THE TREATMENT OF COMPOSITION IN SECONDARY AND EARLY COLLEGIATE MATHEMATICS CURRICULA

By

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ABSTRACT

THE TREATMENT OF COMPOSITION IN SECONDARY AND EARLY COLLEGIATE MATHEMATICS CURRICULA

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Composition has been described as essential for understanding functions (Carlson, Oehrtman, & Engelke, 2010; Cooney, Beckmann, & Lloyd, 2010). Studies of students' understanding of function composition have shown that students use multiplication and other operations in place of composition (Carlson et al., 2010; Horvath, 2010).

While there have been studies of students' knowledge of composition, the teaching of and curricular development of composition has not received as much attention. This dissertation attempted to fill this void by examining the treatment of composition in secondary and early collegiate mathematics curricula. By examining the definitions, explanations, and uses of composition, I was able to describe the kinds of explicit and implicit opportunities that textbooks provide to students with respect to the concept of composition.

This analysis of textbooks of high school algebra 1 and 2, geometry, and precalculus and collegiate precalculus and calculus textbooks utilized multiple frameworks. Mathematically, composition can be viewed as an operation on objects (e.g., functions or relations) or as a recursive sequence of processes where the output of the n^{th} process is the input of the $n+1^{th}$ process. A framework of procedural, conceptual, and conventional knowledge was used to describe the ways that textbooks define, explain, and perform composition. The representation (e.g., algebraic, graphical, tabular) and type of function, relation, or transformation (e.g., polynomial, reflection) is also included in the coding scheme of this study.

The results indicated that composition appeared throughout the secondary and early collegiate curriculum and utilized functions as both objects and processes. Composition content was predominately presented using the algebraic representation and the use of compositive structure in transcendental functions was largely implicit. This examination provided background information for existing studies of student knowledge of composition and provides a framework for future studies of the teaching and learning of composition.

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KEY TO ABBREVIATIONS

GA1	Glencoe *Algebra 1*
GG	Glencoe *Geometry*
GA2	Glencoe *Algebra* 2
GP	Glencoe *Precalculus*
PA1	Center for Mathematics Education (CME) Project *Algebra* 1 published by Pearson
PG	Center for Mathematics Education (CME) Project *Geometry* published by Pearson
PA2	Center for Mathematics Education (CME) Project *Algebra 2* published by Pearson
PP	Center for Mathematics Education (CME) Project *Precalculus* published by Pearson
CP	Stewart, Redlin, & Watson *Precalculus* published by Cengage
CC	Stewart *Calculus* published by Cengage
WP	Connally, Hughes-Hallett, Gleason, et al. *Functions Modleing Change: A Preparation for Calculus* published by Wiley
WC	Hughes-Hallett, Gleason, McCallum, et al. *Calculus* published by Wiley

CHAPTER 1: INTRODUCTION

1.1 Statement of the Problem

Since the release of the report *A Nation at Risk* (1983), there has been an increasing emphasis on preparing students for calculus. Research has reported that many students, even those that receive high grades in calculus, have difficulty learning calculus topics and are only successful on routine problems with which they are familiar (Selden, Selden, & Mason, 1994; Tall, 1993). Research on topics that are prerequisite to calculus (i.e., functions and composition) has indicated that students' foundational knowledge of mathematics is weak, which contributes to the difficulty of learning calculus and advanced mathematics (Carlson, Oehtman, & Engelke, 2010; Ferrini-Mundy & Graham, 1991; Monk, 1994; Oehrtman, Carlson, & Thompson, 2008; Vinner & Dreyfus, 1989).

While the learning of the concept of function has been studied extensively, very few studies have focused on the learning of composition. Engelke, Oehrtman, and Carlson (2005) noted that "student understanding of function composition has not been a primary focus" of research studies (p. 1). The current literature on composition has documented that the learning of composition is complex. Research on the learning of topics built upon composition (i.e., chain rule) has reported that students' difficulties are related to a weak foundation of composition (Clark et al., 1997; Horvath, 2008).

The vast amount of research on the learning and teaching of functions without addressing composition would be like studying the learning and teaching of numbers without addressing arithmetic operations. It is composition that enables functions to be one of the fundamental concept of mathematics. Kawaski (2005) described composition as the characteristic operation of functions and through it nontrivial functions are created and studied. Freudenthal (1983) stated that "the strength of the function concept is rooted in the new operations - composing and inverting functions - which create new possibilities" (p. 523). Understanding the compositive structure of functions is as essential to comprehending functions and advanced mathematics as understanding the additive and multiplicative structure of numbers is essential to comprehending numbers and arithmetic.

Many calculus topics build on and use the operation of composition. For example, the chain rule, graph sketching, optimization, related rates, integration by u- or trig-substitution, and integration by parts all require students to identify different compositive parts of a function, equation, or expression. Given the importance of composition to the calculus curriculum, and the difficulties students have with calculus in general and composition in particular, this study analyzed the curriculum to identify the treatment and development of composition in the written curriculum from its introduction to its use in first year calculus. Studying the curriculum was chosen because curriculum influences learning and a curriculum study could provide context to the aspects of composition with which students have had difficulty and make recommendations for positive changes.

1.2 Study Overview

This study examined the explicit and implicit treatment of composition in secondary and early collegiate mathematics curricula. This analysis of textbooks of high school Algebra 1 and 2, Geometry, and Precalculus and collegiate Precalculus and Calculus textbooks utilizes multiple frameworks. Mathematically, composition can be viewed as an operation on functions as objects or as a sequence of processes where the output of one process is the input of the next process. All mathematical concepts that can be composed, such as functions (including transformations) and relations, were included in this study. The treatment of the concept of compositon was categorized by a framework of procedural, conceptual, and conventional knowledge. Conceptual knowledge includes the definitions and properties of composition and the connecting of composition to other concepts, procedural knowledge includes the doing or performance of composition, and conventional knowledge includes the language and notation of composition. This framework was used to describe the ways that textbooks define, explain, and perform composition. The representation (e.g., algebraic, graphical, tabular) and type of function, or relation was also included in the coding scheme of this study. These aspects were included to gain a more complete picture of the treatment of composition in written curricula.

1.3 Scope and Significance

This study contributes to the field of mathematics education by articulating the explicit and implicit curricular treatment of composition for the benefit of curriculum developers, secondary teachers, and collegiate instructors and by comparing the difference between the high school and college curricular treatment of composition. While this study does not provide information regarding what students actually learn about composition, it does identify the experiences that curricular materials offer to students. By examining the ways that composition is defined, explained, and used, I described the kinds of explicit and implicit opportunities that textbooks provide to students.

This curriculum analysis was motivated by the results of students' performance on composition tasks and contributes to the existing work on student knowledge of composition by studying a component (the written curriculum) that influences student knowledge. Current literature has reported on studies of the outcomes of student learning, while this study will address one of the inputs (i.e., curriculum) that support student learning. This study will provide context for the results of these past studies and direction for future studies of the teaching and learning of the composition concept. Additionally, the framework proposed for this study and its future refinement will be a useful tool for future studies of composition.

The following chapters further discuss the details of this study. In chapter 2, the theoretical and analytical frameworks are discussed. The concept of composition is conceptualized and relevant function and composition literature is reviewed. In chapter 3, the specific methods of the study are presented, including the rationale for selecting textbooks and identifying the curricular material as composition content. Chapters 4, 5, and 6 present the findings of this study. In chapter 4, I discuss the formal definitions of composition and the meanings other terms that relate to the composition content. In chapter 5, I discuss the compositive structure of functions. In chapter 6, I discuss the representations, types of functions, and implicit language used in the composition content. In chapter 7, I discuss these findings and their implications.

CHAPTER 2: THEORETICAL FRAMEWORK AND REVIEW OF THE LITERATURE

This chapter reviews relevant literature on curriculum, the teaching and learning of functions, and the teaching and learning of the composition of functions. I also explain the theoretical framework of how composition is defined. Lastly, I discuss the analytical framework used to study the treatment of composition in textbooks.

2.1 Studies of Written Curriculum

The prevalence of books in schools has provided motivation for studying textbook materials. Most classrooms have students, teacher(s), and textbook(s). Researchers of curriculum have analyzed the mathematical content of textbooks to study the relation of the written curriculum to students and teachers, to make comparisons between mathematics curricula of different countries (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997), and to compare NSF-funded mathematics curricula to curricula developed by private publishers (Huntley, Rasmussen, Villarubi, Sangong, Fey, 2000).

Many have studied curriculum in relation to students and teachers. Curriculum has been used to explain differences in students' performance on tests (Fuson, Stigler, & Bartsch, 1988; Li, 2000). Others have viewed textbooks as a source for teacher learning (Males, 2012; Newton & Newton, 2006; Remillard, 2005). The relationship between the written curriculum and enacted curriculum has also been a focus of curricular research (Newton, 2008).

In order to examine the written curriculum, researchers have focused on the characteristics and mathematical content of textbooks. Charalambous, Delaney, Hsu, and Mesa (2010) classified studies of textbook characteristics as *horizontal* analyses while studies of the mathematical content of textbooks were classified as *vertical* analyses. They noted that horizontal analyses "provided preliminary insights into the treatment of the content in textbooks…, they [did not describe] how concepts were treated within each textbook to structure learning opportunities for students" (p. 120). On the other hand, vertical analyses "overlooked how the treatment of the topic being examined relates to other topics contained in the textbook" (Charalambous et al., 2010, p. 120). A study of curriculum that incorporates both the vertical and horizontal analyses provides both a detailed perspective within a topic and the relation of that topic to other content across the curriculum. Since the combination of both of these analyses affords the most complete picture, this study utilizes this type of framework.

Studies of the mathematics content of curriculum materials have also varied in their sampling within textbooks. Some studies have sampled content within specific lessons. For example, Mesa (2010) sampled all examples within lessons on initial value problems. Other studies have sampled specific content throughout the entire book. Lithner (2004) sampled all examples and exercises throughout a calculus text. Individually, studies of curricular content within specific lessons and studies of curricular content across topics have provided valuable insights into the content of textbooks. Another study that incorporates sampling aspects both within lessons and throughout entire books (as employed in this study) could provide a more complete picture. Of the written curriculum

In summary, the written curriculum influences student learning directly or indirectly through the teacher. Prior research of written curriculum has used horizontal and vertical analyses as a framework to study to the content and structure of textbooks. Sampling within lessons on a specific topic (e.g., initial value problems) or across topics but within a specific feature (e.g., exercises) provides different perspectives of the organization of content within curriculum. The more of these aspects that a study includes, the more comprehensive picture the data will provide.

2.2 What is Composition?

The operation of composition was defined in Royden's (1988) graduate Real Analysis text by the following statement. "If f: X → Y and g: Y→ Z, we define a new function h: X → Z by setting $h(x) = g(f(x))$. The function h is called the composition of g with f and denoted by $g \circ f$ " (p. 10). While this definition refers to the composition of functions, composition is also valid for other mathematical concepts such as relations. For example, the composition of a translation and a reflection is a glide reflection.

Many high school and undergraduate textbooks have also explicitly defined composition as $(g \circ f)(x) = g(f(x))$. The equal sign between $(g \circ f)(x)$ and $g(f(x))$ in these definitions signifies that $(g \circ f)(x)$ and $g(f(x))$ are mathematically equivalent. $(g \circ f)(x)$ can be viewed as the composite function which has a domain, range and the rule $g(f(x))$. The circle ($\circ$) notation between two functions which is similar to other binary operations such as addition (+), subtraction (-), multiplication (× or ·), and division (÷). The circle operation composes two functions, f and g, and results in a new function, $g \circ f$. In this case, functions are the objects being acted upon. On the other hand, the composition rule, $g(f(x))$, denotes a sequence of functions where f corresponds x to $f(x)$ and g corresponds $f(x)$ to $g(f(x))$. Thus, the output of f, $f(x)$, is the input of g and it is the domain elements x and $f(x)$ that are being acted upon. Based on this mathematical difference, I will refer to $g(f(x))$ as a *sequence view of composition* and $(g \circ f)(x)$ as an *operation view of composition*.

The major difference between the sequence and operation view of composition is the way that functions are considered. In the sequence view of composition, functions are *processes*. A *process view of function* is "when the total action can take place entirely in the mind of the subject, or just imagined as taking place, without necessarily running through all of the specific steps" (Breidenbach, Dubinsky, Hawks, & Nichols, 1992, p. 249). For example, an individual with a process view of function could identify that $y = x^2$ results in a real number and that the number is a non-negative number without performing any calculation. Additionally, the function $f(x) = x^2$ can be viewed as the process of corresponding any number to its square. Considering functions as a correspondence is a process view of function.

In contrast, functions are *objects* in the operation view of composition and pairs of functions are acted upon by the composition operator. An object view of function exists when a function is treated as its own entity and not as a process of correspondence (e.g., as a noun instead of a verb). Others who have written about replacing processes with objects include Asiala et al. (1996) using the term encapsulation, Sfard (2008) using the term reification, and Martin (1991) using the term nominalization. The common feature among these theories is that processes are treated as entities which become the objects of other actions and procedures. An example of an object view of function is the composition of the function $g(x) = 2x + 4$ with $f(x) = x^2$, where $f(x)$ replaces the x's in the $g(x)$ function. This results in $g(f(x)) = 2(f(x)) + 4$ or $g(f(x)) = 2(x^2) + 4$. In this situation, $f(x)$ is treated as an object itself and not as a correspondence between its domain and range. Functions are objects in the *operation view of composition* and functions are processes in the *sequence view of composition*. Using functions as objects and processes will be further discussed in the sequence view of composition and operation view of composition sections, respectively.

While the operation view and sequence view of composition are described using the terms *object* and *process*, they describe functions and *not* composition. This is different than much of the literature where the terms action, process, object, and schema are used to describe the main mathematical concept in a study. Composition as an action, process, object, or schema could all exist within either the operation view or the sequence view. The operation and sequence views describe two ways that composition can operate on functions when functions are being used in two different ways.

Figures 2.1 and 2.2 illustrate the difference between these two views of composition. Figure 2.1 showing the sequence view demonstrates that the focus is on points in the sets.

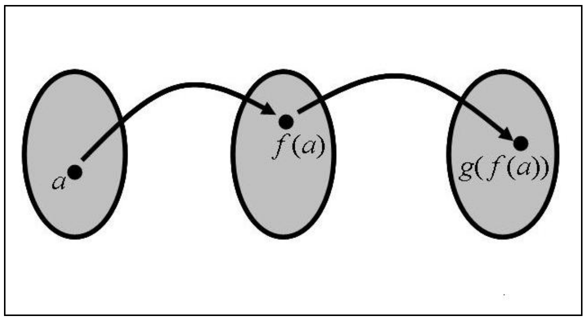

Figure 2.1: Sequence view of composition

On the other hand, the operation view shown in Figure 2.2 focuses on the functions *f* and *g*.

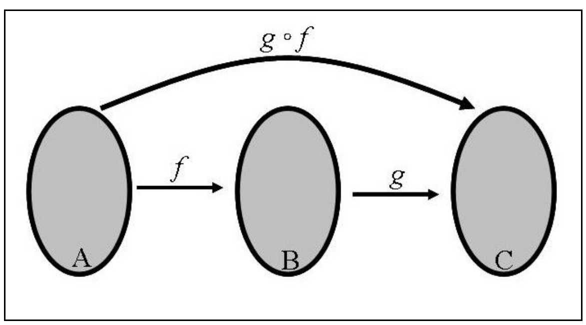

Figure 2.2: Operation view of composition

To clarify the discussion that follows, the terms associated with composition must be discussed. Unlike arithmetic operations, composition does not have specialized names to refer to the objects being composed. Addition has summands, multiplication has factors, subtraction has minuend and subtrahend, and division has dividend and divisor. Even though the components of a composition do not have specialized terms, there are colloquial ways to refer to them. The most common are the first and second functions or the inside and outside functions. In the rest of this paper, I refer to the *g* in ($g \circ f$) as the second or outside function and the *f* as the first or inside function. By definition, $g \circ f$ is read as "*g* composed with *f*" or the second or outside function is composed with the first or inside function.

The rest of this section will further discuss the sequence and operation views of composition and review relevant literature to each. The first section discusses the sequence view of composition and reviews research on student and teacher knowledge of functions and the composition of functions and the influence of representations. The second section discusses the operation view of composition and reviews the literature on student and teacher knowledge of composition related to the operation view.

2.2.1 Composition as a sequence. As stated above, the sequence view of composition describes composition as a sequence of recursive relations (including functions) where the output of the n^{th} term is the input of the $(n+1)^{\text{th}}$ term (or relation). In this view of composition, functions are characterized as processes. Harel and Kaput (1991) referred to this process of "acting on individual elements of [the] domain" and called it a *point-wise operation* (p. 84).

For decades, educational researchers have extensively studied student knowledge of function and have reported that learners of mathematics systematically solve problems or answer questions involving function in standard and nonstandard ways (Carlson, 1998; Even, 1990, 1998; Ferrini-Mundy & Graham, 1991; Leinhardt, Zaslavsky, & Stein, 1990; Monk, 1994; Oehrtman et al., 2008; Vinner & Dreyfus, 1989). A review of the function literature is included as a basis for discussing a recursive sequence of functions. This review defines key terms in the literature and discusses some of the research findings, including the role of representations (algebraic, graphical, tabular) on student knowledge.

2.2.1.1 Function. Much of the research on the learning of function has used the formal Dirichlet-Bourbaki function definition. This definition defines a function as "a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)" (Vinner and Dreyfus, 1989, p. 357). Using this definition, researchers determined whether students' responses to tasks corresponded with this definition. If an individual used the definition (or anything consistent with the definition) in his or her response, it was assumed that the individual had "successfully learned" this concept. However, if the response contained any deviations from the definition, the individual or group may have been described as having a *misconception* or a *limited understanding* of function. These terms include situations where a student may have provided the correct answer, but did not demonstrate knowledge regarding why the answer was reasonable with respect to the definition.

Leinhardt, Zaslavsky, and Stein (1990) defined misconceptions "as incorrect features of student knowledge that are repeatable and explicit" (p. 30). The term "limited understanding" has been used in the literature to describe an individual or group who has not demonstrated "complete understanding". Limited understanding has also been used to describe students that perform procedures but who do not "possess an understanding of why [a] procedure works" (Oehrtman et al., 2008). For example, a student might use the vertical line test to determine if a graph is a function, but is unable to explain why it makes sense to use the vertical line test.

A major portion of the research literature on student knowledge of functions involved the classification of relations as functions or non-functions (Leinhart et al., 1990). These studies have indicated that students do not base their decisions solely on the Dirichlet-Bourbaki definition even though these same students accept this definition (Leinhart et al., 1990; Meel, 1999; Vinner & Dreyfus, 1989). The majority of these studies were designed to first ascertain students' definitions of function and then compare their definitions to their responses on tasks that had them classify relations as functions or non-functions. Researchers used open ended questions (i.e., What is a function?) and multiple-choice tasks to determine students' definitions. Students' responses for defining function have been categorized as a (a) correspondence, (b) dependence relation, (c) rule (with regularity), (d) operation/computational process, (e) formula, (f) representation, (g) set of ordered pairs, and (h) one-to-one (Carlson, 1998; Hitt, 1998; Meel, 1999; Vinner & Dreyfus, 1989). Table 1 includes examples of students' responses for some of these categories.

Table 2.1
Examples of the categorization of students' responses for defining function.

Category	Student Response
correspondence	"A function is a correspondence between two sets that assigns to every element in the first set exactly one element in the second set" (Meel, 1999, p. 3)
dependence relation	"One factor depending on the other one" (Vinner & Dreyfus, 1989, p. 360).
rule (with regularity)	"The result of a certain rule applied to a varying number" (Vinner & Dreyfus, 1989, p. 360).
operation/computational process	"Every function can be expressed by a certain computational formula" (Meel, 1999, p. 5).
Formula	"A function is a formula, algebraic expression, or equation which expresses a certain relation between factors" (Meel, 1999).
Representation	"$y = f(x)$" or "A graph that can be described mathematically" (Vinner & Dreyfus, 1989, p. 360).

Students' definitions were compared to their responses on tasks that had them classify relations as functions or non-functions. Even though the majority of the tasks presented in the literature were expressed verbally and graphically, each of these studies noted that algebraic formulas played a role in a students' decision between function and non-function. Meel (1999) reported that over 20% defined a function as a formula and Vinner & Dreyfus (1989) reported similarly for about 10%. Additionally Hitt (1998) pointed out that one-third of his subjects abandoned the vertical line test when there existed “an algebraic expression with the curve (ellipse, circle, etc.)” (p. 133). Hitt claimed further that this may be due to a curriculum heavy with algebraic symbolism and noted that “teachers have a marked tendency to construct continuous functions defined by a single algebraic expression” (p. 128).

When students were asked to classify functions that deviate from “nice” types of functions (i.e., functions with discontinuities, split domains, and exceptional points), they classified them as non-functions more frequently than “nice” functions (Even, 1993). Even (1990) conjectured that the arbitrary nature of functions like these was rejected because students “expected functions to always be representable by formulas, graphs of functions to be ‘nice’ and ‘reasonable’, or functions to somehow be ‘known’ ” (p. 529). Even related students’ expectations regarding the niceness and reasonableness of functions to the prevalence of “nice” functions in the high school curriculum.

The constant function is another type of function that students frequently classified as a non-function. Vinner and Dreyfus (1989) documented that approximately 55% of their participants incorrectly classified the constant function as a non-function. Carlson (1998) found that 57% of her college algebra participants performed likewise. Both the Vinner and Dreyfus and Carlson studies presented this task verbally. Carlson reasoned that these students may “not be able to translate verbal function language to algebraic function notation” (p. 122). Later in her report, Carlson provided evidence to support this claim with interview data where college algebra participants were unable to explain what the statement “express *s* as a function of *t*” would look like algebraically. If students’ inability to translate verbal function language into algorithmic function notation was a key factor in their classifying functions and non-functions, would students have been more successful if the constant function was represented algebraically?

In summary, each of these studies noted that algebraic formulas affected students' classifications of functions. However, the way(s) in which algebraic formulas affected students' responses is not clear because so few of the tasks were algebraic tasks. Additionally, the role of the mathematics curriculum is not empirically described in these studies. Even though the curriculum was proposed as a plausible explanation for students' responses, it must be noted that these claims were found in studies involving students and not analyses of the curriculum (intended or enacted). By studying student knowledge, these studies focused on a presumed outcome of the curriculum. Since curriculum is among the sources for learning, epistemologically, it was appropriate to assume that features of the curriculum played a role in these outcomes. However, none of these studies closely examined curriculum materials that directly related to and/or supported the claims made about the curriculum.

2.2.1.2 Composition. Unlike function, composition has not been the focus of many research studies. Engelke et al. (2005) noted that "while many authors have noted that composition…problems are challenging for students and these problems are likely tied to a weak function conception, student understanding of function composition has not been a primary focus" (p. 1). This section reviews the literature (the majority of which is conference proceedings or unpublished dissertations) on composition directly related to the sequence view of composition. This literature can be grouped in two categories: student and teacher knowledge on input/output or domain/range and student success rates in different representations.

Research has shown that students have viewed composition as a recursive input and output sequence of functions and that teachers have identified domain and range as prerequisite knowledge to learning composition. For example, a calculus student in Vidakovic's (1996) study evaluated the composition of $g(x) = \begin{cases} |x-1| & \text{if } x > 1 \\ 2 & \text{if } x \le 0 \end{cases}$ with $f(x) = x - 2$ at the point $x = 2$ by first determining $f(2)$ to be 0 and then evaluating g at 0 to obtain the result of 2.[1] Similarly, Carlson (1998) asked students to compute $f(x + a)$ given $f(x) = 3x^2 + 2x - 4$. Students discussed this composition by "describing $x + a$ as the input of the function, or…that they were evaluating the function f at [the new value of] $x + a$, and that the solution after the evaluation was the output to the function" (p. 129). Or in other words, the output of $x + a$ was the input of $f(x)$.

The sequence view of composition appears in the concept of inverse function when an inverse is conceptualized as undoing what the function has done or as reversing a process (Even, 1992; Tall & Razali, 1993). An inverse function is a function f^{-1} such that the composition of f^{-1} and f results in the identity function. Pointwise, this means that the value of the input of the second function is equal to the output of the first function at the end of the composition process. Even (1992) provided students with formulas for f and f^{-1} and asked them to evaluate $(f^{-1} \circ f)(512.5)$. Half of the students that answered this question performed calculations to evaluate $f(512.5)$ and then evaluated f^{-1} at the value of $f(512.5)$. Another 20% made statements related to the properties of inverse functions, but still performed the calculation of $f^{-1}(f(512.5))$. This demonstrates how students can use the sequence view of composition when applying the

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[1] I have adjusted the notation of all problems from cited literature so that all of the composite functions are denoted as $g \circ f$. The mathematics has not been changed, only the naming and notation of functions.

notion of composition to other concepts instead of using the properties of those other concepts, in this case inverse functions.

As with all concepts, students do not always perform composition correctly and may or may not be consistent in their incorrect methods and procedures. A common occurrence is for students to interpret composition as multiplication while using formulas, graphs, and tables (Engelke et al., 2005; Meel, 1999). Engelke et al. (2005) provided the following excerpt from a precalculus student attempting to determine $f(g(3))$ with f and g given by a table.

> I was trying to look for a formula to see what $f(x)$ was but it's not working so I'm doing it wrong. That's not the right way to go about it… $g(3)$ is 0. f of g of 3. Now I'm tempted to say it's f of g times $g(3)$, but I know that's totally wrong, but if I did do that, -2 times 0 would be 0. And I know that $g(3)$ would be 0, so no matter what, my answer is 0, even if I do it wrong. I don't know what, yeah, I'm just gonna say 0 because… (p. 3)

At first, this student attempted to determine a formula from the table values. When unsuccessful in finding a formula, he or she reinterpreted the composition statement of $f(g(3))$ as the multiplication statement of $f(3) \cdot g(3)$.

Students have different success rates on composition problems in different representations. When asked to evaluate $g(f(2))$, Carlson et al. (2010) reported that 94% of students were successful given two algebraic functions, 50% with graphical functions and 47% with tabular functions. Hassani (1998) reported students' success rates as 84%, 10%, and less than 50% for algebraic, graphical, and tabular, respectively. When the task was rephrased to evaluate $(g \circ f)(2)$ the success rates of students in Hassani's study changed to 35%, 25% and 33%, respectively. In an interview with a student in a developmental algebra course, DeMarois

& Tall (1996) reported that the student was able to complete a composition task using the table with considerable guidance from the interviewer. The student was then unable to even begin a graphical composition task. The student was then given an algebraic composition task, at which he was successful with minimal guidance.

Based on this research, algebraic composition tasks appear to be easier for students than other representations. One explanation, such as the one mentioned in the function section, has claimed that this is due to a curriculum that is heavily algebraic and that students have had more exposure and experience with dealing with the algebraic representation. An alternative explanation may be that the algebraic representation provides opportunities in composition tasks that are not available in other representations. An example of one such opportunity is the ability to describe the global behavior of a composite function without performing a pointwise operation or sequence of functions. For instance, a table provides only pointwise information about a function which means that determining a composition of two tabular functions requires the sequencing of functions. Graphing the composition of two graphical functions requires the similar actions. The resulting composite graph, of course, does provide information about the global behavior of the composite function, but the process of composition itself requires a sequencing of functions. The composition of two algebraic functions, however, can be performed pointwise across the domain or by "plugging in" the entire formula of the first (or inside) function into the appropriate variable of the second (or outside) function. Because of this capability within the algebraic representations, $(g \circ f)(2)$ can be determined either by evaluating $g(f(2))$ or by first determining the formula for $(g \circ f)(x)$, calling this composite $h(x)$, and then evaluating $h(2)$. This second method is the operation view of composition that will be described in the next section.

In summary, the *sequence view of composition* is a pointwise operation where the first (or inside) function is evaluated at a point and the second (or outside) function is then evaluated at the output value of the first (or inside) function. This section has noted that the concepts of function, domain, and range are directly related to this view and representation is related to students' success rates on composition tasks.

2.2.2 Composition as an operation. The operation view of composition entails composition as a binary operation on two sets of mathematical entities (e.g., relation, function). These entities are treated as objects and not as processes of correspondence. The result of this operation is also an object of the same type. The main feature of this method is that g is composed with *f* without considering any input/output values while performing composition. For example, given $g(x) = \frac{x}{\sqrt{1-x^2}}$, for $-1 \le x \le 1$ and $f(x) = \sin x$ for all real x, the operation view of composition would evaluate $g(f(x))$ at $x = \frac{\pi}{4}$ by first determining (mostly like via substitution) that $g(f(x)) = \tan x$ and then evaluating $\tan\left(\frac{\pi}{4}\right)$ to equal 1 (Ayers et al., 1988).[2] Only after composition has resulted in a new function is the function evaluated. A glide reflection is a geometric example. A glide reflection is the resulting transformation of the composition of a translation and a reflection. The focus is on the combination of transformations and not what physically happens to a figure. Another example is the composition of two graphs where either the inside or outside function is the graph of the identity function. The composition will yield a graph that is the same as the other graph in the composition. With respect to inverse function, the composition of *f* and f^{-1} results in the identity function. One final example could be the

[2] The incorrect domain of *g* is in the original Ayers et al. article.

composition of two parallel reflection transformations on a geometric shape. The result is a translation transformation with magnitude double the distance between the two lines of reflection. In all of these examples, the result of composition is an object that is understood or described without considering individual points or values.

Students implement the operation view of composition by plugging in or substituting the inside (or first) function for a variable in the outside (or second) function (Ayers, et al., 1988; Carlson, 1998; Horvath, 2010; Uygur & Ozdas, 2007) or by (mis)interpreting composition as a multiplication operation (Horvath, 2010; Meel, 1999). Using the example from Ayers et al. (1988) above, the formula $F(G(x)) = \tan x$ would be determined by first substituting $\sin x$ into all the x's of $F(x)=\frac{x}{\sqrt{1-x^2}}$ resulting in $F(x)=\frac{\sin x}{\sqrt{1-(\sin x)^2}}$. Students have determined the composite function prior to evaluating it at a point or its derivative (Horvath, 2010; Uygur & Ozdas, 2007). Carlson (1998) also showed students' tendencies for such substitution when students interpreted $f(x + a)$ as $f(x) + a$ and explained that they were plugging the expression for $f(x)$ in for x in $f(x + \mathrm{a})$. While this is the incorrect substitution, it does demonstrate students' knowledge of function composition as a substitution operation.

As in the sequence view of composition, students have interpreted composition as multiplication (Horvath, 2010; Meel, 1999). In the operation view of composition, this can be represented symbolically as $(f \circ g)(x) = f(x) \cdot g(x)$. The difference between the sequence view and the operation view is that students not only multiply numbers, but are also multiplying objects such as functions in the latter. Interpreting composition as multiplication may at first appear to be superficial, but after identifying commonalities between the notation and language used for these operations one may wonder why more students do not have this problem. The most basic example is the open circle '∘' for composition and the closed circle, or dot, '·' for multiplication. The word 'of' is another example. 'f of g' means 'f composed with g' and '$\frac{1}{2}$ of $\frac{1}{4}$' means multiply $\frac{1}{2}$ and $\frac{1}{4}$. The parentheses are a commonality with the notation $f(g(x))$ representing composition and $5(3x)$ representing multiply 5 times the quantity $3x$. Even the word composite is used with respect to both multiplication and composition. For instance, a composite number is an integer with more than two factors (i.e., multiplication) while a composite function is a function of a function (i.e., composition).

In Horvath's (2010) study, not all students who interpreted composition as multiplication were consistent with their interpretation when evaluating the derivative of a composite function. For example, some students who interpreted $(f \circ g)(x)$ as $f(x) \cdot g(x)$ did not use the appropriate product rule to find the derivative despite being able to use the product rule on other tasks. Instead they multiplied the derivatives, $f'(x)$ and $g'(x)$, together or performed chain times-ing which is represented as $f'(x) \cdot g(x) \cdot g'(x)$. These inconsistencies may indicate that these students did not know their techniques of derivation. Another explanation may be that students who use multiplication in place of composition know that the operation of composition is different from multiplication, but are uncertain as to the differences. When working on a composition task, they are left to rely on contextual clues found in the notation to decide what to do which leads them to treat composition as a strange form of multiplication.

Students have also interpreted composition as addition (Horvath, 2010). A search of the Common Core State Standards for Mathematics (CCSSM) (2010) on the words compose, decompose, composite, composition, and decomposition has provided some insight to such an interpretation. The search yielded 31 statements. Twenty-three of these indicated an additive structure (e.g., the number 11 is composed of a ten and a one, determine the area of a shape by decomposing into triangles), 3 indicated a multiplicative structure (e.g., a number being composite or prime), and 2 indicated a compositive structure (e.g., verify an inverse function by composition). The idea of interpreting composition as addition is more noteworthy when considering that approximately 74% of the CCSS-M statements with the root "compos" indicated an additive structure of mathematics.

The purpose of the discussion on language and notation is not to advocate the creation of new words or notations; rather to draw attention to issues that may be correlated to the difficulty of learning composition. The fact that students solve composition tasks using other operations that use the same language or notation as composition suggests that teachers and learners of mathematics need to be aware of these similarities and how these similarities have different meaning in each context. Not only are these similarities a potential obstacle, they are a potential strength as well. One example of a potential strength is illustrated in the following transcript. In it the calculus student was defining the inverse function of *f*.

> *Student*: Aaaa...any function such that, any function *g* of *x* such that *f* of *g* is equal to *x*.
>
> *Interviewer*: What do you think would be *g* of *f* in that case?
>
> …
>
> *Student*: Yeah, it's *x*.
>
> *Interviewer*: Why? Do you think that would work for any two functions *f* and *g*?
>
> *Student*: Yes, it should because the inverses multiplied together in any order should be equal [to] one and like any normal numeric, algebraic...so yeah, any two functions that are inverses of each other, *f* of *g* of *x* and *g* of *f* of *x* is equal to *x*. (Vidakovic, 1996, p. 307).

In this episode, the student used information about multiplicative inverses to make claims about compositive inverses (or inverse functions). Analogies like these are essential for the learning of mathematics. The use of similar words and notations supports the creation of such connections across mathematical concepts.

In summary, the operation view of composition treats both the inside and outside functions as objects and the result of the composition is also an object. Students have interpreted composition as a substitution or "plugging in" operation, addition, multiplication, and as an unorthodox quasi-multiplication operation that has some similarities to multiplication. All of these operations use the word "compose" and the parenthetic notation but the meanings are different for each operation.

This section has discussed composition as a sequence of processes and as an operation on objects. It is not the case that one view is more important or useful than another; rather "there are times we have to deal with functions pointwise…and there are times when we deal with functions as entities or objects" (Even, 1990, p. 533-4). There are also times when we have to consider both. For example, given $g(x) = \sqrt[4]{x}$ and $f(x) = x^2$ on the set of real numbers, the operation view of composition would result in $(g \circ f)(x) = \sqrt[4]{x^2}$. It is common for students to then simplify this type of composite function with the exponent rule $(a^b)^c = a^{bc}$, resulting in $(g \circ f)(x) = \sqrt{x}$ without considering the differences in the domains (Lucas, 2006). All real numbers are the domain of $\sqrt[4]{x^2}$, while only non-negative real numbers are the domain of $\sqrt{x}$. Thus, after performing an operation on objects, one must consider which values are valid inputs on the result. Only by viewing functions pointwise and as object can the correct simplification of $(g \circ f)(x) = \sqrt{|x|}$ be attained.

Other main points in this section were that composition was interpreted as other operations that use vocabulary and notation similar to composition and that representation influences students' understanding of both views of composition. It was noted that tables and graphs are more suitable to the sequence view than the operation view while algebraic formulas permit both. These issues raise the questions: How do curricular materials treat composition? How is it addressed explicitly and implicitly? The next section will describe the general theoretical perspective that may be used to study these and other issues. The specific details will be discussed later in the data analysis coding schemes in the methods section.

2.3. Analytical Framework

For almost a century educational researchers have discussed and debated two kinds of knowledge which have been commonly referred to as procedural and conceptual (Baroody, Feil, & Johnson, 2007; Hiebert, 1986; Star, 2005). Other labels for these two types of knowledge have included skill and understanding (Thorndike, 1922); efficiency and understanding (Brownell, 1938); instrumental and relational understanding (Skemp, 1976); and procedural fluency and conceptual understanding (Kilpatrick, Swafford, & Findell, 2001). While there appears to be agreement to distinguish between these two types of knowledge, defining each in a general sense has been difficult (Hiebert & Lefevre, 1986), due in part to the interwoven nature of procedural and conceptual knowledge. Furthermore, the theory regarding procedural and conceptual knowledge and the ways that educational researchers have operationalized these terms have not been aligned (Baroody et al. 2007; Star, 2007).

In a seminal work regarding procedural and conceptual knowledge Hiebert and Lefevre (1986) made a significant effort to define both terms. Procedural knowledge was characterized as consisting of two components: "One part is composed of the formal language, or symbol representation system, of mathematics [and] the other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6). Formal language includes knowledge of the form or style in communicating mathematics and "a key feature of procedures is that they are executed in a predetermined linear sequence" (p. 6). These procedures include symbol manipulation, problem solving strategies and anything involving sequential relations or actions (Baroody et al., 2007; Schneider & Stern, 2005).

> Conceptual knowledge was characterized by Hiebert and Lefevre (1986) "as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. … In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of a conceptual knowledge only if the holder recognized its relationship to other pieces of information." (p. 3-4)

To put this web or network characterization in terms of discrete graph theory, conceptual knowledge is a graph where both the vertices and the edges are essential and of equal importance. Despite their equal importance, however, an isolated vertex (a vertex that is not an endpoint of an edge) is not considered part of the graph.

The graph theory metaphor causes one to ask: What are edges and vertices? It would seem natural to associate vertices with concepts and edges with connections. However, edges and vertices are not mutually disjoint sets. It is possible for an item to be a vertex in some situations and an edge in others. For example, the algebraic equation $y = mx + b$ can be a conceptual edge that connects the concepts of y-intercept and slope. This same equation can also be a conceptual vertex where the y-intercept concept is the edge connecting the b in the equation to a line intersecting the y-axis on a graph or the y-value on a table that is associated with $x = 0$. Defining vertices and edges becomes more difficult with the fact that finding the y-intercept can be classified as a procedure and would fall under procedural knowledge. While this emphasizes how interwoven procedural and conceptual knowledge is, it makes defining conceptual knowledge (i.e., vertices and edges) in a general case to be a difficult endeavor.

Conceptual edges can be interpreted as being superficial or deep. Hiebert and Lefevre (1986) called this the primary and reflective levels. At the primary level "the relationship connecting the information is constructed at the same level of abstractness…than that at which the information itself is represented" (p. 4). An example from differentiation is recognizing that the derivatives of $\sin(x^2)$ and $\tan(2x^3)$ must involve the derivative of the argument. A relationship between two functions is formed, but nothing further. Even the more general notion that all trigonometric functions have this property in common is at a level of equal abstractness. In contrast, at the reflective level "similar core features in pieces of information that are superficially different" are recognized (p. 5). Continuing with an example from differential calculus, a recognition that $\frac{d}{dx}\int_2^{x^2} f(t)dt$ involves the same principle (i.e., chain rule) as $\sin(x^2)$ would be a connection between two superficially different objects.

The exclusion of isolated vertices from conceptual knowledge further complicates matters of defining such knowledge. For example, the formal definition of a function may be known by a student, but if it is not connected to something else (e.g., the vertical line test) it is not conceptual knowledge, according to Hiebert and Lefevre, nor is it a sequential relation (or step-by-step procedure). It might be argued that this can be classified as a procedure with the step: "When asked for the definition respond with the formal definition statement." However, this sort of classification is problematic in two ways. First, it is forcing situations to fit into existing theory instead of creating or modifying theory to fit with existing situations. Second, this makes the procedural knowledge category a “dumping ground” for anything that researchers decide doesn’t “fit” in the ambiguous notion of conceptual. This dumping into procedural knowledge causes procedural knowledge to be regarded as “anything that is not conceptual” and not recognized in its own right.

Characterizing conceptual and procedural knowledge as interwoven, complementary components permits an item, such as an isolated definition, to not fall under either knowledge category. This means that either (a) there are other kinds of knowledge outside of conceptual and procedural, (b) items such as this example are not considered to be knowledge, or (c) the definitions of conceptual and procedural need to be adjusted to include such situations. Despite the ambiguity of the terms procedural knowledge and conceptual knowledge, there is consensus that procedural knowledge alone does not imply conceptual knowledge and conceptual knowledge alone does not imply procedural knowledge; rather the two are interwoven (Hiebert & Lefevre, 1986; Kilpatrick et al., 2001).

The characterization of conceptual knowledge above relies on a person or student explicitly identifying connections (edges) between bits of information (vertices). A curricular analysis based on such a characterization would identify statements of explicit connections as conceptual knowledge elements, step-by-step procedures as procedural knowledge elements, and leave everything else either uncategorized or as implicit connections which would be dependent on a researcher’s explicit connections (or conceptual knowledge) resulting in different outcomes for different researchers. In order to be rigorous and reliable, this study allows isolated bits of information in its definition of conceptual knowledge and includes explicit connections, definitions, and properties. For composition this includes the following elements:

1. Meaning/Definition of composition:
    a. Operation view of composition: The result of a composition of the same class as the inside and outside object (e.g., transformation, function, relation)
    b. Sequence view of composition: The output of one relation is the input of the next relation (but not the actual computation)
2. Properties of composition such as associativity and (non)commutativity
3. Domain and range of $(f \circ g)$
4. Non-uniqueness of decomposition

These items need not be explicitly connected to other items to be classified as conceptual knowledge. In this study, they may exist in isolation.

Procedural knowledge has also been slightly adjusted for this study. Procedures and algorithms (such as substitution) and their demonstrated use will be called *procedural knowledge*, while the vocabulary, formal language, and mathematical symbols will be called *conventional knowledge*. The separation of these two aspects of the classic definition of procedural knowledge was made due to reports in the literature that students interpret the operation of composition as multiplication and that composition and multiplication share similar vocabulary and notation. Having two categories will help identify what opportunities students are given by curricula to learn the similarities and differences between these two operations.

These three categories were also suggested by Van Dormolen (1986) for analyzing mathematical texts. He named his categories *theoretical* for mathematical structure such as theorems, definitions, axioms, and properties, *algorithmical* for ‘how to do’ rules, and *communicative* for conventions. These categories have been used to study the treatment of "angle" in textbooks from different countries (Haggarty & Pepin, 2002) and the treatment of the measurement concepts of length, area, and volume in US textbooks (Smith et al., 2008).

Three categories were used to analyze the texts in addition to the conceptual, procedural, and conventional categories. They are called *representation*, *function type*, and *location*. The representation category captured the type of representation (e.g., algebraic, graphical, tabular, mapping) used to represent the inside function, the outside function, and the result of the composition. Many studies (discussed earlier) of students' knowledge of composition have noted the role of representation. The function type category will capture the type of function used in the different parts of the composition (e.g., polynomial, trigonometric, exponential, piece-wise, etc.). The role of function type was originally noted by Webster (1978) but his study only accounted for four different inside-outside function combinations. This study has been expanded to include sixty-four such combinations. Lastly, the location category will identify each unit of data as being in the exposition, examples, or exercises of the text. This will be used at the end of the analysis to determine where the conceptual, procedural, and conventional categories are typically located in the textbook.

In summary, this section has discussed how the components of conceptual, procedural, and conventional knowledge have been characterized in the mathematics education literature and how this study has slightly adjusted those characterizations. Adjustments have specifically included allowing isolated bits of information to be characterized as conceptual knowledge and dividing procedural knowledge into the two elements of procedural and conventional. The specific coding scheme of each knowledge component will be discussed in detail in the data analysis of the next section.

CHAPTER 3: METHOD

3.1 Specific Research Questions

As previously discussed, composition is (1) an important topic in mathematics, (2) students have difficulties in understanding and using composition, and (3) researchers have presented anecdotal evidence that these difficulties may be related to features of the curriculum. This raises the question, "How does the notion of function composition get developed in school mathematics?" This question could be investigated by studying (a) the written curriculum, (b) the teaching of function composition, and/or (c) the students who are learning about function composition. While the written curriculum does not determine what teachers teach or what students learn, the written curriculum does influence both (Remillard, Herbel-Eisenmann, & Lloyd, 2009; Stein, Remillard, & Smith, 2007). In order to better understand the potential influence of the written curriculum on the opportunities students have to learn, this study analyzed the development of the concept of function composition in written curriculum over the span from Algebra to Calculus.

High school curricula were analyzed to study ways in which students are introduced to composition in high school. The analyzed texts included the curricular series of Algebra 1 and 2, Geometry, and Precalculus. Because the notion of composition is developed further in calculus, which many students study in college, collegiate Precalculus and Calculus texts were also analyzed. The duplication of the precalculus text at both the high school and college level helped to identify differences in the preparation for calculus at these two levels.

The following questions are the focus of this curricular analysis of textbook treatment of composition across high school and collegiate texts.

(1) When is composition formally introduced and how is it originally defined and explained to students?

(2) What vocabulary terms and notations are used with composition and how are they defined, explained, and used?

(3) Which representations and types of functions are used in the composition content of mathematics textbooks?

(4) Which topics are explicitly connected to composition and in what ways?

(5) Is the high school treatment of composition different than the treatment in college precalculus? If so, how?

The purpose of this study was to examine one component (the written curriculum) that is involved in the teaching and learning of composition. The focus of this study was to examine the treatment of composition across the curriculum and not to analyze differences between individual texts. Due to this, only differences between courses will be discussed, not differences between texts within a single course.

The rest of the section will discuss the rationale for selecting specific curricular series, the criteria for textbook material to be included in the study, and the coding scheme used in the data analysis.

3.2 Selection of Curricular Series

In order to study the treatment of composition across the curriculum, entire curricular series were analyzed. At the secondary level, a series consists of textbooks for the courses of Algebra 1, Geometry, Algebra 2, and Precalculus that are produced and marketed by a single publisher. Similarly, at the collegiate level, a series is defined as precalculus and calculus texts marketed by a single publisher with common authors. Two secondary series and two collegiate series were analyzed in this study. This resulted in a total of 12 individual textbooks.

In both the secondary and collegiate levels, one of the selected series was widely adopted by schools, while the other series approached the teaching and learning of composition differently than the more popular series. Analyzing a series with a large user base provided information about the mathematical curriculum that affects many students, while choosing a second series with different features provided a broader view of the treatment of composition across curricula.

The two secondary mathematics curriculum series selected to be analyzed were *Glencoe/McGraw-Hill Mathematics* (2010) and the *CME Project* (2009) published by Pearson. Glencoe/McGraw-Hill Mathematics was chosen due to Dossey, Halvorsen, and McCrone's (2008) report that this series had a large share of the secondary mathematics textbook market. In this series, the mathematical content is arranged according to the traditional sequence of courses: Algebra 1 (2010), Geometry (2010), Algebra 2 (2010), and Precalculus (2011), and follows the format of exposition-examples-exercises within each lesson. The *CME Project* content was also arranged according to the mathematical content of Algebra 1, Geometry, Algebra 2, and Precalculus. Instead of the demonstrative ("how to") examples that Glencoe/McGraw-Hill provided, the CME Project materials mix problems with exposition and examples such as the format of exposition-problems-exposition-example-problems-exercises. With respect to composition, the Glencoe/McGraw-Hill Mathematics series focuses on substituting one expression into another, while the CME Project introduces composition as the linking of function machines and formally defines composition as an operation on functions.

At the collegiate level, a widely used precalculus and calculus series was chosen by surveying approximately 100 Department of Mathematics' websites and identifying the texts used for calculus and precalculus courses. The institutions chosen for the survey were those classified as very research intensive in the Carnegie Classification. This survey was conducted in June 2010.

The process for identifying the textbook that was adopted at each institution involved multiple steps. First, the calculus sequence recognized by the institution to fulfill the calculus requirement for engineering and science (including mathematics) majors was determined. Choosing this sequence allowed the survey to compare texts with similar target audiences. After identifying this sequence at each institution, the text used for the sequence was determined. I identified the textbooks adopted for use during the Spring 2010, Summer 2010, and Fall 2010 academic terms by perusing the mathematics department website, course syllabi, or the student bookstore. In rare cases the text for the engineering and science calculus sequence was in transition with a new text being phased in over multiple semesters. For example, Calculus I would be using a new book, while Calculus II would use the older book. In these situations the new text being phased in was recorded as the textbook adopted by that institution.

Almost 30% of the institutions surveyed used *Calculus: Early Transcendentals* by Stewart and an additional 20% used Stewart's *Calculus* (non early transcendentals). See Appendix A for the complete survey results. The most recent edition (the hybrid edition) of Stewart's *Calculus: Early Transcendentals* (2012) published by Cengage was the edition chosen to be analyzed for this study. The second calculus text selected is *Calculus, 5th edition* (2009) by Hughes-Hallett, Gleason, McCallum, et al which was published by Wiley. The approach in the Hughes-Hallett text differs from the Stewart Calculus text in that the latter utilized more rigorous proof to explain mathematical concepts while the former provides a mixure of justifications and proofs (Askey, 1997). The Hughes-Hallett text incorporated more multiple representations (e.g., graphs, tables) throughout the text than the Stewart book. Additionally, the Hughes-Hallett, et al (2009) text was the most widely adopted text after the Stewart texts (approximately 6% of the institutions).

The process of determining a widely used collegiate precalculus text was not as straightforward as the process for the calculus texts selected. Not all institutions offered precalculus courses. Some institutions offered multiple courses below calculus, but none with the course title of "Precalculus". Occasionally multiple courses at a single institution were described as a preparation for calculus. Some course syllabi named the course as precalculus when the institutional title and course description did not. The first situation of institutions without a precalculus course was resolved by separating the survey into subsets: institutions with both calculus and precalculus courses and others with only calculus courses. Because course descriptions on syllabi varied by instructor, the institution's course description was used to identify the course that was intended to prepare students for the calculus sequence for engineering and science majors. Precalculus courses for other calculus sequences (e.g., Calculus for biology majors) were not included. Once the course number was identified, the associated text was identified using the same approach as that used for the selection of the calculus text.

The results of the precalculus survey identified *Precalculus: Mathematics for Calculus,* by Stewart, Redlin, and Watson as the most widely used precalculus text. The most recent edition available, the *5th edition* (2007), was chosen for this study. The most popular calculus and precalculus texts were from the same publisher (Cengage) and have a common lead author (Stewart). To follow this same pattern, the second precalculus text chosen was *Functions*

Modeling Change: A Preparation for Calculus, 4th edition (2011) by Connally, Hughes-Hallett, Gleason, et al. Both the second calculus and precalculus texts were published by Wiley and authored by Hughes-Hallett and Gleason.

3.3 Data Collection and Analysis

The next step was to identify the content in each book that was related to composition. Studies of curriculum content have analyzed different parts of textbooks. For example some studies of curricular content have analyzed entire units or lessons that the researchers identified as reasonable places for finding specific content (e.g., Cai, Lo, & Watanabe, 2002; Yan, Reys, & Wu, 2010) while other studies have analyzed all the problems and examples in a text (e.g., Ding & Li, 2010; Mesa, 2010). Few studies, however, have analyzed the entire content of a textbook for all the instances of a particular concept (e.g., Ashcraft & Christy, 1996). In order to answer the question regarding the treatment of composition across the secondary and early collegiate curriculum, all content in each textbook was analyzed.

The content in each textbook was analyzed in three different phases. These multiple phases supported the deliberate process of identifying the composition content efficiently while minimizing the possibility of excluding core content throughout each textbook. In this study, content considered core to composition included (1) the compositive combination of functions such as the sequencing of functions where the output of the first function is the input of the second function and the composite operation on functions, (2) the decomposition of a function into multiple parts which, when composed, result in the original function (3) the circle or nested parenthetic notation of composition, (4) the language associated with composition such as substitution, replacing, plugging in, and inside or outside function, (5) taking or raising a function such as taking sin or raising to the power of *e*, (e.g., e^x) and (6) other topics that textbooks explicitly connected to composition.

The criteria of the first phase were intentionally broad with the goal to err on the side of inclusion rather than exclusion. The criteria of the second phase then reduced the amount of data collected from the first phase and refined the grain size of the unit of analysis. In the third phase, the composition coding scheme was applied to the remaining content. The rest of this section provides details for each of these phases, the associated criteria, and the unit of analysis of each phase.

3.3.1 Phase One: Data Collection. The first phase, called the Data Collection phase, analyzed every page of each textbook. The criteria for this phase were designed to be inclusive with regard to identifying the pages that contained text on or related to composition. A broad and large scope was intended to reduce the possibility that core composition content was left out of the data set.

The Data Collection phase had five criteria. If a page met one of these criteria, it was included in the data set. These criteria were:

1. The page was listed in the index under *composition*, or the entire lesson that was explicitly titled to be about composition;
2. Any form of the word *composition* was located on the page ;
3. The page was located in a lesson that used or could have used the concept of composition;
4. The page contained notation or language that meant composition;
5. An entire lesson on a concept that included notation and language similar to composition.

Additionally, if during Phase One, any topic or lesson was identified in only one of the textbooks, the other curricula were then double checked to verify that the topic or lesson was not overlooked.

The first criterion was straightforward and included lessons with the titles such as: "Composition of Functions," "Algebra with Functions," "New Functions from Old Functions," and "Compositions of Transformations." The second criterion was similarly straightforward as the page contained a form of the word *composition* (e.g., compose, composite) or it did not. The remaining three criteria were used to identify composition located in other sections of the textbooks and deserve a further discussion of the kinds of topics or concepts they encompassed.

The third criterion included the following topics and concepts closely related to the concept of composition: inverse function, iteration, recursion, solving transcendental functions, solving via substitutions such as u- or trig-substitutions, and the chain rule. Table 3.1 summarizes topics and concepts encompassed by criterion 3.

Table 3.1
Topics and concepts related to criterion 3 for inclusion in the Data Collection phase

Criteria 3	Relation to composition
Inverse Functions	A function composed with its inverse results in the identity function.
Iteration	A "process of repeatedly composing a function with itself" (GA2, p. 716).
Recursion	Uses the previous output as the input
Solving Transcendental Equations	Composition with the transcendental function's inverse is needed to remove the unknown from the argument of the function.
Solving via u-substitution	Decomposition
Chain Rule	Derivatives of composite functions

By definition inverse functions are functions that result in the identity function when composed with the original function, e.g., $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$. Iteration is the "process of repeatedly composing a function with itself" (GA2, p. 716). Recursion expresses new output in terms of previous outputs (from the same function) which is connected to the sequence view of composition. Solving transcendental functions with the variable in the argument requires either (1) composition with the inverse function to "remove" the variable from the argument or (2) using a definition or property. For example, solving $e^{2x} = 5$ could be performed using composition by taking the natural logarithm of both sides giving $\ln(e^{2x}) = \ln(5)$ which becomes $2x = \ln 5$. It could also be solved by using the definition of logarithms which says that $e^{2x} = 5$ means that $2x = \ln 5$. Solving via substitution is also closely related to the concept of decomposition. For example, solving the equation $x^4 + 2x^2 + 1 = 0$, can be solved by letting $u = x^2$, so that the equation becomes $u^2 + 2u + 1 = 0$. Composing this new equation with $u = x^2$ would result in the original equation. Any page that included these concepts was included by criteria 3.

The fourth criterion included the notation and language used for composition: substitution, replacing, and plugging in; combining, sequencing, or linking functions; referring to an inside or outside function; "taking" a function or "raising" it to a power; and using the circle or nested parenthetic notation. Frequently, the procedure of composition is described as substituting, replacing, or plugging one function in for all of the *x*'s of another. Composition is also described as the combining of functions or the linking of function machines or as a sequencing of functions where the output of the first function, or "inside function" is used as the input of the second, or "outside function." The idea of "taking" the square root, logarithm or a trigonometric function is language that means to compose functions. For example, if the equation is $\sin \theta = 0.4$, the instructions "Take $\sin^{-1}$ of each side" results in $\sin^{-1}(\sin \theta) = \sin^{-1} 0.4$. The instructions to “take” means to compose $\sin^{-1} x$ with $\sin \theta$ and apply the $\sin^{-1} x$ function to 0.4. Similarly, the term “raise” in the instructions “raise e to each side” means to compose e with the expressions on both sides of an equation. For example, raising e to both sides of the equation $\ln(x + 2) = 3$ means to compose e with $\ln(x + 2)$ and e with 3 resulting in $e^{\ln(x + 2)} = e^3$.

The notations of the circle, $\circ$, and nested parentheses, $f(g(x))$, are used to denote composition. The nested parentheses can indicate the functions being composed within the notation itself or in separate statements. For instance, the functions being composed in the notation of $f(g(x))$ must be defined in separate statements. The notation $f(x + 5)$ defines the $g(x)$ function within the notation and the $f(x)$ function in a separate statement. Conversely, $3g(x) – 7$ defines the $f(x)$ function within the notation and the $g(x)$ function in a separate statement.

The fifth criterion included the entire lesson of topics and concepts that used notation and language similar to composition. Those topics included difference quotient, even and odd functions, graph transformations, and symmetry which all utilize the parenthetic notation such as $f(x + h)$ or $f(-x)$ and solving systems of equations which uses the language of substitution. Table 3.2 summarizes topics and concepts included by criterion 5.

Table 3.2
Topics and concepts related to criterion 5 for inclusion in the Data Collection phase

Criterion 5	Relation to composition
Difference Quotient	Composition notation of $f(x + h)$.
Even and Odd Functions	Composition notation of $f(-x)$.
Graph Transformations	Composition notation of $f(x + h)$.
Symmetry	Composition notation of $f(-x)$.
Systems of Equations	The term substitution

In summary, the Data Collection phase evaluated every page of each textbook to determine, in a broad sense, the pages which explicitly or implicitly contained text on composition.

3.3.2 Phase Two: Data Reduction. The purpose of Phase Two was to further reduce the data set by identifying more precisely the data that was central to the composition concept. This refinement of the data occurred by changing the unit of analysis from the page level to the sentence or problem level. All text or images on a page were analyzed individually. This included sentences, problem statements, graphs, tables, and other images along with their captions. I will refer to these items as *elements* of the data set. For a sentence to be included in the Phase Two data set, it had to directly connect to composition and not be just related to composition. An element was admitted in the Phase Two data set if it met one of the following seven criteria:

1. A sequencing of functions or transformations where the output of the first is the input of the second
2. The operation of composition on functions
3. The circle or nested parenthetic notation
4. The language associated with composition of substitution, replacing, plugging in, and inside or outside function
5. Taking or raising a function such as "taking $\sin^{-1}$" or "raising to the power of *e*"
6. Other statements explicitly connected to composition
7. The decomposition of a function into parts that result in the original function when composed (e.g., u-substitution)

These criteria, unlike those in the Data Collection phase, required an element to be connected to composition. Affiliation to a topic such as inverse functions or any other topics described in criteria 3 and 5 of the Data Collection phase were no longer sufficient to be included in the data set. Each element had to have its own connection to composition. For example, an inverse function can be described in three different ways: First, an inverse function *switches* a function's domain with its range; second, the inverse function *undoes* the original function; third, the inverse function composed with the original function results in the identity function. Only an element found in the third presentation of inverse functions would be admitted into the Phase Two data set. The first two presentations do not meet any of the criteria for Phase Two, while the third presentation meets criteria 2 and 3.

Even though the concepts of even and odd functions, symmetry, and graph transformations use parenthetic notation, criterion 3 alone was not sufficient for them to be included in the Phase Two data set. The treatment of symmetry and even and odd functions in these textbooks involves computing $f(-x)$ or replacing the x's or y's in a function with $-x$ or $-y$, respectively. If an element of these topics only met criterion 3, it was not included. On the other hand, if an element met other criteria, it was included. The topic of graph transformations also connects to composition by parenthetic notation such as $f(ax + h) + k$. The treatment of graph transformations in these textbooks, however, mainly focuses on the value (positive or negative) of a, h, and k and how that affects the graph of a function. Similarly, the treatment of the graph transformations of trigonometric functions focuses on the amplitude, period, and phase shift. Because the focus was on the specific values and not on the entire expression in the parenthesis, $(ax + h)$, criterion 3 was not a sufficient condition to include an element in the graph transformation section in the data set. It had to meet additional criteria. See Appendix B for a list of inclusions and exclusions by mathematical topic.

Table 3.3 shows the total number of pages in the lessons explicitly about composition (criterion 1 of Phase One) and the number of pages identified in Data Collection and Data

Reduction phases. It also shows the total number of elements analyzed in Phase Three or the Data Analysis phase of the study.

Table 3.3
Number of pages included in the three analysis phases of this study.

	Total Number of **Pages** in Textbook	Total Number of **Pages** in the			Total Number of **Elements** in the Data Analysis Phase	
		Composi-tion Lesson(s)	Data Collection Phase	Data Reduction Phase	Composition Content	Substitution Content
Algebra 1						
Glencoe	846	0	177	112	27	378
Pearson	763	5	151	83	182	102
Geometry						
Glencoe	955	10	283	248	140	545
Pearson	711	14	55	38	81	25
Algebra 2						
Glencoe	978	14	354	198	456	237
Pearson	777	35	289	191	794	304
Precalculus						
Glencoe	836	10	287	166	683	235
Pearson	703	0	143	74	63	80
Cengage	888	9	324	182	578	188
Wiley	608	14	229	96	592	35
Calculus						
Cengage	65+	25	67	50	226	20
Wiley	68+	15	80	51	218	16
Total	8198	151	2439	1489*	4040	2165

+ The Calculus texts were only analyzed through the chain rule lesson in single variable Calculus.

*732 of these pages included composition content. The remaining 757 only contained substitutions terms.

In summary, the Data Reduction phase, reduced the amount of data collected in the first phase by focusing on the more explicit elements of composition in the textbooks. It also identified the specific sentences and problems that would be analyzed in the third phase, data analysis. As shown in Table 3.3, the amount of composition content that would have been missed by only looking at composition located in the composition lesson(s) of each textbook is considerable.

3.3.3 Phase Three: Data Analysis. The third phase of the analysis applied the composition coding scheme to each element in the Phase Two data set. The coding scheme contained categories of conceptual, procedural, and conventional knowledge of composition, the type of representation(s) and type of function(s), and the location of the data element within a textbook. The coding scheme also consisted of a category for the terms of *substitution*, *replace*, and *plug in*. The rest of this section describes major/general aspects of each of these categories. Examples are provided in chapters 4 and 5 and the complete coding scheme is found in Appendix C.

Both the conceptual and procedural composition codes captured content of domain and range of composite functions, properties such as associativity and commutativity, decomposition, geometric compositions, connections to other topics such as inverse functions and trigonometric functions, application problems, and other various ideas. The procedural codes captured the *how to* aspects of the curricula while the conceptual codes focused on the general definitions, principles, and connections of composition to other topics. For example, elements on the non-uniqueness of decomposition were conceptual while elements on the actual process of how to decompose a function were procedural. Another example would be elements that discuss the non-commutativity of composition were conceptual while finding both $f \circ g$ and $g \circ f$ were procedural.

The conventional composition codes comprised the language and notation used with composition. Elements such as "$(f \circ g)(x)$ is read as 'f of g' or 'f circle g' ", or those referring to inside and outside functions were captured by this category. Describing composition as a combination of functions or as substitution was also included in the conventional codes. The notations component of this category included the circle and nested parentheses notations.

If a data element included a function, the type of representation used to represent the function and the type of function was coded. There were eight representation codes and 15 different types of functions. The representation codes were the representations of algebraic, graphical, tabular, mappings on sets of domain and range, lists of ordered pairs, function machines, verbal names such as REC for $\frac{1}{x}$ or ABS for $|x|$, and geometry for pictures or figures not on a coordinate graph. The types of functions included polynomials (including its order and the number of terms), rational functions, functions defined as ordered pairs, exponential functions, logarithmic functions, piecewise-defined functions, square and cubic roots, absolute value, the greatest integer function, trigonometric and inverse trigonometric functions, reflections, rotations, translations, and glide reflections.

In addition to these composition categories, the location of each element was also coded. The location codes included codes for content found in the student or teachers edition as well as in exposition, examples, homework and review problems, chapter tests, comments made in the margins, and the answers to the problems found in the teachers guide.

The terms of substitution, replace, and plug in were included to study the treatment of these terms throughout the texts. These were categorized into four main categories: Equality, Values, Expressions, and Methods. Equality involved substitution of something of equal quantity, Values involved substituting a value for a variable, Expressions involved substituting an expression for a variable, and Methods involved the term substitution being used as part of the name of a procedure or algorithm such as *direct substitution*.

While an initial coding scheme was created prior to data analysis, it was necessary to create and adjust codes during the actual coding of this study. Afterwards, a second coder was given the coding scheme to determine the inter-rater reliability of the composition coding scheme. This second coder coded a random sample of 300 elements. Overall there was 83% agreement between the coders.

The next chapter discusses the formal definitions of composition and the meanings and uses of the terms of substitution, replace, and plug in. Chapter 5 then presents the findings from the conceptual, procedural, conventional, representation type, and function type codes.

CHAPTER 4: DEFINITIONS OF COMPOSITION AND MEANINGS OF SUBSTITUTION

The definitions of composition and the meanings of other words associated with composition are essential in studying the treatment of composition in the written curriculum. In this chapter, I first discuss the definitions of composition in the secondary and early collegiate curriculum and how these definitions develop over time. The second part of this chapter discusses the meanings of the word *substitution*, one of the key words used to describe the procedure of composing. The discussion focuses on the different meanings and their relation to the concept of composition.

4.1 Definitions of composition

The focus of this section is on the definitions of composition located in glossaries and lessons where composition is formally defined. These definitions can be characterized by four aspects of composition: The sequence view of composition, the requirements of the domain and range of composite functions, the operation view of composition, and the notation of the composition rule $(g \circ f)(x) = g(f(x))$. These definitions became more mathematically abstract as the mathematics courses progress (see Table 4.1).

Table 4.1
Four principles of composition definitions arranged by course.

	Sequence View	Domain & Range	Operation View	Notation
Algebra 1	x			
Geometry	x			
Algebra 2		x	x	x
HS Precalculus		x		x
College Precalculus			x	x
College Calculus			x	x

It is important to keep in mind that this section focuses on the definitions of composition and does not wholly describe the textbooks' treatments of composition.[3] The following provides examples of definitions for each aspect of composition.

4.1.1 Sequence view of composition. The main principle of the sequence view of composition is that it is a pointwise operation. Singular points, values, and objects, such as polygons, are mapped through a chain of processes. Initially, composition is defined as a succession of processes where the output of the first process is the input of the second process. This occurs in the context of geometric transformations (Example 4.A) and functions (Example 4.B).

Example 4.A When a transformation is applied to a figure and then another transformation is applied to its image, the result is called a composition of transformations (GG, p. 641).

Example 4.B A composition of functions is a combination of two or more functions (PA1, glossary).

Example 4.A processes figures through multiple transformations. Example 4.B combines functions by linking function machines together, which causes values to be processed through one function and then the result to be processed through another (see Figure 4.1).

[3] When additional information was needed to understand the meaning of a definition, context was only taken from the exposition in the section where the definition was located.

You can link function machines to make more complex functions. The diagram illustrates an example.

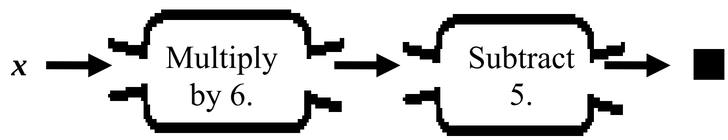


Figure 4.1. Combination of function machines (PA1*,* p. 423).

4.1.2 Domain & Range. By Algebra 2, definitions of composition progress from referring to a rule that maps particular values to referring to entire sets of values such as the domain and range. The principles of domain and range appeared in these definitions in two ways. First, the range of the first function must be in the domain of the second function (Example 4.C). Second, the composition of functions has its own domain and range (Examples 4.D and 4.E).

Example 4.C Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composition function $f \circ g$ can be described by $[f \circ g](x) = f[g(x)]$. (GA2, p. 411)

Example 4.D The composition of function f with g is defined by $[f \circ g](x) = f[g(x)]$. The domain of $f \circ g$ includes all x-values in the domain of g that map to $g(x)$-values in the domain of f. (GP, p. 58)

Example 4.E For two functions $f: A \to B$ and $g: B \to C$, the composite function meets the following conditions.

- $g \circ f : A \to C$
- $g \circ f(x) = g(f(x))$ (PA2, p. 110)

In contrast to Examples 4.A and 4.B which use particular inputs and outputs, Example 4.C refers to sets of inputs and outputs (domain and range). While this example still describes a pointwise operation, there is a shift from singular values to sets of values. Example 4.D refers to

$f \circ g$ as having a domain and range of its own and Example 4.E discusses the composite $f \circ g$ as a function (or object) itself with its own domain and range. Thus, similar to the examples in the sequence view, these examples refer to functions as processes, where the domain is mapped to the range, but they introduce the idea that the composite $f \circ g$ is a function itself, defined by its domain and range. Even though the functions here can be viewed as processes, the understanding that a function results from a composition is important in helping students make the shift from viewing functions as processes to viewing functions as objects, as required in the context of composition. This understanding is an important bridge between the sequence view and operation view of composition.

4.1.3 Operation View. By collegiate Precalculus, definitions of composition treat functions as objects. Composition operates on functions (e.g., formulas) yielding another function (or new formula). Examples 4.F and 4.G illustrate that the composition of functions results in a function.

Example 4.F For two functions $f(t)$ and $g(t)$, the function $f(g(t))$ is said to be a composition of f with g. The function $f(g(t))$ is defined by using the output of the function g as the input to f (WP, p. 86).

Example 4.G The composite of f and g is the function $f(g(x))$; f is the outside function, g the inside function (WC, p. 666).

While Example 4.F defines the composite function using outputs and inputs, the context of the surrounding text indicated that the goal of composition was to determine the function or a representation of the function and not individual values. Example 4.G took the operation view a step further by emphasizing in the second clause that the objects being operated on (the outside and inside object) are both functions and did not refer to inputs and outputs.

4.1.4 Notation. The final aspect defines composition by its rule (Example 4.H).

Example 4.H Given two functions f and g, the composite function $f \circ g$ (also called the composition of f and g) is defined by $(f \circ g)(x) = f(g(x))$ (CP, p. 193).

Defining composition solely by its notation rule was the most abstract of all the definitions, but is the most common among all the courses. In one entire collegiate series, this completely defined composition.

This analysis has illustrated how the development of the definitions of composition begins with the sequence view in Algebra 1 and Geometry and moves to the operation view by collegiate Precalculus. The appearance of the sequence view and operation view in textbook definitions, indicates that these views effectively frame the treatment of composition in the written curriculum and may be useful in framing the teaching and learning of composition. These two views could aid teachers in identifying how their students view composition and assist students in understanding composition both sequentially and operationally.

4.2 Meanings of Substitution

In addition to formal definitions, textbooks provide explanations and illustrations on how to perform a composition. Three of the four curricula analyzed in this study use the term *substitution* to explain the procedures of evaluating a composition. In the composition section of these textbooks, substitution indicates that one function is being placed into another or that an expression is being put in place of a variable or function notation. Example 4.I and Figure 4.2 are two instances of the use of substitution which appear on the same page as a formal definition of composition in their respective textbooks.

Example 4.I The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. (CC, p. 33)

For $f(x) = 2a - 5$ and $g(x) = 4a$, find $[f \circ g](x)$ and $[g \circ f](x)$, if they exist		
a. $[f \circ g](x) = f[g(x)]$	Composition of functions	$[g \circ f](x) = g[f(x)]$
$= f(4a)$	Substitute	$= g(2a - 5)$
$= 2(4a) - 5$	Substitute again	$= 4(2a - 5)$
$= 8a - 5$	Simplify	$= 8a - 20$

Figure 4.2. GA2, p. 411, Example 3b.

In Example 4.I the entire process of function composition is summarized as substitution, while Figure 4.2 uses substitution to describe small steps within the process of composition.

Other terms used in ways similar to substitution include *replace* and *plug in*. Figure 4.3 and Examples 4.J and 4.K demonstrate this.

Given $f(x) = x^2 + 1$ and $g(x) = x - 4$, find each of the following.	
a. $[f \circ g](x)$	
$[f \circ g](x) = f[g(x)]$	Definition of $f \circ g$
$= f(x - 4)$	Replace $g(x)$ with $x - 4$
$= (x - 4)^2 + 1$	Substitute $x - 4$ for x in $f(x)$.
$= x^2 - 8x + 16 + 1$ or $x^2 - 8x + 17$	Simplify

Figure 4.3. GP, p. 58, Example 2

Example 4.J For Exercise 60, $[(f \circ g)(x)]$, remind students that they will be finding $f(x^2 + 8)$. So every x in $x^2 + 2x - 8$ will be replaced with $x^2 + 8$. (GA2, p. 415)

Example 4.K *Plugging in* is the same process as *substitution*. (PA1, p. 364)

The use of *replace* in Figure 4.3 is identical to the use of the first *substitute* in Figure 4.2. Example 4.J uses the term *replace* to explain in detail the same process that Example 4.I uses the term *substitution* to describe. Meanwhile, Example 4.K explicitly defines the terms of plugging in as substitution.

Since *substitute*, *replace*, and *plug in* are used to explain the procedure(s) of composition, a further discussion of how textbooks define and use these words is pertinent to the development of the composition concept. The rest of this section discusses the ways that substitution is defined and used throughout the texts. It also compares and contrasts these uses to the use of substitution that occurs in composition.

Substitution appears throughout the secondary and early collegiate mathematics curriculum with various definitions and uses. In total, the terms *substitution*, *replace*, and *plug in* appear 2165 times in these four curricular series. For the rest of this section, the term substitution will indicate any of the three terms *substitution*, *replace*, and *plug in*. As seen in Figure 4.4, substitution occurs most frequently in the early high school curriculum and declines after the geometry course. The majority of the instances of substitution occur in worked out proofs and examples in the student edition (SE) or in the problem answers located in the teachers edition (TE).

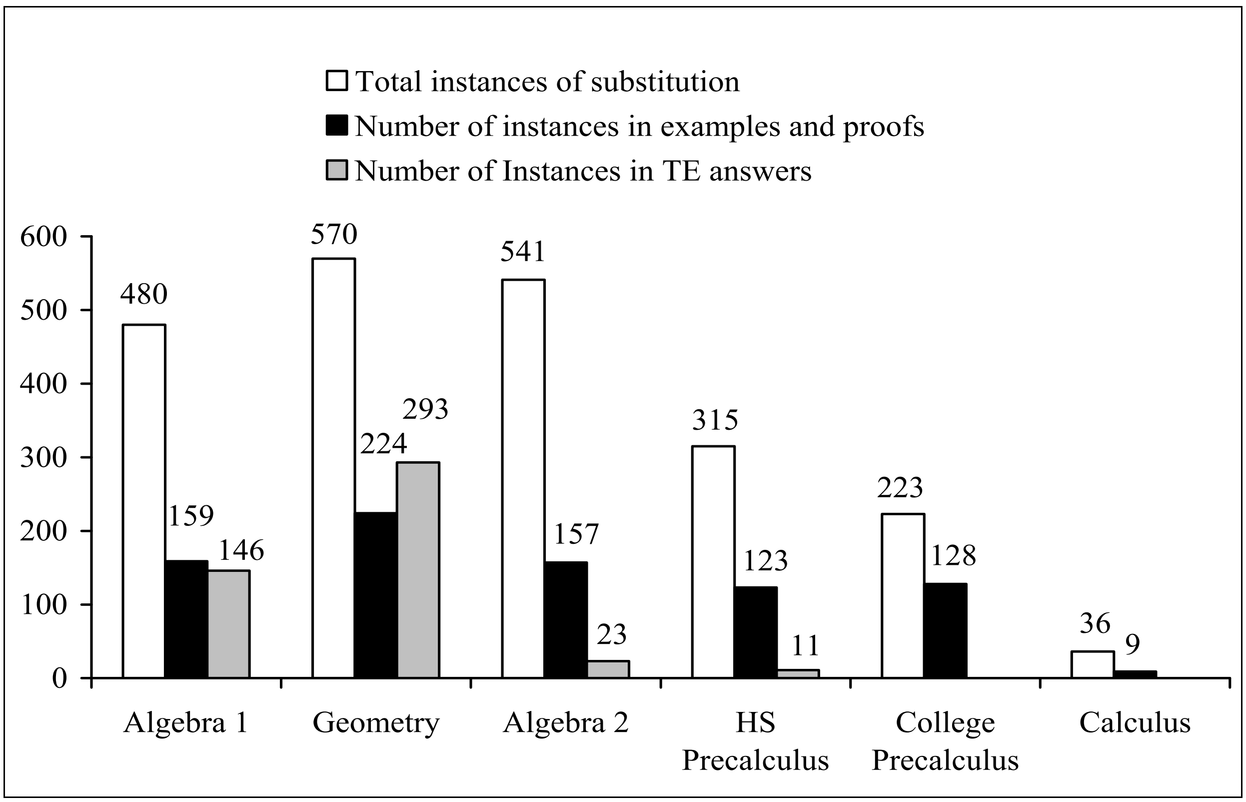


Figure 4.4. Distribution of the terms substitution, replace, plug in by course.

Each instance of substitution involves putting a numerical value or expression in place of something else in a formula, but the reasons and explanations for each substitution can be divided into four different categories. These categories were (1) *Equality*, such as substituting -1 for $\cos\pi$, (2) *Values*, replacing a single variable with a numerical value such as evaluating a function or checking an answer for correctness, (3) *Expressions*, replacing a single variable with an expression, such as happens in the composition of functions, or replacing an expression as a single variable such as in u-substitution, and (4) *Methods*, the word substitution was used as part of the name of particular methods, such as solving systems of equations via substitution or synthetic substitution. Table 4.2 provides brief descriptions, examples, and the total number of instances for each of these categories.

Table 4.2
Descriptions, examples, and the total number of instances for the categories of substitution.

Category	Total Instances	Description	Example
Equality	622	Equal values, quantities, or measurements are "substituted" for each other.	A quantity may be substituted for its equal in any expression. (GA1 p. 16) Given: m∠JKL = 8*x*+13, m∠NKL = 6x+11 Proof: m∠JKL = m∠NKL; 8*x*+13 = 6x+11 by substitution (GG, p. 40)
Values	579	A numerical value is substituted for a variable.	Substituting $t = 20$, $Q = 88.2$ and $t = 23$, $Q = 91.4$ gives two equations for $Q(0)$ and a: (WC, p. 12)
Check	(64)*	Substitution is used to check the correctness of an answer.	Check Your Answer: Substituting x = [final answer] into the original equation and using a calculator, we get [a true statement]. (CP, p. 332)
Evaluate	(28)*	An expression is evaluated by substituting a value for the variable.	The value f(-6) is found by substituting -6 for each x in the equation. (GA2, p. 64)
Expressions	447	Replacing a single variable or function notation with an expression, or vice versa.	Substitution involves substituting an expression from one equation for a variable in the other. (GA1, p. 388) Replace $g(x)$ with $x^2 - 9$ (GP, p. 59) Since $x = \cos t$ and $y = \sin t$, we can substitute x and y into this equation: ... giving .... (WP, p. 569)

Table 4.2 (Cont'd)

Methods	302	Substitution is part of the name of the method or algorithm.	Similarly, direct substitution provides the correct answer in part (b). (CP, p. 851) Solve systems of linear equations with two variables using substitution and elimination (PA1, p. 403)

*Numbers in parentheses are included in the 579 elements in the Values category.

The substitution that occurs with composition is included in the Expressions category. This type of substitution is different from the other categories because it changes functions and creates new ones. The other categories do not change the function. As the name of the Equality category suggests, one quantity is substituted with another representation of equal value. The Values category uses substitution to identify parameters of a function or the value of a function at specific point(s). In all cases the Values category is used to obtain more or specific information about functions and not to change them. The substitution terms identified by the Methods category also do not change functions. These are simply key words to indicate what procedures are used to solve the problem. The rest of this section describes these four categories of substitution with detailed examples, the main locations of each category, and how the meaning (explicit or implicit) of substitution changes over time. Figure 4.5 shows the distribution of these categories by course.

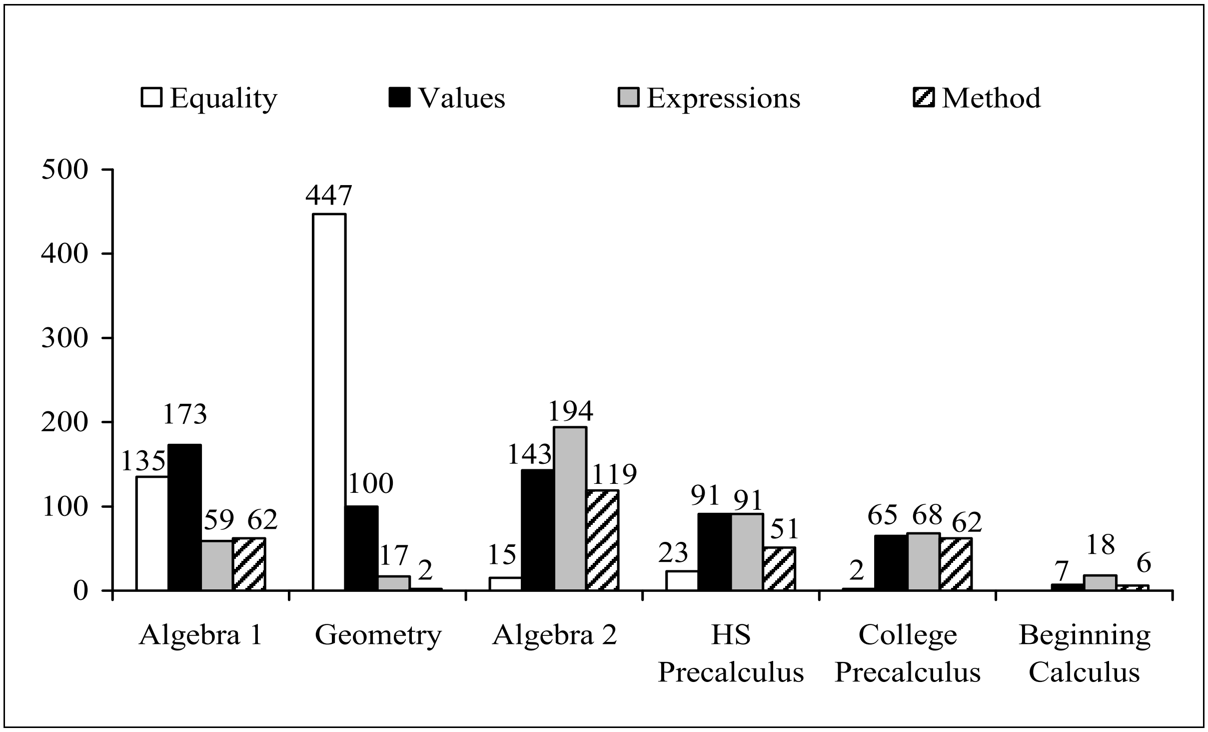


Figure 4.5. Distribution of the substitution categories by course.

4.2.1 Equality. The Equality category contains the most instances of substitution and was the first substitution category to occur in the high school curriculum. Substitution is originally called the *Substitution Property of Equality* and defined in the following manner. "A quantity may be substituted for its equal in any expression. If $a = b$, then a may be replaced by b in any expression. [For example,] if $n = 11$, then $4n = 4 \cdot 11$" (GA1, p. 16). This definition indicates that given the value of an unknown variable, that value may be substituted into an equation because the value and the unknown variable represent the same quantity. This same definition is used in another text of the same curricular series as shown in the example in Figure 4.6.

Solve -5(x + 4) = 70. Write a justification for each step.

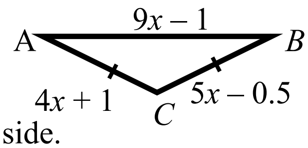
$-5(x + 4)$	$= 70$	Original equation or Given
$-5 \cdot x + (-5) \cdot 4$	$= 70$	Distributive Property
$-5x - 20$	$= 70$	Substitution Property of Equality
$-5x - 20 + 20$	$= 70 + 20$	Addition Property of Equality
$-5x$	$= 90$	Substitution Property of Equality
$\frac{-5x}{-5}$	$= \frac{90}{-5}$	Division Property of Equality
x	$= -18$	Substitution Property of Equality

Figure 4.6. GG, p. 134, Example 1

The example in Figure 4.6 uses the substitution property of equality as the reason for replacing (–5) · 4 with –20 and 70 + 20 with 90 since (–5) · 4 equals –20 and 70 + 20 equals 90. Generalizing from this example, all arithmetic could be explained with the reason of the substitution property of equality. Even though the entire name of the property is used at first, this only lasts for nine pages where it is noted in the text that "The Substitution Property of Equality is often just written as *Substitution*" (GG, p. 143). This abbreviation explicitly connects substitution indicating equality to every other use of the word substitution. A total of 622 instances (about 29% of all the instances of *substitution*) were classified in the Equality category. The majority of those in the Equality category (413) are located in the answers sections in the teachers' edition. An additional 167 instances in the Equality category are reasons located in the worked out steps of examples or proofs. Figure 4.5 shows that the number of Equality instances is highest in geometry and algebra 1 and almost negligible in the remaining courses.

Although the substitution property of equality is formally defined, other terms which are not explicitly connected to this property are used in its place. For example, the textbook example in Figure 4.7 (GG, p. 237) could have used the reason of substitution on lines 9, 11, 13 and 14.

(1) **Find the measures of the sides of isosceles triangle ABC.**



(2) Step 1 Find x.			
(3)	AC	$= CB$	Given
(4)	$4x + 1$	$= 5x - 0.5$	Substitution
(5)	1	$= x - 0.5$	Subtract $4x$ from each side.
(6)	1.5	$= x$	Add 0.5 to each side.
(7) Step 2 Substitute to find the length of each side.			
(8)	AC	$= 4x + 1$	Given
(9)		$= 4(1.5) + 1$ or 7	$x = 1.5$
(10)	CB	$= AC$	Given
(11)		$= 7$	$AC = 7$
(12)	AB	$= 9x - 1$	Given
(13)		$= 9(1.5) - 1$	$x = 1.5$
(14)		$=12.5$	Simplify

Figure 4.7. GG, p. 237, Example 5

Lines 9, 11, and 13 simply denote the items that are equal. Line 14 is very similar to the use of the substitution property of equality in Figure 4.6.[4] However, the term *simplify* is never formally defined or informally explained in any of the texts that formally define substitution. This makes it impossible to compare and contrast the authors intended meanings of the terms *simplify* and *substitution*.

The purpose of drawing attention to these terms is that these undefined terms are used in place of the defined term of substitution. It may be the case that the meaning of the term substitution shifts from equality to something else. Such a shift in the meaning of substitution, however, is not explicitly explained in any of the texts. The categories of Values and Expressions represent uses of substitution that shift from the meaning of equality to something closer to the substitution that happens in composition.

4.2.2 Values. The Values category is the first of two categories that entails substitution different from equality. In this case, a single variable is replaced by a numerical value. This

[4] Lines 9, 11, 13, and 14 were not included in the total number of instances for substitution since they were not explicitly labeled as substitution.

occurs by substituting a value into a general formula (Example 4.L), checking the answer to a problem (Example 4.M), or evaluating a function at a specific value (Example 4.N).

Example 4.L So we replace z by 9 in the equation of the sphere and get.... (CP, p. 601)

Example 4.M Check your work by substituting $x = 5$ and $y = 2$ into each equation. (PA1, p. 362)

Example 4.N The value $f(-6)$ is found by substituting -6 for each x in the equation. (GA2, p. 64)

Examples 4.L, 4.M, and 4.N all substitute a numerical value in place of a variable. These are different from the equality category above because the variables in each of these examples attain the numerical value that is being substituted for them, but they are not equal to that value at all times. If the letters being replaced by numbers were unknowns, these examples would be coded as Equality; however, in each case these are variables which attain many values. A total of 579 instances (about 27% of all the instances of *substitution*) were classified in the Values category. The majority of those in the Values category (402) are reasons located in the worked out steps of examples or proofs. Figure 4.5 shows that the number of Values instances is greatest in Algebra 1 and 2 and decreases in Precalculus and Calculus.

4.2.3 Expressions. The Expressions category also entailed substitution that differs from equality. In contrast to the Values category, this category substitutes expressions, function notation, and variables in place of other expressions, notations, or variables. This is illustrated in Figure 4.3 and Examples 4.O, and 4.P.

Example 4.O Substitute $x + 3$ for x in the equation of the parabola. (PG, p. 553)

Example 4.P Now we can substitute $\sin t = x$ from the first equation to get $y = 1 + x^2$. (CP, p. 566)

In Figure 4.3, the expression $x - 4$ replaces the function notation $g(x)$. In Example 4.O, the single variable x is replaced with the expression $x + 3$. In Example 4.P the expression $\sin t$ is replaced with the single variable x. In each case, an expression either replaces something or an expression is the object being replaced. A total of 447 instances (about 21% of all the instances of *substitution*) were classified in the Expressions category. The majority of those in the Expressions category (147) are reasons located in the worked out steps of examples or proofs. Figure 4.5 shows that compared to the instance of the other substitution categories, the percentage of Expressions instances increases over time.

Two textbooks provide explicit statements that explain substitution in a way that was coded in the Expressions category.

Example 4.Q Substitution involves substituting an expression from one equation for a variable in the other. (GA1, p. 388, Exercise #5)[5]

Example 4.R The key idea in using these formulas (or any other formula in algebra) is the Principle of Substitution: We may substitute any algebraic expression for any letter in a formula. (CP, p. 27)

Examples 4.Q and 4.R both use the word substitution to define substitution. Example 4.Q is a review problem in the Substitution method for solving systems of equations and is located at the end of the chapter on systems of equations. Example 4.R explains how to use special product formulas (e.g., $(A + B)^2 = A^2 + 2AB + B^2$). Both mention substitution using expressions and neither indicates equality. It is also interesting that the statement in Example 4.Q is from the same book as the Substitution Property of Equality mentioned in the description of the Equality

[5] In terms of coding, Example 4.Q received a code of Method for the first instance of "substitution" and a code of Expressions for the second instance of "substituting."

category. This is interesting because a single book provides multiple definitions for the word substitution without discussing (even in the teachers edition) the similarities and/or differences of these definitions.

Another curricular series attempts to define the term *substitution* as replacing a variable with an expression and the term *plugging in* as replacing a variable with a number or value.

Example 4.S *Plugging in* is the same process as *substitution*. When you "plug in numbers," you replace a variable with a number. When you "substitute," you replace a variable with an *expression* that (usually) involves variables. In fact, "plugging in" in [this curriculum], is just a special case of substitution when the expression you are substituting is just a number. (PA1, p. 364, TE)

This example says that plugging in is for values and substitution is for expressions. However the following examples, from the same series, demonstrate that this convention is not always consistent.

Example 4.T Some students have difficulty with plugging expressions into functions such as $f(x + 2)$. (PA1, p. 432)

Example 4.U We can just plug those new expressions in for x and y. (PG, p. 549)

Example 4.V Substitute $x = 3$ into (1) to find that $A = f(3) = -46$. (PP, p. 192)

Examples 4.T and 4.U have expressions plugged in and Example 4.V had values substituted. Similar to the Equality category, substitution is specifically defined in one way and used in another.

4.2.4 Methods. In this analysis, seven methods or algorithms included the terms *substitution* or *replacing* as part of the methods' name: (1) solving a system of equations via substitution, (2) linear substitution, (3) synthetic substitution for evaluating functions, (4) direct substitution, (5) trigonometric substitution, (6) back-substitution with matrices and solving systems of equations, and (7) replacing-the-axes with graph transformations. Even though these instances of substitution were coded separately from the other substitution categories, each of these methods relate in some way to one or more of the other three categories. Solving a system of equations via substitution relates to Equality if the system is dependent, but relates to Expressions if the system is independent or inconsistent. Synthetic substitution, direct substitution, and back-substitution are related to Values. Trigonometric substitution and replacing-the-axes are related to Expressions.

A total of 302 instances (about 14% of all the instances of *substitution*) were classified in the Methods category. The majority of those in the Method category (84) are reasons located in the worked out steps of examples or proofs.

In summary, substitution is a key word used to describe the procedure of composition. Throughout the curriculum, the word substitution is used to mean equality, the act of substituting values and expressions, and as the name of specific methods or algorithms. The substitution used in composition was categorized in the Expressions category, which was the only category that allowed for the creation of new functions. Figure 4.8 shows the frequencies and main locations of each category.

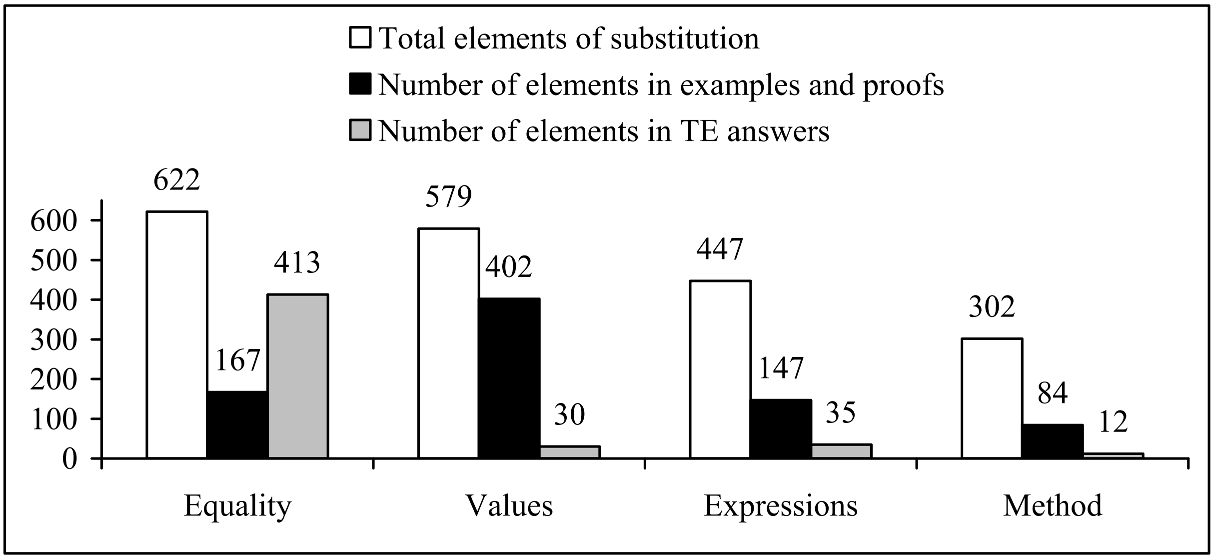


Figure 4.8. Distribution of the substitution categories and their location.

This figure shows that two-thirds of the Equality category was located in the problem answers in the teachers edition with most of the remaining third located in worked out examples and proofs in the student edition. Also about two-thirds of the Values category and over one-third of the Expressions category was located in worked out examples and proofs.

CHAPTER 5:

CONCEPTUAL AND PROCEDURAL TREATMENT OF COMPOSITION

Composition provides functions with structure, similar to the arithmetic operations provide structure to numbers. Simple functions can be combined to create more complicated functions and complicated functions can be broken down or decomposed into simple or familiar ones. This chapter on the conceptual and procedural treatment of composition in curricula is organized around the compositive structure of functions. First, I discuss the descriptions and informal definitions of composition. Second, I discuss the treatment of the properties of composition. This is followed by the conceptual and procedural treatment of evaluating a composition and decomposing a composite function.

5.1 Definitions of Composition

The definitions of composition discussed in the previous chapter are the definitions textbooks identify as the "formal definition." Frequently this is indicated by shading the word being defined or by putting a box around the entire definition. Within each text, there are also statements that informally define or describe composition. Additionally, there are definitions and descriptions of other concepts that use composition. This section discusses both the informal describing of composition and the formal and informal descriptions of concepts that use composition.

There are nine informal descriptions of composition in these textbooks. Three of these are located in review problems (such as "What is composition?"). One informal definition refers to a composition of transformations being a sequence of transformations and another simply states that "The function $f(g(t))$ is said to be a composition of f with g" (WP, 398). Describing composition as the linking of two or more function machines occurs four times, including once in college Precalculus. This demonstrates that even though formal definitions of composition tend toward the operation view, the sequence view is still a part of the composition content in higher grade levels.

Composition is used in 35 elements to explicitly define the mathematical concepts of translations, rotations, glide reflections, and affine transformations. Nine elements define translations as the composition of two reflections over parallel lines. Similarly, nine elements define rotations as the composition of two reflections over intersecting lines. An additional five elements focus on classifying a composition of reflections as either a translation or rotation. Nine elements define glide reflections as the composition of a translation and a reflection. Three elements define affine transformations as a composition of a translation and a dilation. All but one of the translation, rotation, and glide reflection elements appear in Geometry while all three affine transformation elements appear in Algebra 2.

Composition can also be used to define inverse functions. However, in these texts, inverse functions are defined as swapping the domain and range or as “reversing” or “undoing” a function. Instead of appearing in the definition, the composition concept usually appears as the inverse function property or cancellation property of inverse function. These properties state that the composition of a function and its inverse result in the identity function. Since composition appears as a property of inverse functions instead of in the definition, a detailed discussion of composition and inverse functions will appear in the section on the properties of composition (Section 5.2.3).

5.2 Composite Structure of Functions

Functions are closed under the operation of composition. This means that composing two or more functions results in another function. There are a total of 24 conceptual elements that discuss this general principle of functions (Example 5.A).

Example 5.A So far, we have used composition to build complicated functions from simpler ones. (CP, 195)

An additional 71 elements consist of statements and problems of the composite structure of functions. These include the composition of functions with special properties and how those properties influence the properties of the resulting composite function. This would include connecting the function notation of a composite function to its formula (Example 5.B), the composition of two continuous functions (Example 5.C), compositions of even and odd functions (Example 5.D), compositions of increasing and decreasing functions (Example 5.E), the period of a composite function (Example 5.F), the composition of one-to-one functions (Example 5.G), and other principles related to the structure of composite functions (Example 5.H). Also, the derivative of a composite function is defined by its compositive parts and the informal justifications of the chain rule is related to principles of composition (Example 5.I).

Example 5.B Let $u(x) = \sin x$ and $f(x) = \sin^2 x$. Is $f(x)$ equal to $u(u(x))$ or to $(u(x))^2$? (WP, 427)

Example 5.C If f and g are continuous, and if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval. (WC, 57)

Example 5.D Determine whether $(f \circ g)(x)$ is *even*, *odd*, *neither*, or *not enough information* if f is even and g is odd (GP, 63).

Example 5.E Let $f(x)$ be an increasing function. Is $f(f(x))$ increasing, decreasing, or is it impossible to tell? (WP, 429)

Example 5.F True or False. If $f(x)$ is a periodic function with period k, then $f(g(x))$ is periodic with period k for every function $g(x)$. (WC, 66)

Example 5.G Prove that, if functions f, g: $\mathbf{R} \to \mathbf{R}$ are one-to-one, then $f \circ g$ is one-to-one. (PA2, 123)

Example 5.H True or False. If $f(x)$ and $g(x)$ are quadratic, then $f(g(x))$ is quadratic. (WP, 430)

Example 5.I If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. (CC, 157)

Table 5.1 displays the distribution of these elements across the curriculum and shows that the majority of the elements on the composite structure of function happen in college texts. The lone exception is the Algebra 2 text that develops graph transformations as affine transformations, which are defined as compositions.

Table 5.1
The distribution of elements on the composite structure of functions.

	Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
Conceptual						
Composite Notation	-	-	-	-	13	2
Closure	2	-	13	-	4	5
Continuity	-	-		-	-	11
One-to-one	-	-	2	-	-	-
Other	-	-	2	-	5	1
Derivative	-	-	-	-	-	11
Procedural						
Continuity	-	-	-	-	-	11
Even/Odd	-	-	-	4	3	2
Inc/Dec	-	-	-	-	5	1
Period	-	-	-	-	-	2
Derivative	-	-	-	-	-	1

5.2.1 Domain and Range. Since the composition of functions yields a function, the concept of domain and range are relevant to composition. Because the domain and range of a composite function is dependent on the domain and range of the functions being composed, the elements with conceptual and procedural codes refer to the domain and range of the inside function, the outside function, and the composite function.

A total of 234 elements involve the domain and range of composite functions; 29% conceptual and 71% procedural. The elements with a conceptual code include elements that (1) use the Sequence view and Operation view, (2) use the terms "input" and "output" to refer to domain and range, and (3) restrict the domain of composite functions composed with inverse trigonometric functions. The elements with a procedural code include elements that have students find the domain and range of the outside function, the inside function, or the composite function.

All of the elements with a conceptual code referred to the domain of a composite function, an inside function, or an outside function. The range of a composite function, however, never appeared in the elements with a conceptual code. The only mention of range was the range of an inside function of a composition.

The Operation view and Sequence view of composition can be applied to conceptual elements of domain and range of composite functions. In the Operation view, the focus is on the domain or range of the composite function $f \circ g$. In the Sequence view, the attention is on the range of the inner function being a subset of the domain of the outer function. The Operation view is focused on the result of a composition (Example 5.J), while the Sequence view is focused on the relationship of the output of the inner function and the input of the outside function (Example 5.K).

Example 5.J The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (CP, 193)

Example 5.K Suppose f and g are functions such that the range of g is a subset of the domain of f. (GA2, 411)

The Operation view appears in 12 elements and the Sequence view appears in 15 elements. Note that not every composition element on domain and range could be coded as either Operation view or Sequence view (Examples 5.L). An additional 18 elements use the terms "input" and "output" in ways similar to domain and range (Examples 5.M and 5.N).

Example 5.L Emphasize how the domain changes when you switch the order of the composition. (PA1, 433)

Example 5.M In the composition $(g \circ f)$, the output $f(x)$ is used as input for g. (GA2, 410)

Example 5.N If $f: A \rightarrow B$ and $g: B \rightarrow C$, you can define a new function that maps inputs in A to outputs in C with the following rule: $x \rightarrow g(f(x))$. (PA2, 109)

Lastly, 12 elements with a conceptual code involve explicit statements of the restricted domain of composite functions involving inverse trigonometric functions. While all textbooks that included sections on inverse trigonometric functions discuss the restriction of their domains, only one high school precalculus text explicitly discusses the affect that these domain restrictions have on composite functions (Example 5.O).

Example 5.O Because the inverse trigonometric functions are defined only on an interval, rather than for all values, compositions of functions involving inverse trigonometric functions may or may not exist, depending on the *x*-value. (GP, 286)

Table 5.2 shows how the elements with a conceptual domain and range code were distributed across the curriculum.

Table 5.2
The distribution of the conceptual Domain and Range elements related to compositions across the curriculum.

	Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
Domain of a Composite Function (Operation view)	-	-	2	5	4	1
Domain of a Composite Function (Sequence view)	-	-	7	3	3	2
Domain of a Composite Function (no view)	2	-	4	3	2	2
Terms of "Input" and "Output"	4	-	5	-	9	-
Restricting the Domain of a Composite Function	-	-	-	12	-	-

The 167 elements with procedural codes direct students to find the domain and range of the outside function, the inside function, or the composite function. Finding the domain occurs in

98% of the elements, while finding the range occurs only in 10%. Table 5.3 shows the distribution for finding the domain and range across the curriculum.

Table 5.3
The distribution of elements with Domain and Range procedural codes across the curriculum.

Total		Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
	Find domain						
96%	Composite	1%	-	1%	34%	52%	8%
5%	Inside	-	-	-	2%	2%	1%
6%	Outside	-	-	1%	2%	1%	1%
	Find range						
7%	Composite	< 1%	-	-	6%	< 1%	-
1%	Inside	-	-	1%	-	-	-
1%	Outside	-	-	< 1%	-	< 1%	-

One can see that finding the domain of the composite is heavily emphasized in precalculus.

5.2.2 Associative and Non-commutative Properties of Composition. The properties of associativity and commutativity are also relevant to the compositive structure of functions. The operation of composition is associative, but not commutative. There are some functions such as iterations, inverse functions, and any function with the identity function that are commutative under composition, but composition is not commutative in general.

The associative property of composition appears a total of ten times. All ten of these appear in a single Algebra 2 text. Four are conceptual (i.e., a statement that $f \circ (g \circ h) = (f \circ g) \circ h$) and six are procedural (i.e., given formulas for f, g, and h, compute $f \circ (g \circ h) = (f \circ g) \circ h$.

In contrast to the appearances of the associative property, the principle of commutativity appears 280 times and in every course. Approximately 31% of these are conceptual (see Examples 5.P and 5.Q).

Example 5.P Notice that in most cases, $f \circ g \neq g \circ f$. Therefore, the order in which two functions are composed is important. (GA2, 412)

Example 5.Q When two rotations are performed on a single image, does the order of the rotations sometimes, always, or never affect the location of the final image? Explain. (GG, 648)

Another 37% are procedural code elements that compute both $f \circ g$ and $g \circ f$, and 31% are procedural code elements that compute $f \circ g$ and $g \circ f$ in adjacent problems or statements. The remaining 2% find functions or values that are commutative (see Examples 5.R and 5.S).

Example 5.R Find two functions f and g such that $f(g(x)) = g(f(\mathrm{x}))$ for every number x. (PA1, 428)

Example 5.S Find all numbers b, such that $f(g(b) = g(f(b))$. (PA2, 108)

The distribution of these elements across the curriculum is shown in Figure 5.1.

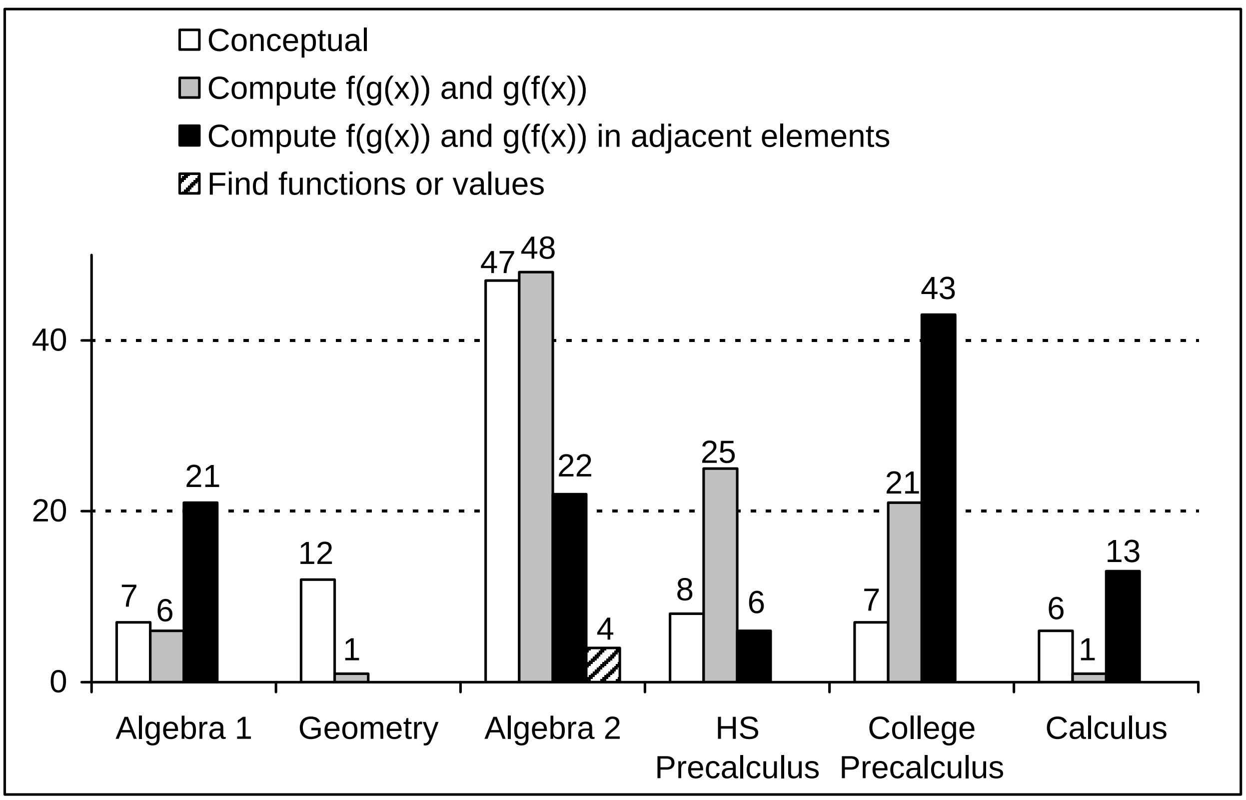


Figure 5.1. Distribution of the non-commutative property elements across the curriculum.

Figure 5.1 shows that the non-commutative property of composition appears most frequently in Algebra 2 and College Precalculus. Filtering out the elements on the non-commutativity of affine transformations, the Algebra 2 course still contains the most element with 63 conceptual code elements and 70 procedural code elements. In college Precaclulus, over two-thirds of the procedural code elements are elements that are computations of $f \circ g$ and $g \circ f$ in adjacent problems. These were included in this study as *implicit* non-commutativity.

5.2.3 Compositive Identity and Compositive Inverse. The function $f(x) = x$ is the compositive identity for the set of functions. The compositive identity appears in the composition content in two ways. First, the composition of the compositive identity with any other function, $g(x)$, results in the function $g(x)$. Second, the composition of a function with its inverse yields the compositive identity. This section discusses the compositive structure of functions through both of these instances as well as the elements on finding the inverse of a composite function.

There are a total of 25 elements on composing the function $f(x) = x$ with another function. Of these, 80% are conceptual (Examples 5.T and 5.U) and 20% are procedural. All of the procedural elements involve composing the compositive identity with other functions. All of the procedural elements and 75% of the conceptual elements appear in a single Algebra 2 text, while the remaining conceptual elements appear in both college Precalculus texts.

Example 5.T Is there an identity function when doing composition; in other words, is there some function f so that for any function g, $f \circ g = g \circ f = g$? (PA2, 109)

Example 5.U Show that, for any function f, we have $f \circ I = f$, $I \circ f = f$, and $f \circ f^{-1} = f^{-1} \circ f = I$. (CP, 207)

As illustrated in Example 5.U, the compositive identity is related with compositive inverses in that any function composed with its inverse results in the identity function. In total, 437 elements refer to compositive inverses. Of these, 37% are conceptual and 63% are procedural. All of these elements were classified into four categories: (1) general inverse functions, (2) verifying two functions are inverses, (3) the inverse relationship between exponentials and logarithms, and (4) the inverse relationship between trigonometric and inverse trigonometric functions. The inverse relationship between exponentials and logarithms occurs in 43% of the compositive inverse elements. The majority of the conceptual code elements are statements such as $\log_b b^x = x$ and $e^{\ln x} = x$ and the majority of the procedural code elements involve using those statements in a problem such as "[Use] a calculator and the fact that $\ln(e^{5k}) = 5k$" (WC, 27). Verifying that two functions are inverses appears in 27% of the compositive inverse elements. The majority of these elements are procedural and have students show that both $f \circ f^{-1}$ and $f^{-1} \circ f$ are equal to the compositive identity. The general inverse functions category accounts for 21% of the compositive inverse elements. Examples of general inverse functions conceptual code elements are shown in Examples 5.V and 5.W and an example of procedural code elements is found in Examples 5.X.

Example 5.V Notice that the composition of f and f^{-1} is always the identity function. (GP, 68)

Example 5.W This property tell us that composing a function and its inverse function returns the original value as the end result. (WP, 409)

Example 5.X $f^{-1}(f(x)) = f^{-1}(x + 5) = x + 5 - 5 = x$ (PA2, 120)

The inverse relationship between trigonometric and inverse trigonometric functions appears in 9% of the compositive inverse elements. The elements with conceptual and procedural codes in this category are similar to those in the exponentials and logarithms category. Many of the conceptual elements state that the relationships of $\cos(\cos^{-1} x) = x$ is valid for $-1 \leq x \leq 1$ and

$\cos^{-1}(\cos x) = x$ is valid for $0 \leq x \leq \pi$. The procedural code elements then use that information in a problem or example such as "using the cancellation property $\cos(\cos^{-1} x) = x$ …) (CP, 467).

Table 5.4 shows the distribution of elements in all the Compositive Inverse categories. This table shows, as other tables have shown, that compositions with inverse functions appear more frequently in College Precalculus than in High School Precalculus.

Table 5.4
Distribution of elements in the Compositive Inverse categories.

Total		Algebra 2	HS Precalculus	College Precalculus	Calculus
	Inverse Function				
15%	Conceptual	6%	3%	4%	2%
6%	Procedural	6%	-	-	-
	Verify an Inverse				
3%	Conceptual	1%	1%	1%	-
25%	Procedural	10%	9%	6%	-
	Exponential-Logarithm				
13%	Conceptual	1%	5%	5%	3%
30%	Procedural	6%	5%	17%	2%
	Trig-Inverse Trig				
7%	Conceptual	< 1%	3%	3%	1%
2%	Procedural	-	-	2%	-

In addition to the principle of compositive inverses, another 15 elements are associated with finding the inverse of a composite function. Fourteen of these appear in a single Algebra 2 text while the remaining element is located in both college Precalculus texts. All of these elements use the sequence view of composition as illustrated in Example 5.Y.

Example 5.Y If $A_{(c,d)}$ is putting on your socks and $A_{(a,b)}$ is putting on your shoes, then the composition $A_{(a,b)} \circ A_{(c,d)}$ represents putting on first your socks and then your shoes. To undo this process, you first take off your shoes (represented by $(A_{(a,b)}{}^{-1})$ and then take off your socks (represented by $(A_{(c,d)}{}^{-1})$. (PA2, T847)

In summary, complicated functions can be built from simple ones through the operation of composition. The properties of those simple functions may provide information on the properties of the composite function. Similarly, the domain and range of a composite function are directly related to the domain and range of the composing functions. The associative property rarely appears across the entire curriculum and the non-commutative property of composition is explicitly mentioned more in the High School courses than the collegiate courses. The majority of elements on the compositive inverse focus on the inverse relationships between logarithmic and exponential functions and trigonometric and inverse trigonometric functions. Within each of the topics in this section, (1) there are more procedural code elements than conceptual code elements and (2) College Precalculus has more composition elements than any other course.

5.3 Evaluating a Composition and Decomposing Functions

Creating a new function through composition is sometimes called evaluation. The operation of composition, like arithmetic operations, can be reversed. The reversing or undoing of a composition is called decomposition. This section discusses the treatment of evaluating compositions and decomposing composite functions across the curriculum.

5.3.1 Evaluate a composition. There are 1749 elements on evaluating a composition. All of these are procedural. This total includes evaluating a composite function at a numerical value (e.g., $(f \circ g)(5)$), evaluating a function at an expression (e.g., $f(a), f(5a)$, or $f(x+a)$), evaluating an expression at a function (e.g., $3[f(t)]+2$), a combination of evaluating a function at an expression and evaluating an expression at a function (e.g., $f(x-1)-f(x)$), and evaluating a composition (e.g., $(f \circ g)(x)$). Also included in this total is evaluating a composition of geometric transformation (e.g., reflect the figure across line l and then rotate it 90 degrees), and writing a composition of geometric transformations or affine transformations as a single transformation (e.g., a reflection across two parallel lines as a translation). In addition to these different forms of evaluating compositions, there are elements that work through the steps involved in evaluating a composition. For example, the first step in evaluating $[g \circ f](x)$ for the functions $f(x) = 2x - 5$ and $g(x) = 4x$ involves substituting the expression $2x - 5$ into the notation for $f(x)$, resulting in $g(2x - 5)$. The second step involves substituting the expression $2x - 5$ into the x's of the expression for $g(x)$, resulting in $4(2x - 5)$.

Table 5.5
Distribution of evaluating a composition across the curriculum.

Total		Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
33%	Evaluate a composite function	3%	-	8%	7%	13%	3%
9%	Evaluate a composite geometric transformation	-	8%	< 1%	-	-	-
17%	Evaluate a function at an expression	2%	-	4%	3%	7%	1%
3%	Combination: $f(x-1) - f(x)$	-	-	1%	< 1%	1%	1%
2%	Evaluate an expression at a function	< 1%	-	< 1%	-	2%	< 1%
35%	Evaluate a composition	< 1%	-	16%	5%	12%	2%
5%	Write a composition of transformations as a single transformation	-	3%	2%	-	-	-
12%	Steps involved in evaluating a composition	-	-	4%	1%	5%	2%

Table 5.5 shows the distribution of all of the evaluation categories across the curriculum. The most frequent categories are Evaluating a composition (35%) and Evaluating a composite function (33%). The categories of Evaluating a composition and Writing a composite transformation as a single transformation are similar. In each case you have two objects (functions or transformations) and the goal is to define or describe the combination as a single object. The uniting of these two categories results is 40% of the evaluation elements. Similarly, the categories of Evaluating a composite function and Evaluating a composite geometric transformation can be united, resulting in 42% of the evaluation elements. The table also shows a difference in the amount of composition content between high school precalculus and college precalculus.

5.3.2 Decomposition. Decomposition is the opposite of evaluating a composition and involves the breaking apart into composite parts. While evaluating a composition creates a composite function, decomposition begins with a composite function with the goal to find two functions whose composition is the given composite function. There are a total of 218 decomposition elements; 6% are conceptual and 94% are procedural.

Elements with conceptual decomposition codes include defining decomposition (Example 5.Z), the non-uniqueness of decompositions (Example 5.AA) and the composite structure of functions (Example 5.BB). There is an even distribution among these three categories with each accounting for approximately one-third of the conceptual elements.

Example 5.Z To decompose a function h, you need to find two functions whose composition is h. (GP, 59)

Example 5.AATrue or False. There is more than one way to write $h(x) = (3x^2 + 1)^3$ as a composition $h(x) = f(g(x))$. (WP, 430)

Example 5.BB But in Calculus it is often useful to be able to decompose a complicated function into simpler ones. (CC, 34)

The procedural elements are more diverse. They include decomposing a function into two or three functions (71%), providing a composite function, *h*, and one of either *f* or *g* and requesting students to determine the remaining function (27%), putting a restriction on the type of functions allowed to be used in the decomposition (e.g., neither function can be the identity function) (14%) , finding a second composition that results in the same given function (procedural of the non-uniqueness principle) (1%), and numerical decomposition such as in a table or graph where students must use the numerical value of the composite function to determine the value(s) of the functions being composed (see Figure 5.2) (10%).

Complete the table given $h(x) = g(f(x))$.

X	$f(x)$	$g(x)$	$f(g(x))$
0	2		
1		0	0
2		3	2
3	0		1
4	3	2	4

(WP, 402)

Figure 5.2. Example of numerical decomposition.

In summary, evaluating compositions occurs eight times more than decomposing composite functions. The majority of the evaluating and decomposing elements focused on finding formulas and values resulting from a composition. In total, over 99% were coded with procedural codes.

This chapter has discussed the descriptions and explanations of composition, the compositive structure of functions, the associative and non-commutative properties of composition, the compositive identity and compositive inverse, the evaluating compositions and the decomposition of composite functions. Composition occurs in every course from Algebra 1 in high school through college Calculus and was predominately found in college Precalculus. Of the 3081 elements referred to in this section, 6% of them were coded as conceptual and 94% were coded as procedural. While the conceptual elements are distributed across every section except for evaluating a composition, the procedural elements were clustered in the domain and range, non-commutatitvity, compositive inverse, evaluating compositions, and decomposition sections.

CHAPTER 6: REPRESENTATIONS AND TYPES OF FUNCTIONS

This chapter discusses the types of representations and types of functions used in the composition content across the secondary through calculus curriculum. I also discuss other terms, such as taking and raising, that indicate composition.

6.1 Representations of Composition

Algebraic, graphical, and numerical tables are common representations of mathematics and have been referred to as the "Big Three" (Kaput, 1998). This section discusses these and other representations used in the treatment of composition and how they are distributed across the secondary and early collegiate curriculum. In addition to single representations, the use of multiple representations also appears in the treatment of composition. This section also discusses which representations are used together, how they are used together, and where they are located.

The composition content in these textbooks used eight different types of representations. In addition to the Big Three, the representations of numerical (not tables), verbal, function machines, mapping diagrams, and geometric figures are also used to represent composition. The numerical representations (not tables) include ordered pairs and numerical functional relationships such as in Examples 6.A and 6.B.

Example 6.A For each pair of functions, find $[f \circ g](x)$ and $[g \circ f](x)$, if they exist.

$f = \{(1,8), (0, 13), (15, 11), (14, 9)\}$, $g = \{(8, 15), (5, 1), (10, 14), (9, 0)\}$

(WP, 100)

Example 6.B Suppose that $j(x) = h^{-1}(x)$ and that both j and h are defined for all values of x. Let $h(4) = 2$ and $j(5) = -3$. Evaluate, if possible, $j(h(4))$. (GA2, 411)

Verbal representations include written out descriptions that are usually denoted in mathematical symbols such as "subtract 3" (for $x - 3$) or "ABS" to indicate the absolute value function ($|x|$).

Example 6.C f is the rule "square" and g is the rule "subtract 3." The function $f \circ g$ *first* subtracts 3 and *then* squares; the function $g \circ f$ *first* squares and *then* subtracts 3. (CP, 193)

Function machines are figures which illustrate that a function machine produces exactly one output for every input and composition is the linking of two or more of these machines (see Figure 6.1).

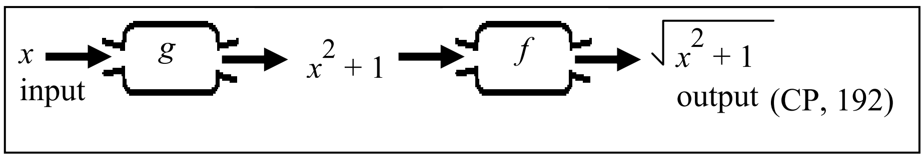


Figure 6.1. A function machine representation of composition.

Mapping diagrams map particular points through multiple functions or through a single composite function (see Figure 6.2).

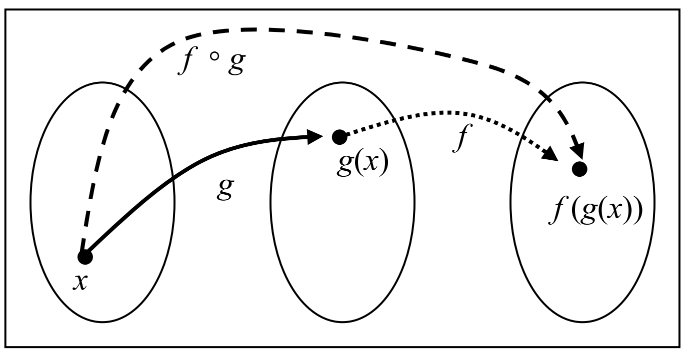


Figure 6.2. A mapping diagram representation of composition

Lastly, geometric figures include geometric shapes and pictures that are transformed via composite transformations (see Figure 6.3).

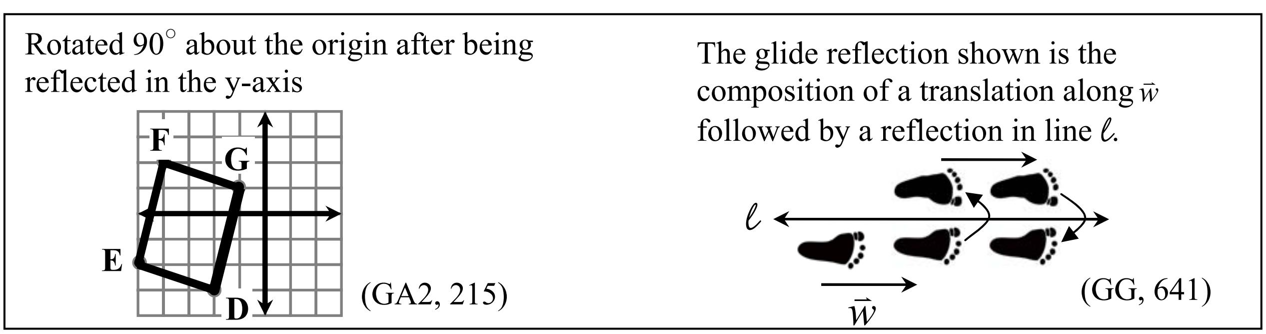


Figure 6.3. Examples of a geometric figure representation.

A total of 2769 elements received a representation code. Of these elements, over 81% contained an algebraic equation or expression. Table 6.1 shows the location and relative frequency of each representation across the curriculum.

Table 6.1.
The distribution of each type of representation across the composition curriculum.

Total		Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
81.3%	Algebraic	4.4%		17.3%	20.2%	29.3%	10.1%
6.7%	Geometric Figures		6.2%	0.4%	< 0.1%		
6.0%	Graphical	0.4%	0.8%	0.8%	0.9%	2.0%	1.2%
4.6%	Verbal	1.1%		1.9%	0.2%	0.8%	0.6%
2.7%	Tables	< 0.1%			0.1%	2.2%	0.3%
1.6%	Numerical			1.1%		0.2%	
1.1%	Machine	0.8%		0.1%		0.1%	0.1%
0.1%	Mapping Diagrams			< 0.1%	< 0.1%	< 0.1%	

Algebraic, graphical, and verbal representations appear throughout the curriculum. Numerical and function machine representations are predominately found in secondary texts, while tables predominately appear in college texts. Additionally, Table 6.1 illustrates that, in every type of representation, composition is emphasized more in college precalculus than in high school precalculus.

Of the 2769 elements, 110 (4%) were coded with more than one representation. Of these, 46 contained both algebraic and graphical representations, 37 contained both algebraic and verbal representations, 8 contained algebraic, function machine, and verbal representations, and 5 contained both graphical and tabular representations. Table 6.2 displays the distribution of elements with multiple representations across the composition curriculum.

Table 6.2
The distribution of elements with multiple representations across the composition curriculum.

Total				Algebra 1	Algebra 2	HS Precalculus	College Precalculus	Calculus
46	Algebraic	Graph		10	8	7	9	12
37	Algebraic	Verbal		15		2	11	9
5	Graph	Table				2	3	
4	Algebraic	Table				2	2	
2	Table	Machine		2				
4	Algebraic	Numerical					1	3
1	Algebraic	Machine					1	
1	Mapping Diagram	Machine			1			
8	Algebraic	Verbal	Machine	6			2	
2	Algebraic	Verbal	Graph					2

The predominance of algebraic representation also exists in the elements with multiple representations. Of the 110 elements with multiple representations, 102 (93%) include an algebraic representation. The graphical and verbal representations are contained in 53 and 47 elements respectively, while function machine figures are contained in 12. Also of note, the college precalculus course contained more than twice the number of elements with multiple representations than the high school precalculus course (29 and 13, respectively.) Expanding the comparison to the entire high school and college curriculum, the amount of the composition content using multiple representations contained in four years of high school is equal to the amount in the one year of college precalculus and calculus (55).

These elements with multiple representations have been classified into two categories: Display and Create. The Display category included instances where two or more representations are displayed either in the same element or explicitly linked together such as in example 6.D. The Create category includes instances where one representation is provided and then prompts for another representation to be created such as in Example 6.E.

Example 6.D. For example, the rule “square, then subtract” is expressed as the function

$f(x) = x^2 - 5$. (CP, 149)

Example 6.E. Suppose $f(x) = 2x + 3$ and $g(x) = x^2$. Find a way to construct the graph of

$f \circ g$ from the graphs of f and g. (PA2, 114)

Of the 110 multiple representation elements, 62 displayed two or more representations and 43 requested that another representation be created. The remaining five elements both displayed multiple representations and requested that another representation be created.

The majority of elements in the Create category were graphs being created from algebraic functions. The most common representation pairings in the Display category were verbal-algebraic representations and algebraic-graphical representations. The five elements in both the Create and Display categories all provided both the algebraic and verbal representations and requested that a function machine be created. Tables 6.3 and 6.4 show all the representations that were used by elements in Create and Display categories, respectively. Please note that the five elements in both categories are NOT included on either table.

Table 6.3
The combinations of representations in the Create category.

Representation Given	Representation to Create	Total
Verbal	Algebraic	7
Algebraic	Graph	30
Algebraic	Table	3
Table	Graph	1
Machine	Table	2

Table 6.4
The combinations of representations in the Display category.

Combinations of Representations			Total
Algebraic	Verbal		30
Algebraic	Graph		18
Algebraic	Numerical		4
Algebraic	Verbal	Machine	3
Algebraic	Verbal	Graph	2
Table	Graph		2
Algebraic	Table		1
Algebraic	Machine		1
Mapping Diagram	Machine		1

The majority of elements with multiple representations are located in problems for students such as homework exercises and review problems. In fact, 100% of the Create elements and 58% of the Display elements are located in such problems. The remaining Display elements are located in the exposition (16%), examples (15%), teachers edition (6%), and figures (5%). Table 6.5 summarizes the locations of all elements.

Table 6.5
The locations of elements with multiple representations broken down by category.

Location	Create Another Representation	Display Multiple Representations	Both
Exposition	-	10	-
Example	-	9	-
Problem for student to complete	43	36	5
Figure	-	3	-
TE	-	4	-

In summary, algebraic representation received the greatest amount of attention in the composition content. Approximately 81% of the elements with representations codes and 93% of the elements with multiple representations codes were algebraic. The instances of multiple representations included translations between representations (Create) and the connections among representations (Display) with the majority of the connections being between the algebraic and verbal representations.

6.2 Types of Functions Used in Composition

The composite structure of functions allows complicated functions to be made from simple ones. This section discusses the types of functions used in the composition content of textbooks and how they are distributed across the curriculum.

The composition content in these textbooks use eleven different types of functions; polynomials, trigonometric, root functions (e.g., square root, cube root, n^{th} root), rational functions, exponential, logarithmic, absolute value functions, piecewise-defined functions, geometric transformations (such as rotations, reflections, and translations of figures), and affine transformations (such as translations and dilations of graphs). A total of 2597 elements contained at least one of these types of functions. The majority are polynomials (50%) followed by trigonometric (12%), root functions (12%), and exponential (12%). Table 6.6 shows the distribution of each function type across the curriculum.

Table 6.6
The distribution of each type of function across the composition curriculum.

Total*		Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
50%	Polynomials	3.7%	--	15.5%	8.6%	17.5%	4.9%
12%	Trigonometric	--	--	0.8%	3.9%	6.1%	2.0%
12%	Root Functions	0.3%	--	2.7%	3.5%	4.4%	1.2%
12%	Exponential	--	--	1.8%	1.5%	6.5%	2.1%
10%	Logarithmic	--	--	1.5%	1.6%	5.0%	1.5%
10%	Affine Transformations	--	--	10.4%	--	--	--
8%	Geometric Transformations	--	7.1%	0.8%	0.2%	--	--
7%	Rational Functions	0.5%	--	0.5%	2.1%	3.4%	0.7%
2%	Piecewise-defined	0.1%	--	--	0.4%	0.8%	0.3%
1%	Absolute Value	0.5%	--	< 0.1%	< 0.1%	0.6%	

* The percentage totals do not sum to 100 because many elements contained more than one type of function.

While most types of functions occur throughout the curriculum, geometric transformations predominately appear in Geometry and affine transformations only appear in Algebra 2. All other types of functions are emphasized more in college Precalculus than in high school Precalculus.

The compositions of transcendental functions predominately involve the composition of inverse functions. Seventy-one percent of the trigonometric functions compose trigonometric and inverse trigonometric functions. Similarly, 62% of the compositions involving exponential functions and 77% of those involving logarithmic functions are the composition of exponentials and logarithms with a focus on the cancellation property of inverse functions.

Of the 2597 elements with a function type code, 2414 (93%) are the composition of two functions, 159 (6%) are the composition of three functions, 12 are the composition of four functions (7 of which were iterations), and the remaining 12 are compositions of more than four functions (all of which were iterations).

The coding of polynomial degree and the number of terms of polynomials revealed that 63% of the polynomials are linear, 31% are quadratic, 4% are cubic polynomials, and 1% have a degree higher than 3. With regard to the number of terms of each polynomial 25% are single term or monomials, 66% are binomials, and 8% are trinomials. Tables 6.7 and Tables 6.8 show the distribution of degrees and number of terms among inner, middle, and outer polynomials functions of a composition. The term "middle function" refers to the function(s) between the inner and outer functions in a composition of three or more functions. For example, in the composition of four functions $(f \circ g \circ h \circ k)(x)$, the functions g and h are middle functions.

Table 6.7
The distribution of the polynomial orders across the outer, inner, and middle functions of a composition.

Total		Linear	Quadratic	Cubic	Higher Order
42%	Outer	23.0%	16.3%	2.1%	0.7%
5%	Middle	2.8%	1.3%	0.3%	0.2%
49%	Inner	34.5%	12.2%	1.7%	0.3%

Table 6.8
The distribution of the number of terms of polynomials across the outer, inner, and middle functions of a composition.

Total		Monomial	Binomial	Trinomial	Higher Number of Terms
42%	Outer	9.2%	27.5%	5.3%	--
5%	Middle	0.8%	3.1%	0.7%	0.2%
49%	Inner	14.0%	33.0%	1.7%	0.1%

In addition to showing the distribution of the degrees of polynomials and the number of terms of polynomials, Tables 6.7 and 6.8 indicate that the majority of polynomials are the inner function of a composition. Merging the data on the degree and number of terms of polynomials reveals that about half of all polynomials being composed are linear binomials. This means that about 25% of the secondary and early collegiate composition curriculum involves linear two-term polynomials (see Table 6.9).

Table 6.9
The distribution of the number of terms of polynomials of a given order.

	1 Term	2 Terms	3 Terms	Higher number of Terms
Linear	13.9%	49.0%	--	--
Quadratic	8.8%	14.7%	7.8%	--
Cubic	1.5%	2.0%	0.5%	0.3%
Higher Order	0.9%	0.6%	--	--

With the majority of polynomials being simple and the transcendental functions focusing on inverse properties, compositions with rational functions and piecewise functions are one of the more challenging types of functions that students encounter in the composition content of textbooks. There are 189 elements involving rational functions and 40 elements involving piecewise-defined functions. The majority (129) of the elements with rational functions have a rational function as the outer function of the composition. Having a rational function as the outside function in a composition is important since it appears later in the curriculum with the derivative of the natural logarithmic function and the chain rule. With regard to the piecewise function elements, two are represented with tables, two algebraically (both of which are iterations), and the remaining 36 are represented graphically such as in Figure 6.4.

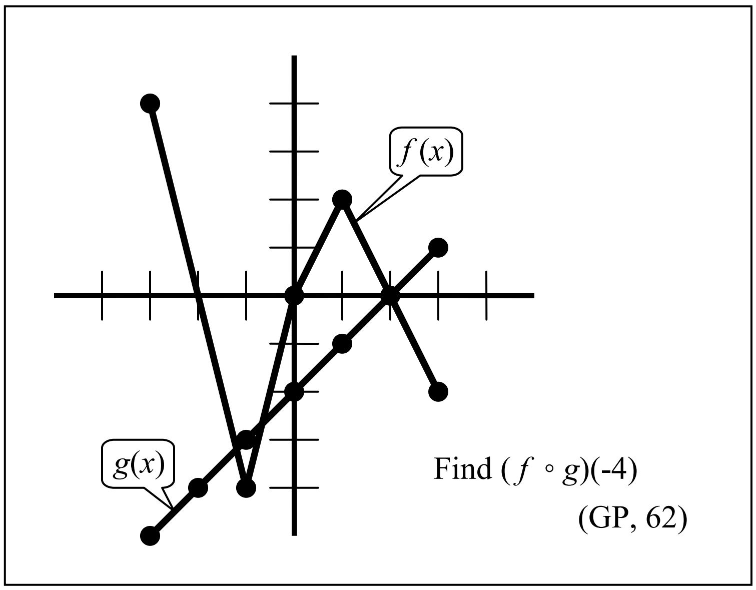


Figure 6.4. Example of a graphical piecewise-defined function composition element.

In summary, the overwhelming majority of composition content in textbooks uses polynomials with only one- or two-terms. The majority of transcendental functions focus on the cancellation properties of inverse functions. Piecewise functions appear most commonly in graphs and the small amount of rational functions appears as the outer function of a composition. While most types of function appear throughout the curriculum, trigonometric, exponential, logarithmic, and piecewise-defined functions occur most disproportionally in collegiate Precalculus.

6.3 The Language of Composition

When solving equations, the words *take*, *raise*, and *apply* indicate composition. (Examples 6.F and 6.G) These terms occur 171 times across the curriculum. They are most commonly used with root functions (57), power functions (33), logarithmic functions (53), exponential functions (15), and trigonometric functions (15).

Example 6.F Add 1, then take the cosine of the result. (PP, 57)

Example 6.G We solve for l by raising e to both sides: (WP, 211)

Throughout the curriculum in this study, the principle of composition is never explicitly connected to the terms *take*, *raise*, and *apply*. Occasionally, this principle is described as the One-to-One Property of certain types of functions. The One-to-One Property is described as $\log x = \log y$ if and only if $x = y$. Similarly, it is described for exponential functions as $e^x = e^y$ if and only if $x = y$. In one high school precalculus text, the term *take* is explicitly connected to the One-to-One Property of logarithmic and exponential function (Example 6.H).

Example 6.H This application of the One-to-One Property is called *taking the logarithm of each side* of an equation. (GP, 192)

Table 6.10 shows the distribution of these terms for different types of function across the curriculum.

Table 6.10
Distribution of "taking" or "raising" a function across the curriculum.

Total		Algebra 1	Geometry	Algebra 2	HS Precalculus	College Precalculus	Calculus
53	Take log	-	-	2	11	33	7
57	Take root function	-	-	28	4	25	-
33	Take a power	-	-	12	6	15	-
15	Take trig function	-	-	1	3	11	-
15	Raise to a number	-	5	2	5	2	1
8	Take other function	2	-	3	1	2	-

Forty-six percent of these instances involve transcendental functions and half involve taking a root function or a power (e.g., squaring). Even though the same language, *take*, was used for all of these different types of functions, there was no explicit presentation on how taking a power is not necessarily a one-to-one function (unlike the explicit content on the One-to-One Property of transcendental functions).

Taking is valid mathematically because of the compositive structure of functions. The composition of two functions is a function and more specifically, the composition of two one-to-one functions is a one-to-one function. However this is rarely explicitly connected to the principles of *taking* and *raising*.

This chapter has discussed the types of representations and types of functions used in the written curriculum to treat the concept of composition. Algebraic representations appeared most frequently across the curriculum while multiple representations appeared sparingly. Linear polynomials occur most frequently in the composition content, while rational functions and piecewise-defined functions occur least. Transcendental functions predominately focus on the cancellation properties of inverse functions. Lastly, when solving equations with transcendental functions, composition is commonly referred to as *taking*.

CHAPTER 7: DISCUSSION

This study was motivated by research results of students' performance on composition tasks. The treatment of composition in the written curriculum in this study was examined to identify the opportunities students have to interact with composition. The following research questions guided my study: In mathematics textbooks,

(1) when is composition formally introduced and how is it originally defined and explained to students?

(2) what vocabulary terms and notations are used with composition and how are they defined, explained, and used?

(3) which representations and types of functions are used in the composition content?

(4) which topics are explicitly connected to composition and in what ways?

(5) is the high school treatment of composition different than the treatment in college precalculus? If so, how?

In this chapter, I return to these questions and discuss possible implications the results might have on student learning.

7.1 Definitions and Terms of Composition

Composition appears formally defined in every course from Algebra 1 through Calculus. Beginning in Algebra 1, composition is formally defined using the Sequence view of composition. In higher level courses, formal definitions use the Operation view of composition. The main difference between these two views is the set of objects that are being operated on. In the sequence view the objects are numbers, points, and figures and in the operation view the objects are functions.

Even though the formal definitions progress from the Sequence view to the Operation view, the informal descriptions and explanations of composition do not follow the same trajectory. All courses use both views to explain how to perform composition. Since the Sequence and Operation views are used throughout curriculum, while the Operation view is more mathematically abstract, it is not necessarily more sophisticated than the Sequence view. The relationship between these two views is not hierarchical. Both views are important to understanding composition just like the rule of a function (the sequence view) is important to understanding the function itself (the operation view).

In all courses, many problems and examples focus on determining the formula that results from a composition. Since the focus is on the resulting function, the majority of these elements subscribe to the Operation view. This may result in students habitually finding a formula whenever a problem involves composition. For example, Horvath (2010) showed that students routinely determined a formula for $h(x) = (f \circ g)(x)$ and $h'(x)$ when asked to find the value of the derivative of $h(x)$ at a point. The students used the Operation view to determine $h(x)$ and then used derivative rules to find $h'(x)$. If their understanding of composition was more flexible, they could use the Sequence view with the chain rule (requiring less algebra).

Language is another aspect that influences student learning. The term substitution is frequently used to describe the act of composing, but is used across the curriculum to mean equality, substituting a value for a letter or variable (e.g., evaluation), substituting an expression for a variable, and a name for a method. Although the change from the formal definition to other meanings occurs within a single text, neither the teachers' guide or the student edition notifies the user of this change in meaning.

It may be that part of the difficulty of composition for students is associated with these multiple meanings of the term substitution. The shift from substitution meaning equality to substitution meaning evaluation is subtle but mathematically important. This shift parallels the difference between a letter representing an unknown fixed quantity or value and a letter representing a variable quantity (Clement, 1982) or generalized number (Graham & Thomas, 2000). Evaluation requires the substitution of a single valid value, but the value and the variable are not equivalent and are not interchangeable. Symbolically, substitution meaning equality could be represented as $x \equiv 3$, and substitution meaning evaluation could be represented as $x = 3$.

7.2 Representations and Types of Functions

The overwhelming majority of composition content is algebraic or symbolic. Of the few elements with multiple representations, many of them involve displaying more than one representation or creating a second representation different than the one provided. Despite all the research on multiple representations (e.g., Elia, et al., 2007; Leinhardt, Zaslavsky, & Stein, 1990), multiple representations seldom appears in the composition content of the curriculum. A few exceptions include a single problem requesting the composition of two graphs. The benefit of problems such as this is not to make students proficient in composing graphs, but to see compositive structure in a representation other than algebraic. This could support students' understanding of the structure composition provides to functions and not just the operation of composing two or more functions. For example, students could sketch the composite graph pointwise (using the sequence view) or they could use the operation view by composing a curve with a line by using graph transformations. An increase in the percentage of non-algebraic representations will allow students to experience compositive structure in another setting and connect different aspects of composition across multiple representations. This could provide students an opportunity to experience a broader picture of the compositive structure of functions.

Since the majority of composition content is algebraic, current literature on students understanding of functions can be reframed. For example, Meel (1998) and Carlson and colleagues (2010) found that students were successful with compositions represented algebraically. Knowing that the majority of composition content is algebraic, we can interpret Meel's results as supporting Selden and colleagues (1994) finding that students were successful with familiar problems. Not surprisingly, students are successful with problems they have previously experienced.

The majority of the functions used in composition content is what Even (1993) called "nice" types of functions. Approximately 50% of all composition content consists of linear polynomials (32%) or the inverse relationship of transcendental functions (16%). "Non-nice" functions such as piecewise-defined functions, seldom appear and many of these are written in ways that simplify the demands on students. For example, many of the composition elements with piecewise-defined functions have students graphically evaluate a composition at a specific value. This simplifies the problem because the issues of domain and range associated with the algebraic representation of piecewise-defined functions are removed and the difficulty of finding the resulting graph from the composition of two graphs is also averted by using only a specific value.

Interestingly, some types of functions that appear less frequently in the composition curriculum have been difficult for students even though they are "nice" functions. For example, rational functions occur as the outside function in a composition in only 5% of the textbook elements. Horvath (2010) showed that students have difficulty performing compositions with the rational function on the outside of the composition. While “nice functions” allow students to solve problems quickly, they do not provide students opportunities to grapple with compositive structure in complicated or non-routine situations, which support students understanding of the composition concept.

7.3 Compositive Structure

The compositive structure of functions occurs throughout the curriculum both explicitly and implicitly. This section discusses the potential benefits of explicitly comparing and contrasting composition to the arithmetic operations and explicitly discussing the role of composite structure in transcendental functions.

7.3.1. Compare & Contrast Composition to Arithmetic Operations. When compared to the arithmetic operations, composition receives little attention. Comparing and contrasting the new operation of composition to the familiar operations of arithmetic has the potential to improve students’ understanding of composition. For example, the limit laws of addition, subtraction, multiplication, and division behave similarly. The limit of the sum/difference/product/quotient is the sum/difference/product/quotient[6] of the limits. The limit of the composition, however, is not the composition of the limits. A discussion identifying this difference could help students understand both composition and limits and could include the limit of continuous functions and the cases when one or both functions in a composition are continuous. This could lead to “If *f* is continuous at the limit of *g*, then the limit of the composition equals the value of the outside function evaluated at the limit of the inside function” (CC, 102). Comparing and contrasting composition to the arithmetic operations could help

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[6] As long as the denominator in a quotient is not zero.

students see the structure of each operation and how these structures have similar and different behavior.

Comparing and contrasting the operations of multiplication and composition may be particularly beneficial to students. Previous research has documented that students confuse these two operations the most (e.g., Engelke et al., 2005; Horvath, 2010). Multiplication and composition both use parenthetic notation and the terms *of* and *cancellation*; however these notations and terms have different meanings in the different structures. For example, when parentheses indicate multiplication, one can use the distributive property to write the expression $4(x + y)$ as $4x + 4y$. When parentheses indicate composition such as $\sin(x + y)$, the outside function cannot be distributed across the inner function multiplicatively. Instead, one has to use the angle sum formula to attain an equivalent expression such as $\sin(x)\cos(y) + \cos(x)\sin(y)$. Another example is the term *of*. The value of one-half of one-fourth can be determined by multiplying $\frac{1}{2}$ and $\frac{1}{4}$ while "*f* of *g*" and "take the natural log *of* both sides" indicates composition and not multiplication. Cancellation also occurs in both multiplication and composition. In multiplication, cancellation happens rationally with the multiplicative inverse. The written curriculum, however, rarely mentions the compositive structure explicitly with respect to compositive cancellation. It is typically referred to as a property of inverse functions. Understandably, using similar terms for different operations could be confusing. Explicitly comparing and contrasting the similar terminology of multiplication and composition may help students differentiate between these two structures.

7.3.2. Transcendental Functions. Composite structure is embedded in transcendental functions. For example, we express these functions verbally as "sine of *x* squared" or "take the natural log of both sides." Both of these statements use the word "of" just like in "*f* of *g*" which means to compose *f* with *g*. Composition also exists with the cancellation of transcendental functions. Typically the cancellation is a compositive cancellation involving inverse functions. Despite these connections, the composite structure of transcendental functions is rarely made explicit in the written curriculum.

Explicitly connecting compositive structure to transcendental functions has the potential to improve students' understanding of transcendental functions. It could help students understand the importance of having arguments with transcendental functions. For example, *cos* without an argument is meaningless. Using the words "compose log x with both sides" instead of "take log of both sides" could help students identify that they are using the compositive structure when they solve equations and "take" functions.

Explicitly connecting compositive structure to transcendental functions could also increase students' understanding of function in general. For example, some may argue that using multiplicative cancellation with transcendental functions, such as $\frac{\log(x+1)}{\log x} = \frac{x+1}{x}$ or $\frac{\sin x}{\cos x} = \frac{\sin}{\cos}$, is due to a lack of knowledge about function. However, this mathematical error is associated with the compositive *structure* of functions and not solely with the *definition* of functions.

Current literature on students using multiplicative cancellation with transcendental functions has described this type of behavior as the linear extrapolation error (Matz, 1980) and treating the name of a function as a variable (Liang & Wood, 2005; Yen, 1999). All of these interpretations involve claims of students generalizing (incorrectly) from more familiar situations. Although many of these authors suggest that the definitions of specific functions should be emphasized, such as "$\log_a b$ is the exponent required on the base *a* to produce the value *b*," I would advocate emphasizing the compositive structure of transcendental functions and contrasting it with multiplicative structure. This connects transcendental functions to mathematical concepts beyond its definitions. Additionally, composite structure provides a natural connection among the different types of transcendental functions. It may be that implicitly using compositive structure, students over-generalize the multiplicative structure because they view composition as something to do and not as a structure of functions. Providing students with an increased number of explicit experiences with composition prior to and during the learning of transcendental function, may improve their understanding of composition specifically and of functions more generally.

More explicit exposure throughout the curriculum, where appropriate, has the potential to provide gains for students in understanding compositive structure without increasing the amount of time needed to teach and learn the material and without the need to create a new developmental sequence.

7.4 Comparing the Composition Content in Secondary and Collegiate Courses

It may not be surprising that more composition elements appear in college texts, than in high school texts since more advanced mathematical concepts build on prior concepts. The difference in the composition content in precalculus courses at the different levels *is* surprising. In all of the aspects of composition discussed in previous chapters, the college Precalculus texts have more composition elements than the secondary texts in all but one instance. On the largest scale, the college precalculus texts have 78 more composition elements in 43 fewer total pages. In some aspects, such as Evaluating a Composition and Multiple Representations, there are twice as many elements in collegiate precalculus. In other aspects, such as the Types of Representations, there is only a 13% increase. The one aspect that has more elements in high school precalculus is the conceptual elements on domain and range. Table 7.1 summarizes previous tables in this dissertation and aggregates all of the high school and college Precalculus statistics from each table. The purpose of this table is to compare values across rows to see the difference in the composition content in secondary and collegiate Precalculus. To keep the connection to previous tables, if the previous table used percents, then the row in Table 7.1 associated with that table also uses percents. The same is true regarding raw counts.

Table 7.1
Comparison between the amount of composition content in secondary and collegiate precalculus.

Table/Figure Number	Description of Table	HS Precalculus	College Precalculus	Amount More in College Precalculus
Table 3.3	Number of Textbook Pages	1539	1496	-43
Table 3.3	Number of Total Elements	118	196	+78
Table 5.1	Composite Structure	4	30	+26
Table 5.2	Conceptual Domain & Range	23	18	-5
Table 5.3	Procedural Domain & Range	44%	57%	+13%
Figure 5.1	Non-Commutative	39	71	+32
Table 5.4	Compositive Inverse	26%	38%	+12%
Table 5.5	Evaluate a Composition	16%	40%	+24%
Table 6.1	Types of Representations	31%	35%	+4%
Table 6.2	Multiple Representations	13	29	+16
Table 6.6	Types of Functions	22%	44%	+22%
Table 6.9	"Taking" or "Raising"	30	88	+58

7.5 Limitations & Implications

7.5.1 Limitations. As with any study involving curriculum, it is possible, in fact likely, that the enacted curriculum will be very different from the written curriculum. Teachers make individual choices on what content to teach and when and how to supplement textbooks. The purpose of this study was to identify what opportunities the written curriculum provides teachers and students with respect to the concept of composition. Additionally, this study was did not analyze an exhaustive list of written curriculum. While the twelve selected textbooks were carefully chosen, these findings are limited to the content of these twelve texts. I also acknowledge limitations associated with the methods used in this study. Despite analyzing each book multiple times to identify composition content, some relevant content may have been overlooked based on the range of interpretations of composition. My study uses my perspective of where to draw the line with respect to implicit composition content (see Appendix B). Although I based my perspective of composition on many conversations with colleagues and study of advanced mathematical texts and prior research, others might choose to draw the line somewhere else.

7.5.2 Implications for Teaching & Curriculum Development. Composition occurs throughout the secondary and early collegiate curriculum and provides an underlying mathematical structure for functions. If compositive structure was made more explicit at the secondary and collegiate levels, it would likely help students develop a more robust and flexible conception of composition and improve their understanding of other concepts associated with composition.

As discussed previously, research has shown that students struggle to understand composition and transcendental functions. This study has shown that connections to composition are not frequently made explicit in the written curriculum. Teachers and curriculum developers might consider using composition and the compositive structure to connect mathematical ideas across the curriculum. Including explicit composition content with transcendental functions could provide structure to connect the different types of transcendental functions. Additionally, including composition, where appropriate, provides students with the opportunity to experience and identify how composition and compositive structure appears in multiple contexts.

Compositive cancellation is another aspect that could be made explicit. Compositive cancellation is only connected to inverse functions and the compositive structure is not explicitly discussed in these texts. Contrasting compositive cancellation with multiplicative cancellation could help students see how these two structures are different even though they use the same language and notation.

Lastly, an increase in the percentage of non-algebraic representations and multiple representations could help students understand composition. In these texts, many of the elements with multiple representations involve the display of more than one representation or the creation of a second representation. A task that requires students to perform the composition of two graphs is one such example of a non-algebraic task. This could be accomplished using the sequence view of plotting points or by the operation view with graphic transformations. The purpose of such a task would not be solely to help students become proficient in composing graphs, rather to help them see compositive structure in another setting to get a broader picture of how this structure works. Other benefits include connecting commutative composition to commutative graph transformations.

7.5.3 Implications for Research. While there are many possible implications for research from this study, this section mentions only a few. First, research on the teaching and learning of function could be reconceptualized to include the compositive structure of function. Carlson and colleagues (2010) included composition as an essential topic in preparing students for calculus and this dissertation has pointed out various parts of the curriculum where compositive structure played an essential role in the mathematics. Research on number sense has benefited from studies on the arithmetic operations and it is reasonable to expect that research on function sense would similarly benefit from studies of composition. Second, future research on composition could use the sequence view and operation view of composition as a framework to study students understanding of composition. For example, a study could focus on which view students use when solving a variety of problems in a variety of contexts. Third, teaching experiments could be conducted to study the suggested changes to the curriculum and interviews could be conducted to examine how students interpret some of the ideas that implicitly relate to composition. This could include a study of students on what it means to them to "take the natural log" and determine why they think it is a valid mathematical move. Composition is a rich and important topic with many possibilities for future work.

APPENDICES

APPENDIX A: SURVEY OF CALCULUS AND PRECALCULUS TEXTBOOKS

Author(s) & *Title* of textbook
Name of Institution | Course Number | Course Title

Calculus Textbooks

Author(s) & *Title* / Name of Institution	Course Number	Course Title
Anton, Bivens, & Davis *Calculus: Early Transcendentals*		
Case Western Reserve University	Math 121	Calculus for Science and Engineers
University of Colorado at Denver	Math 1401	Calculus I
Apostol *Calculus*		
California Institute of Technology	Math 1a	Calculus of One Variable
Massachusetts Institute of Technology	18.014	Calculus I with Theory
Briggs & Cochran *Calculus*		
Tulane University	Math 1150	Long Calculus
University of Virginia	Math 1310	Calculus I
Briggs & Cochran *Calculus: Early Transcendentals*		
Louisiana State University	Math 1550	Calculus I
University of Connecticut	Math 1125Q	Calculus Ia
Edwards *Calculus: Early Transcendentals*		
Case Western Reserve University	Math 121	Calculus for Science and Engineers
Ellis & Gulick *Calculus*		
University of Maryland	Math 140	Calculus I
Goldstein, Lay, Schneider & Asmar *Calculus & It's Applications*		
University of California, Berkeley	Math 16A	Analytic Geometry & Calculus
Hass, Weir, & Thomas *University Calculus*		
University of Nebraska	Math 106	Analytic Geometry and Calculus I
Virginia Polytechnic Institute and State University	Math 1205	Calculus I
University of Georgia	Math 2250	Calculus I for Science and Engineering
University of Hawaii	Math 241	Calculus I
Hughes-Hallet, et al *Calculus*		
Duke University	Math 31	Introductory Calculus
Ohio State University	Math 151.02	Calculus and Analytic Geometry I
University of Arizona	Math 124/125	Calculus I
University of Colorado at Boulder	Math 1300	Analytic Geometry/Calculus I
University of Michigan	Math 115	Calculus I
Yeshiva University	MAT 1412	Calculus I
Kreider, Lahr & Diesel *Principles of Calculus Modelling: An Interactive Approach*		
Dartmouth	Math 3	Introduction to Calculus
Larson, Hostetler, & Edwards *Calculus: Early Transcendental Functions*		
University of South Florida	Math 2281	Engineering Calculus I
University of South Florida	Math 2311	Calculus I
Mueller & Brent *Just-In-Time: Algebra and Trigonmetry for Students of Calculus*		
Ohio State University (supplemental)	Math 15.01	Calculus and Analytic Geometry

Rogawski *Calculus*		
University of California, Los Angelos	Math 31A	Differential and Integral Calculus
Rogawski *Calculus: Early Transcendentals*		
Louisiana State University	Math 1550	Calculus I
Rutgers	Math 151	Calculus I for the Mathematical and Physical Sciences
University of California, San Diego	Math 20A	Calculus for Science and Engineering
University of Cincinnati	15 Math 251	Calculus I
University of Illinois, Chicago	Math 180	Calculus I
Salas, Hille, & Etgen *Calculus: One & Several Variables*		
Georgia Institute of Technology	Math 1501	Calculus I
Simons *Calculus with Analytic Geometry*		
Massachusetts Institute of Technology	Math 18.01	Calculus I
Smith & Minton *Calculs: Early Transcendentals*		
Arizona State University	Math 270	Calculus with Analytic Geometry I
Stewart *Calculus*		
University of Missouri	Math 1500	Analytic Geometry & Calculus I
Kansas State University	Math 220	Analytic Geometry & Calculus I
University of New Mexico	Math 162	Calculus I
University of Pennsylvania	Math 103	Introduction to Calculus
University of Texas	M408C	Calculus I
University of Washington	Math 124	Calculus with Analytic Geometry I
Vanderbilt University	Math 155a	Accel Single-Vari Calc I
University of California, Irvine	Math 2A	Single-Variable Calculus
Emory University	Math 111	Calculus I
University of Iowa	22M:025	Calculus I
Pennsylvania State University	Math 140	Calculus with Analytic Geometry I
John Hopkins University	110:108	Calculus I
University of Notre Dame	Math 10550	Calculus I
University of California, Berkeley	Math 1A	Calculus
University of California, Riverside	Math 9A	First Year of Calculus
University of Albany (SUNY)	AMAT 112	Calculus I
Indiana University	Math M211	Calculus I
Stewart *Calculus: Early Transcendentals*		
Columbia University	Math V1101	Calculus I
University of Illinois, Urbana-Champaign	Math 220	Calculus
Ohio State University	Math 151.01	Calculus and Analytic Geometry
Syracuse University	Mat 295	Calculus I
Tulane University	Math 1210	Calculus I
University of Florida	MAC 2311	Analytic Geometry and Calculus I
Rice University	Math 101	Single Variable Calculus I
Wayne State University	MAT 2010	Calculus I
University of Kentucky	Math 113	Calculus I
University of California, Santa Cruz	Math 19A	Calculus for Science , Engineering, and Mathematics

Tufts University	Math 11	Calculus I
Montana State University	M171Q	Calculus I
Florida State University	MAC 2311	Calculus I
Ohio State University	Math 151.01	Calculus and Analytic Geometry
University of Minnesota	Math 1271	Calculus I
University of Rochester	Math 161	Calculus IA
Purdue University	MA 161	Plane Analytic Geometry and Calculus I
University of South Carolina	Math 141	Calculus I
University of North Carolina at Chapel Hill	Math 231	Calculus Function One Variable I
University of Oregon	Math 251	Calculus I
Brown University	Math 0090	Introductory Calculus, Part I
University of California, Santa Barbara	Math 3A	Calculus with Applications, first course
University of Toronto	MAT 135Y	Calculus I
Northwestern University	Math 212	Single Variable Calculus I
Northwestern University	Math 220	Differential Calculus of One Variable Functions
University of Kentucky	MA 113	Calculus I
University at Buffalo (SUNY)	Math 141	College Calculus I
Yale University	Math 112	Calculus of Functions of One Variable I
McGill University	Math 140	Calculus I
University of Massachusetts	Math 131	Calculus I

Stewart *Calculus: Early Vectors*

Texas A&M University	Math 171	Analytic Geometry and Calculus

Stewart *Essential Calculus*

University of Pittsburgh	Math 0220	Analytic Geometry and Calculus I
University of Southern California	Math 125	Calculus I
New York University	V63.0121	Calculus I
Oregon State University	Math 251	Differential Calculus

Stewart *Essential Calculus: Early Transcendentals*

Carnegie Mellon University	21-120	Calculus I
Washinton State University	Math 171	Calculus I
University of Alabama	Math 125	Calculus I
Georgetown University	Math 035	Calculus I

Stewart *Calculus: Concepts and Contexts*

Brandeis University	Math 10a	Methods and Techniques of Calculus
Washington University in St. Louis	Math 131	Calculus I
North Carolina State University	MA 141	Calculus I
Boston University	CAS MA 123	Calculus I
Stony Brook University (SUNY)	Math 125	Calculus A
Stanford University	Math 19	Calculus
Stanford University	Math 41	Calculus
Harvard University	Math 1a	Calculus I

University of Tennessee Math 141 | Calculus I
University of Kansas Math 121 | Calculus I

Stewart *Custom edition*

University of Delaware Math 241 | Analytic Geometry and Calculus A

Thomas, Weir, & Hass *Thomas' Calculus*

Michigan State University Mth 132 | Calculus I

Thomas, Weir, & Hass *Thomas' Calculus: Early Transcendentals*

Cornell University Math 1110 | Calculus I
University of California, Davis Math 021A | Calculus
Colorado State University Math 160 | Calculus for Physical Scientists I

Thomas, Weir, & Hass *Thomas' Calculus with Second Order Differential Equations*

University of Wisconsin, Madison Math 221 | Calculus and Analytic Geometry

Varburg *Calculus*

Iowa State University Math 165 | Calculus I

Varburg *Calculus with Differential Equations*

University of Utah Math 1210 | Calculus I

Precalculus Textbooks

Aufmann *College Algebra & Trigonometry*		
Boston University	CAS MA 118	College Algebra and Trigonometry
Axler *Precalculus: A Prelude to Calculus*		
Georgia Institute of Technology	Math 1113	Precalculus
University of Wisconsin	Math 114	Algebra and Trigonometry
Axler *Precalculus*		
Michigan State University	Mth 116	College Algebra & Trigonometry
University of Oregon	Math 112	Elementary Functions
Barnett, Ziegler, & Byleen *Precalculus*		
University of California, Irvine	Math 1A-1B	Precalculus
Beecher, Penna, & Bittinger *Algebra & Trigonometry*		
University of Pittsburgh	Math 0032	Trigonometry and Functions
Beecher, Penna, & Bittinger *College Algebra*		
Kansas State University	Math 100	College Algebra
Blitzer *Precalculus*		
Stony Brook University (SUNY)	Mat 123	Introduction to Calculus
Arizona State University	MAT 170	Precalculus Mathematics
Blitzer *Algebra and Trigonometry*		
University of Alabama	Math 115	Precalculus Algebra and Trigonometry
University at Buffalo (SUNY)	Math 115	Survey of Algebra and Trigonometry
Coburn *Precalculus*		
University of Illinois, Urbana-Champaign	Math 115	Preparation for Calculus
University of Texas	M305G	Preparation for Calculus
University of California, Los Angelos	Math 1	Precalculus
University of California, San Diego	Math 4C	Precalculus for Science and Engineering
University of California, Berkeley	Math 32	Precalculus
Cohen, Lee, & Sklar *Precalculus: A Problem-Oriented Approach*		
University of California, Santa Cruz	Math 3	Precalculus
University of California, Davis	Math 012	Precalculus
Cohen, Lee, & Sklar *Precalculus*		
University of Arizona	Math 120R	Calculus Preparation
Collingwood & Prince *Precalculus* (Online Book)		
University of Washington	Math 120	Precalculus
Connally *Functions Modeling Change: A Preparation for Calculus*		
Syracuse University	Mat 194	Precalculus
Connally, Hughes-Hallet, Gleason, et al. *Functions Modeling Change: A Preparation for Calculus*		
University of California, Santa Barbara	Math 15	Precalculus
Yeshiva University	MAT 1160	Precalculus
Dugopolski *College Algebra & Trigonometry*		
University of California, Riverside	Math 5	Introduction to College Mathematics

University of California, Riverside Math 8A | Introduction to College Mathematics for Science

Dugopolski *Precalculus: Functions & Graphs*

University of California, Riverside Math 8B | Introduction to College Mathematics for Science

University of Colorado, Boulder Math 1150 | Precalculus Mathematics

University of South Carolina Math 115 | Precalculus Mathematics

Estry *Precalculus*

Montana State University M 151Q | Precalculus

Faires & DeFanza *Precalculus*

New York University V63.0009 | Algebra & Calculus

University of Cincinnati 15 Math 250 | Calculus 0

University of Connecticut Math 1060Q | Precalculus

Goodman & Hirsch *Precalculus: Understanding Functions*

University of Delaware Math 117 | Pre-Calculus for Scientists and Engineers

Hungerford *Contemporary Precalculus*

University of Pittsburgh Math 0200 | Prep for Scientific Calculus

Keedy *Algebra & Trigonometry*

Vanderbilt University Math 133 | Precalculus

Larson & Hostetler *Precalculus*

Georgia Institute of Technology Math 1113 | Precalculus

University of Utah Math 1050 | College Algebra

University of Utah Math 1060 | Trigonometry

Larson & Hostetler *Precalculus with Limits*

University of Florida MAC 1147 | Precalculus Algebra and Trigonometry

Larson & Hostetler *Precalculus: A Concise Course*

University of Colorado, Denver Math 1130 | Precalculus Mathematics

Larson, Hostetler, & Edwards *Precalculus: Functions and Graphs*

University of Kansas Math 104 | Precalculus

Lial, Greenwell, & Ritchey *Calculus with Applications*

Cornell University Math 1101 | Calculus Preparation

Mueller & Brent *Just-In-Time Algebra and Trigonmetry for Calculus*

Carnegie Mellon University 21-105 | Pre Calculus

Online Departmental Texts (Different for each institution)

University of Washington Math 120 | Precalculus

Texas A&M University Math 150 | Functions, Trigonometry, and Linear Systems

Washington State University Math 107 | Precalculus

University of Cincinnati (calculus supplement) 15 Math 250 | Calculus 0

Stony Brook University (SUNY) (calculus supplement) Mat 123 | Introduction to Calculus

Ratti & McWaters *Precalculus*		
University of South Florida	MAC 1147	Precalculus, Algebra, and Trigonometry
Rockswold *Algebra & Trigonometry with Modeling and Visualization*		
Oregon State University	Mth 112	Elementary Functions
Stewart *Calculus*		
University of California, Riverside	Math 8B	Introduction to College Mathematics for Science
Stewart, Redlin, & Watson *Algebra and Trigonometry*		
Pennsylvania State University	Math 041	Trigonometry and Analytic Geometry
Stewart, Redlin, & Watson *Precalculus*		
Brandeis University	Math 5a	Precalculus Mathematics
Ohio State University	Math 150	Elementary Functions
Rutgers University	Math 640:115	Precalculus College Mathematics
University of Hawaii	Math 140	Precalculus
University of Maryland	Math 115	Pre-Calculus
University of Missouri	Math 1160	Precalculus Mathematics
University of New Mexico	Math 150	Pre-Calculus Mathematics
University of Tennessee	Math 130	Precalculus
Wayne State University	MAT 1800	Elementary Functions
Sullivan *Algebra & Trignometry*		
Iowa State University	Math 142	Trigonometry & Analytic Geometry
Louisiana State University	Math 1023	College Algebra and Trigonometry
University of Nebraska	Math 103	College Algebra & Trigonometry
University of Miami	Math 107/108	Precalculus Mathematics I/II
Sullivan *College Algebra*		
University of Miami	Math 105	Algebra and Trigonometry
Sullivan *Precalculus*		
Case Western Reserve University	Math 120	Elementary Functions and Analytic Geometry
Iowa State University	Math 142	Trigonometry & Analytic Geometry
University of Illinois, Chicago	Math 121	Precalculus Mathematics
University of Iowa	22M:009	Elementary Functions
University of Minnesota	Math 1155	Intensive Precalculus
University of Minnesota	Math 1051/1151	Precalculus I/II
Sullivan *Precalculus: Enhanced with Graphing Utilities*		
Florida State University	MAC 1140	Precalculus Algebra
Sullivan & Sullivan *Precalculus: Concepts Through Fuctions*		
North Carolina State University	MA 111	Precalculus Algebra and Trigonometry
Swokkowski & Cole *Precalculus: Functions and Graphs*		
Cornell University	Math 1009	Precalculus Mathematics
Georgetown University	Math 001	Pre-Calculus
University of Georgia	Math 1113	Precalculus
University of North Carolina	Math 130	Precalculus Mathematics

Swokowski & Cole *Algebra and Trigonometry with Analytic Geometry*		
Indiana University	Math M027	| Pre-Calculus Mathematics with Trigonometry
Purdue University	Math 159	| Precalculus
Swokowski & Cole *Fundamentals of Trigonometry*		
Kansas State University	Math 150	| Plane Trigonometry
Warner & Costenoble *Applied Calculus*		
Virginia Polytechnic Institute and State University	Math 1015	| Elementary Calculus with Trigonometry (Precalculus)
Young *Algebra & Trigonometry*		
Tufts University	Math 4	| Fundamentals of Mathematics

APPENDIX B: PHASE TWO INCLUSIONS AND EXCLUSIONS OF IMPLICIT COMPOSITION

Table B.1

Data Elements Included and Excluded by Phase Two Criteria

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Chain Rule	The chain rule is defined using composition, $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ and it is used to find the derivative of a composite function.	The derivative of a composite function $f[g(x)]$ is the derivative of the outer function f evaluated at the inner function g multiplied by the derivative of the inner function g (GP, 799). [Criteria 2, 3, & 6]	
Difference Quotient	The Difference Quotient, $\frac{f(x+h)-f(x)}{h}$, uses parenthetic notation. Also, simplification of the Difference Quotient requires the evaluation of $f(x + h)$.	Substitute $f(x) = \ln x$ into the difference quotient (GP, 217). [Criteria 4] True or False. If $f(x) = x^2 + x$, then $\frac{f(x+h)-f(x)}{h} = 2x + h$ (WP, 430). [Criteria 3]	
Even and Odd Functions	Determining if a function is even or odd and the procedure of composition are both referred to as substitution or plugging in. To determine whether a function is even or odd, $-x$ is plugged into the x's of $f(x)$. Only elements that explicitly connected composition to even and odd functions or the procedure of determining whether a function is even or odd to terms of substitution were included in Phase Two.	Determine whether $(f \circ g)(x)$ is even, odd, neither, or not enough information for each of the following (GP, 63). [Criteria 2]	Even Function: For every x in the domain of f, $f(-x) = f(x)$ (GP, 18). Odd Function: For every x in the domain of f, $f(-x) = -f(x)$ (GP, 18).

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Evaluating functions at specific values	Evaluating functions numerically could be thought of as the composition of a function with a constant function. This view, however, was never used in any of the texts analyzed. Numerically evaluating functions was not sufficient for an element to be included in the Phase Two data set. It must also contain other content that satisfied the criteria of Phase Two such as having the terms of *substitution*, *replace*, or *plug in* present.	When you have an equation such as $4x - 3y = 11$, you can find one point easily by plugging in 0 for x and solving for y (PA1, 344). [Criteria 4]	Given $f(x) = 2x^2 - 8$, find $f(6)$ (GA1, 64).
Graph Trans-formations	The notation used with graph transformations is similar to the parenthetic notation used in composition. Connections between these two concepts were rarely made explicit. Frequently, the function notation, $f(ax + h) + k$, is typically used in the box features of the texts. Equations, such as $(x + h)^2 + (y + k)^2 = r^2$, are used elsewhere. The main focus of graph transformations content is to identify the a, h, and k in the equation and describe how this shifts and transforms the graph and not on principles of composition. Graph transformations with trigonometric functions typically focus on identifying the amplitude, period, and phase shift of the transformed function. Content on graph transformations was included in Phase Two if it speaks about these transformations in Sequence view language (first one transformation and THEN another) or function notation is used.	For example, if we apply a horizontal shift followed by a horizontal stretch, we may get a different result than if we first applied the horizontal stretch followed by the horizontal shift (WP, 256). [Criteria 1] Write a formula for the trans-formation of $f(x) = \sqrt{x}$ given by $6f(x-8)$ (WP, 246). [Criteria 3] Let $f(x) = \frac{1}{x}$. $r(x) = \frac{2}{x-3} = 2\left(\frac{1}{x-3}\right) = 2f(x-3)$ (CP, 279). [Criteria 3]	The graph of $y = (x - h)^2 + k$ is the graph of $y = x^2$ translated $\lvert h\rvert$ units left if h is negative or $\lvert h\rvert$ units right if h is positive and $\lvert k\rvert$ units up if k is positive or $\lvert k\rvert$ units down if k is negative (GA2, 320). The general form of the tangent function, which is similar to that of the sinusoidal functions, is $y = a\tan(bx + c) + d$, where a produces a vertical stretch or compression, b affects the period, c produces a phase shift, and d produces a vertical shift (GP, 269).

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Higher Order Derivatives	Second and higher order derivatives are found by repeatedly applying the derivative. Each time, the result of the previous derivative is used to determine the next derivative. This implicit idea of composition was not included in Phase Two.		We see that the successive derivatives occur in a cycle of length 4 and in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4 (CC, 155).

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Inverse Functions	Inverse functions can be defined as (1) switching the domain and range so that the range maps to the domain, (2) the function whose composition with the original function results in the identity function, and (3) undoing a function. Interchanging the domain and range was not included in Phase Two. Explicit connections between inverse functions and composition, such as denoting $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$ or referring to this principle as the "inverse function property" or "cancellation property of inverse function" was included in Phase Two. Lastly, when *undoing* a function referred to processing the output of the original function through the inverse function (a Sequence view of composition), it was included in the study by Criteria 1. Other forms of *undoing* were not sufficient to be included in Phase Two.	When $f(g(x)) = g(f(x)) = x$, f and g are inverses of each other (GP, p. 641). [Criteria 3] By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started (CP, 201). [Criteria 1] The functions f and f^{-1} are called inverses because they "undo" each other when composed (WP, 90). [Criteria 2] Answer: They are inverse functions. They "undo" each other, giving you back your input (PA1, 434). [Criteria 1] The inverse of a function is a rule that acts on the output of the function and produces the corresponding input. So the inverse "undoes" or reverses what the function has done (CP, p. 199). [Criteria 1]	The inverse relation is the set of ordered pairs obtained by exchanging the coordinates of each ordered pair (GA2, 417). When a relation is expressed as an equation, its inverse relation can be found by interchanging the independent and dependent variables (GP, 65). This exercise points out that if f and g are inverses and the point (a, b) is on the graph of f, the point (b, a) is on the graph of g (PP, 258). Extend the idea of finding an inverse of a function by undoing its steps in reverse order to find the inverse of an affine transformation (PA2, 532). The operation of taking a logarithm "undoes" the exponential function; the logarithm and the exponential are inverse functions (WP, 181).

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Iteration	Iterations can be explained as performing a function, then applying the function to the output value and repeating that process as many times as needed. The nested parenthetic notation such as $g(g(g(x)))$ is also used with iterations.	Iteration is the process of repeatedly composing a function with itself (GA2, 716). [Criteria 6] For Problem 3, when students choose a positive number and apply g repeatedly, their number tends towards the number 1.680. (PA2, 101). [Criteria 1]	
Parametric Equations	Parametric equations also have implicit connections to composition. However, only elements that connected parametric equations to substitution were included in Phase Two.	Write $y = t^2 + 2$ and $x = 3t - 1$ in rectangular form by substituting $x = \frac{x+1}{3}$ for t in the equation for y (GP, 465). [Criteria 4]	
Periodic Functions	Periodic functions are similar to graph transformation in that it uses parenthetic notation, but the major focus of periodic functions is identifying the value of p in the equation $f(x + p) = f(x)$. Content on periodic functions was included if there was an explicit connection to composition.	True or False. If $f(x)$ is a periodic function with period k, then $f(g(x))$ is periodic with period k for every function $g(x)$ (WC, 66). [Criteria 3]	The period of a periodic function is the least positive number p so that $f(x + p) = f(x)$ always (PP, p. 57).
Polar Coordinates	Polar coordinates also have implicit connections to composition. However, only elements that connected polar coordinates to substitution were included in Phase Two.		

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Raising to a power	Raising numbers and functions to a power use parenthetic notation. Simply raising to a power, however, does not always emphasize composition. For example, the equation of a circle, for instance, squares $(x+h)$, but the focus is on the general form of the equation and not on the squaring the output of $x+h$. Thus, this alone was not sufficient to be included in the data set of Phase Two and required other content to be present in an element that satisfied the Phase Two criteria.	A verbal description near $(x + 7)^3$, such as "add seven and then cube the result." [Criteria 1] $(f(x))^2$ [Criteria 3]	$(x + 7)^3$ $(4^2)^3$
Recursion	Recursion is implicitly connected to the Sequence view of composition. It involves using previous output(s) to determine future outputs. Also, functions can be expressed in recursive form and closed form. All instances of recursion were originally included, and then later excluded from the Phase Two data set. The categorizing of recursion as implicit composition and recursion as function form became problematic. The differences need to be examined as the primary focus of study, rather than as a secondary component.		

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Add, subtract, multiply, or divide both sides of an equation	Adding 4 to both sides of the equation $5x - 4 = 4x^2 - 3$ can be considered a composition. If $g(x) = 5x - 4$ and $f(x) = x + 4$, then the result of the composition $f(g(x))$ is the same as adding 4 to the left hand side of the equation. Since this was never presented as composition in any of these textbooks, this interpretation was not used to include the arithmetic operations in the Phase Two data set.		
Substitu-tion	Substitution a term often used to refer to the procedure of composition. To evaluate the composition $f(g(x))$, $g(x)$ is substituted into the x's of $f(x)$. U-substitution and trig-substitution is also used to solve equations. This is related to decomposing a composite function.	Substitute x - 4 for x in $f(x)$ (GP, 58). [Inclusion by Criteria 4] Sasha goes to the board and covers up the sin x [in the equation 4 sin $x + 5 = 7$] with her hand: 4 + 5 = 7 (PP, 18). [Inclusion by Criteria 7]	
Taking a function of both sides of an equation.	When *taking* a function of both sides of an equation, a function is being applied to a function which is composition. For example "take sin of both sides" means to compose the sine function with the expressions on both sides of an equation.	This application of the One-to-One Property is called taking the logarithm of each side of an equation (GP, 192). [Inclusion by Criteria 5]	

Table B.1 (Cont'd)

Topic	Rationale for Inclusion	Examples - Included	Examples - Excluded
Symmetry	Similar to the topic of even and odd functions, symmetry about the origin, *x*-axis, and *y*-axis is related to composition through parenthetic notation and the substitution terms. Elements containing the substitution terms were included in the Phase Two data set, but statements such as $f(-x) = f(x), f(x) = -f(x), f(-x) = -f(x)$ were not sufficient for inclusion.	Algebraic Test: Replacing *x* with -*x* produces an equivalent equation (GP, 16). [Criteria 4]	Graphical Test: The graph of a relation is symmetric with respect to the y-axis if and only if for every points (*x*, *y*) on the graph, the point (-*x*, *y*) is also on the graph (GP, 16).
Systems of Equations	The topic of systems of equations connects to composition through the term *substitution*. Substitution is the name of a method for solving systems of equations and the term often used to refer to the procedure of composition. The results of each of these concepts however are different. Solving systems results in values while composition results in functions.		

APPENDIX C: COMPOSITION CODING SCHEME

MC: Expressions that are mathematically problematic (incomplete, not well-defined, etc.) about composition.

S: The element is a statement. This includes exposition, explanation in the solution of a worked example.
P: The element is a problem that the student is given to solve.
IP: The element is instructions to a problem that the student will solve more than once for different functions or given information. *Note: If the student is to solve it only once, it would just be coded as P.*
Q: Question in the TE.
F: In a Figure

Conceptual Composition Codes
CC 01 Domain of $(f \circ g)(x)$ (operation): The values of x in the domain of g such that $g(x)$ is an element in the domain of f, the domain of $(f \circ g)$ consists of just those values of x. *Note: the focus is on the final composed function.*
CC 02 Domain (sequence view): $g(x)$ must be in the domain of f *Note: the focus is on the range of the inside function being a subset of the domain of the outside function*
CC 02a Domain (first half of CC 02): x maps to $g(x)$
CC 03 Domain of the composition (no view): Domain that is not CC 01 or CC 02
CC 04 Output of the "inside", is the input of the "outside": Similar to CC 02, but no reference to domain and range, only input and output
CC 05 The composition f(g(x)) is undefined if g(x) is not an element of the domain of f(x):
CC 06 Associativity: $(f \circ g) \circ h = f \circ (g \circ h)$:
CC 07 Commutativity: In general, $f \circ g \neq g \circ f$:
CC 08 Some pairs of f and g are commutative:
CC 09 Composition of 2 one-to-one functions is one-to-one:
CC 10 Composite is defined where both $g(x)$ and $f(x)$ are defined:

CC 11 Composition describe via the Sequence view: First one function and then another, then another, output used as input, etc.
CC 12 Composition described via the Operation view: The result of a composition is a function
CC 13 Definition of Composition: $(f \circ g)(x) = f(g(x))$, and others
CC 14 Composition is linking 2 function machines (SV):
CC 15 Difference/Similarities of composition with the arithmetic operations:
CC 16 Define a new function using composition:

CC 19 Other Properties of composite functions: such as lim of composite = $f(\lim g)$

CC 24 Compute by substitution:
CC 25 Inverse functions "undo" each other when composed: Explicit link to composition required.

CC 29 Trig inverse Domain Stuff:

CC 30 Identity function: Such as compose with the identity and other identity function text connected to composition.

CC 31 Non-uniqueness of decomposition: Decomposition of a function is not unique.
CC 32 Definition of Decomposition: two (or more) functions whose composition is *h*
CC 33 You can decompose a function:

CC 35 Conceptual of PC 35:

CC 41 Define a composition of geometric transformations or describe its properties: Define a glide reflection (p. 299, Glencoe Geometry) (GG, p. 588 #39), translation, rotation, etc.
CC 42 Composition of Geometric Transformations via sequence view: First one transformation and then another, then another, etc.
CC 43 Operation view (transformation): The result of a composition is a transformation
CC 44 Composition of reflections stuff:

CC 51-# Number of elements in a composition: # indicates how many elements

CC 61 Statements explicitly connecting composition to other items: When *f* and *g* are compositively commutative and equal *x* (the identity function), they are inverses of each other.
CC 62 Statements explicitly connecting other items (i.e., inverse functions) to composition:

CC 65 Cancellation property Log/Exp: Example $\log_{10} 10^3 = 3$ or $\ln e^x = x$
CC 66 Cancellation property Trig: Example $\sin(\sin^{-1} x) = x$
CC 67 Composition of a function and its inverse is the identity function: This also includes the cancellation property of inverse functions not listed in CC 65 or CC 66.
CC 68 Conceptual part of verify f and f^{-1} are inverses:

CC 71 Important skill/concept in future mathematics (i.e., calculus):

CC 74 Conceptual u-substitution stuff: i.e., Talking about “lumping”
CC 75 Monic in “2x” or something else nontrivial:
CC 76 Plug the u back in:
CC 77 Conceptual PC 77 stuff:
CC 78 Statement about compositive structure: Build more complicated/complex functions from simple ones, etc.

CC 80 Composition of more than 2 functions is possible:
CC 81 Decompose into more than 2 functions is possible:

CC 83 Define an affine transformation:

CC 88 Chain rule allows the derivative of a composite function:

CC 91 Composition in real-world situations
CC 96 Seems conceptual about composition, but doesn't fit elsewhere
CC 97 units of application problems:

Procedural Composition Codes
PC 01o Domain of the "outside": Find the domain of the outside function
PC 01i Domain of the "inside": Find the domain of the inside function
PC 02 Domain of a composed function: Find the domain of a composed function (including finding "any restrictions" requests)
PC 02a Assertion of the domain: Just the answer

PC 03o Range of the "outside": Find the range of the outside function
PC 03i Range of the inside: Find the range of the inside function
PC 04 Range of composed function: Find the range of a composed function

PC 06 Associativity: Compute both $(f \circ g) \circ h$ and $f \circ (g \circ h)$
PC 07 Commutativity: Compute both $f \circ g$ and $g \circ f$.
PC 07-b: the second part of computing both $f \circ g$ and $g \circ f$ *Note: I created a second line to capture the different FC codes*
PC 08 Find 2 functions such that $f(g(x) = g(f(x))$:
PC 09 Reverse the order of function machines:
PC 10 Given f find g such that $f(g) = g(f)$:

PC 11 Evaluate a composite $f \circ g$ at a numerical point: $(f \circ g)(4)$ not dependent on view
PC 12 Evaluating a function "at a variable": The variable must be different than the one used to define the function. Example: Given f, Find $f(a)$ [Ex: $f(t) = 2t^3$, $f(g) = 2g^3$]
PC 12-c The variable has a coefficient on it: Given f, Find $f(3a)$
PC 13 Evaluating a function "at an expression": Given f, Find $f(x + h)$
PC 14 A function put "inside" a formula or expression: Key feature/difference between PC 26 is that the argument is just a single variable. Example: $3[f(t)] + 2$
PC 15 Evaluate a composition where f and g are defined external to the request for composition: Given the outside and inside function (formula, graph, table, machine, etc.) provide an expression for the composition (Notation has f and g as objects) (similar to PC 42)
PC 16 Compose a function with the identity function:
PC 17 Evaluate a composition via the sequence view: Given f (outside) and g (inside), evaluate $(f \circ g)$ at a given value a (or numerical value), by first evaluating $g(a)$ and **then** evaluating $f(g(a))$.
PC 18 Evaluate a composition via the operation view: Given f (outside) and g (inside), evaluate $(g \circ f)$ at a given value a (or numerical value), by first representing $(g \circ f)$, then evaluating at a.

PC 19 Define a composite mapping (or ordered pairs): Given two mappings f and g, find $(f \circ g)(x)$.

PC 20 Define a composition of affine transformations as a single affine transformation:

PC 21 Evaluate a composition of trig functions (using inverse trig functions which is the same as the original function) at a numerical value: $\sin(\sin^{-1} ¾)$ or sin(arcsin ¾)

PC 22 Evaluate a composition of trig functions (using inverse trig functions which is the different to the original function) at a numerical value: $\cos(\tan^{-1} 1)$ or tan(arcos 1)

PC 23 Given the output, find the input of the composition:

PC 24 (operation view) Perform composition by plugging in/substitution/replacing an expression for a variable:

PC 25 Find [something] of the result: (SV)

PC 26 Evaluating a function "at an expression (PC 13) is used within another formula or expression (PC 14): Key feature/difference between PC 14 is that the argument is just an expression. This includes things like "$f(x + 1) - f(x)$". If "$f(x + 1) - f(5)$" instead of $f(x)$ it is PC 14. $f(x)$ is a "variable" while $f(5)$ is a number.

PC 27 Worked out step of composition where the inside is plugged into the $f(g(x))$ formula (step 1)

PC 28 Worked out step of composition where the g(x) is plugged into the formula for $f(x)$ (step 2)

PC 84 When a PC 27 or PC 28 code is done at the same time was a numerical value is evaluated for x (see GA2 p. 412 "Solve" section of Ex 4)

PC 29 Trig functions that have multiple functions as the angle (or argument):

Ex. $\tan[\cos^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2}]$

PC 30 Graph a composition

PC 31 Check (or verify) your work of decomposition by composing the functions:

PC 32-# Decomposition: Given a function, decompose it into # of functions. (usually 2 functions)

PC 33 Restriction on the decomposition: (i.e., Neither function can be the identity function)

PC 34 Given $f(x)$ (outer) and $(f \circ g)(x)$, find $g(x)$ (inner):

PC 35 Given $g(x)$ (inner) and $(f \circ g)(x)$, find $f(x)$ (outer):

PC 36 Given one function, find the other function such that $(f \circ g)(x) = x$: Finding a one-sided inverse function

PC 37 Given one function and a composition, Find 2 other function such that composing the 3 function results in the given composite function:

PC 38 Restriction on PC 37: (i.e., Use only linear functions)

PC 39 Given one function, find the other function such that $(f \circ g)(x) = (g \circ f)(x) = x$: Find a two-sided inverse function

PC 40 Another decomposition where the composition results in the original/given function:

PC 41 Two Geometric Transformations (sequence view): First one transformation and **then** the second. (elements with "this THEN that" would be coded here)

PC 42 Write Two Geometric Transformations with a single rule: (Note: Similar to PC 15)

PC 43 Given a single transformation, produce 2 transformations that result in the original when composed: (decompose idea)

PC 51-# Number of elements to be composed: i.e., $(f \circ g \circ h)(x)$ Note: # indicates how many objects

PC 60 Draw a function machine:
PC 62 Write composition into function Notation: See SP p. 279
PC 63 Find the inverse of a composition:
PC 65 Using the cancellation property Log/Exp: Ex. $\log 10^3 = 3$ or $\ln e^x = x$
PC 66 Using the cancellation property Trig: Ex. $\sin(\sin^{-1}x) = x$
PC 67 Doing the CC 67 principle: Compose a function with its inverse to get the identity
PC 68 Verify / Show the inverse function: Verify that a function g is the inverse function f^{-1} by verifying that $f \circ g = x$ and $g \circ f = x$

PC 71 Write a composite trig function algebraically without involving trig functions:
PC 72 Write an algebraic expression as a composite trig (or inverse trig) function:
PC 73 Lumping with graphing:
PC 74 u-subsitution: Let variable = an expression in the equation or expression and then substitute that variable into the equation to assist in solving it.
PC 75 Do monic polynomial in something nontrivial:
PC 76 Plug back in for u:

PC 77 Both $f(g)$ and $g(f)$ are found in subsequent problems/exercises/examples/sentences/statements: Ex. Problem 3 finds $f(g)$ and problem 4 finds $g(f)$ and nothing is noted about it commutativity or that they are finding both.
PC 78 Using composite structure to solve problems:
PC 79 Work from the inside out:

PC 80 Numerical Decomposition:

PC 81 Repeated substitution: Ex: $y(x) = 5x$, $z(y) = 2y + 1$, find $z(x)$ Ex: PA2, p. 35 #6
PC 82 Graph Transformation that uses composition similar to PC 14: (GP p. 54 #64-71)
PC 83 Define f and g so that $f(g$(specified value) = another specified value
PC 84 When a PC 27 or PC 28 code is done at the same time was a numerical value is evaluated for x (see GA2 p. 412 "Solve" section of Ex 4)
PC 85 Evaluating pretend students' work:
PC 86 Find values such that $f(g(b)) = g(f(b))$:
PC 87 Find a value such that $f(g(a))$ = specified value:
PC 88 Differentiate a composite function:
PC 89 Compose with the Identity Function:

PC 91 Contextual Word Problem, Solving Real-World Problems/Functions
PC 92 What does the composite function represent (in the context of the problem)
PC 93 Does ($f(g(x))$ or $g(f(x))$ represent the context of the problem?
PC 94 Use a graphing calculator to graph the composite:
PC 95 Use a graphing calculator to help determine the domain:

PC 96 Use Sketchpad to graph a composite:
PC 97 What are the units of the composite function?:

Other Composition Related Codes

Iteration

OC 01 Iteration (the inside and outside function are the same): Example: Find $f(f(1)$
OC 02 Find a function that is its own inverse: $f(f(x)) = x$
OC 03 Find a function that when composed with itself 3 times (or 3 iterations) the result is x:
OC 04 More than 3 iterations:
OC 05 Definition of Iteration: (CC)
OC 06 Repeatedly apply the function:
OC 07 Iteration convergence or end behavior: (CC)
OC 08 Find if the function converges or find the end behavior:
OC 09 Other PC convergence or end behavior: (PC)
OC 10 Other iteration stuff: (PC)
OC 11 Other iteration stuff: (CC)
OC 12 Notation:

Function Operation

OC 21 Find the domain of functions added, subtracted, multiplied, or divided:
OC 22 Find the range of functions added, subtracted, multiplied, or divided:
OC 23 Function operation (addition):
OC 24 Function operation (subtraction):
OC 25 Function operation (multiplication):
OC 26 Function operation (division):

Difference Quotient

OC 31 Evaluate the difference quotient: Not given VC code.
OC 32 Statement of the difference quotient: Not given VC code. (if shown, even in an example)
OC 33 limit of the difference quotient: Not given VC code.
OC 34 The difference quotient must be used to solve a problem:
OC 35 The limit of the difference quotient must be used to solve a problem: (this is when it is not shown)

Take & Raise

OC 41 Take the log (or ln) of both sides:
OC 42 Raise both sides to the *e* or 10:
OC 43 Square, cube, 4^{th} power, etc. both sides:
OC 44 Take the square root of both sides:
OC 45 Take a trig function to both sides: Ex. Take the sin of both sides
OC 46 Take or Raise both sides to something not listed in OC 41-45: such as the opposite, the reciprocal, etc.
OC 47 Take the absolute value:
OC 48 Take inverse:

OC 49 Take *f* (or some other function):
OC 50 Take limit:

OC 51 "untaking" log:
OC 52 "take log" happens in worked out steps, but not explicitly stated in the text:
OC 53 set exponents equal in an exponential equation:
OC 54 Square root property: (similar to OC 53)
OC 55 "Take *e*" or "Raise to the *e*": (similar to OC 51/52/53)

OC 63 n^{th} root n>2:
OC 64 raise # to each side:
OC 65 raise to *x* (exponent) the each side:

OC 61 Worked out steps shows log applied to both sides of the equation:
OC 62 Worked out steps shows square root applied to both sides of the equation:
OC 66 Worked out steps shows trig (or trig inverses) applied to both sides of the equation:

Affine Transformations
OC 71 PC
OC 72 CC
OC 73 Find k such that $T(a) \circ T(b) = T(k)$
OC 74 Transforming Equations:
OC 75 Write a single Affine Transformation:

Graph Transformations (graphing)
OC 81 PC
OC 82 CC

Conventional Composition Codes
VC 01 Spoken mathematics (of): $(f \circ g)(x)$ is read as "*f* of *g*"
VC 02 Spoken mathematics (circle): $(f \circ g)(x)$ is read as "*f* circle *g*"
VC 03 Spoken mathematics ("composed with" or "composition"): $(f \circ g)(x)$ is read as "*f* composed with *g*"
VC 04 Spoken mathematics (other): $(f \circ g)(x)$ is read as _____________ (fog)
VC 05 In $f(g(x))$, *f* is called the "outside" function:
VC 06 In $f(g(x))$, *g* is called the "inside" function (or innermost):
VC 47 a function is called the "middle" function:
VC 48 The "variable" is a function:
VC 49 a function "of" a function:
VC 50 In $f(g(x))$, *g* is called the "first" function:
VC 51 In $f(g(x))$, *f* is called the "second" function:
VC 52 In $f(g(x))$, *g* is called the "input":
VC 53 "output":

VC 07 Parenthetic notation with f and g defined external of the composition: composition is denoted as $f(g(x))$, where $f(x) =$ …. and $g(x) =$ …. There can be more than one VC 07 code if more than one expression of parentheses is printed in the problem/statement. (Note: one expression with 3 sets of parentheses is only one VC 07 code.)

VC 08 Parenthetic notation using function names (trig, log, *e*) and not *f* and *g*:
sin(arcsin *x* + arccos *x*)
VC 18 (old VC 08-e): The exponential function is used, but no parentheses are used.
VC 19 The log function is used, but no parentheses are used.
VC 20 The exponential function with an real number base, but no parentheses are used.
VC 20→(not used every time—this should be deleted)

VC 09 Parenthetic notation *f* (outside) is defined external and *g* (inside) is defined internal of the composition: Example $f(x + h)$, $f(4x)$, etc. There can be more than one VC 09 code if more than one expression of parentheses is printed in the problem/statement. (Note: one expression with 3 sets of parentheses is only one VC 09 code.)

VC 10 Parenthetic notation *f* (outside) is defined internal and *g* (inside) is defined external of the composition: Example $f(t) = 2t^3$, Find $3[f(t)] + 2$

VC 11 Circle notation: composition is denoted as $(f \circ g)(x)$ There can be more than one VC 11 code if more than one circle is printed in the problem/statement. (Note: one expression with 3 circles is only one VC 11 code.)
VC 11-multi: If an expression has more than one circle in it.

VC 12 Definition / Notation of Composition: $(f \circ g)(x) = f(g(x))$

VC 14 Neither VC 07 or VC 11, just told to evaluate the variable at the given value:

VC 15 Parentheses related to the order of operations: (working from the inside of parentheses out)

VC 16 sequence or chain of variables: $y = f(u) =$ expression(u) and $u = g(x) =$ expression(x) (see CC p. 33)

VC 17 Double prime notation used to indicate the second image of a transformation (single prime for first image):

VC 18 (old VC 08-e): The exponential function is used, but no parentheses are used.
VC 19 The log function is used, but no parentheses are used.
VC 20 The exponential function with an real number base, but no parentheses are used.

VC 21 Composition is described as “substitution”:
VC 22 Composition is described as “replacing”:
VC 23 Composition is described as “plugging in”:
VC 24 Composition is described as “combining”:
VC 25 Composition is described as “linking function machines”:

VC 31 Caution/Warning about not confusing composition with multiplication such as: Do not mix up $f(g(x))$ with *f* times *g*

VC 32 f(x) is read as “*f* of *x*”: similarly, $f(5)$ is read “*f* of 5”

VC 33 The word “of” means multiply with a fraction or decimal:

VC 34 Other notation comment:

VC 41 The word ‘combine’ is used:
VC 46 the “next” function:
VC 47 a function is called the “middle” function:
VC 48 The “variable” is a function:
VC 49 a function “of” a function:
VC 50 In $f(g(x))$, g is called the “first” function:
VC 51 In $f(g(x))$, f is called the “second” function:
VC 52 In $f(g(x))$, g is called the “input”:
VC 53 “output”:

VC 57 Parentheses, f(g(x)), are used: (such as in an explanation) and there is not an explicit formula attached to it.

VC 74 “lumping” technique:

VC 97 The word *decompose* is in the element
VC 98 The word *composition* is in the element:
VC 99 The word *composition* is in the element, but it doesn’t fit in another code:

Representation Composition Codes
RC ##γ (where γ indicates the part of the composition - outside (o), middle (m), or inside (i))
- RC ##o (outside):
- RC ##m (inside):
- RC ##n (middle):
- RC ##i (second middle)
- RC ##a (answer or result):
- RC ##+ (anything over 4 things composed):
- RC ##d (the representation of the thing to be decomposed):

RC 01 Algebraic Representation: This will also include instances similar to $\sin^{-1}(\sin\frac{\pi}{3})$.

RC 02 Graphical Representation: Coordinate graph
RC 03 Tabular Representation:
RC 04 Mapping Representation:
RC 05 Ordered-Pairs Representation:
RC 08 Function Machine:
RC 09 Verbal names: such as REC for $\frac{1}{x}$ and ABS for |x|, or just the words square root written out.
RC 10 Geometry: Picture or figure not on a coordinate graph or anything
RC 11 Numerical: $h(4) = 2$ (no other info given)

Function Type Composition Codes

FC ##γ (where γ indicates the part of the composition - outside (o), middle (m), or inside (i))

This is for FC 01, FC 04-FC 20 (basically all <u>except</u> FC 02 and FC 03)

<u>FC ##o Polynomial (outside)</u>:

<u>FC ##m Polynomial (middle)</u>:

<u>FC ##n Polynomial (second middle, if any)</u>:

<u>FC ##i Polynomial (inside)</u>:

<u>FC ##d Polynomial (decompose)</u>: If the problem asks students to decompose a function, then the type of function that is given to the student is given the d gamma code.

<u>FC ##do Polynomial (decompose)</u>: The given outside function (problem asks the student to provide the inside function)

<u>FC ##di Polynomial (decompose)</u>: The given inside function (problem asks the student to provide the outside function)

<u>FC 02γ-# The number of terms in the polynomial</u>:

γ indicates the outside (o), middle (m), or inside (i) part of the composition

means the number of terms in the polynomial

Ex: $(ax^2 + bx + c)^3$ is coded as FC 02o-1 and FC 02i-3 (the outer function has on term (x^3) and the inner function has three terms (ax^2, bx, and c)

FC 03: *Order of the polynomial*

<u>FC 03γ-0 Constant term</u>: $x = a$

<u>FC 03γ-1 Linear polynomial: $ax + b$</u>:

<u>FC 03γ-2 Quadratic polynomial: $ax^2 + bx + c$</u>:

<u>FC 03γ-3 Cubic polynomial: $ax^3 + bx^2 + cx + d$</u>:

<u>FC 03γ-4 Polynomial of order 4</u>:

<u>FC 03γ-5 Polynomial of order 5</u>:

….

<u>FC 01 Polynomial</u>:

<u>FC 04 Rational Function/Expression</u>: Note: If the function is written as $\frac{x+5}{7}$ with the denominator being a constant and not a polynomial with degree>0 is considered a polynomial (or FC 01). The example would then be coded in FC 02 and FC 03 as $\frac{1}{7}x + \frac{5}{7}$.

<u>FC 05 Defined as ordered pairs or elements (could be mapping defined for each element)</u>:

<u>FC 06 Exponential</u>:

<u>FC 07 Logarithmic</u>:

<u>FC 08 Piecewise-defined function</u>:

<u>FC 09 Square root function</u>:

<u>FC 10/## Absolute Value /## is the type of function inside the absolute value</u>:

<u>FC 11 Greatest Integer Function</u>:

<u>FC 12 Cubic Root</u>:

<u>FC 13 Identity Function $f(x) = x$</u>:

<u>FC 14 Trigonometric</u>:

FC 15 Inverse Trig Function (trig^{-1} notation):
FC 16 Inverse Trig Function (arc notation) :
FC 17 Non-simple or factored polynomial: Example $2(x - 5)^2$

FC 18 Reflection (Geometric situation)
FC 18a write a double (composition) reflection composition as a single transformation
FC 19 Rotation (Geometric situation)
FC 19a write a double (composition) rotation as a single transformation
FC 20 Translation (Geometric situation)
FC 20a write a double (composition) translation as a single transformation

FC 21 Glide Reflection
FC 22 Dilation
FC 23 n'th root function (greater than cube root):
FC 24 Affine Transformation:
FC 25 Affine Transformation – Translation:
FC 26 Affine Transformation – Dilation:

Location Composition Codes

LC 01 Preparation Material/Diagnoses Problems: Before a lesson to review prerequisite skills (Examples here are still LC 06)
LC 02 Exposition: The location of a sentence is in the explanation
LC 03 Pop up Box: The location of a sentence is in a pop up box or material in the margin(s) of the Student Manual.
LC 04 Key Concept / Blue Box: The location of a sentence is in an "important" box.
LC 05 Proof Box: The location of a sentence is located inside a proof in the exposition (the proof may or may not be enclosed in a box).
LC 06 Example: The location of a sentence is in an example (prep/diagnosis example goes here)
LC 07 Example Exercise or Mid-lesson Exercise: The problem is immediately after an example in the text or part of the lesson for students to work through.
LC 08 Exercise: The location of a sentence is in an exercise
LC 09 Review Problems / Maintaining Skills/Skills Refresher: Problems in another section to review previously learned material (e.g., Spiral Review, Standardized Test Practice, Skills Review)
LC 10 Chapter Summary/Review or Reflection (text, not problems):
LC 11 Chapter Review Problems/Test or Mid-Chapter Test (end of investigation test/reflection problem):
LC 12 Chapter Review Problems Example:
LC 13 Cumulative Review Problems:
LC 14 Activity or Project: The location of the content is in an activity section
LC 15 Exercises that are indicated (in Stewart) to be Example Exercises: Indicated by a pencil in the Exercises list and noted after every example
LC 16 Lesson/unit goal statement:
LC 17 Caption on a picture, image, or figure:
LC 18 In the figure, graph, itself: A curve (or function) on a graph with the equation labeled as in WP p. 77, Fig 2.12

LC 21 TE Exposition: The location of a sentence is in the Teacher Guide.
LC 22 TE box feature: The location of a sentence is in a feature (box) in the TE. (e.g., Differentiated Instruction)
LC 23 TE Example: The problem is an additional example located in the TE.
LC 24 Question and Answer are both in the TE: (see p. 271 Glencoe Geometry)
LC 25 TE Exposition on exercises, problems, examples:
LC 26 TE lesson/unit goal statement:

LC 37 (LC 28) TE Answer to an LC 07:
LC 38 (LC 27) TE Answer to Exercise (LC 08):
LC 39 (LC 30) TE Answer to an LC 09:

LC 31 (LC 31) TE Answer to an LC 11:
LC 32 TE Answer to an LC 13:
LC 33 (LC 29) TE Answer to an LC 23:
LC 34 TE Answer to LC 14:

Substitution / Replace / Plug in Codes - The word(s) substitution, replace, or plug in must be used.

Equality: Equal values are "substituted" in for values such as $\cos(\frac{\pi}{3}) = \frac{1}{2}$. It also indicates two expressions are equal such as $x = \log_a b$ or $h = c\sin A$.

Ex: A quantity may be substituted for its equal in any expression. (GA1 p. 16)
If $a = b$, then a may be replaced by b in any expression. (GA1 p. 16)
$(5 \cdot 2) = 10$ by substitution
Given: m<JKL = 8*x* + 13   Proof: m<JKL = 8*x* + 13 by substitution (GG, p. 40)

Check: Substitution is used to check if the answer is correct.

Evaluate: An expression is evaluated by substituting a value for the variable.
Ex: The value $f(-6)$ is found by substituting -6 for each x in the equation. (GA2, p. 64)

Values: A numerical value is substituted for a variable. *Note: Evaluating at an expression goes under SV->Expres*
Ex: Substitute $x = -5$ into (2) and solve for A. (PP, p. 268)
Substituting $t = 20$, Q = 88.2 and t = 23, Q = 91.4 gives two equations for Q(0) and a: (WC, p. 12)

Method: Substitution is part of the name of the method or algorithm. These include: (1) solving a system of equations via substitution, (2) synthetic substitution, and/or (3) direct substitution. (4) trigonometric substitution, (5) back-substitute, and (6) replacing-the-axes. *Note*: *Even if the method is to substitute values (i.e., back-substitution), I put it here.*
Ex: Similarly, direct substitution provides the correct answer in part (b). (CP, p. 851)

Solve systems of linear equations with two variables using substitution and elimination (PA1, p. 403)

<u>Expressions</u>: One of the following situations.

SV->Expres: Replacing a single variable with an expression, even when the expression is another single variable.

Ex: Substitution involves substituting an expression from one equation for a variable in the other. (GA1, p. 338)

Replacing x by (x - h) moves a graph to the right by h (to the left if h is negative) (WC, p. 18)

Expres->SV: Replacing an expression with a single variable (such as u-substitution)

Ex: When you find two values for H, replace H by sin x and then solve for x. (PP, p. 20)

Since $x = \cos t$ and $y = \sin t$, we can substitute x and y into this equation: ... giving .... (WP, p. 569)

FN->Expres: Function notation is replaced with an expression

Ex: Replace $g(x)$ with $x^2 - 9$ (GP, p. 59)

FN->SV: Function notation is replaced with a single variable

Ex: Replace $f(x)$ with y (GP, p. 67)

SV->FN: A single variable is replaced with function notation

Ex: Replace y with $f^{-1}(x)$. (GP, p. 67)

<u>None</u>: Problems associated with instructions that refer to substitution in some way, but the word substitution is not used in the problem statement itself. Ex: Problem #16 (PA2, p. 314)

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