SOLUTIONS OF ENGINEERING PROBLEMS BY RELAXATION OF LINEAR MATRICES WITH THE DIGITAL COMPUTER

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James T. Anderson
Major professor

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SOLUTIONS OF ENGINEERING PROBLEMS BY RELAXATION

OF LINEAR MATRICES WITH THE DIGITAL COMPUTER

By

ELMER LOUIS LE BAY

AN ABSTRACT

Submitted to the School of Graduate Studies of Michigan State College of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

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Many problems in the fields of engineering and the applied sciences find their most direct and natural expression as a system of linear equations. Thus, there is a perpetual search for better and faster methods of solving them, and the combination of Southwell's Relaxation method with the IBM Calculating Punch Type 602-A appears to be a practical approach with many advantages. The relaxation method is inherently accurate, offers much in the way of flexibility and opportunity for computational skill, and is easily checked. The IBM 602-A Calculating Punch is a versatile machine, requiring a reasonable amount of set-up time, and is available at a moderate cost.

Southwell's Relaxation method is an iterative, successive approximation process that works especially well with diagonal matrices (diagonal terms large compared to the other coefficients). Fortunately, most physical problems take the form of an incomplete, diagonal matrix which is a type well adapted to solution by the computing machine.

The IBM 602-A Calculating Punch is one unit of a wide variety of electrically operated machinery designed for use with punched cards. It is operated by a plug-in wired electrical control panel, and can read data punched in cards, automatically perform one or more calculations (addition, subtraction, multiplication, division), and automatically punch the results.

It appears that the break-even point, when machine relaxation is compared with hand relaxation, and with some other methods of solution with the aid of an ordinary desk calculator, would be somewhere in the neighborhood of matrices of order 10. Skill at machine relaxation will do much to hasten convergence of the solutions, and there is considerable opportunity for the individual relaxer to employ technique and ingenuity to speed convergence.

Several improvements have been devised during the development of this process, one of which is the use of two six-pole 12-position gang switches to progressively shift the various reading and punching fields on the punched cards. It is hoped that this is the beginning of a better method for solving large linear matrices.

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INTRODUCTION

Many problems in the fields of the engineering sciences find their most direct and natural expression as systems of linear equations. Among these are plate stresses, structural frameworks, heat conduction networks, and mass spectrography.

Large systems of linear equations (order ten and greater) present a serious challenge to the engineer and the mathematician. Often, the accuracy of the final results is questionable due to the growth of rounding error and other forms of inaccuracy. Thus, there is a perpetual search for better and faster methods of solving these matrices.

Automatic computing machines offer new approaches to the solution of these problems, and many methods of solving linear matrices have been developed. The combination of the digital computer and Sir Richard Southwell's Relaxation method seems to provide an inherently accurate method, and one which is easily checked for errors.

RELAXATION

The first application of relaxation methods was made in 1935 by Sr. Richard Southwell, F.R.S., then Professor of Engineering Science in the University of Oxford. Since then he has been energetically and almost continuously devoted to its growth as a powerful computational weapon for the solution of engineering and physical problems.

Originally, it was developed as an iteration process to be applied directly to the solution of pin-connected and stiff-jointed structural framework problems, heat conduction networks, and many other types of physical problems. However, it was found to be a very versatile device in the realm of mathematical science.

Perhaps the most widespread use of the relaxation method has been to obtain particular solutions of partial differential equations in two dimensions. It is, also, important in the solution of a wide variety of eigenvalue problems, and linear simultaneous equations.

As applied to linear simultaneous equations relaxation is an iterative successive approximation process. A linear system of equations may be written:

$$R_1 = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1j}X_k + C_1$$

$$R_2 = a_{21}X_1 + a_{22}X_2 + \cdots + a_{2j}X_k + C_2$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$R_n = a_{i1}X_1 + a_{i2}X_2 + \cdots + a_{ij}X_k + C_n$$

where $i,j,k,n = 1,2,3, \ldots$

The functions have calculable values, called residuals $R_1 R_2, \ldots R_n$, for any values of the variables X_k. To solve the matrix it is necessary to find the particular values of the variables which make all of the residuals zero or approximately so.

Relaxation is begun by selecting an initial set of values for the variables. Ordinarily, these values are taken to be zero, residuals equal to the constant terms C_n, unless there is some definite reason otherwise. For instance, if all constant terms were exactly zero, one might well begin with the variables equal to one, residuals equal to the summation of the coefficients a_{ij}.

BASIC RELAXATION

The procedure is then to successively reduce the currently largest residual to zero, by successive relaxation of the variables. It can immediately be seen that if variable X_1 is relaxed by one, it would change residual one R_1 by a_{11}, residual two R_2 by a_{21}, and the last residual R_n by a_{i1}. Thus, an operations table may be constructed for the unit relaxation of each variable using the coefficients of the variables.

OPERATIONS TABLE

operation	R	R		R
$X_1 = 1$	a_{11}	a_{12}	. . .	a_{1j}
$X_2 = 1$	a_{21}	a_{22}	. . .	a_{2j}
. . .	. . .	. . .	. . .	
$X_k = 1$	a_{i1}	a_{i2}	. . .	a_{ij}

Relaxation proceeds by taking increments of the variables called <u>operators</u>, which are multiples of the respective unit operations; progressively liquidating all of the residuals to approximately zero.

The relaxation process is accomplished by means of a relaxation table which is a running account of the residuals and the increments of the variables.

RELAXATION TABLE

operators	R_1	R_2	R_n
$X_1 \cdot \cdot X_n = 0$	C_1	C_2	C_n
$X_1 = m$	$ma_{11} + C_1$	$ma_{12} + C_2$	$ma_{1j} + C_n$
$X_n = h$			

A summation of the operators gives the desired solutions.

EXAMPLE OF RELAXATION

Solve to two decimal places the pair of equations

$$- 9X_1 + 4X_2 + 74 = 0$$

$$3X_1 - 8X_2 + 129 = 0$$

The operations table is

	R_1	R_2
$X_1 - 1$	$- 9$	3
$X_2 - 1$	4	$- 8$

The relaxation table is

	R_1	R_2
$X_n = 0$	74	129
$X_2 = 16$	138	1
$X_1 = 15$	3	46
$X_2 = 5$	23	6
$X_1 = 2$	5	12
$X_2 = 1$	9	4
$X_1 = 1$	0	7
$X_2 = .8$	3.2	.6
$X_1 = .3$	.5	1.5
$X_2 = .2$	1.3	- .1
$X_1 = .1$	.4	.2
$X_1 = .04$	.04	.32
$X_2 = .04$	.20	0
$X_1 = .02$	.02	.06
$X_2 = .01$	- .06	- .02
$X_1 = .01$	- .03	.01
$X_1 = 18.47$ $X_2 = 23.05$	- .03	.01

The last step is the checking operation.

CHECKING AND ERRORS

The solutions are checked very simply and easily by substituting into the equations. If theresiduals found in this manner differ from those resulting from the last operation, there is an error in the relaxation process.

One of the important advantages of the relaxation method is that it is not critically dependent upon arithmetical accuracy in the computation. It is unnecessary to discover a mistake. Merely accept the correct residuals, enter them into the relaxation table and continue the liquidation process.

CONVERGENCE

Convergence is the rate at which the summation of the increments approaches the solutions. In general, matrices throughout which the magnitudes of the coefficients are well scattered are slow to converge. Therefore, it is highly desirable that the matrix, and thus the operations table, be symmetrical about the leading diagonal. That is, the coefficients a_{11}, a_{22}, a_{33}, ... are larger than the other terms in the matrix.

It is fortunate, indeed, that most engineering problems and most physical problems are of this form, however, there are acceptable methods of converting an unsymmetrical matrix.

OVERRELAXATION

Often the unit operations are such that when used to reduce one residual, another is automatically increased. Much work can be saved in this situation by anticipating the reaction, since this property is a hindrance to convergence. Knowing that a later step is going to automatically increase the magnitude of a residual, it is apparent that there would be advantage to be gained from over-shooting zero.

This device is known as overrelaxation. As a general rule, the overrelaxation should not exceed one and one-half times the relaxation required to reduce the residual to zero. However, a judicious use of overrelaxation may be used to advantage in speeding convergence.

Repeating the sample problem using overrelaxation demonstrates its utility in reducing the number of operational steps. The relaxation table is

	R_1	R_2
$X_n = 0$	74	129
$X_2 = 20$	154	- 31
$X_1 = 18$	- 8	23
$X_2 = 3$	4	- 1
$X_1 = .5$	- .5	.5
$X_2 = .06$	- .26	.02
$X_1 = - .03$	.01	- .07
$X_2 = - .01$	- .03	.01
$X_1 = 18.47$ $X_2 = 23.05$	- .03	.01

There are other variations of the relaxation method which do not appear to be applicable here, one of which would be useful in the eventuality of a divergent matrix. However, divergent matrices are rarely, if ever, found in practical problems.

IBM PUNCHED CARD MACHINES

The International Business Machines Corporation manufactures a wide variety of machinery which operates with standardized punched cards. Among these are the keypunch, the reproducing punch, and the 602A computer. These machines were originally designed for accounting work, but lend themselves readily to many other types of determinations.

IBM CARDS

IBM cards measure 3-1/4 inches by 7-3/8 inches with the upper surface divided into 80 columns and 12 rows. The twelve rows are designated 0 down through 9 at the bottom of the card, and the X or 11 row with the 12 row above it at the top of the card. Ordinarily the 12 row is not used, the X row is for code or minus sign, and the others are for digits, one punch to a column, although an X punch may occur in a column with a digit punch.

Adjacent columns are often grouped into fields of several columns. To punch a five-digit number requires a field of five columns with an X punch somewhere in the field if it is a negative value.

KEYPUNCH

The keypunch is an electrically operated machine for the punching of IBM cards. It has a keyboard similar to that of a typewriter, and automatically feeds the cards through as they are punched. A program card is used to control the passage of the blank cards through the punching position. It is mounted on a rotating cylinder, and punching position is synchronized with the rotation of the program card past the reading brushes. An X-punch in the program card triggers skipping which continues by means of 12-punches until an unpunched column is reached which halts skipping for punching to begin. After punching takes place, skipping may be repeated or used to skip the card out of the machine.

REPRODUCING PUNCH

The reproducing punch, commonly known as a reproducer, automatically reproduces a punched card in its entirety or in part into any of the columns ofthe blank c ard. A plug-in wired control panel controls the operation of the machine. Cards to be reproduced are fed from one hopper, and blank cards from a nother. The machine stops automatically when the last card is fed in, requiring the start key to be depressed until it has been run out.

CALCULATING PUNCH TYPE 602A

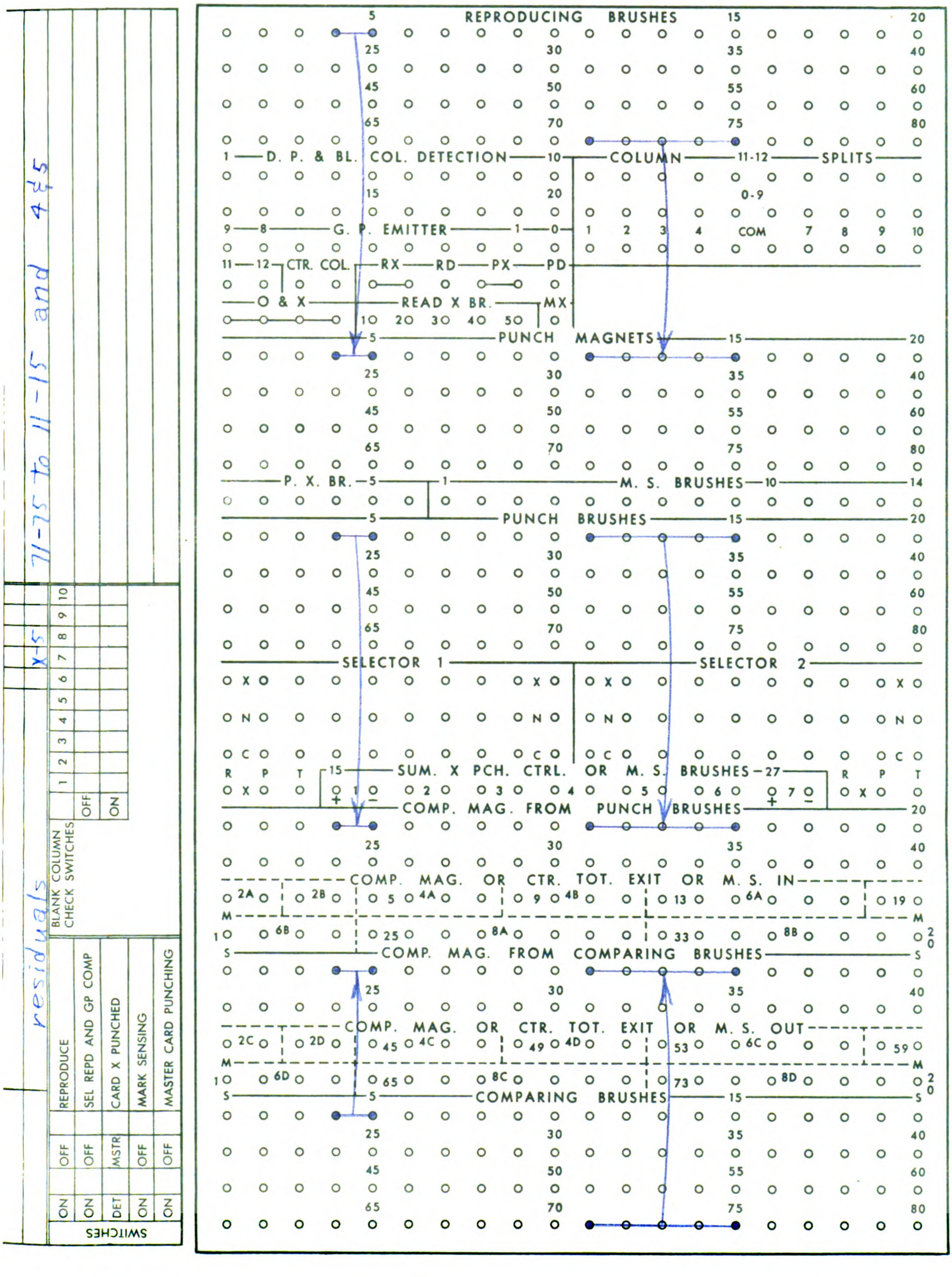
The 602A calculating punch r eads data from punched cards, automatically performs one or more calculations, and punches the results in the last data card or cards that follow. This machine performs multiplications, divisions, additions and subtractions singly or in various combinations. Some of the operations may occur simultaneously as for example, calculating

IBM
TRADE MARK

INTERNATIONAL BUSINESS MACHINES CORPORATION

AUTOMATIC REPRODUCING PUNCH, TYPE 513–514 CONTROL PANEL

FOR SUMMARY PUNCHING – ALPHABETIC ACCOUNTING MACHINE



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DEPT. ______ NO. ______

NAME ______ USE ______

and punching.

The machine also has a memory system permitting it to store data for later operations or group calculations. Thus, factors read from the first of a group of cards may be used for calculations involving each of the following cards. Positive and negative number conventions are maintained throughout any calculation.

OPERATION OF CALCULATOR

The 602A calculator has an electrically wired control panel through which all calculations and operations are triggered and sequenced. Some experience is necessary in programming the machine operations and wiring of the control panel, although it is not difficult to wire a control panel from the wiring diagram. The calculator is wired internally for X pickup and has a limited capacity of 20 positions which may be connected for any 20 of the 80 columns.

Punching placement is controlled by an adjustable skip bar. It contains 80 slots, one for each card column, in which a small insert is placed wherever punching is to begin.

After the cards are placed in the feed hopper the machine is started by depressing the start key. The stop key halts the machine instantly even when in the middle of a calculation, however the calculation may be rèsumed without interruption by starting the machine again. If at anytime the calculations are unable to continue due to incorrect card punching or control panel wiring, the machine stops automatically, and the comparing light flashes on. The machine will start again after the reset key has been depressed. Cards held within the machine at any time may be run out by opening the control panel cover and starting the machine.

SORTER AND COLLATOR

Additional machinery which is often used with IBM punched cards is the sorter and collator. The sorter separates a pack of cards according to the punchings a column at a time. By this method it may be used to arrange a pack of cards in numerical order if they are serially numbered in one field. The column representing the lowest numerical magnitude is sorted first, followed by sorting on the other columns in their order, stacking them in ascending order from top to bottom after each sorting operation.

The sorter may also be used to find the largest value of a group in a single field by sorting the column of greatest numerical magnitude first, and resorting only the pack with the highest numerical punches each time.

The collator combines two packs of cards in any specified order. This machine is operated by an electrically wired control panel the same size the reproducer panel which is half the size of the 602A calculator control panel. It would be useful only for the larger matrices.

MACHINE RELAXATION

Considerable planning is necessary before a matrix can be solved with the IBM Calculating Punch type 602-A. The use of the limited card space must be planned, and the use of the limited number of X pickups must be arranged. Then the control panel planning chart should be filled out, and the wiring arrangement sketched in on the control panel diagram. After the control panel has been wired and the cards have been punched relaxation can be carried out on the machine.

The calculation may be formulated as follows:

$(\pm A) \times (\pm B) \quad (\pm C) = (\pm D)$ where,

A is the operator (group multiplier)	code:	X in col. 1
B is the operation	"	X in col. 3
C is the current residual	"	X in col. 5
D is the new residual	"	X in col. 5

CARD PLANNING

Card planning is an important part of putting any computation on the 602-A calculator especially if the entire card is to be utilized. In this instance the operators, operations table and the residuals must be converted to punched cards. On the planning card five columns have been assigned to code and serial numbering, while the remainder of the card is divided into fields of five columns for punching five digit values. Of course, only the necessary code and numbering columns, and the necessary numerical fields would be used on any one card.

X-80 neg.		#14
75		#13
70		#12
65		#11
60	Master Planning Card	#10
55	operator X-1 operation X-3	#9
50	residual X-5	#8
45		#7
40		#6
35		#5
30		#4
25		#3
20		#2
15	residual value	#1
X-8 neg.	operator-operation value	
	residual no.	
	operation no.	

The columns have been designated as follows:

Col. 1	X for operator code
Col. 2 - 3	X in 3 for operations code operations serial numbers
Col. 4 - 5	X in 5 for residual code residual serial numbers
Col. 6 - 10	Operator or operations values X in 8 negative
Col. 11 - 15	Residual field number 1 X in 15 negative
Col. 16 - 80	Residual fields 2 through 14 X in last column of field negative

The internal wiring for X pickup would be for columns 1, 3, 5, 8, 15, 20, 25, 30, 35, ... , 80.

CARD FILES

A standard file of operator cards, both positive and negative, is adequate for most relaxation problems. The file would contain the values:

.001, .002, .003, ...009

.01, .02, .03, ...09

.10, .15, .20, .25, ...95

1, 1.5, 2, 2.5, 3, ... 9.5

11, 12, 13, 14, ...20

Other operator values, if needed, can be punched up as required.

The file of operations cards contains (n) packs with (n) cards in each pack; that is, one pack for each operation, and within each pack one card to relax each residual. These cards must be serially numbered and kept in their proper order.

There is one pack of (n) residual cards, and these too must be kept in their proper order. At he outset, the initial residual will appear in residual field number 1, columns 11 - 15, and the new residuals will be punched progressively across the card until the last field is reached, and the card is reproduced.

COMPUTER CONTROL PANEL

Considerable knowledge of the calculating machine and the operations controls section of the control panel is necessary to understand the functioning of the machine. However, the machine operator need only be familiar with the reading and punching of residual values for the relaxation process.

In reading, five wires are used to read-in the value of the residual, and one for the X (negative) pickup. For the first step the five are plugged into reading hubs (holes) 11 - 15, and the X pickup is in control reading hub 6. As relaxation progresses the wires for reading must be moved, between calculations, from the reading hubs for the current residual field to the reading hubs for the next residual field. The negative pickup wire must also be moved along from control reading hub 6 for residual field 1 to the corresponding hub for the position of the reading wires. This wire is moved along one hub at a time.

Five punching wires must also be indexed along between calculations, followed by a skip control wire. For the first step the punching wires would be plugged into punching hubs 16 - 20 for residual field number 2, followed by the skip wire in hub 21. The skip wire is necessary for the residual card to

APPLICATION ______________________

PROBLEM $(\pm A) \times (\pm B) + (\pm C) = (\pm D)$

IBM TRADE MARK

PLANNING CHART: TYPE 602A CALCULATING PUNCH WITH 8 COUNTERS & STORAGE UNITS

PROG. STEP	OPERATION	STORAGE UNIT 1L	STORAGE UNIT DIVR.-MULT. 1R	COUNTERS DIVIDEND 1	COUNTERS DIVIDEND 2	COUNTERS DIVIDEND 3	4	5	6	7	8	STORAGE UNITS 2L	2R	3L	3R	4L	4R	5L	5R	PUNCH UNITS 6L	6R *	7L	7R *	8L
	READ CYCLE		RI X-1 Group mult (A) 6-10	X-5 NX-5	NX-15 + X-15 − 11-15 (C)							RI X-3 (B) 6-10												
1	Mult			+ + + − − + − −	+ − − +							RO											RI Punch (D)	
2					RO RE																			
3																								
4																								
5																								
6																								
7																								
8																								
9																								
10																								
11																								
12																								

*UNITS POSITION MUST BE WIRED TO PUNCH

X-1 skip out go to read
X-3 " " " " "

Pilot selector 1, picked up by X in col. 1 in operator card from control brushes
Pilot selector 2, picked up by X in col. 3 in operation card " " "
Pilot selector 3, picked up by X in col. 5 in residual card " " "
Pilot selector 4, picked up by X for negative residual value " " "
Pilot selector 5, picked up by X for negative operator value from read brushes
Pilot selector 6, picked up by X for negative operation value " " "

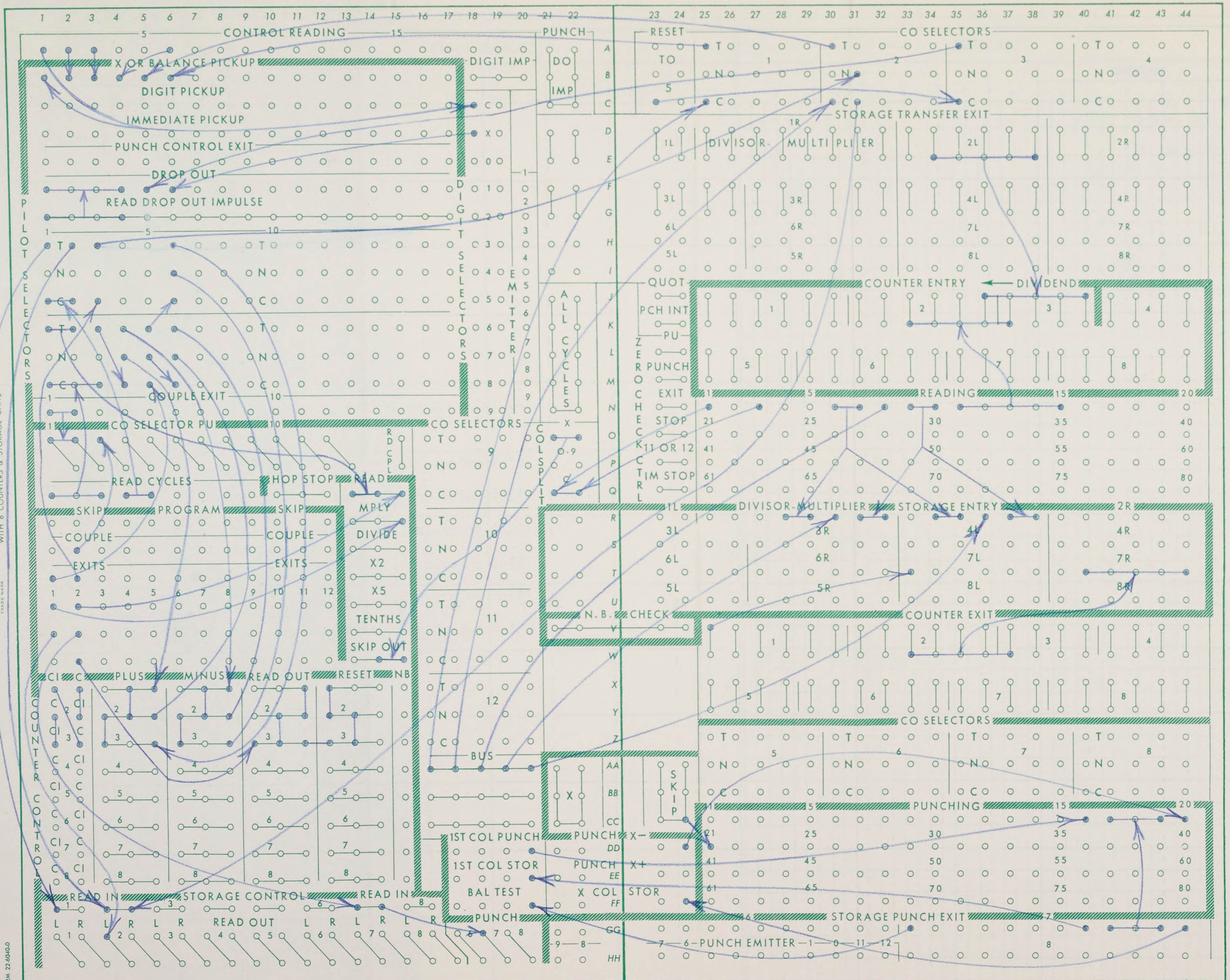
INTERNATIONAL BUSINESS MACHINES CORPORATION

IBM CALCULATING PUNCH

TYPE 602 A CONTROL PANEL

WITH 8 COUNTERS & STORAGE UNITS

PRINTED IN U.S.A.



ELECTRO NO.	CARD NAME OR FUNCTION	X OR D CODE	NOTES:
	Operator	X-1	± Group multiplier
	Operation	X-3	± multiplicand
	Residual	X-5	± addendum

Name ______ Dept. ______

Use ______ No. ______

FORM 22-6040-0

be skipped out of the machine and the next calculation to begin. These wires are moved along until they reach field number 14 where the skip wire is plugged into punching hub number 1, and field number 14 must be reproduced into field number 1 in a new pack in order to continue.

Although reading and punching is ordinarily done in adjacent fields, this is not essential to the operation of the machine. If for any reason a field is incorrect, it may be by-passed by leaving the reading wires where they are, and moving the punching wires to the next field. Of course, the reading wires must also by-pass the incorrect field for the following step.

If the reading and punching circuits are wired through two six pole gang switches, moving from field to field can be accomplished by a flick of the switches. Six pole, 12 position radio gang switches have been found to be ideal for this purpose. Field 14 is not used. The reading switch is wired for fields 1 through 12, and the punching switch is wired for fields 2 through 12. The punching hubs that are also used for skipping must be double wired to the gang switch.

The wiring arrangement and programming used here is simple and functionally satisfactory, however it is not singularly unique. Many of the control circuits could be arranged differently, but a skilled operator would be required to make the changes. Sometimes, due to malfunctioning of certain controls within the machine it is convenient to make minor changes in the control panel wiring, and use other control circuits.

MECHANICS OF MACHINE RELAXATION

Relaxation as put to the computer is in a sense an imitation of the hand process, and is not fully automatic. The largest residual must be discovered in some manner whether by inspection or by sorting. The relaxer would then select the proper operation and operator, and collate the packs of cards representing these quantities by hand or by machine. After the cards have been run through the calculator they must be separated again by hand or by machine. This process would be repeated until the variables were determined to the desired accuracy keeping a running account of the relaxation accomplished in each step.

The reading and punching wiring must also be changed between steps, and the skip stop in the skip bar must be inserted in the slot corresponding to the column where punching is to begin. Two skip bars may be used for convenience, since they must be removed from the machine to move the skip stop.

A relatively efficient sequence of events for one step in the machine relaxation process is:

Find the largest residual

Choose operator value

Enter data into log

Collate operator and residual cards

Place operator card at front of pack

Insert pack into feed hopper

Change reading and punching circuits

Change skip stop or skip bar

Begin calculation, and move skip stop in the
extra skip bar if one is being used

Remove operator card from pack after calculation

Separate residual and operation packs

Check computation on the largest residual

When all residual fields have been punched, the residual pack must be reproduced in order to carry on with the relaxation process.

For the larger matrices, visual scanning of the cards for the largest residual, and hand collating and separating of the operations and residual packs will become too tedious and time consuming. Depending upon the character of the matrix, the skill of the relaxer, and the time required to do these operations by machinery, there will be an optimum size matrix or point of division between the two methods.

As the cards have been set up, the values of the various factors are limited to five digits, however, this should be adequate for most problems in view of the fact that the accuracy of the coefficients in the original matrix will generally not exceed five places. The calculator has no sense of decimal location allowing the relaxer to assign it anywhere within or outside of the fields as long as all fields are alike in operator, operations and residual cards.

DISCUSSION

Methods of solving simultaneous linear equations may be conveniently divided into two groupings: (1) direct methods, and (2) indirect methods. Direct methods are usually lengthy but yield results of considerable accuracy if a sufficiently large number of significant figures is retained throughout. Solution by determinants, and solution by elimination as in the Gauss process, Cholesky scheme and the Jordan process are examples of the direct method. Indirect methods are successive approximation in nature, and the approximate solutions become progressively more accurate as the process is continued. The Gauss-Seidel Iteration process and the Southwell Relaxation process are two of the better known indirect methods.

The solution of a series of algebraic linear equations of order $\underline{n}$, when $\underline{n}$ is small, is usually done by one of the direct methods. However, as $\underline{n}$ increased the time consumed by these methods increases roughly with n^3.(5). Thus, as the order n of a matrix approaches 10, the time and labor required become prohibitive, and solution impractical by these methods.

For certain types of matrices in which the diagonal terms are large compared to the other coefficients in each equation (diagonal matrix) the successive approximations methods can be employed. Fortunately, these systems of linear equations are among the most important and most frequently encountered problems in applied mathematics. Although the time required for successive approximation methods is also considerable, it increases roughly with $\underline{n}^2$.(5). It can be seen immediately that the time required for relaxation solution of algebraic simultaneous linear equations as compared with direct methods varies inversely with n. Thus, relaxation appears to have a considerable advantage over the direct methods.

The main virtue of the relaxation method is its freedom from rigid routine and it would seem to have much more flexibility and opportunity for computational skill than any of the other successive approximation methods. The mathematical skill and the physical intuition of the computer may be used in many ways to accelerate convergence, perhaps reducing the proportional time for solution a preciably below n^2.

The time required for setting-up a problem to be solved with the punched card computer is somewhat more than for hand solutions with or without the aid of the ordinary desk calculating machine. It can easily be seen that to convert the operations table to punched cards requires the punching of n^2 operations cards. A pack of n residual cards must also be punched out. However, a skilled keypunch operator can punch from 10 to 30 cards per minute. Since,the set up time varies as $n^2 + n$ or approximately n^2 with the higher values of n, there will be a value of n above which it would be impractical to use this application of the relaxation method.

Fortunately, most physical problems take the form of an incomplete matrix in which the more distant terms from the diagonal are zero. Since it is unnecessary to have a card for zero operations values, the theoretical number n^2 of operations cards may be reduced to the order of $1/2\ n^2$. This also reduces the number of cards to be collated and separated which, in effect, raises the value of n for which the use of the collator and sorter becomes expedient.

It is suggested that the abbreviated pack of operations be made continuous. That is, a card with the code X in column 3 should be inserted for any zero values which occur within the group of terms making up the abbreviated operations pack. This will simplify collating and aid in preventing errors. Of course, all of the residual cards, including those for which there are no operations cards, must be contained in the pack run through the calculator in the order that the residuals in these cards be reproduced into the new field. When computing is to be carried out with the abbreviated operations packs, the calculator should be cleared previous to any computations. This need only be done once, and may be accomplished by running the clear card through separately or including it as the first card of the first pack. The clear card should have X punches in columns 1, 3 and 5.

The time required by the calculator to make the computations is about 3 seconds for each operation. Thus, the time elapsed during one pass of the cards through the machine would be something less than 3n seconds or about 1 minute for each twenty residuals. However, the machine time is a small portion of the total time required to solve a series of equations. Therefore, a more rapidly cycling machine would not appreciably reduce the total time necessary to work out acceptable solutions.

A matrix in its initial state is often not in a good form to be converted into the operations table. It is sometimes

convenient to adjust the diagonal terms to the same relative magnitude or convert them all to a value of one. This, of course, can be accomplished by multiplying or dividing through any of the equations of the series by a required constant. Reducing the diagonal terms to a value of one will most likely result in some of the other coefficients becoming continuous decimals which must be rounded off to five digits. The errors thus introduced can be eliminated at any time by computing the residuals with the accomplished approximations and the original coefficients; and proceeding with the relaxation process by operating on the corrected residuals mentioned above.

In the instance of a diagonal matrix (diagonal terms large compared to the other coefficients) the terms farthest removed from the diagonal terms may be quite small in comparison. In the operations table these small values have little effect on the relaxation process, and it may prove expedient to assume them to be zero. Thus, the number of cards which must be punched for the abbreviated operations pack is further reduced, reflecting a corresponding reduction in the number of cards to be collated and separated. Errors introduced in this manner can be eliminated at any time by <u>checking</u>, and replacing the erroneous residuals by the correct ones as was done in eliminating rounding off errors.

Perhaps the greatest advantage of relaxation as carried out with the automatic computer is its overall relative freedom from arithmetical error. Relaxation is fundamentally an iteration process, and when carried out with pencil and paper involves columns and rows of digits which appear to the human mind as an endless stream of unrelated numbers. It may be monotonous and hypnotic to the extent that the human computer becomes quite susceptible to arithmetical errors, and although the errors can be discovered it would take a series of steps to correct for them. Machine computation should contribute nothing in the way of arithmetic errors, excepting for the case where erroneous data has been submitted to the calculator. In general, errors in machine solution can be detected by checking the relaxation performed on the largest residual as a regular part of each step in the relaxation process.

Due to the great variance of convergence rates among linear matrices, it is very difficult to make any concrete comparison of the solution times required for various sizes of matrices. An artificial, perfectly symmetrical series of matrices was devised in which the symmetrically corresponding coefficients and the constant terms were identical in each equation. The non-zero terms were 1,3,7,3,1 and the constant term had a value of 1. The solutions of each proved to be symmetrical as did the relaxation tables, however, the convergence rates do not appear to be the same, and thus they could not be used as a basis for comparison.

Some comparison of the times required for machine solution, and for pencil and paper solution with the aid of a slide rule can be made. A symmetrical matrix of order 8 representing a stress network was solved. It took approximately 6-1/2 hours to solve with the computer, and approximately 4 hours to solve with hand relaxation. In general, it appears that the break-even point between hand relaxation solution and machine solution would be somewhere in the neighborhood of matrices of order 10.

The same matrix was also solved by the Gauss method with the aid of a desk calculator. This method took approximately 6-1/2 hours which is the same time as for solution with the automatic computer (relaxation method), however, set up time would increase the total time for the punched card computer method. In general, it appears that the break-even point between these two methods would also be in the neighborhood of matrices of order 10.

CONCLUSIONS

The application of the IBM 602A Calculating Punch to relaxation methods for solving algebraic simultaneous linear equations appears to have several distinct advantages. This machine is the "work-horse" of automatic digital computing machinery, and despite the development of faster and more complete-ely automatic computers, it is yet the most versatile and easily adapted machine. It has become relatively available at a moderate cost as compared to the ultra high speed computers, and therefore is no longer out of reach of the lesser financially endowed institutions. As a general purpose machine in the fields of accounting and miscellaneous calculations, the cost of the machine may easily be justified by the saving of labor costs it can produce.

Of all the automatic digital computing machinery available today, the IBM 602A Calculating Punch probably requires the least skill and knowledge of its operation. Much of the later computing machinery is impractical for many types of problems due to the prohibitive set-up time required. Even then, cost and availability put them beyond the reach of all but the wealthiest groups.

Relaxation solutions of engineering and other physical problems need be carried no further than would be indicated by the accuracy of the data from which the matrices have been constructed. Skill at machine relaxation will do much to hasten convergence of the solutions to the required accuracy, and there is considerable opportunity for the individual relaxer to employ every technique he has discovered to speed convergence. This method offers a practical solution to problems which have been heretofore shunned by mathematicians and engineers.

Considering the availability at moderate cost of the IBM 602A calculator, the inherent accuracy of relaxation methods, and the opportunity for the individual ingenuity of the operator, here is perhaps the beginning of a better method of solving large linear matrices.

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