FEEDBACK AMPLIFIER NETWORK ANALYSIS

Thesis for the Degree of M. S.

MICHIGAN STATE COLLEGE

John J. La Rue

1954

This is to certify that the

thesis entitled

"Feedback Amplifiers Network Analysis"

presented by

John J. LaRue

has been accepted towards fulfillment
of the requirements for

M.S. degree in E.E.

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Date May 17, 1954

FEEDBACK AMPLIFIER NETWORK ANALYSIS

By

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A THESIS

Submitted to the School of Graduate Studies of Michigan
State College of Agriculture and Applied Science
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

Department of Electrical Engineering

1954

ACKNOWLEDGMENT

The author wishes to express his thanks to Doctor J. A. Strelzoff for his assistance in the development of this thesis.

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INTRODUCTION

The following paper is based on a portion of the book Network Analysis and Feedback Amplifier Design, by H. W. Bode.

Mr. Bode's book may be roughly divided into three sections. The first section, on which this work is based, consists of basic circuit theory and defining parameters of feedback amplifiers. The second portion of Bode's book discusses the basic properties of the parameters defined in section one and their use in the design of feedback amplifiers. The third portion of Bode's book deals with specific design problems.

This paper is intended to enlarge and illustrate the theorems of the first section of Bode's book by presenting more detailed derivations and specific examples of some of the theorems and definitions given in the book.

CHAPTER I

1.1 Introduction

The networks considered in this chapter will consist of linear passive circuit elements and vacuum tubes. The discussion will be confined to steady state analysis for simplicity, however, the extension to transient analysis by the Laplace transformation is immediate and direct.

It is assumed that the reader is familiar with basic circuit theory, network theorems, and equivalent circuits for vacuum tubes. The following discussion develops some of the fundamental concepts of circuit theory in determinant form for use in later chapters of the paper.

1.2 Definitions of Terms

Electric networks are composed of active and passive elements. Active elements are energy sources such as voltage or current generators. Passive elements are elements where energy is stored or dissipated, such as inductors, capacitors and resistors.

The terminals of any element are nodes, i.e. a single element has two nodes. When two or more terminals are connected together they form a single node.

A number of network elements in series form a branch, while any continuous closed path forms a loop.

1.3 Application of Ohm's Law and Kirchoff's Laws To a Simple Circuit

Kirchoff's laws may be stated as follows:

1. Voltage law -- The summation of the instantaneous voltage drops around any closed path is zero.

2. Current law -- The summation of the instantaneous currents flowing away from any node (flowing to any node) is zero.

For steady state analysis Kirchoff's laws may be restated, omitting the word "instantaneous."

Ohm's law for steady state:

The voltage drop across an impedance is equal to the product of the current flowing through the impedance and the impedance.

The above three laws in mathematical form are:

$$\Sigma E = 0 \qquad (1\text{-}1)$$

$$\Sigma I = 0 \qquad (1\text{-}2)$$

$$E = IZ \qquad (1\text{-}3)$$

Ohm's law deals with voltage drops or potential differences between the terminals of any element rather than the individual potentials at the terminals. Hence, in any network a reference potential must be assumed at some point in the circuit. A convenient value for the reference potential is zero.

In Figure 1.1 the resistors in series represent a voltage dividing network. E_1, E_2, E_3, E_4 are the potentials at the terminals indicated. Considering the resistor R_1 for example, by Ohm's law the voltage across R_1 is $E_3 - E_2 = IR_1$. It is evident either E_3 or E_2 may be assumed zero or the reference voltage.

If $E_2 = 0$

$E_3 = IR_1$

If $E_3 = 0$ $\quad -E_2 = IR_1$

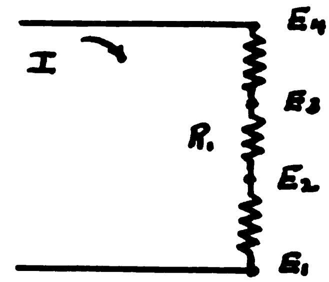

Fig. 1.1

The corresponding circuits are

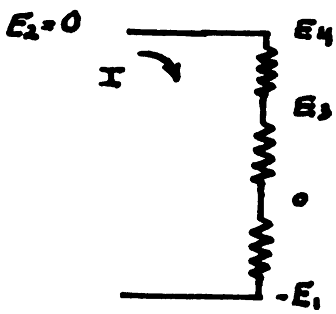

Fig. 1.2

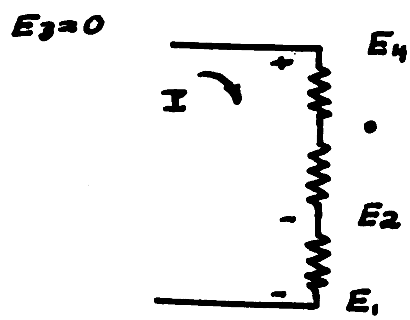

Fig. 1.3

The polarity signs refer to the given reference voltage.

The current law will provide one more equation than is required for a solution of a problem. In any network containing n nodes, only n -1 independent current equations may be written if continuity of current is to hold as stated by the current law.

The discussion of the number of independent loop equations is somewhat more involved but may be simplified by writing the branch equations. Thus, one voltage equation may be written for each branch of the network. The voltage law may then be written as the sum of the iz drops through any given branch, plus the potential difference between the terminals of the branch which must be zero.

The following example will illustrate the preceding theory.

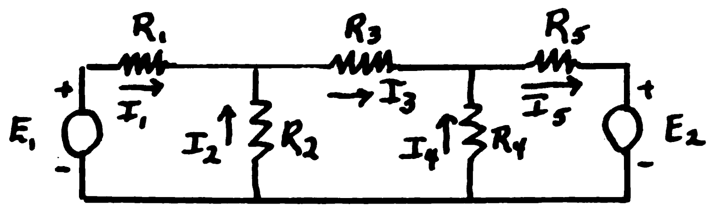

Fig. 1.4

E_1 and E_2 represent constant voltage generators. I_1, I_2, I_3, I_4 and I_5 represent branch currents. E_a, E_b, E_c are the potentials at the nodes, indicated with respect to an arbitrary reference potential.

Choosing $E_a = 0$ as the reference for this network, the voltage equations are

$$E_1 - I_1R_1 - E_b = 0 \quad (1\text{-}4)$$

$$I_2R_2 + E_b = 0 \quad (1\text{-}5)$$

$$I_3R_3 - E_b + E_c = 0 \quad (1\text{-}6)$$

$$I_4R_4 + E_c = 0 \quad (1\text{-}7)$$

$$E_c - I_5R_5 - E_2 = 0 \quad (1\text{-}8)$$

The current equations are

$$I_1 + I_2 - I_3 = 0 \quad (1\text{-}9)$$

$$I_3 + I_4 - I_5 = 0 \quad (1\text{-}10)$$

We thus obtain seven equations in seven unknowns.

1.4 Mesh Equations

In the preceding equations it should be noted that the voltages E_b and E_c can be eliminated from the voltage equations by use of equations (1-5) and (1-7).

There results three voltage equations:

$$E_1 - I_1R_1 + I_2R_2 = 0 \qquad (1\text{-}4a)$$

$$I_3R_3 + I_2R_2 - I_4R_4 = 0 \qquad (1\text{-}6a)$$

$$+ E_2 + I_5R_5 + I_4R_4 = 0 \qquad (1\text{-}8a)$$

These three equations are loop equations as shown in Figure 1.5.

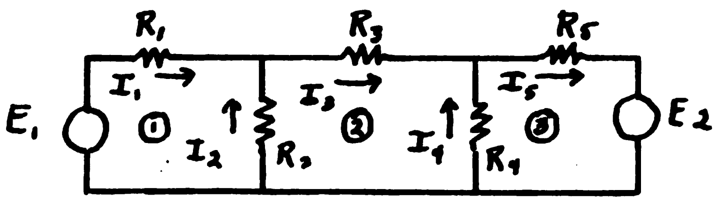

Fig. 1.5

Two currents may now be eliminated from equations (1-4a), (1-6a), and (1-8a) by the use of equations (1-9) and (1-10). The currents eliminated must be the currents flowing in the mutual impedance branches, i.e. I_2 and I_4. From (1-9) and (1-10)

$$I_2 = I_3 - I_1$$

$$I_4 = -I_3 + I_5$$

Substituting these values of I_2 and I_4 in (1-4a), (1-6a) and (1-8a), we obtain

$$E_1 = I_1 (R_1 + R_2) - I_3R_2 \qquad (1\text{-}4b)$$

$$0 = -I_1R_2 + I_3 (R_3 + R_2 + R_4) - I_5R_5 \qquad (1\text{-}6b)$$

$$-E_2 = -I_3R_5 + I_5 (R_5 + R_4) \qquad (1\text{-}8b)$$

Equations (1-4b), (1-6b) and (1-8b) are the familiar mesh equations for the network shown. They may be rewritten as

$$E_1 = I_1Z_{11} - I_3Z_{13} \qquad (1\text{-}4b)$$

$$0 = -I_1Z_{31} + I_3Z_{33} - I_5Z_{35} \qquad (1\text{-}6b)$$

$$-E_2 = -I_3Z_{53} + I_5Z_{55} \qquad (1\text{-}8b)$$

Where Z_{ii} is the sum of all passive elements in the i th mesh, and Z_{ij} represent the sum of all passive elements common to the i th and j th meshes.

For a network containing n meshes, the equations would take the form

$$E_1 = \sum_{j=1}^{n} I_j Z_{1j}$$

$$E_2 = \sum_{j=1}^{n} I_j Z_{2j}$$

$$\vdots$$

$$E_n = \sum_{j=1}^{n} I_j Z_{nj}$$

As a result of the above example we may state the following theorem:

Theorem:1[1] : In any conductively united network, the number of independent closed meshes is one less than the difference between the number of branches and the number of nodes.

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[1] Hendrik W. Bode, Network Analysis and Feedback Amplifier Design, D. Van Nostrand Company, Inc., New York. Seventh Printing, Sept. 1951, p. 3.

1.5 Node Equations

The node equations or current equations can be developed from equations (1-4) (1-10) by eliminating the voltage equations. This is done by solving equations (1-4) (1-8) for the currents I_1 and I_5 and substituting in equations (1-9) and (1-10).

Performing the indicated operations, we obtain

$$\frac{E1}{R_1} - \frac{E_b}{R_1} - \frac{E_b}{R_2} - \frac{E_b}{R_3} + \frac{E_c}{R_3}$$

$$\frac{E_b}{R_3} - \frac{E_c}{R_3} - \frac{E_c}{R_4} - \frac{E_c}{R_5} + \frac{E_2}{R_5} = 0$$

Since E_1 and E_2 are known voltages, we collect the known terms on one side of the equations

$$\frac{E_1}{R_1} = + E_b \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - \frac{E_c}{R_3}$$

$$\frac{E_2}{R_5} = -\frac{E_b}{R_3} + E_c \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right)$$

Using the relation $Gi = \frac{1}{R_i}$ we obtain

$$E_1G_1 = E_b\ (G_1 + G_2 + G_3) - E_c\ G_3 \qquad (1\text{-}9b)$$

$$E_2G_5 = -E_bG_3 + E_c\ (G_3 + G_4 + G_5) \qquad (1\text{-}10b)$$

Drawing the network for the above equations, it may be noted that the products E_1G_1, and E_2G_5 represent constant current generators.

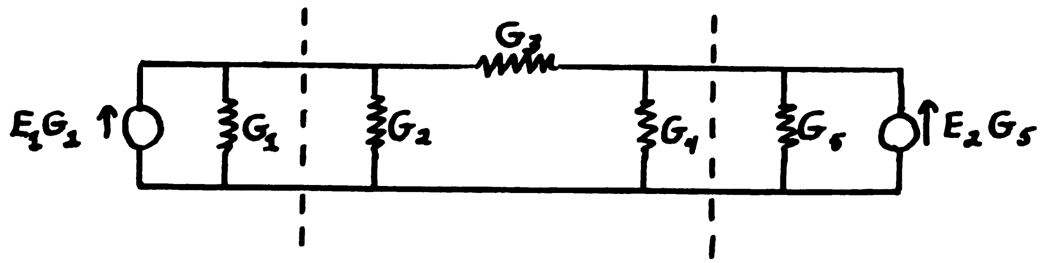

Fig. 1.6

Equations (1-9b) and (1-10b) represent the node equations for the original circuit. Figure 1.6 is an equivalent circuit for the original circuit where the constant voltage generators have been replaced by constant current generators.

The above result leads to the following theorem:

Theorem II[2] : In a conductively coupled network, the number of independent node equations is one less than the number of nodes.

1.6 Constant Current and Constant Voltage Generators

The two energy sources in Figure 1.4, E_1 and E_2, were referred to as constant voltage generators. In a similar fashion the energy sources in Figure 1.6, E_1G_1 and E_2G_2, are referred to as constant current generators. Although such devices are physically impossible, they occur quite often in theoretical analysis. If both types of energy sources occur in the same circuit, considerable confusion can result. Consider the case of the terminals of a constant voltage generator connected to the terminals of a constant current generator, Figure 1.7a. Although such an extreme case will never exist practically, it illustrates a very important point.

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[2] Ibid., p. 12.

The voltage across the terminals of a constant current generator is determined by the voltage generator or, in general, the voltage across the terminals of a constant current generator is determined by the circuit connected to the terminals of the current generator.

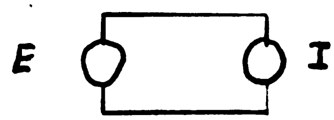

Fig. 1.7a

No voltage equations can be written considering the constant current generator as an independent branch. A similar situation exists for the constant voltage generator. The current through a constant voltage generator is determined by the external circuit, and no node equation can be written for it. In the example shown, no equations can be written. This is obvious since no impedance is shown in the circuit.

The best way to handle such problems is to replace all constant current generators by constant voltage generators for a mesh analysis, and to replace all constant voltage generators by constant current generators for a node analysis.

However, in order to illustrate the point involved, the basic equations for both the node analysis and the mesh analysis are presented here for the circuit of Figure 1.4. The voltage generator E_2 has been replaced by a constant current generator.

The following rules may be stated for interchange of sources:[3]

[3] Murray F. Gardner and John L. Barnes, Transients in Linear Systems, John Wiley and Sons, New York, ~~Eighth Printing~~, p. 43.

1. To replace a constant voltage generator by a constant current, there must be at least one impedance in series with the constant voltage generator.

2. To replace a constant current generator by a constant voltage generator, there must be at least one impedance in parallel with the constant current generator.

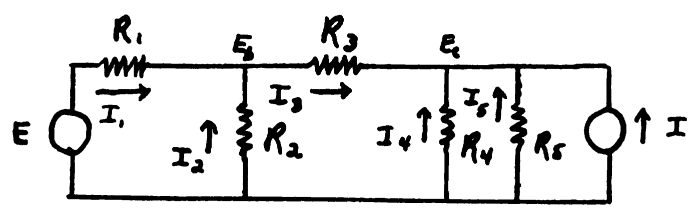

Fig. 1.7

1.7 Driving Point and Transfer Impedence

The driving point impedance of the i^{th} mesh may be defined as the ratio of voltage produced by a constant voltage generator inserted in the i^{th} mesh to the current flowing in the i^{th} mesh with all other energy sources replaced by their internal impedance.

In the network shown in Figure 1.8

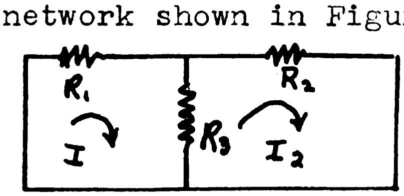

Fig. 1.8

the mesh equations may be written as

$$E_1 = I_1 (R_1 + R_3) - I_2R_3$$

$$0 = -I_1R_3 + I_2 (R_2 + R_3)$$

The impedance determinant is $\Delta = \begin{vmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{vmatrix}$

$$Z_{11} = \frac{E_1}{I_1} = \frac{E_1}{\dfrac{\begin{vmatrix} E_1 & -R_3 \\ 0 & R_2 + R_3 \end{vmatrix}}{\Delta}} = \frac{E_1 \Delta}{E_1 (R_2 + R_3)} = \frac{\Delta}{R_2 + R_3} = \frac{\Delta}{\Delta_{11}}$$

where Δ and Δ_{11} denote the determinant and its cofactor.

The general development of the driving point impedance as the ratio of two determinants follows:

The general mesh equations for an n mesh network are

$$E_1 = I_1 Z_{11} + I_2 Z_{12} + - - + I_n Z_{1n}$$

$$E_i = I_1 Z_{i1} + I_2 Z_{i2} + - - - + I_n Z_{in}$$

$$E_n = I_1 Z_{n1} + I_2 Z_{n2} + - - - + I_n Z_{nn}$$

Assume a generator E_i is placed in the i th mesh and all other voltage sources are replaced by their internal impedance. The current in the i th mesh is

$$I_i = E_i \frac{\Delta_{ii}}{\Delta}$$

and $$Z_{ii} = \frac{E_i}{I_i} = \frac{E_i}{E_i \dfrac{\Delta_{ii}}{\Delta}} = \frac{\Delta}{\Delta_{ii}} \qquad (1\text{-}11)$$

It should be noted that the equation (1-11) does not give the driving point impedance when the generator Ei is connected in series with an impedance which has two or more mesh currents flowing in that branch. Hence, it is not a defining equation for driving point impedance. For example, suppose it is desired to compute the driving point impedance when a generator is placed in series with R_3 of Figure 1.8. The simplest solution for this problem is to rearrange the meshes so only one current flows in R_3. One possible re-arrangement is shown in Figure 1.9.

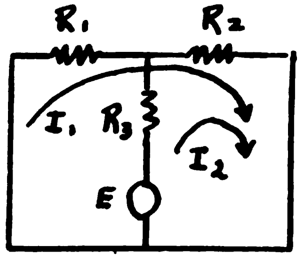

Figure 1.9

We then obtain

$$E = I_1R_2 + I_2\,(R_2 + R_3)$$

$$0 = I_1\,(R_1 + R_2) + I_2R_2$$

$$\Delta = \begin{vmatrix} R_2 & R_2 + R_3 \\ R_1 + R_2 & R_2 \end{vmatrix}$$

$$Z_{22} = \frac{\begin{vmatrix} R_2 & R_2 + R_3 \\ R_1 + R_2 & R_2 \end{vmatrix}}{R_2} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2}$$

Z_{22} could be computed, or the driving point impedance in series with R_3 could be computed from the definition of driving point impedance, using the circuit of Figure 1.8 as

$$Z = \frac{E}{I_1 - I_2}$$

$$I_1 = \frac{E\,(R_2 + R_3)}{\Delta}$$

$$I_2 = \frac{E\,R_3}{\Delta}$$

$$\therefore \quad z = \frac{E}{I_1 - I_2} = \frac{E}{\frac{E\,(R_2 + R_3 - ER_3}{\Delta}} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_2}$$

In general, it will be easier to rearrange the mesh currents in a network than to compute a driving point impedance if there is more than one current flowing through the branch. Since in an n mesh network, if the currents are rearranged properly, it is only necessary to evaluate two determinants, one of order n and one of order $n-1$, while if the currents are not rearranged, it will be necessary to evaluate one determinant of order n and as many of order $n-1$ as there are mesh currents flowing through the branch in question.

The transfer impedance between the i^{th} and j^{th} meshes is defined as the ratio of a voltage E placed in the i^{th} mesh to the current j flowing in the j^{th} mesh, with all other voltage sources replaced by their internal impedance.

Consider the network of Figure 1.8.

The transfer impedance from mesh one to mesh two, z_{12}, is then

$$Z_{12} = \frac{E}{I_2}$$

$$I_2 = \frac{E_1 R_3}{\Delta}$$

$$Z_{12} = \frac{R_1R_2 + R_1R_3 + R_2R_3}{R_3}$$

A general expression for the transfer impedance from the i^{th} to the j^{th} mesh for an n mesh network can be obtained as follows:

The general mesh equations are

$$E_1 = I_1 Z_{11} + I_2 Z_{12} + - - - + I_n Z_{1n}$$

$$E_i = I_1 Z_{i1} + I_2 Z_{i2} + - - - + I_n Z_{in}$$

$$E_n = I_1 Z_{n1} + I_2 Z_{n2} + - - - + I_n Z_{nn}$$

If E_i is placed in the i^{th} mesh and all voltages replaced by their internal impedance

$$I_j = E_i \frac{\Delta_{ij}}{\Delta}$$

and

$$Z_{ij} = \frac{E_i}{I_j} = \frac{\Delta}{\Delta_{ij}} \qquad (1\text{-}12)$$

Here again, as in the driving point impedance, the voltage generator E_i cannot be placed in a branch common to two or more meshes if the determinant expression is to hold.

A simple example will illustrate the above statement. Consider the circuit of Figure 1.10 with currents cho~~o~~sen as shown

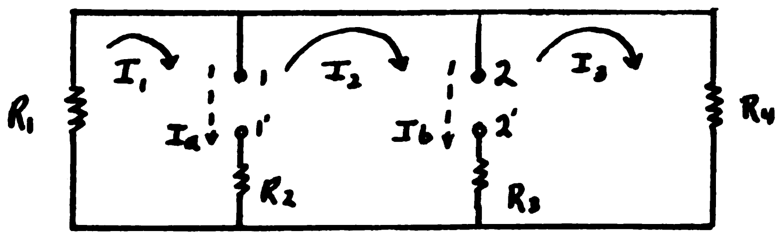

Fig. 1.10

Ia and Ib are branch currents. If we attempt to compute the transfer impedance between branch a and branch b by the determinant equation $Z_{ij} = \frac{\Delta}{\Delta_{ij}}$ it is not clear just how we should apply this equation. Definition Z_{ij} is the ratio of the voltage in mesh i to the current in mesh j. With all other voltage sources replaced by their internal impedance.

But in the circuit shown, branch a is common to meshes 1 and 2 and branch b is common to meshes 2 and 3 .

We note, however, that we can compute the transfer impedance if we define the transfer impedance as the ratio of the voltage in branch a to the current in branch b with all other sources of energy replaced by their internal impedance.

This definition is somewhat more general than that given in terms of mesh voltages and currents, because it is independent of the choice of mesh currents. By this definition the transfer impedance Z_{ab} is

$$Z_{ab} = \frac{E}{I_2 - I_3}$$

where E is a voltage generator of zero internal impedance inserted between the terminals $1\,1'$ and $I_2 - I_3$ is the current flowing in branch b when the terminals $2\,2'$ are shorted. In terms of determinants Z_{ab} becomes

$$Z_{ab} = \frac{E}{\frac{-E\Delta_{12} + E\Delta_{22}}{\Delta} - \frac{-E\Delta_{13} + E\Delta_{23}}{\Delta}} = \frac{\Delta}{\Delta_{22} + \Delta_{13} - \Delta_{12} - \Delta_{23}}$$

In this particular case we must compute one third order determinant and four second order determinants in order to compute the transfer impedance.

If the currents in Figure 1.10 were rearranged as in Figure 1.11, the mesh definition of transfer impedance could be used.

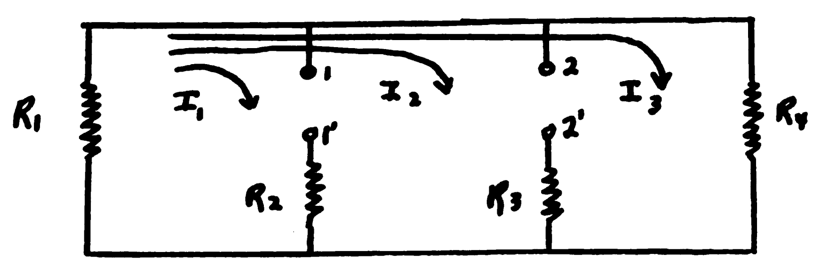

Fig. 1.11

In this case a generator E placed between terminals 11' is an element of the first mesh only and the current flowing between terminals 22' when they are shorted is simply I_2. The desired transfer impedance is then

$$z_{12} = \frac{E}{I_2} = \frac{E}{E\frac{\Delta_{12}}{\Delta}} = \frac{\Delta}{\Delta_{12}}$$

General expression can be written for the transfer impedance between any two branches of an n mesh network regardless of the choice of mesh currents in terms of the determinants of the network and its cofactors. However, such expressions would be of little practical value due to the large number of cofactors which must be computed. It will always be easier to rearrange the mesh currents and then compute the transfer impedance.

If the transfer impedance is desired between a branch common to two or more meshes and another branch common to two or more meshes, then the currents or meshes must be rearranged or a complicated expression for the transfer impedance will result.

1.8 Driving Point and Transfer Admittance

The driving point admittance for the i^{th} node, is defined as, the ratio of the current of a constant current generator connected to the i^{th} node, and to the voltage across the node with all other current sources replaced by their internal admittance.

The general node equations for an n node network are

$$I_1 = E_1Y_{11} + E_2Y_{12} + - - - + E_nY_{1n}$$

$$I_i = E_1Y_{i1} + E_2Y_{i2} + - - - + E_nY_{in}$$

$$I_n = E_1Y_n + E_2Y_{n2} + - - - + E_nY_{nn}$$

If a generator Ii is placed in the i^{th} node then

$$Ei = Ii \frac{\Delta_{ii}}{\Delta}$$

and $$Yii = \frac{Ii}{Ei} = \frac{\Delta}{\Delta_{ii}}$$

It should be noted at this point that the determinant expression for Y_{ii} is the same as the determinant expression for Zii; further, the definition of driving point admittance is the same as that for driving point impedance, except that the terms voltage and current are interchanged.

This is true in general and the symbol Δ will be used for either an impedance determinant or an admittance determinant. The name immittance will be used to refer to either impedance or admittance.

On the basis of the above, the definition of transfer admittance is the ratio of the current at the i^{th} node to the voltage at the j^{th} node. The determinant expression for Yij is

$$Y_{ij} = \frac{\Delta}{\Delta_{ij}}$$

1.9 Variable Impedance

If a variable impedance is placed in the i^{th} mesh (variable admittance in the i^{th} node), this can be greatly simplified as follows:

The determinant Δ would have the form

$$\Delta = \begin{vmatrix} z_{11} & z_{12} & - - - - - - - - - - - - - - - & z_{1n} \\ z_{21} & z_{22} & - - - & z_{2n} \\ z_{i1} & z_{i2} & + - - - + (z_{ii} + z) - - - & z_{in} \\ z_{n1} & z_{n2} & & z_{nn} \end{vmatrix}$$

where z_{ii} represents the sum of all the impedance in the i^{th} mesh except the variable impedance z.

The determinant of z may be expanded in the following form by elements of the i^{th} row.

$$\Delta = z_{i1}\Delta_{i1} + z_{i2}\Delta_{i2} + --- + (z_{ii} + z)\Delta_{ii} + --- + z_{in}\Delta_{in}$$

$$= \sum_{j=1}^{n} z_{ij}\Delta_{ij} + z\Delta_{ii}$$

If we denote the original determinant with the variable element z by Δ' and let Δ represent the determinant with z removed, then we obtain the expression

$$\Delta' = \Delta + z\Delta_{ii}$$

If the letter W is used to represent immittance, then a general expression for node and mesh equation is

$\Delta' = \Delta + W\Delta_{ii}$ where W is the variable immittance.

As in the case of driving point and transfer immittance, the variable immittances must be in the i^{th} mesh (node) alone, and not a mutual element between two or more meshes (nodes) in order for the above equations to hold.

The result of the above discussion may be used to simplify the expressions for driving point and transfer immittances.

Let W_{ij} represent a driving point or transfer immittance and W be a variable immittance in the K^{th} mesh or node. Then W_{ij}[4] is given by

$$W_{ij} = \frac{\Delta'}{\Delta'_{ij}} = \frac{\Delta + W\Delta_{KK}}{\Delta_{ij} + W\Delta_{ijKK}} \qquad (1\text{-}13)$$

The variable element W does not appear in Δ, Δ_{KK}, Δ_{ij} and Δ_{ijKK}. Thus these four quantities may be computed for a given network and W_{ij} may then be computed for any value of the variable immittance W by evaluating the right hand side of equation (1-13).

To illustrate equation (1-13) consider the circuit of Figure 1.12.

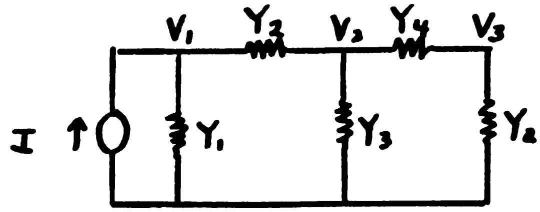

Fig. 1.12

[4] Bode, Op. cit., p. 11, Equations (1-11) and (1-12).

Let Y_L be the variable element. The node equations are

$$I = V_1 (Y_1 + Y_2) - V_2Y_2$$

$$0 = -V_1Y_2 + V_2 (Y_2 + Y_3 + Y_4) - V_3Y_4$$

$$0 = \qquad - V_2 Y_4 \qquad +V_3 (Y_4 + Y_L)$$

The driving point admittance for node 1 is

$$2. \quad = \frac{\Delta'}{\Delta'_{11}} = \frac{\Delta + Y_L \Delta_{33}}{\Delta_{11} + Y_3 \Delta_{11\,33}} \qquad (1\text{-}14)$$

where

$$3. \quad \Delta = \begin{vmatrix} Y_1+Y_2 & -Y_2 & 0 \\ -Y_2 & Y_2+Y_3+Y_4 & -Y_4 \\ 0 & -Y_4 & Y_4 \end{vmatrix} \qquad (1\text{-}15)$$

$$4. \quad \Delta_{33} = \begin{vmatrix} Y_1 + Y_2 & -Y_2 \\ -Y_2 & Y_2+Y_3+Y_4 \end{vmatrix} \qquad (1\text{-}16)$$

$$5. \quad \Delta_{11} = \begin{vmatrix} Y_2+Y_3+Y_4 & -Y_4 \\ -Y_4 & Y_4 \end{vmatrix} \qquad (1\text{-}17)$$

$$6. \quad \Delta_{11\,33} = Y_2 + Y_3 + Y_4 \qquad (1\text{-}18)$$

Note that equations (1-15), (1-16), (1-17) and (1-18) are all constants, that is, they are not functions of the variable admittance Y_L . Therefore they need only be computed once and Y_{11} may be evaluated from equation (1-14) for any value of Y_2. If the variable element W appears as a unilateral coupling element[5] such as a vacuum tube (see equation (1-10) and (1-11) equation (1-13) becomes

$$W_{ij} = \frac{\Delta + W\Delta_{kl}}{\Delta_{ij} + W\Delta_{ijkl}} \qquad (1\text{-}14)$$

5 Ibid., p. 10, Equations (1-13) and (1-11).

1.10 Mesh Equations For a Circuit Containing Vacuum Tubes

The circuit of Figure 1.13 is a portion of an n mesh network. The current flowing in the grid circuit is Ii, and the current flowing in the plate circuit is Ij.

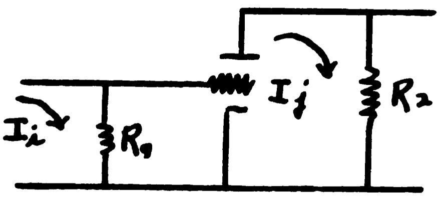

Fig. 1.13

The equivalent circuit for Figure 1.13 is shown in Figure 1.14.

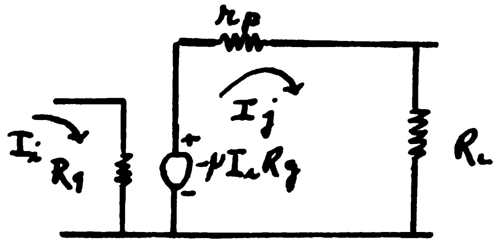

Fig. 1.14

The vacuum tube appears as a voltage generator of voltage $-\mu I_i R_g$ and internal impedance of r_p. The equation for the j^{th} mesh may be written as

1.13 $-\mu I_i R_g = I_1 Z_{j1} + - - - + I_i Z_{ji} + --- I_n Z_{jn}$

Equation (1-10) may be rewritten as

1.14 $0 = I_1 Z_{j1} + - - - + I_i (\mu R_g + Z_{ji}) ---- + I_n Z_{jn}$

by transposing the term $-\mu I_i R_g$ to the right side of the equation. Thus an unsymmetrical determinant is formed $(Z_{ij} \neq Z_{ji})$.

A similar situation will exist for the node equations. Examples of both mesh and node equations for circuit containing vacuum tubes are given in later chapters of this paper.

CHAPTER II

2.1 Elementary Theory of Feedback Circuits

A feedback amplifier consists of a standard amplifier without feedback, or a μ circuit and a network to feed a portion of the output of the μ circuit back to the input of the μ circuit. This is the feedback or β circuit. If the voltage feedback to the μ circuit is in phase with the input of the μ circuit, the feedback is positive or regenerative. This type of feedback is unstable, and the circuit may break into oscillations. The type of feedback to be discussed in this paper is negative, where the voltage feedback to the input of the μ circuit is 180° out of phase with the input voltage to the μ circuit.

The properties of a negative feedback amplifier are easily studied, since the behavior of the system is completely determined by the voltages at the terminals. Consider the block diagram shown in Figure 2.1. The input voltage to the system is denoted by E_{in}, the output by E_{out}, the voltages input to the μ circuit by E_o and the feedback voltage by E_f.

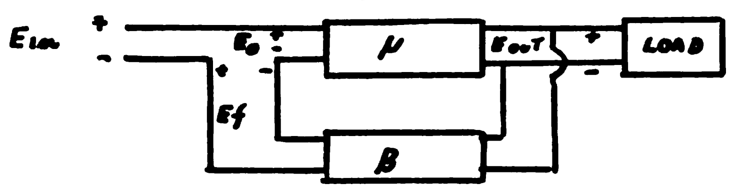

Fig. 2.1

The following equations may be written

$$E_o = E_{in} + E_f \qquad (2\text{-}1)$$

$$E_{out} = \mu \; E_o \qquad (2\text{-}2)$$

$$E_f = \beta E_{out} \qquad (2\text{-}3)$$

Eliminating E_o and E_f from the equations (2-1), (2-2), and (2-3) and solving for the gain of the amplifier, $\frac{E_{out}}{E_{in}}$, we obtain

$$\theta = \frac{E_{out}}{E_{in}} = \frac{\mu}{1-\mu B} \qquad (2\text{-}4)$$

In equation (2-4), θ is the logarithmic gain of the circuit. Since μ represents the gain of the amplifier without feedback, the following theorem[1] may be stated:

Theorem III: Feedback reduces the gain of an amplifier by the factor $1 - \mu\beta$. The quantity $\mu\beta$ is known as the feedback factor and represents the transmission around the loop ($\mu\beta$ loop) from the input of the circuit, through the μ circuit, through the β circuit, and back to the input of the μ circuit.

In ordinary practice, the product $\mu\beta$ is much larger than unity, and equation (2-4) may be rewritten as

$$\theta = \frac{E_R}{E_o} = \frac{\mu B}{\mu B - 1}\left(-\frac{1}{B}\right) \approx -\frac{1}{\beta} \quad \text{for } |\mu\beta| >> 1 \qquad (2\text{-}5)$$

From equation (2-5) we can conclude that if the product $|\mu\beta| >> 1$, the gain of the amplifier varies inversely with β and is independent of μ, or the gain is approximately

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1 Op. Cit., Bode, P. 32.

proportional to the β circuit loss. The error in this conclusion, due to the departure of $\left|\frac{\mu\beta}{1-\mu\beta}\right|$ from unity, will be called the $\mu\beta$[2] effect or the $\mu\beta$ error in subsequent discussion.

In order to show more clearly the independence of the gain of the amplifier from the μ circuit, we differentiate E_R of equation (2-4) with respect to μ keeping β constant and obtain

$$\frac{dE_R}{E_0 d\mu} = -\frac{\mu(-\beta)-(1-\mu\beta)}{(1-\mu\beta)^2} = -\frac{-\mu\beta-1+\mu\beta}{(1-\mu\beta)^2} = \frac{1}{(1-\mu\beta)^2}$$

Dividing by (2-4) and rearranging terms

$$\frac{dE_R}{E_R} = \frac{d\mu}{\mu}\frac{1}{1-\mu\beta} \qquad (2\text{-}6)$$

or $$d\ln E_R = \frac{1}{1-\mu\beta}\, d\ln\mu$$

From equation (2-6) we can state the following theorem[3]:

Theorem IV: The variation in the final gain characteristic in decibels, per decibel change in the gain of the μ circuit, is reduced by feedback in the ratio $1-\mu\beta : 1$.

The basic feedback diagram in Figure 2.1 can be redrawn in a single line diagram as shown in Figure 2.2. It should be noted that all of the theorems regarding feedback developed so far are for the particular case when the output of the μ circuit is fedback to the input of the μ circuit through

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2 Ibid., p. 33.
3 Ibid., p. 33.

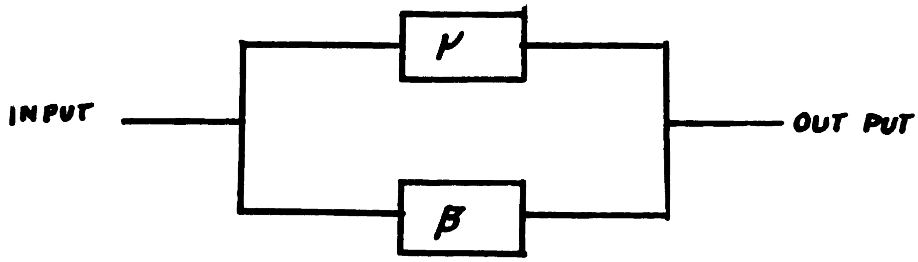

Fig. 2.2

the β circuit, and are not valid under other conditions. For the circuit shown in Figure 2.3, the total gain is given by $e^{\theta} = \frac{\mu'\mu}{1-\mu B}$, and the preceding theorems do not apply.

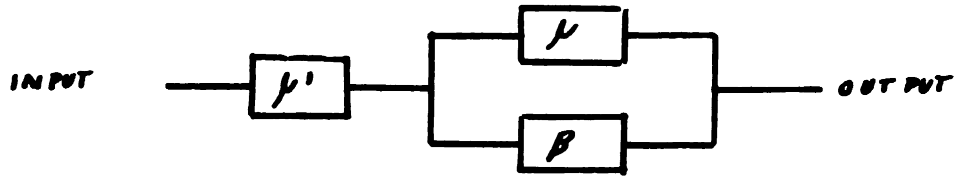

Fig. 2.3

2.2 Return Ratio, Return Difference and Sensitivity[4]

The preceding section presented an elementary theory of feedback amplifiers. From this discussion, there emerged two main ideas: first, the idea of loop transmission or return of voltage; and second, the idea of a feedback amplifier being independent of the μ circuit. In a normal feedback amplifier the μ circuit is composed of ordinary passive elements and vacuum tubes, while the β circuit is composed entirely of passive elements. If the β circuit is composed of passive elements, then the second idea may be restated that in a feedback amplifier there is a reduction in the effects of tube variation. In normal circuits these two

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[4] Ibid., p. 47.

ideas are related by simple mathematical laws so that the term feedback may be applied to both.

In exceptional circuits, the correlation between the two ideas breaks down and the one corresponding to the normal function of feedback is the idea of loop transmission or the product $\mu\beta$. In order to prevent any confusion the negative of the loop transmission will be called the return ratio and denoted by the symbol T .

$$T = -\mu\beta$$

The quantity $1 - \mu\beta$ will be given the name return difference and denoted by the symbol F .

$$F = 1 - \mu\beta = 1 + T$$

The concept of reduction in tube variation will be called sensitivity and denoted by the symbol S .

The quantities T, F and S are analogous to $-\mu\beta$, $1 - \mu\beta$, and $1 - \mu\beta$ respectively. However, properly defined, they are much more general than their analogous quantities and as a result are much more useful in theoretical analysis.

2.3 Return Ratio and Return Difference

Consider the circuit shown in Figure 2.4. For mesh equation the equivalent circuit appears in Figure 2.5.

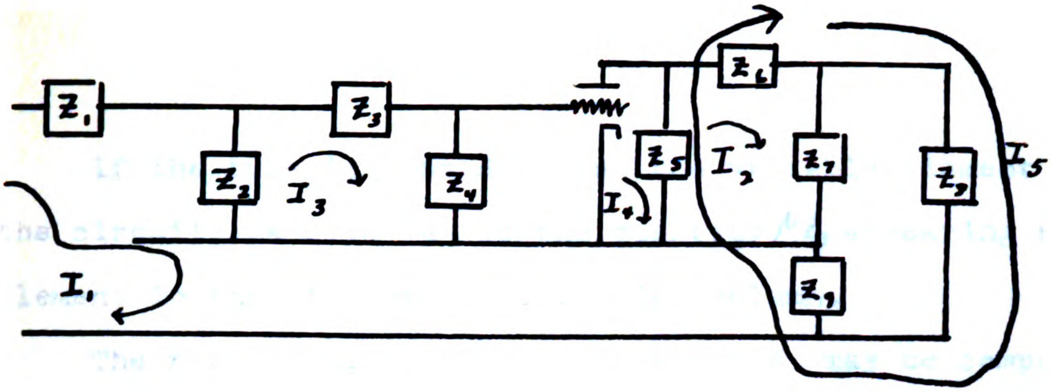

Fig. 2.4

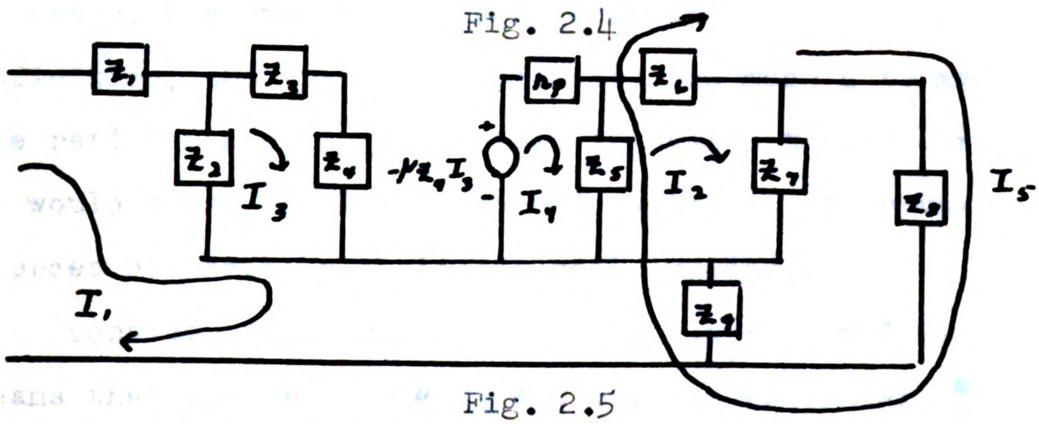

Fig. 2.5

$$
\begin{aligned}
\mu E_1 &= I_1(Z_1+Z_2+Z_9) - I_3 Z_2 - I_5 Z_9 \\
0 &= I_2(Z_5+Z_6+Z_7) - I_4 Z_5 + I_5 Z_5 \\
0 &= -I_1 Z_2 + I_3(Z_2+Z_3+Z_4) \\
-\mu Z_4 I_3 &= -I_2 Z_5 + I_4(r_p+Z_5) - I_5 Z_5 \\
0 &= -I_1 Z_9 + I_2 Z_5 - I_4 Z_5 + I_5(Z_5+Z_6+Z_9+Z_8)
\end{aligned}
\qquad (2\text{-}7)
$$

The mesh equations for the circuit of Figure 2.5 are given in (2-7). The determinant of (2-7) is given in (2-8).

$$
\Delta = \begin{vmatrix}
Z_1+Z_2+Z_9 & 0 & -Z_2 & 0 & -Z_9 \\
0 & Z_5+Z_6+Z_7 & 0 & -Z_5 & Z_5 \\
-Z_2 & 0 & Z_2+Z_3+Z_4 & 0 & 0 \\
0 & -Z_4 & \mu Z_9 & r_p+Z_5 & -Z_5 \\
-Z_9 & Z_5 & 0 & -Z_5 & Z_5+Z_6+Z_7+Z_9
\end{vmatrix}
\qquad (2\text{-}8)
$$

If the tube is considered as the variable element in the circuit, we note W is the quantity μZ_g appearing as the element in the 4th row and the third column.

The return ratio, for the element W may be computed by use of the transfer impedance defined in Chapter I.[5] By definition, the transfer impedance from mesh 4 to mesh 3 is the ratio of a voltage inserted in mesh 4 to the current it would cause to flow in mesh 3, with all other voltage sources replaced by their internal impedance.

Consider the circuit of Figure 2.5 when W = 0. This means that the product $\mu Z_g = 0$ or, in particular, that the μ of the tube is zero. This is only a mathematical concept since the plate resistance and interelectrode impedance remain in the circuit. If we then place a constant voltage generator E_4 in mesh 4 and indicate the determinant with $W = \mu Z_g = 0$ by Δ° then, by definition

$$\frac{E_4}{I_3} = Z_{43} \quad \text{or} \quad \frac{E_4}{Z_{43}} = I_3$$

The voltage at the grid of the tube is obtained by multiplying both sides of the above equation by Z_g; hence,

$$E_g = I_3 Z_g = \frac{E_4}{Z_{43}} Z_g$$

The ratio $\dfrac{E_g}{E_4} = \dfrac{Z_g}{Z_{43}}$. The return ratio may then be obtained by multiplying by the μ of the tube; hence,

---

5 Chapter I, Section 1.7.

$$T = \mu \frac{Z_g}{Z_{43}}$$

The superscript ° on Δ^{o}_{43} is unnecessary since the cofactor Δ_{43} does not contain the element W.

$$T = \frac{W \Delta^{1}_{43}}{\Delta^{o}} \qquad (2\text{-}9)$$

The return difference

$$F = 1 + T = 1 + \frac{W\Delta_{43}}{\Delta^{o}} = \frac{\Delta^{o} + W\Delta_{43}}{\Delta^{o}}$$

By determinant theory of Chapter I,

$$F = \frac{\Delta}{\Delta^{o}} \qquad (2\text{-}10)$$

As a result of (2-10) we may state the following definition:[6]

The return difference, or feedback, for any element in a complete circuit is equal to the ratio of the values assumed by the circuit determinant, when the specified element has its normal value and 0

The equations for the return ratio and return difference may be developed in terms of the node equations. The equivalent circuit of Figure 2.4 is shown in Figure 2.6.

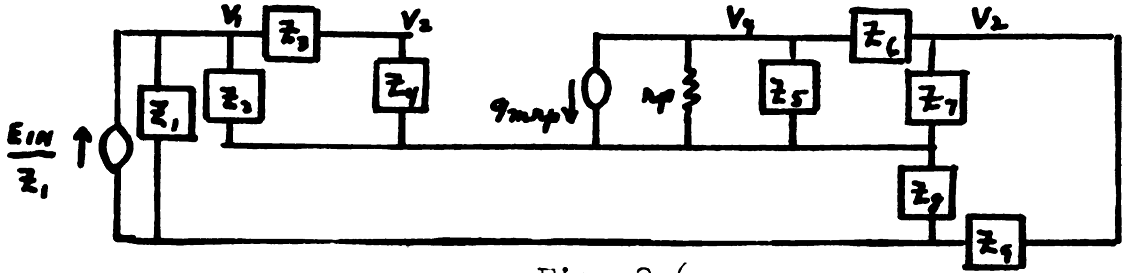

Fig. 2.6

6 Op. cit., Bode, p. 49.

The node equations for Figure 2.6 are given by equations (2-11) and the admittance determinant by (2-12).

$$
\begin{aligned}
\frac{E_{IN}}{Z_1} &= V_1\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right) - \frac{V_3}{Z_3} - \frac{V_5}{Z_1} \\
0 &= V_2\left(\frac{1}{Z_6} + \frac{1}{Z_7} + \frac{1}{Z_8}\right) - \frac{V_4}{Z_6} - \frac{V_5}{Z_8} \\
0 &= -\frac{V_1}{Z_3} + V_3\left(\frac{1}{Z_3} + \frac{1}{Z_4}\right) \\
-g_m V_3 &= -\frac{V_2}{Z_6} + V_4\left(\frac{1}{r_p} + \frac{1}{Z_5} + \frac{1}{Z_6}\right) \\
0 &= -\frac{V_1}{Z_1} - \frac{V_2}{Z_8} + V_5\left(\frac{1}{Z_1} + \frac{1}{Z_9} + \frac{1}{Z_8}\right)
\end{aligned}
\qquad (2\text{-}11)
$$

$$
\Delta = \begin{vmatrix}
\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3} & 0 & -\frac{1}{Z_3} & 0 & -\frac{1}{Z_1} \\
0 & \frac{1}{Z_6}+\frac{1}{Z_7}+\frac{1}{Z_8} & 0 & -\frac{1}{Z_6} & -\frac{1}{Z_8} \\
-\frac{1}{Z_3} & 0 & \frac{1}{Z_3}+\frac{1}{Z_4} & 0 & 0 \\
0 & -\frac{1}{Z_6} & g_m & \frac{1}{r_p}+\frac{1}{Z_5}+\frac{1}{Z_6} & 0 \\
-\frac{1}{Z_1} & -\frac{1}{Z_8} & 0 & 0 & \frac{1}{Z_1}+\frac{1}{Z_8}+\frac{1}{Z_9}
\end{vmatrix}
\qquad (2\text{-}12)
$$

It should be noted that the variable element W appears as the element in the fourth row and third column and $W = g_m$.

The return ratio may be computed in a similar manner as for the mesh equations. The return ratio is a voltage ratio; hence, as in the mesh computation, we introduce a voltage E_4 in place of the voltage generator $\mu Z_g I_3$ in Figure 2.5. The constant voltage generator E_4 is then replaced by a current generator $\frac{E_4}{r_p}$. The final circuit is the same as that of Figure 2.6 except that the current generator $g_m V_3$ is replaced by $\frac{E_4}{r_p}$.

$$\frac{I_4}{V_3} = 1 \frac{E_4}{r_p V_3} = \frac{\Delta^\circ}{\Delta_{43}}$$

Multiplying by μ, we obtain the complete expression as

$$T = \frac{V_3}{E_4} = \frac{\mu \Delta_{43}}{r_p \Delta^\circ} = g_m \frac{\Delta_{43}}{\Delta^\circ} = W \frac{\Delta_{43}}{\Delta^\circ}$$

As before

$$F = 1 + T = 1 + W \frac{\Delta_{43}}{\Delta^\circ} = \frac{\Delta^\circ + W \Delta_{43}}{\Delta^\circ} = \frac{\Delta}{\Delta^\circ}$$

2.4 Return Difference for a General Reference

In Article 2.4 an expression for the return difference for the reference value W = 0 was developed. This result can be generalized for any reference value W = k. The gain of the tube with respect to this reference value k is then W -K; hence, the return ratio for the reference value k is

$$T_k = (W - k) \frac{\Delta_{43}}{\Delta^k}$$

and the return difference F_k is

$$F_k = 1 + T_k = 1 + (W - k)\,\frac{\Delta_{43}}{\Delta^k} = \frac{\Delta^k + W\Delta_{43} - k\Delta_{43}}{\Delta^k}$$

$$F_k = \frac{\Delta^{\circ} + k\Delta_{43} + W\Delta_{43} - k\Delta_{43}}{\Delta^k} = \frac{\Delta^{\circ} + W\Delta_{43}}{\Delta^k} = \frac{\Delta}{\Delta^k} \qquad (2\text{-}13)$$

If the right hand side of equation (2-13) is multiplied by $\frac{\Delta^{\circ}}{\Delta^{\circ}}$ we obtain

$$F_k\,(W) = \frac{\Delta}{\Delta^k}\,\frac{\Delta^{\circ}}{\Delta^{\circ}} = \frac{\Delta}{\Delta^{\circ}} \times \frac{1}{\frac{\Delta^k}{\Delta^{\circ}}} = \frac{F(W)}{F(K)} \qquad (2\text{-}14)$$

Stated in words, this result is the theorem:

Theorem V[7]: The return difference of W for any reference is equal to the ratio of the return differences, with zero reference which would be obtained if W assumed first its normal value, and second, the chosen reference value.

2.5 Return Difference For a Bilateral Element

The equation for the return difference $F = \frac{\Delta}{\Delta^{\circ}}$ was developed for a unilateral element. However, it is obvious that the equation is valid mathematically for a bilateral element. A physical meaning for a bilateral element may be determined by considering the circuit of Figure 2.8. Let $W = R_i$ be the internal resistance of the generator E.

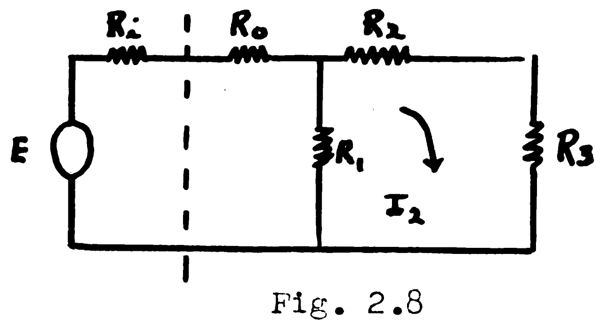

Fig. 2.8

7 Ibid., p. 50.

The circuit equations are

$$E = I_1 \ (W + R_1 + R_o) \ -I_2R_1$$

$$0 = -I_1R_1 + (R_2 + R_3)$$

The determinant of Δ is then

$$= \begin{vmatrix} W + R_1 + R_o & -R_1 \\ -R_1 & R_2 + R_3 \end{vmatrix}$$

The return ratio is

$$T = W \ \frac{\Delta_{11}}{\Delta^{\circ}} = \frac{W}{\frac{\Delta^{\circ}}{\Delta_{11}}}$$

The expression $\frac{\Delta^{\circ}}{\Delta_{11}}$ is the driving point impedance in mesh one if the internal impedance R_i is omitted. Hence, the return ratio T is the ratio of the impedance W to the impedance presented to W by the rest of the circuit.

The return difference

$$F = 1 + W\frac{\Delta_{11}}{\Delta^{\circ}} = \frac{\Delta}{\Delta^{\circ}} = \frac{\frac{\Delta}{\Delta_{11}}}{\frac{\Delta^{\circ}}{\Delta_{11}}}$$

is the ratio of the driving point impedance of the complete circuit to the driving point impedance when $W = 0$. This represents a measure of the effectiveness of a generator with internal impedance in driving its external circuit. If E in Figure 2.8 is a unit voltage, then a voltage drop across R_i can be thought as the return voltage, and the difference between E and the return voltage is the return difference.

An exactly analogous situation exists for the node analysis.

2.6 Definition of Sensitivity[8]

The sensitivity, S, for a given element W is defined as

$$S = \frac{1}{\frac{\partial \theta}{\partial \ln W}} \qquad \text{where } e^{\theta} = \frac{E_{out}}{E_{in}} \qquad (2\text{-}15)$$

If θ is a function of W alone, then the partial derivatives may be replaced by ordinary derivatives.

$$S = \frac{1}{\frac{d\theta}{d\ln W}} = \frac{d\ln W}{d\theta} = \frac{1}{W}\frac{dW}{d\theta}$$

Letting E_{in} be a unit voltage

$$e^{\theta} = E_{out}$$

$$e^{\theta}\, d\theta = dE_{out}$$

Substituting these equations in the equation for S, we obtain

$$S = \frac{dW}{W} \times \frac{E_{out}}{dE_{out}} \qquad \text{or} \qquad \frac{dE_{out}}{E_{out}} = \frac{1}{S}\frac{dW}{W} \qquad (2\text{-}16)$$

Thus S is analogous to the quantity $1 - \mu B$, in equation (2-6).

The gain of an amplifier may be obtained by dividing the output immittance by the transfer immittance from the input to the output; thus, $e^{\theta} = \frac{\Delta_{12}}{\Delta} W_R$ where W_R represents the output immittance.

If we consider the plate and grid as part of the fourth and third meshes (nodes) respectively, then by determinant theory

$$e^{\theta} = \frac{\Delta_{12}}{\Delta} W_R = \frac{\Delta^{\circ}_{12} + W\Delta_{1243}}{\Delta^{\circ} + W\Delta_{43}} W_R \qquad (2\text{-}17)$$

The transmission when $W = 0$ known as the direct transmission, can be obtained from (2-17) as

8 *Ibid.*, p. 52.

$$e^{\theta_0} = \frac{\Delta^\circ_{12}}{\Delta^\circ} \qquad (2\text{-}18)$$

The sensitivity S may be obtained from (2-17) by applying the definition

$$S = \frac{1}{\frac{\partial\theta}{\partial \ln W}} = \frac{1}{W\frac{\partial\theta}{\partial W}}$$

from (2-17)

$$\theta = \ln(\Delta^\circ_{12} + W\Delta_{1243}) + \ln W_R - \ln(\Delta^\circ + W\Delta_{43})$$

$$\frac{\partial\theta}{\partial W} = \frac{\Delta_{1243}}{\Delta^\circ_{12} + W\Delta_{1243}} - \frac{\Delta_{43}}{\Delta^\circ + W\Delta_{43}}$$

Therefore

$$S = \frac{1}{W\left[\frac{\Delta_{1243}}{\Delta^\circ_{12} + W\Delta_{1243}} - \frac{\Delta_{43}}{\Delta^\circ + W\Delta_{43}}\right]} = \frac{1}{W}\left[\frac{(\Delta^\circ_{12} + W\Delta_{1243})(\Delta^\circ + W\Delta_{43})}{(\Delta^\circ + W\Delta_{43})(\Delta_{1243}) - (\Delta^\circ_{12} + W\Delta_{43})}\right]$$

$$S = \frac{1}{W}\left[\frac{(\Delta^\circ_{12} + W\Delta_{1243})(\Delta^\circ + W\Delta_{43})}{\Delta^\circ\Delta_{1243} - \Delta^\circ_{12}\Delta_{43}}\right] \qquad (2\text{-}19)$$

Equation (2-19) may be simplified by the general formula[9]

$$\Delta\Delta_{abcd} = \Delta_{ab}\Delta_{cd} - \Delta_{ad}\Delta_{cb}$$

Let $a = 1,\ b = 2,\ c = 3,\ d = 3$

$$\Delta\Delta_{1243} = \Delta_{12}\Delta_{43} - \Delta_{13}\Delta_{42}$$

$$S = \frac{1}{W}\frac{\Delta_{12}\Delta}{\Delta_{13}\Delta_{42}} \qquad (2\text{-}20)$$

9 Mimeographed pamphlet 5185, Department of Electrical Engineering, Michigan State College.

2.7 Return Difference and Sensitivity in the Case of Zero Direct Transmission

If the transmission is to be zero when $W = 0$, we see from equation (2-18) that Δ°_{12} must also be zero. Substituting this result in equation (2-18), we obtain

$$S = \frac{W\Delta_{1243}(\Delta^{\circ} + W\Delta_{43})}{\Delta^{\circ}\,\Delta_{1243}} = W\frac{\Delta}{\Delta^{\circ}} \qquad (2\text{-}21)$$

Equation (2-21) is the same as the equation for the return difference; thus we have the theorem:

> The sensitivity and return difference are equal for any element whose vanishing leads to zero direct transmission through the circuit as a whole.

2.8 General Relationship Between Sensitivity and Return Difference

According to the theorem V of Article 2.5, the return difference and sensitivity are equal for a prescribed element if the transmission through the circuit is zero when the element is zero.

Since the quantity e^{θ_o} represents the transmission when the given element is zero, we might expect the sensitivity of the difference $e^{\theta} - e^{\theta_o}$ for the element W will be equal to the return difference for the element W. Defining the quantity $e^{\theta^1} = e^{\theta} - e^{\theta_o}$ we have

$$e^{\theta'} = W_R\left[\frac{\Delta^{\circ}_{12} + W\Delta_{1243}}{\Delta^{\circ} + W\Delta_{43}} - \frac{\Delta^{\circ}_{12}}{\Delta^{\circ}}\right]$$

$$e^{\theta'} = W_R \left[\frac{(\Delta^{\circ}_{12} + W\Delta_{1243})\Delta^{\circ} - \Delta^{\circ}_{12}(\Delta^{\circ} + W\Delta_{43})}{\Delta^{\circ}(\Delta^{\circ} + W\Delta_{43})}\right]$$

$$= W_R \left[\frac{\Delta^{\circ}\Delta^{\circ}_{12} + W\Delta^{\circ}\Delta_{1243} - \Delta^{\circ}_{12}\Delta^{\circ} - W\Delta^{\circ}_{12}\Delta_{43}}{\Delta^{\circ}(\Delta^{\circ} + W\Delta_{43})}\right]$$

$$W_R W \left[\frac{\Delta^{\circ}\Delta_{1243} - \Delta^{\circ}_{12}\Delta_{43}}{\Delta^{\circ}(\Delta^{\circ} + W\Delta_{43})}\right]$$

$$\theta' = \ln W_R + \ln W + \ln(\Delta^{\circ}\Delta_{1243} - \Delta^{\circ}_{12}\Delta_{43}) - \ln \Delta^{\circ} - \ln(\Delta^{\circ} + W\Delta_{43})$$

$$S^1 = \frac{1}{W\frac{d\theta}{dW}} = \frac{1}{W\left[\frac{1}{W} - \frac{\Delta_{43}}{\Delta^{\circ} + W\Delta_{43}}\right]} = \frac{\Delta^{\circ} + W\Delta_{43}}{\Delta^{\circ} + W\Delta_{43} - W\Delta_{43}} = \frac{\Delta}{\Delta^{\circ}} \qquad (2\text{-}22)$$

As a result of (2-22) we may state the following theorem:[10]

The sensitivity of the difference $e^{\theta} - e^{\theta_o}$, between the normal output and the direct transmission for any element W, is equal to the return difference for W.

Another useful relation may be derived by dividing the return difference for the element W by the sensitivity for the element W.

$$\frac{F}{S} = \frac{\frac{\Delta}{\Delta^{\circ}}}{-\frac{1}{W}\frac{\Delta\Delta_{12}}{\Delta_{13}\Delta_{42}}} = -W\frac{\Delta_{13}\Delta_{42}}{\Delta_{12}\Delta^{\circ}}$$

If the right hand side of the above equation is multiplied by $\frac{\Delta W_R}{\Delta W_R}$ and the terms rearranged, we obtain

$$\frac{F}{S} = -W\frac{\Delta_{13}\Delta_{42}}{\Delta_{12}\Delta^{\circ}}\,\frac{\Delta W_R}{\Delta W_R} = -\frac{W\Delta_{13}\Delta_{42}W_R}{\Delta^{\circ}\Delta}\left[\frac{\Delta}{\Delta_{12}W_R}\right]$$

$$\frac{F}{S} = \frac{e^{\theta} - e^{\theta_o}}{e^{\theta}} = 1 - \frac{e^{\theta_o}}{e^{\theta}} \qquad (2\text{-}23)$$

Equation (2-23) is useful in determining if the difference

10 Op. Cit., Bode, P. 50.

between F and S is great enough to be considered in making calculations.

Reference Value for W

Definition:[11] The reference value of any element is, that value which gives zero transmission through the circuit as a whole, when all other elements of the circuit have their normal values.

We can compute the reference value for W, W_o, by setting equation (2-17) equal to zero.

Hence, the reference value for W is given by

$$W_o = -\frac{\Delta^{o}_{12}}{\Delta_{1243}} \qquad (2\text{-}24)$$

If we let W^1 represent the departure $W - W_o$ from the reference value $W^1 = W - W_o$

$$W = W^1 + W_o = W^1 - \frac{\Delta^{o}_{12}}{\Delta_{1243}} \qquad (2\text{-}25)$$

Substituting this value in the equation for the gain, we obtain

$$e^{\theta} = \frac{\Delta^{o}_{12} + \left(W' - \frac{\Delta^{o}_{12}}{\Delta_{1243}}\right)\Delta_{1243}}{\Delta^{o} + \left(W' - \frac{\Delta^{o}_{12}}{\Delta_{1243}}\right)\Delta_{43}} = \frac{W'\Delta_{1243}}{\Delta^{o} - \frac{\Delta^{o}_{12}\Delta_{43}}{\Delta_{1243}} + W'\Delta_{43}} W_R \qquad (2\text{-}26)$$

When $W^1 = 0$ we note from (2-25) that $W = -\frac{\Delta^{o}_{12}}{\Delta_{1243}}$; therefore, the determinant will have the value

$$\Delta' = \Delta^{o} + W\frac{\Delta_{43}}{\Delta_{1243}} = \Delta^{o} - \frac{\Delta^{o}_{12}\,\Delta_{43}}{\Delta_{1243}}$$

The value of Δ' contains the first two terms of the denominator of (2-26). Substituting Δ' in (2-26) we obtain

$$e^{\theta} = \frac{W'\Delta_{1243} \quad W_R}{\Delta' + W'\Delta_{43}} = \frac{W'\Delta_{1243}}{\Delta'} W_R \qquad (2\text{-}27)$$

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11 *Ibid*., p. 61.

Equation (2-27) is an expression for the gain, with $W - W_0$ as the reference value of the variable element in the circuit. From (2-2) we can compute the sensitivity based on the same reference value of W. This will be designated as S^1 and known as the relative sensitivity.

$$S^1 = \frac{1}{\frac{\partial\theta}{\partial \ln W'}} = \frac{1}{W'\frac{\partial\theta}{\partial W'}} \tag{2-28}$$

$$= \frac{1}{W'\left[\frac{\Delta_{1243}}{W'\Delta_{1243}} - \frac{\Delta_{43}}{\Delta' W'\Delta_{43}}\right]} = \frac{1}{1 - \frac{W'\Delta_{43}}{\Delta' + W'\Delta_{43}}}$$

$$S^1 = \frac{\Delta' + W'\Delta_{43}}{\Delta'} = \frac{\Delta + W_0\Delta_{43} + (W - W_0)\Delta_{43}}{\Delta'} = \frac{\Delta}{\Delta'} \tag{2-29}$$

From the theorem of (2-9) we may state that the relative sensitivity for any element W is equal to the return difference of W for the reference.

The proof of the above theorem is obvious, since the relative sensitivity is computed on the basis of W_0 as a reference value.

CHAPTER III

3.1

This chapter continues the development of the theory of feedback network analysis on the basis of the theorems and definitions developed in the preceding chapters.

3.2 Impedance of an Active Circuit

The definition of driving point impedance specified that all energy sources be replaced by their internal impedance. This is a passive impedance. The actual input circuit of a feedback amplifier contains currents due to the vacuum tubes in the circuit. Thus, the input impedance of a feedback amplifier will be quite different from that obtained by the usual definition. Because of this reason we shall define an active driving point impedance for the nth mesh as the ratio of a constant voltage generator inserted in the nth mesh to the current flowing in the nth mesh.

The active driving point impedance for the nth mesh becomes

$$Z = \frac{\Delta}{\Delta_{nn}} \qquad (3\text{-}1)^{1}$$

If we choose any arbitrary impedance, W, equation 3-1 may be rewritten as

$$Z = \frac{\Delta}{\Delta_{nn}} \times \frac{\Delta^{\circ}}{\Delta^{\circ}} \times \frac{\Delta_{nn}}{\Delta^{\circ}_{nn}} = \frac{\Delta_{nn}}{\Delta^{\circ}_{nn}} \times \frac{\Delta}{\Delta^{\circ}} \times \frac{1}{\frac{\Delta_{nn}}{\Delta^{\circ}_{nn}}} \qquad (3\text{-}2)^{2}$$

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1 Op, Cit., Bode, p. 67.
2 Ibid., p. 67.

The ratio $\frac{\Delta^{\circ}}{\Delta^{\circ}_{nn}}$ may be considered as the passive impedance of the nth mesh with respect to the element W, since W does not appear in the equation. However, this may not be a passive impedance in the normal sense of the word, since, if W is considered the impedance of a vacuum tube and if feedback is still present when W = 0, then $\frac{\Delta^{\circ}}{\Delta^{\circ}_{nn}} = Z_0$ is a passive impedance with respect to the element W only.

The ratio $\frac{\Delta}{\Delta^{\circ}}$ is the return difference with W = 0 as a reference.

The ratio $\frac{\Delta_{nn}}{\Delta^{\circ}_{nn}}$ can be considered as the return difference with respect to W when the self-impedance of the nth mesh is made infinite, or the terminals between which Z is measured are open.

Denoting the passive impedance by Z_0, the normal feedback by F (0) -- (since this is the return ratio when the terminals across which Z is measured are short circuited) and the return difference with Z open circuited by F (∞), (3-2) may be written as

$$Z = Z_0 \frac{F(0)}{F(\infty)} \qquad (3\text{-}3)^{3}$$

In a similar manner we can obtain an expression for the driving point admittance

$$Y = Y_0 \frac{F(0)}{F(\infty)} \qquad (3\text{-}4)^{4}$$

3 *Ibid.*, p. 67.
4 *Ibid.*, p. 67.

If, in equation (3-1) we let W represent the transimpedance of a tube and give it a definite value W_1, we obtain

$$Z_{W1} = \frac{\Delta_{W_1}}{\Delta_{W_1,nn}} \times \frac{\Delta^{\circ}_{nn}}{\Delta^{\circ}_{nn}} \times \frac{\Delta^{\circ}}{\Delta^{\circ}} = \frac{\Delta^{\circ}}{\Delta^{\circ}_{nn}} \times \frac{\Delta_{W_1}}{\Delta^{\circ}} \times \frac{1}{\dfrac{\Delta_{W_1,nn}}{\Delta^{\circ}_{nn}}} = Z_0 \frac{F_{W_1}(0)}{F_{W_1}(\infty)} \qquad (3\text{-}5)$$

Making the same computation for a different value of W, $W = W_2$, we obtain

$$Z_{W2} = \frac{\Delta_{W_2}}{\Delta_{W_2,nn}} \times \frac{\Delta^{\circ}}{\Delta^{\circ}} \times \frac{\Delta^{\circ}_{nn}}{\Delta^{\circ}_{nn}} = \frac{\Delta^{\circ}}{\Delta^{\circ}_{nn}} \times \frac{\Delta_{W_2}}{\Delta^{\circ}} \times \frac{1}{\dfrac{\Delta_{W_2,nn}}{\Delta^{\circ}_{nn}}} = Z_0 \frac{F_{W_2}(0)}{F_{W_2}(\infty)} \qquad (3\text{-}6)$$

The ratio of the impedances for the two values of W is

$$\frac{Z_{W1}}{Z_{W2}} = \frac{F_{W_1}(0)\ F_{W_2}(\infty)}{F_{W_2}(0)\ F_{W_1}(\infty)} \qquad (3\text{-}7)$$

If W_1 is considered as the normal operating value of W and W_2 is a prescribed reference, the following theorem may be stated:

Theorem VI;[5] The ratio of the impedances seen at any point of a network when a given element W is assigned two different values, is equal to the ratio of the return differences for W when the terminals between which the impedance is measured are first short-circuited and then open-circuited; if the return differences are computed by letting the first value of W be the operating value, and the second be the reference.

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[5] *Ibid*., p. 68.

The relation between feedback and impedance can be stated in another way. Suppose an arbitrary impedance Z_n is added in series with the nth mesh and let Δ^1 represent the determinant with Z_n included and $\Delta^{o'}$ represent the impedance when $Z_n = 0$. The return difference for the circuit containing Z_n is-

$$F = \frac{\Delta'}{\Delta^{o'}} = \frac{\Delta + Z_n \Delta_{nn}}{\Delta^{o} + Z_n \Delta^{o}_{nn}} \qquad (3\text{-}8)$$

Where the symbols Δ and Δ_{nn} represent the circuit determinant and its cofactor when the circuit is in the normal condition, that is, Z_n is not present.

Choose Z_n such that $F = 0$

$$Z_n = -\frac{\Delta}{\Delta_{nn}} \qquad (3\text{-}9)$$

By the definition of active impedance, equation (3-9) may be written as

$$Z_n = \frac{\Delta}{\Delta_{nn}} = -Z_n \qquad (3\text{-}10)$$

As a result of (3-10) we can state the following theorem:

Theorem VII:[6] The impedance seen in any mesh is the negative of the impedance whose insertion in that mesh would give zero return difference for an arbitrarily chosen element in the circuit.

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6 *Ibid.*, p. 69.

3.3 Examples of Active Impedance

Consider the circuit of Figure 3.1.

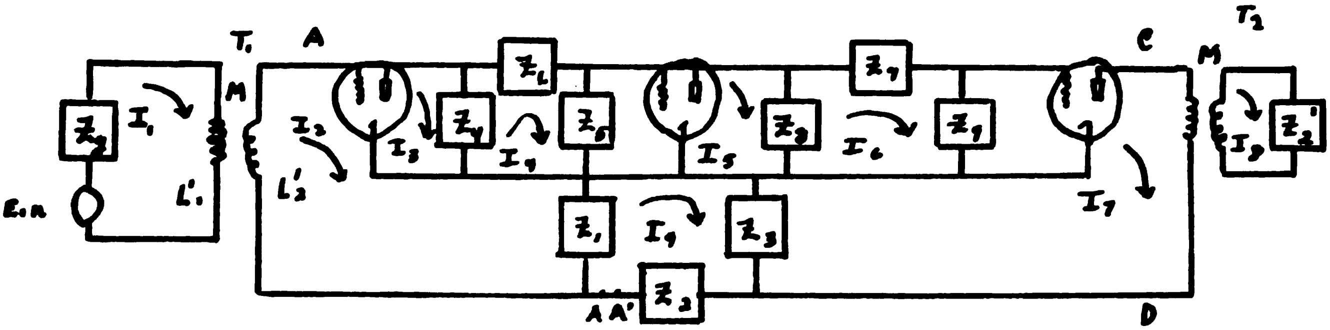

Fig. 3.1

If we assume there is no grid current flowing and that T_1 and T_2 are ideal transformers, the input circuit to the left of AB may be redrawn as follows.

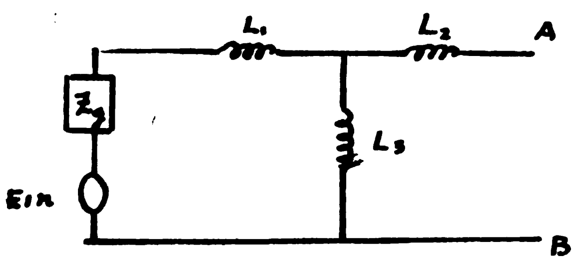

Fig. 3.2

where $L_1 = L_1' \not{-} M$

$L_2 = L_2' \not{-} M$

$L_3 = M$

By applying Thevinen's theorem to the circuit to the left of the points AB, we obtain the circuit

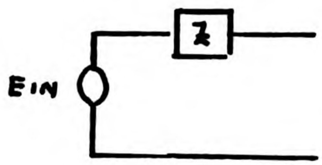

Fig. 3.3

where $E_{in} = \dfrac{j\, E_{in}\, W\, L_3}{Z_g + jw\,(L_1 + L_3)}$

$$Z = \frac{(Z_g + jwL_1)(jwL_3)}{Z_g + jW\,(L_1 + L_3)} + jwL_2$$

Assuming T_2 is an ideal transformer, the circuit to the right of CD may be redrawn as shown in Figure 3.4.

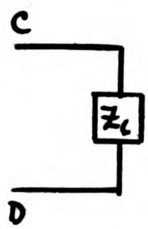

Fig. 3.4

where $Z_L = \dfrac{Z_L'}{a^2}$ and a is the turn's ratio of T_2.

Replacing the vacuum tubes in Figure 3.1 by their equivalent circuits, we obtain the circuit of Figure 3.5.

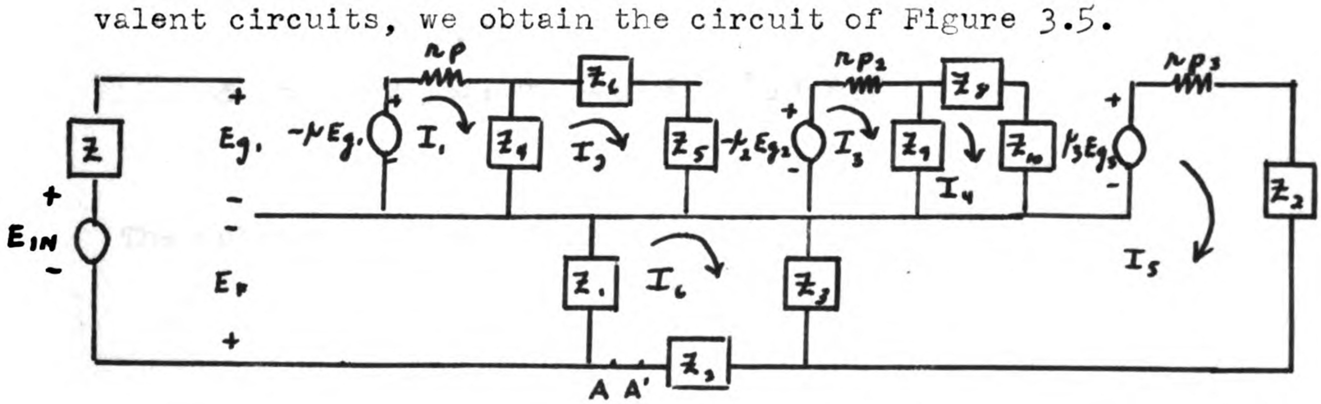

Fig. 3.5

The mesh equations for the circuit of Figure 3.5 are

$$-\mu_1 E_{g1} = I_1(r_{p1} + Z_4) - I_2 Z_4$$
$$0 = -I_1 Z_4 + I_2(Z_4 + Z_5 + Z_6)$$
$$-\mu_2 E_{g2} = I_3(r_{p2} + Z_9) - I_4 Z_9$$
$$0 = -I_3 Z_9 + I_4(Z_9 + Z_8 + Z_{10})$$
$$-\mu_3 E_{g3} = I_5(r_{p3} + Z_2 + Z_3) - I_6 Z_3$$
$$0 = -I_5 Z_3 + I_6(Z_1 + Z_2 + Z_3)$$

In addition the following equations may be written:

$$EF + E_{in} = E_{g1}$$
$$EF = I_6 Z_1$$
$$I_2 Z_5 = E_{g2}$$
$$I_4 Z_{10} = E_{g3}$$

Combining these equations we obtain

$$-\mu_1 E_{g1} = I_1(r_{p1} + Z_4) - I_2 Z_4 + I_6 \mu_1 Z_1$$
$$0 = -I_1 Z_4 + I_2(Z_4 + Z_5 + Z_6)$$
$$0 = I_2 \mu_2 Z_5 + I_3(r_{p2} + Z_9) - I_4 Z_9$$
$$0 = -I_3 Z_9 + I_4(Z_8 + Z_9 + Z_{10})$$
$$0 = I_4 \mu_3 Z_{10} + I_5(r_{p3} + Z_2' + Z_3) - I_6 Z_3$$
$$0 = -I_5 Z_3 + I_6(Z_1 + Z_2 + Z_3)$$

The determinant of the preceding mesh equation is

$$\Delta = -\frac{1}{\mu_1}\begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & 0 & \mu_1 Z_1 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & -Z_{34} & 0 & 0 \\ 0 & 0 & -Z_{43} & Z_{44} & 0 & 0 \\ 0 & 0 & 0 & \mu_3 Z_{10} & Z_{55} & -Z_{56} \\ 0 & 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix}$$

where $Z_{ij} = Z_{ji}$ except where indicated.

The active impedance between the terminals AA^1, or the active driving point impedance Z_{66} may be computed in several ways. The first method will be by applying formula (3-3) of Article 3.2 and using the determinants.

$$Z_{66} = Z_o \frac{F\ (0)}{F\ (\infty)}$$

where Z_o is the passive impedance of the 6th mesh. F (0) is the return difference with the terminals AA^1 short circuited or the circuit in the normal condition, and F (∞) is the return differences with the terminals AA^1 open circuited.

The element W in this example will be taken as $\mu_3 Z_{10}$.

By determinate theory

$$Z_{o,66} = \frac{\Delta^\circ}{\Delta^\circ_{66}}$$

$$\frac{\Delta^\circ}{\Delta^\circ_{66}} = \begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & 0 & \mu_1 Z_1 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & -Z_{34} & 0 & 0 \\ 0 & 0 & -Z_{43} & Z_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_{55} & -Z_{56} \\ 0 & 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix} \qquad (3\text{-}11)$$

$$\Delta^{o}_{66} = \begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & 0 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & -Z_{34} & 0 \\ 0 & 0 & -Z_{43} & Z_{44} & Z_{55} \\ 0 & 0 & 0 & 0 & -Z_{65} \end{vmatrix}$$

Expanding the numerator and denominator of (3-11) by a Laplace development of the second and third columns

$$\frac{\Delta^{o}}{\Delta^{o}_{66}} = \frac{\begin{vmatrix} Z_{33} & -Z_{34} \\ -Z_{43} & Z_{44} \end{vmatrix} \begin{vmatrix} Z_{11} & -Z_{12} & 0 & \mu_1 Z_1 \\ -Z_{21} & Z_{22} & 0 & 0 \\ 0 & 0 & Z_{55} & -Z_{56} \\ 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix}}{\begin{vmatrix} Z_{33} & -Z_{34} \\ -Z_{43} & Z_{44} \end{vmatrix} \begin{vmatrix} Z_{11} & -Z_{12} & 0 \\ -Z_{21} & Z_{22} & 0 \\ 0 & 0 & Z_{55} \end{vmatrix}} \tag{3-12}$$

Again expanding

$$\frac{\Delta^{o}}{\Delta^{o}_{66}} = \frac{\begin{vmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{vmatrix} \begin{vmatrix} Z_{55} & -Z_{56} \\ -Z_{65} & Z_{66} \end{vmatrix}}{\begin{vmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{vmatrix} Z_{55}} = \frac{\begin{vmatrix} Z_{55} & -Z_{56} \\ -Z_{65} & Z_{66} \end{vmatrix}}{Z_{55}} \tag{3-13}$$

Substituting values from the mesh equations in equation (3-13), we obtain

$$Z_{o,\,66} = \frac{\begin{vmatrix} rp_3 + Z_2' + Z_3 & -Z_3 \\ -Z_3 & Z_1 + Z_2 + Z_3 \end{vmatrix}}{rp_3 + Z_2' + Z_3}$$

$$= Z_1 + Z_2 + Z_3 - \frac{Z_3^2}{rp_3 + Z_2' + Z_3} \tag{3-14}$$

It should be noted that the Z_o given by (3-14) is not the value of Z_o we would expect from an examination of the original circuit in Figure 3.1.[7] The reason for the discrepancy

7 Bode, p. 67.

appears very clearly in the equivalent circuit of Figure 3.5. With the W of the third tube C, there is still a complete path for the current I_5 and, thus, we should expect a term involving Z_2' to appear in the expression for $Z_{AA}^{\ 1}$ or Z_{66}.

If it is assumed that the output impedance is large compared with the β circuit impedance, then equation (3-14) becomes

$$Z_{o,\ 66} = Z_1 + Z_2 + Z_3 \qquad (3\text{-}15)$$

A similar difficulty may exist in the input circuit. In order that Z_o be the passive impedance, as used by Bode, it is necessary to make the additional assumption that input impedance be large compared to the β circuit.

$$F(0) = \frac{\Delta}{\Delta^\circ} = 1 + \mu_3 Z_{10} \frac{\Delta_{54}}{\Delta^\circ}$$

$$\frac{\Delta_{54}}{\Delta^\circ} = \frac{\begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & \mu_1 Z_1 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & 0 & 0 \\ 0 & 0 & -Z_{43} & 0 & 0 \\ 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix}}{\begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & 0 & \mu_1 Z_1 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & -Z_{34} & 0 & 0 \\ 0 & 0 & -Z_{43} & Z_{44} & 0 & 0 \\ 0 & 0 & 0 & \mu_3 Z_{10} & Z_{55} & -Z_{56} \\ 0 & 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix}} \qquad (3\text{-}16)$$

By a Laplace development (3-16) becomes

$$= \frac{\begin{vmatrix} 0 & \mu_1 Z_1 \\ -Z_{65} & Z_{66} \end{vmatrix} (-Z_{21})(\mu_2 Z_5)(-Z_{43})}{\begin{vmatrix} Z_{33} & -Z_{34} \\ -Z_{43} & Z_{44} \end{vmatrix} \begin{vmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{vmatrix} \begin{vmatrix} Z_{55} & -Z_{56} \\ -Z_{65} & Z_{66} \end{vmatrix}}$$

$$= \frac{\mu_1 Z_1 Z_{21} \mu_2 Z_5 Z_{43}}{(Z_{33} Z_{44} - Z_{34}^2)(Z_{11} Z_{22} - Z_{12}^2)(Z_{55} Z_{66} - Z_{56}^2)} \qquad (3\text{-}17)$$

$$F\ (0) = 1 + \frac{\mu_3 Z_{10} \mu_1 Z_1 Z_{21} \mu_2 Z_5 Z_{43}}{(Z_{33} Z_{44} - Z_{34}^2)(Z_{11} Z_{22} - Z_{12}^2)(Z_{55} Z_{66} - Z_{56}^2)} \qquad (3\text{-}18)$$

Equation (3-18) gives the complete expression for F (0) or the return difference when the circuit is in the normal condition.

F (0) may also be written as

$$F\ (0) = 1 - \mu\beta. \qquad (3\text{-}19)$$

$$F(\infty) = \frac{\Delta_{66}}{\Delta^\circ_{66}} = \frac{\Delta^\circ_{66} + \mu_3 Z_{10} \Delta_{6654}}{\Delta^\circ_{66}} = 1 + \frac{\mu_3 Z_{10} \Delta_{6654}}{\Delta^\circ_{66}}$$

$$\Delta_{6654} = -\frac{1}{\mu'} \begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 \\ -Z_{21} & Z_{22} & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & 0 \\ 0 & 0 & -Z_{43} & 0 \end{vmatrix} = 0$$

$$\Delta^\circ_{66} = -\frac{1}{\mu'} \begin{vmatrix} Z_{11} & -Z_{12} & 0 & 0 & 0 \\ -Z_{21} & Z_{22} & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & Z_{33} & -Z_{34} & 0 \\ 0 & 0 & -Z_{43} & Z_{44} & 0 \\ 0 & 0 & 0 & 0 & Z_{55} \end{vmatrix} \neq 0$$

Therefore, F (∞) is given by

$$F\ (\infty) = 1 + \frac{\mu_3 Z_{10}\ (0)}{\Delta^\circ_{66}} = 1 \qquad (3\text{-}20)$$

The complete expression for the active impedance of Z_{AA}^1 is given by the product of (3-14), (3-18) and (3-20).

$$Z_{66} = \left[Z_1 + Z_2 + Z_3 - \frac{Z_2^2}{r\mu_3 + Z_2 + Z_3} \right] (1 - \mu\beta) \qquad (3\text{-}21)$$

Under the condition that the output impedance is much greater than the β circuit impedance, (3-21) reduces to

$$Z_{66} = (Z_1 + Z_2 + Z_3)(1 - \mu B) \qquad (3\text{-}22)$$

The active impedance Z_{66} is, therefore, smaller than the passive impedance $Z_{o,\ 66}$.

The preceding calculations could have been simplified, somewhat. If the assumption that the input impedance and output impedance are much greater than the β circuit impedance, the passive impedance $Z_{o,\ 66}$ can be computed immediately as

$$Z_{o,66} = Z_1 + Z_2 + Z_3$$

F (0) is the normal return difference and must be computed.

$F(\infty) = 1 - \mu B$ when the terminals AA^1 are open. In this condition the product $\mu\beta = 0$, therefore

$$Z_{66} = (Z_1 + Z_2 + Z_3)(1 - \mu B)$$

3.4 Exact Formula for External Gain With Feedback

In Chapter II, it was stated that the gain of an amplifier was reduced by the amount of feedback.[8] This statement seems simple enough, but in attempting to apply it, it is not clear just what the gain before feedback is. Since there is energy lost in the B circuit, it is not clear if the B circuit should be considered in computing the gain before feedback. In addition, the circuits containing appreciable direct transmission, the question arises as to whether we should include the direct transmission as part of the gain.

8 See Article 2.1, p.

The last problem presents a point of departure. We shall define the gain before feedback or fractionated gain as follows:

$$e^{\theta_F} = F\,(e^{\theta} - e^{\theta_o}) \quad [9] \qquad (3\text{-}23)$$

It should be noted that in defining $e^{\theta F}$, the direct transmission term is subtracted from the final gain e^{θ} and is not considered as a gain term in the definition.

Equation (3-23) may be rewritten as

$$e^{\theta} - e^{\theta_o} = \frac{1}{F}\, e^{\theta F} \qquad (3\text{-}24)$$

From equations (2-17) and (2-18) we obtain

$$e^{\theta} - e^{\theta_o} = \left(\frac{\Delta_{12}}{\Delta} - \frac{\Delta^\circ_{12}}{\Delta}\right) WR \qquad (3\text{-}25)$$

$$= \left(\frac{\Delta^\circ_{12} + W\Delta_{1243}}{\Delta^\circ + W\Delta_{43}} - \frac{\Delta^\circ_{12}}{\Delta^\circ}\right) WR$$

$$= \frac{\Delta^\circ\Delta^\circ_{12} + \Delta^\circ W\Delta_{1243} - \Delta^\circ\Delta^\circ_{12} - W\Delta^\circ_{12}\Delta_{43}}{\Delta^\circ(\Delta^\circ + W\Delta_{43})}\, WR$$

$$= \frac{W(\Delta^\circ\Delta_{1243} - \Delta^\circ_{12}\Delta_{43})}{\Delta^\circ(\Delta^\circ + W\Delta_{43})}\, WR$$

The last equation of (3-25) may be simplified by the general determinant relation of Chapter II, and the expression becomes

$$e^{\theta} - e^{\theta_o} = \frac{-W\Delta_{13}\Delta_{42}}{\Delta\,\Delta^\circ}\, WR \qquad (3\text{-}26)$$

Multiplying the right side of (3-26) by $\dfrac{\Delta^\circ}{\Delta^\circ}$ we obtain

$$e^{\theta} - e^{\theta_o} = -\frac{W\Delta_{13}\Delta_{42}}{F\,\Delta^\circ\Delta^\circ}\, WR \qquad (3\text{-}27)$$

9 *Op. Cit.*, Bode, p. 81.

Comparing equation (3-24) and (3-27) we find

$$e^{\theta F} = - W \frac{\Delta_{13}}{\Delta^{\circ}} \frac{\Delta_{42}}{\Delta^{\circ}} WR \qquad (3\text{-}28)$$

The formula for the fractionated gain (3-28) can be considered as the product of three factors $\frac{\Delta_{13}}{\Delta^{\circ}}$ which represents the transmission from the input circuit to the grid of the element W, $\frac{\Delta_{42}}{\Delta^{\circ}}$ WR which represents the transmission from the plate of the tube to the output circuit, and W the transmittance of the tube in question. The first two terms $\frac{\Delta_{13}}{\Delta^{\circ}}$ and $\frac{\Delta_{42}}{\Delta^{\circ}}$ WR contain the B circuit impedance.

An example of the calculation of the fractionated gain is furnished by the circuit of Figure 3.1. The actual calculation will be based on the equivalent circuit of Figure 3.5. The element W will again be taken as the third tube and $W = \mu_3' Z_{10}$.

In the original derivation of gain in Chapter II, the subscripts 1, 2, 4, and 3 referred to the input, output, plate and grid of the circuit and tube under discussion respectively. In the equivalent circuit of Figure 3.5, the subscript 1 refers to the input circuit, the subscript 4 refers to the grid of the third tube and the subscript 5 refers to both the plate of the third tube and to the output circuit. With this change in notation, (3-28) becomes

$$e^{\theta F} = -\mu_3 Z_{10} \frac{\Delta_{13} \; \Delta_{42}}{(\Delta^{\circ})^2} Z \qquad (3\text{-}29)$$

Substituting in (3-29), we obtain

$$e^{\theta_F} = \frac{\mu_3 Z_{10}\left(\frac{1}{\mu_1}\right)\begin{vmatrix} -Z_{21} & Z_{22} & 0 & 0 & 0 \\ 0 & \mu_2 Z_5 & -Z_{34} & 0 & 0 \\ 0 & 0 & Z_{44} & 0 & 0 \\ 0 & 0 & \mu_3 Z_{10} & Z_{55} & -Z_{56} \\ 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix}\left(-\frac{1}{\mu_1}\right)\begin{vmatrix} Z_{11} & 0 & 0 & 0 & \mu_1 Z_1 \\ -Z_{21} & 0 & 0 & 0 & 0 \\ 0 & Z_{33} & -Z_{34} & 0 & 0 \\ 0 & 0 & \mu_3 Z_{10} & Z_{55} & -Z_{56} \\ 0 & 0 & 0 & -Z_{65} & Z_{66} \end{vmatrix} Z_2'}{\frac{1}{\mu_1^2}\left\{\begin{vmatrix} Z_{33} & -Z_{34} \\ -Z_{43} & Z_{44} \end{vmatrix}\begin{vmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{vmatrix}\begin{vmatrix} Z_{55} & -Z_{56} \\ -Z_{65} & Z_{66} \end{vmatrix}\right\}^2} \qquad (3\text{-}30)$$

$$= -\mu_3 Z_{10}\,\frac{\begin{vmatrix} -Z_{21} & Z_{22} & 0 \\ 0 & \mu_2 Z_5 & -Z_{34} \\ 0 & 0 & Z_{44} \end{vmatrix}\begin{vmatrix} Z_{33} & -Z_{34} \\ 0 & \mu_3 Z_{10} \end{vmatrix}\begin{vmatrix} Z_{11} & 0 & \mu_1 Z_1 \\ -Z_{21} & 0 & 0 \\ 0 & -Z_{56} & Z_{66} \end{vmatrix}}{\begin{vmatrix} Z_{55} & -Z_{56} \\ -Z_{65} & Z_{66} \end{vmatrix}\left\{\begin{vmatrix} Z_{33} & -Z_{34} \\ -Z_{43} & Z_{44} \end{vmatrix}\begin{vmatrix} Z_{11} & -Z_{12} \\ -Z_{21} & Z_{22} \end{vmatrix}\right\}^2}$$

$$= \frac{\mu_3 Z_{10} Z_{21} \mu_2 Z_5 Z_{44} Z_{33} \mu_3 Z_{10} Z_{21} Z_{56} \mu_1 Z_1}{(Z_{55}Z_{66} - Z_{56}^2)(Z_{33}Z_{44} - Z_{34}^2)^2(Z_{11}Z_{22} - Z_{12}^2)^2}$$

The third expression in equation (3-30) represents the complete expression for the fractionated gain of the circuit of Figure 3.1.

The circuit of Figure 3.1 presents another problem in the use of the determinant formulas. Consider the expression for the gain e^{θ} in equation (3-17)

$$e^{\theta} = \frac{\Delta_{12}}{\Delta} WR = \frac{\Delta^{\circ}_{12} + W\Delta_{1243}}{\Delta^{\circ} + W\Delta_{43}} WR \qquad (3\text{-}17)$$

Changing the subscripts to agree with Figure 3.1, equation (3-17) becomes

$$e^{\theta} = \frac{\Delta_{15}}{\Delta} WR = \frac{\Delta^{\circ}_{15} + W\Delta_{1545}}{\Delta^{\circ} + W\Delta_{45}} \qquad (3\text{-}31)$$

The second order cofactor Δ_{1545} seems to represent an unsymmetrical determinant or to have no meaning. The apparent discrepancy does not exist; however, for an examination of the original determinant, it is noted that the element W does not appear in the cofactor Δ_{15}. Hence, the last term on the right of equation (3-31) does not exist, and the correct expression for the gain e^{θ} is simply

$$e^{\theta} = \frac{\Delta_{15}}{\Delta} WR = \frac{\Delta_{15}}{\Delta^{\circ} + W\Delta^{\circ}_{45}} \qquad (3\text{-}32)$$

The direct transmission gain equation (3-18) becomes

$$e^{\theta} = \frac{\Delta_{15}}{\Delta^{\circ}} WR$$

In many cases an unsymmetrical array will appear and seem to indicate an error in the determinant theory. However, all of these cases will be similar to equation (3-31), and the term containing the unsymmetrical array actually does not exist and may be dropped from the equation.

CHAPTER IV

4.1

This chapter will be devoted to the computation of W_o. Since many of the calculations involving feedback amplifiers can be carried out easier in terms of W^1 and S^1 and these quantities in turn depend on W_o, it is of some importance to be able to compute W_o easily.

4.2 Simplified Computation of W_o

If the amplifier in question belongs to a class indicated by Figure 4.1, 4.2, 4.3 and 4.4, the computation of W_o can be simplified.[1]

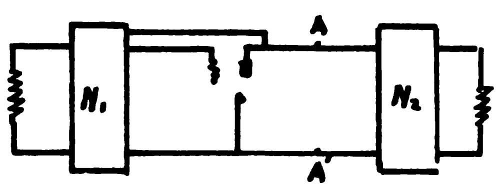

Fig. 4.1

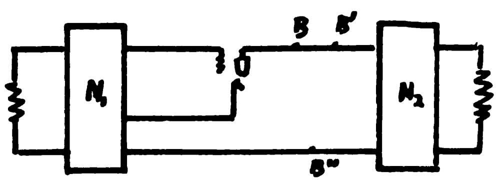

Fig. 4.2

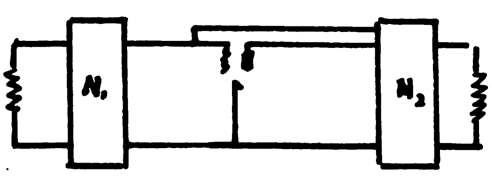

Fig. 4.3

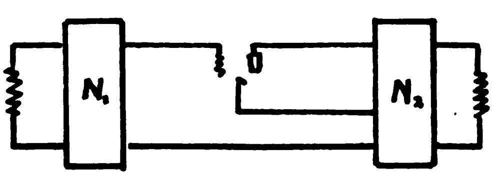

Fig. 4.4

1 Op. Cit., Bode, p. 89.

By definition W_o is the reference value of any element which gives zero transmission through the circuit as a whole when all other elements have their normal values.

When the circuit is in the reference condition, the output is zero; thus, the total voltage across the load or the current through the load must be zero. This demands that the voltage or current supplied to the load by the tube must cancel the voltage or current supplied to the load by the rest of the circuit. This calculation might still be difficult, for in order to evaluate the voltage or current supplied to load by the tube, one must take into account the feedback of the circuit. However, if the feedback path can be interrupted, the complete calculation can be simplified.

Consider the case of circuit such as that in Figure 4.1. Since there is no voltage in the load, it follows that there is no voltage between the terminals AA^1 when the tube is in the reference condition. Hence, in order to compute W_o, the terminals AA^1 may be shorted provided that we define the reference condition as that which supplies zero current through the short circuit. With the terminals AA^1 short-circuited, the feedback path is interrupted, and W_o may be computed as a transmittance between the grid and the plate with the tube dead. The tube must have this transmittance in order that the current be zero through the short circuit.

The circuit of Figure 4.5 provides an example of this type of calculation. We shall first compute W_o by requiring that the gain be zero and then by the simplified method discussed above.[2]

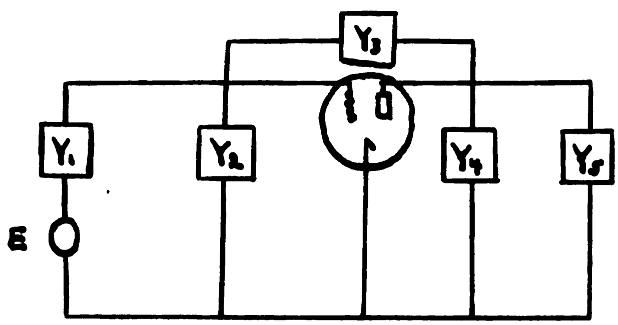

Fig. 4.5

The equivalent circuit on the node basis for the circuit of Figure 4.5 is shown by Figure 4.6.

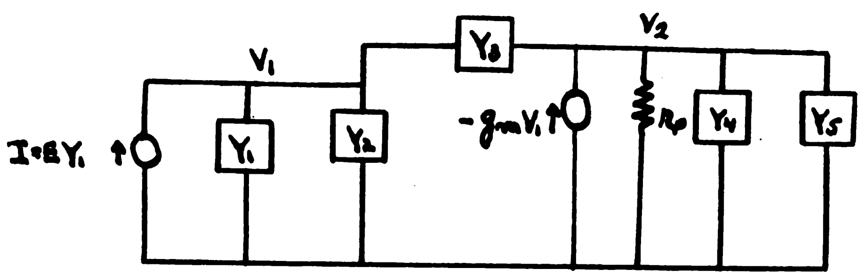

Fig. 4.6

The node equations are

$$EY_1 = V_1 \; (Y_1 + Y_2 + Y_3) - V_2 Y_3$$
$$- gmV_1 = -V_1 Y_3 + V_2 \; (Y_4 + Y_5 + \frac{1}{rp}) \qquad (4\text{-}1)$$

where W is the transconductance of the tube gm. W_o may be computed by requiring that V_2 be zero. Therefore

$$\begin{vmatrix} Y_1 + Y_2 + Y_3 & EY_1 \\ g_m - Y_3 & 0 \end{vmatrix} = 0 \qquad (4\text{-}2)$$

2 Ibid., p. 90.

Hence, $EY_1 (gm - Y_3) = 0$. Since $Y_1 \neq 0$, $W_o = gm_o = Y_3$.

By the simplified method of computation the admittances Y_4 and Y_5 are shorted. W_o is then the ratio of the current through Y_3 divided by voltage between the grid and cathode. By inspection this is

$$W_o = \frac{V_1 Y_3}{V_1} = Y_3 \qquad (4\text{-}4)$$

The value of W_o obtained by the two methods is identical.

A similar situation occurs in the circuit of Figure 4.2. However, if the terminals B, B" are shorted, the feedback path is not interrupted. This may be accomplished by opening the terminals B B" and requiring that the voltage between these terminals be zero in the reference conditions.

An example of this type of calculation is furnished by the circuit of Figure 4.7.[3]

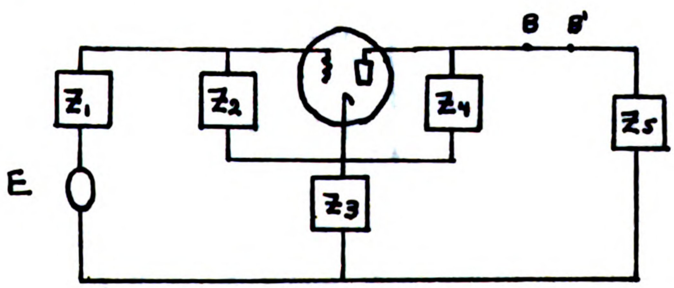

Fig. 4.7

As in the previous example, we shall compute W_o first by requiring that the gain be zero. The equivalent circuit on the mesh basis is shown by Figure 4.8.

3 Ibid., p. 90.

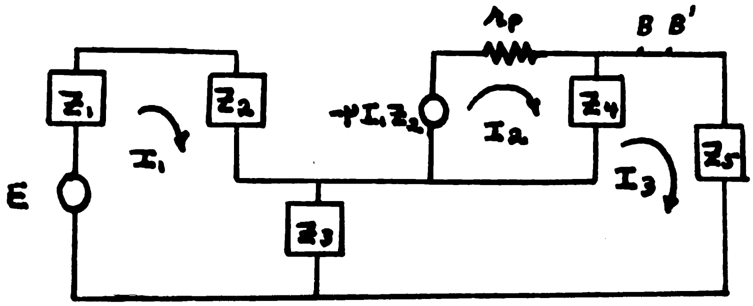

Fig. 4.8

The mesh equations are

$$E = I_1\ (Z_1 + Z_2 + Z_3) - I_2Z_3 \qquad 0 \qquad (4\text{-}5)$$
$$0 = -I_1Z_3 + I_2\ (Z_3 + Z_4 + Z_5) - I_3Z_4$$
$$0 = \mu\ Z_2I_1 \qquad -I_2Z_4 \qquad + I_3\ (rp + Z_4)$$

W in the above equation is μZ_2; thus we shall compute a value $W_o = \mu_o Z_2$ which will make the current $I_3 = 0$.

Thus

$$\begin{vmatrix} -Z_3 & -Z_4 \\ \mu Z_2 & rp + Z_4 \end{vmatrix} = 0 \qquad (4\text{-}6)$$

and

$$W_o = \mu_o Z_2 = Z_3 \frac{(rp + Z_4)}{Z_4} \qquad (4\text{-}7)$$

This is in contrast to the value of W_o given by Bode's equation (6-9).[4]

W_o may also be computed by the node equations. The equivalent circuit of Figure 4.7 on the node basis is given by Figure 4.9.

4 Ibid., p. 90.

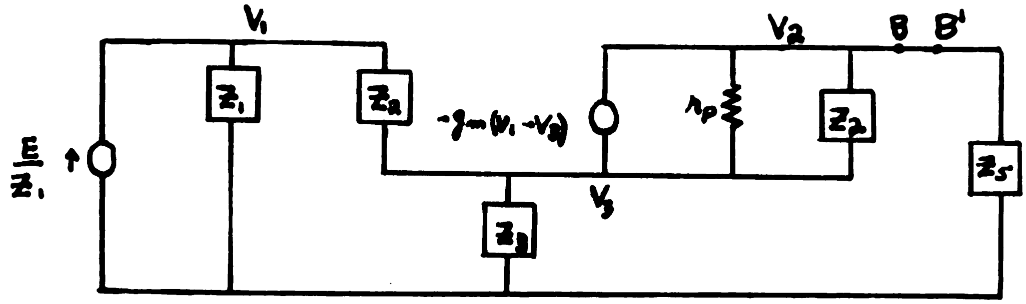

Fig. 4.9

The node equations are

$$\frac{E}{Z_1} = V_1\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) \qquad -\frac{V_3}{Z_2} \qquad (4\text{-}8)$$

$$-gm\,(V_1 - V_3) = V_2\left(\frac{1}{rp} + \frac{1}{Z_4} + \frac{1}{Z_5}\right) - V_3\left(\frac{1}{rp} + \frac{1}{Z_4}\right)$$

$$gm(V_1 - V_3) = V_1\left(\frac{1}{Z_2} - gm\right) - V_2\left(\frac{1}{rp} + \frac{1}{Z_4}\right) + V_3\left(\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{rp} + gm\right)$$

The determinant of (4-8) is

$$\begin{vmatrix} Z_1\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right) & 0 & -\frac{Z_1}{Z_2} \\ gm & \left(\frac{1}{rp} + \frac{1}{Z_4} + \frac{1}{Z_5}\right) & -\left(\frac{1}{rp} + \frac{1}{Z_4} + gm\right) \\ -\frac{1}{Z_2} - gm & -\left(\frac{1}{rp} + \frac{1}{Z_4}\right) & \left(\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{rp} + gm\right) \end{vmatrix} \qquad (4\text{-}9)$$

The condition that V_2 be zero is given by

$$\begin{vmatrix} gm & -\left(\frac{1}{rp} + \frac{1}{Z_4} + gm\right) \\ -(1/Z_2 + gm) & \left(\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{rp} + gm\right) \end{vmatrix} \qquad (4\text{-}10)$$

$$gm\left(\frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} + \frac{1}{rp} + gm\right) = \left(\frac{1}{Z_2} + gm\right)\left(\frac{1}{rp} + \frac{1}{Z_4} + gm\right)$$

$$W_o = gm_o = \frac{Z_3\ (rp + Z_4)}{Z_2 Z_4} \qquad (4\text{-}11)$$

Equation (4-11) is in contrast to Bode's equation (6-10).[5] W_o is essentially a function of the tube parameter μ_o or gm_o.

If the equation

$$gm_o = \frac{Z_3\ (rp + Z_4)}{Z_2 Z_4\ rp}$$

is multiplied by rpZ_2, we obtain

$$\mu_o = \frac{Z_3\ (rp + Z_4)}{Z_4} \qquad (4\text{-}12)$$

The value of μ_o obtained in equation (4-12) is the same as that in equation (4-7).

The value of W_o may be computed by the simplified method discussed previously. The condition that the voltage across BB^1 be zero in Figure 4.8 requires that the voltage across Z_4 be equal and opposite to that across Z_3. W_o, under these conditions is the ratio of the voltage across Z_4 to the current in the grid circuit. Assuming a current I_1 the voltage across Z_3 is $I_1 Z_3$.

$$V_{Z3} = I_1 Z_3 \qquad (4\text{-}13)$$

The voltage across Z_4 is

$$V_{Z4}\ \frac{-\mu I_1\ Z_2 \times Z_4}{rp + Z_4} \qquad (4\text{-}14)$$

Equating (4-13) and (4-14) and changing the sign we obtain

$$I_1 Z_3 = \frac{\mu\ I_1 Z_2 \times Z_4}{rp + Z_4} \qquad (4\text{-}15)$$

5. *Ibid.*, p. 90.

W_o is $\mu_o Z_2$

Thus

$$W_o = \mu_o Z_2 = \frac{Z_3 (rp + Z_4)}{Z_4} \qquad (4\text{-}16)$$

or

$$\mu_o = \frac{Z_3 (rp + Z_4)}{Z_2 Z_4} \qquad (4\text{-}17)$$

This agrees with the value of μ_o given by equations (4-7) and (4-12).

On the node basis, Figure 4.9, the calculation is similar to that of the previous method except the W_o = the ratio of the current in Z_4 to the voltage across the grid and cathode of the tube. Thus, as before, assume a current I_1 in Z_2. The voltage across Z_3 is

$$V_{Z3} = I_1 Z_3 \qquad (4\text{-}18)$$

The voltage across Z_4 is

$$V_{Z4} - gm\ I_{Z2} \times \frac{rp\ Z_4}{rp + Z_4} \qquad (4\text{-}19)$$

Equating 4-18 and the negative of 4-19, we obtain

$$I_1 Z_3 = gm_o \frac{I_1 Z_2 rp\ Z_4}{rp + Z_4}$$

Thus W_o = gm is

$$W_o = gm_o = \frac{Z_3 (rp + Z_4)}{rp\ Z_2 Z_4} \qquad (4\text{-}20)$$

This is the same value as obtained from equation (4-11).

The circuits of Figures 4.3 and 4.4 differ slightly from those just discussed.

Consider the circuit of Figure 4.3. As in the previous cases when the tube is in the reference condition, there is no voltage or current in the load. However, we cannot short circuit the plate and cathode as in Figure 4.1 since this condition would still leave the feedback path open. We can make the output zero by shorting the grid to the cathode and interrupting the feedback path. This can be done by supposing a voltage generator of zero internal impedance is connected between the grid and cathode. The reference value W_o may then be computed as the ratio of a current generator in the plate circuit to the voltage in the grid circuit when both sources are adjusted to produce the same current in the load with the tube dead.

This type of calculation can be illustrated by the circuit of Figure 4.5 and its equivalent circuit, Figure 4.6. If a voltage generator is placed between the grid and cathode of Figure 4.6, a voltage V will appear in the load equal to

$$V_2 = E \left[\frac{Y_3 \left(\frac{1}{rp} + Y_4 + Y_5 \right)}{\frac{1}{rp} + Y_3 + Y_4 + Y_5} \right] \times \frac{1}{\left(\frac{1}{rp} + Y_4 + Y_5 \right)} \qquad (4\text{-}21)$$

A current generator I placed across the load will cause a voltage V_L across the load equal to

$$V_L = I \left(rp + \frac{1}{Y_3} + \frac{1}{Y_4} + \frac{1}{Y_5} \right) \qquad (4\text{-}22)$$

Equating (4-21) and (4-22), we obtain

$$W_o = \frac{I}{E} = Y_3 \qquad (4\text{-}23)$$

The network of Figure 4.4, may be analyzed in a similar fashion except that in this case the feedback path is interrupted by connecting a current generator of infinite internal impedance in series with the grid. W_o is then equal to the ratio of a constant voltage generator connected in the plate circuit to the current generator in the grid circuit when both are adjusted to produce the same output with the tube dead.

An example is furnished by the circuit of Figure 4.7 and its equivalent circuit Figure 4.8. The equivalent circuit of Figure 4.8 with a constant current generator in the grid circuit is shown in Figure 4.10.

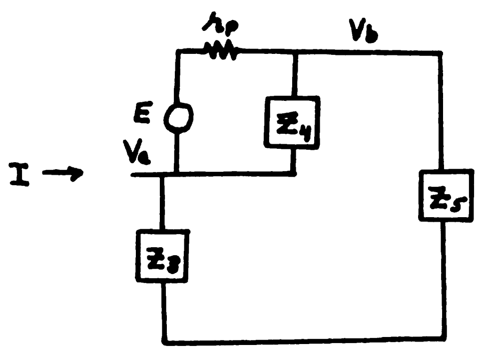

Fig. 4.10

The constant current generator I has an infinite internal impedance and therefore interrupts the feedback path. By superposition we can compute the voltage across Z_5 due the constant current generator I and the constant voltage generator E. W_o will then be the ratio of $\frac{E}{I}$.

The node equations for the circuit of Figure 4.10 are

$$I = V_a\left(\frac{1}{Z_3} + \frac{1}{\dfrac{rpZ_4}{rp + Z_4}}\right) - V_b\left(\frac{1}{\dfrac{Z_4 rp}{rp + Z_4}}\right) \qquad (4\text{-}24)$$

$$0 = -V_a\left(\frac{1}{\dfrac{rpZ_4}{rp + Z_4}}\right) + V_b\left(\frac{1}{Z_5} + \frac{1}{\dfrac{rpZ_4}{rp + Z_4}}\right)$$

The voltage across Z_5 is

$$V_b = \frac{\dfrac{I\ (rp + Z_4)}{rpZ_4}}{\dfrac{1}{Z_3Z_5} + \dfrac{1}{Z_3\ \dfrac{rpZ_4}{rp + Z_4}} + \dfrac{1}{Z_5\ \dfrac{rpZ_4}{rp + Z_4}}} \qquad (4\text{-}25)$$

$$= \frac{I\ (rp + Z_4)\ Z_3Z_5}{rpZ_4 + rpZ_5 + Z_4Z_5 + rpZ_3 + Z_3Z_4}$$

The mesh equations for Figure 4.11 are

$$E = I_1\ (rp + Z_4) - I_2Z_4 \qquad (4\text{-}26)$$

$$0 = -\ I_1Z_4 + I_2\ (Z_3 + Z_4 + Z_5)$$

The voltage across Z_5 may be computed as

$$V_b = \frac{EZ_4\ Z_5}{rpZ_3 + rpZ_4 + rpZ_5 + Z_3Z_4 + Z_4Z_5} \qquad (4\text{-}27)$$

Equating equations (4-26) and (4-28) we obtain

$$IZ_3Z_5\ (rp + Z_4) = EZ_4Z_5 \qquad (4\text{-}28)$$

Thus W_o is

$$W_o = \frac{E}{I} = Z_3 \frac{(rp + Z_4)}{Z_4} \qquad (4\text{-}29)$$

Equation (4-30) gives the same result for W_o as obtained before. However, in this case, the computation by the simplified method is more tedious than simply equating the gain to zero as in equation (4-10). Thus in using the simplified method on simple circuits, a certain amount of judgment must be used in selecting the method of calculation.

BIBLIOGRAPHY

Bode, Hendrik W. Network Analysis and Feedback Amplifier Design. D. Van Nostrand Company, New York, 1945.

Cruft Electronics Staff. Electronic Circuits and Tubes. McGraw-Hill Book Company, New York, 1947.

Gardner, Murray F. and Barnes, John L. Transients in Linear Systems. John Wiley and Sons, New York, 1942.

Guillemin, E. A. Communication Networks, Vol. 1. John Wiley and Sons, New York, 1949.

Guillemin, E. A. Communication Networks, Vol. 2. John Wiley and Sons, New York, 1949.

Guillemin, E. A. The Mathematics of Circuit Analysis. John Wiley and Sons, New York, 1951.