the eigenvalue obtained by the matrix method for the TE_{10} mode is $K_c^2 = .26794919$. This value is the exact value of the lowest order eigenvalue for this ten by ten matrix. Any other method should approach this value as the correct one. It is not the correct value for the wave guide, in fact, it is in error by some 2% as was mentioned before, but it is the correct eigenvalue for the ten by ten matrix. Therefore, if the iteration method is as accurate, one should get the same answer.

The eigenvalue obtained by the iteration method, again for the TE_{10} mode, is $K_c^2 = .2679473$. Even greater accuracy is possible since the accuracy constant of the iteration program was not set at its minimum. It is just as well, for this accuracy is much more than is needed. Greater accuracy would only require more computing time. The answer above was obtained in nineteen iterations.

In the first example only ten points were used. In this next example a total of three hundred forty-one points were used.

Since a circular wave guide does not fit a rectangular grid system very well it was felt by trying to solve the circular wave guide one might gain insight into the errors involved. An outline of the grid system placed on the circular wave guide is shown in Figure 4.4. The TM_{01} mode was the mode investigated. The E_z component

was used since it is zero everywhere on the boundary.

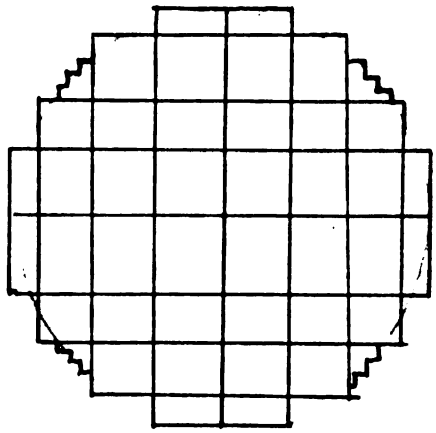

Figure 4.4

The initial values used somewhat approximated a sine wave, with maximum values toward the center and small values toward the edge.

Seventy iterations were required for convergence. This is to be expected since the percentage of boundary points is much lower than in the first example.

In this example the cut-off frequencies will be compared rather than the characteristic values. The cut-off frequency of the TM_{01} mode in a cylindrical wave guide (1, p. 376) of radius a is given in equation (4-4).

$$f_c = \frac{.383}{a\sqrt{\mu\epsilon}} \qquad (4\text{-}4)$$

For a cylindrical wave guide with a radius of 11 cm. the cut-off frequency for the TM_{01} mode is calculated from equation (4-4) to be 2,090 mcs.

The eigenvalue computed by the program was

K_c = .2228. From it the cut-off frequency can be obtained by using equation (1-5). The cut-off frequency as calculated from the iteration program is f_c = 2,130 mcs.

This answer should be more exact for the polygon approximating the circle than for the cylindrical wave guide, but this answer still only differs from the value for the cylindrical wave guide by 1.8%. This is sufficient accuracy for numerous engineering applications. Slightly greater accuracy could be obtained by using four hundred points rather than three hundred forty-one.

CONCLUSION

While the program described here is far from perfect it does describe a very practical method of solving this type of problem. Many improvements for the program have been mentioned.

A battery of three or four programs using different variations could fit mesh points into almost any type of configuration. The accuracy one can obtain is limited only by the ingenuity of the program writer and most important by the size of the digital computer memory. The greater the memory the greater the number of nodes which can be used.

This thesis has dealt exclusively with the two dimensional wave equation. There is no reason why the same procedure can not be extended to the three dimensional wave equation. Again the major problem would be to obtain sufficient memory space. It would then be possible to consider discontinuities in wave guides or wave guides of non-uniform cross section. Finding such things as the resonant frequency of a resonant cavity could be easily done. By using a different type of boundary condition the field patterns of antennas could be studied.

It has been tacitly assumed that the dielectric medium contains no electric charges and no conduction currents. It is possible by modifying the wave equation that even these effects could be studied by this same procedure. Problems of this type would require computers with quite extensive memories, however.

APPENDIX I

THE ITERATION PROGRAM

LOCATION	ORDER	NOTES
0	K5F	Unpack interconnection addresses
	4268L	
1	5017F	
	J080L	
2	S5F	
	109F	
3	468L	
	102F	
4	428L	
	5017F	
5	J081L	
	S5F	
6	469L	
	102F	
7	429L	
	L582L	
8	L4F	Compute first E value using
	L4F	equation (3-9)
9	L4F	
	L4F	

LOCATION	ORDER	NOTES
10	4089L	
	5089L	
11	7J90L	
	5082L	
12	L537L	
	4213L	
13	2213L	
	L5F	
14	4084L	Add to sum representing
	L491L	numerator and denominator of
15	4091L	equation (3-11)
	L5100L	
16	4031L	
	508L	
17	J097L	
	S3F	
18	3619L	
	2620L	
19	F531L	
	4031L	
20	508L	
	J099L	
21	S3F	
	3222L	

LOCATION	ORDER	NOTES
22	2223L	
	F531L	
23	4031L	
	509L	
24	J097L	
	S3F	
25	3626L	
	2627L	
26	F531L	
	4031L	
27	509L	
	J099L	
28	S3F	
	3229L	
29	2230L	
	F531L	
30	4031L	
	L582L	
31	2231L	
	2632L	
32	L484L	
	2633L	
33	L484L	
	2634L	

LOCATION	ORDER	NOTES
34	L484L	
	2635L	
35	L484L	
	L492L	
36	4092L	
	2637L	
37	L584L	
	4017F	
38	2640L	
	00F	
39	00F	
	00F	
40	F54L	Reset to compute second and
	404L	following E values by equation
41	L51L	(3-9)
	L485L	
42	401L	
	F537L	
43	4037L	
	2645L	
44	00F	
	00F	
45	F586L	
	4086L	

LOCATION	ORDER	NOTES
46	L03F	
	3247L	
47	261L	
	2648L	
48	L569L	Reset for new iteration
	4012L	
49	L577L	
	4013L	
50	L54L	
	L03F	
51	404L	
	0020F	
52	461L	
	L587L	
53	2654L	
	00F	
54	4237L	
	L582L	
55	4086L	
	F588L	
56	4088L	
	2657L	
57	5092L	Compute new δ by equation (3-11)
	7J90L	

LOCATION	ORDER	NOTES
58	5082L	
	6691L	
59	S5F	
	4083L	
60	L582L	
	4091L	
61	L582L	
	4092L	
62	L593L	Check convergence of δ
	4094L	
63	L783L	
	4093L	
64	L593L	
	L094L	
65	4095L	
	L795L	
66	L096L	
	361L	
67	2267L	
	L583L	
68	4016F	
	22F	
69	6683L	Program constants
	S5F	

LOCATION	ORDER	NOTES
70	00F00F	
71	00F00F	
72	00F00F	
73	00F00F	
74	00F00F	
75	00F00F	
76	00F00F	
77	2614L00F	
78	00F00F	
79	00F00F	
80	3L3584FLL2048F	
81	00511F002044F	
82	00F00F	
83	00F00400000000000J	(Location of δ)
84	00F00F	
85	001F00F	
86	00F00F	(Counts E's computed during one iteration)
87	0017F0017F	
88	00F00F	(iteration count)
89	00F00F	
90	00F00100000000000J	
91	00F00F	
92	00F00F	
93	00F00F	

LOCATION	ORDER	NOTES
94	00F00F	
95	00F00F	
96	00F00100000J	(Accuracy constant)
97	004095F00F	
98	00F00F	
99	00F004095F	
100	2231L2632L	

The following is required before entering the routine:

1. n must be in location 3
2. n must be added to the left hand order in cell 1L
3. location zero (0F) must be cleared
4. n initial conditions must be stored in location 17F through (17 + n - 1) F
5. n interconnections must be stored in locations (17 + n) F through (17 + 2n - 1) F

where n = the number of nodes in the grid system

APPENDIX II

PACKING ROUTINE

LOCATION	ORDER	NOTES
0	K5F	Pack one word
	4215L	
1	5124L	
	8012F	
2	0029F	
	L417L	
3	4017L	
	L52L	
4	L018L	
	402L	
5	L520L	
	L023L	
6	4020L	
	361L	
7	L517L	Reset for following words
	4017F	
8	L524L	
	4017L	

LOCATION	ORDER	NOTES
9	F57L	
	407L	
10	L52L	
	L422L	
11	402L	
	L516L	
12	4020L	
	F521L	
13	4021L	
	L03F	
14	3215L	
	2615L	
15	261L	
	22F	
16	00F003F	Program constants
17	00F00F	
18	009F00F	
19	00F00F	
20	00F003F	
21	00F00F	
22	0036F00F	
23	00F001F	
24	00F00F	

The following is required before entering the sub-routine:

1. n must be stored in 3F
2. 7L must have n added to it before entering
3. n sets of interconnections as shown on page 32 must be in the reader

where n = the number of nodes in the grid system

APPENDIX III

INTERCONNECTION PROGRAM

LOCATION	ORDER	NOTES
0	L598FL486L	
1	4086L262L	
2	F588L40100F	
3	4088LL086L	
4	3615LF52L	
5	402L262L	
6	00F00F	
7	00F00F	
8	00F00F	
9	00F00F	
10	00F00F	
11	00F00F	
12	00F00F	
13	00F00F	
14	00F00F	
15	F589L4089L	
16	L0600F3622L	
17	L5101F2226L	
18	4090LL589L	

LOCATION	ORDER	NOTES
19	L087L3624L	
20	L591LL490L	
21	0016F2225L	
22	L591L2628L	
23	4090L2218L	
24	L599FL490L	
25	0016F8224F	
26	2630L5091L	
27	0012F2618L	
28	5091L0012F	
29	2623L00F	
30	L5700FL089L	
31	3633LL591L	
32	4090L2640L	
33	L5700FL089L	
34	L091L3231L	
35	L592LL0700F	
36	4237L2237L	
37	00FL5F	
38	0012F4090L	
39	2640L00F	
40	L5600FL089L	
41	L0701F3648L	
42	L5601FL4600F	
43	L089LL0701F	

LOCATION	ORDER	NOTES
44	3645L2648L	
45	L592LL4701F	
46	0020F4647L	
47	L5F2649L	
48	2248LL591L	
49	L490L4090L	
50	L590L0016F	
51	8224F92131F	
52	2653L00F	
53	F592LL493L	
54	4092LL517L	
55	L493L4017L	
56	L524LL493L	
57	4024LF594L	
58	4094LL0600F	
59	3664L2615L	
60	00F00F	
61	00F00F	
62	00F00F	
63	00F00F	
64	F558L4058L	
65	4242L0020F	
66	4616L4640L	
67	L093L4634L	
68	F558L0020F	

LOCATION	ORDER	NOTES
69	4642LL530L	
70	L493L4030L	
71	4633LF543L	
72	4043L4245L	
73	0020F4641L	
74	L591L4089L	
75	4094LF535L	
76	4035LF562L	
77	4062LL099F	
78	3679L2615L	
79	0FF00F	
80	00F00F	
81	00F00F	
82	00F00F	
83	00F00F	
84	00F00F	
85	00F00F	
86	00F0016F	
87	00F002F	
88	00F0016F	
89	00F00F	
90	00F00F	
91	00F00F	
92	L5100FL5100F	
93	001F00F	

LOCATION	ORDER	NOTES
94	00F00F	

The following specification tape must immediately follow the program:

0098K	
00F00nF	where n = total number of nodes
00F00aF	where a = total number of horizontal rows
00600K	
00F00b_1F	where b_a = number of points in
00F00b_2F	horizontal row number a
⋮	
00F00b_aF	
00700K	
00F00F	where X_a = the number one must
00F00X_1F	add to a point in row a to
00F00X_2F	obtain the point below it in row
⋮	(a + 1).
00F00X_{a-1}F	
00F00F	
243N	

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