EXISTENCE, CONSISTENCY, AND OBLIQUE DISCOURSE

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EXISTENCE, CONSISTENCY, AND OBLIQUE DISCOURSE

By

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AN ABSTRACT

This thesis develops a system of logic containing both modal operators and quantifiers. This system contains C. I. Lewis' system S4, but it does not contain restrictions of type theory. In it, certain departures from similar systems of Ruth Barcan, Rudolf Carnap, and Frederic Fitch are proposed. One such departure is the inclusion in the system of a notation for singular existence as this has been done by Henry Leonard.

The thesis also includes an outline of a second system which is an attempt to codify G. Frege's notion of the oblique occurrence of a term in a context. This system is applied to a treatment of the paradoxes of the theory of types, in order to justify abandoning type theoretical restrictions in the first system.

ACKNOWLEDGEMENT

The author wishes to express grateful thanks to Professor Henry S. Leonard, Head of the Department of Philosophy, under whose guidance this thesis was written. His courses in logic, and especially, his recent paper, "The Logic of Existence", have been the source of this thesis.

I wish also to thank Dr. Richard S. Rudner, Assistant Professor of Philosophy, for his many discussions of various positions that are maintained here.

CONTENTS

CHAPTER I

INTRODUCTION

1.1 Professor Henry S. Leonard has recently published a paper entitled "The Logic of Existence"[1] which modifies the logical system of *Principia Mathematica*[2] in order to deal with questions of existence of which there is no treatment in the latter system.

The alterations in logic proposed by Professor Leonard consist, in part, in a notation for singular existence which takes variables as arguments, recognition of certain laws governing existence, not expressible in *Principia Mathematica*, and the introduction into logic of terms which do not denote.

The following two systems are based upon Professor Leonard's paper.

1.2 Throughout his paper, "The Logic of Existence", Professor Leonard calls attention to the importance of considering the topic of modal logic and its bearing upon questions of existence. This emphasis of Professor Leonard's paper has influenced the following formulations in many ways.

In the first of the following systems, existence is not, as in Professor Leonard's system, introduced by

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1. Henry S. Leonard, "The Logic of Existence," *Philosophical Studies*, Vol. VII, Number 4, (June 1956), pp. 49-64.

2. Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, The Cambridge University Press, First Edition 1910.

definition. It is rather taken as primitive. The postulates of System I are used to characterize this primitive.

The considerable bearing which modal logic has upon existence is illustrated in System I in that, only in a modal system such as System I, can a primitive "existence" be adequately characterized. Had System I been an entirely material logic, the resulting system would not have been sufficiently rich in connections with the primitive "existence" to have specified the interpretation intended for it.

One such connection, between deducibility and existence, which is called to our attention in "The Logic of Existence", consists in the invalidity of the following argument:

Santa Claus lives at the North Pole. (1)

∴Someone lives at the North Pole. (2)

The argument from (1) to (2) is invalid because, in addition to premise (1), a premise to the effect that Santa Claus exists is required in order that (2) might be inferred.

Because of this consideration, System I contains only the formula:

$$fx.E!x \prec (\exists y)fy \qquad (3)$$

rather than the stronger:

$$fx \prec (\exists y)fy \qquad (4)$$

Systems of modal logic which leave existence out of account and contain the invalid formula (4), contain untrue theorems such as:

$$\sim\Diamond\sim(\exists x)(fx \vee \sim fx) \qquad (5)$$

Difficulties occasioned by results such as (5) in systems of quantified modal logic, like those of Ruth Barcan Marcus[3] and Rudolf Carnap,[4] have caused much controversy among logicians.

Unlike Mr. Leonard's system, System I contains as a law, the formula: "E!x", and thereby also contains the restriction that only terms which denote are allowed as suitable for substitution. For this reason, System I is so to speak, a logic of denotation.

A material logic which, unlike Mr. Leonard's system, does not introduce terms that do not denote, does not require a notation for singular existence. It will be maintained in what follows, that a modal logic requires consideration of singular existence even though that logic does not allow of terms that do not denote.

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3. Ruth C. Barcan, "A Functional Calculus of the First Order Based on Strict Implication." The Journal of Symbolic Logic, vol. 11 (1946) p. 1.

________________, "The Deduction Theorem in a Functional Calculus of First Order Based on Strict Implication", The Journal of Symbolic Logic, vol. 11 (1946) p. 115.

________________, "The Identity of Individuals in a Strict Functional Calculus of Second Order", The Journal of Symbolic Logic, vol. 12 (1947) p. 12.

4. Rudolf Carnap, "Modalities and Quantification", The Journal of Symbolic Logic, vol. 11 (1946), p. 33.

________________, Meaning and Necessity, The University of Chicago Press, 1947.

1.3 To a certain extent, System II is based upon a criticism of "The Logic of Existence." However, before considering this criticism, it might be well to review certain traditional difficulties concerning existence.

Parmenides made claims to the effect that everything that we believe in or speak of must exist; or put in other words, we cannot believe in or speak of a thing that does not exist. A fair sample of such doctrines is to be found in Plato's The Sophist:[5]

> Stranger. The truth is, my friend, that we are faced with an extremely difficult question. This "appearing" or "seeming" without really being, and the saying of something which yet is not true--all these expressions have always been and still are deeply involved in perplexity. It is extremely hard, Theaetetus, to find correct terms in which one may say or think that falsehoods have a real existence, without being caught in a contradiction by the mere utterance of such words.
> Theaetetus. Why?
> Stranger. The audacity of the statement lies in its implication that "what is not" has being; for in no other way could a falsehood come to have being. But my young friend, when we were of your age the great Parmenides from beginning to end testified against this, constantly telling us what he also says in his poem:
>
> > 'Never shall this be proved--that things that are not are; but do thou, in thy inquiry, hold back thy thought from this way.'
>
> So we have the great man's testimony, and the best way to obtain a confession of the truth

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5. Plato, The Sophist, 236D-237B.

> may be to put the statement itself to a mild degree of torture. So, if it makes no difference to you, let us begin by studying it on its own merits.

In System I, a theorem affirms that everything we speak of exists, or that it is impossible to speak of a thing that does not exist.

It can be proved that:

$$(x)E!x \supset (x)(aSx \supset E!x) \tag{6}$$

(where 'xSy' abbreviates 'x speaks of y')

and further that:

$$(x)E!x \tag{7}$$

and thereby:

$$(x)(aSx \supset E!x) \tag{8}$$

Since it can also be obtained that (8) is analytic, it follows that it is impossible, in the sense of inconsistency, that anyone speaks of a thing that does not exist.

However, Parmenides' injunction may rather be to the effect: "Do not speak with terms that do not denote". Putting the rule in such a terminology of mention rather than of use changes the rule from a necessarily true, and hence unbreakable one, to one which is breakable, and in fact is broken.

Thus for instance, we might ask, is it true that "Santa Claus wears a red suit"? Or even, is it true that "Santa Claus does not exist"? Apparently, each of these sentences is not true, since were they to be true, the term

"Santa Claus" must denote something having respectively the properties of wearing a red suit and of not existing.

The point of the second of "Parmenides' rules" would then seem to be to prevent us from asserting sentences which must be automatically untrue because they contain terms which denote nothing.

Sensible though this second rule seems, the acceptance of it raises a particularly vexing problem of how an assertion of non-existence can be true, since such an assertion seemingly must break the rule if it is to be true.

1.4 The last difficulty was left unresolved. However, before proceeding to a discussion of any of the several ways of resolving it that will be recognised here, a way of avoiding it will be examined that will not be followed here.

This way of avoiding the difficulty might be called "the logic of possibles".

This approach will recognise some things that, while they do not exist, nevertheless are at least "possibles", and claim that every term denotes something and that terms such as "Santa Claus" merely denote possibles rather than actuals.

"Possible logic" will recognise two systems of quantifiers. Square brackets might be adopted as a notation for generalization in an inclusive sense over both possibles and actuals: "[x]fx" to mean: f is true of everything, possible or actual. The more usual sort of generalization

over everything that exists might be defined:
$(x)fx =Df [x](E!x \supset fx)$. A weak form of existential generalization might be defined as: $[\exists x]fx =Df \sim[x]\sim fx$. The usual strong form of existential generalization might be given the definition: $(\exists x)fx =Df [\exists x](E!x.fx)$.

Mr. W. V. Quine has discussed this problem and criticised the position of "the logic of possibles" outlined here.[6] He makes the well taken point that while "existence" is a free word and hence there is nothing to stop us from using it in such a way as to apply to only a special class of things rather than to everything unrestrictedly, to do so is nevertheless to take away the word's usual meaning.

However, it is not only the case that the "logic of possibles" takes away the meaning of the term "existence," but it takes away the meaning of quantifiers as well. In fact the "logic of possibles" reinterprets the whole of logic as applying to only a restricted class of things, with a new set of quantifiers and kind of existence for the things that are left over. In short, an objection to "the logic of possibles" is that what was meant by parenthetic quantifiers in the first place is the meaning which the "logic of possibles" gives to bracket quantifiers, after the parenthetic quantifiers have been suitably misinterpreted.

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6. Willard Van Orman Quine, "On What There Is", in, From A Logical Point of View, Harvard University Press, 1953.

The logic of possibles achieves its extension of the usual logic by misconstruing that logic in a narrow way.

1.5 A clue to a resolution of the problem of how there can be true statements of non-existence is to be found in one standard means of making such statements--by employing Russell's theory of definite descriptions.

The sentence "the man who lives at the North Pole does not exist" can be true without incurring any paradox. This is so since it can be read as saying that the property of being a man living at the North Pole has either no instances or more than one instance. A feature of the sentence that is important to note is that though the definite description occurring in it does not denote anything, it nevertheless does refer to something; namely the property of living at the North Pole.

Frege has proposed the term "oblique" to apply to linguistic expressions which, although they do refer, do not do so in the usual mode of denotation. The definite description in the example in question would seem to be occurring in a way which perhaps could be described by Frege's "oblique".

In any case, the term "oblique" will be taken in all of the following discussion to apply to any reference made by a term which is not a reference by denotation. The term "oblique" in its present meaning might be defined as follows: ("xRy" abbreviates "x refers to y", "xDy abbreviates "x denotes y", and "xOy" abbreviates "x obliquely refers to y").

$$xOy =_{Df} xRy . \sim xDy \qquad (9)$$

Such oblique usages of terms allow non-denoting terms to be used in true sentences because such terms may refer in some oblique way to something of which the sentence says something true.

So to speak, the last statement of Parmenides' rule should be reformulated so as to read: "Do not speak using terms that do not denote and are not used obliquely". Any such non-oblique or denotative mode of reference will be referred to hereafter as "direct reference".

As was mentioned above, System I is a logic of denotation, and therefore also a logic of direct reference.

System II on the other hand, is a logic of connotation and of oblique reference. System II is based upon a mode of reference in which the referents are connotata rather than, as in System I, denotata. Angle brackets ("$\langle\rangle$") are introduced in System II with the meaning that an expression together with angle brackets enclosing it shall be taken to name the connotatum of the expression enclosed in such brackets. Although the usage of angle brackets is characterized in System II by postulates and rules of transformation, two points concerning the interpretation might be mentioned here rather than deferred to Chapter IV.

First, the relation of connotation is analogous to that of denotation in that in either mode of reference, a term has at most one referent, but differs from the latter in that every referent in the mode of connotation is a characteristic or property which is, so to speak, a definitional criterion by which one identifies the associated denotatum. This is to say, the possession of, or failure to possess, the connotatum of a given term is a test by which a thing may be respectively accepted or rejected as the denotatum of that term. Secondly, sentences as well as terms may be enclosed in angle brackets, and if the former is the case, then the indicated connotatum is a definitional characteristic of a state of affairs.

However, the meaning of the angle bracket notation will be explained in more detail later. The last point to be made in the present section is the criticism of "The Logic of Existence", which was mentioned at the opening of Section 1.3 but was deferred until a consideration could be made of the difficulties which lead to Parmenides' injunction in one or another of its forms.

That criticism is that "The Logic of Existence" allows statements of non-existence to appear in the system without an explicit notation indicating the mode of reference in which such statements are to be interpreted. The absence of such a notation becomes even more serious in the interpretation of propositional logic than in the interpretation of a functional logic such as "The Logic of Existence". It will be maintained later, that many formulas which are valid laws of propositional logic if interpreted in a direct mode of reference, are not valid when interpreted in another mode of reference.

1.6 Every denotative logic, such as System I, allows the substitution only of terms which denote in any inference carried out within that logic. As a consequence of this, any argument which is purportedly carried out within such a logic, but which contains steps which make substitutions of terms which do not denote for free variables in formulas of the logic, is an invalid argument, and since it is not carried out within the rules of the logic in question, is in fact not an inference of that system of logic at all. One of the purposes of developing System II is to show that at least some of the paradoxes of the theory of types require such arguments in order that they might be inferred.

In particular, the term "k" defined:

$$k =_{Df} \hat{f}(\sim f_f) \tag{11}$$

would seem not to denote a property. Speaking obliquely, the property of non-self-application does not exist. But if such is the case, then the argument which leads to Russell's paradox is not an inference within either System I or *Principia Mathematica*.

Russell proscribes in general, all reference to "illegitimate totalities". It will be suggested that the phrase "illegitimate totality" might be interpreted to mean "non-existent totality". Parmenides' injunctions then provide a means for avoiding the paradoxes of the theory of types which does not depend upon the reason given by Russell in formulating the theory of types, namely that unrestricted generalization is not possible.

In fact, since no term banned from use by Parmenides' injunction denotes something, it follows that the injunction, while it does restrict the vocabulary of a language to which it is applied, does not correspondingly place any limitation upon the range of subject matters which can be discoursed about within that language. Or in other words, avoidance of paradox, and unrestricted generalization, are together possible.

Put differently, since there are no non-existents, and hence no non-existent totalities, we may restrict generalization to "legitimate totalities", and also, generalize quite unrestrictedly to everything.

As a summary of the position on existence outlined above, the following quotation from a tract which Cornford suggests was written in approximately 400 B.C., speaks for itself:[7]

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7. Francis Macdonald Cornford, Plato's Theory of Knowledge, Routledge & Kegan Paul Ltd., p. 209.

"It seems to me in general that there is no art that is not, for it is irrational to think that something which is, is not. For what 'being' have things that are not, which one could look at and say of it that 'it is'? For if it is possible to see things that are not, as you can see things that are, I do not understand how one can regard them as not being, when you can see them with your eyes and think of them in your mind that they are..."

CHAPTER II

DIRECT DISCOURSE AND DENOTATIVE LOGIC: SYSTEM I

2.1 Formation rules and Nomenclature.

2.11 A purpose of Chapter II is to develop a quantified modal logic. This quantified modal logic will be referred to hereafter as System I.

2.12 The primitives of System I are those of the propositional modal logic of C. I. Lewis,[1] and in addition, five primitives peculiar to this system. Two of the latter primitives are signs of grouping.

2.121 The primitives of Lewis are the curl ('~'), the dot ('.'), and the diamond ('◇').

2.122 The first of three additional primitives that are not signs of grouping is the predicate of universal instantiation ('A'). To assert the sentence 'A is true of f', or, 'A_f' is to assert that 'f is a property possessed by everything'. (The expression 'A' is adopted to suggest a contraction of 'all'.) The more usual notation '(x)fx' will later be introduced by a definition involving 'A'.

In Principia Mathematica '(x)fx' is not interpreted to be synonymous with 'f is true of everything unrestrictedly'. The theory of types calls for a limitation upon the

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1. Clarence Irving Lewis and Cooper Harold Langford, Symbolic Logic, The Century Co., 1932.

universality of a generalization.[2] However in the present system, the interpretation of 'A_f' is intended in the unrestricted sense.

2.123 The second primitive in addition to those of C. I. Lewis is the predicate of singular existence ('E!').

2.124 The third primitive is the cap ('$\wedge$'). The cap is placed over variables preceding propositional formulas. The resulting formulas signify <u>properties</u>. Although the cap is often taken to signify classes, a double cap ('$\wedge\!\!\wedge$') will be used for this purpose in the present system. The double cap is introduced by a definition which is essentially the <u>Principia</u> <u>Mathematica</u> definition of indefinite descriptions.[3]

2.125 The two signs of grouping of the present system are the left parenthesis ('('), and the right parenthesis (')'). C. I. Lewis uses a dot system of grouping in his systems of propositional modal logic. The use of parentheses rather than dots is a departure of the present notation from that of C. I. Lewis. (Other departures, all minor, will be noted in due course.)

2.13 Two kinds of variables occur in System I. 'p', 'q', 'r', or one of these variables followed by a numerical subscript will be employed as <u>propositional variables</u>. 'x',

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2. A. N. Whitehead and Bertrand Russell, <u>Principia</u> <u>Mathematica</u> V.1 Chapter II.

3. Ibid., *20.01, V. 1 p. 188.

'y', 'z', 'w', 'f', 'g', 'h', or one of these letters followed by a numerical subscript, will be used as <u>non-propositional</u> <u>variables</u>.

2.14 'F' and 'F' followed by a numerical subscript will be referred to as <u>formula</u> <u>variables</u>. These will be the only meta-linguistic variables used. They do not occur in System I.

2.15 A <u>variable-sequence</u> is defined to be any formula satisfying all of the following conditions.

(1) The first sign of the formula is a left parenthesis, and the last sign of the formula is a right parenthesis;

(2) Every sign of the formula that is neither the first nor last sign of the formula, is a non-propositional variable;

(3) At least two signs of the formula are variable tokens.

2.16 A <u>first</u> <u>formula</u> is defined to be any one of the following expressions.

(1) $(\sim p)$

(2) $(p.q)$

(3) $(\Diamond p)$

(4) E!

(5) A

2.17 A <u>well</u> <u>formed</u> <u>formula</u> is defined to be any first formula, any variable-sequence, or any expression obtainable from first formulas and variable-sequences by means of one or more successive applications of the following <u>Formation Rules</u>. (Not all first formulas and not all well formed formulas are assertable, or propositional.)

FR1. A well formed formula may be formed by substituting in any well formed formula, F, any propositional or non-propositional variable or well formed formula for any occurrence of any variable which is respectively propositional or non-propositional, provided that occurrence is free in F.[4]

FR2. A well formed formula may be formed by prefixing any series of one or more capped non-propositional variables not containing two variable tokens of the same variable type, to any well formed formula F, such that F contains at least one occurrence which is free in F of each variable in the series being prefixed to F.

2.18 F_1 is bound in F_2 if and only if:

(1) F_1 is a variable, and F_2 is a well formed formula; and

(2) There is a formula, F_3, such that:

(a) F_1 is contained in F_3 and F_3 is contained in F_2;

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4. "Propositional" and "non-propositional" are defined below, in 2.111; "free in F" is defined below, in 2.19.

(b) F_3 can be formed by an application of FR2; and

(c) F_3 contains in its initial series of capped variables, a variable of the same variable type as F_1.

2.19 F_1 is free in F_2 if and only if:

(1) F_1 is a variable and F_2 is a well formed formula; and

(2) F_1 is contained in F_2 and F_1 is not bound in F_2.

2.110 F_1 binds F_2 if and only if:

(1) F_1 and F_2 are tokens of the same variable type; and

(2) There is a formula F_3, a formula F_4, and a formula F_5; such that:

(a) F_1 and F_2 are contained in F_3; and

(b) F_4 is a series of capped variables, and F_5 a propositional formula, such that F_3 can be formed in accordance with FR2, by prefixing F_4 to F_5; and

(c) F_1 is contained in F_4, and F_2 is not bound in F_5.

2.111 A _propositional formula_ is any well formed formula the first sign of which is a left parenthesis. A _non-propositional_ formula is any well formed formula the first sign of which is not a left parenthesis.[5]

2.2 Definitions.

2.21 A well formed non-propositional formula, F, is <u>proper</u> if it satisfies either of the following conditions.

(1) No occurrences in F of any variable are free in F, and the substitution of F for 'x' in the formula 'E!x' yields a true sentence.

(2) There is at least one occurrence of a variable which is free in F, and the substitution of F for 'x' in the formula 'E!x', and subsequent existential generalization of the resulting formula with respect to every free variable occurring in it yields a true sentence.

2.22 C. I. Lewis' definitions of the modal and truth functional connectives are adopted in this system, with the exception of the definition of strict equivalence. A double arrow ('$\Leftrightarrow$') is here used as a sign of strict equivalence.

System I contains the following additional definitions.

D1. $\exists!f =_{Df} \sim(A\hat{x}(\sim(fx))$

D1 defines the predicate of plural existence.

D2 defines the <u>Principia</u> notation for universal instantiation.

D2. $(x)fx =_{Df} A\hat{x}(fx)$

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5. For example, the first three first formulas are propositional, and the last two are non-propositional.

D3 defines the <u>Principia</u> notation for plural existence.

D3. $(\exists x)fx =Df\ \exists!\hat{x}(fx)$

D4 defines identity.

D4. $x=y =Df\ (f)(fx \supset fy)$

D5 defines the definite description.

D5. $f(\iota x)gx =Df\ (\exists x)(fx.gx).(x)(y)((gx.gy) \supset x=y)$

Finally, D6 defines class abstraction.

D6. $f\hat{x}(gx) =Df\ (\exists h)(f_h.(x)(gx \equiv hx)$

Definitions 5 and 6 are essentially the <u>Principia</u> definitions, *14.01, and *20.01.

2.3 Postulates.

2.31 The propositional logic of System I is C. I. Lewis' System S4. System I therefore contains Lewis' postulates B1-B4, B6, B7, and Becker's postulate.[6]

2.32 Four postulates concerning quantification are assumed in System I.

P1 $(x)fx.E!y \prec fy$

P2 $(x)(p \prec (E!x \supset fx)) \prec (p \prec (x)fx)$

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6. These are:

B1. $(p.q) \prec (q.p)$
B2. $(p.q) \prec p$
B3. $p \prec (p.p)$
B4. $((p.q).r) \prec (p.(q.r))$
B6. $((p \prec q).(q \prec r)) \prec (p \prec r)$
B7. $(p.(p \prec q)) \prec q$
Becker's Postulate. $\sim\Diamond\sim p \prec \sim\Diamond\sim\sim\Diamond\sim p$

(B5 of Lewis' original postulate set for S4, has been shown by McKinsey to be reducible to those above. See: J. C. C. McKinsey, "A Reduction in Number...", <u>American Mathematical Society Bulletin</u>, vol. 40 (1947), p. 432.)

P3 $\Diamond \sim (\exists x)E!x$

P4 $E!x$

2.4 Transformation rules.

2.41 Some expressions involve predicates formed with capped variables in such a way that the capped variable expression as a whole, so to speak, "reduces to" an expression not involving capped variables. For example, the expression:

$$(\hat{x}(Mx)a) \tag{1}$$

which says of a, that it is an x such that M is a property of x, reduces to:

$$(Ma) \tag{2}$$

which says of a, that it possesses M.

Again, predicates involving more than one capped variable may also "reduce to" simpler expressions. The expression:

$$(\hat{x}\hat{y}(Rxy)ab) \tag{3}$$

reduces to:

$$(Rab) \tag{4}$$

The following definition is an attempt to codify the relation of reducing to. This definition is not itself a transformation rule, although whenever a first line reduces to a second line, the second line is deducible from the first. The corresponding transformation rule--the rule of reduction--will be defined later.

2.42 F_1 reduces to F_2 if and only if:

There is a variable sequence, F_3, containing n/1 variables (n>0) and a well formed formula, F_4, consisting of a sequence of n capped variables followed by a propositional formula, F_5, such that:

(1) F_1 can be obtained from F_3 by substituting F_4 for the first variable occurrence in F_3, and some non-propositional variable or non-propositional formula for every other variable occurrence in F_3; and

(2) F_2 can be obtained from F_5 by substituting for each variable occurrence in F_5 which is bound by a variable in the kth position in the series of capped variables preceding F_5 in F_4, the expression substituted for the k/1st variable occurrence in F_3 in the series of substitutions prescribed in (1).

2.43 If a formula or variable is substituted in another formula for one or more expressions to yield a resultant formula, some of the signs in the resultant formula are obtained by exchange of a substitute for an expression in the formula upon which substitution was carried out, while other signs in the resultant formula are simply copied from the formula upon which substitution is carried out. Signs of the former sort will be said to _result from exchange_, and those of the latter sort, to _result from copying_.

2.44 For the sake of a more usual notation for quantifiers, and other variable binders, definitions 2, 3, 5, and 6 were given in section 2.22. However, System I recognises only caps

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There is a variable sequence, F_3, containing n≠1 variables (n>0) and a well formed formula, F_4, consisting of a sequence of n capped variables followed by a propositional formula, F_5, such that:

(1) F_1 can be obtained from F_3 by substituting F_4 for the first variable occurrence in F_3, and some non-propositional variable or non-propositional formula for every other variable occurrence in F_3; and

(2) F_2 can be obtained from F_5 by substituting for each variable occurrence in F_5 which is bound by a variable in the kth position in the series of capped variables preceding F_5 in F_4, the expression substituted for the k≠1st variable occurrence in F_3 in the series of substitutions prescribed in (1).

2.43 If a formula or variable is substituted in another formula for one or more expressions to yield a resultant formula, some of the signs in the resultant formula are obtained by exchange of a substitute for an expression in the formula upon which substitution was carried out, while other signs in the resultant formula are simply copied from the formula upon which substitution is carried out. Signs of the former sort will be said to <u>result from exchange</u>, and those of the latter sort, to <u>result from copying</u>.

2.44 For the sake of a more usual notation for quantifiers, and other variable binders, definitions 2, 3, 5, and 6 were given in section 2.22. However, System I recognises only caps

as "official" binding signs, hence it is assumed that all defined expressions introduced by D2, D3, D5, and D6, are eliminated prior to application of all except the last of the following transformation rules. The last transformation rule is a rule for introducing such defined expressions subsequent to an inference not involving them. The Transformation Rules of System I are the following.

TR1. (<u>Adjunction</u>, abbreviated 'Adj') If F_1 and F_2 are postulates or inferred lines, then the line $F_1.F_2$ may be inferred.

TR2. (<u>Detachment</u>, abbreviated 'Detach') If F_1 and $F_1 \dashv F_2$ are previously obtained lines, then F_2 may be inferred.

TR3. (<u>Exchange</u>) If F_1 is an obtained line, containing F_2, and $F_2 \Leftrightarrow F_3$ is an obtained line, then F_3 may be substituted for F_2 in F_1 to yield an inferred line F_4; <u>provided</u> that every variable type is such that, there is a token of that type which is free in F_2 but bound in F_1, if and only if, there is a token of that type which is free in F_3 and bound in F_4.

The proviso to TR3 is necessary because without it, an inference of the following sort would be sanctioned by TR3.

$(A\hat{x}(fx \vee {\sim}fx))$ (5)

$(fx \vee {\sim}fx) \Leftrightarrow (fy \vee {\sim}fy)$ (6)

$(A\hat{x}(fy \vee {\sim}fy))$ (7)

Line (7), inferred from (5) and (6), is not well formed. However, this "inference" violates the requirement that every variable type of which there is an occurrence free in the substitute, but bound in the line in which the substitute occurs, must have an occurrence in the substituted expression which is free in the substituted expression, but bound in the line in which the substituted expression occurs.

TR4. (Free variable substitution) Any variable which is propositional or non-propositional may be substituted for every free occurrence in any previously obtained line of any variable which is respectively propositional or non-propositional provided; every sign in the resulting line which results from exchange is free in the line as a whole.

TR5. (Bound variable substitution) If F_1 is a capped variable in any obtained line, then any non-propositional variable may be substituted for F_1 and every variable bound by F_1 provided; that in the resulting line, every variable F_2 is such that F_2 is bound by the variable substituted for F_1 if and only if F_2 results from exchange.

Examples of the need for the proviso to TR5 are the following.

$$(A\hat{x}(fyv{\sim}fx)) \supset (fyv(A\hat{x}({\sim}fx))) \qquad (8)$$

$$(A\hat{y}(fyv{\sim}fy)) \supset (fyv(A\hat{x}({\sim}fx))) \qquad (9)$$

If its proviso is ignored, TR5 would sanction the inference of the invalid (9) from the valid (8). The part of the proviso to TR5 which is violated in going from (8) to (9), is the requirement that every variable in the inferred line which is bound by a capped variable resulting from exchange must itself result from exchange. The further demand of the proviso to TR5 that all bound variables in the inferred line which result from exchange must be bound by the capped variable which results from substitution, is violated in the following invalid argument.

$$(A\hat{x}(A\hat{y}(fxy))) \supset (fxy) \qquad (10)$$

$$(A\hat{y}(A\hat{y}(fyy))) \supset (fxy) \qquad (11)$$

(10) is a valid formula of System I, while (11) is not even well formed.

TR6. (Formula substitution.) Any propositional or non-propositional formula, F_1, may be substituted for every free occurrence of any variable which is respectively propositional or non-propositional in any obtained line, provided; every variable in the resulting line which is free in an occurrence of F_1 resulting from exchange, is also free in the resulting line as a whole, and every non-propositional formula in the resulting line is proper.

The following argument illustrates the need for a part of the proviso to TR6.

$$(A\hat{x}(p \supset fx)) \supset (p \supset (A\hat{x}(fx))) \tag{12}$$

$$(A\hat{x}((fx) \supset (fx)) \supset ((fx) \supset (A\hat{x}(fx))) \tag{13}$$

(12) is a valid law of System I, while (13) is not valid. The proviso to TR6 is violated in that, not every variable in the resulting line which is free in an occurrence of the substitute which results from exchange, is free in the line as a whole. The latter part of the proviso to TR6 is worded so as to apply only to occurrences of the substitute in the resulting line which result from exchange. The purpose of this restriction is to allow some perfectly valid inferences which would be proscribed were this part of the proviso made to apply to every occurrence of the substitute in the resulting line.

$$(A\hat{x}(p \supset (fx))) \supset (p \supset (A\hat{y}(fy))) \tag{14}$$

$$(A\hat{x}((fy) \supset (fx))) \supset ((fy) \supset (A\hat{y}(fy))) \tag{15}$$

The inference from (14) to (15) is a substitution of '(fy)' for 'p' in (14) to yield (15). Both the inference itself and the formulas involved are valid. However, because (15) contains in its consequent an occurrence of the substituted '(fy)', which contains an occurrence of 'y' which is free in that substitute but not free in (15) as a whole, this valid inference would not be allowed by TR6 if its proviso were to be so worded as to apply to any occurrence of the substitute in the resulting line, rather than to only those occurrences of the substitute which result from exchange.

That part of the proviso to TR6 which demands that every formula in the resulting line be proper, has to do with the avoidance of the paradoxes of the theory of types, and its relevance will be discussed later.

TR7. (Reduction, abbreviated 'Reduc') If F_1 is any obtained line which contains F_2 and F_2 reduces to F_3, then a new line may be inferred by substituting F_3 for F_2 in F_1.

TR8. (Universal Generalization, abbreviated 'U.G.') If F_1 is any postulate or theorem, containing at least one occurrence of some variable which is free in F_1, then a new line may be inferred by substituting 'A' for the first variable of some two variable variable-sequence and a formula which can be formed in accordance with FR2 by following a capped occurrence of the variable in question by F_1, for the second variable of the variable-sequence; provided that in the resulting line, every non-propositional well formed formula is proper.

A ninth transformation rule will often be appealed to in the course of proofs. This rule governs exchange in accordance with definitions. Although listed as one of the rules of the formalized part of System I, TR9 will only on occasion be involved in formal proofs. Often TR9 will only be used to introduce contextually defined terms and expressions such as the classical notations for quantifiers, which must be eliminated prior to application of the other transformation rules. Therefore, TR9 will frequently be somewhat informal in application, its proper use being often left in part to intuition.

TR9. (Definitional exchange) Any expression may be exchanged with its definitional equivalent in any postulate or theorem. Definitional exchanges may be preceded by one or more substitutions of variables or formulas for each occurrence of some free variable in the definition, subject to the restrictions of quantifier control. Such substitutions may be followed by one or more reductions, also prior to exchange.

2.5 Theorems.

2.51 Proof annotations in the derivations to follow will follow the method of C. I. Lewis in Symbolic Logic. The proofs of Section I will be given in full. The proofs of later sections will be abbreviated in accordance with conventions which will be introduced prior to the use of these abbreviations. Certain theorems of Lewis' system S4 not proved in Symbolic Logic will be stated without proof. The theorems of section 0 are the latter theorems of S4. The theorems of section 1 are dependent upon the first postulate, those of section 2 upon the first two postulates, and those of section 3 upon the first three postulates, and those of section 4 upon

upon all four postulates, while those of section 5 are miscellaneous items dependent upon various combinations of the postulates.

2.52 Section 0.

T.0.1 $\sim\Diamond\sim p \prec \sim\Diamond\sim\sim\Diamond\sim p$

T.0.2 $\sim\Diamond\sim p \prec (q \prec \sim\Diamond\sim p)$

T.0.3 $(p \prec q) \prec (r \prec (p \prec q))$

T.0.4 $\sim\Diamond\sim\sim\Diamond\sim(p \vee \sim p)$

T.0.5 $((p \vee \sim p) \prec q) \prec \sim\Diamond\sim q$

T.0.6 $(p \prec q) \prec \sim\Diamond\sim(p \prec q)$

2.53 The following is a translation of the four postulates (given in §2.32) for functional logic in System I, into standard notation.

P1 $(((Af).(E!x)) \prec (fx))$

P2 $((A\hat{x}(p \prec ((E!x) \supset (fx)))) \prec (p \prec (Af)))$

P3 $(\Diamond(\sim(\exists!E!)))$

P4 $(E!x)$

2.54 Section 1.

T1.1 $((Af) \prec ((E!x) \supset (fx)))$

(From 14.26 L and L; $(Af)/p$; $(E!x)/q$; $(fx)/r$)

$((((Af).(E!x)) \prec (fx)) \Leftrightarrow ((Af) \prec ((E!x) \supset (fx))))$ (1)

(From (1), and P.1, by Exchange) QED (2)

In more usual notation, T1.1 might be written:

$(x)fx \prec (E!x \supset fx)$,

and could have been proven in this form. However, had T1.1

been proven in the latter form, some steps in the proof would not be explicitly sanctioned by the transformation rules. In particular, the rule for exchange in accordance with definition would have been somewhat informally applied.

T1.2 $(((fx).(E!x)) \mathrel{-\!\!3} (\exists!f))$

(From 12.43 L and L: $(Af)/p$; $((E!x)\supset(fx))/q$)

$(((Af) \mathrel{-\!\!3} ((E!x)\supset(fx))) \mathrel{-\!\!3} ((\sim((E!x)\supset(fx))) \mathrel{-\!\!3} (\sim(Af))))$ (1)

(From T1.1, and (1) by Detach)

$((\sim((E!x)\supset(fx))) \mathrel{-\!\!3} (\sim(Af)))$ (2)

((2), by Formula Substitution: $\hat{x}(\sim(fx))/f$)

$((\sim((E!x)\supset(\hat{x}(\sim(fx))x))) \mathrel{-\!\!3} (\sim(A\hat{x}(\sim(fx)))))$ (3)

((3), by Reduc)

$((\sim((E!x)\supset(\sim(fx)))) \mathrel{-\!\!3} (\sim(A\hat{x}(\sim(fx)))))$ (4)

The formula '$(\hat{x}(\sim(fx))x)$' in (3) reduces to '$(\sim(fx))$'. Applying the definition of 'reduce to' (see section 2.42) with n=1, and:

F_1; '$(\hat{x}(\sim(fx))x)$'

F_2; '$(\sim(fx))$'

F_3; '(xx)'

F_4; '$\hat{x}(\sim(fx))$'

F^5; '$(\sim(fx))$'

it can be seen that the conditions of the definition are met.

(From 14.12 L and L: $(E!x)/p$; $(\sim(fx))/q$)

$((\sim((E!x)\supset(\sim(fx)))) \Leftrightarrow ((E!x).(\sim(fx))))$ (5)

(From (4), and (5), by Exchange)

$(((E!x).(\sim(\sim(fx)))) \dashv (\sim(A\hat{x}(\sim(fx)))))$ (6)

(From 12.3 L and L: $(fx)/p$)

$((fx) \Leftrightarrow (\sim(\sim(fx))))$ (7)

(From (6), and (7), by Exchange)

$(((E!x).(fx)) \dashv (\sim(A\hat{x}(\sim(fx)))))$ (8)

(From (8), and D1, by exchange in accordance with def.)

$(((E!x).(fx)) \dashv (\exists!f))$ (9)

Although step (9) was obtained by means of Exchange in Accordance with Definition, it is not actually a part of the unformalized development of System I. Only those definitions which must be eliminated prior to application of the Transformation Rules are incompletely formalized. To have explicitly given the conditions under which definitions not eliminated prior to application of the Transformation Rules, may be exchanged for their definitional equivalents, would have required two rules for definitional exchange; one applicable only to definitions eliminated prior to such application, and another, applicable only to definitions not so eliminated. Because of its inconvenience, such a procedure is not followed here. However, the definition called for in (9) need not be eliminated prior to application of the Transformation Rules, and therefore (9) is at least in principle, capable of being formalized.

(From 12.15, L and L: $(E!x)/p$; $(fx)/q$)

$(((E!x).(fx)) \Leftrightarrow ((fx).(E!x)))$ (10)

(From (9) and (10) by exchange) QED. (11)

2.55 In subsequent proofs, various abbreviations will be used in order to simplify exposition. Parentheses will be omitted if grouping is evident from context. Some lines of proofs, easily supplied by the reader, will also be omitted. Although the more conventional notation for quantifiers is no more compact than the primitive notation for quantifiers introduced above, the former will for the most part replace the latter. If signs of grouping are omitted, the scope of a first connective will extend over the scope of a second, if the first precedes the second in the following list: '$\Leftrightarrow$', '$\dashv$', '$\equiv$', '$\supset$', '$\vee$', '$.$', '$\Diamond$', '$\sim$', '$=$'. Following the proofs there is a catalogue of all formulas assumed to be proper.

2.56 Section 2.

T2.1 $(p \dashv (x)fx) \dashv (x)[p \dashv (E!x)fx)]$

(T1.1, To.3)

$(p \dashv (x)fx) \dashv [(p \dashv (x)fx).((x)fx \dashv (E!x)fx)]$ (1)

((1), 11.6 L&L) $(p \dashv (x)fx) \dashv [p \dashv (E!x)fx)]$ (2)

(15.2 L&L) $[p \dashv (E!x)fx)] \dashv [E!x)[p \dashv (E!x)fx)]]$ (3)

((2), (3), 11.6 L&L)

$(p \dashv (x)fx) \dashv [E!x)[p \dashv (E!x)fx)]]$ (4)

((4), U.G., R2) QED (5)

T2.2 $(x)[p \prec (E!x)fx] \Leftrightarrow (p \prec (x)fx)$

(R2, T2.1, 11.03 L&L) QED (1)

T2.3 $(x)[(E!x.fx) \prec p] \Leftrightarrow [(\exists x)fx \prec p]$

(T2.2, 12.44 L&L) $(x)[\sim(E!x)fx) \prec \sim p] \Leftrightarrow (\sim(x)fx \prec \sim p)$ (1)

((1), $\sim p/p$) QED (2)

T2.4 $(x)(p \prec fx) \prec (p \prec (x)fx)$

(R1) $[(x)(p \prec fx).E!x)] \prec (p \prec fx)$ (1)

(15.2 L&L) $fx \prec (E!x)fx)$ (2)

((1), (2), T0.3)

$[(x)(p \prec fx).E!x] \prec [(p \prec fx).[fx \prec (E!x)fx)]]$ (3)

((3), 11.6 L&L) $[(x)(p \prec fx).E!x] \prec [p \prec (E!x)fx)]$ (4)

((4), U. G., 14.26 L&L)

$(x)[(x)(p \prec fx) \prec (E!x)[p \prec (E!x)fx)]]$ (5)

((5), R2) QED (6)

Theorem 2.4 illustrates both a way in which quantified modal logic differs from quantified material logic, and also the need for including existence in modal logic. While the analogue of 2.4 with material implication replacing strict implication is true biconditionaly, the converse of 2.4 is not true (see comment after T2.7), and in order to obtain a law analogous to T2.4 the main connective of which is an equivalence relation, the antecedent of 2.4 must be weakened by introducing 'E!' as in T2.2.

T2.5 $(x)(fx ⥽ gx) ⥽ [(x)fx ⥽ (x)gx]$

(P1) $(x)(fx ⥽ gx) . E!x ⥽ (fx ⥽ gx)$ (1)

((1), P1, T0.3)

$(x)(fx ⥽ gx) . E!x ⥽ ((x)fx . E!x ⥽ fx) . (fx ⥽ gx)$ (2)

((2), 11.6 L&L, 14.26 L&L)

$(x)(fx ⥽ gx) ⥽ E!x \supset [(x)fx ⥽ (E!x \supset gx)]$ (3)

((3), U. G., P2) QED (4)

T2.6 $\sim\lozenge\sim(x)fx \Leftrightarrow (x)\sim\lozenge\sim(E!x \supset fx)$

(T2.2: $p\vee\sim p/p$) $(x)(p\vee\sim p ⥽ E!x \supset fx) \Leftrightarrow (p\vee\sim p ⥽ (x)fx)$ (1)

(T0.5) $(p\vee\sim p ⥽ E!x \supset fx) \Leftrightarrow \sim\lozenge\sim(E!x \supset fx)$ (2)

(T0.5) $(p\vee\sim p ⥽ (x)fx) \Leftrightarrow \sim\lozenge\sim(x)fx$ (3)

((1), (2), (3)) QED (4)

T2.7 $(x)\sim\lozenge\sim fx ⥽ \sim\lozenge\sim(x)fx$

(T2.4: $p\vee\sim p/q$) $(x)(p\vee\sim p ⥽ fx) ⥽ (p\vee\sim p ⥽ (x)fx)$ (1)

(T0.5: fx/q) $(p\vee\sim p ⥽ fx) \Leftrightarrow \sim\lozenge\sim fx$ (2)

(T0.5: $(x)fx/q$) $(p\vee\sim p ⥽ (x)fx) \Leftrightarrow \sim\lozenge\sim(x)fx$ (3)

((1), (2), (3)) QED (4)

The converse of T2.7 is not valid. An exception to the converse of T2.7 can be obtained by substituting 'E!' for 'f' in such a supposed law. The "law" fails because, while it is necessarily the case that everything exists (see T2.15), it is not the case that everything necessarily exists, and in fact, of anything it is contingent that it exists (see T3.1).

Had the converse of T2.4 been valid, then the converse of T2.7 would have followed. Hence the invalidity of

this converse, as exhibited above, also exhibits the invalidity of that converse.

T2.8 $(x)(fx \prec gx) \prec \sim\Diamond\sim(x)(fx \supset gx)$

(T2.7) $(x)\sim\Diamond\sim(fx \supset gx) \prec \sim\Diamond\sim(x)(fx \supset gx)$ (1)

((1), 18.7 L&L) QED (2)

T2.9 $(\exists x)\Diamond(E!x.fx) \Leftrightarrow \Diamond(\exists x)fx$

(T2.6, 12.11 L&L: ~p/p) $\sim(x)\sim\Diamond\sim(E!x \supset fx) \Leftrightarrow \sim\sim\Diamond\sim(x)fx$ (1)

((1), D3, 12.3 L&L) QED (2)

T2.10 $\Diamond(\exists x)fx \prec (\exists x)\Diamond fx$

(T2.7, 12.43 L&L) $\sim\sim\Diamond\sim(x)fx \prec \sim(x)\sim\Diamond\sim fx$ (1)

((1), 12.3 L&L, D.3) QED (2)

The counter instance given earlier to disprove the converse of T2.7 also disproves the converse of T2.10.

T2.11 $\Diamond(x)fx \prec (x)\Diamond(E!x \supset fx)$

(T1.1, T0.6) $\sim\Diamond\sim[(x)fx \prec (E!x \supset fx)]$ (1)

(18.53 L&L) $[[(x)fx \prec (E!x \supset fx)].\Diamond(x)fx] \prec \Diamond(E!x \supset fx)$ (2)

((1), (2), 18.61 L&L) $\Diamond(x)fx \prec \Diamond(E!x \supset fx)$ (3)

T2.12 $(\exists x)\sim\Diamond\sim(E!x.fx) \prec \sim\Diamond\sim(\exists x)fx$

(12.43 L&L, T2.11) $\sim(x)\Diamond(E!x \supset fx) \prec \sim\Diamond(x)fx$ (1)

((1), 12.3 L&L, 14.01 L&L, D.3) QED (2)

The converses of T2.11 and T2.12 can be proved only as material implications in this system (see T3.5). However, these converses could have been obtained as strict

implications had Lewis' postulate C11 been assumed, or had P3 of the present system been replaced with '$\sim\Diamond\sim\Diamond\sim(\exists x)E!x$'. For if it is necessarily contingent that something exists, then as a result of the so called "paradoxical" properties of strict implication, the theorems in question could be obtained. In any case, both antecedent and consequent of T2.11 are always true, while both antecedent and consequent of T2.12 are always false.

T2.13 is an alternative form of T2.6.

T2.13 $\sim\Diamond\sim(x)fx \Leftrightarrow (x)(E!x \prec fx)$

(T2.6, 18.7 L&L) QED (1)

T2.14 $\sim\Diamond\sim(x)E!x$

(T2.13, E/f, 12.1 L&L) QED (1)

T2.15 $\sim\Diamond\sim(x)(fx \supset E!x)$

(15.2 L&L) $E!x \prec (fx \supset E!x)$ (1)

((1), U.G.) $(x)[E!x \prec (fx \supset E!x)]$ (2)

(T2.13, (2)) QED (3)

It is on account of T2.15 that Parmenides' claims that everything which one talks about, thinks about, etc., exists, can be affirmed as so, and indeed, as logically necessary.

Since however, the converse of T2.8 is invalid, $(f)\sim\Diamond\sim(x)(fx \supset E!x)$ does not imply $(f)(x)(fx \prec E!x)$. In fact it is not the case that $(f)(x)(fx \prec E!x)$. While all properties are extensionally included in the property of being an existent, existence is not in the intension of every property--that is, not all properties "imply existence". For example, no necessary properties imply existence.

T2.16 $(x)(E!x \supset fx) ⥽ (x)fx$

(P1)	$[(x)(E!x \supset fx) . E!x] ⥽ (E!x \supset fx)$	(1)
((1), 14.26 L&L, 12.5 L&L, 12.7 L&L)		
	$(x)(E!x \supset fx) . E!x ⥽ fx$	(2)
(14.26 L&L, (2), U.G., T2) QED		(3)

T2.17 $(x)(E!x \supset fx) \Leftrightarrow (x)fx$

(15.2 L&L)	$fx ⥽ (E!x \supset fx)$	(1)
((1), U.G., T2.5)	$(x)fx ⥽ (x)(E!x \supset fx)$	(2)
((2), T2.16, 11.03 L&L) QED		(3)

T2.17 affirms in effect, that a generalization about everything that exists is a generalization about everything. As a result of T2.17, a generalization about all "legitimate," or existent, totalities is an unrestrictedly universal generalization.

T2.18 $(\exists x)fx \Leftrightarrow (\exists x)(E!x . fx)$

(11.3 L&L, T1.2)	$fx . E!x ⥽ (\exists x)(E!x . fx)$	(1)
((1), U.G., T2.3)	$(\exists x)fx ⥽ (\exists x)(E!x . fx)$	(2)
(T1.2, T2.3)	$(\exists x)(E!x . fx) ⥽ (\exists x)fx$	(3)
((2), (3)) QED		(4)

T2.19 $(x)(fx \supset \sim E!x) \Leftrightarrow \sim(\exists x)fx$

(T2.17)	$(x)(\sim\sim fx \supset \sim E!x) \Leftrightarrow \sim\sim(x)\sim fx$	(1)
((1), 12.3 L&L, D.3) QED		(2)

T2.19 asserts that, none of a certain kind of thing exists, is equivalent to, that kind of thing does not have plural existence.

T2.20 $(x)(y)fxy \prec (y)(x)fxy$

(P1) $(x)(y)fxy . E!x \quad (y)fxy$ (1)

(T1.1) $(y)fxy \prec (E!y \supset fxy)$ (2)

((1), (2), 11.6 L&L, 14.26 L&L, 12.15 L&L)

$(x)(y)fxy . E!y . E!x \prec fxy$ (3)

((3), 14.26 L&L, U.G., P2) $(x)(y)fxy . E!y \prec (x)fxy$ (4)

((4), 14.26 L&L, U.G., P2) QED (5)

T2.21 $(x)(y)fxy \Leftrightarrow (y)(x)fxy$

(By proof similar to that for T2.20)

$(y)(x)fxy \prec (x)(y)fxy$ (1)

((1), T2.21, 11.03 L&L) QED (2)

T2.22 $(x)(fx \prec gx) . (x)(gx \prec hx) \prec (x)(fx \prec hx)$

(P1) $(x)(fx \prec gx) . E!x \prec (fx \prec gx)$ (1)

((1): g/f, h/g, 19.68 L&L)

$(x)(fx \prec gx) . E!x . (x)(gx \prec hx) \prec (fx \prec gx) . (gx \prec hx)$ (2)

((2), 11.6 L&L, 14.26 L&L, U.G., P2) QED (3)

T2.23 $(x)(fx \supset gx) . (x)(gx \supset hx) \prec (x)(fx \supset hx)$

(Proof similar to that for T2.22)

2.57 Section 3.

T3.1 $\Diamond \sim E!x$

(T1.2: $E!/f$, 12.76 L&L) $E!x \prec (\exists x)E!x$ (1)

((1), P3, 18.52 L&L) QED (2)

T3.2 $\Diamond\sim(\exists x)fx$

(T1.2, $E!/f$, 12.76 L&L) $E!x \prec (\exists x)E!x$ (1)

((1), 19.51 L&L) $E!x.fx \prec (\exists x)E!x$ (2)

((2), U.G., T2.3) $(\exists x)fx \prec (\exists x)E!x$ (3)

((3), 18.52 L&L) QED (4)

T3.3 $\Diamond\sim\Diamond(\exists x)fx$

(T3.2) $\Diamond\sim(\exists x)\Diamond fx$ (1)

((1), T2.10) QED (2)

There are many true cases of plural existence. Since it is also the case that the formula '$p \prec \Diamond p$' is valid, there are also many cases of consistent plural existence. By T3.3, these cases of consistent plural existence are also cases of contingently consistent plural existence. These are results of the view that plural existence is always contingent, and of some of the above laws governing commutation of modal operators with quantifiers.

Because of these results, the present system is inconsistent with postulate C11 of Lewis and Langford Symbolic Logic. This postulate is one of several speculations of Professor Oskar Becker. These appear in Appendix II of Symbolic Logic, as alternative assumptions concerning iterated modalities. These alternative assumptions are the following:

C10: $\sim\Diamond\sim p \mathrel{-\!\!\!3} \sim\Diamond\sim\sim\Diamond\sim p$

C11: $\Diamond p \mathrel{-\!\!\!3} \sim\Diamond\sim\Diamond p$

C12: $p \mathrel{-\!\!\!3} \sim\Diamond\sim\Diamond p$

and in addition to these, one further alternative:

C13: $\Diamond\Diamond p$

Lewis shows that in a system such as the present one, which assumes postulate C10, postulate C13 has exceptions. Furthermore, if C12 be added to such a system, C11 becomes a theorem. Since the present system is inconsistent with C11, it is also inconsistent with C12.

Hence in the present system, all of the above speculations are decidable. C11, C12, and C13 all have exceptions, while C10 is a postulate.

T3.4 $\Diamond(x)fx$

(T3.2, D3) $\Diamond\sim\sim(x)\sim fx$	(1)
(1) $\Diamond\sim\sim(x)\sim\sim fx$	(2)
((2), 12.3 L&L) QED	(3)

T3.4 affirms that all generalizations are consistent. This view is a result of assuming that it is contingent that something exists and that a generalization to everything that exists is a generalization to everything unrestrictedly. That it is contingent that something exists is equivalent to its being consistent that nothing exists. But a generalization to everything that exists would be guaranteed true were it the case that nothing existed, since the antecedent of that generalization would be always false. Hence any generalization to everything that exists--and therefore any generalization--is consistent. Although T3.4 affirms as consistent, even a universal generalization over an inconsistent property, as in '◇(x)(fx.~fx)', nevertheless, System 1 does not allow as valid '◇[(∃x)E!x.(x)(fx.~fx)]'. That is to say, it is true that everything is (say) red and not red, only provided that nothing exists.

T3.5 ~◇~(fx)⥽~(fx⥽E!x)

(18.52 L&L, T3.1, 12.44 L&L) QED (1)

T3.6 (x)◇fx⊃◇(x)fx

(T3.4, 15.2 L&L) QED (1)

T3.7 ◇(x)fx⊃(x)◇(E!x⊃fx)

(T2.11) QED (1)

T3.8 (x)◇(E!x⊃fx))⊃◇(x)fx

(T3.6) (x)◇(E!x⊃fx))⊃◇(x)(E!x⊃fx) (1)

((1), T2.17) QED (2)

T3.9 ◇(x)fx ≡ (x)◇(E!x⊃fx)

(T3.7, T3.8) QED (1)

T3.10 ◇~(∃x)fx ≡ (x)◇~(E!x.fx)

(T3.9) QED (1)

T3.11 ~◇(x)fx ≡ (∃x)~◇(E!x⊃fx)

(T3.9) QED (1)

2.58 Section 4.

2.581 For the most part, the theorems of section four depend upon the postulate 'E!x'. This postulate is so to speak, tacit, in the system of Principia Mathematica, since while Principia Mathematica contains the restriction that only terms which denote are allowed in the system, P4 of the present system does not appear in Principia Mathematica. The logic of Principia Mathematica is contained in the present system. T4.4 and T4.5 of System I correspond respectively to *10.1 and *10.21 of Principia. However, the inference of these formulas characteristic of Principia Mathematica depends upon the postulate 'E!x'.

2.582 The first three theorems and the fifth are presented in section four because they are key theorems in the inference of the postulates of Principia Mathematica. Unlike the remaining theorems of section four, they do not depend upon P4, and might have given in section two.

T4.1 (x)(p⊃fx) -3 p⊃(x)fx

(P1) (x)(p⊃fx).E!x -3 p⊃fx (1)

((1), 14.26 L&L) (x)(p⊃fx).p -3 E!x⊃fx (2)

((2), U.G., P1) (x)(p⊃fx).p -3 (x)fx (3)

((3), 14.26 L&L) QED (4)

T4.2 p⊃(x)fx -3 (x)(p⊃fx)

(11.7 L&L) p.(p⊃(x)fx) -3 (x)fx (1)

((1), T1.1) p.(p⊃(x)fx) -3 E!x⊃fx (2)

((2), 14.26 L&L) $p \supset (x)fx \dashv\exists E!x \supset (p \supset fx)$ (3)

((6), U.G., P2) QED (4)

T4.3 $(x)(p \supset fx) \Leftrightarrow p \supset (x)fx$

(T4.1, T4.2, 11.03 L&L) QED (1)

T4.4 $(x)fx \supset fx$

(P1, 14.26 L&L, P4) QED (1)

T4.5 $(x)(p \supset fx) \supset (p \supset (x)fx)$

(T4.1, 14.1 L&L) QED (1)

Theorems such as T4.5 and T2.2 are sometimes referred to as "confinement" laws. Such confinement laws can be validly formulated for material connectives without introducing the predicate 'E!'. This fact, plus the validity of T4.4 in a logic allowing only of terms which denote, makes possible in such a logic, a quantified material calculus that does not contain 'E!'.

T4.6 $(\exists x)E!x$

(T1.2) $E!x.E!x \dashv\exists (\exists x)E!x$ (1)

((1), P4) QED (2)

While T4.6 is often taken to be an assumption of logic, in a system such as <u>Principia Mathematica</u> such a theorem cannot be obtained, since a notation for singular existence is not available.

2.59 Section 5.

T5.1 $x=y \supset (fx \dashv\exists fy)$

((T4.4, f/x) $(f)(fx \supset fy) \supset (fx \supset fy)$ (1)

(D4, (1)) $x=y \supset [(fx \prec fx) \supset (fx \prec fy)]$ (2)

((2), 15.8 L&L, 12.1 L&L) QED (3)

T5.2 $x=y \supset \sim\Diamond\sim x=y$

(19.52 L&L: fx/q, fy/r, $E!f/p$, 14.1 L&L)

$(fx \prec fy) \supset (E!f.fx \prec fy)$ (1)

((1), T5.1) $x=y \supset (E!f.fx \prec fy)$ (2)

((2), 14.26 L&L, U.G., 18.7 L&L, T4.3)

$x=y \supset (f)\sim\Diamond\sim(E!f \supset (fx \supset fy))$ (3)

(T2.6, (3), D4) QED (4)

T5.3 $x=y \equiv \sim\Diamond\sim x=y$

(18.42 L&L, T5.2) (1)

T5.3 is true if modal terms are interpreted as terms of the object language. If a is identical with b then everything true of a is also true of b. Hence modal terms of the object language which apply to a also apply to b.

On the other hand, if modal terms are construed as terms of the metalanguage, then those that apply to 'a' may not also apply to 'b'. This is so because 'a' and 'b' are not identical, and therefore not every property of 'a' is a property of 'b'.

For example, the sentence '$\sim\Diamond\sim(Ra \vee \sim Ra)$' meaning, a is red or a is not red, is true; and would be a theorem of the present system, were the constants 'a' and 'R' added to the list of first formulas for System I. Similarly, ''(Rav~Ra)' is analytic' is also true.

However, if an occurrence of 'b' is substituted for one but not for two of the occurrences of 'a' in each of these quoted sentences, and further a is identical with b, then the result of substitution on the first of these sentences is true while the result of substitution on the second sentence is a sentence which would in many languages be false.

In particular, in the language of System I, if 'analytic' were defined to mean a sentence which is a substitution instance of a theorem of System I, then ''(Rav~Rb)' is analytic' would be false.

But '~◇~(Rav~Rb)' makes an assertion about the same thing as does '~◇~(Rav~Ra)', and moreover makes the same claim concerning that thing as does the latter. Hence both of these last two sentences are true.

T5.3 is not valid if definite descriptions of the Russellian sort are allowed to replace the variables in T5.3. No assertion of necessity containing a definite description the scope of which is the sentence or formula to which the sign of necessity is prefixed, is respectivly true or valid. This is because every statement containing a definite description is analysed by Russell into a statement of plural existence; and none of these are analytic. For example, '~◇~(ιx)(fx)=(ιx)(fx)' which is equivalent to, '~◇~((∃x)(x=x.fx).(x)(y)((fx.fy) x=y))' is not only invalid but contravalid as well.

Definite descriptions are involved with T5.3 in the usual formulations of certain "paradoxes" such as the paradox of morning star and evening star and the paradox of analysis.[7] A. F. Smullyan was the first writer to notice that in systems using Russell's analysis of definite descriptions, these "paradoxes" could be traced to certain fallacies involving scopes of definite descriptions,[8] although suggestions of such a solution can be found in the earlier writings of Alonzo Church.[9] W. V. Quine has suggested some of the latest versions of such paradoxes,[10] while Frederic B. Fitch has given what is perhaps one of the latest and most comprehensive analyses of the fallacies involved.[11]

In addition to the above reasons, there is at least one other reason why T5.3 may seem paradoxical. Some logical writing (for instance Frege's) and perhaps informal discourse, employs a sense of identity which is apparently quite different from that of '=' in System I. By 'a is identical with b' is meant something is named by 'a' and by 'b'. Under this interpretation, the

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7. W. V. Quine, "Reference and Modality", in From a Logical Point of View.
__________, "The Problem of Interpreting Modal Logic", The Journal of Symbolic Logic, vol. 12 (1947) p. 43.

8. Arthur Francis Smullyan, Review of "The Problem of Interpreting Modal Logic", The Journal of Symbolic Logic, vol. 12 (1947) p. 139.
__________, "Modality and Description," The Journal of Symbolic Logic", vol. 13 (1948), p. 31.

9. Alonzo Church, The Journal of Symbolic Logic, vol. 7 (1942), p. 100.

10. See footnote 7.

11. Frederic Brenton Fitch, "The Problem of Morning Star and Evening Star", Philosophy of Science, vol. 16, pp. 137-141.

term '=' is taken to involve surreptitous mention of expressions taken as arguments to it. These statements are also statements of plural existence and so are contingent. In consequence of this, T5.3 again fails to be valid with '=' so interpreted.

If definite descriptions are not substituted for variables in T5.3 and the sense of '=' is that of D4, and '$\sim\Diamond\sim$' is given the interpretation of 'necessarily' rather than the interpretation of 'is analytic', then T5.3 loses its paradoxical features. Or rather, T5.3 is a "paradox of necessity" in the same sense that '$\sim p \supset (p \supset q)$' is a "paradox of material implication".

T5.4 $\sim\Diamond\sim x=x$

(12.9 L&L, U.G.)	$(f)(fx \supset fx)$	(1)
((1), D4)	$x=x$	(2)
((2), T5.2)	QED	(3)

T5.5 $E!(\iota x)(fx) \Leftrightarrow (\exists x)(fx).(x)(y)[(fx.fy) \supset x=y]$

(D5, 12.11 L&L)

$E!(\iota x)(gx) \Leftrightarrow (\exists x)(E!x.gx).(x)(y)[(gx.gy) \supset x=y]$ (1)

((1): f/g; T2.18) QED (2)

If the primitive 'E!' takes a definite description as an argument then T5.5 shows that the resultant statement is equivalent as a theorem to a condition which is equivalent to the condition taken in Principia Mathematica to be equivalent by definition to '$E!(\iota x)(fx)$'. Or, put differently, what is essentially the Principia's general definition for '$f(\iota x)(gx)$' reduces in System 1 to what is essentially the <u>Principia's</u> definition for '$E!(\imath x)(fx)$' when '$E!$' is substituted for 'f'.

This last fact is the justification for using the same notation for singular existence in System I as is used in <u>Principia Mathematica</u>.

In T5.6 as in other uses in System I of definite descriptions, the <u>Principia</u> convention that scopes are taken to be the smallest possible when not explicitly indicated may be followed. It is however, unnecessary to introduce scope operators in System I, provided that the rules of transformation are exactly followed. Perhaps this can best be made clear by a consideration of the example used in <u>Principia</u> to justify the introduction of scope operators.

The example chosen in <u>Principia</u> is, except for minor notational differences, the following:[12]

$$f(\imath x)gx \supset p$$

This may be either:

$$[(\exists x)(fx.gx).(x)(y)((gx.gy)\supset x=y)] \supset p$$

or:

$$(\exists x)((fx\supset p).gx).(x)(y)((gx.gy)\supset x=y).$$

But if $\sim E!(\imath x)(gx)$, then the first of these is true and the second is false. It would therefore, seem to be necessary to introduce some such device as the <u>Principia</u> scope operator in order to distinguish between these two cases.

12. <u>Principia Mathematica</u>, *14, <u>Summary</u>.

However, even prior to the introduction of scope operators, only the first of the above translations of the first statement containing the definite description, can be made in System I.

The second is rather a case of '$\hat{x}(fx \supset p)(\iota x)(gx)$'. It might seem plausible that the latter could be inferred from '$f(\iota x)(gx) \supset p$' by substituting upon '$p \mathrel{-\!\!3} p$' to obtain:

$$\hat{x}(fx \supset p)(\iota x)(gx) \mathrel{-\!\!3} \hat{x}(fx \supset p)(\iota x)(gx)$$

then reducing this to obtain:

$$f(\iota x)(gx) \supset p \mathrel{-\!\!3} \hat{x}(fx \supset p)(\iota x)(gx)$$

But the rule of reduction (and the definition of 'reduce to') has been so formulated as not to allow reduction of a sentence consisting of a predicate followed by a definite description.

'$f(\iota x)(gx) \supset p$' can however, be inferred from '$\hat{x}(fx \supset p)(\iota x)(gx)$'.

That ambiguity in sentences and formulas containing definite descriptions can be avoided without the introduction of scope operators is relevant to the modal "paradoxes" as they have been treated by Smullyan and Fitch. Although no explicit rules of reduction appear in <u>Principia</u>, scope operators appear to have been introduced with the intent of distinguishing between sentences which involve definite descriptions as arguments to predicates and are obtainable by one or more reductions on each other.

However, to allow such reductions prior to the introduction of scope operators is to allow invalid rules of inference, as the above inference from a true premise to a false conclusion illustrates. Hence the need for some such device as the scope operator.

In order to obtain the modal paradoxes, such invalid reductions must be allowed in addition to subsequent invalid use of definite descriptions without their then required scope operators.

T5.6 $E!\hat{x}(fx)$

(12.1 L&L, 14.1 L&L, U.G.) $(x)(fx \equiv fx)$ (1)

((1), T4.4) $(\exists g)(x)(fx \equiv gx)$ (2)

((2), T2.18) $(\exists g)(E!g.(x)(fx \equiv gx))$ (3)

((3), D4) Q.E.D.

In spite of T5.6, System I is compatible with Russell's thesis that classes are fictions. T5.6, as well as any other statement containing ambiguous descriptions, is analysed in such a way that the ambiguous descriptions occurring in it are syncategorematic. If for example, the predicate 'R' used above is substituted for 'f' in T5.6, the result of substitution is '$E!\hat{x}(Rx)$'. But this resultant sentence does not literally assert existence of a class, it is rather an assertion with a complex notation for a relation between existence and redness.

Similar remarks concerning scopes and scope operators apply to ambiguous descriptions as were mentioned immediately preceding T5.6 for definite descriptions. If System I is formally followed, then no scope operators need be used, but if reduction is taken to be applicable to ambiguous descriptions, then scope operators are necessary.

The following is a catalog of formulas assumed to be proper in the course of the above proofs.

C1. $E!$

C2. A

C3. $\exists!$

C4. $\hat{x}(fx)$

C5. $\hat{x}(\sim fx)$

C6. $\hat{x}[p ⥽ (E!x \supset fx)]$

C7. $\hat{x}[p ⥽ (x)fx) ⥽ (E!x \supset [p ⥽ (E!x \supset fx)])]$

C8. $\hat{x}(p ⥽ fx)$

C9. $\hat{x}[(x)(p ⥽ fx) ⥽ (E!x \supset [p ⥽ (E!x \supset fx)])]$

C10. $\hat{x}(fx ⥽ gx)$

C11. $\hat{x}((x)(fx ⥽ gx) ⥽ E!x \supset [(x)fx ⥽ (E!x \supset gx)])$

C12. $\hat{x}[(x)fx ⥽ (E!x \supset gx)]$

C13. $\hat{x}(p \vee \sim p ⥽ fx)$

C14. $\hat{x}(p \vee \sim p ⥽ fx)$

C15. $\hat{x}(fx \supset gx)$

C16. $\hat{x}\sim\Diamond\sim(fx \supset gx)$

C17. $\hat{x}\sim\Diamond\sim(E!x \supset fx)$

C18. $\hat{x}[\Diamond\sim(E!x \supset fx)]$

C19. $\hat{x}[\Diamond(E!x \supset fx)]$

C20. $\hat{x}[\Diamond(x)fx \text{⥽} \Diamond(E!x \supset fx)]$

C21. $\hat{x}[\Diamond(E!x \supset \sim fx)]$

C22. $\hat{x}[\Diamond\sim(E!x . fx)]$

C23. $\hat{x}[\sim\Diamond\sim(E!x . fx)]$

C24. $\hat{x}[E!x \text{⥽} fx]$

C25. $\hat{x}(E!x \text{⥽} E!x)$

C26. $\hat{x}[E!x \text{⥽} (fx \supset E!x)]$

C27. $\hat{x}[fx \supset E!x]$

C28. $\hat{x}(E!x \supset fx)$

C29. $\hat{x}[(x)(E! \supset fx) \text{⥽} E!x \supset fx]$

C30. $\hat{x}[fx \text{⥽} (\hat{x}(E!x \supset fx)x)]$

C31. $\hat{x}(E!x \supset (\hat{x}(\sim fx)x))$

C32. $\hat{x}[fx \text{⥽} (E!x \supset fx)]$

C33. $\hat{x}(E!x \supset \sim fx)$

C34. $\hat{x}(\sim\sim fx \supset \sim E!x)$

C35. $\hat{x}((y)fxy)$

C36. $\hat{x}(\hat{x}((y)fxy)x)$

C37. $\hat{y}(fxy)$

C38. $\hat{x}[(x)(y)fxy . E!y \text{⥽} E!x \supset fxy]$

C39. $\hat{x}[(x)(y)fxy . E!y \text{⥽} (\hat{x}(E!x \supset fxy)x)]$

C40. $\hat{y}[(x)(y)fxy \text{⥽} E!y \supset (x)fxy]$

C41. $\hat{y}[(x)(y)fxy ⥽ (\hat{y}(E!y \supset (x)fxy)y)]$

C42. $\hat{y}((x)fxy)$

C43. $\hat{x}[(x)(fx ⥽ gx).(x)(gx ⥽ hx) ⥽ E!x \supset (fx ⥽ Hx)]$

C44. $\hat{x}[(x)(fx ⥽ gx).(x)(gx ⥽ hx) ⥽ (\hat{x}(E!x \supset (fx ⥽ hx))x)]$

C45. $\hat{x}[(x)(fx \supset gx).(x)(gx \supset hx) ⥽ E!x \supset (fx \supset hx)]$

C46. $\hat{x}[(x)(fx \supset gx).(x)(gx \supset hx) ⥽ (\hat{x}(E!x \supset (fx \supset hx))x)]$

C47. $\hat{x}(\sim E!x)$

C48. $\hat{x}[E!x.fx ⥽ (\exists x)E!x]$

C49. $\hat{x}(\sim(\hat{x}(\sim fx)x)$

C50. $\hat{x}(\Diamond fx)$

C51. $\hat{x}(p \supset fx)$

C52. $\hat{x}[(x)(p \supset fx).p ⥽ E!x \supset fx]$

C53. $\hat{x}[(x)(p \supset fx).p ⥽ (\hat{x}(E!x \supset fx)x)]$

C54. $\hat{x}[p \supset (x)fx ⥽ E!x \supset (p \supset fx)]$

C55. $\hat{f}(fx \supset fy)$

C56. $\hat{y}(fx \supset fy)$

C57. $\hat{f}[x=y \supset \sim\Diamond\sim(E!f \supset (fx \supset fy))]$

C58. $\hat{f}[\sim\Diamond\sim(E!f \supset (fx \supset fy))]$

C59. $\hat{f}[x=y \supset (\hat{f}(\sim\Diamond\sim(E!f \supset (fx \supset fy)))f)]$

C60. $\hat{f}(fx \supset fx)$

C61. $\hat{x}(\sim E!x.gx)$

C62. $\hat{x}[(y)[(gx.gy) \supset x=y]]$

C63. $\hat{y}[(gx.gy) \supset x=y]$

C64. $\hat{x}(fx \equiv fx)$

C65. $\hat{g}(\sim(x)(fx \equiv gx))$

C66. $\hat{x}(fx \equiv gx)$

C67. $\hat{g}(\sim(E!g.(x)(fx \equiv gx)))$

CHAPTER III

OBLIQUE DISCOURSE AND CONNOTATIVE LOGIC: SYSTEM II

3.1 The paradoxes of the theory of types were avoided in System I by the theory of prerequisites together with the assumption that "$\hat{x}(\sim(xx))$" and other "paradoxical predicates" do not denote anything.

While this assumption may be looked upon as to some degree justified simply by its avoidance of paradox and by a certain intuitive appeal, nevertheless in the absence of more conclusive evidence, it seems to have the rather unsatisfactory appearance of having been introduced ad hoc.

3.2 The purposes of the present section are to suggest a system of logic within which the above questions may be more critically investigated and to apply the resulting system to the investigation of the particular issue of whether or not "$\hat{x}(\sim(xx))$" denotes something.

3.3 The description of the following logic is not intended to be complete. The following is an account of some of the more salient features of a logic suitable to the above purposes.

3.4 The questions concerning singular existence which are prerequisites in System I, cannot themselves be investigated in System I, because these are issues which must be

settled prior to an application of System I. This is why a new system of logic must be developed to investigate these questions of singular existence.

3.5 Even if a given term does not denote any existent thing, the term itself exists. This fact suggests a metalinguistic approach to the investigation of the prerequisites to System I. The central question concerning prerequisites in connection with a term, would be whether or not the term denoted something. If investigation revealed that a term "a" denoted something, then the sentence "E!a" would be true; and if "a" did not denote anything, then "E!a" would not be true--or false.

This approach to the investigation of prerequisites would be quite in accord with previous suggestions to the effect that an oblique mode of discourse is necessary to carry out those investigations.

To discourse in such a way as to mention terms is to use those terms obliquely since a term in quotes is not used in order to talk about something denoted by the term, but rather as a syncategorematic part of a larger expression consisting of the term in question enclosed in quotes, which is used to talk about the term itself.

3.6 Though such a metalinguistic approach may seem promising, it will not be followed here. The drawback of this approach as far as the present system is concerned consists in its adoption of semantical terms such as "denotation" as technical terms of the system. The explication of the meaning of such terms is beyond the scope of the present discussion.

3.7 Although the present system will avoid semantical terms as formal devices, it nevertheless will be a system of oblique reference. The mode of reference will be that of connotation. The meanings of "connotation" and of "denotation" that are intended, are those which were introduced informally in Chapter 1. Although no extremely precise explication of these terms will be attempted, since these terms will be used only to talk about the system rather than in it, some discussion of the present usage, in part by way of review of the discussion of Chapter 1, would seem to be appropriate.

3.8 Both connotation and denotation are modes of reference. Each mode of reference is analogous to the relation of naming in that, just as any given term names at most one thing, so there is at most one thing denoted or connoted by any term. Of these two modes of reference, denotation is most similar to the relation of naming, and in fact is here taken to be synonymous with it.

The single things which are denoted and connoted by a term will be referred to respectively as that term's denotatum and connotatum. The connotatum of any term or of any sentence is always a property.

The connotatum of a term is a characteristic which is so to speak, a definitional criterion by means of which one identifies the denotatum of the term. A candidate for a denotatum of a term may be accepted or rejected as that term's denotatum, accordingly as it possesses or fails to possess the connotatum of that term.

3.9 The primary purpose for which System II will be used will be to investigate questions of singular existence. A system of connotative logic could investigate the subject of connotative discourse generally, after the analogy of Russell's general analysis of definite descriptions in any context in which they might occur, by means of the contextual definition:

$$f(\iota x)gx =Df\ (\exists x)(y)((gy \equiv x=y).fx). \qquad (1)$$

But rather than this, the present investigation will be concerned only with the analysis of connotative assertions of existence, after the analogy of Russell's analysis of this particular context for definite descriptions:

$$E!(\iota x)fx =Df\ (\exists x)(y)((fy \equiv x=y) \qquad (2)$$

The reason for this restriction is of course, that the primary purpose for which System II is used here, is to investigate the prerequisites of System I.

3.10 An expression enclosed in angle brackets such as "$\langle a \rangle$" will be taken to be a name of the connotatum of the expression enclosed in such brackets. Since the

connotatum of an expression is a characteristic possessed only by a denotatum of the expression, a statement of singular existence in the connotative mode of interpretation will be expressed by asserting plural existence to be a property of a terms connotatum:

$$\exists! \langle a \rangle \tag{3}$$

"Santa Claus does not exist." or, "There is no Santa Claus." might be written:

$$\sim \exists! \langle s \rangle \tag{4}$$

Sentences as well as terms may be enclosed in such angle brackets and in this case, the connotatum named by the bracketed expression together with the brackets enclosing it, is a characteristic satisfied only by a state of affairs which the enclosed sentence denotes.

In general, a criterion for the truth of a sentence of the form:

$$\exists! \langle F \rangle \tag{5}$$

where F is a sentence, will be taken to be whether or not F denotes a state of affairs.

3.11 Just as denotative logic assumes that every term used in the logic denotes something, so connotative logic assumes that every term used in it connotes something.

Rather than being developed as an independent system, the following logic will be formulated as an application of System I.

Because of the above two points, the following system cannot be formulated with just one kind of propositional and non-propositional variables. The variables of System I will be employed in System II with their previous restriction that only terms which denote may be involved in any formulas substituted for them. In addition, in System II, the letters "u", and "v" will be used as propositional variables, while the letters "a", and "b" will be used as non-propositional variables, each with the restriction that only terms which connote may be substituted for them.

3.12 The following is a listing of principles of System II. This listing is not a list of postulates for connotative logic, but rather a combination of what might be both postulates and theorems of connotative logic, in a development which proceeded more rigorously than the present one.

P1: $\exists! \langle (u_1 \supset u_2) \rangle \cdot \exists! \langle (u_2 \supset u_3) \rangle \supset \exists! \langle (u_1 \supset u_3) \rangle$

Principle 1 might be called "The Principle of Connotative Transitivity".

P2: $\exists! \langle u \rangle \cdot \exists! \langle u \supset v \rangle \supset \exists! \langle v \rangle$

Principle 2 might be called "The Principle of Modus Ponens".

P3: $\exists! \langle u \rangle \supset \sim (\exists! \langle \sim u \rangle)$

Principle 3 might be called "The Principle of

Double Negation". The converse of Principle 3 is not valid. Some sentences are such that neither they nor their negates denote a state of affairs. For example, neither the sentence "Santa Claus wears a red suit." nor its negate, "Santa Claus does not wear a red suit" indicates a state of affairs about someone denoted by the term "Santa Claus".

P4: $$\exists!\langle u.v\rangle \equiv (\exists!\langle u\rangle . \exists!\langle v\rangle)$$

Principle 4 will be called "The Principle of Conjunctive Distribution".

P5: $$\exists!\langle u \vee v\rangle \supset (\exists!\langle u\rangle \vee \exists!\langle v\rangle)$$

Principle 5 will be called "The Principle of Disjunctive Distribution". Unlike the Principle of Conjunctive Distribution, which is true biconditionally, the converse of the Principle of Disjunctive Distribution is not true. Let "R_c" abbreviate "Santa Claus wears a red suit". Let "W" abbreviate "Snow is white.". It is true that "$\exists!\langle R_c\rangle \vee \exists!\langle W\rangle$", because it is true that "$\exists!\langle W\rangle$" and that everything which functions as a term in the sentence denotes something, and therefore the disjunction as a whole is true. However, the sentence "$\exists!\langle R_c \vee W\rangle$" is false (and not merely untrue), because "c" does not denote anything, and therefore the sentence "$R_c \vee W$" cannot denote a state of affairs about something denoted by "c".

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P6: $\exists!\langle u\rangle \equiv \exists!\langle\sim\sim u\rangle$

Principle 6 is another form of a principle of double negation.

P7: $\exists!\langle u\rangle \vee \sim\exists!\langle u\rangle$

Principle 7 is a law of excluded middle. Principle 7 is actually not a special principle of System II, but is rather merely a direct application of the usual law of excluded middle of System I. P7 is mentioned here mainly in order to allow comparison of it with the invalid "law of excluded middle": "$\exists!\langle u\rangle \vee \exists!\langle\sim u\rangle$".

P8: $(\exists!\langle a\rangle . \exists!\langle b\rangle) \supset (\exists!\langle ab\rangle \vee \exists!\langle\sim ab\rangle)$

Principle 8 is another law of excluded middle which is very similar to the invalid "law" mentioned immediately above P8. Through the use of non-propositional variables, the fact can be expressed that the invalid "law" above is true under a restricting condition. Principle 8 expresses the fact that so to speak, there are no exceptions "in nature" to the law of excluded middle; or in other words, that everything either possesses or fails to possess any given property.

In addition to stating such particular principles of connotative logic as those above, certain general meta-linguistic rules can be formulated which describe how valid laws of denotative logic can be transformed into valid laws of connotative logic.

Rule 1: Form the negate of any valid formula of the material propositional calculus. Replace each occurrence of a denotative propositional variable in the resulting formula with an occurrence of some connotative propositional variable. Enclose the resulting formula in angle brackets, and precede the whole with the expression "$\llcorner\exists!$". The resulting formula will be a valid formula of connotative logic.

Rule 2: If $F_1 \supset F_2$ is any valid formula of the propositional calculus, such that every variable an occurrence of which is in F_2, also has at least one occurrence in F_1, then form $\exists!\langle F_1\rangle \supset \exists!\langle F_2\rangle$. Replace each occurrence of any denotative propositional variable in the resulting formula with an occurrence of some connotative propositional variable. The resulting formula will be a valid formula of connotative logic.

The above 8 principles and two rules suffice to determine a considerable amount of material connotative logic.

3.13 Since System II is being developed within System I, the methods of inference of System I are also available in System II.

An additional rule for System II is a rule for substitution upon free connotative variables, analogous to the rules of System I for substitution upon free denotative variables.

Two further rules, or perhaps extensions of previous rules, are a rule of reduction, and a rule for exchange in accordance with definitions, when the expressions which are respectively reduced or exchanged, occur within angle brackets and hence may not denote anything.

3.14 If the above principles and methods of connotative logic are employed, the formula "$\sim\exists!\langle k\rangle$" may be proved to be a theorem of System II (where "k" is Russell's predicate) as follows:

(From P3: k_k/u) $\exists!\langle k_k\rangle \supset \sim(\exists!\langle\sim k_k\rangle)$ (1)

((1), and exchange in accordance with the definition: $k =_{Df} \hat{f}(\sim f_f)$)

$\exists!\langle k_k\rangle \supset \sim(\exists!\langle\sim \hat{f}(\sim f_f)k\rangle)$ (2)

(From (2) by reduction) $\exists!\langle k_k\rangle \supset \sim(\exists!\langle\sim\sim k_k\rangle)$ (3)

(From P6) $\sim\exists!\langle\sim\sim(k_k)\rangle \supset \sim\exists!\langle k_k\rangle$ (4)

(From (3), and (4)) $\exists!\langle k_k\rangle \supset \sim\exists!\langle k_k\rangle$ (5)

(From (5)) $\sim\exists!\langle k_k\rangle$ (6)

(From (6) by exchange in accordance with the definition of "k")

$\sim\exists!\langle\hat{f}(\sim f_f)k\rangle$ (7)

(From (7) by reduction) $\sim\exists!\langle\sim k_k\rangle$ (8)

(From (6), and (8) by adjunction)

$\sim\exists!\langle k_k\rangle \cdot \sim\exists!\langle\sim k_k\rangle$ (9)

(From P8: k/a; k/b) $(\exists!\langle k\rangle \cdot \exists!\langle k\rangle) \supset (\exists!\langle kk\rangle \vee \exists!\langle\sim kk\rangle)$ (10)

(From (10)) $\exists!\langle k\rangle \supset (\exists!\langle kk\rangle \vee \exists!\langle \sim kk\rangle)$ (11)

(From (9), and (11)) $\sim \exists!\langle k\rangle$ (12)

3.15 Principle 8 and its use in the above proof suggest that investigation of principles of connotative logic which involve non-propositional, and possibly, quantified variables, might prove fruitful.

However, it seems very unlikely that connotative logic will admit of a calculus of non-propositional variables which will have an importance in connotative logic, comparable to the importance of the theory of quantifiers in denotative logic.

One of the difficulties lies in finding an interpretation for bound variables which are enclosed in angle brackets and bound by capped variables or quantifiers which are outside the angle brackets. A similar problem of interpretation arises in all oblique modes of speech.

An expression occurs obliquely in a given sentence if and only if:

(1) The expression in question purports to denote something (or is a term), and occurs in the sentence.

and, (2) When taken as asserted, the sentence does not purport to denote a state of affairs concerning, or to discourse concerning, something purportedly denoted by the expression.

A frequent oblique mode of speech is mention. The term "water" in the sentence, ""Water" has five letters." satisfies the two conditions above, and therefore occurs obliquely in this sentence.

Because of these difficulties of interpretation, such fundamental methods of inference as Universal and Existential generalization, are unavailable for connotative variables. The absence of these methods is in itself a serious limitation upon connotative logic insofar as it treats of non-propositional variables.

3.16 As a final attempt to give some explication of the use of angle brackets in this chapter, it may be helpful to indicate that $\hat{g}(f)(gf \equiv {\sim}ff)$ is plausible as a connotatum of "$\hat{f}({\sim}(ff))$". Or, put differently, that:

$$\langle \hat{f}({\sim}ff) \rangle = \hat{g}(f)(gf \equiv {\sim}ff)$$

(The present author hopes that "$\hat{g}(f)(gf \equiv {\sim}ff)$" denotes something.)

CHAPTER IV

APPLICATIONS

4.1: Paradoxes of Logic

4.1 Russell's paradox can be avoided on the grounds that,

$$\sim \exists ! \langle k \rangle, \tag{1}$$

and (1) was proven as a theorem in the course of developing System II. In this paradox, 'k' is defined:

$$k =_{Df} \hat{x}(\sim xx). \tag{2}$$

Paradoxes very similar to Russell's are given in Principia Mathematica. These paradoxes involve relations which are analogous to Russell's monadic predicate. One such relation is there defined:

$$R =_{Df} \hat{x}\hat{y}(\sim xxy). \tag{3}$$

Substitution of 'RRR' for 'p' in 'p $\equiv$ p' yields:

$$(RRR) \equiv (RRR). \tag{4}$$

Substitution on (4) in accordance with (3), and subsequent reduction, yields:

$$(RRR) \equiv \sim (RRR). \tag{5}$$

Although it will not be done here, a proof similar to that for (1) can be given in System II for:

$$\sim \exists ! \langle R \rangle. \tag{6}$$

By this means, the paradoxical (5) can be avoided in much the same way as Russell's paradox. Relations with definitions similar to (2) and (3) can be given for any number of arguments. Each such relation gives rise to a paradox. All such paradoxes are avoidable by a proof of non-existence for the relation on which each depends. Each such proof can be given in System II.

Grelling's paradox is a paradox rather different from any of these. We might examine how this paradox can be dealt with in System II.

Grelling's paradox arises from considerations such as the following. Some terms are applicable to themselves, others are not. For instance 'short' is applicable to itself, as is 'word', while 'elephant' is not. Let us define 'Heterological', (abbreviated 'Het') to mean non-self-applicable. If 'Het' is heterological, then it is not heterological; and if 'Het' is not heterological, it is heterological.

Abbreviating 'is applicable to' by 'App', 'Het' may be defined as:

$$\text{Het} =_{\text{Df}} \hat{x}(\sim x\text{App}x). \qquad (7)$$

Grelling's paradox may be obtained as follows. Taking,

$$\text{'Het'App'Het'} \equiv \text{Het'Het'}, \qquad (8)$$

as a postulate, we can from,

$$\text{Het'Het'} \equiv \text{Het'Het'}, \qquad (9)$$

by means of a definitional exchange in accordance with

(7), subsequent reduction and use of (8), obtain:

$$\text{Het'Het'} \equiv \sim\text{Het'Het'}. \tag{10}$$

(10) is the contradiction of Grelling's paradox.

Another rule in addition to Rules 1 and 2 of System II, governing transformation of valid laws of denotative logic into valid laws of connotative logic, might be formulated as follows.

Rule 3: If F_2 is any valid formula of material logic that contains no propositional variables, and F_1 is a conjunction of formulas which are substitution instances of '$\exists!\langle a\rangle$' by replacement of 'a' with a variable, and F_1 contains at least one instance of every variable of which F_2 contains an instance; then $F_1 \supset \exists!\langle F_2\rangle$ is a valid formula of connotative logic.

If Rule 3 is added to System II, then,

$$\sim(\exists!\langle\text{Het}\rangle .\ \exists!\langle\text{'Het'}\rangle . \exists!\langle\text{'Het'App'Het'} \equiv \text{Het'Het'}\rangle), \tag{11}$$

can be proven as follows.

(By Rule 3) $(\exists!\langle a\rangle . \exists!\langle b\rangle) \supset \exists!\langle ab \equiv ab\rangle$ (a)

((a): Het/a; 'Het'/b)

$(\exists!\langle\text{Het}\rangle . \exists!\langle\text{'Het'}\rangle) \supset \exists!\langle\text{Het'Het'} \equiv \text{Het'Het'}\rangle$ (b)

((b), exchange in accordance with (7), reduction)

$(\exists!\langle\text{Het}\rangle . \exists!\langle\text{'Het'}\rangle) \supset \exists!\langle\text{Het'Het'} \equiv \sim\text{Het'App'Het'}\rangle$ (c)

(By Rule 2)

$\exists!\langle aba \equiv ca\rangle \supset \exists!\langle \sim aba \equiv \sim ca\rangle$ (d)

((d): 'Het'/a; App/b; Het/c)

$\exists!\langle \text{'Het'App'Het'} \equiv \text{Het'Het'}\rangle \supset$

$\exists!\langle \sim\text{'Het'App'Het'} \equiv \sim \text{Het'Het'}\rangle$ (e)

((c), (e)) $(\exists!\langle \text{Het}\rangle . \exists!\langle \text{'Het'}\rangle . \exists!\langle \text{'Het'App'Het'}$

$\equiv \text{Het'Het'}\rangle) \supset (\exists!\langle \text{Het'Het'} \equiv \sim \text{'Het'App'Het'}\rangle .$

$\exists!\langle \sim \text{'Het'App'Het'} \equiv \sim \text{Het'Het'}\rangle)$ (f)

Rule 2) $\exists!\langle (ab \equiv \sim bcb).(\sim bcb \equiv \sim ab)\rangle \supset$

$\exists!\langle ab \equiv \sim ab\rangle$ (g)

((g): Het/a; 'Het'/b; App/c)

$\exists!\langle (\text{Het'Het'} \equiv \sim \text{'Het'App'Het'}).(\sim \text{'Het'App'Het'}$

$\equiv \sim \text{Het'Het'})\rangle \supset \exists!\langle \text{Het'Het'} \equiv \sim \text{Het'Het'}\rangle$ (h)

((f), (h), by Conjunctive Distribution)

$(\exists!\langle \text{Het}\rangle . \exists!\langle \text{'Het'}\rangle . \exists!\langle \text{'Het'App'Het'} \equiv$

$\text{Het'Het'}\rangle) \supset \exists!\langle \text{Het'Het'} \equiv \sim \text{Het'Het'}\rangle$ (i)

(By Rule 1) $\sim \exists!\langle \sim (ab \equiv \sim ab)\rangle$ (j)

((j): Het/a; 'Het'/b)

$\sim \exists!\langle \sim (\text{Het'Het'} \equiv \sim \text{Het'Het'})\rangle$ (k)

(From P6 of System II) $\exists!\langle \text{Het'Het'} \equiv \sim \text{Het'Het'}\rangle$

$\supset \exists!\langle \sim (\text{Het'Het'} \equiv \sim \text{Het'Het'})\rangle$ (l)

((k), (l)) $\sim \exists!\langle \text{Het'Het'} \equiv \sim \text{Het'Het'}\rangle$ (m)

((i), (m)) QED (n)

In order that the Grelling paradox be avoided, it is not necessary to maintain that any particular one of the factors to the conjunction in (i) is false. Since

$\exists!\langle aba \equiv ca\rangle \supset \exists!\langle \sim aba \equiv \sim ca\rangle$ (d)

((d): 'Het'/a; App/b; Het/c)

$\exists!\langle \text{'Het'App'Het'} \equiv \text{Het'Het'}\rangle \supset$

$\exists!\langle \sim\text{'Het'App'Het'} \equiv \sim \text{Het'Het'}\rangle$ (e)

((c), (e)) $(\exists!\langle \text{Het}\rangle . \exists!\langle \text{'Het'}\rangle . \exists!\langle \text{'Het'App'Het'}$

$\equiv \text{Het'Het'}\rangle) \supset (\exists!\langle \text{Het'Het'} \equiv \sim\text{'Het'App'Het'}\rangle .$

$\exists!\langle \sim\text{'Het'App'Het'} \equiv \sim\text{Het'Het'}\rangle)$ (f)

Rule 2) $\exists!\langle (ab \equiv \sim bcb).(\sim bcb \equiv \sim ab)\rangle \supset$

$\exists!\langle ab \equiv \sim ab\rangle$ (g)

((g): Het/a; 'Het'/b; App/c)

$\exists!\langle (\text{Het'Het'} \equiv \sim\text{'Het'App'Het'}).(\sim\text{'Het'App'Het'}$

$\equiv \sim \text{Het'Het'})\rangle \supset \exists!\langle \text{Het'Het'} \equiv \sim\text{Het'Het'}\rangle$ (h)

((f), (h), by Conjunctive Distribution)

$(\exists!\langle \text{Het}\rangle . \exists!\langle \text{'Het'}\rangle . \exists!\langle \text{'Het'App'Het'} \equiv$

$\text{Het'Het'}\rangle) \supset \exists!\langle \text{Het'Het'} \equiv \sim\text{Het'Het'}\rangle$ (i)

(By Rule 1) $\sim\exists!\langle \sim (ab \equiv \sim ab)\rangle$ (j)

((j): Het/a; 'Het'/b)

$\sim\exists!\langle \sim (\text{Het'Het'} \equiv \sim\text{Het'Het'})\rangle$ (k)

(From P6 of System II) $\exists!\langle \text{Het'Het'} \equiv \sim\text{Het'Het'}\rangle$

$\supset \exists!\langle \sim (\text{Het'Het'} \equiv \sim \text{Het'Het'})\rangle$ (l)

((k), (l)) $\sim\exists!\langle \text{Het'Het'} \equiv \sim\text{Het'Het'}\rangle$ (m)

((i), (m)) QED (n)

In order that the Grelling paradox be avoided, it is not necessary to maintain that any particular one of the factors to the conjunction in (l) is false. Since

the derivation of the paradox depends on each of these factors being true, to show that their conjunction is false is sufficient to avoid the paradox.

Another paradox for which interesting results can be obtained by application of the foregoing logic is the following.

fa and ~fa appear to share such consequences as E!a and $(\exists x)(fx \vee \sim fx)$, but since these consequences are contingent, it follows that $\Diamond(fa . \sim fa)$, or, that fa and ~fa are not contradictories.

W. V. Quine has made some comments to this paradox in a review of a discussion of it by Everett J. Nelson.[1] Since Mr. Quine's review raises several important questions concerning both this paradox and other closely allied paradoxes, the following discussion of the above paradox will follow this review in the points raised in it.

Mr. Quine opens his discussion by making the well taken point that "the supposed existential consequences of 'fa' and '~fa'" should be questioned. However, Mr. Quine goes on to suggest as evidence for this that the entity a is not a "constituent of the propositions fa and ~fa", and as evidence for this in turn:

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1. Everett J. Nelson, "Contradiction and the presupposition of existence", Mind, n.s.v. 55 (1946), pp. 319-27. Review by W. V. Quine, The Journal of Symbolic Logic, vol. 12, p. 52.

> ... consider the propositions Hesperus is the Evening Star and Hesperus is the Morning Star. Being respectively analytic and synthetic, these propositions are distinct: yet the supposed constituents Evening Star and Morning Star wherein alone they can differ are one and the same thing. Clearly we must view the constituents not as the Evening Star and the Morning Star (i.e., the planet named by 'Evening Star' and 'Morning Star'), but rather as the respective meanings of 'Evening Star' and 'Morning Star'.

Without here investigating all of the very complex questions raised by this comment (such as what 'being a constituent of a proposition' means, under what conditions propositions are distinct, whether or not the morning star and the evening star are the only constituents of the propositions in question wherein they can differ, and whether named things or meanings are involved in propositions); it will only be argued here that the contention that Hesperus is the Evening Star and Hesperus is the Morning Star are respectively analytic and synthetic, is highly questionable.

Some of the questions involved in the problem of when statements of identity are analytic and when contingent, were raised during the elaboration of the system above. In application of these results, it is the case that, if the phrases 'Evening Star' and 'Morning Star' are meant to be definite descriptions in the sense of, say, Russell; then both Hesperus is the Evening Star and

Hesperus is the Morning Star are contingent for the reasons given in the discussion in the system. On the other hand, if these phrases are meant to be names then both Hesperus is the Evening Star and Hesperus is the Morning Star are analytic. But in neither case will one of these be analytic and the other contingent.

In any case, the original point still stands that fa does not always entail E!a.

Mr. Quine mentions in passing (as they will also be mentioned here) some further paradoxes such as that while 9 is 9, is analytic; 9 is the number of planets, is contingent.

Mr. Quine next examines certain questions concerning the contrast between meaning and naming and the conditions under which sentences of the form 'E!x' can be inferred from premises of the form 'fx'. These issues lie outside the scope of the present discussion since they are topics involving metalogic rather than modal connections. Yet in this discussion Mr. Quine does suggest certain conditions under which fx may entail E!x. He suggests that logical functions of E! entail existence, but that other functions might be regarded as unknowable as to whether or not they entail existence. The positive part of this comment would certainly seem to be correct, and indeed to constitute one of the soundest means available for determining if fx entails $E!x$. On the other hand the negative part of the suggestion is not always true, since it can be proven that analytic properties do not entail existence (Theorem 3.5).

Finally, Mr. Quine very nearly recognises the position that is here being proposed when he suggests that "the usual convenient techniques of logic which allow proof of '$(\exists x)(fx \vee {\sim}fx)$' and inference of '$(\exists x)fx$' from '$(x)fx$' can be accepted as a semi-logical amalgam, comprising pure logic plus a true extra-logical premise to the effect that there is something." The only inaccuracy in this statement as a description of the present position is that the true extra-logical premise is not that 'something exists' but rather the formula '$E!x$', the result that something exists being a theorem, not a postulate.

4.2: Identity of States of Affairs.

Truth and falsity are ways of being representative. A sentence is true or false accordingly as it does or does not have a denotatum. As mentioned earlier, the denotata of sentences are here taken to be states of affairs (and the term "proposition" is here taken to be synonymous with "state of affairs").

Often, in spite of what has just been said, the terms "true" and "false" are applied to states of affairs (or propositions); rather than sentences. Moreover, it is clear from an examination of such instances of usage that the meanings of "true" and "false" are not species representativeness. Some authors have suggested that such usages should be dismissed as mère confusion; on the grounds that only sentences can be true or false while propositions or states of affairs simply either exist or do not exist.

But putting the matter this way actually serves to bring to light the quite legitimate meaning which "truth" and "falsity" have in application to states of affairs.

Using "P" for the property of being a proposition, the terms "true" and "false" might be defined in this second sense as follows:

$$Tx =_{Df} Px.E!x \tag{1}$$

$$Fx =_{Df} Px.\sim E!x \tag{2}$$

Among the consequences of these definitions are the facts that there are no false states of affairs, but rather, all states of affairs (or propositions) are true:

$$\sim(\exists x)(Px.Fx) \tag{3}$$

$$(x)(Px \supset Tx) \tag{4}$$

(3) and (4) are consequences of Th. 2.8; the principle that everything exists.

Furthermore, these results are consonant with the view stated above that true sentences have denotata, while false sentences do not. If a true sentence is a sentence that denotes a true state of affairs; then all sentences that are not true (e.g. false sentences) do not denote a true state of affairs. But sentences that do not denote a true state of affairs do not denote any state of affairs (again because everything exists). Therefore false sentences do not denote states of affairs.

This view of true sentences as posessing denotata, while false sentences do not, raises an apparent problem concerning the connectives of logic.

When we assert a material conditional:

$$A \supset B \tag{5}$$

it would seem natural to interpret such a statement as asserting that the state of affairs denoted by "A" stands in a certain relation (material implication) to another state of affairs denoted by "B". However such a view cannot be the case when "A" and "B" are false; since in this case there are no states of affairs denoted by "A" and "B" to stand in the relation of material implication (or any other relation) to one another. Obviously, similar remarks could be made about other logical connectives as well--what can the connective "~" be applying to in the sentence "~A" if "A" is false?

The answer to this difficulty suggested here is that the connectives of propositional logic are all relations with twice as many argument places as propositional variables which they connect. These arguments are properties and individuals, rather than states of affairs. For instance, rather than writing a material implication as:

$$p \supset q \qquad (6)$$

we might more appropriately write this formula as:

$$(f,x) \supset (g,y) \qquad (7)$$

construing "$\supset$" as denoting a relation of four argument places rather than two. If this is done, then of course the above problem about what is being related by the relation of material implication when sentences which are arguments to "$\supset$" are false disappears. What is being related are properties and perhaps individuals, but in any case things which exist even if they do not result in a state of affairs.

Again, similar comments apply to the other connectives such as "$\sim$". The curl can be construed as a dyadic relation, and the formula "$\sim p$" written as:

$$\sim(f,x) \qquad (8)$$

Such a construal of the propositional connectives need not involve the abandonment of the use of propositional variables however. In fact, such a construal need not involve any alterations in the usual propositional logic. Propositional variables can be viewed as, so to speak, "dummy" variables and the expressions which are substituted for them as determining the arguments to the connective in question though not denoting these arguments.

In short, the change being suggested is only a change in interpretation of the usual propositional calculus rather than a change in the calculus itself.

What this suggested change amounts to is the view that sentences which occur as arguments to propositional connectives are not terms (although the same sentence might very well be a term in other contexts).

However, if this view be a correct one, it has some significant consequences for the problem of under what circumstances states of affairs or propositions are identical with one another.

An important question in this regard is: if propositions are such that each is deducible from the other, then are they always identical with one another?

Suppose we were to attempt to formulate this question as follows: "Is it the case that:

$$p \Leftrightarrow q : \supset : p = q \quad ?" \tag{9}$$

The sign of strict equivalence in the antecedent of (9) however does not relate states of affairs denoted by substitution instances of "p" and "q" while the sign of identity in the consequent of (9) does. For this reason the problem which has just been eliminated in the case of propositional connectives is raised again in (9) in the case of the sign of identity occurring in it. If false sentences are substituted for "p" and "q" in (9) then since these sentences will occur as terms, the resulting sentence will be purporting to say something about their denotata when in fact there are no such denotata.

Evidently, (9) is not a satisfactory formulation of the problem. However, the fact that true sentences do denote states of affairs might suggest that it could be satisfactorily formulated under the restriction that the substitution instances of "p" and "q" be true. (Of course this restriction cannot be formulated merely by asserting (9) to be materially implied by the condition that "p.q" be true:

$$(p.q) \supset (p \Leftrightarrow q : \supset : p = q) \tag{10}$$

for (10) involves the same problem as does (9); namely that since substitution instances of "p" and "q" occur as terms in the sentence as a whole, the sentence purports to denote some fact or state of affairs about things denoted by those terms which do not denote anything.)

The question might however, be formulated as follows. Let the term "P" be defined as:

xPy, f : x is a proposition analysable into y's having the property f.

Now the restricted form of the problem might be put:

$$(fx) \Leftrightarrow (gy) .\ zPx,f.\ wPy,g :\supset: z = w \tag{11}$$

This last formulation avoids the difficulties raised above in that any substitution instance of (11)

will not contain terms that do not denote provided that only denoting terms are substituted for free variables in (11).

A perhaps interesting fact; which, though it is by no means conclusive evidence against (11), might nevertheless lead one to question it; is that if the usual logical assumption that any given property and any given subject of that property determine at most one state of affairs, is extended to the further assumption that any given state of affairs and any given property determine at most one subject of that property; then (11) is false.

Suppose it to be the case that:

$$xPz,f.\ yPw,f.\ x = y :\supset: z = w \qquad (12)$$

Let "s" name the sun, and "m" name the moon. Since:

$$s \neq m \qquad (13)$$

we can infer from (12) and (13) that:

$$xPs,f.\ yPm,f :\supset: x \neq y \qquad (14)$$

Further, let "H" abbreviate "heavy or not heavy". Since both Hs and Hm are analytic, it follows that:

$$Hs \Leftrightarrow Hm \qquad (15)$$

Yet from (14) it follows that:

$$xPs,H.\ yPm,H :\supset: x \neq y \qquad (16)$$

It is also true that since H is an analytic property; Hs and Hm are true, or determine states of affairs, and that therefore:

$$(\exists x)(xPs,H) . (\exists y)(yPm,H) \qquad (17)$$

But from (15), (16), and (17), the denial of the principle expressed in formula (11) is deducible.

As was mentioned earlier, this result is not conclusive evidence against (11). However the result does make it seem doubtful that the intuitive acceptability of (11) is in itself enough to settle the matter.

In any case, the primary purpose of the present argument is not to examine the question of under what conditions propositions are identical. The purpose of the present discussion is rather to show that an adequate treatment of this question will be involved with the logical problems raised by terms that do not denote.

4.3: A Problem in the Theory of Perception

There is a further problem; in the theory of perception, which calls for a logic of terms which do not denote. Many authors have discussed this, or related problems; however, the present account will make primary reference to a paper on perception by H. A. Prichard.[2]

Prichard asks the question: "What do we see?", and he concludes that although the naive answer is "bodies" this is not true because it cannot account for illusions

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2. H. A. Pritchard, "Perception", in <u>Knowledge and Perception</u> Oxford University Press, 1950, P. 52.

and that in fact, in doing what we ordinarily call "seeing a body" we actually are seeing sensa (as, colors) and mistaking these for bodies. The purpose of the present section is to show how formulations of this point, and others which Prichard makes, require a logic of terms which do not denote.

To take an example which Prichard himself takes, suppose that I ask what it is that I see when I have an experience of seeing the moon as being yellow. If I describe my experience as 'seeing the moon as being yellow'; I would appear to be asserting that I am instantiating a certain triadic relation which might be symbolized as: "P(x,y,f)", and described in general in this manner:

P(x,y,f) : x sees (or perceives) y as possessing the property f.

The instance in question would then be describable as:

"a sees m as possessing the property yellow"

or: "P(a,m,Y)".

But Prichard maintains that the patch of color which is the sensum of which I am aware when I see the moon as yellow, is not the moon (nor presumably, the property of being yellow) and therefore is not included among the arguments to P in P(a,m,Y). We might therefore introduce a further relation--that of sensing--in order to have a notation for "a senses the-moon-as-yellow", where the phrase "the-moon-as yellow", names the sensum in question. However, before doing this, it is as well to notice immediately a certain inadequacy in formulation of the phrase "the-moon-as-yellow". Prichard will wish to maintain that sensa are relative to percipients. The phrase "the-moon-as-yellow" (say); and thereby the original sentence amended to: "a senses a-perceiving-the-moon-as-yellow".

If we adopt the letter "S" as a notation for the dyadic relation of sensing:

xSy : x senses y.

then the above sentence might be expressed as:

aS(a-perceiving-the-moon-as-yellow).

Or alternatively as:

aS(a-perceiving-m-as-Y).

The question naturally arises of how to interpret the English phrase "a-perceiving-m-as-Y".

A suggestion made by several investigators is to interpret such hyphenated phrases as names for states of affairs. As the reader may have already guessed, such a suggestion will be followed here by regarding the phrase "a-perceiving-m-as-Y" as denoting the state of affairs that a perceives m as Y. The original sentence "a senses the moon as yellow" will therefore be taken to be equivalent to "aS(P(a,m,Y))".

Although a does not experience either the moon, or the property yellow in perceiving the moon as yellow; he nevertheless does experience the state of affairs (or proposition) that he perceives the moon as yellow. In fact the state of affairs which a experiences (or senses) is the patch of yellow color which is his sensum.

The purpose of the present section is not to argue for this analysis of perceiving and sensing, but rather to show how this analysis calls for a logic of non-denoting terms.

Suppose that within this theory, we wished to give formal expression to what might be regarded as a plausible law of perception: "Whenever x perceives y as possessing the property f; then x senses the state of affairs produced by x's perceiving y as f.". It might seem to be possible to express this "law" in the following formula:

$$(x)(y)(f)(P(x,y,f) \supset xS(P(x,y,f))) \qquad (1)$$

However this "law" is not even an admissably formulated one in a logical system such as <u>Principia</u> <u>Mathematica</u>. Suppose that a does not perceive b as possessing G. From (1) the following can be inferred:

$$P(a,b,G) \supset aS(P(a,b,G)) \qquad (2)$$

but since the state of affairs that a perceives b as possessing G does not exist; (2) contains in its consequent the term "P(a,b,G)" which does not denote anything.

(Of course the expression "P(a,b,G)" is not a term occurrence in the antecedent of (2) and therefore this occurrence does not give rise to difficulty.) Ordinarily a sentence which is a material conditional with a false antecedent is true. However this principle presupposses that all of the terms which appear in the material conditional denote--as does every principle of traditional logic; and indeed, every sentence in which the terms appearing in the sentence all occur directly in the sentence. Since (2) contains such a non-denoting term, it is unacceptable as it stands; and moreover, the difficulties of formulation which it raises can be given no simple solution in any logic which permits only terms which denote. Therefore in spite of the fact that (2) is a material conditional with a false antecedent, it must be regarded as either false or meaningless.

Prichard's point that we ordinarily mistake our sensa for bodies; runs afoul of similar difficulties.

Suppose this point were to be formulated as follows: "If x perceives y as f and x's perceiving y as f is a case of ordinary, unreflective, perception; then x mistakes P(x,y,f) for y." then "P(x,y,f)" occurs as a term in this sentence. For the same reasons as in the earlier example, this formulation also will break down in cases of x not perceiving y as f.

Many more examples such as these of "principles" of perception which call for some such logic as will be proposed in the previous chapter could be found. These examples would for the most part, occur in theories of perception which take sensa to be states of affairs or propositions rather than bodies; however theories which make such proposals are today to be widely found in the literature on perception. The formulation of such theories in classical logical systems is not wholly impossible. However, such formulations would involve considerable circumlocution at best.

Perhaps enough has been said in this last example of application, and in the preceding ones; to show that there are uses of a logic allowing terms which do not denote over and above whatever intrinsic interest such a system might have.

CHAPTER V

MISCELLANEOUS COMMENTS AND FURTHER TOPICS OF INVESTIGATION

Sections 5.1 - 5.6 are a series of rather disconnected comments on several topics. All of these comments are intended to illustrate views of their topics which reflect the position of earlier chapters. Sections 5.7 - 5.11 briefly indicate some further problems of this treatment of that earlier position.

5.1 In System I, 'E!' is a primitive. Two theorems of System I fix the interpretation of 'E'; in the sense that they limit the interpretations which can be made of 'E' in such a way as to prevent it from having a great many unintended interpretations. These are:

$$\sim\Diamond\sim(x)E!x \tag{1}$$

and,

$$(x)\Diamond\sim E!x \tag{2}$$

Interpretations of 'E!' can easily be found which will satisfy one of these conditions but not the other. However, there are few alternative interpretations which will satisfy both (1) and (2).

For instance, it has often been suggested that 'existence' can be adequately explicated by self identity. And indeed, self identity does satisfy (1). However, self identity does not satisfy (2); and therefore, is not adequate as an explication of 'E!'.

Again, it is easy to find interpretations which give conditions that are contingent of everything. But few of these interpretations will also satisfy (1).

5.2 It has often been maintained that syntactical criteria of inference are adèquate to the purposes of logic. There would seem to be evidence that this view is unwise.

One of the conditions necessary to the correctness of an inference isthat no false conclusion can be inferred from a true premise. However, no sentence containing a subject term that does not denote something is true, and syntactical criteria alone never suffice to show whether or not a term denotes. Purely syntatical criteria of inference that are adequate, may not be as easily formulated as was at first supposed.

5.3 It is sometimes thought that all uses of modalities as terms of the object language can be replaced by uses of the modals as terms of the metalanguage. More often than not, the reasons for such a supposition are that for any assertion of modal connection in the object language there can always be found a parallel assertion of metalinguistic connection.

This last claim would indeed, seem to be true. So that for instance, rather than saying '$\sim\Diamond\sim(fx \vee \sim fx)$', we might rather say 'analytic'(fxv~fx)''. However the first claim of the preceding paragraph is much stronger and more dubious. Moreover, acceptance of the second claim of the preceding paragraph by no means entails acceptance of the first.

One of the reasons for this is that while any statement of modality may be translated into the metalanguage in such a way as to be well formed, often larger contexts containing statements of modality are not such that those contained statements of modality can be translated into the metalanguage in such a way that the larger context makes sense. For instance if the expressions '$\hat{x}\sim\Diamond\sim(fx \vee \sim fx)$' and '$(\exists x)\Diamond\sim(fx \vee \sim fx)$' are translated by replacing their contained assertions of necessity as in the preceding paragraph, they result in '$\hat{x}$ analytic'(fxv~fx)'', and '(∃x)analytic'(fxv~fx)''. But these last statements are difficult to allow as well formed.

5.4 Sometimes the law '$(\exists x)fx \Leftrightarrow (\exists x)(E!x.fx)$' is called into question on the grounds that there are fictitious characters in fiction. The argument is often put that since there are fictitious characters, but no fictitious characters that exist, the above law fails in this case.

Actually, this last claim that there are fictitious characters, is only a special case of what might be called truth within a myth. In a sense, it is true as claimed that in the myth of Oz, the tin woodman and the cowardly lion exist, and that in reality these things are not so. But to say this is only to say that given those things asserted in the story of Oz as premises, one can deduce that the tin woodman and the cowardly lion exist. This might be put symbolically by using 'Oz' as an abbreviation for the conjunction of those things asserted in the story of Oz, 'Tx' for 'x is a tin woodman', and 'Cx' for 'x is a cowardly lion'. The last statements then become:

$$Cz \dashv E!(\iota x)(Tx) \tag{3}$$

and,

$$Oz \dashv E!(\iota x)(Cx) \tag{4}$$

In other words, to be in a myth is to be implied by that myth. But furthermore, the tin woodman does not exist. We might therefore assert that in Oz the tin woodman exists, but he nevertheless does not exist:

$$(Oz \dashv E!(\iota x)(Tx)) \sim E!(\iota x)(Tx) \tag{5}$$

But from (5) we should not infer that there are non-existent tin woodmen.

5.5 The ontological argument for the existence of God proceeds by defining the term'God' in such a way that 'existence' enters into its definition. The argument then continues with a line of reasoning to the effect that with the term so defined, it is an essential feature of God that he exist. Since anything must possess its essential features, continues the argument, it must then be the case that God exists.

This version of the ontological argument might be put into the notation of System I as follows.

We might introduce the term 'God' by a definition such as:

$$\text{God} =_{DF} (\iota x)(E!x . Ox). \tag{6}$$

Here, 'O' is used to abbreviate some further qualifying condition such as 'is omnibenevolent', or, 'is omnipotent'; over and above existence itself, as in the definition of 'God'. What this further condition means need not be examined for the purposes of the present remark. The present comments will be concerned solely with the logical properties of 'E!', rather than with the interpretation of 'O'. It may be that 'O' intensionally contains 'E!'. This is to say, it may be that to assert 'O' of anything, is to imply that 'E!' is applicable to that thing. In case this is so, (6) contains 'E' in a way that is superfluous. But again, whether or not this is so, while it will be relevant to whether or not (6) is as economical as possible, need not be of concern for the present point.

The argument above infers 'E!God', presumably by appeal to some such principle as '$f(\iota x)(fx)$'.

But in System I, this principle is not available, though,

$$E!(\iota x)(fx) \prec f(\iota x)(fx), \tag{7}$$

is available. However, from (7) and (6) the desired conclusion does not follow without begging the question, since we require,

$$E!(\iota x)(E!x . Ox) \tag{8}$$

as a premise in order to conclude 'E!God' by means of (6) and (7).

5.6.1 Concerning the previously mentioned topic of truth within a myth, the myth that nothing exists is a particularly fruitful one. In one sense, there would be no truths if nothing were to exist. This is true both in the sense that there would be no true sentences and also in the sense that there would be no true propositions, were it the case that nothing exists. These facts are consequences of

$$\sim(\exists x)E!x \prec \sim(\exists x)fx \tag{9}$$

by substituting "is a sentence" and "is a proposition" for "f".

On the other hand, some things are true in an empty universe, in the sense that there are some things that are implied by nothings existing. This might be put:

$$(\exists x)(\exists f)(\sim(\exists x)E!x \mathbin{-\!\!3} fx) \tag{10}$$

5.6.2 These facts are more than mere curiosities, for they indicate (as we see immediately below) that certain formulas of the quantified material calculus, which are sometimes called laws of "confinement" and of which

$$(\exists x)(p \supset fx) \supset (p \supset (\exists x)fx) \tag{11}$$

is an example, are such that if the horseshoes appearing in them are replaced with flowers, the resulting formulas are not valid laws.

(11) and its converse

$$(p \supset (\exists x)fx) \supset (\exists x)(p \supset fx) \tag{12}$$

are both valid laws of the quantified material calculus which govern "confinement" of existential quantifiers over material implication. If the central connective in (12) is replaced with a flower, the resulting formula

$$(p \supset (\exists x)fx) \mathbin{-\!\!3} (\exists x)(p \supset fx) \tag{13}$$

is invalid in System 1. An exception to (13) can be obtained by substituting '$(\exists x)fx$' for 'p' in (13):

$$((\exists x)fx \supset (\exists x)fx) \mathbin{-\!\!3} (\exists x)((\exists x)fx \supset fx) \tag{14}$$

But in System 1, the antecedent of (14) is analytic, and the consequent of (14) contingent, and hence (14) itself, contravalid. On the other hand, the converse of (13) is a theorem of System 1.

"Confinement" of existential quantifiers over flowers rather than over horseshoes might also be investigated.

If the horseshoes in the antecedent and consequent of (11) are replaced with flowers, the result is:

$$(\exists x)(p \prec fx) \supset (p \prec (\exists x)fx). \qquad (15)$$

But (15) would seem to have exceptions on the grounds of the argument concerning the empty universe given at the outset of 5.6. And of course, if (15) is rejected, then the formula obtained by replacing the horseshoe in (15) with a flower, must also be rejected.

The converse of (15) can as a strict implication, be rejected on grounds similar to those which lead to a rejection of (13). Substituting '$(\exists x)fx$' for 'p' in

$$(p \prec (\exists x)fx) \prec (\exists x)(p \prec fx), \qquad (16)$$

yields

$$((\exists x)fx \prec (\exists x)fx) \prec (\exists x)((\exists x)fx \prec fx), \qquad (17)$$

which is contravalid in System I on the same grounds as is (14).

The converse of (15) is, however, a more difficult case to decide. It may be that, as with (11), the converse of (15) is valid as a material, but not as a strict, implication.

This converse,

$$(p \prec (\exists x)fx) \supset (\exists x)(p \prec fx), \qquad (18)$$

can be shown to be equivalent in System I to

$$(x)\Diamond(p.fx) \supset \Diamond(p.(x)fx). \qquad (19)$$

(19) may have the following exception, substituting '$(\exists x){\sim}Rx$' for 'p' and 'R' for 'f' in (19), (where 'Rx' abbreviates 'x is red') yields

$$(x)\Diamond((\exists x){\sim}Rx.Rx) \supset \Diamond((\exists x){\sim}Rx.(x)Rx) \qquad (20)$$

While it would seem to be so that of everything it is true that it is consistent both that it be red and that something fail to be red, it would seem not to be the case that it is consistent both that something is not red and that everything is red. But if these last things are so, then (20) is not true.

These tentative results may be summarised as follows:

$(\exists x)(p \supset fx) \supset (p \supset (\exists x)fx)$	valid	(21)
$(p \supset (\exists x)fx) \supset (\exists x)(p \supset fx)$	"	(22)
$(\exists x)(p \supset fx) \mathrel{-\!\!3} (p \supset (\exists x)fx)$	"	(23)
$(p \supset (\exists x)fx) \mathrel{-\!\!3} (\exists x)(p \supset fx)$	invalid	(24)
$(\exists x)(p \mathrel{-\!\!3} fx) \supset (p \mathrel{-\!\!3} (\exists x)fx)$	"	(25)
$(p \mathrel{-\!\!3} (\exists x)fx) \supset (\exists x)(p \mathrel{-\!\!3} fx)$	"	(26)
$(\exists x)(p \mathrel{-\!\!3} fx) \mathrel{-\!\!3} (p \mathrel{-\!\!3} (\exists x)fx)$	"	(27)
$(p \mathrel{-\!\!3} (\exists x)fx) \mathrel{-\!\!3} (\exists x)(p \mathrel{-\!\!3} fx)$	"	(28)

5.6.3 The laws of the above list that are given as valid can be shown to be theorems of System I. The author is not aware of a proof for any of the formulas listed as invalid. There are, however, theorems of System I which are analogues, respectively, of (24)-(28):

$$(p \supset (\exists x)fx) . E!x \mathrel{-\!\!3} (\exists x)(p \supset fx) \tag{29}$$

$$(\exists x)(\sim\Diamond\sim E!x . p \mathrel{-\!\!3} fx) \supset (p \mathrel{-\!\!3} (\exists x)fx) \tag{30}$$

$$(p \mathrel{-\!\!3} (\exists x)fx) . \sim\Diamond\sim E!x \supset (\exists x)(p \mathrel{-\!\!3} fx) \tag{31}$$

$$(\exists x)(\sim\Diamond\sim E!x . (p \mathrel{-\!\!3} fx)) \mathrel{-\!\!3} (p \mathrel{-\!\!3} (\exists x)fx) \tag{32}$$

$$(p \mathrel{-\!\!3} (\exists x)fx) . \sim\Diamond\sim E!x \mathrel{-\!\!3} (\exists x)(p \mathrel{-\!\!3} fx) \tag{33}$$

The only law of this list that is of serious interest is (29), the other laws being trivialized by their containing a counter-analytic condition in their antecedents.

The equivalence laws which are consequences of (30)-(33) and laws which are similar to their converses, lose interest for similar reasons. However, the equivalence law:

$$(p \mathrel{-\!\!3} (\exists x)fx) . (\exists x)E!x \Leftrightarrow (\exists x)(p \mathrel{-\!\!3} fx) \tag{34}$$

is a theorem of System 1, not so trivialized. So to speak, the formulas '$(p \mathrel{-\!\!3} (\exists x)fx)$', and '$(\exists x)(p \mathrel{-\!\!3} fx)$', are not equivalent because the latter makes a "surplus assumption" of existence which the former does not make. This "surplus assumption" is the denial of the myth that nothing exists.

5.7 This and the remaining sections of Chapter 5, suggest problems which will not be investigated in this thesis, but which might be studied if the topics discussed here were to be carried further.

The Tarski pardigm of truth does not apply to all sentences. Some of those to which it does not apply are those sentences which have terms which do not denote anything as subjects.

It might be possible to extend the Tarski paradigm to cover the latter sentences if some such codification as the following were adopted.

Let 'S' name the sentence 'X'. The following might then be taken as a criterion of the truth of S:

$$S \text{ is true} \equiv \exists! \langle X \rangle \tag{35}$$

The Tarski paradigm does not apply to sentences containing subject terms which do not denote, because these terms occur in the direct mode of discourse on one side of the biconditional equivalence which formulates the test of truth. However, such terms cannot so occur in (35), and so such a criterion as this might extend the Tarski test to sentences containing terms which do not denote as subjects.

But this method leads to an ambiguity in the case of falsity. There are available two alternative ways of defining falsehood. Letting as before, 'S' be a name of a sentence 'X', falsehood of S might be tested by the criterion:

$$S \text{ is false} \equiv \sim\exists! \langle X \rangle \tag{36}$$

or, on the other hand, by the criterion:

$$S \text{ is false} \equiv \exists! \langle \sim X \rangle \tag{37}$$

In order that a sentence be false, (36) so to speak, requires that the sentence not say what is so, while (37) requires rather, that the sentence say what is not so.

If (36) is taken as a criterion of falsity, sentences with subject terms that do not denote are false. If (37) is taken as a criterion of falsity, then such sentences are neither true nor false.

5.8 One of the difficulties of the connotative logic sketched in Chapter 4 is that the notion of an oblique usage of a term is not adequately explicated. The explication of the notion of oblique discourse is another topic which might be investigated further.

An important feature of this notion is that it, like the notions of bound and free variable, are ways in which constituent expressions are related to larger expressions of which they are parts. For this reason, it is misleading to speak as has been done above, of an expression simply as "oblique" or as "direct", without including some specification of the context in which the term is intended to be asserted to be oblique or direct.

The sort of thing that is meant by saying that a term is oblique in a sentence of which the term is a part, is that the sentence does not make an assertion about something denoted by that term. However, this definition must wait upon a clarification of the semantical terms that are involved in it before it can be regarded as an adequate explication. In this respect the notions of oblique and direct differ from those of free and bound; the latter being terms which can be syntactically defined.

Some of the advantages that go with the syntactical definitions of free and bound can also be enjoyed by the notions of oblique and direct, provided that a syntactical criterion of oblique and direct such as the following is adopted.

A term occurs obliquely in a sentence of System II, if and only if, there is some pair of angle brackets in the sentence which enclose the term.

Although this criterion does have much of the advantageous immediacy of the definitions of free and bound, it nevertheless also has two serious drawbacks.

First, the definition provides a criterion of obliquity only for sentences of System II. There are many cases of obliquity which are, for instance, instances of expressions which occur within quotes, and which are not instances of obliquity in System II.

Second, and perhpas more serious, the above criterion of obliquity provides only an extensional test of obliquity, not an intensional condition of obliquity. Because of this last difficulty, the criterion cannot be taken as a definition of oblique discourse.

The first of the above conditions of obliquity would not seem to share either of these last two disadvantages.

5.9 The formation rules of System II were not specified. There is however, an important condition that an adequate set of formation rules for System II should satisfy. No variable should be bound in such a way as to occur obliquely in the propositional formula following its binder. In other words, no bound variable should occur enclosed in a pair of angle brackets that do not enclose its binder.

It is a general restriction on all oblique discourse, that sentences which involve "binding over" oblique contexts do not make sense. That is, no variable which is oblique in a context and bound by a binder which is outside of that context, occurs in such a way that there is a sentence containing this binder and context, which makes sense.

This difficulty was mentioned previously in Section 5.3. Metalanguages which make use of expressions enclosed in quotes, are examples of languages of oblique discourse, other than the connotative language of System II.

The logical use of bound variables is such an effective technique that it is possible that oblique discourse will never be as useful as direct discourse, simply because this restriction on oblique discourse is so strong.

5.10 Various syntactical criteria of analyticity have been proposed. For instance, a criterion due essentially to Quine, is that a sentence is analytic if and only if, it is a substitution instance of a law of logic. Sometimes it has been further maintained that these criteria are adequate as definitions for 'analytic'. Useful though such criteria may be, they are, however, very likely not adequate as definitions for 'analytic', because they are not intensionally equivalent to analyticity.[1]

The following might be considered as an intensional criterion of analyticity. S is analytic, is equivalent to, the expression formed by prefixing S with a sign of logical necessity, is a true sentence.

5.11 The paradoxes of the theory of types are avoided in <u>Principia Mathematica</u> by proscribing altogether, the substitution of some expressions, and restricting substitution of other expressions to special contexts.

The former sort of restriction might be called an absolute proscription, and the latter, a relative proscription, of substitution.

In System I, much more reliance was put upon absolute than upon relative proscriptions of substitution, than is the case in <u>Principia Mathematica</u>. Yet in both

1. This suggestion is due to Mr. Leonard.

systems, some relative proscriptions of substitution are necessary. Relative proscriptions are, for instance, necessary to avoid allowing substitution of non-propositional expressions for propositional variables. A further topic of investigation would be that of examining to what extent absolute restrictions on substitution are sufficient to avoid paradox, and to what extent relative restrictions are necessary for this purpose.

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