THE DIFFERENTIAL GEOMETRY OF A GENERAL SURFACE IN S_4

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CONTENTS

THE DIFFERENTIAL GEOMETRY OF A GENERAL SURFACE IN S_4

I. INTRODUCTION

The purpose of this paper will be to study a general surface in a space of four dimensions by means of an orthogonal net upon it. We first set up a defining system of partial differential equations. Associated with each point of the surface is a unique plane containing all of the normals to the surface. We define certain unique normals and pairs of normals to the surface and characterize them geometrically. For this purpose we study the sustaining surfaces of the orthogonal projections of the given net onto certain geometrically defined spaces of three dimensions. We call these surfaces <u>normal projection surfaces</u>.* A normal determines a unique normal projection surface. Among the normal projection surfaces there are two, one possessing maximum total curvature, the other minimum total curvature. The normals determining these particular projection surfaces are perpendicular. They are called the

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* V. G. Grove, <u>Differential Geometry of a Certain Surface in S_4</u>, Transactions of the American Mathematical Society, Vol. 39 (1936), p. 62. Hereafter referred to as Grove, <u>Geometry</u>.

principal normals.* A necessary and sufficient condition that the given surface be immersed in three dimensions is deduced, followed by theorems relating to change of net upon the surface.

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* Grove, Geometry, p. 64.

II. THE DEFINING DIFFERENTIAL EQUATIONS

Let the parametric equations of the given surface S_x be

$$x_i = x_i(u,v) \qquad\qquad i = 1, 2, 3, 4.$$

We assume that the non-homogeneous cartesian coordinates (x_1, x_2, x_3, x_4) of a point P_x on S_x are analytic functions of the two variables (u, v) and that it is not possible to express the x_i as functions of a single variable.

The square of the element of arc for a curve C lying upon the given surface is given by

$$(1) \qquad ds^2 = \sum_{i=1}^{4} dx_i^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 ,$$

where the <u>first fundamental coefficients</u> E, F, G are defined by the relations

$$(2) \qquad E = \sum x_u^2 , \quad F = \sum x_u x_v , \quad G = \sum x_v^2 .$$

Let the curves of the given orthogonal net N_x be taken as the parametric curves. We will then have

$$(3) \qquad F = \sum x_u x_v = 0 .$$

We shall call the plane containing all of the normals to S_x at x <u>the normal plane</u> to S_x at x. Select in the normal plane two perpendicular lines λ and μ with direction cosines $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\mu_1, \mu_2, \mu_3, \mu_4)$ respectively. It follows that the functions λ and μ satisfy the equations

$$\sum \lambda X_u = 0\,, \quad \sum \lambda X_v = 0\,, \quad \sum \mu X_u = 0\,, \quad \sum \mu X_v = 0\,,$$

(4)

$$\sum \lambda^2 = 1\,, \quad \sum \mu^2 = 1\,, \quad \sum \lambda\mu = 0\,.$$

The <u>second fundamental coefficients</u> D_1, D_2, D_1', D_2', D_1'', D_2'' are defined by

$$D_1 = \sum \lambda X_{uu}\,, \quad D_1' = \sum \lambda X_{uv}\,, \quad D_1'' = \sum \lambda X_{vv}$$

(5)

$$D_2 = \sum \mu X_{uu}\,, \quad D_2' = \sum \mu X_{uv}\,, \quad D_2'' = \sum \mu X_{vv}\,.$$

From equations (2), (3), (4), (5), we obtain the following relationships

(a) $\sum X_u X_{uv} = \frac{1}{2}E_v$, $\sum X_v X_{uv} = \frac{1}{2}G_u$, $\sum \lambda X_{uv} = D_1'$, $\sum \mu X_{uv} = D_2'$,

(b) $\sum X_u X_{uu} = \frac{1}{2}E_u$, $\sum X_v X_{uu} = -\frac{1}{2}E_v$, $\sum \lambda X_{uu} = D_1$, $\sum \mu X_{uu} = D_2$,

(c) $\sum X_u X_{vv} = -\frac{1}{2}G_u$, $\sum X_v X_{vv} = \frac{1}{2}G_v$, $\sum \lambda X_{vv} = D_1''$, $\sum \mu X_{vv} = D_2''$,

(6) (d) $\sum \lambda \lambda_u = 0$, $\sum X_u \lambda_u = -D_1$, $\sum X_v \lambda_u = -D_1'$, $\sum \mu \lambda_u = -\sum \lambda \mu_u = -A_2$,

(e) $\sum \lambda \lambda_v = 0$, $\sum X_v \lambda_v = -D_1''$, $\sum X_u \lambda_v = -D_1'$, $\sum \mu \lambda_v = -\sum \lambda \mu_v = -B_2$,

(f) $\sum \mu \mu_u = 0$, $\sum X_u \mu_u = -D_2$, $\sum X_v \mu_u = -D_2'$, $\sum \lambda \mu_u = -\sum \mu \lambda_u = -A_1$,

(g) $\sum \mu \mu_v = 0$, $\sum X_v \mu_v = -D_2''$, $\sum X_u \mu_v = -D_2'$, $\sum \lambda \mu_v = -\sum \mu \lambda_v = -B_1$.

The relationships (6) constitute seven sets of four linear equations each, in the unknowns X_{uu}, X_{uv}, X_{vv}, λ_u, λ_v, μ_u and μ_v respectively. The determinant of the coefficients in each of the seven sets is the same, namely

$$(7) \qquad H = \begin{vmatrix} x_{1u}, & x_{2u}, & x_{3u}, & x_{4u} \\ x_{1v}, & x_{2v}, & x_{3v}, & x_{4v} \\ \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \end{vmatrix}$$

From equations (2), (4), (7), we may write

$$(8) \quad H^2 = \begin{vmatrix} x_{1u}, & x_{2u}, & x_{3u}, & x_{4u} \\ x_{1v}, & x_{2v}, & x_{3v}, & x_{4v} \\ \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \end{vmatrix} \cdot \begin{vmatrix} x_{1u}, & x_{1v}, & \lambda_1, & \mu_1 \\ x_{2u}, & x_{2v}, & \lambda_2, & \mu_2 \\ x_{3u}, & x_{3v}, & \lambda_3, & \mu_3 \\ x_{4u}, & x_{4v}, & \lambda_4, & \mu_4 \end{vmatrix} = \begin{vmatrix} E & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = EG \neq 0.$$

Since, for a real surface with real values for u, v, the determinant of the coefficients H is not zero, we may solve uniquely for the unknowns $X_{uu}, X_{uv}, X_{vv}, \lambda_u, \lambda_v, \mu_u, \mu_v$ to obtain the system of defining differential equations for the surface S_x, namely,

$$x_{uu}=\alpha x_u+\beta x_v+D_1\lambda+D_2\mu\ ,\quad \lambda_u=m_1x_u+p_1x_v+A_1\mu\ ,$$

$$(9)\qquad x_{uv}=a x_u+b x_v+D_1'\lambda+D_2'\mu\ ,\quad \lambda_v=q_1x_u+n_1x_v+B_1\mu\ ,$$

$$x_{vv}=\gamma x_u+\delta x_v+D_1''\lambda+D_2''\mu\ ,\quad \mu_u=m_2x_u+p_2x_v+A_2\lambda\ ,$$

$$\mu_v=q_2x_u+n_2x_v+B_2\lambda\ ,$$

wherein

$$\alpha=\frac{1}{2}\frac{E_u}{E}\ ,\ \beta=-\frac{1}{2}\frac{E_v}{G}\ ,\quad D_1=\sum\lambda x_{uu}\ ,\quad D_2=\sum\mu x_{uu}\ ,$$

$$a=\frac{1}{2}\frac{E_v}{E}\ ,\ b=\frac{1}{2}\frac{G_u}{G}\ ,\quad D_1'=\sum\lambda x_{uv}\ ,\quad D_2'=\sum\mu x_{uv}\ ,$$

$$(10)\qquad \gamma=-\frac{1}{2}\frac{G_u}{E}\ ,\ \delta=\frac{1}{2}\frac{G_v}{G}\ ,\quad D_1''=\sum\lambda x_{vv}\ ,\quad D_2''=\sum\mu x_{vv}\ ,$$

$$m_1=-\frac{D_1}{E}\ ,\ m_2=-\frac{D_2}{E}\ ,\quad n_1=-\frac{D_1''}{G}\ ,\quad n_2=-\frac{D_2''}{G}\ ,$$

$$p_1=-\frac{D_1'}{G}\ ,\ p_2=-\frac{D_2'}{G}\ ,\quad q_1=-\frac{D_1'}{E}\ ,\quad q_2=-\frac{D_2'}{E}\ .$$

The equations (9) represent twenty-eight equations since x itself represents (x_1, x_2, x_3, x_4). The derivation of the equation: $x_{1uu}=\alpha x_{1u}+\beta x_{1v}+D_1\lambda_1+D_2\mu_1$ is given below, the other twenty-seven being derived similarly.

Referring to (7), let us multiply the first column of H by x_{1u}, and add to its elements the products of the elements of the second, third, and fourth columns by x_{2u}, x_{3u}, and x_{4u}

respectively. We obtain

$$(11)\qquad H\cdot X_{1u}=\begin{vmatrix} E, & X_{2u}, & X_{3u}, & X_{4u} \\ 0, & X_{2v}, & X_{3v}, & X_{4v} \\ 0, & \lambda_2, & \lambda_3, & \lambda_4 \\ 0, & \mu_2, & \mu_3, & \mu_4 \end{vmatrix} = E\cdot\begin{vmatrix} X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix}\ .$$

Hence

$$(12)\qquad \begin{vmatrix} X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix} = H\cdot\frac{X_{1u}}{E}\ .$$

By precisely the same process it is easily shown that

$$(13)\qquad \begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix} = -H\,\frac{X_{1v}}{G}\,,\quad \begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ X_{2v}, & X_{3v}, & X_{4v} \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix} = H\lambda_1\,,\quad \begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_2, & \lambda_3, & \lambda_4 \end{vmatrix} = -H\mu_1\ .$$

Formally solving equations (6b) for X_{1u}, we obtain

$$(14)\qquad X_{1u}=\frac{\begin{vmatrix} \frac{1}{2}E_u, & X_{2u}, & X_{3u}, & X_{4u} \\ -\frac{1}{2}E_v, & X_{2v}, & X_{3v}, & X_{4v} \\ D_1, & \lambda_2, & \lambda_3, & \lambda_4 \\ D_2, & \mu_2, & \mu_3, & \mu_4 \end{vmatrix}}{\begin{vmatrix} X_{1u}, & X_{2u}, & X_{3u}, & X_{4u} \\ X_{1v}, & X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_1, & \mu_2, & \mu_3, & \mu_4 \end{vmatrix}} = \frac{\frac{1}{2}E_u\begin{vmatrix} X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix} + \frac{1}{2}E_v\begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ \lambda_2, & \lambda_3, & \lambda_4 \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix}}{H} +$$

$$+\ \frac{D_1\cdot\begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ X_{2v}, & X_{3v}, & X_{4v} \\ \mu_2, & \mu_3, & \mu_4 \end{vmatrix} - D_2\begin{vmatrix} X_{2u}, & X_{3u}, & X_{4u} \\ X_{2v}, & X_{3v}, & X_{4v} \\ \lambda_2, & \lambda_3, & \lambda_4 \end{vmatrix}}{H}\ .$$

Substituting from (12) and (13) into the right member of (14), gives

$$(15)\qquad X_{1uu} = \frac{1}{2}\frac{E_u}{E}\cdot X_{1u} - \frac{1}{2}\frac{E_v}{G}\cdot X_{1v} + D_1\lambda_1 + D_2\mu_1 \ .$$

If the notations in (10) be used equation (15) may be written in the form

$$X_{1uu} = \alpha X_{1u} + \beta X_{1v} + D_1\lambda_1 + D_2\mu_2 \ .$$

III. INTEGRABILITY CONDITIONS

The coefficients in system (9) are not independent, but satisfy certain integrability conditions. We must have, for example

$$(16)\qquad \begin{aligned} &(a) \quad X_{uuv} = X_{uvu} \ , \\ &(b) \quad X_{vvu} = X_{uvv} \ . \end{aligned}$$

With the aid of equations (9), (16) may be written

$$(17)\qquad \begin{aligned}
&(a)\ X_{uuv} = (\alpha_v + a\alpha + \beta\gamma + D_1 g_1 + D_2 q_2) X_u + (\alpha b + \beta_v + \beta\delta + D_1 n_1 + D_2 n_2) X_v \\
&\qquad + (\alpha D_1' + \beta D_1'' + D_{1v} + B_2 D_2)\lambda + (\alpha D_2' + \beta D_2'' + B_1 D_1 + D_{2v})\mu \ , \\
&(b)\ X_{uvu} = (a_u + a\alpha + ab + D_1' m_1 + D_2' m_2) X_u + (a\beta + b_u + b^2 + D_1' p_1 + D_2' p_2) X_v \\
&\qquad + (a D_1 + b D_1' + D_{1u}' + A_2 D_2')\lambda + (a D_2 + b D_2' + D_{2u}' + A_1 D_1')\mu \ , \\
&(c)\ X_{vvu} = (\gamma_u + \alpha\gamma + a\delta + D_1'' m_1 + D_2'' m_2) X_u + (\beta\gamma + \delta_u + b\delta + D_1'' p_1 + D_2'' p_2) X_v \\
&\qquad + (\gamma D_1 + \delta D_1' + D_{1u}'' + A_2 D_2'')\lambda + (\gamma D_2 + \delta D_2' + D_{2u}'' + A_1 D_1'')\mu \\
&(d)\ X_{uvv} = (a_v + a^2 + b\gamma + D_1' q_1 + D_2' q_2) X_u + (ab + b_v + b\delta + D_1' n_1 + D_2' n_2) X_v \\
&\qquad + (a D_1' + b D_1'' + D_{1v}' + B_2 D_2')\lambda + (a D_2' + b D_2'' + D_{2v}' + B_1 D_1')\mu \ .
\end{aligned}$$

Equating coefficients of X_u, X_v, λ, and μ, first in (a) and (b), then in (c) and (d) we get:

$$
(18)\quad
\begin{aligned}
&(a)\quad G_u\left(\frac{E_u}{E}+\frac{G_u}{G}\right)+E_v\left(\frac{E_v}{E}+\frac{G_v}{G}\right)=2\left(E_{vv}+G_{uu}\right)+4\left(D_1D_1''+D_2D_2''-D_1'^2-D_2'^2\right),\\
&(b)\quad \tfrac{1}{2}E_v\left(\frac{D_1}{E}+\frac{D_1''}{G}\right)=\tfrac{1}{2}D_1'\left(\frac{E_u}{E}-\frac{G_u}{G}\right)+D_{1v}+B_2D_2-A_2D_2'-D_{1u}' \quad ,\\
&(c)\quad \tfrac{1}{2}E_v\left(\frac{D_2}{E}+\frac{D_2''}{G}\right)=\tfrac{1}{2}D_2'\left(\frac{E_u}{E}-\frac{G_u}{G}\right)+D_{2v}+B_1D_1-A_1D_1'-D_{2u}' \quad ,\\
&(d)\quad \tfrac{1}{2}G_u\left(\frac{D_1}{E}+\frac{D_1''}{G}\right)=D_{1u}''+A_2D_2''-\tfrac{1}{2}\left(\frac{E_v}{E}-\frac{G_v}{G}\right)D_1'-D_{1v}'-B_2D_2' \quad ,\\
&(e)\quad \tfrac{1}{2}G_u\left(\frac{D_2}{E}+\frac{D_2''}{G}\right)=D_{2u}''+A_1D_1''-\tfrac{1}{2}\left(\frac{E_v}{E}-\frac{G_v}{G}\right)D_2'-D_{2v}'-B_1D_1' \quad .
\end{aligned}
$$

In addition to equation (16), there is also the condition

$$
(19)\qquad \lambda_{uv} = \lambda_{vu} \quad .
$$

Using equations (9), we may write (19) as follows:

$$
(20)\quad
\begin{aligned}
\lambda_{uv} &= \left(m_{1v}+a m_1+p_1\gamma+A_1q_2\right)X_u+\left(p_{1v}+b m_1+p_1\delta+A_1n_2\right)X_v\\
&\quad +\left(D_1'm_1+D_1''p_1+A_1B_2\right)\lambda+\left(D_2'm_1+D_2''p_1+A_{1v}\right)\mu ,\\
\lambda_{vu} &= \left(q_{1u}+\alpha q_1+n_1a+B_1m_2\right)X_u+\left(n_{1u}+\beta q_1+n_1b+B_1p_2\right)X_v\\
&\quad +\left(D_1q_1+D_1'n_1+A_2B_1\right)\lambda+\left(D_2q_1+D_2'n_1+B_{1u}\right)\mu \quad .
\end{aligned}
$$

The only new integrability condition obtained is found by equating coefficients of μ, namely,

$$
(21)\qquad A_{1v}-B_{1u}=\left(\frac{D_1}{E}-\frac{D_1''}{G}\right)D_2'-\left(\frac{D_2}{E}-\frac{D_2''}{G}\right)D_1' \quad .
$$

Also from relations (6 d,e,f,g) is obtained the set of conditions

$$
(22)\qquad A_1+A_2=0 \quad , \quad B_1+B_2=0 \quad .
$$

Equations (18), (21), (22) constitute the integrability conditions for the system of equations (9).

IV. POWER SERIES EXPANSIONS FOR THE SURFACE

If we use the tangent lines to C_u and to C_v, and the lines λ and μ for the axes of a local system of reference, we find that the coordinates of a point $\underline{y}$ with general coordinates (y_1, y_2, y_3, y_4) will have local coordinates $(\xi_1, \xi_2, \xi_3, \xi_4)$ defined by the expression

$$y = x + \frac{\xi_1 x_u}{\sqrt{E}} + \frac{\xi_2 x_v}{\sqrt{G}} + \xi_3 \lambda + \xi_4 \mu . \tag{23}$$

Let $\underline{y}$ be a point on the surface S_x with curvilinear coordinates $(u+\Delta u, v+\Delta v)$ where (u,v) are the curvilinear coordinates of x. The coordinates of $\underline{y}$ are of the form

$$y = x + x_u \Delta u + x_v \Delta v + \frac{1}{2}\left(x_{uu}\Delta u^2 + 2x_{uv}\Delta u \Delta v + x_{vv}\Delta v^2\right) + \cdots . \tag{24}$$

If use be made of equations (9), (24) may be written

$$\begin{aligned} y = x &+ \left\{\Delta u + \tfrac{1}{2}(\alpha \Delta u^2 + 2a\,\Delta u\,\Delta v + \gamma \Delta v^2) + \cdots\right\} x_u + \left\{\Delta v + \tfrac{1}{2}(\beta \Delta u^2 + 2b\,\Delta u\,\Delta v + \delta \Delta v^2) + \cdots\right\} x_v \\ &+ \left\{\tfrac{1}{2}(D_1 \Delta u^2 + 2D_1' \Delta u\,\Delta v + D_1'' \Delta v^2) + \cdots\right\} \lambda + \left\{\tfrac{1}{2}(D_2 \Delta u^2 + 2D_2' \Delta u\,\Delta v + D_2'' \Delta v^2) + \cdots\right\} \mu . \end{aligned} \tag{25}$$

Hence the local coordinates of $\underline{y}$ are

$$\begin{aligned} \xi_1 &= \sqrt{E}\left[\Delta u + \tfrac{1}{2}(\alpha \Delta u^2 + 2a\,\Delta u\,\Delta v + \gamma \Delta v^2) + \cdots\right] , \\ \xi_2 &= \sqrt{G}\left[\Delta v + \tfrac{1}{2}(\beta \Delta u^2 + 2b\,\Delta u\,\Delta v + \delta \Delta v^2) + \cdots\right] , \\ \xi_3 &= \tfrac{1}{2}\left[D_1 \Delta u^2 + 2D_1' \Delta u\,\Delta v + D_1'' \Delta v^2\right] + \cdots , \\ \xi_4 &= \tfrac{1}{2}\left[D_2 \Delta u^2 + 2D_2' \Delta u\,\Delta v + D_2'' \Delta v^2\right] + \cdots . \end{aligned} \tag{26}$$

We may express ξ_3 and ξ_4 each as power series in ξ_1 and ξ_2 obtaining

$$(27)\qquad \begin{aligned} \xi_3 &= \tfrac{1}{2}\cdot \frac{D_1}{E}\,\xi_1^2 + \frac{D_1'}{\sqrt{EG}}\,\xi_1\xi_2 + \tfrac{1}{2}\frac{D_1''}{G}\,\xi_2^2 + \cdots\cdots, \\ \xi_4 &= \tfrac{1}{2}\frac{D_2}{E}\,\xi_1^2 + \frac{D_2'}{\sqrt{EG}}\,\xi_1\xi_2 + \tfrac{1}{2}\frac{D_2''}{G}\,\xi_2^2 + \cdots\cdots . \end{aligned}$$

Equations (27) may be interpreted as follows. The first equation and $\xi_4 = 0$ are the equations of the sustaining surface of the orthogonal projection of the given net onto the S_3 determined by the tangent plane and the normal λ. A similar statement holds for the second equation with $\xi_3 = 0$. We shall call these surfaces <u>the normal projection surfaces of</u> S_x <u>determined by</u> λ <u>and</u> μ respectively, and shall denote them by S_λ and S_μ respectively.

Let us denote the first fundamental coefficients for the surface S_λ by $E_\lambda, F_\lambda, G_\lambda$. Similarly, let us represent the second fundamental coefficients by D_λ, D_λ', D_λ''. The equations defining S_λ may be written

$$(28)\qquad \begin{aligned} &\xi_1 = u, \quad \xi_2 = v, \quad \xi_4 = 0, \\ &\xi_3 = \tfrac{1}{2}\frac{D_1}{E}u^2 + \frac{D_1'}{\sqrt{EG}}uv + \tfrac{1}{2}\frac{D_1''}{G}v^2 + \cdots\cdots . \end{aligned}$$

Hence we may write

$$(29)\qquad \begin{aligned} &E_\lambda = \sum \xi_u^2 = 1 + \left(\frac{D_1}{E}\right)^2 u^2 + \cdots, \quad F_\lambda = \sum \xi_u \xi_v = \frac{D_1}{E}\cdot\frac{D_1'}{\sqrt{EG}}\,uv + \cdots\cdots, \\ &G_\lambda = \sum \xi_v^2 = 1 + \frac{D_1'^2}{EG}u^2 + \frac{D_1''^2}{G^2}v^2 + \cdots\cdots . \end{aligned}$$

<u>At the point</u> P_x with local coordinates $(0,0,0,0)$, $E_\lambda = 1$, $F_\lambda = 0$, $G_\lambda = 1$. Also, at the point P_x, we have

$$\xi_{1uu} = 0, \quad \xi_{2uu} = 0, \quad \xi_{3uu} = \frac{D_1}{E}, \quad \xi_{1u} = 1, \quad \xi_{2u} = 0, \quad \xi_{3u} = 0,$$

$$(30) \qquad \xi_{1vu} = 0, \quad \xi_{2uv} = 0, \quad \xi_{3uv} = \frac{D_1'}{\sqrt{EG}}, \quad \xi_{1v} = 0, \quad \xi_{2v} = 1, \quad \xi_{3v} = 0,$$

$$\xi_{1vv} = 0, \quad \xi_{2vv} = 0, \quad \xi_{3vv} = \frac{D_1''}{G}, \quad H_\lambda = \sqrt{E_\lambda G_\lambda - F_\lambda^2} = 1.$$

The second fundamental coefficients* have therefore the following values at the point P_x:

$$(31) \qquad D_\lambda = \frac{1}{H_\lambda} \begin{vmatrix} \xi_{1uu}, & \xi_{2uu}, & \xi_{3uu} \\ \xi_{1u}, & \xi_{2u}, & \xi_{3u} \\ \xi_{1v}, & \xi_{2v}, & \xi_{3v} \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 0 & \frac{D_1}{E} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \frac{D_1}{E}.$$

Similarly, we see that

$$D_\lambda' = \frac{D_1'}{\sqrt{EG}}, \qquad D_\lambda'' = \frac{D_1''}{G}.$$

The principal radii of normal curvature R_m at P for the surface S_λ are roots of**

$$(32) \quad (D_\lambda D_\lambda'' - D_\lambda'^2) R_m^2 - (E_\lambda D_\lambda'' - 2F_\lambda D_\lambda' + G_\lambda D_\lambda) R_m + E_\lambda G_\lambda - F_\lambda^2 = 0.$$

Using (29), (31), equation (32) becomes

$$\left(\frac{D_1}{E} \cdot \frac{D_1''}{G} - \frac{D_1'^2}{EG} \right) R_m^2 - \left(\frac{D_1}{E} + \frac{D_1''}{G} \right) R_m + 1 = 0,$$

or

$$(33) \qquad (D_1 D_1'' - D_1'^2) R_m^2 - (E D_1'' + G D_1) R_m + EG = 0.$$

In the same way we may show that the principal radii of normal curvature R_m for the surface S_μ at the point P_x are given as roots of

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* L. P. Eisenhart, Differential Geometry of Curves and Surfaces, New York, Ginn and Company, 1909, p. 115. Hereafter referred to as Eisenhart, Geometry.

** Eisenhart, Geometry, p. 120.

$$(34)\quad (D_2 D_2'' - D_2'^2) R_m^2 - (E D_2'' + G D_2) R_m + EG = 0 \ .$$

The total curvatures of S_λ and S_μ are respectively

$$(35)\qquad K_1 = \frac{D_1 D_1'' - D_1'^2}{EG} \ , \qquad K_2 = \frac{D_2 D_2'' - D_2'^2}{EG}$$

and the mean curvatures are respectively

$$(36)\qquad M_1 = \frac{D_1}{E} + \frac{D_1''}{G} \ , \qquad M_2 = \frac{D_2}{E} + \frac{D_2''}{G} \ .$$

The sum of the total curvatures of S_λ and S_μ is

$$(37)\qquad K_1 + K_2 = \frac{D_1 D_1'' + D_2 D_2'' - D_1'^2 - D_2'^2}{EG} \ .$$

Referring to integrability condition (18a) it is readily seen that the sum $K_1 + K_2$ in (37) may be expressed entirely in terms of the first fundamental coeffients E, G and their partial derivatives, which have a constant value at the point P_x. Hence we may state that <u>the sum of the total curvatures of the normal projection surfaces determined by any two perpendicular normals is a constant at a point of the surface</u>.

V. A CANONICAL FORM OF THE DEFINING DIFFERENTIAL EQUATIONS

Let us now make the transformation

$$(38)\qquad \begin{aligned} \lambda &= A\bar{\lambda} + B\bar{\mu} , \\ \mu &= -B\bar{\lambda} + A\bar{\mu} , \qquad A^2 + B^2 = 1 , \end{aligned}$$

on the system of defining differential equations (9). This transformation is equivalent to a rotation of axes in the normal plane. Let the coefficients of the transformed differential equations be denoted by $\bar{\alpha}, \bar{\beta}, \ldots\ldots.$ Substituting the expressions for λ and μ into the first of equations (9), we have

$$x_{uu} = \alpha x_u + \beta x_v + D_1(A\bar{\lambda} + B\bar{\mu}) + D_2(-B\bar{\lambda} + A\bar{\mu}) ,$$

or

$$(39)\quad x_{uu} = \alpha x_u + \beta x_v + (AD_1 - BD_2)\bar{\lambda} + (BD_1 + AD_2)\bar{\mu} .$$

Hence

$$\bar{\alpha} = \alpha , \quad \bar{\beta} = \beta , \quad \bar{D}_1 = AD_1 - BD_2 , \quad \bar{D}_2 = BD_1 + AD_2 .$$

Similarly

$$\bar{a} = a , \quad \bar{b} = b , \quad \bar{\gamma} = \gamma , \quad \bar{\delta} = \delta , \quad \bar{D}_1' = AD_1' - BD_2' ,$$
$$\bar{D}_2' = BD_1' + AD_2' , \quad \bar{D}_1'' = AD_1'' - BD_2'' , \quad \bar{D}_2'' = BD_1'' + AD_2'' .$$

To obtain the transformed coefficients $\bar{m}_1$, $\bar{p}_1$, $\bar{A}_1$, etc., let us first write down the inverse transformation of (38), namely

$$(39)\qquad \begin{aligned} \bar{\lambda} &= A\lambda - B\mu , \\ \bar{\mu} &= B\lambda + A\mu , \qquad A^2 + B^2 = 1 . \end{aligned}$$

Now

$$\bar{\lambda}_u = (A\lambda - B\mu)_u = A_u\lambda - B_u\,\mu + A\lambda_u - B\mu_u \quad .$$

Using the expressions for λ_u and μ_u from (9), we obtain

$$\bar{\lambda}_u = A_u\lambda + A(m_1 x_u + p_1 x_v + A_1\mu) - B_u\mu - B(m_2 x_u + p_2 x_v + A_2\lambda) \ ,$$

$$= (Am_1 - Bm_2)x_u + (Ap_1 - Bp_2)x_v + (A_u - A_2 B)\lambda + (AA_1 - B_u)\mu \quad ,$$

(40) $$= \bar{m}_1 x_u + \bar{p}_1 x_v + (A_u - A_2 B)(A\bar{\lambda} + B\bar{\mu}) + (AA_1 - B_u)(-B\bar{\lambda} + A\bar{\mu}) \quad ,$$

$$= \bar{m}_1 x_u + \bar{p}_1 x_v + (-AA_1B + BB_u + AA_u - ABA_2)\bar{\lambda} + (A^2A_1 - AB_u + A_uB - A_2B^2)\bar{\mu} \ ,$$

$$= \bar{m}_1 x_u + \bar{p}_1 x_v + \left\{-AB(A_1 + A_2) + \tfrac{1}{2}(A^2+B^2)_u\right\}\bar{\lambda} + \left\{A_1(A^2+B^2) - AB_u + A_uB\right\}\bar{\mu} \ ,$$

Referring to (22), (38), we note that

$$A_1 + A_2 = 0 \quad , \quad (A^2 + B^2)_u = 0 \quad .$$

Hence (40) becomes

(41) $$\bar{\lambda}_u = \bar{m}_1 x_u + \bar{p}_1 x_v + (A_1 - AB_u + A_uB)\bar{\mu} \quad .$$

Therefore

$$\bar{m}_1 = Am_1 - Bm_2 \ , \quad \bar{p}_1 = Ap_1 - Bp_2 \ , \quad \bar{A}_1 = A_1 - AB_u + A_uB \ .$$

The transformed coefficients $\bar{q}_1$, $\bar{n}_1$, ... may be found in a similar manner. We list here all of the transformed coefficients for convenience of reference.

(42)
$$\begin{gathered}
\bar{\alpha} = \alpha\,,\ \bar{\beta} = \beta\,,\ \bar{a} = a\,,\ \bar{b} = b\,,\ \bar{\gamma} = \gamma\,,\ \bar{\delta} = \delta\ , \\
\bar{D}_1 = AD_1 - BD_2\ ,\ \bar{D}_2 = BD_1 + AD_2\ ,\ \bar{D}_1' = AD_1' - BD_2'\ ,\ \bar{D}_2' = BD_1' + AD_2'\ , \\
\bar{D}_1'' = AD_1'' - BD_2''\ ,\ \bar{D}_2'' = BD_1'' + AD_2''\ ,\ \bar{m}_1 = Am_1 - Bm_2\ ,\ \bar{p}_1 = Ap_1 - Bp_2\ , \\
\bar{m}_2 = Bm_1 + Am_2\ ,\ \bar{p}_2 = Bp_1 + Ap_2\ ,\ \bar{q}_1 = Aq_1 - Bq_2\ ,\ \bar{n}_1 = An_1 - Bn_2\ , \\
\bar{q}_2 = Bq_1 + Aq_2\ ,\ \bar{n}_2 = Bn_1 + An_2\ ,\ \bar{A}_1 = A_1 + A_uB - AB_u\ , \\
\bar{B}_1 = B_1 + A_vB - AB_v\ ,\ \bar{A}_2 = A_2 + AB_u - A_uB\ ,\ \bar{B}_2 = B_2 + AB_v - A_vB\ .
\end{gathered}$$

The total curvature $\bar{K}_1$ of the surface of normal projection $S_{\bar{\lambda}}$ is determined by the expression

$$(43) \qquad EG\,\bar{K}_1 = (AD_1 - BD_2)(AD_1'' - BD_2'') - (AD_1' - BD_2')^2 \quad .$$

This surface has maximum or minimum total curvature if

$$(44) \qquad \frac{d\bar{K}_1}{dA} = 0 \quad .$$

Now

$$\frac{d\bar{K}_1}{dA} = \frac{1}{EG}\left\{\left(D_1 - D_2\frac{dB}{dA}\right)(AD_1'' - BD_2'') + (AD_1 - BD_2)\left(D_1'' - D_2''\frac{dB}{dA}\right) - 2(AD_1' - BD_2')\left(D_1' - D_2'\frac{dB}{dA}\right)\right\} .$$

Since

$$A^2 + B^2 = 1 \quad ,$$

then

$$\frac{dB}{dA} = -\frac{A}{B} \quad .$$

Whence

$$(45) \quad EG\frac{d\bar{K}_1}{dA} = \left\{\left(D_1 + \frac{AD_2}{B}\right)(AD_1'' - BD_2'') + (AD_1 - BD_2)\left(D_1'' + \frac{AD_2''}{B}\right) - 2(AD_1' - BD_2')\left(D_1' + \frac{AD_2'}{B}\right)\right\} .$$

From (45) we see that $\frac{d\bar{K}_1}{dA}$ will be zero if

$$(46) \quad (AD_2 + BD_1)(AD_1'' - BD_2'') + (AD_1 - BD_2)(AD_2'' + BD_1'') - 2(AD_1' - BD_2')(AD_2' + BD_1') = 0 \quad .$$

Rewriting (46), it follows that the surface of normal projection $S_{\bar{\lambda}}$ has maximum or minimum total curvature if and only if A and B satisfy the quadratic equation

$$(47) \qquad L A^2 + 2MAB - LB^2 = 0 \quad ,$$

wherein

$$(48) \quad L = D_1 D_2'' + D_1'' D_2 - 2D_1' D_2' \quad , \quad M = D_1 D_1'' - D_2 D_2'' + D_2'^2 - D_1'^2 \quad .$$

Equation (47) determines two values for $\frac{A}{B}$, each of which fixes a particular normal line $\bar{\lambda}$. Since the product of these two values is minus one the two normals determined by them are perpendicular. These normals are called the principal normals* of S_x at x, and the normal projection surfaces determined by the principal normals, the principal normal projection surfaces. The equation corresponding to (47) for the surface $S_{\bar{\mu}}$ is

$$L A^2 - 2 M AB - L B^2 = 0 \quad . \tag{49}$$

Since the roots of (49) as a quadratic in $\frac{A}{B}$ are the negatives of the roots of (47), and since their product is minus one, the normals determined by (49) coincide with the normals determined by (47). We may sum up our results with the following statement: Through the point x there exist two normals with the property that the normal projection surfaces determined by them have maximum and minimum total curvatures.

Let us suppose that the transformation (38) with values of A and B determined by (47) has been effected on the system of equations (9). Then using (42) we can easily verify that

$$\bar{L} = \bar{D}_1 \bar{D}_2'' + \bar{D}_1'' \bar{D}_2 - 2 \bar{D}_1' \bar{D}_2'$$

is identically zero.

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* Grove, Geometry, p. 64.

The resulting differential equations assume a canonical form in which

$$(50) \qquad D_1 D_2'' + D_1'' D_2 = 2 D_1' D_2' \quad .$$

For this form λ and μ are the principal normals.

VI. OTHER UNIQUE NORMALS

The general coordinates of the principal centers of normal curvature of the surface of normal projection $S_{\bar{\lambda}}$ determined by

$$\bar{\lambda} = A\lambda - B\mu$$

are

$$(51) \qquad X + \bar{R}_m \bar{\lambda} \quad , \qquad m = 1, 2 .$$

If $\bar{\lambda}$ be replaced by $A\lambda - B\mu$, equation (51) may be written

$$(52) \qquad X + A\bar{R}_m \lambda - B\bar{R}_m \mu \quad , \qquad m = 1, 2 .$$

The local coordinates of the points (52) are

$$(53) \quad \xi_1 = 0 , \quad \xi_2 = 0 , \quad \xi_3 = A\bar{R}_m , \quad \xi_4 = -B\bar{R}_m \quad , \qquad m = 1, 2 .$$

From (53) we obtain

$$(54) \quad \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = A^2\bar{R}_m^2 + B^2\bar{R}_m^2 = (A^2 + B^2)\bar{R}_m^2 = \bar{R}_m^2 \quad , \quad m = 1, 2 .$$

Hence

$$(55) \qquad \bar{R}_m = \sqrt{\xi_3^2 + \xi_4^2} \quad , \qquad m = 1, 2 .$$

It follows immediately from (53) and (55) that

$$(56) \quad A = \frac{\xi_3}{\sqrt{\xi_3^2 + \xi_4^2}} \quad , \quad B = -\frac{\xi_4}{\sqrt{\xi_3^2 + \xi_4^2}} \quad , \quad \bar{R}_m = \sqrt{\xi_3^2 + \xi_4^2} \quad .$$

Referring to equation (33) we see that the principal radii of normal curvature $\bar{R}_m$ at the point P_x for the surface $S_{\bar{\lambda}}$ are given as roots of

$$(57) \quad (\bar{D}_1 \bar{D}_1'' - \bar{D}_1'^2)\bar{R}_m^2 - (E\bar{D}_1'' + G\bar{D}_1)\bar{R}_m + EG = 0 \; .$$

Using (42), equation (57) becomes

$$(58) \left\{(AD_1 - BD_2)(AD_1'' - BD_2'') - (AD_1' - BD_2')^2\right\}\bar{R}_m^2 - \left\{E(AD_1'' - BD_2'') + G(AD_1 - BD_2)\right\}\bar{R}_m + EG = 0 \; .$$

We may obtain the locus of the centers of principal normal curvature for all normal projection surfaces $S_{\bar{\lambda}}$ by eliminating A, B, and $\bar{R}_m$ from (58) by using (56). As the equation of our locus we find

$$(59) \quad \begin{aligned} &\xi_1 = 0 , \quad \xi_2 = 0 , \\ &(D_1D_1'' - D_1'^2)\xi_3^2 + (D_1''D_2 + D_1D_2'' - 2D_1'D_2')\xi_3\xi_4 + (D_2D_2'' - D_2'^2)\xi_4^2 \\ &\qquad - (ED_1'' + GD_1)\xi_3 - (ED_2'' + GD_2)\xi_4 + EG = 0 \; . \end{aligned}$$

Exactly the same equation (59) is obtained when the surface $S_{\bar{\mu}}$ is considered. Hence we may state that <u>the locus of the centers of principal normal curvature for all normal projection surfaces is a conic in the normal plane</u> with equation given by (59). We shall call this conic <u>the central conic of S_x at x</u>.

Equation (59) may be written in the form

$$(60) \qquad \xi_1 = 0\,, \quad \xi_2 = 0\,, \quad (D_1\xi_3 + D_2\xi_4 - E)(D_1''\xi_3 + D_2''\xi_4 - G) - (D_1'\xi_3 + D_2'\xi_4)^2 = 0\,.$$

If our given orthogonal net is a conjugate net*, that is if

$$D_1' = D_2' = 0 \qquad ,$$

then (60) becomes

$$(61) \qquad \xi_1 = 0\,, \quad \xi_2 = 0\,, \quad (D_1\xi_3 + D_2\xi_4 - E)(D_1''\xi_3 + D_2''\xi_4 - G) = 0 \quad .$$

Hence <u>if S_x sustains an orthogonal conjugate net, the locus of the principal centers of normal curvature for all normal projection surfaces is a pair of straight lines in the normal plane</u>** whose equations are

$$(62) \qquad \begin{array}{l} \xi_1 = 0\,, \quad \xi_2 = 0\,, \quad D_1\xi_3 + D_2\xi_4 = E\,, \\ \xi_1 = 0\,, \quad \xi_2 = 0\,, \quad D_1''\xi_3 + D_2''\xi_4 = G\,. \end{array}$$

The form (60) of the equation of the central conic shows clearly that the lines (62) are tangent to the conic at their points of intersection with the line

$$(63) \qquad \xi_1 = 0\,, \quad \xi_2 = 0 \quad , \quad D_1'\xi_3 + D_2'\xi_4 = 0\,.$$

The line (63) lies, of course, in the normal plane and passes through the point P_x, and is thus a normal to the surface S_x at the point P_x. The normal line perpendicular to (63) has

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* E. P. Lane, <u>Projective Differential Geometry of Curves and Surfaces</u>, Chicago, University of Chicago Press, 1932, p. 122.

** Grove, <u>Geometry</u>, p. 64.

for its equation

$$(64) \qquad \xi_1 = 0 \ , \quad \xi_2 = 0 \ , \quad D_2' \, \xi_3 - D_1' \, \xi_4 = 0 \quad .$$

The normal line (64) passes through the point with local coordinates $(0,0,D_1',D_2')$ and the point $(0,0,0,0)$, or, speaking in terms of general coordinates, through the points

$$(65) \qquad X + D_1' \lambda + D_2' \mu \quad , \quad X \quad .$$

The direction cosines of the normal (64) are therefore proportional to the differences of the coordinates (65), namely

$$(66) \qquad D_1' \lambda + D_2' \mu \quad .$$

Referring to the transformation (38) and equations (42), we recall that

$$(67) \qquad \bar{D}_1' = A D_1' - B D_2' \quad , \quad \bar{D}_2' = B D_1' + A D_2' \quad .$$

If we should choose $\frac{A}{B} = \frac{D_2'}{D_1'}$, then from (67) and (38) we see that

$$(68) \qquad \bar{D}_1' = 0 \ , \quad \bar{D}_2' = \sqrt{D_1'^2 + D_2'^2} \ , \quad A = \frac{D_2'}{\sqrt{D_1'^2 + D_2'^2}} \quad , \quad B = \frac{D_1'}{\sqrt{D_1'^2 + D_2'^2}}$$

The transformation (38) then becomes

$$(69) \qquad \begin{aligned} \bar{\lambda} &= \frac{D_2' \lambda - D_1' \mu}{\sqrt{D_1'^2 + D_2'^2}} \quad , \\ \bar{\mu} &= \frac{D_1' \lambda + D_2' \mu}{\sqrt{D_1'^2 + D_2'^2}} \quad , \qquad A^2 + B^2 = 1 \ . \end{aligned}$$

From (64), (66), (69) we readily observe that the normal line, with direction cosines proportional to $D_1'\lambda + D_2'\mu$, is the normal $\bar{\mu}$ for $\bar{\mu}$ given by (69). It is easily shown that the normal line (63) has the direction cosines $\bar{\lambda}$. Since, from (67) it is plain that not both $\bar{D}_1'$ and $\bar{D}_2'$ can be made zero simultaneously, and since $\bar{D}_1' = 0$ implies that the second fundamental coefficient $\bar{D}_\lambda'$ for the normal projection surface $S_{\bar{\lambda}}$ is zero, we have the following characterization for the normal line given by (63): <u>The normal line (63) is uniquely determined as the only line through P_x in the normal plane which has upon its normal projection surface an orthogonal conjugate net.</u> Hence the normal line (64) with direction cosines proportional to $D_1'\lambda + D_2'\mu$ is characterized uniquely in that it is perpendicular to (63). We shall call it the <u>conjugate normal to S_x at x</u>. We also note that the pole of the unique normal line (63) with respect to the conic (59) is the point of intersection of the lines (62).

Let us define a geodesic as an extremal curve of the integral

$$(70) \qquad \int (E u'^2 + G v'^2)^{1/2}\, dt \,, \qquad u' = \frac{du}{dt} \,, \quad v' = \frac{dv}{dt} \,.$$

Equation (70) is easily changed to the form

$$(71) \qquad \int (E + G \mathrm{v}'^2)^{1/2}\, du \,, \qquad \mathrm{v}' = \frac{dv}{du} \,.$$

Euler's equation for the extremals of the integral in (71) is

(72) $\frac{d}{du} f_{\mathbf{v}'} = f_{\mathbf{v}}$, $f(u, \mathbf{v}, \mathbf{v}') = (E + G\mathbf{v}'^2)^{1/2}$, $\mathbf{v} = v$.

Performing the indicated differentiation in (72) we obtain

(73) $f_{\mathbf{v}'\mathbf{v}'} \frac{d^2\mathbf{v}}{du^2} + f_{\mathbf{v}\mathbf{v}'} \frac{d\mathbf{v}}{du} + f_{u\mathbf{v}'} = f_{\mathbf{v}}$.

Calculating the various partial derivatives in (73) we have

(74)
$$f_{\mathbf{v}'\mathbf{v}'} = \frac{EG}{(E + G\mathbf{v}'^2)^{3/2}} , \quad f_{u\mathbf{v}'} = \frac{2(EG_u - \frac{1}{2}E_uG)\mathbf{v}' + GG_u\mathbf{v}'^3}{2(E + G\mathbf{v}'^2)^{3/2}} ,$$
$$f_{\mathbf{v}} = \frac{E_{\mathbf{v}} + G_{\mathbf{v}}\mathbf{v}'^2}{2(E + G\mathbf{v}'^2)^{1/2}} , \quad f_{\mathbf{v}'\mathbf{v}} = \frac{2(EG_{\mathbf{v}} - \frac{1}{2}E_{\mathbf{v}}G)\mathbf{v}' + GG_{\mathbf{v}}\mathbf{v}'^3}{2(E + G\mathbf{v}'^2)^{3/2}} .$$

By making the necessary substitutions from (74) into (73) and reducing the result, we obtain

(75) $\mathbf{v}'' = -\frac{G_u}{2E}\mathbf{v}'^3 + \left(\frac{E_{\mathbf{v}}}{E} - \frac{G_{\mathbf{v}}}{2G}\right)\mathbf{v}'^2 + \left(\frac{E_u}{2E} - \frac{G_u}{G}\right)\mathbf{v}' + \frac{E_{\mathbf{v}}}{2G}$.

Using (10), equation (75) may be written

(76) $\mathbf{v}'' = \gamma\mathbf{v}'^3 + (2a - \delta)\mathbf{v}'^2 + (\alpha - 2b)\mathbf{v}' - \beta$.

Remembering that $\mathbf{v}' = \frac{dv}{du}$, it is easily seen that

(77) $\mathbf{v}' = \frac{v'}{u'}$, $\mathbf{v}'' = \frac{u'v'' - u''v'}{u'^3}$, $u' = \frac{du}{dt}$, $v' = \frac{dv}{dt}$.

From (76) and (77) we find that the differential equation of the geodesics on S_x is

(78) $u'v'' - u''v' = \gamma v'^3 + (2a - \delta)u'v'^2 + (\alpha - 2b)u'^2v' - \beta u'^3$.

Consider now the curve

$$u = u(t) \quad , \quad v = v(t)$$

on S_x. Let us find the equations in local coordinates of the osculating plane of the curve at the point P_x. The equations of any plane in homogeneous coordinates are

$$(79) \qquad \begin{aligned} A_1' X_1 + B_1' X_2 + C_1' X_3 + D_1' X_4 + E_1' X_5 &= 0 \quad , \\ A_2' X_1 + B_2' X_2 + C_2' X_3 + D_2' X_4 + E_2' X_5 &= 0 \quad . \end{aligned}$$

Since the plane (79) must pass through (0,0,0,0,1), we have $E_1 = E_2 = 0$. Taking suitable linear combinations of (79), we may write them in the form

$$(80) \qquad \begin{aligned} A_1 X_1 + B_1 X_2 + C_1 X_3 &= 0 \quad , \\ A_2 X_1 + B_2 X_2 + D_2 X_4 &= 0 \quad . \end{aligned}$$

In addition to the point (0,0,0,0,1), the osculating plane passes through the points

$$X' = X_u u' + X_v v' \quad , \quad X'' = X_{uu} u'^2 + 2X_{uv} u'v' + X_{vv} v'^2 + X_u u'' + X_v v'' .$$

The local coordinates of these points are

$$(81) \qquad \begin{aligned} &(u'\sqrt{E} \; , \; v'\sqrt{G} \; , \; 0 \; , \; 0 \; , \; 0) \quad , \\ &\Big\{ (\alpha u'^2 + 2a u'v' + \gamma v'^2 + u''), \; (\beta u'^2 + 2b u'v' + \delta v'^2 + v'') \; , \\ &\qquad (D_1 u'^2 + 2D_1' u'v' + D_1'' v'^2), \; (D_2 u'^2 + 2D_2' u'v' + D_2'' v'^2), \; 0 \Big\} . \end{aligned}$$

Substituting the coordinates (81) in equation (80) we obtain

$$
(82)\qquad
\begin{aligned}
&A_1\sqrt{E}\,u' + B_1\sqrt{G}\,v' = 0 \quad ,\\
&A_1(\alpha u'^2+2au'v'+\gamma v'^2+u'')+B_1(\beta u'^2+2bu'v'+\delta v'^2+v'')+C_1(D_1u'^2+2D_1'u'v'+D_1''v'^2)=0 \,,\\
&A_2\sqrt{E}\,u' + B_2\sqrt{G}\,v' = 0 \,,\\
&A_2(\alpha u'^2+2au'v'+\gamma v'^2+u'')+B_2(\beta u'^2+2bu'v'+\delta v'^2+v'')+D_2(D_2u'^2+2D_2'u'v'+D_2''v'^2)=0
\end{aligned}
$$

Assuming $B_1 \neq 0$, $B_2 \neq 0$, we may solve (82) for the ratios

$$\frac{A_1}{B_1}\ , \quad \frac{C_1}{B_1}\ , \quad \frac{A_2}{B_2}\ , \quad \frac{D_2}{B_2}\ ,$$

and make the proper substitutions in equations (80). We thus obtain as the equations of the osculating plane, (after replacing the x's by ξ's to preserve the local coordinate notation,) the equations

$$
(83)\qquad
\begin{aligned}
&\sqrt{G}\,v'(D_1u'^2+2D_1'u'v'+D_1''v'^2)\xi_1-\sqrt{E}\,u'(D_1u'^2+2D_1'u'v'+D_1''v'^2)\xi_2+\sqrt{EG}\,J\,\xi_3=0\,,\\
&\sqrt{G}\,v'(D_2u'^2+2D_2'u'v'+D_2''v'^2)\xi_1-\sqrt{E}\,u'(D_2u'^2+2D_2'u'v'+D_2''v'^2)\xi_2+\sqrt{EG}\,J\,\xi_4=0\ .
\end{aligned}
$$

wherein

$$(84)\quad J = u'v''-u''v'-\gamma v'^3+(\delta-2a)u'v'^2-(\alpha-2b)u'^2v'+\beta u'^3\ .$$

The equations of the normal plane to S_x at x are

$$(85)\qquad \xi_1 = 0\ , \qquad \xi_2 = 0\ .$$

We may think of equations (83) and (85) as four homogeneous linear equations in the unknowns ξ_1, ξ_2, ξ_3 and ξ_4. The condition therefore that these four equations possess a solution other than (0,0,0,0), or, in other words, the condition that the osculating plane (83) and the normal plane (85) intersect in a line is

$$(86)\qquad \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ \sqrt{G}\, v'(D_1 u'^2 + 2D_1' u'v' + D_1'' v'^2), & -\sqrt{E}\, u'(D_1 u'^2 + 2D_1' u'v' + D_1'' v'^2), & \sqrt{EG}\, J, & 0 \\ \sqrt{G}\, v'(D_2 u'^2 + 2D_2' u'v' + D_2'' v'^2), & -\sqrt{E}\, u'(D_2 u'^2 + 2D_2' u'v' + D_2'' v'^2), & 0, & \sqrt{EG}\, J \end{vmatrix} = 0 .$$

Simplifying (86) we find the condition to be

$$(87)\qquad J = 0 .$$

From equations (78), (87) we readily see that <u>the osculating plane to the curve $u = u(t)$, $v = v(t)$ on S_x at a point P_x intersects the normal plane to the surface at P_x in a line if and only if the curve is a geodesic</u>.

It follows from (78) by putting $v =$ constant that the curve C_u is a geodesic if and only if

$$\beta = -\frac{1}{2}\frac{E_v}{G} = 0 ,$$

Similarly the curve C_v is a geodesic if and only if

$$\gamma = -\frac{1}{2}\frac{G_u}{E} = 0 .$$

Hence we may state that <u>the parametric curves C_u, C_v are geodesics if and only if</u>

$$E_v = G_u = 0 .$$

Moreover we see from integrability condition (18a) that if

$$E_v = G_u = 0 \quad ,$$

then

(88) $$(D_1 D_1'' - D_1'^2) = -(D_2 D_2'' - D_2'^2) \quad .$$

Using (88) we notice that the discriminant of the central conic (59) becomes

(89) $$(D_1'' D_2 + D_1 D_2'' - 2 D_1' D_2')^2 + 4(D_1 D_1'' - D_1'^2)^2 \quad .$$

Since the expression in (89) is definitely positive it follows that <u>the central conic is an hyperbola if the curves of the given net N_x on S_x are geodesics</u>.

The equations of the osculating plane to the curve C_u at the point P_x may be written immediately from equations (83) by putting $v = \text{const}$. They are

(90) $$\begin{aligned} D_1 \xi_2 - \beta\sqrt{G}\, \xi_3 &= 0 \quad , \\ D_2 \xi_2 - \beta\sqrt{G}\, \xi_4 &= 0 \quad . \end{aligned}$$

Each of the equations in (90) represents a three-space, their intersection being the plane in question. Any linear combination of these equations also represents a three-space containing this plane. If we should multiply the first of (90) by $-D_2$, the second by D_1 and add, we obtain

(91) $$D_2 \xi_3 - D_1 \xi_4 = 0 \quad .$$

which represents a space of three dimensions passing through the plane (90). The equations of the tangent plane to the surface S_x at the point P_x are

$$(92) \qquad \xi_3 = 0 \ , \quad \xi_4 = 0 \quad .$$

It is evident from (92) that the tangent plane to S_x at P_x lies in the three-space (91). Hence we may say that the osculating plane to the curve C_u at x and the tangent plane to S_x at x determine the space of three dimensions (91). This space and the normal plane intersect in the line

$$(93) \qquad \xi_1 = 0 \ , \quad \xi_2 = 0 \ , \quad D_2 \xi_3 - D_1 \xi_4 = 0 \ .$$

The direction cosines of the line (93) are proportional to

$$D_1 \lambda + D_2 \mu \quad .$$

In exactly the same fashion we may show that the tangent plane to S_x at x and the osculating plane to the curve C_v at x determine a space of three dimensions which intersects the normal plane in the line

$$(94) \qquad \xi_1 = 0 \ , \quad \xi_2 = 0 \ , \quad D_2'' \xi_3 - D_1'' \xi_4 = 0 \quad .$$

The direction cosines of the line (94) are proportional to

$$D_1'' \lambda + D_2'' \mu \ .$$

The two normals (93), (94) to S_x at x with directions defined by

$$D_1 \lambda + D_2 \mu$$

and

$$D_1'' \lambda + D_2'' \mu$$

respectively we shall call the <u>intersector</u> <u>normals</u> of the C_u and C_v curves. The intersector normals will be orthogonal if

$$\sum (D_1 \lambda + D_2 \mu)(D_1'' \lambda + D_2'' \mu) = 0 \ .$$

Using (4) this condition reduces to

(95) $$D_1 D_1'' + D_2 D_2'' = 0 \ .$$

From (93), (94) we see that the intersector normals will coincide if

(96) $$D_1 D_2'' - D_1'' D_2 = 0 \ .$$

Again from (64) we can say that the conjugate normal coincides with each of the intersector normals if

(97) $$D_1 D_2' - D_1' D_2 = 0 \quad , \quad D_1'' D_2' - D_1' D_2'' = 0 \ .$$

Combining equations (96), (97), we may state that <u>the conjugate normal and the two intersector normals all coincide if the matrix</u>

(98) $$\Delta = \begin{pmatrix} D_1 & D_1' & D_1'' \\ D_2 & D_2' & D_2'' \end{pmatrix}$$

<u>is of rank one</u>.

VII. CONDITIONS FOR S_x TO BE IMMERSED IN THREE DIMENSIONS

Let us suppose that in the system of defining differential equations (9) for S_x the following condition is satisfied: the matrix Δ in (98) is of rank one. If we make the transformation

$$\bar{\lambda} = \frac{D_1 \lambda + D_2 \mu}{\sqrt{D_1^2 + D_2^2}} \quad , \quad \bar{\mu} = \frac{-D_2 \lambda + D_1 \mu}{\sqrt{D_1^2 + D_2^2}} \quad , \tag{99}$$

on equations (9), we obtain immediately from (42) with

$$A = \frac{D_1}{\sqrt{D_1^2 + D_2^2}} \quad , \quad B = \frac{-D_2}{\sqrt{D_1^2 + D_2^2}} \quad ,$$

the following transformed system:

$$X_{uu} = \alpha X_u + \beta X_v + \sqrt{D_1^2 + D_2^2}\; \bar{\lambda}$$

$$X_{uv} = a X_u + b X_v + \frac{D_1 D_1' + D_2 D_2'}{\sqrt{D_1^2 + D_2^2}} \bar{\lambda} + \frac{D_1 D_2' - D_1' D_2}{\sqrt{D_1^2 + D_2^2}} \bar{\mu} \quad ,$$

$$X_{vv} = \gamma X_u + \delta X_v + \frac{D_1 D_1'' + D_2 D_2''}{\sqrt{D_1^2 + D_2^2}} \bar{\lambda} + \frac{D_1 D_2'' - D_1'' D_2}{\sqrt{D_1^2 + D_2^2}} \bar{\mu} \quad ,$$

$$\bar{\lambda}_u = -\frac{\sqrt{D_1^2 + D_2^2}}{E} X_u - \frac{D_1 D_1' + D_2 D_2'}{G\sqrt{D_1^2 + D_2^2}} X_v + \left[A_1 + \frac{D_1 D_{2u} - D_2 D_{1u}}{D_1^2 + D_2^2}\right] \bar{\mu} \quad , \tag{100}$$

$$\bar{\lambda}_v = -\frac{D_1 D_1' + D_2 D_2'}{E\sqrt{D_1^2 + D_2^2}} X_u - \frac{\sqrt{D_1''^2 + D_2''^2}}{G} X_v + \left[B_1 + \frac{D_1 D_{2v} - D_2 D_{1v}}{D_1^2 + D_2^2}\right] \bar{\mu} \quad ,$$

$$\bar{\mu}_u = \frac{D_1' D_2 - D_1 D_2'}{G\sqrt{D_1^2 + D_2^2}} X_v + \left[A_2 - \frac{D_1 D_{2u} - D_2 D_{1u}}{D_1^2 + D_2^2}\right] \bar{\lambda} \quad ,$$

$$\bar{\mu}_v = \frac{D_1' D_2 - D_1 D_2'}{E\sqrt{D_1^2 + D_2^2}} X_u - \frac{D_1 D_2'' - D_1'' D_2}{G\sqrt{D_1^2 + D_2^2}} X_v + \left[B_2 - \frac{D_1 D_{2v} - D_2 D_{1v}}{D_1^2 + D_2^2}\right] \bar{\mu} \quad .$$

If we make use of the fact that the matrix Δ is of rank one, we see immediately that in the expressions for x_{uv} and x_{vv} the coefficients of $\bar{\mu}$ vanish; likewise in the expressions for $\bar{\mu}_u$ and $\bar{\mu}_v$ the coefficients of x_v and x_u also vanish. We shall show presently that the coefficients of $\bar{\mu}$ for the last four of equation (100) are zero.

Referring to the integrability conditions (18), p. 5, let us multiply equation (c) by D_1 and subtract from it the result of multiplying equation (b) by D_2. We obtain

$$(101) \quad D_1 D_{2v} - D_2 D_{1v} + B_1(D_1^2 + D_2^2) - A_1(D_1 D_1' + D_2 D_2') + D_2 D_{1u}' - D_1 D_{2u}' = 0 .$$

Since Δ is of rank one it follows that

$$D_1' = \frac{D_1 D_2'}{D_2}, \quad D_1'' = \frac{D_1' D_2''}{D_2'}, \quad D_{1u}'' = \frac{D_{1u}' D_2''}{D_2'} + \frac{D_1' D_{2u}''}{D_2'} - \frac{D_1' D_2'' D_{2u}'}{D_2'^2} .$$

Substituting into (101) and simplifying we obtain

$$(102) \quad (D_1' D_{2u}' - D_2' D_{1u}')\frac{D_2''}{D_2'} + A_1(D_1'^2 + D_2'^2)\frac{D_2''}{D_2'} + D_2' D_{1v}' - D_1' D_{2v}' - B_1(D_1'^2 + D_2'^2) = 0 .$$

Putting $D_1' = \frac{D_1 D_2'}{D_2}$ in (102) and simplifying again, there results

$$(103) \quad \frac{D_2''}{D_2}\left\{D_1 D_{2u}' - D_2 D_{1u}' + \frac{D_2'}{D_2} A_1(D_1^2 + D_2^2)\right\} + \frac{D_2'^2}{D_2^2}\left\{D_2 D_{1v} - D_1 D_{2v} - B_1(D_1^2 + D_2^2)\right\} = 0 .$$

By (101), the second bracket is equal to

$$-A_1(D_1 D_1' + D_2 D_2') + D_2 D_{1u}' - D_1 D_{2u}' .$$

Hence (103) may now be written in the form

$$(104)\qquad \left(\frac{D_2 D_2'' - D_2'^2}{D_2^2}\right)\left\{A_1\left(D_1^2+D_2^2\right)\frac{D_2'}{D_2} + \left(D_1 D_{2u}' - D_2 D_{1u}'\right)\right\} = 0 .$$

Since $D_2 D_2'' - D_2'^2$ is not invariant under rotation we may assume it not zero. Finally let us substitute

into the second bracket in (104) to obtain

$$(105)\qquad A_1\left(D_1^2 + D_2^2\right)\frac{D_2'}{D_2} + \frac{D_2'}{D_2}\left(D_1 D_{2u} - D_2 D_{1u}\right) = 0 .$$

From (105) it follows that

$$(106)\qquad A_1 + \frac{D_1 D_{2u} - D_2 D_{1u}}{D_1^2 + D_2^2} = 0 .$$

The coefficient

$$B_1 + \frac{D_1 D_{2v} - D_2 D_{1v}}{D_1^2 + D_2^2}$$

is shown to be zero by a similar demonstration.

Equations (100) under the conditions imposed now become

$$X_{uu} = \alpha X_u + \beta X_v + \sqrt{D_1^2 + D_2^2}\;\bar{\lambda} ,$$

$$X_{uv} = a X_u + b X_v + \frac{D_1 D_1' + D_2 D_2'}{\sqrt{D_1^2 + D_2^2}}\;\bar{\lambda} ,$$

$$X_{vv} = \gamma X_u + \delta X_v + \frac{D_1 D_1'' + D_2 D_2''}{\sqrt{D_1^2 + D_2^2}}\;\bar{\lambda} ,$$

$$(107)\qquad \bar{\lambda}_u = -\frac{\sqrt{D_1^2 + D_2^2}}{E} X_u - \frac{D_1 D_1' + D_2 D_2'}{G\sqrt{D_1^2 + D_2^2}} X_v ,$$

$$\bar{\lambda}_v = -\frac{D_1 D_1' + D_2 D_2'}{E\sqrt{D_1^2 + D_2^2}} X_u - \frac{\sqrt{D_1''^2 + D_2''^2}}{G} X_v ,$$

$$\bar{\mu}_u = 0 , \quad \bar{\mu}_v = 0 .$$

The equations (107) are the ordinary "Gauss Equations" of a surface in three dimensions.* Hence, *a necessary and sufficient condition that S_x be immersed in a space of three dimensions is that the matrix*

$$\Delta = \begin{pmatrix} D_1 & D_1' & D_1'' \\ D_2 & D_2' & D_2'' \end{pmatrix}$$

be of rank one.

From (98) we may state the above result as follows: *A necessary and sufficient condition that S_x be immersed in a space of three dimensions is that the conjugate normal and the two intersector normals all coincide*.

From equations (36) and (42) we note that the surface of normal projection $S_{\bar\lambda}$ defined by (38) is a minimal surface** if and only if

$$(108) \quad \bar{M}_1 = \frac{\bar{D}_1}{E} + \frac{\bar{D}_1''}{G} = \frac{A D_1 - B D_2}{E} + \frac{A D_1'' - B D_2''}{G} = A M_1 - B M_2 = 0 \ .$$

Since M_1 and M_2 represent the mean curvatures of the surfaces of normal projection S_λ and S_μ it follows that *if two surfaces of normal projection taken in perpendicular directions are both minimal surfaces, all surfaces of normal projection are minimal*. *If not both M_1 and M_2 are zero, there exists just one normal to the surface at x which determines a minimal surface of normal projection*.

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* Eisenhart, *Geometry*, p. 154.
**Eisenhart, *Geometry*, p. 129.

If the surface S_x is not immersed in a space of three dimensions, then from the point x there may be drawn two normals each tangent to the central conic. Each of these normals determines a surface of normal projection for which the two principal radii of normal curvature are equal. Hence the point x is an umbilical point* for each of these surfaces of normal projection.

Equation (43) may be rewritten

$$(109)\quad (D_1 D_1'' - D_1'^2)A^2 - (D_1 D_2'' + D_1'' D_2 - 2D_1' D_2')AB + (D_2 D_2'' - D_2'^2)B^2 = EG\bar{K}_1 \quad .$$

The surface of normal projection S will be developable if

$$\bar{K}_1 = 0 \quad .$$

Hence if

$$(110)\quad (D_1 D_1'' - D_1'^2)A^2 - (D_1 D_2'' + D_1'' D_2 - 2D_1' D_2')AB + (D_2 D_2'' - D_2'^2)B^2 = 0 \quad .$$

For the canonical form of equations (9), the coefficient of AB is zero. Equation (110) may therefore be written:

$$(111)\qquad (D_1 D_1'' - D_1'^2)A^2 + (D_2 D_2'' - D_2'^2)B^2 = 0 \, .$$

The latter form indicates that there are exactly two surfaces of normal projection which are developables. The normals determining these developable surfaces are called <u>the developable normals</u>.** It may be readily verified that the developable

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* Eisenhart, <u>Geometry</u>, p. 120.
**Grove, <u>Geometry</u>, p. 67.

normals separate the principal normals harmonically.

VIII. THE RELATION R AND THE CONJUGATE NORMAL

If we should construct tangents to the curves C_u at the points where they meet a fixed C_v curve, we obtain a ruled surface which we may denote by $R^{(v)}$. Similarly by constructing tangents to the curves C_v at points where they meet a fixed C_u curve we obtain a ruled surface $R^{(u)}$.

Let l_2 be any line lying in the tangent plane to S_x at x, but not passing through x. Let l_1 be a line passing through x but not lying in the tangent plane at x. The line l_2 intersects the tangents to the C_u and C_v curves in points r and s respectively. If the tangent planes to $R^{(v)}$ and $R^{(u)}$ at r and s respectively intersect in the line l_1 the given lines l_1 and l_2 will be said to be <u>in relation R</u>* with respect to the parametric net N_x.

The points r and s are defined by expressions of the form

$$(112) \qquad r = x - \frac{1}{p} x_u \quad , \quad s = x - \frac{1}{q} x_v \quad .$$

The tangent planes to R and R at r and s intersect in a

* E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, University of Chicago Press, 1932, p. 82.

line joining x to y defined by

(113) $$y = x + k\,(x_{uv} - q\,x_u - p\,x_v) \,,$$

where k is proportional to the distance between the points x and y. Using the second of equations (9) we may rewrite (113) in the form

(114) $$y = x + k\left\{(a-q)x_u + (b-p)\,x_v + D_1'\lambda + D_2'\mu\right\} .$$

Since $y_i - x_i$ are proportional to the direction cosines of the line xy, the line xy will be a normal line to S_x at x if

(115) $$\sum x_u (y-x) = a - q = 0 \;, \quad \sum x_v (y-x) = b - p = 0 \;.$$

Hence if

$$a = q \;, \quad b = p \;.$$

The direction cosines of this normal will then be proportional to

$$D_1'\lambda + D_2'\mu \;.$$

From (64) we readily verify that this normal is the conjugate normal. We may therefore characterize <u>the conjugate normal as the only normal line to S_x at x which is in relation R with a given line in the tangent plane with respect to the net</u> N_x.

IX. CHANGE OF NET UPON THE SURFACE

Any net of curves on S_x may be defined by a differential equation of the form*

$$(116) \qquad (dv - \theta\, du)(du - \omega\, dv) = 0\,,$$

or

$$\theta\, du^2 - (1 + \theta\omega)\, du\, dv + \omega\, dv^2 = 0\,,$$

where θ and ω are functions of u,v. If the discriminant of (116) vanishes, that is, if $1 - \theta\omega$ is zero, then there is only a one parameter family of curves on the surface and not a net. If $1 + \theta\omega = 0$, the net is a conjugate net. If $\theta = \omega = 0$, the net is our given parametric net. Hence we shall suppose that

$$(117) \qquad \theta\omega\,(1 - \theta^2\omega^2) \neq 0\,.$$

The net (116) will be orthogonal if the harmonic invariant of the two forms

$$\theta\, du^2 - (1 + \theta\omega)\, du\, dv + \omega\, dv^2 = 0\,,$$

$$E\, du^2 \qquad\qquad + G\, dv^2 = 0\,,$$

vanishes.**

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* V. G. Grove, A General Theory of Nets on a Surface, Transactions of the American Mathematical Society, Vol. 29 (1927), p. 802.

**Eisenhart, Geometry, p. 80.

This condition is

$$\theta G + \omega E = 0 . \tag{118}$$

Let us make the net (116) parametric by the transformation of variables defined by

$$\bar{u} = \varphi(u,v) \quad , \quad \bar{v} = \psi(u,v) . \tag{119}$$

We must have

$$d\bar{u} = \varphi_u\, du + \varphi_v\, dv = 0 \; , \quad d\bar{v} = \psi_u\, du + \psi_v\, dv = 0 .$$

Employing (116) we may write

$$\frac{dv}{du} = - \frac{\varphi_u}{\varphi_v} = \theta \quad , \quad \frac{dv}{du} = - \frac{\psi_u}{\psi_v} = \frac{1}{\omega} .$$

Whence come the relationships

$$\varphi_u = -\theta\, \varphi_v \quad , \quad \psi_v = -\omega\, \psi_u . \tag{120}$$

From (119) we may now write, after formal differentiation, the equations

$$\begin{aligned}
& x_u = x_{\bar{u}}\, \varphi_u + x_{\bar{v}}\, \varphi_v \quad , \quad x_v = \varphi_v\, x_{\bar{u}} + \psi_v\, x_{\bar{v}} \; , \\
& x_{uu} = \varphi_u^2\, x_{\bar{u}\bar{u}} + 2 \varphi_u \psi_u\, x_{\bar{u}\bar{v}} + \psi_u^2\, x_{\bar{v}\bar{v}} + \varphi_{uu}\, x_{\bar{u}} + \psi_{uu}\, x_{\bar{v}} \; , \\
& x_{uv} = \varphi_u \varphi_v\, x_{\bar{u}\bar{u}} + (\varphi_v \psi_u + \varphi_u \psi_v)\, x_{\bar{u}\bar{v}} + \psi_u \psi_v\, x_{\bar{v}\bar{v}} + \varphi_{uv}\, x_{\bar{u}} + \psi_{uv}\, x_{\bar{v}} \; , \\
& x_{vv} = \varphi_v^2\, x_{\bar{u}\bar{u}} + 2 \varphi_v \psi_v\, x_{\bar{u}\bar{v}} + \psi_v^2\, x_{\bar{v}\bar{v}} + \varphi_{vv}\, x_{\bar{u}} + \psi_{vv}\, x_{\bar{v}} \; , \\
& \lambda_u = \lambda_{\bar{u}}\, \varphi_u + \lambda_{\bar{v}}\, \varphi_v \quad , \quad \lambda_v = \varphi_v\, \lambda_{\bar{u}} + \psi_v\, \lambda_{\bar{v}} \; , \\
& \mu_u = \mu_{\bar{u}}\, \varphi_u + \mu_{\bar{v}}\, \varphi_v \quad , \quad \mu_v = \varphi_v\, \mu_{\bar{u}} + \psi_v\, \mu_{\bar{v}} .
\end{aligned} \tag{121}$$

Substituting from (121) into equations (9) we find

$$
\begin{aligned}
&\varphi_u^2 x_{\bar u\bar u} + 2\varphi_u\psi_u x_{\bar u\bar v} + \psi_u^2 x_{\bar v\bar v} = (\alpha\varphi_u + \beta\varphi_v - \varphi_{uu})x_{\bar u} + \\
&\qquad\qquad (\alpha\psi_u + \beta\psi_v - \psi_{uu})x_{\bar v} + D_1\lambda + D_2\mu, \\
&\varphi_u\varphi_v x_{\bar u\bar u} + (\varphi_v\psi_u + \varphi_u\psi_v)x_{\bar u\bar v} + \psi_u\psi_v x_{\bar v\bar v} = (a\varphi_u + b\varphi_v - \varphi_{uv})x_{\bar u} + \\
&\qquad\qquad (a\psi_u + b\psi_v - \psi_{uv})x_{\bar v} + D_1'\lambda + D_2'\mu, \\
&\varphi_v^2 x_{\bar u\bar u} + 2\varphi_v\psi_v x_{\bar u\bar v} + \psi_v^2 x_{\bar v\bar v} = (\gamma\varphi_u + \delta\varphi_v - \varphi_{vv})x_{\bar u} + \\
&\qquad\qquad (\gamma\psi_u + \delta\psi_v - \psi_{vv})x_{\bar v} + D_1'\lambda + D_2'\mu, \\
&\varphi_u\lambda_{\bar u} + \varphi_v\lambda_{\bar v} = (m_1\varphi_u + p_1\varphi_v)x_{\bar u} + (m_1\varphi_v + p_1\psi_v)x_{\bar v} + A_1\mu, \\
&\varphi_v\lambda_{\bar u} + \psi_v\lambda_{\bar v} = (q_1\varphi_u + n_1\varphi_v)x_{\bar u} + (q_1\varphi_v + n_1\psi_v)x_{\bar v} + B_1\mu, \\
&\varphi_u\mu_{\bar u} + \varphi_v\mu_{\bar v} = (m_2\varphi_u + p_2\varphi_v)x_{\bar u} + (m_2\varphi_v + p_2\psi_v)x_{\bar v} + A_2\lambda, \\
&\varphi_v\mu_{\bar u} + \psi_v\mu_{\bar v} = (q_2\varphi_u + n_2\varphi_v)x_{\bar u} + (q_2\varphi_v + n_2\psi_v)x_{\bar v} + B_2\lambda.
\end{aligned}
\tag{122}
$$

Consider the first three equations of (122) as linear equations in the unknowns $x_{\bar u\bar u}$, $x_{\bar u\bar v}$, $x_{\bar v\bar v}$. The determinant of the coefficients may be written, by using (120), in the form

$$
\begin{vmatrix}
\theta^2\varphi_v^2, & -2\theta\varphi_v\psi_u, & \psi_u^2 \\
-\theta\varphi_v^2, & (1+\theta\omega)\varphi_v\psi_u, & -\omega\psi_u^2 \\
\varphi_v^2, & -2\omega\varphi_v\psi_u, & \omega^2\psi_u^2
\end{vmatrix}
= \left\{\varphi_v\psi_u(\theta\omega-1)\right\}^3 .
$$

By (117) we note that this determinant is not zero since

$$\theta\omega - 1 \neq 0 .$$

Hence we may solve for $X_{\bar u\bar u}, X_{\bar u\bar v}, X_{\bar v\bar v}$ obtaining

$$X_{\bar u\bar u} = \bar\alpha X_{\bar u} + \bar\beta X_{\bar v} + \left\{\frac{D_1\omega^2 + 2D_1'\omega + D_1''}{\varphi_v^2(\theta\omega-1)^2}\right\}\lambda + \left\{\frac{D_2\omega^2 + 2D_2'\omega + D_2''}{\varphi_v^2(\theta\omega-1)^2}\right\}\mu,$$

(123) $$X_{\bar u\bar v} = \bar a X_{\bar u} + \bar b X_{\bar v} + \left\{\frac{D_1\omega + D_1'(1+\theta\omega) + D_1''\theta}{\varphi_v\psi_u(\theta\omega-1)^2}\right\}\lambda + \left\{\frac{D_2\omega + D_2'(1+\theta\omega) + D_2''\theta}{\varphi_v\psi_u(\theta\omega-1)^2}\right\}\mu,$$

$$X_{\bar v\bar v} = \bar\gamma X_{\bar u} + \bar\delta X_{\bar v} + \left\{\frac{D_1 + 2D_1'\theta + D_1''\theta^2}{\psi_u^2(\theta\omega-1)^2}\right\}\lambda + \left\{\frac{D_2 + 2D_2'\theta + D_2''\theta^2}{\psi_u^2(\theta\omega-1)^2}\right\}\mu,$$

where $\bar\alpha, \bar\beta, \ldots\ldots, \bar\delta$ are new functions of u,v corresponding to $\alpha, \beta, \ldots\ldots, \delta$ in (9). Solving the last four equations of (122) for $\lambda_{\bar u}, \lambda_{\bar v}, \mu_{\bar u}, \mu_{\bar v}$ there results

$$\lambda_{\bar u} = \bar m_1 X_{\bar u} + \bar p_1 X_{\bar v} + \frac{A_1\psi_v - B_1\varphi_v}{\varphi_u\psi_v - \varphi_v^2}\mu,$$

$$\lambda_{\bar v} = \bar q_1 X_{\bar u} + \bar n_1 X_{\bar v} + \frac{A_1\varphi_v - B_1\varphi_u}{\varphi_v^2 - \varphi_u\psi_v}\mu,$$

(124)

$$\mu_{\bar u} = \bar m_2 X_{\bar u} + \bar p_2 X_{\bar v} + \frac{A_2\psi_v - B_2\varphi_v}{\varphi_u\psi_v - \varphi_v^2}\lambda,$$

$$\mu_{\bar v} = \bar q_2 X_{\bar u} + \bar n_2 X_{\bar v} + \frac{A_2\varphi_v - B_2\varphi_u}{\varphi_v^2 - \varphi_u\psi_v}\lambda.$$

Equations (123) may be written

$$X_{\bar u\bar u} = \bar\alpha X_{\bar u} + \bar\beta X_{\bar v} + \bar D_1\lambda + \bar D_2\mu,$$

$$X_{\bar u\bar v} = \bar a X_{\bar u} + \bar b X_{\bar v} + \bar D_1'\lambda + \bar D_2'\mu,$$

$$X_{\bar v\bar v} = \bar\gamma X_{\bar u} + \bar\delta X_{\bar v} + \bar D_1''\lambda + \bar D_2''\mu,$$

wherein

$$\bar{D}_1 = k_1(D_1\omega^2 + 2D_1'\omega + D_1''), \quad \bar{D}_2 = k_1(D_2\omega^2 + 2D_2'\omega + D_2''), \quad k_1 = [\varphi_v(\theta\omega - 1)]^{-2},$$

(125) $$\bar{D}_1' = k_2[D_1\omega + D_1'(1+\theta\omega) + D_1''\theta], \quad \bar{D}_2' = k_2[D_2\omega + D_2'(1+\theta\omega) + D_2''\theta], \quad k_2 = [\varphi_v\psi_u(\theta\omega-1)^{+2}]^{-1},$$

$$\bar{D}_1'' = k_3(D_1 + 2D_1'\theta + D_1''\theta^2), \quad \bar{D}_2'' = k_3(D_2 + 2D_2'\theta + D_2''\theta^2), \quad k_3 = [\psi_u(\theta\omega - 1)]^{-2}.$$

For the surface defined by equations (123), (124), the intersector normals will coincide if

(126) $$\bar{D}_1\bar{D}_2'' - \bar{D}_1''\bar{D}_2 = 0 .$$

From (125), we may write (126) as follows

$$(\omega^2 D_1 + 2\omega D_1' + D_1'')(\theta^2 D_2'' + 2\theta D_2' + D_2) - (\theta^2 D_1'' + 2\theta D_1' + D_1)(\omega^2 D_2 + 2\omega D_2' + D_2'') = 0 .$$

Upon expanding and collecting terms the latter form becomes

(127) $$\begin{aligned}&(D_1D_2'' - D_1''D_2)\omega^2\theta^2 + 2(D_1D_2' - D_1'D_2)\omega^2\theta + 2(D_1'D_2'' - D_1''D_2')\omega\theta^2 + \\ &2(D_1'D_2 - D_1D_2')\omega + 2(D_1''D_2' - D_1'D_2'')\theta + (D_1''D_2 - D_1D_2'') = 0 .\end{aligned}$$

Imposing the condition (118) that the net (116) be orthogonal, equation (127) becomes

(128) $$[G(D_1D_2'' - D_1''D_2)]\theta^2 + 2[G(D_1D_2' - D_1'D_2) - E(D_1'D_2'' - D_1''D_2')]\theta - E(D_1D_2'' - D_1''D_2) = 0 .$$

If in addition to having the intersector normals coincident, the net were conjugate, then the matrix Δ becomes

$$\begin{pmatrix} \bar{D}_1 & 0 & \bar{D}_1'' \\ \bar{D}_2 & 0 & \bar{D}_2'' \end{pmatrix} ,$$

which is of rank one. Referring to our previous results, we see that S_x would then be immersed in a space of three dimensions. We shall suppose therefore that

$$\bar{D}_1' \neq 0 , \quad \bar{D}_2' \neq 0 .$$

Since (128) is a quadratic equation in θ, <u>there exists on a surface not sustaining a conjugate net two orthogonal nets whose intersector normals coincide. We shall call these nets the intersector nets of</u> S_x.

After dividing out $\theta\omega-1$ which is not zero, equation (127) becomes

$$(\theta\omega+1)(D_1D_2''-D_1''D_2)+2\omega(D_1D_2'-D_1'D_2)+2\theta(D_1'D_2''-D_1''D_2')=0 .$$

The latter equation when solved for θ gives

$$\text{(129)} \qquad \omega = \frac{A\theta+B}{-B\theta+D} ,$$

wherein

$$A = 2(D_1''D_2'-D_1'D_2'') , \quad B = (D_1''D_2-D_1D_2'') , \quad D = 2(D_1D_2'-D_1'D_2) .$$

From the form of (129) we may state that <u>the tangents to the curves of non-orthogonal nets whose intersector normals coincide are projectively related</u>. If in (129) we should put $\theta' = \frac{1}{\theta}$, we obtain

$$\omega = \frac{B\theta' + A}{D\theta' - B} \quad ,$$

which is the equation of an involution.

Let us refer the surface S_x to one of the two inter-sector nets. In addition let us make the transformation

$$(130) \qquad \bar{\lambda} = \frac{\bar{D}_1 \lambda + \bar{D}_2 \mu}{\sqrt{\bar{D}_1^2 + \bar{D}_2^2}} \quad , \quad \bar{\mu} = \frac{-\bar{D}_2 \lambda + \bar{D}_1 \mu}{\sqrt{\bar{D}_1^2 + \bar{D}_2^2}} \quad ,$$

on equations (123), (124), giving us equations of the form (100) but with

$$\bar{D}_1 \bar{D}_2'' - \bar{D}_1'' \bar{D}_2 = 0 \; .$$

<u>Hence the surface S_x (not sustaining a conjugate net) may be defined by a system of differential equations of the form</u>

$$(131) \qquad \begin{aligned} x_{uu} &= \alpha x_u + \beta x_v + D_1 \lambda \quad , & \lambda_u &= m_1 x_u + p_1 x_v + A_1 \mu \, , \\ x_{uv} &= a x_u + b x_v + D_1' \lambda + D_2' \mu \, , & \lambda_v &= q_1 x_u + n_1 x_v + B_1 \mu \, , \\ x_{vv} &= \gamma x_u + \delta x_v + D_1'' \lambda \quad , & \mu_u &= m_2 x_u + p_2 x_v + A_2 \lambda \, , \\ & & \mu_v &= q_2 x_u + n_2 x_v + B_2 \lambda \, , \end{aligned}$$

where the letters $\alpha, \beta, \cdots, D_1, D_1', \cdots, m_1, p_1, \cdots, A_1, B_1, \cdots \lambda, \mu$ represent the transformed quantities corresponding to the letters in equations (123), (124). The matrix Δ for the system (131) is

$$\begin{pmatrix} D_1 & D_1' & D_1'' \\ 0 & D_2' & 0 \end{pmatrix} \tag{132}$$

<u>Hence the surface S_x represented by (131) will be immersed in a space of three dimensions if and only if</u>

$$D_2' = 0 .$$

For the system (131) the parametric net on S_x is one of the intersector nets, and the normal λ represents the two coincident intersector normals for curves of that net. The mean curvature M_2 for the surface of normal projection S_μ, where μ represents the normal perpendicular to the normal λ, is

$$M_2 = \frac{D_2}{E} + \frac{D_2''}{G} \equiv 0 , \tag{133}$$

since from (132) we have

$$D_2 = D_2'' = 0 .$$

Hence the normal projection surface S_μ is a minimal surface.

The equation of the central conic for the canonical system (131) now becomes

$$\xi_1 = 0,\ \xi_2 = 0,\ (D_1 D_1'' - D_1'^2)\xi_3^2 - 2D_1'D_2'\,\xi_3\xi_4 - D_2'^2\xi_4^2 - (ED_1'' + GD_1)\xi_3 + EG = 0. \tag{134}$$

The condition that the quadratic expression in ξ_3, ξ_4 in (134) be factorable, in other words the condition that the central conic be degenerate is

$$(135) \qquad \begin{vmatrix} D_1 D_1'' - D_1'^2 , & -D_1' D_2' , & -\tfrac{1}{2}(E D_1'' + G D_1) \\ -D_1' D_2' , & -D_2'^2 , & 0 \\ -\tfrac{1}{2}(E D_1'' + G D_1), & 0 , & E G \end{vmatrix} = 0 .$$

Equation (135) may be written in the simple form

$$(136) \qquad \left\{ E G D_2' \left(\frac{D_1''}{G} - \frac{D_1}{E} \right) \right\}^2 = 0 .$$

Referring to integrability condition (21), we may write (136) as follows

$$(137) \qquad \left\{ E G \left(A_{1v} - B_{1u} \right) \right\}^2 = 0 .$$

Therefore, <u>a necessary and sufficient condition that the central conic be degenerate is</u>

$$(138) \qquad A_{1v} - B_{1u} = 0 .$$

From (136) two cases arise:

Case I : $\quad D_2' = 0 \quad ,$

Case II : $\quad \dfrac{D_1}{E} - \dfrac{D_1''}{G} = 0 .$

If $D_2' = 0$, we see from (132) that S_x is immersed in a space of three dimensions. To investigate the geometric meaning of Case II, let us find the principal directions* in the tangent plane to the surface S_λ at the point P_x. They are given by

$$(139) \quad (E_\lambda D'_\lambda - F_\lambda D_\lambda)\,du^2 + (E_\lambda D''_\lambda - G_\lambda D_\lambda)\,du\,dv + (F_\lambda D''_\lambda - G_\lambda D'_\lambda)\,dv^2 = 0 .$$

Using (29), (31) equation (139) may be written

$$(140) \quad \frac{D_1'}{\sqrt{EG}}\,du^2 + \left(\frac{D_1''}{G} - \frac{D_1}{E}\right)du\,dv - \frac{D_1'}{\sqrt{EG}}\,dv^2 = 0 .$$

The principal directions given by (140) separate harmonically the tangents to the curves of the intersector net

$$(141) \quad 2\,du\,dv = 0 .$$

on S_x, if and only if

$$\frac{D_1''}{G} - \frac{D_1}{E} = 0 .$$

Hence when the latter relationship obtains the central conic is degenerate. In summary we may therefore state that the central conic is degenerate if $D_2' = 0$ in which case the surface is immersed in a space of three dimensions; or if $\frac{D_1''}{G} - \frac{D_1}{E} = 0$, when the principal directions in the tangent plane to the surface S_λ at the point P_x separate harmonically the tangents to the curves of the intersector net on S_x.

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* Eisenhart, *Geometry*, p. 121.

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