THE ANALYSIS
OF THE NON-SYMMETRICAL VIERENDEEL TRUSS
BY THE ITERATION METHOD OF LEAST WORK

By

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A THESIS

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Contents

Introduction

A Vierendeel truss is a statically indeterminate rigid frame composed of a series of rectangular or trapezoidal panels without diagonal members. It can be analyzed by any of the standard methods —— Least Work, Virtual Work, slope-deflection and moment distribution. However, in all except the simplest cases, the application of these methods is extremely laborious.

Various improved methods[1] have been proposed, among which the panel method[2] adapted by Prof. L. C. Maugh is perhaps the best. But all these methods are far from being satisfactory, especially in the analysis of the non-symmetrical Vierendeel truss.

In this thesis, the writer's original method, the Iteration method of Least Work, is used to analyze the non-symmetrical Vierendeel truss. Original formulae are derived, and several typical non-symmetrical Vierendeel trusses are analyzed by this method as illustrative examples. This original method, is, to the writer's

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1. Dana Young, "Analysis of Vierendeel Trusses", A.S.C.E. Transactions, 1937, p.869.
   Louis Baes, "Rigid Frames Without Diagonals", A.S.C.E. Transactions, 1942, p.1215.
   Rathbun and Cunningham, "Continuous Frame Analysis by Elastic Support Action", A.S.C.E. Proceedings, Apr. 1947.
2. Engineering News Record, Mar. 14, 1935, p.379.

best knowledge, the best existing method in the analysis of the Vierendeel truss. A comparison between the writer's simple solution of his illustrative example (1) on page 16 and the laborious solution of the same problem appeared in the Apr. 1947 issue of the A.S.C.E. Proceedings (Example 6, "Continuous Frame Analysis by Elastic Support action" , by Rathbun and Cunningham) clearly shows the former's superiority.

This new method involves no new principle. It is simply the application of the age-old principle of iteration to the solution of a set of carefully formed simultaneous Least-Work Equations.

In this method, the total internal work of the Vierendeel truss is first expressed in terms of the statically redundant internal moments. Then the partial derivatives of the internal work with respect to these redundant moments are each placed equal to zero to obtain the simultaneous Least-Work equations. These equations are similar in form to the equations of the Theorem of Three Moments, and a single general formula is sufficient to formulate all of them.

The actual solution of a problem consists of two steps, the formation of the simultaneous equations by

using the general formula, and the solution of these simultaneous equations by iteration.

The greatest obstacle to the general acceptance of the Vierendeel truss has been its reputation for necessitating laborious methods of stress analysis. With the removal of this obstacle by his simple method presented in this thesis, the writer believes the Vierendeel truss is destined to play a prominent role in the bridges of the future.

Sign Convention

All counterclockwise moments are taken as positive.

All clockwise panel shears are taken as positive.

Part I. Non-symmetrical parallel-chord Vierendeel truss.

Derivation of the general formula.

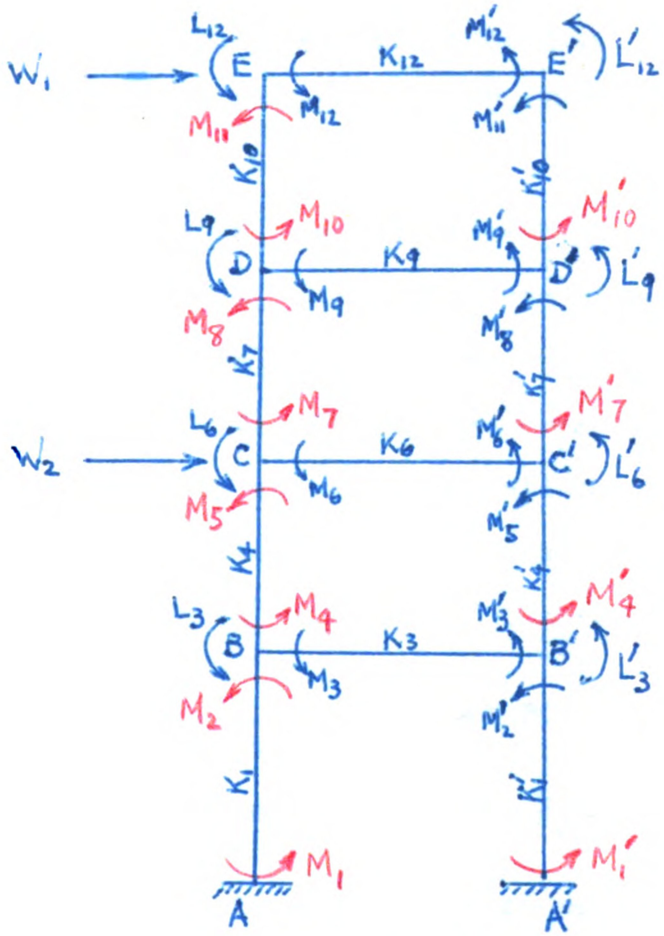


Figure (1)

In the above figure, W_1 and W_2 are external forces acting on the Vierendeel truss. L_3, L_6, L_9, L_{12}, L'_3, L'_6, L'_9 and L'_{12} are external moments acting on the joints B, C, D, E, B′, C′, D′ and E′ respectively.

The problem is to find the internal moments $M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}, M_{12}$, $M'_1, M'_2, M'_3, M'_4, M'_5, M'_6, M'_7, M'_8, M'_9, M'_{10}, M'_{11}$ and M'_{12}.

For each joint, one joint equation can be written. The joint equations of joints B, C, B′ and C′ are as follows:

$$M_2 + M_3 + M_4 = L_3 \text{ ------------------ } (1)$$

$$M_5 + M_6 + M_7 = L_6 \text{ ------------------ } (2)$$

$$M'_2 + M'_3 + M'_4 = L'_3 \text{ ------------------ } (3)$$

$$M'_5 + M'_6 + M'_7 = L'_6 \text{ ------------------ } (4)$$

For each panel, one shear equation can be written. The shear equation of panel BC is as follows:

$$M_4 + M_5 + M'_4 + M'_5 = H\,h \text{ ----------- } (5)$$

where

H = panel shear of panel BC, clockwise as positive.

h = panel length of panel BC.

Let W_{BC} = the internal work of member BC.

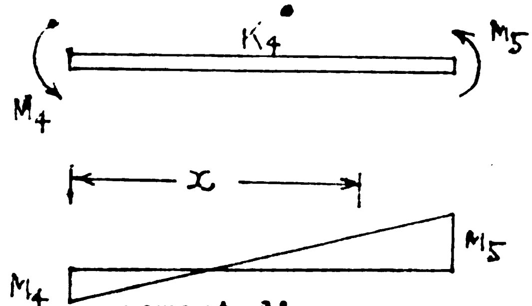
$W_{B'C'}$ = the internal work of member B′C′.

$W_{CC'}$ = the internal work of member CC′.

etc.

Then $W_{BC} = \int_0^l \frac{M^2\,dx}{2EI}$

(<u>The axial stress item is neglected.</u>)



moment diagram

Figure (2)

From figure (2),

$$M = -M_4 + (M_4 + M_5)\frac{x}{l}$$

Hence

$$W_{BC} = \int_0^l \frac{1}{2EI}\left[-M_4 + (M_4 + M_5)\frac{x}{l}\right]^2 dx = \frac{l}{6EI}(M_4^2 - M_4 M_5 + M_5^2)$$

or

$$W_{BC} = \frac{1}{6EK_4}(M_4^2 - M_4 M_5 + M_5^2)$$

also

$$\frac{\partial W_{BC}}{\partial M_5} = \frac{1}{6EK_4}\left[(2M_4 - M_5)\frac{\partial M_4}{\partial M_5} + (2M_5 - M_4)\right]$$

similarly,

$$W_{B'C'} = \frac{1}{6EK_4'}(M_4'^2 - M_4'M_5' + M_5'^2)$$

$$W_{CC'} = \frac{1}{6EK_6}(M_6^2 - M_6 M_6' + M_6'^2)$$

etc.

$$\frac{\partial W_{B'C'}}{\partial M_5} = \frac{1}{6EK_4'}\left[(2M_4' - M_5')\frac{\partial M_4'}{\partial M_5} + (2M_5' - M_4')\frac{\partial M_5'}{\partial M_5}\right]$$

$$\frac{\partial W_{CC'}}{\partial M_5} = \frac{1}{6EK_6}\left[(2M_6 - M_6')\frac{\partial M_6}{\partial M_5} + (2M_6' - M_6)\frac{\partial M_6'}{\partial M_5}\right]$$

etc.

Let W be the total internal work of the whole truss, then

$$W = W_{BC} + W_{B'C'} + W_{CC'} + W_{CD} + \text{------} \quad \text{etc.}$$

The given unsymmetrical truss is statically indeterminate to the twelfth degree. $M_1, M_2, M_4, M_5, M_7, M_8, M_{10}, M_{11}, M_1', M_4', M_7'$ and M_{11}' (marked red in figure 1) are chosen as the statically redundant elements.

From the principle of Least Work, the partial derivatives of W with respect to each of the above redundants should be equal to zero.

The partial derivatives of all the internal moments, except M_5, M_6, M_5' and M_6', with respect to M_5 are equal to zero.

By definition,

$$\frac{\partial M_5}{\partial M_5} = 1$$

From equation (2),

$$\frac{\partial M_6}{\partial M_5} = -1$$

From equation (5),

$$\frac{\partial M_5'}{\partial M_5} = -1$$

From equations (4) and (5),

$$M_6' + M_7' + H\,h - (M_4 + M_5 + M_4') = L_6'$$

or $$\frac{\partial M_6'}{\partial M_5} = 1$$

Hence

$$\frac{\partial W_{BC}}{\partial M_5} = \frac{1}{6EK_4} \left(2 M_5 - M_4 \right)$$

$$\frac{\partial W_{B'C'}}{\partial M_5} = \frac{-1}{6EK'_4} \left(2 M'_5 - M'_4 \right)$$

$$\frac{\partial W_{CC'}}{\partial M_5} = \frac{3}{6EK_6} \left(M'_6 - M_6 \right)$$

$$\frac{\partial W_{CD}}{\partial M_5} = \frac{\partial W_{AB}}{\partial M_5} = \text{-----} = 0$$

and

$$\frac{\partial W}{\partial M_5} = \frac{1}{6EK_4}(2M_5 - M_4) - \frac{2M'_5 - M'_4}{6EK'_4} + \frac{3(M'_6 - M_6)}{6EK_6}$$

since $\frac{\partial W}{\partial M_5} = 0$, we have

$$\frac{2M_5 - M_4}{K_4} - \frac{2M'_5 - M'_4}{K'_4} + \frac{3(M'_6 - M_6)}{K_6} = 0 \text{---------------} \quad (6)$$

similarly we have,

$$\frac{\partial W}{\partial M_4} = 0$$

or

$$\frac{2M_4 - M_5}{K_4} - \frac{2M'_5 - M'_4}{K'_4} + \frac{2M'_6 - M_6}{K_6} - \frac{2M_3 - M'_3}{K_3} = 0 \text{-----------} \quad (7)$$

$$\frac{\partial W}{\partial M'_4} = 0$$

or

$$\frac{3(M'_4 - M'_5)}{K'_4} + \frac{2M'_6 - M_6}{K_6} - \frac{2M'_3 - M_3}{K_3} = 0 \text{----------------} \quad (8)$$

Eliminating M'_5, M_6 and M'_6 from equations (2), (4), (5) and (6), we have

$$2\,t\,M_5 = \left(\frac{2}{K'_4} + \frac{3}{K_6}\right) H\,h - 3\left(\frac{1}{K'_4} + \frac{1}{K_6}\right) M'_4 - \left(\frac{2}{K'_4} + \frac{3}{K_6} - \frac{1}{K_4}\right) M_4 + \frac{3}{K_6}\left(M'_7 - M_7 + L_6 - L'_6\right) \quad \text{----(9)}$$

where

$$t = \frac{1}{K_4} + \frac{3}{K_6} + \frac{1}{K'_4}$$

This equation of M_5 is a useful general equation. It can be used to find $M_2, M_8, M_{11}, M_2', M_5', M_8'$ and M_{11}'. For instance,

$$2\,t\,M_5' = \left(\frac{2}{K_4'}+\frac{3}{K_6}\right) H\,h - 3\left(\frac{1}{K_4'}+\frac{1}{K_6}\right) M_4$$
$$-\left(\frac{2}{K_4'}+\frac{3}{K_6}-\frac{1}{K_4}\right) M_4' + \frac{3}{K_6}(M_7 - M_7' + L_6' - L_6) \text{ ------- } (9)_a$$

$$2\,t_1 M_2 = \left(\frac{2}{K_1}+\frac{3}{K_3}\right) H_1 h_1 - 3\left(\frac{1}{K_1'}+\frac{1}{K_3}\right) M_1'$$
$$-\left(\frac{2}{K_1'}+\frac{3}{K_3}-\frac{1}{K_1}\right) M_1 + \frac{3}{K_3}(M_4' - M_4 + L_3 - L_3') \text{ ------- } (9)_b$$

$$2\,t_1 M_2' = \left(\frac{2}{K_1}+\frac{3}{K_3}\right) H_1 h_1 - 3\left(\frac{1}{K_1}+\frac{1}{K_3}\right) M_1$$
$$-\left(\frac{2}{K_1}+\frac{3}{K_3}-\frac{1}{K_1'}\right) M_1' + \frac{3}{K_3}(M_4 - M_4' + L_3' - L_3) \text{ ------- } (9)_c$$

where

$$t_1 = \frac{1}{K_1} + \frac{3}{K_3} + \frac{1}{K_1'}$$

H_1 = shear in panel BC, clockwise as positive.

h_1 = panel length of panel BC.

Substracting equation (8) from equation (7), we have

$$\frac{2\,M_4 - M_5}{K_4} - \frac{2\,M_4' - M_5'}{K_4'} + \frac{3(M_3' - M_3)}{K_3} \text{ ------- } (10)$$

Eliminating M_3, M_3', M_5 and M_5' from equations (1), (3), (9), $(9)_a$, $(9)_b$, $(9)_c$ and (10), we have

$$\left\{\frac{1}{K_3}\left(1-\frac{3}{t_1K_3}\right)+\frac{1}{2t}\left[\frac{1}{K_4}\left(\frac{1}{K_4}+\frac{1}{K_4'}\right)+\frac{1}{K_6}\left(\frac{5}{K_4}-\frac{1}{K_4'}\right)\right]\right\} M_4$$
$$-\left\{\frac{1}{K_3}\left(1-\frac{3}{t_1K_3}\right)+\frac{1}{2t}\left[\frac{1}{K_4'}\left(\frac{1}{K_4}+\frac{1}{K_4'}\right)+\frac{1}{K_6}\left(\frac{5}{K_4'}-\frac{1}{K_4}\right)\right]\right\} M_4'$$
$$=\left\{\frac{\frac{1}{K_6}\left(\frac{1}{K_4}-\frac{1}{K_4'}\right)}{2t}\right\} H h + \left\{\frac{\frac{1}{K_1}-\frac{1}{K_1'}}{t_1K_3}\right\} H_1 h_1 + \left\{\frac{\frac{2}{K_1'}-\frac{1}{K_1}}{t_1K_3}\right\} M_1'$$
$$-\left\{\frac{\frac{2}{K_1}-\frac{1}{K_1'}}{t_1K_3}\right\} M_1 + \left\{\frac{\frac{1}{K_4}+\frac{1}{K_4'}}{2tK_6}\right\}(M_7'-M_7) + \left\{\frac{\frac{1}{K_4}+\frac{1}{K_4'}}{2tK_6}\right\}(L_6-L_6')$$
$$+\frac{1}{K_3}\left(1-\frac{3}{t_1K_3}\right)(L_3-L_3') \text{ ------------------------------ } (11)$$

Eliminating $M_3, M_3', M_6, M_6', M_5$ and M_5' from equations (1), (2), (3), (4), (8), (9), $(9)_a$, $(9)_b$, and $(9)_c$, we have

$$\left\{\frac{1}{K_3}\left(\frac{9}{2t_1K_3}-1\right)+\frac{1}{2t}\left[\frac{1}{K_6}\left(\frac{7}{K_4}+\frac{3}{K_6}+\frac{7}{K_4'}\right)+\frac{9}{K_4K_4'}\right]\right\}M_4$$

$$+\left\{\frac{1}{K_3}\left(2-\frac{9}{2t_1K_3}\right)+\frac{1}{2t}\left[\frac{1}{K_6}\left(\frac{4}{K_4}+\frac{3}{K_6}+\frac{22}{K_4'}\right)+\frac{3}{K_4'}\left(\frac{4}{K_4}+\frac{1}{K_4'}\right)\right]\right\}M_4'$$

$$=\frac{1}{2t}\left\{\frac{1}{K_6}\left(\frac{4}{K_4}+\frac{3}{K_6}+\frac{7}{K_4'}\right)+\frac{6}{K_4K_4'}\right\}Hh$$

$$-\frac{1}{2t_1K_3}\left(\frac{4}{K_1}+\frac{3}{K_3}-\frac{2}{K_1'}\right)H_1h_1$$

$$+\frac{1}{2t_1K_3}\left(\frac{7}{K_1}+\frac{3}{K_3}-\frac{2}{K_1'}\right)M_1+\frac{1}{2t_1K_3}\left(\frac{4}{K_1}+\frac{3}{K_3}-\frac{5}{K_1'}\right)M_1'$$

$$+\frac{1}{2tK_6}\left(-\frac{2}{K_4}+\frac{3}{K_6}+\frac{7}{K_4'}\right)M_7+\frac{1}{2tK_6}\left(\frac{4}{K_4}+\frac{3}{K_6}-\frac{5}{K_4'}\right)M_7'$$

$$+\frac{1}{K_3}\left(\frac{9}{2t_1K_3}-1\right)L_3+\frac{1}{K_3}\left(2-\frac{9}{2t_1K_3}\right)L_3'$$

$$+\frac{1}{K_6}\left(1-\frac{9}{2tK_4}-\frac{9}{2tK_6}\right)L_6+\frac{1}{K_6}\left(\frac{9}{2tK_4}+\frac{9}{2tK_6}-2\right)L_6'$$

---------------------(12)

Interchanging all the L, M and K with the respective L′, M′, and K′ (i.e., change M_4 into M_4', M_1' into M_1, K_1 into K_1', K_1' into K_1, etc.) we get a new equation, say, equation (13). Adding equation (13) and equation

(12), we have,

$$\left\{ \frac{1}{K_3} + \frac{\frac{1}{K_6}\left(\frac{29}{K_4} + \frac{6}{K_6} + \frac{11}{K_4'}\right) + \frac{3}{K_4}\left(\frac{1}{K_4} + \frac{7}{K_4'}\right)}{2t} \right\} M_4$$

$$+ \left\{ \frac{1}{K_3} + \frac{\frac{1}{K_6}\left(\frac{11}{K_4'} + \frac{6}{K_6} + \frac{29}{K_4}\right) + \frac{3}{K_4'}\left(\frac{1}{K_4'} + \frac{7}{K_4}\right)}{2t} \right\} M_4'$$

$$= \frac{1}{2t}\left\{ \frac{1}{K_6}\left(\frac{11}{K_4} + \frac{6}{K_6} + \frac{11}{K_4'}\right) + \frac{12}{K_4K_4'} \right\} Hh$$

$$- \frac{H_1h_1}{K_3} + \frac{M_1 + M_1'}{K_3} + \frac{1}{2tK_6}\left(-\frac{7}{K_4} + \frac{6}{K_6} + \frac{11}{K_4'}\right) M_7$$

$$+ \frac{1}{2tK_6}\left(-\frac{7}{K_4'} + \frac{6}{K_6} + \frac{11}{K_4}\right) M_7' + \frac{1}{K_3}\left(L_3 + L_3'\right)$$

$$+ \frac{1}{K_6}\left(-1 + \frac{9}{2tK_4} - \frac{9}{2tK_4'}\right) L_6 + \frac{1}{K_6}\left(-1 + \frac{9}{2tK_4'} - \frac{9}{2tK_4}\right) L_6' \text{-------(14)}$$

Eliminating M_4' from equations (11) and (14), we have our general formula. (on next page)

The general formula.

$$(v'u + u'v)\,M_4 = G + \beta k_3\left(v - \frac{2B_1}{\beta_1}u\right)M_1$$

$$+ \beta k_3\left(v + \frac{2B_1'}{\beta_1}u\right)M_1' + (E\,v - \alpha\,u)\,M_7 + (E'v + \alpha\,u)\,M_7'$$

where

$$\alpha = k_4 + k_4'$$

$$\beta = 2\,(\alpha + 3)$$

$$\gamma = 11\,\alpha + 12\,k_4 k_4' + 6$$

$$A = k_4 - k_4'$$

$$B = A + k_4$$

$$C = 3k_4'\left(7k_4 + k_4' + \frac{29}{3}\right) + 11\,k_4 + 6$$

$$D = k_4'\,(\alpha + 5) - k_4$$

$$E = 11\,k_4' - 7k_4 + 6$$

$$G = (Au + \gamma\,v)\,H\,h + \beta k_3\left(\frac{2A_1}{\beta_1}u - v\right)H_1 h_1 + J$$

$$J = (v - D)\,u\,(L_3 - L_3') + \beta k_3 v\,(L_3 + L_3')$$

$$+ \alpha\,u\,(L_6 - L_6') - v\,(EL_6 + E'L_6')$$

$$u = \beta k_3 + C$$

$$v = \beta k_3\left(1 - \frac{6}{\beta_1}\right) + D$$

$$k_4 = \frac{K_6}{K_4}$$

$$k_4' = \frac{K_6}{K_4'}$$

$$k_3 = \frac{K_6}{K_3}$$

In the above general formula, A' is obtained by interchanging all k-values with k'-values, and A_1 is obtained by using the k-values of the panel immediately below the panel under consideration. Thus

$$A' = k_4' - k_4$$
$$A_1 = k_1 - k_1'$$

The above relation holds true for all the constants.

Note that the values of α, β, γ and k_3 do not change by interchanging k-values with k'-values.

Note also that k is not the stiffness, but is the stiffness ratio.

In analyzing any non-symmetrical parallel-chord Vierendeel truss, there are five steps:

First step. Calculate for each panel all the constants.

Second step. Formulate two equations for each panel by using the general formula.

Third step. Solve these simultaneous equations by the principle of iteration.

Fourth step. Solve for M_{BA}, M_{CB}, M_{DC}, M_{ED}, $M_{B'A'}$, $M_{C'B'}$, $M_{D'C'}$ and $M_{E'D'}$ by using the general equation of M_5 shown below:

$$3M_5 = (2k_4' + 3)Hh - (B' + 3)M_4 - 3(k_4' + 1)M_4' + 3(M_7' - M_7 + L_6 - L_6')$$

The above equation of M_5 is exactly the same as equation (9) on page 9, except that it is rearranged.)

__Fifth step__. Solve for $M_{BB'}$, $M_{CC'}$, $M_{DD'}$, $M_{EE'}$, $M_{B'B}$, $M_{C'C}$, $M_{D'D}$ and $M_{E'E}$ by the joint equations. (Four of the joint equations are shown on page 6)

Illustrative example (1)*

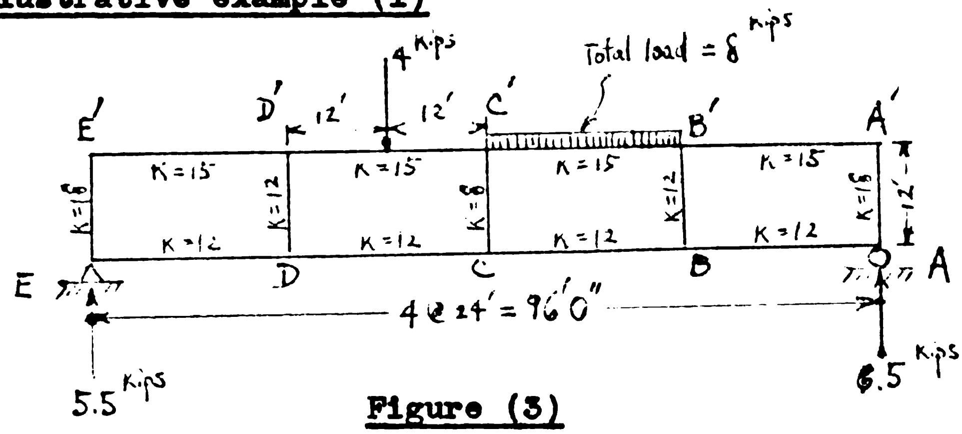


Figure (3)

The fixed-end moment at C′ and D′ due to the four kips load at the middle is $\frac{P}{8}\ell$ or 12 ft-kips.

The fixed-end moment at C′ and B′ due to the distributed load of 8 kips is $\frac{W}{12}\ell$ or 16 ft-kips.

The given loading is therefore equivalent to the following two loadings:

* Same problem as example 6, "Continuous Frame Analysis by Elastic Support Action" by J. Charles Rathbun and C. W. Cunningham, A.S.C.E. Proceedings, Apr. 1947.

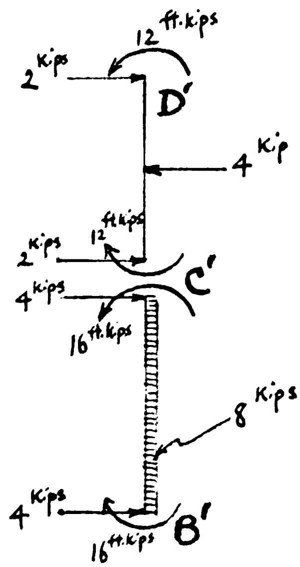


Figure (3)a

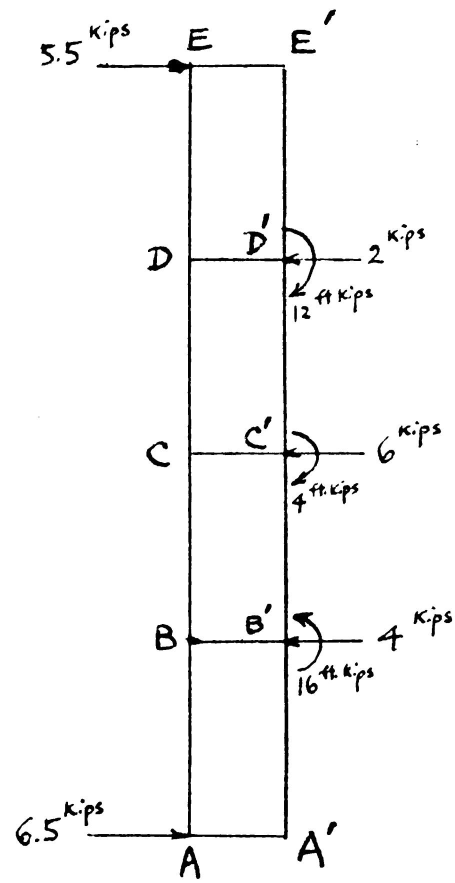


Figure (3)b

The solution of loading A, shown in figure $(3)_a$ gives

$M_{B'C'} = - 16$ ft-kips.

$M_{C'B'} = 16$ ft-kips.

$M_{C'D'} = - 12$ ft-kips.

$M_{D'C'} = 12$ ft-kips.

(All counterclockwise moments are taken as positive.)

All other members have zero moments.

The solution of loading B, shown in figure $(3)_b$ will be given in tabular form below:

Table I   Computation of constants.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
	L_3	H h	k_4	α	β	γ	βk_3	A	B
M_{DE}	0	132	1.5	2.7	11.4	57.4	17.1	0.3	1.8
$M_{D'E'}$	-12		1.2					-0.3	0.9
M_{CD}	0	84	1	1.8	9.6	35.4	14.4	0.2	1.2
$M_{C'D'}$	- 4		0.8					-0.2	0.6
M_{BC}	0	- 60	.667	1.2	8.4	23.5	5.6	.133	0.8
$M_{B'C'}$	16		.533					-.133	0.4
M_{AB}	0	-156	1	1.8	9.6	35.4	6.4	0.2	1.2
$M_{A'B'}$			0.8					-0.2	0.6

Table I (continued from the preceding page)

	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
	C	D	E	u	v	J	Coefficients of		
							M_4	Hh	$H_1 h_1$
M_{DE}	99.3	7.73	8.7	116.4	14.13	6030	3710	845	-158.0
$M_{D'E'}$	107.2	10.30	14.1	124.3	16.70	-12950		921	-373.0
M_{CD}	58.9	4.44	7.8	73.3	8.54	3460	1405	317	- 89.4
$M_{C'D'}$	63.6	6.00	11.4	78.0	10.10	-2160		342	-181.0
M_{BC}	36.5	2.63	7.2	42.1	4.73	- 600	456	117	- 16.7
$M_{B'C'}$	40.0	3.60	9.6	45.6	5.70	2040		127	- 42.5
M_{AB}	58.9	4.44	7.8	65.3	10.84	-3850	1566	396	——
$M_{A'B'}$	63.6	6.00	11.4	70.0	12.40	- 240		425	——

Table I. (continued from the preceding page)

	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)
	Coefficients of				G	First Approx.	Iteration Factor			
	M_1	M_1'	M_7	M_7'			M_1	M_1'	M_7	M_7'
M_{DE}	-256	489	—	—	103700	28.0	-.0688	.132	—	—
$M_{D'E'}$	20.5	818	—	—	77800	21.0	.0055	.221	—	—
M_{CD}	- 77.3	223	-65.4	229	35400	25.2	-.0550	.159	-.0465	.163
$M_{C'D'}$	38.4	360	-25.0	219	37400	26.7	.0273	.257	-.0178	.156
M_{BC}	- 32.3	56	-16.4	96	- 5020	-11.0	-.0707	.123	-.0360	.210
$M_{B'C'}$	-.056	96	0	96	1050	2.3	-.0001	.210	0	.210
M_{AB}	—	—	-32.5	242	-65600	-41.9	—	—	-.0207	.154
$M_{A'B'}$	—	—	15.0	223	-66500	-42.4	—	—	.0096	.143

Table II Iteration.

	M_{AB}	$M_{A'B'}$	M_{BC}	$M_{B'C'}$	M_{CD}	$M_{C'D'}$	M_{DE}	$M_{D'E'}$
Iteration Factors	-.0707	.210	.123	-.0001	-.0207	.143	-.0550	.257	.154	.0096	.159	.0273	-.036	.210	-.0688	.221	.210	0	.132	.0055	-.0465	.156	.163	-.0178
First Approx.	-41.9	-42.4	-11.0	2.3	25.2	26.7	28.0	21.0
Iteration			3.0 -5.2 -0.9 5.6 8.5	-8.8 — 5.3 — -1.2			-1.7 3.5 24.8	5.5 0.1 26.6
	0.2 -0.1	-1.2 —			0.5 — -1.4 4.3 3.4	-2.2 — 4.7 -0.5 2.0		
			-0.1 -0.1 0.4	— 0.7 —			-0.2 0.3	0.8 —
Total	-41.8	-43.6	-8.3	-0.5	28.6	28.7	29.9	27.4

Table III. Computation of M_5.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Coefficients of			H h	M_4	$3(M'_7 - M_7 + L_6 - L'_6)$	Final moments due to Loading B
	H h	M_4	M'_4				
M_{ED}	5.4	-3.9	-6.6	132	29.9	——	36.5
$M_{E'D'}$	6.0	-4.8	-7.5		27.4	——	38.3
M_{DC}	4.6	-3.6	-5.4	84	28.6	28.5	16.2
$M_{D'C'}$	5.0	-4.2	-6.0		28.7	-28.5	10.3
M_{CB}	4.066	-3.4	-4.6	- 60	- 8.3	12.3	-24.0
$M_{C'B'}$	4.333	-3.8	-5.0		- 0.5	-12.3	-27.2
M_{BA}	4.6	-3.6	-5.4	-156	-41.8	-24.6	-36.9
$M_{B'A'}$	5.0	-4.2	-6.0		-43.6	24.6	-33.5

Table IV. To check the values of M_5 found in table III by means of the shear equation.

Panel	$M_4 + M_4' + M_5 + M_5'$	H h	
DE	132.1	132	checked
CD	83.8	84	checked
BC	- 60.0	- 60	checked
AB	-155.8	-156	checked

Table V. To check the values of M_4 and M_5 found in table II and table III by means of equation (8) on page 9.

Panel	sum of Positive terms in equation (8)	sum of negative terms in equation (8)	
DE	15.8	-15.8	checked
CD	10.9	-10.9	checked
BC	9.8	- 9.8	checked
AB	17.4	- 17.3	checked

Table VI. Final solution.

	(1)	(2)	(3)	(4)
Member	Moments due to Loading A	Loading B	Final moments ft-kips	Moments by Rathbun and Cunningham
AB	——	-41.8	-41.8	-41.8
A′B′	——	-43.6	-43.6	-43.6
BC	——	- 8.3	- 8.3	- 8.4
B′C′	-16.0	- 0.5	-16.5	-16.5
CD	——	28.6	28.6	28.6
C′D′	-12.0	28.7	16.7	16.6
DE	——	29.9	29.9	29.9
D′E′	——	27.4	27.4	27.3
BA	——	-36.9	-36.9	-37.1
B′A′	——	-33.5	-33.5	-33.5
CB	——	-24.0	-24.0	-24.0
C′B′	16.0	-27.2	-11.2	-11.2
DC	——	16.2	16.2	16.3
D′C′	12.0	10.3	22.3	22.4
ED	——	36.5	36.5	36.5
E′D′	——	38.3	38.3	38.3
AA′			41.8	41.8
A′A			43.6	43.6
BB′			45.2	45.2
B′B			50.0	50.0
CC′			- 4.6	- 4.7
C′C			- 5.5	- 5.5
DD′			-46.1	-46.3
D′D			-49.7	-49.7
EE′			-36.5	-36.5
E′E			-38.3	-38.3

Explanation of Table I.

(1) The first column records the external moments applied at the lower joints of the panel under consideration. Counterclockwise external moments are positive.

Thus for M_{BC}, $L_3 = 0$, $L_3' = 16$.

for $M_{B'C'}$, $L_3 = 16$, $L_3' = 0$.

Note that the value of L_3 for M_{BC} is also the value of L_3' for $M_{B'C'}$, and the value of L_3' for M_{BC} is also the value of L_3 for $M_{B'C'}$. Hence all values of L_3' can be found under the first column (marked L_3). Throughout this method, the above relation will be utilized in recording all the other coefficients.

(2) The second column records the product of panel shear and panel length. Clockwise panel shear is taken as positive.

Thus for panel AB,

$$H\,h = -6.5 \times 24 = -156$$

(3) The third column records the ratio of K_6 to K_4.

Thus for M_{AB},

$$k_4 = \frac{K_6}{K_4} = \frac{12}{12} = 1$$

(4) The fourth column records α,

Thus for the last panel,

$$\alpha = k_4 + k_4' = 1 + 0.8 = 1.8$$

(5) The fifth column records β,

Thus for the last panel,

$$\beta = 2\ (\alpha + 3) = 2\ (1.8 + 3) = 9.6$$

(6) The sixth column records γ,

Thus for the last panel,

$$\gamma = 11\alpha + 12 k_4 k_4' + 6 = 11 \times 1.8 + 12 \times 1 \times 0.8 + 6 = 35.4$$

(7) The seventh column records βk_3.

Thus for the last panel,

$$\beta k_3 = 9.6 \left(\frac{6}{9} \right) = 6.4$$

(8) The eighth column records values of A.

Thus for M_{AB},

$$A = k_4 - k_4' = 1 - 0.8 = 0.2$$

(9) The ninth column records values of B.

Thus for M_{AB},

$$B = A + k_4 = 0.2 + 1 = 1.2$$

(10) The tenth column records values of C.

Thus for M_{AB},

$$C = 3k_4'(7k_4 + k_4') + (11k_4 + 29k_4' + 6)$$

$$= 3\ (0.8)\ (7 \times 1 + 0.8) + (11 \times 1 + 29 \times 0.8 + 6)$$

$$= 58.9$$

(11) The eleventh column records values of D.

Thus for M_{AB},

$$D = k_4'(\lambda + 5) - k_4 = 0.8(1.8 + 5) - 1 = 4.44$$

(12) The twelfth column records values of E.

Thus for M_{AB},

$$E = 11 k_4' - 7 k_4 + 6 = 11 \times 0.8 - 7 \times 1 + 6 = 7.8$$

(13) The thirteenth column records values of u.

Thus for M_{AB},

$$u = \beta k_3 + C = 6.4 + 58.9 = 65.3$$

(14) The fourteenth column records values of v.

Thus for M_{AB},

$$v = \beta k_3 (1 - \frac{6}{\beta_1}) + D = 6.4 \ (1 - 0) + 4.44 = 10.84$$

for M_{BC},

$$v = \beta k_3 (1 - \frac{6}{\beta_1}) + D = 5.6 \ (1 - \frac{6}{9.6}) + 2.63 = 4.73$$

Note that the value of β_1 is taken from the value of β of the panel immediately below. Throughout this method, this relation will be utilized in recording other coefficients such as B_1, $H_1 h_1$, etc.

Note also that all the β_1 terms are omitted in the computations for M_{AB} and M_{AB}', since there is no panel below panel AB.

(15) The fifteenth column records the values of J in the general formula on page 13.

Thus for M_{AB},

$$J = (v - D)\,u\,(L_3 - L_3') + 3k_3 v\,(L_3 + L_3')$$
$$+ \lambda\,u\,(L_6 - L_6') - v\,(E\,L_6 + E'\,L_6')$$
$$= (v - D)\,u \times 0 + 3k_3 v \times 0$$
$$+ 1.8 \times 65.3\,(-16) - 10.84\,(11.4 \times 16)$$
$$= -3850$$

Note that L_3' for panel BC, 16, is also L_6' for panel AB. This relation will be utilized in finding all values of L_6 and L_6'.

(16) The sixteenth column records the coefficient of M_4 in the general formula on page 13.

Thus for panel AB,

$$v'u + u'v = 12.40 \times 65.3 + 70.0 \times 10.84 = 1566$$

(17) The seventeenth column records the coefficient of $H\,h$. in the equation of G on page 13.

Thus for M_{AB},

$$A\,u + r\,v = 0.2 \times 65.3 + 35.4 \times 10.84 = 396$$

(18) The eighteenth column records the coefficient of $H_1 h_1$ in the equation of G on page 13.

Thus for M_{BC},

$$3k_3\left(\frac{2A_1}{B_1}u - v\right) = 5.6\left(\frac{2 \times 0.2}{9.6} \times 42.1 - 4.73\right) = -16.7$$

For panel AB, the coefficient of $H_1 h_1$ needs not be computed, since there is no $H_1 h_1$ term at all.

(19) The ninteenth column records the coefficient of M_1 in the general formula.

Thus for M_{BC},

$$\beta k_3 \left(v - \frac{2B_1}{\beta_1} u\right) = 5.6 \left(4.73 - \frac{2 \times 1.2}{9.6} \times 42.1\right) = -32.3$$

(20) The twentieth column records the coefficient of M_1' in the general formula.

Thus for M_{BC},

$$\beta k_3 \left(v + \frac{2B_1'}{\beta_1} u\right) = 5.6 \left(4.73 + \frac{2 \times 0.6}{9.6} \times 42.1\right) = 56$$

(21) The twenty first column records the coefficient of M_7 in the general formula.

Thus for M_{AB},

$$E\, v - \alpha\, u = 7.8 \times 10.84 - 1.8 \times 65.3 = -32.5$$

(22) The twenty second column records the coefficient of M_7' in the general formula.

Thus for M_{AB},

$$E'\, v + \alpha\, u = 11.4 \times 10.84 + 1.8 \times 65.3 = 242$$

(23) The twenty third column records the value of G in the general formula.

Thus for M_{AB},

$$G = (A\, u + \gamma\, v)\, H\, h + \beta k_3 \left(\frac{2A_1}{\beta_1} u - v\right) H_1 h_1 + J$$

$$= 396\ (-156) + 0 - 3850 = -65600$$

Note that the coefficients of H h and $H_1 h_1$ are taken from columns (17) and (18).

At this stage, all the coefficients of the eight simultaneous equations have been calculated. These eight equations are:

$$
\begin{aligned}
1566\ M_{AB} &= -65600 - 32.5\ M_{BC} + 242\ M_{B'C'} \\
1566\ M_{A'B'} &= -66500 + 15.0\ M_{B'C'} + 223\ M_{BC} \\
456\ M_{BC} &= -5020 - 32.3\ M_{AB} + 56\ M_{A'B'} - 16.4\ M_{CD} + 96\ M_{C'D'} \\
456\ M_{B'C'} &= 1050 - .056\ M_{A'B'} + 96\ M_{AB} + 96\ M_{CD} \\
1405\ M_{CD} &= 35400 - 77.3\ M_{BC} + 223\ M_{B'C'} - 65.4\ M_{DE} + 229\ M_{D'E'} \\
1405\ M_{C'D'} &= 37400 + 38.4\ M_{B'C'} + 360\ M_{BC} - 25.0\ M_{D'E'} + 219\ M_{DE} \\
3710\ M_{DE} &= 103700 - 256\ M_{CD} + 489\ M_{C'D'} \\
3710\ M_{D'E'} &= 77800 + 20.5\ M_{C'D'} + 818\ M_{CD}
\end{aligned}
$$

These equations can be solved by the usual method of elimination. But a better method to solve them is the method of iteration.

We first divide each equation by the coefficient of the left hand term. The transformed equations are:

$$M_{AB} = -41.9 \qquad\qquad\qquad\qquad - .0207\, M_{BC} + .154\, M_{B'C'}$$

$$M_{A'B'} = -42.4 \qquad\qquad\qquad\qquad + .0096\, M_{B'C'} + .143\, M_{BC}$$

$$M_{BC} = -11.0 - .0707\, M_{AB} + .123\, M_{A'B'} - .0360\, M_{CD} + .210\, M_{C'D'}$$

$$M_{B'C'} = 2.3 - .0001\, M_{A'B'} + .210\, M_{AB} \qquad + .210\, M_{CD}$$

$$M_{CD} = 25.2 - .0550\, M_{BC} + .159\, M_{B'C'} - .0465\, M_{DE} + .163\, M_{D'E'}$$

$$M_{C'D'} = 26.7 + .0273\, M_{B'C'} + .257\, M_{BC} - .0178\, M_{D'E'} + .156\, M_{DE}$$

$$M_{DE} = 28.0 - .0688\, M_{CD} + .132\, M_{C'D'}$$

$$M_{D'E'} = 21.0 + .0055\, M_{C'D'} + .221\, M_{CD}$$

Columns (24) to (28) record the coefficients of the above transformed equations. They are obtained by dividing columns (19) to (23) by column (16).

Explanation of Table II.

In the first of these transformed equations, we assume M_{BC} and $M_{B'C'}$ are each equal to zero. Hence we get the first approximation of M_{AB} as -41.9. Similarly the first approximations of all the other moments can be found. These first approximations are taken from column (24) of Table I and recorded in the second row of Table II.

The first approximations of M_{AB}, $M_{A'B'}$, M_{CD} and $M_{C'D'}$ are substituted into the third of these transformed equations to obtain the first correction of M_{BC}. Thus

$$\begin{aligned} \text{First correction of } M_{BC} &= -.0707\ (-41.9) + .123\ (-42.4) \\ &\quad -.036\ (25.2) + .210 \times 25.2 \\ &= 3.0 - 5.2 - 0.9 + 5.6 \\ &= -\ 8.5 \end{aligned}$$

These four correction moments are recorded in rows (3) to (6) under M_{BC} of table II.

The four iteration factors on row (1) under M_{CD} are recorded in such a manner that when they are multiplied by M_{CD}, they go to M_{BC}, $M_{B'C'}$, M_{DE} and $M_{D'E'}$ as correction moments in the same relative position. Thus

$-.036 \times 25.2 = -0.9$ goes to M_{BC}

$.210 \times 25.2 = 5.3$ goes to $M_{B'C'}$

$-.0688 \times 25.2 = -1.7$ goes to M_{DE}

$.221 \times 25.2 = 5.5$ goes to $M_{D'E'}$

All these iteration factors are recorded in this manner.

Note that the iteration factors are transferred from columns (25) to (28) of table I to the first row of table II in a diagonal manner. Thus the top four iteration factors of columns(25) and (26) of table I are arranged in the order of -.0707, .210, .123, -.0001 in row (1) under M_{AB} and $M_{A'B'}$ of table II. This is because M_1 in the equation of $M_{B'C'}$ refers to $M_{A'B'}$ and not to M_{AB}.

The iteration process is carried forward as shown in table II until the desired degree of accuracy is obtained.

Note that the iteration factors average only about 0.1, which makes the convergence very rapid. This is one of the advantages of this "Iteration Method of Least-Work."

<u>Explanation of table III.</u>

(1) The first column records the coefficient of H h of the general equation of M_5 on page 14.

Thus for M_{BA},

$$2k'_4 + 3 = 2 \times 0.8 + 3 = 4.6$$

(2) The second column records the coefficient of M_4 of the general equation of M_5 on page 14.

Thus for M_{BA},

$$-(B' + 3) = -(0.6 + 3) = -3.6$$

(3) The third column records the coefficient of M'_4.

Thus for M_{BA},

$$-3\,(k'_4 + 1) = -3\,(0.8 + 1) = -5.4$$

(4) The fourth column records the values of H h. They are taken from column (2) of table I.

(5) The fifth column records the values of M_4. They are taken from the last row of table II.

(6) The sixth column records the values of $3(M'_7 - M_7 + L_6 - L'_6)$

Thus for M_{BA},

$$3\,(-0.5 + 8.3 + 0 - 16) = -24.6$$

(7) The seventh column records the final value of M_5 as computed by the general formula of M_5 on page 14.

Thus for M_{BA},

$$\frac{1}{\beta}\Big[(2k''_4 + 3)\,H\,h - (B' + 3)\,M_4 - 3\,(k''_4 + 1)\,M'_4 + 3\,(M'_7 - M_7 + L_6 - L'_6)\Big]$$

$$= \frac{1}{9.6}\Big[4.6\,(-156) - 3.6\,(-41.8) - 5.4\,(-43.6) - 24.6\Big]$$

$$= -36.9$$

Explanation of Table IV.

Equation (5) on page 6 is used as a check on the moments found in table III.

Thus for panel AB,

$$M_4 + M_4' + M_5 + M_5' = -41.8 - 43.6 - 36.9 - 33.5 = -155.8$$

The value of H h, taken from column (2) of table I, is -156. The check is close enough for slide-rule computations.

Explanation of Table V.

Equation (8) on page 9 is used as a check on the moments found in table II and table III.

Thus for panel AB,

$$M_4' = -43.6$$

$$M_5' = -33.5$$

$$M_6' = 16 - (-0.5 - 33.5) = 50.0$$

$$M_6 = -(-8.3 - 36.9) = 45.2$$

$$M_3' = 43.6$$

$$M_3 = 41.8$$

$$K_4' = 15$$

$$K_6 = 12$$

$$K_3 = 18$$

Equation (8) is

$$\frac{3(-43.6+33.5)}{15}+\frac{100-45.2}{12}-\frac{87.2-41.8}{18}$$

or $$-8.72+6.70+8.34-3.77-4.84+2.32$$

The sum of the positive terms is

$$6.70+8.34+2.32=17.4$$

The sum of the negative terms is

$$-8.72-3.77-4.84=-17.3$$

The check is close enough for slide-rule computations.

Explanation of Table VI.

(1) The first column records the moments due to loading A as found on page 17.

(2) The second column records the moments due to loading B as found in tables II and III.

(3) The third column records the sum of column (1) and column (2).

The final moments of members $A'A$, AA', $B'B$, etc., are found by solving the joint equations.

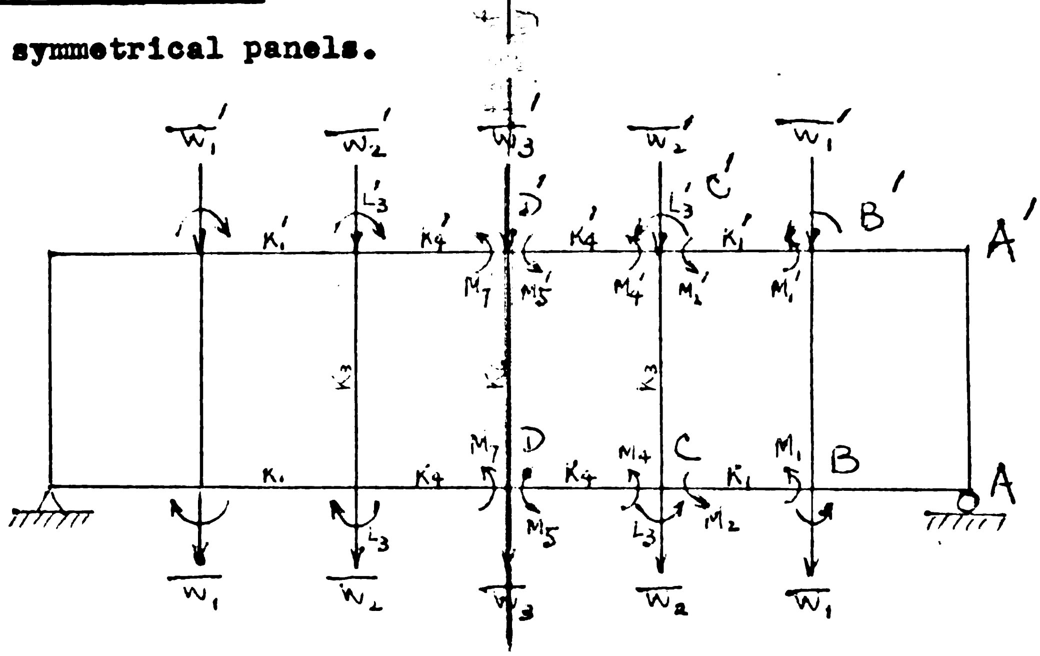
Thus for $M_{AA'}$,

$$M_{AA'}+M_{AB}=0$$

or $$M_{AA'}=-M_{AB}=41.8$$

(4) The fourth column records the moments computed by Rathbun and Cunningham in their paper, "Continuous Frame Analysis by Elastic Support Action", A.S.C.E. Proceedings, Apr. 1947. The results agree close enough for slide-rule computations.

<u>Special case (1)</u>. Symmetrical loadings on even number of symmetrical panels.



<u>Figure (4)</u>

In figure (4), both the Vierendeel truss and the loading are symmetrical with respect to DD'.

Hence for panel CD,

$L_6 = L_6' = 0$ (no external moment at D or D)

$M_5 = - M_7$

$M_5' = - M_7'$

Substituting $M_5 = - M_7$ into the equation of M_5 on page 14, we have

$$-\beta M_7 = (2k_4' + 3) H h - 3(k_4'^{2} + 1) M_4' - (B' + 3) M_4 + 3(M_7' - M_7) \quad \text{------------------ (15)}$$

similarly,

$$-\beta M_7' = (2k_4 + 3)\, H\, h - 3(k_4 + 1)\, M_4 - (B + 3)\, M_4'$$
$$3(M_7 - M_7') \text{ ---------------------- } (16)$$

Eliminating M_7' from equations (15) and (16), we have

$$2\alpha M_7 = -2k_4' H\, h + (2k_4' - k_4)\, M_4 + 3k_4' M_4' \text{ ------------ } (17)$$

similarly,

$$2\alpha M_7' = -2k_4 H\, h + (2k_4 - k_4')\, M_4' + 3k_4 M_4 \text{ ------------ } (18)$$

Substituting values of M_7 and M_7' as found in equations (17) and (18) into equations (11) and equation (14) on pages 10 and 12 respectively, we get two new equations, say, (11)$'$ and (14)$'$.

Eliminating M_4' from equations (11)$'$ and (14)$'$, we have our general formula which is the same as that shown on page 13 except that there are no M_7, M_7', L_6 and L_6' terms. Also four of the coefficients Γ, A, C and D each take a new definition:

$$\Gamma = 12k_4k_4'(1+\frac{3}{\alpha})$$

$$A = 0$$

$$C = 3k_4'(\alpha + 6k_4)(1+\frac{3}{\alpha})$$

$$D = \frac{\beta k_4'}{2}$$

Note that A is no longer equal to $k_4 - k_4'$.

Substituting $M_7 = - M_5$ into equation (17), we have the new general equation of M_5:

$$2\alpha M_5 = 2k_4'H\,h - B'M_4 - 3k_4'M_4' \text{ ---------- } (19)$$

Note that these new formulae apply to panel CD only. The original general formula and the original formula of M_5 should be used for panels AB and BC.

Illustrative example (2)*

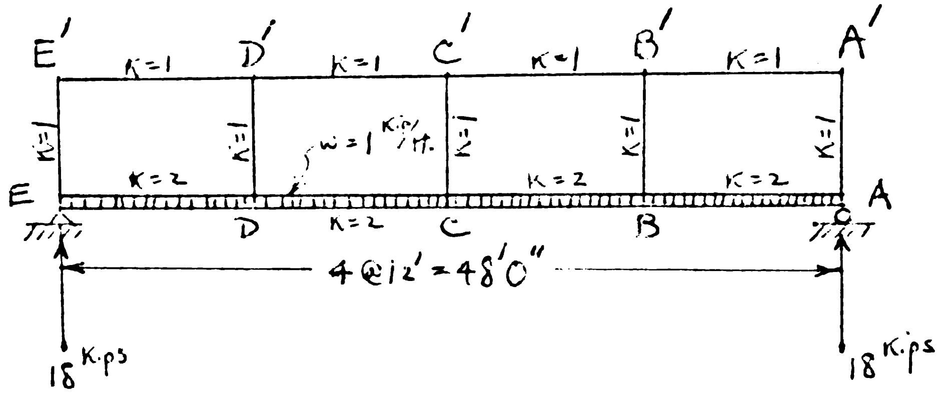


Figure (5)

The fixed-end moment at each end of the bottom chord is equal to $\frac{w}{12}l$ or 12 ft-kips.

The given loading is therefore equivalent to the following two loadings:

* Same problem as example 2, p.130, "Analysis of Rigid Frames" by A. Amirikian.

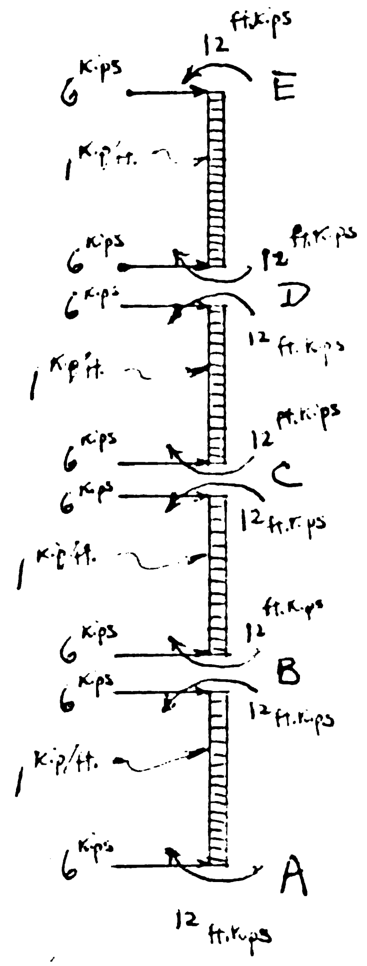


Figure (5)a

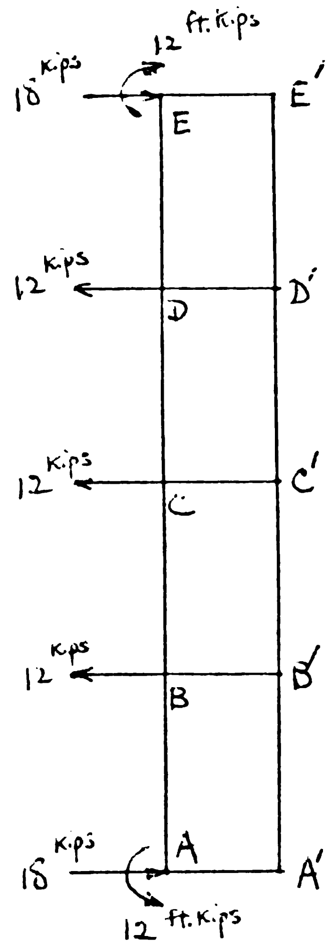


Figure (5)b

The solution of loading A, shown in figure $(5)_a$, gives

$$M_{AB} = M_{BC} = M_{CD} = M_{DE} = -12 \text{ ft-kips.}$$

$$M_{BA} = M_{CB} = M_{DC} = M_{ED} = 12 \text{ ft-kips.}$$

The solution of loading B, shown in figure $(5)_b$, will be given in tabular form below:

Table VI. Computation of constants.

	L_3	H h	k_4	α	β	γ	$3k_3$	A	B
M_{BC} $M_{B'C'}$	0	- 72	0.5 1	1.5	9	18	9	0	0 1.5
M_{AB} $M_{A'B'}$	12 0	-216	0.5 1	1.5	9	28.5	9	-0.5 0.5	0 1.5

Table VI. (continued)

	C	D	E	u	v	J	Coefficients of		
							M_4	Hh	H_1h_1
M_{BC} $M_{B'C'}$	40.5 33.8	4.50 2.25	13.5 4.5	49.5 42.8	7.50 5.25	0	581	135.0 94.4	117.0 - 4.5
M_{AB} $M_{A'B'}$	54.0 42.8	6 2.25	13.5 4.5	63.0 51.8	15 11.3	8450 -4400	1484	395.5 346.9	—— ——

Table VI. (continued)

	Coefficients of M_1	M_1'	M_7	M_7'	G	First Approx.	Iteration factors M_1	M_1'	M_7	M_7'
M_{BC}	67.5	216	—	—	15550	26.7	.116	.371	—	—
$M_{B'C'}$	-81.0	47.3	—	—	- 5830	-10.0	-.139	.0815	—	—
M_{AB}	—	—	108	162	-76850	-51.8	—	—	.0728	.109
$M_{A'B'}$	—	—	- 27	229	-79200	-53.3	—	—	-.0182	.155

Table VII. Iteration.

	M_{AB}		$M_{A'B'}$		M_{BC}		$M_{B'C'}$	
Iteration factors	.116	.0815	.371	-.139	.0728	.155	.109	-.0182
First approx.	-51.8		-53.3		26.7		-10.0	
Iteration	.1 - .8		— .1		- 6.0 -19.7	1.0	- 4.2 7.4	6.8
Total	-52.5		-53.2		1.0		- 6.8	

Table VIII. Computation of M_5.

	Coefficients of Hh	Coefficients of M_4	Coefficients of M_4'	H h	M_4	$3(M_7'-M_7+L_6-L_6')$	Final moments due to Loading B
M_{CB}	2	-1.5	- 3	- 72	1.0	——	-41.7
$M_{C'B'}$	1	0	-1.5		-6.8	——	-24.5
M_{BA}	5	-4.5	- 6	-216	-52.5	-23.4	-60.7
$M_{B'A'}$	4	- 3	-4.5		-53.2	23.4	-49.4

Table IX. To check the values of M_5 found in table VIII by means of the shear equation.

Panel	$M_4+M_4'+M_5+M_5'$	H h	
BC	- 72.0	- 72	checked
AB	-215.8	-216	checked

Table X. To check the values of M_4 and M_5 found in table VII and table VIII by means of equation (8) on page 9.

Panel	sum of positive terms in equation (8)	sum of negative terms in equation (8)	
BC	133.2	-132.8	checked
AB	325.1	-325.7	checked

Table XI. Final solution.

Member	Moments due to Loading A	Moments due to Loading B	Final Moments (ft-kips)	Moments by Amirikian
AB	-12.0	-52.5	-64.5	-64.7
A′B′	———	-53.2	-53.2	-53.1
BC	-12.0	1.0	-11.0	-11.1
B′C′	———	- 6.8	- 6.8	- 6.9
BA	12.0	-60.7	-48.7	-49.0
B′A′	———	-49.4	-49.4	-49.4
CB	12.0	-41.7	-29.7	-29.5
C′B′	———	-24.5	-24.5	-24.5
AA′			64.5	64.6
A′A			53.2	53.1
BB′			59.7	60.0
B′B			56.2	56.3
CC′			0	0
C′C			0	0

Explanation.

The calculations are similar to that to Illustrative Example (1) on page 16 with the following exceptions:

(1) For panel BC,

$$r = 12 k_4 k_4' (1 + \frac{3}{\alpha})$$

$$A = 0$$

$$C = 3 k_4' (\alpha + 6 k_4) \ (1 + \frac{3}{\alpha})$$

$$D = k_4' \ (\alpha + 3)$$

Thus for M_{BC},

$$r = 12 \times 1 \times 0.5 \ (1 + \frac{3}{1.5}) = 18$$

$$A = 0$$

$$C = 3 \times 1 \ (1.5 + 6 \times 0.5) \ (1 + \frac{3}{1.5}) = 40.5$$

$$D = 1 \ (1.5 + 3) = 4.5$$

(2) There are no M_7 and M_7' terms for panel BC.

(3) For panel BC, the equation of M_5 is

$$2\alpha M_5 = 2 k_4' H h - B' M_4 - 3 k_4' M_4'$$

Thus in table VIII, the computations for M_{CB} are:

Coefficient of H h = $2 k_4' = 2 \times 1 = 2$

Coefficient of $M_4 = -B' = -1.5$

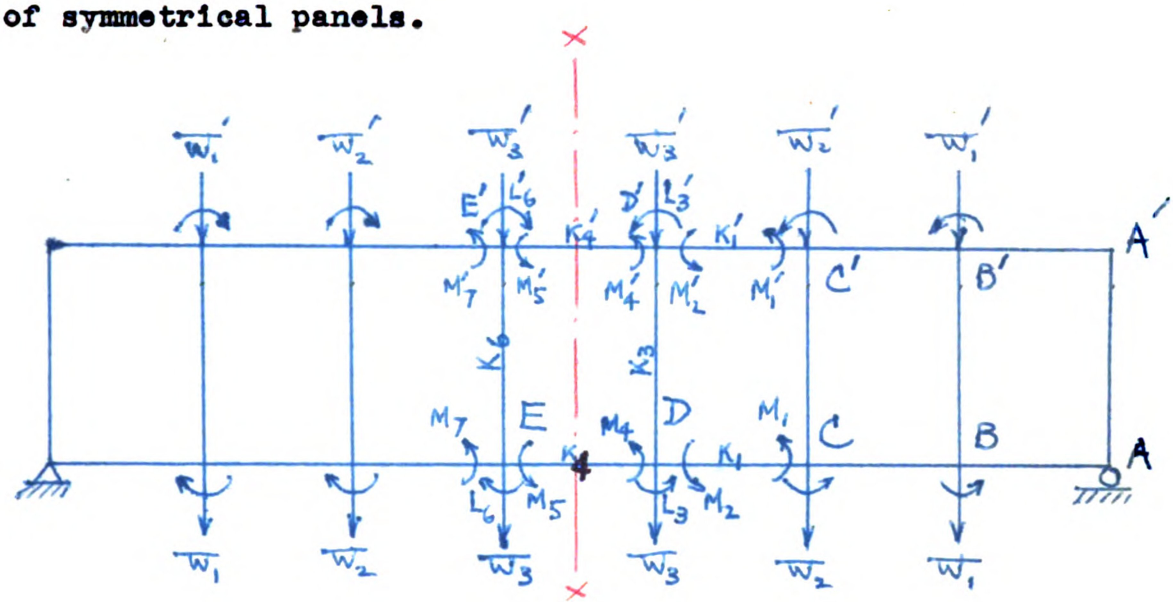
Coefficient of $M_4' = -3 k_4' = -3 \times 1 = -3$

Final $M_{CB} = \frac{1}{2 \times 1.5} \left[2(-72) + (-1.5) \times 1 - 3(-6.8) \right] = -41.7$

The last column of table XI gives the moments computed by A. Amirikian on page 130 of his book, "Analysis of Rigid Frames".

The check is close enough for slide-rule computations.

<u>Special case (2).</u> Symmetrical loadings on odd number of symmetrical panels.



<u>Figure (6)</u>

In figure (6), both the Vierendeel truss and the loading are symmetrical with respect to XX.

Hence for panel DE,

$$M_5 = - M_4$$

$$M_5' = - M_4'$$

$$H h = 0$$

$$L_6 = - L_3$$

$$L_6' = - L_3'$$

$$K_6 = K_3$$

$$k_3 = 1$$

Substituting $M_5 = - M_4$ into the equation of M_5 on page 14, we have

$$-\beta M_4 = -(B' + 3)\ M_4\ -3(k_4' + 1)\ M_4' + 3(M_7' - M_7 + L_3' - L_3) \text{ --- } (20)$$

Substituting $M_2 = - M_7$ and $M_2' = - M_7'$ into the shear equation of panel CD, we have

$$M_1 + M_1' - (M_7 + M_7') = H_1 h_1 \text{ ---------------------- } (21)$$

Eliminating M_7' from equations (21) and (22), we have

$$2M_7 = (k_4 + 1)M_4 - (k_4' + 1)M_4' + (M_1 + M_1' - H_1 h_1 + L_3' - L_3) \text{ ------}(22)$$

similarly

$$2M_7' = (k_4' + 1)M_4' - (k_4 + 1)M_4 + (M_1 + M_1' - H_1 h_1 + L_3 - L_3') \text{ ------}(23)$$

Substituting values of M_7 and M_7' of equations (21) and (22) into equation (11) and equation (14) on pages 10 and 12 respectively, we get two new equations, say, $(11)''$ and $(14)''$.

Eliminating M_4' from equations $(11)''$ and $(14)''$, we get the following modified general equation:

$$(v'u + u'v)\ M_4 = G + \beta\left(2v - \frac{2B_1}{\beta_1}u\right) M_1 + \beta\left(2v + \frac{2B_1'}{\beta_1}u\right) M_1'$$

where

$$G = \beta\left(\frac{2A_1}{\beta_1}u - 2v\right) H_1 h_1 + J$$

$$J = (v - D)\ u\ (L_3 - L_3') + \beta\ v\ (L_3 + L_3')$$

$$9Av(L_3 - L_3') + v(EL_3 + E' L_3')$$

$$C = 2k_4'(6\alpha + 19) + 2(k_4 + 3)$$

$$D = \beta k_4'$$

All other constants and coefficients have the same definition as that of the general formula on page 13.

Note that there are no H h, M_7 and M'_7 terms.

The general equation of M_5 for panel DE is, of course, $M_5 = - M_4$, which will not be needed since only half of the symmetrical truss will be analyzed.

Care should be taken to apply the modified formula to panel DE only. The original general formula should be used for panels AB, BC and CD.

Part II. Non-symmetrical inclined-chord Vierendeel truss.

Derivation of the general formula.

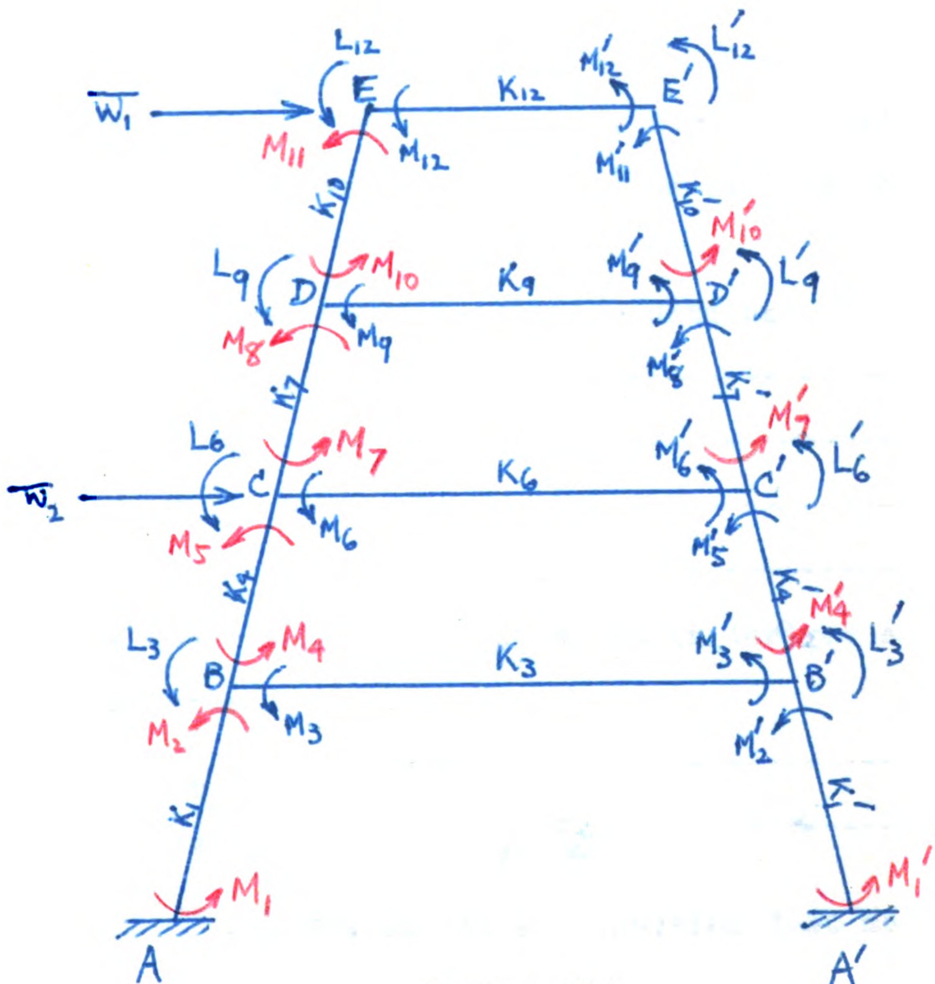


Figure (7)

In the above figure, W_1 and W_2 are external forces acting on the Vierendeel truss. L_3, L_6, L_9, L_{12},

L_3', L_6', L_9' and L_{12}' are external moments acting on the joints B, C, D, E, B', C', D' and E' respectively.

The problem is to find the internal moments M_1, $M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}, M_{12}, M_1', M_2', M_3', M_4', M_5', M_6', M_7'$, $M_8', M_9', M_{10}', M_{11}'$ and M_{12}'.

The derivation of the general formula is similar to that of the parallel-chord Vierendeel truss shown in Part I.

Thus we have the same joint equations:

$$M_2 + M_3 + M_4 = L_3 \text{ ----------------- } (24)$$

$$M_5 + M_6 + M_7 = L_6 \text{ ----------------- } (25)$$

$$M_2' + M_3' + M_4' = L_3' \text{ ----------------- } (26)$$

$$M_5' + M_6' + M_7' = L_6' \text{ ----------------- } (27)$$

The shear equation of panel BC is derived as follows:

First, we have

$$Q_4 + M_4 + M_4' + V\ell_3 = 0 \text{ -------------- } (28)$$

$$Q_4 = Q_7 - H\,h + L_6 + L_6' \text{ ---------------- } (29)$$

where

Q_4 = The moment at a section just above BB', caused by all the external loads above this section, counterclockwise as positive.

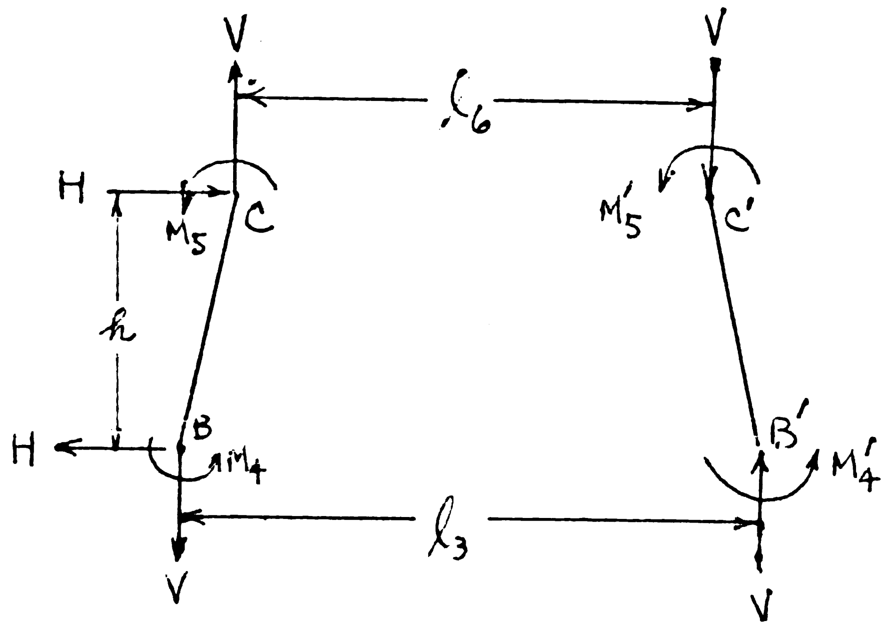
Q_7 = The moment at a section just above CC', caused by all the external loads above this section, counterclockwise as positive.

l_3 — length of BB.

H = panel shear of panel BC, clockwise as positive.

h = panel length of panel BC.

V = vertical component of the axial stress in BC, tension as positive.



From the equilibrium of BC and B'C' as shown in the above figure, we have

$$M_4 + M_5 + M_4' + M_5' + V(l_3 - l_6) = H\,h \text{ ----------- } (30)$$

where

l_6 = length of CC.

Eliminating V and H h from equations (28), (29) and (30), we have our shear equation for panel BC:

$$M_5 + M_5' + n(M_4 + M_4') = Q_7 - nQ_4 + L_6 + L_6' \quad \text{--------} \quad (31)$$

where

$$n = \frac{\ell_6}{\ell_3}$$

Again choosing $M_1, M_2, M_4, M_5, M_7, M_8, M_{10}, M_{11}, M_1', M_4', M_7'$ and M_{11}' (marked red in figure 7) as the statically redundant elements, and noting that

$$\frac{\partial M_5'}{\partial M_4} = \frac{\partial M_5'}{\partial M_4'} = -n$$

we have

$$\frac{\partial W}{\partial M_5} = 0$$

or

$$\frac{2M_5 - M_4}{K_4} - \frac{2M_5' - M_4'}{K_4'} + \frac{3(M_6' - M_6)}{K_6} = 0 \quad \text{-----} \quad (32)$$

$$\frac{\partial W}{\partial M_4} = 0$$

or

$$\frac{2M_4 - M_5}{K_4} - \frac{2M_5' - M_4'}{K_4'}n + \frac{2M_6' - M_6}{K_6}n - \frac{2M_3 - M_3'}{K_3} = 0 \text{--} \quad (33$$

$$\frac{\partial W}{\partial M_4'} = 0$$

or

$$\frac{2M_4' - M_5'}{K_4'} - \frac{2M_5' - M_4'}{K_4'}n + \frac{2M_6' - M_6}{K_6}n - \frac{2M_3' - M_3}{K_3} = 0 \text{---} (34$$

Eliminating M_5', M_6 and M_6' from equations (25), (27), (31) and (32), we have the general equation of M_5:

$$\beta M_5 = (2k_4'+3)(Q_7 - nQ_4 + L_6 + L_6') - \left[(2k_4'+3)n - k_4\right]M_4$$
$$- \left[(2k_4'+3)n + k_4'\right]M_4' + 3(M_7' - M_7 + L_6 - L_6') \quad \text{-----------} \quad (35)$$

where $$k_4 = \frac{K_6}{K_4}$$

$$k_4' = \frac{K_6}{K_4'}$$

$$\xi = 2(k_4 + k_4' + 3)$$

Proceeding in the same manner as in the case of the parallel-chord Vierendeel truss, we can derive the following general formula: (on next page)

The general formula.

$$(v'u+u'v)\,M_4 = G + \beta k_3\left[n_1 v - \frac{2u}{\beta_1}(A_1 n_1 + k_1)\right] M_1$$

$$+\beta k_3\left[n_1 v + \frac{2u}{\beta_1}(A_1' n_1 + k_1')\right] M_1'$$

$$+\left[(E'n+3A')v - \alpha u\right] M_7 + \left[(En+3A)v + \alpha u\right] M_7'$$

where

$$\alpha = k_4 + k_4'$$

$$\beta = 2(\alpha+3)$$

$$r = 8(\alpha + k_4 k_4') + 6$$

$$\delta = 3\alpha + 4k_4k_4'$$

$$A = k_4 - k_4'$$

$$B = 3(\alpha + 2k_4') + 8k_4k_4'$$

$$C = k_4'(3\alpha + 2k_4 + 12)$$

$$D = k_4'(\alpha+4)$$

$$E = 4(2k_4 - k_4') + 6$$

$$G = \left[Au + (rn+\delta)v\right](Q_7 - nQ_4 + L_6 + L_6')$$
$$+\beta k_3\left(\frac{2A_1}{\beta_1}u - v\right)(Q_4 - n_1 Q_1 + L_3 + L_3') + J$$

$$J = \beta k_3\left(1 - \frac{6}{\beta_1}\right)u(L_3 - L_3') + \beta k_3 v(L_3 + L_3')$$
$$+\alpha u(L_6 - L_6') - \left[(E'n+3A')L_6 + (En+3A)L_6'\right]v$$

$$u = \beta k_3 + rn^2 + Bn + C$$

$$v = \beta k_3\left(1 - \frac{6}{\beta_1}\right) - An + D$$

$$k_4 = \frac{K_6}{K_4}$$

$$k_4' = \frac{K_6}{K_4'}$$

$$k_3 = \frac{K_6}{K_3}$$

$$n = \frac{l_6}{l_3}$$

Illustrative example (3)

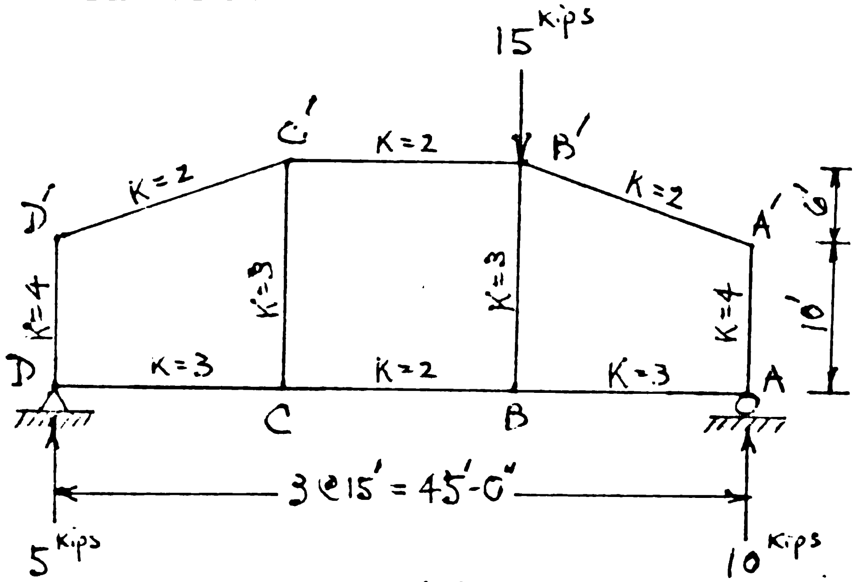


Figure (8)

The solution is given in tabular form below:

Table XII. Computation of constants.

	Q_4	n	$Q_7 - nQ_4$	k_4	α	β	γ	δ	βk_3	A	B
M_{CD} $M_{C'D'}$	75	.625	46.8	1.333 2	3.333	12.67	54	20.7	16.9	-.667 .667	43.3 39.3
M_{BC} $M_{B'C'}$	150	1	75	1.5	3	12	48	18	12	0	36
M_{AB} $M_{A'B'}$	0	1.6	-150	1 1.5	2.5	11	38	13.5	8.25	-0.5 0.5	28.5 25.5

Table XII. (continued)

	C	D	E	u	v	Coefficient of				
						M_4	$a_7 - n_1 a_1$	$a_4 - n_1 a_1$	M_1	M_1'
M_{CD}	49.4	14.67	8.67	114.4	23.5	4310	1204	-397	- 86	877
M_{CD}'	34.6	9.76	16.7	97.1	17.8		1035	-301	-109	708
M_{BC}	36.0	10.50	12.0	132.0	16.0	4220	1055	-386	250	970
M_{BC}'								- 48	-354	365
M_{AB}	32.2	9.75	8.0	183.2	18.8	5740	1300	—	—	—
M_{AB}'	22.5	6.50	14.0	168.7	14.0		1122	—	—	—

Table XII. (continued)

	Coefficient of		G	First Approx.	Iteration factors			
	M_7	M_7'			M_1	M_1'	M_7	M_7'
M_{CD}	—	—	26500	6.2	-.0200	.204	—	—
M_{CD}'	—	—	25900	6.0	-.0253	.164	—	—
M_{BC}	-204	588	129700	30.8	.0593	.230	-.0485	.139
M_{BC}'			86500	20.5	-.0840	.0865		
M_{AB}	- 9	671	-195000	-34.0	—	—	-.0016	.117
M_{AB}'	-262	756	-168500	-29.3	—	—	-.0456	.132

Table XIII. Iteration.

	M_{AB}	$M_{A'B'}$	M_{BC}	$M_{B'C'}$	M_{CD}	$M_{C'D'}$
Iteration factors	.0593 .0865	.230 -.084	-.0016 .132 -.0200 .164	.117 -.0456 .204 -.0253	-.0485 .139	.139 -.0485
First approx.	-34.0	-29.3	30.8	20.5	6.2	6.0
Iteration			- 2.0 - 6.7 - .3 .8 22.6	- 2.9 2.5 .8 - .3 20.6		
	— 2.4	3.0 -1.0 2.0			- .5 4.2 3.7	3.7 - .5 3.2
			.1 .5 - .2 .5 0.9	.2 - .2 .5 - .1 0.4		
	—	.1			— .1	.1 —
Total	-31.6	-27.2	23.5	21.0	10.0	9.3

Table XIV. Computation of M_5.

	Coefficients of $Q_7 - nQ_4$	Coefficients of M_4	Coefficients of M'_4	$Q_7 - nQ_4$	M_4	$3(M'_7 - M_7 + L_6 - L'_6)$	Final moments
M_{DC}	7	-3.04	-6.37	46.8	10.0	——	18.8
M'_{DC}	5.67	-1.54	-4.87		9.3	——	15.9
M_{CB}	6	-4.50	-7.50	75	23.5	-2.1	15.4
M'_{CB}					21.0	2.1	15.2
M_{BA}	6	-8.60	-11.1	-150	-31.6	-7.5	-30.4
M'_{BA}	5	-6.50	- 9		-27.2	7.5	-25.5

Table XV. To check the values of M_5 found in table XIV by means of the shear equation.

Panel	$M_5 + M'_5 + n(M_4 + M'_4)$	$Q_7 - n\ Q_4$	
CD	46.8	46.8	checked
BC	75.1	75.0	checked
AB	-149.9	-150.0	checked

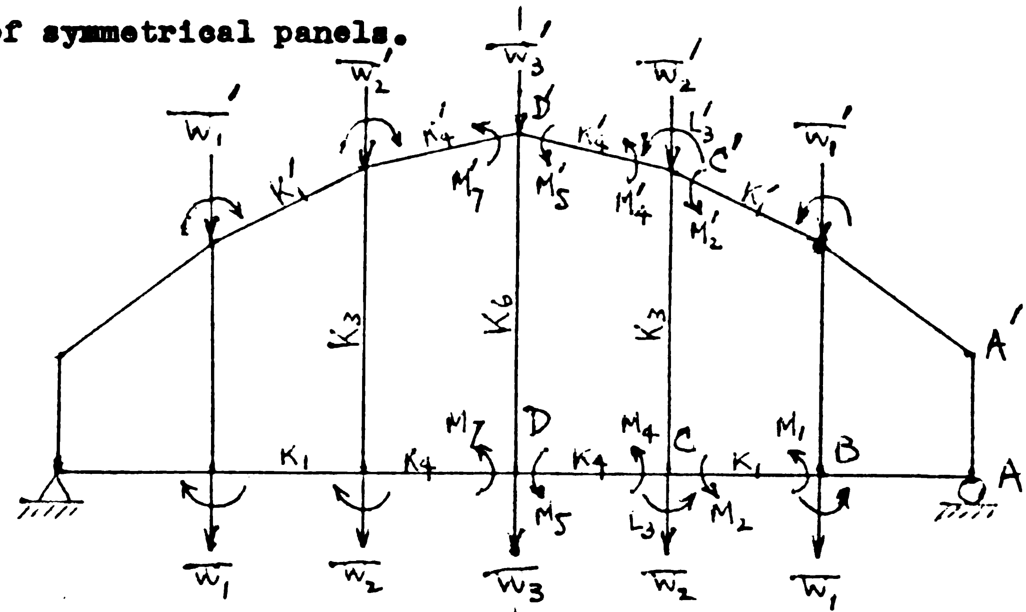
Table XVI. To check the values of M_4 and M_5 found in table XIII and table XIV by means of equation (34) on page 55.

Panel	sum of Positive terms in equation (34)	sum of Negative terms in equation (34)	
CD	31.4	-31.4	checked
BC	42.2	-42.1	checked
AB	66.3	-66.3	checked

Table XVII. Final solution.

Member	Moment (ft-kips)
AB	-31.6
A'B'	-27.2
BC	23.5
B'C'	21.0
CD	10.0
C'D'	9.3
BA	-30.4
B'A'	-25.5
CB	15.4
C'B'	15.2
DC	18.8
D'C'	15.9
AA'	31.6
A'A	27.2
BB'	6.9
B'B	4.5
CC'	-25.4
C'C	-24.5
DD'	-18.8
D'D	-15.9

<u>Special case (1).</u> Symmetrical loadings on even number of symmetrical panels.



<u>Figure (9)</u>

In figure (9), both the Vierendeel truss and the loading are symmetrical with respect to DD'.

Hence for panel CD,

$$L_6 = L_6' = 0$$

$$M_5 = -M_7$$

$$M_5' = -M_7'$$

Substituting $M_5 = -M_7$ into the equation of M_5 on page 55, we have

$$-\beta M_7 = (2k_4'+3)(Q_7 - n\,Q_4 + L_6 + L_6') - \left[(2k_4'+3)n - k_4\right]M_4 - \left[(2k_4'+3)n + k_4'\right]M_4' + 3(M_7' - M_7) \text{ ---------------- (36)}$$

similarly

$$-\beta M_7' = (2k_4+3)(Q_7 - n\ Q_4 + L_6 + L_6') - \left[(2k_4+3)n - k_4'\right]M_4' - \left[(2k_4+3)n + k_4\right]M_4 + 3(M_7 - M_7') \text{ -------------- } (37)$$

Eliminating M_7' from equations (36) and (37), we have

$$2\alpha\, M_7 = -2k_4'(Q_7 - n\ Q_4) + (2k_4'n - k_4)M_4 + (2n+1)k_4'M_4' \text{ -----}(38)$$

similarly

$$2\alpha\, M_7' = -2k_4(Q_7 - n\ Q_4) + (2k_4 n - k_4')M_4' + (2n+1)k_4 M_4 \text{ ------ } (39)$$

Eliminating M_7, M_7' and M_4' in a manner similar to that of the symmetrical parallel-chord Vierendeel truss shown on page 39, we get our general formula which is the same as that shown on page 57 except that there are no M_7, M_7', L_6 and L_6' terms. Also six of the coefficients γ, δ, A, B, C and D each take a new definition:

$$\gamma = 8k_4k_4'\left(1+\frac{3}{\alpha}\right)$$

$$\delta = 4k_4k_4'\left(1+\frac{3}{\alpha}\right)$$

$$A = 0$$

$$B = 8k_4k_4'\left(1+\frac{3}{\alpha}\right) = \gamma$$

$$C = k_4'(3\alpha + 2k_4)\left(1+\frac{3}{\alpha}\right)$$

$$D = \frac{\beta k_4'}{2}$$

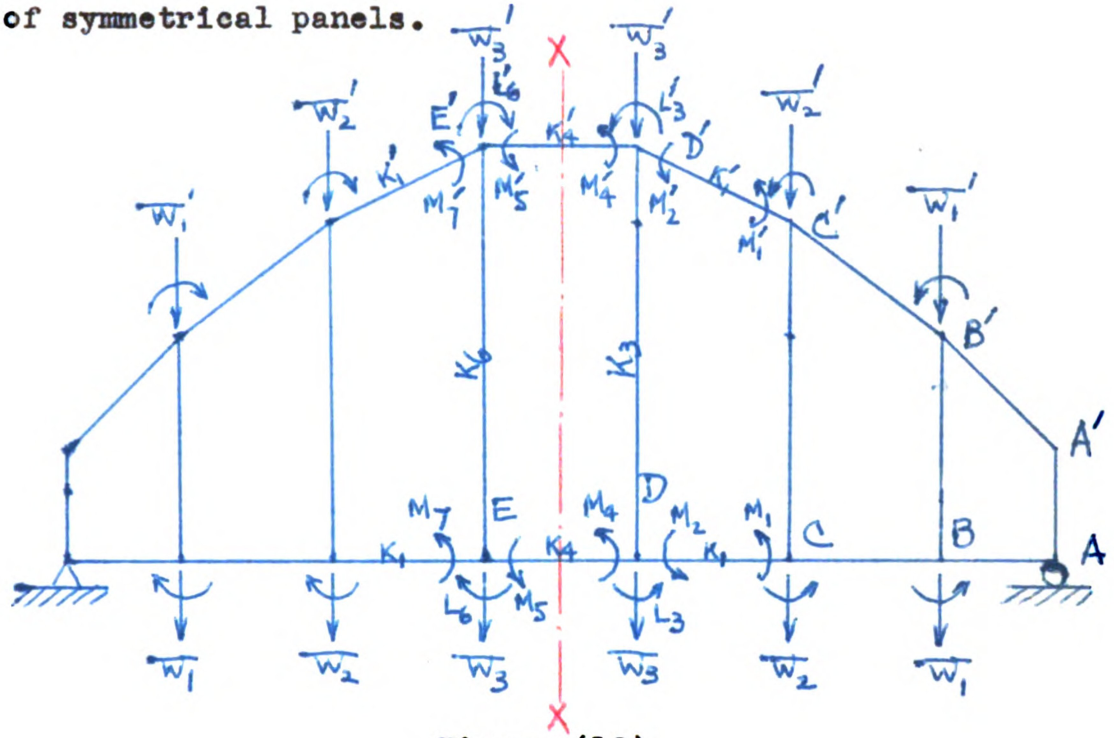
Note that A is no longer equal to $k_4 - k_4'$.

The new equation of M_5 is:

$$2\alpha\, M_5 = 2k_4'(Q_7 - n\ Q_4) - (2k_4'n - k_4)M_4 - (2n+1)k_4'M_4'$$

Note that these new formulae apply to panel CD only. The original general formula and the original formula of M_5 should be used for panels AB and BC.

<u>Special case (2).</u> Symmetrical loadings on odd number of symmetrical panels.



<u>Figure (10)</u>

In figure (10), both the Vierendeel truss and the loading are symmetrical with respect to XX.

Hence for panel DE,

$$M_5 = - M_4$$

$$M'_5 = - M'_4$$

$$L_6 = - L_3$$

$$L'_6 = - L'_3$$

$$H\,h = 0$$

$$n = 1$$

$$k_3 = 1$$

From equation (29) on page 53, we have

$$Q_7 - nQ_4 + L_6 + L_6' = H\,h = 0$$

Proceeding in the same manner as that of the parallel-chord Vierendeel truss shown on page 50, we have the following modified general formula:

$$(v'u + u'v)\,M_4 = G + \beta\left[2n_1 v - \frac{2u}{\beta_1}(A_1 n_1 + k_1)\right] M_1$$
$$+ \beta\left[2n_1 v + \frac{2u}{\beta_1}(A_1' n_1 + k_1')\right] M_1'$$

where

$$G = \beta\left(\frac{2A_1}{\beta_1}u - 2v\right)(Q_4 - n_1 Q_1 + L_3 + L_3') + J$$
$$J = \beta\left(1 - \frac{6}{\beta_1}\right)u(L_3 - L_3') + \beta\, v(L_3 + L_3') + 9Av(L_3 - L_3')$$
$$+ v\left[(11k_4' - 7k_4 + 6)L_3 + (11k_4 - 7k_4' + 6)L_3'\right]$$
$$u = \beta + 2k_4'(6\lambda + 19) + 2(k_4 + 3)$$
$$v = \beta\left(1 - \frac{6}{\beta_1}\right) + \beta k_4'$$

All other constants and coefficients have the same definition as that of the general formula on page 57.

Note that there are no M_7, M_7' and $(Q_7 - nQ_4 + L_6 + L_6')$ terms.

The general equation of M_5 for panel DE is, of course, $M_5 = -M_4$, which will not be needed since only half of the symmetrical truss will be analyzed.

Care should be taken to apply the above modified formula to panel DE only. The original general formula should be used for panels AB, BC and CD.

Special case (3). Triangular panels.

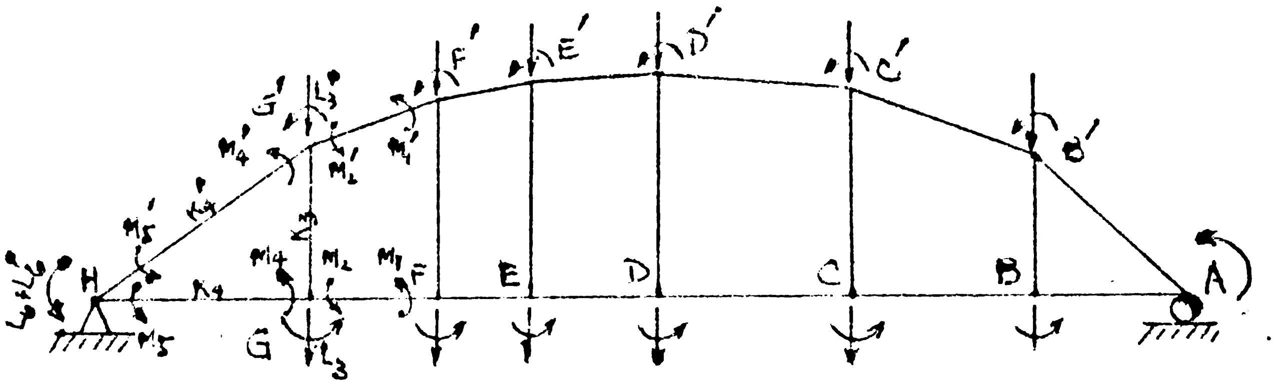


Figure (11)

(A) For panel GH, we have

$$n = 0$$

$$Q_y = 0$$

$$K_6 = \infty$$

$$M_5 + M_5' = L_6 + L_6'$$

Dividing equation (35) on page 55 by K_6 we have the equation of M_5 for panel GH:

$$2\alpha M_5 = k_4 M_4 - k_4' M_4' + 2k_4'(L_6 + L_6') \quad \text{------------ (40)}$$

where
$$k_4 = \frac{1}{K_4}$$

$$k_4' = \frac{1}{K_4'}$$

$$\alpha = k_4 + k_4'$$

Dividing the general formula on page 57 by K_6^{4}, we have the general equation for panel GH:

$$
\begin{aligned}
(v'u+u'v)\,M_4 = {} & 2\alpha k_3\left(\frac{2A_1}{\beta_1}u - v\right)(Q_4 - n_1Q_1 + L_3 + L_3') \\
& + 2\alpha k_3\left[n_1 v - \frac{2u}{\beta_1}(A_1 n_1 + k_1)\right] M_1 \\
& + 2\alpha k_3\left[n_1 v + \frac{2u}{\beta_1}(A_1' n + k_1')\right] M_1' \\
& + 4k_4k_4' v(L_6 + L_6') + 2\alpha k_3\left(1 - \frac{6}{\beta_1}\right)u(L_3 - L_3') \\
& + 2\alpha k_3 v(L_3 + L_3')
\end{aligned}
$$

where

$$k_4 = \frac{1}{K_4}$$

$$k_4' = \frac{1}{K_4'}$$

$$k_3 = \frac{1}{K_3}$$

$$u = 2\alpha k_3 + k_4'(3\alpha + 2k_4)$$

$$v = 2\alpha k_3\left(1 - \frac{6}{\beta_1}\right) + \alpha k_4'$$

All other coefficients have the same definition as that of the general formula on page 57. Thus

$$\alpha = k_4 + k_4'$$

$$A = k_4 - k_4'$$

$$k_1 = \frac{K_3}{K_1}$$

$$k_1' = \frac{K_3}{K_1'}$$

$$A_1 = k_1 - k_1'$$

$$\beta_1 = 2(k_1 + k_1' + 3)$$

etc.

Note that $k_4 = \frac{1}{K_3}$ and $k_1 = \frac{K_3}{K_1}$.

(B) Still refering to panel GH of figure (11), and noting

$n = 0$ and $K_6 = \infty$, equation (33) on page 55 becomes:

$$k_4(2M_4 - M_5) = k_3\left[2(L_3 - M_2 - M_4) - (L_3' - M_2' - M_4')\right] \text{-----(41)}$$

where
$$k_4 = \frac{1}{K_4}$$
$$k_3 = \frac{1}{K_3}$$

Eliminating M_4' from equations (40) and (41), we have

$$\left(\frac{2k_4}{k_3} + 2 - \frac{k_4}{k_4'}\right)M_4 = -\left(\frac{2k_4'}{k_4} + 2 - \frac{k_4'}{k_3}\right)M_5 + 2(L_3 - M_2 + L_6 + L_6') - (L_3' - M_2') \text{ ------------------(42)}$$

similarly,
$$\left(\frac{2k_4'}{k_3} + 2 - \frac{k_4'}{k_4}\right)M_4' = -\left(\frac{2k_4'}{k_4} + 2 - \frac{k_4'}{k_3}\right)M_5' + 2(L_3' - M_2' + L_6 + L_6') - (L_3 - M_2) \text{ ------------------(43)}$$

Comparing the notations of panel GH of figure (11) with that of panel AB of figure (12) below, we find

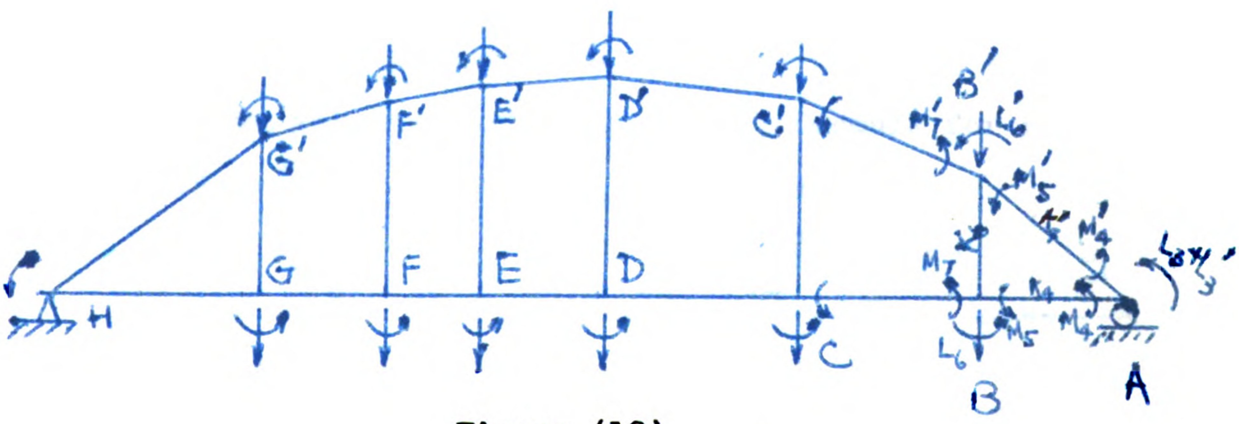
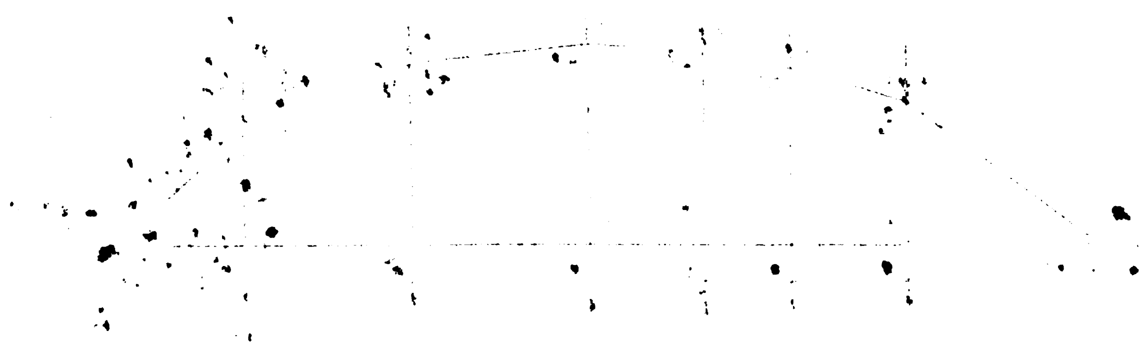


Figure (12)



K_3 corresponds to K_6

M_4 corresponds to M_5

M'_4 corresponds to M'_5

M_5 corresponds to M_4

M'_5 corresponds to M'_4

M_1 corresponds to M_7

M'_1 corresponds to M'_7

L_3 corresponds to L_6

L'_3 corresponds to L'_6

L_6 corresponds to L_3

L'_6 corresponds to L'_3

Hence equation (42) can be transformed into the following equation of M_5 for panel AB:

$$\left(\frac{1}{R}\right) M_5 = -\frac{S}{R} M_4 + (M'_7 - 2M_7) + (2L_6 - L'_6) + 2(L_3 + L'_3) \quad \text{----}(44)$$

Eliminating M_4, M'_4 from equations (40), (42) and (43), and transforming the notations of panel GH into that of panel AB, we get the following general formula for panel AB:

$$\begin{aligned}(k_4 S + k'_4 S' + 2\alpha) M_4 = &-(2k_4 R + k'_4 R')(M_7 - L_6) \\ &+ (k_4 R + 2k'_4 R')(M'_7 - L'_6) + (2k_4 R - 2k'_4 R' + k'_4 S' + 2k'_4)(L_3 + L'_3)\end{aligned}$$

$$\text{------------------------}(45)$$

In both equations (44) and (45),

$$k_4 = \frac{K_6}{K_4} \qquad k'_4 = \frac{K_6}{K'_4} \qquad \alpha = k_4 + k'_4$$

$$R = \frac{1}{2(k_4 + 1) - \frac{k_4}{k'_4}} \qquad S = \left(2 - k_4 + \frac{2k_4}{k'_4}\right) R$$

(C) The equation of M_5 for panel BC in figure (13) is the same as the general equation of M_5 on page 55.

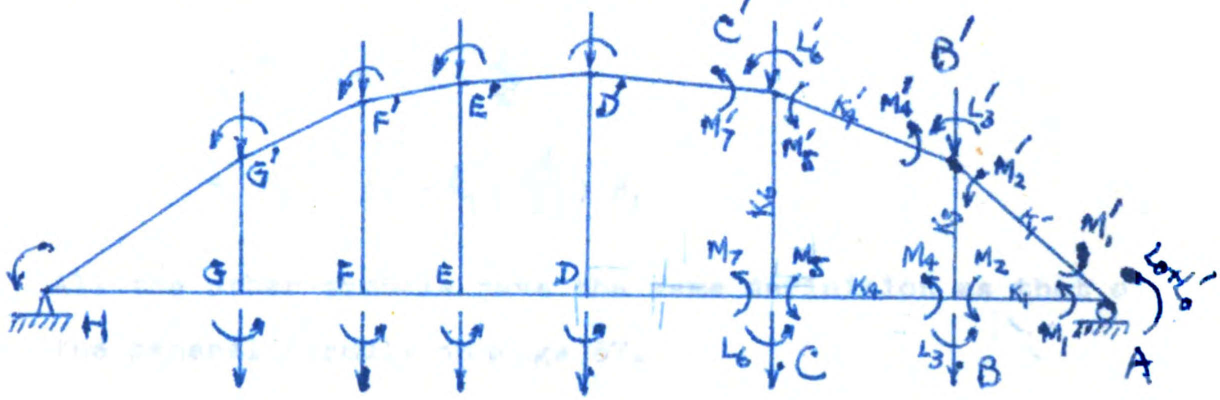


Figure (13)

Proceeding in the same manner as in the derivation of the general formula of page 57, and using ~~the~~ equation (44) as the equation of M_5 for panel AB, we can get the following general formula for panel BC:

$$(v'u+u'v)M_4 = G + \beta k_3 S_1 (v+u) M_1 + \beta k_3 S_1' (v-u) M_1'$$
$$+ \left[(E'n+3A')v - \alpha u\right] M_7 + \left[(En+3A)v + \alpha u\right] M_7'$$

where

$$G = \left[Au + (rn+\delta)v\right](Q_7 - nQ_4 + L_6 + L_6') + J$$
$$J = 2\beta k_3 (R_1' - R_1) u (L_0 + L_0') - 2\beta k_3 (R_1' + R_1) v (L_0 + L_0')$$
$$+ \left[(1 - 2R_1 - R_1')u + (1 + R_1' - 2R_1)v\right] \beta k_3 L_3$$
$$+ \left[(R_1 + 2R_1' - 1)u + (1 + R_1 - 2R_1')v\right] \beta k_3 L_3'$$
$$- m\beta v (L_6 + L_6') + \alpha u (L_6 - L_6')$$

$$u = \rho k_3(1 + R_1 - 2R_1') + \Gamma n^2 + Bn + C$$

$$v = \rho k_3(1 - R_1 - 2R_1') - An + D$$

$$R_1 = \frac{1}{2(k_1 + 1) - \frac{k_4}{k_1}}$$

$$S_i = \left(2 - k_1 + \frac{2k_1}{k_1'}\right) R_1$$

All the other symbols have the same definition as that of the general formula on p age 57.

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