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NETWORK SYNTHESIS, PROBLEMS AND
EXAMPLES ILLUSTRATING THE
USE OF THEOREMS

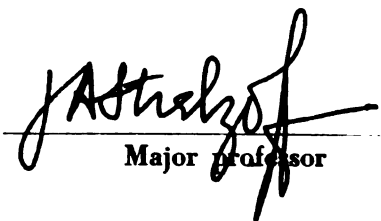
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Noah Herbert Kramer
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NETWORK SYNTHESIS. PROBLEMS AND EXAMPLES
ILLUSTRATING THE USE OF THEOREMS.

BY
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N. H. Kramer

ABSTRACT

An R-C Low- Pass Filter whose Power Insertion Ratio approximates a prescribed function of frequency is synthesized.

The Filter is in the form of a symmetrical lattice with prescribed, but equal, terminating resistors.

The problem of approximation is first considered.

The realization of a lattice that approximates a trapezoidal pattern is then shown.

A few realization conditions, peculiar to this thesis are then discussed and some avenues for further study are made available.

ASTUSZOFF

I. Introduction.

The purpose of this thesis is to present some examples of some theorems on Network Synthesis.

Most of the theorems will be taken from the book 'Network Analysis and Feedback Amplifier Design' by Doctor Hendrik W. Bode.¹

Generous use has been made of two books by Doctor Ernst A. Guillemin,^{2,3} particularly the second volume of 'Communication Networks'.

The amount of help derived from the last named book is not sufficiently noted by references alone, for the book was used to get alternate approaches to problems presented by Dr. Bode.

After a short discussion to explain and define terms, the thesis will follow a standard pattern.

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¹ Bode, H.W. Network Analysis and Feedback Amplifier Design. 1945 New York: D. Van Nostrand.

² Guillemin, Ernst A. Communication Networks Vol. II 1935 New York: John Wiley

³ Guillemin, Ernst A. Communication Networks Vol. I 1935 New York: John Wiley

There will be presented a portion of a theorem from Dr. Bode's book. This may or may not be paraphrased.

Then a demonstrating example will be formed, and the necessary work will be done to show the mechanism of the theorem.

If a conclusion is deemed advisable, it too will be presented.

2. The problem of Synthesis.

The study of circuits is most generally a problem of analysis. This is simply; given a circuit, find its behavior.

In this thesis the problem is not analysis, but synthesis. Given a desired behavior of a circuit, design a network that will have that behavior characteristic.

The latter problem may lead to circuits that are not readily formed with present laboratory equipment. The analysis problem has generally a unique solution, while the synthesis problem seldom leads to a single result.

3. The 'p' notation.

It is necessary to find some symbolic notation to describe the circuit that is under study. Operational Calculus⁴ provides such a handy method. In Operational Calculus the operator d/dt is noted as 'p' and the operator $\int dt$ is written as $1/p$. Thus the common expression $L di/dt + (1/c) \int i dt$ can be written as $(Lp + 1/c)i$.

Since $j\omega L i$ is the voltage across the inductance L , it follows that p is equivalent to $j\omega$, where ω is the angular velocity $2\pi f$, and j is the familiar operator $\sqrt{-1}$.

This gives another form of expression $(Lp + 1/c)i = j\omega L i + -j/\omega c i$.

Guillemin^o uses λ instead of p .

4. The 'p' equations.

The p equations can be easily formed in an analysis problem. Either the mesh or the nodal equations can be written for the network, in terms of p . Then the problem would call for the solution of an unknown current or voltage. This can be done by Cramer's Rule.^{5,6}

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⁴ Carter, G.W. The Simple Calculation of Electric Transients, 1945 New York: MacMillan

^o Ibid 2

This rule gives a result which is the quotient of two polynomials in p . The form is $\frac{A(p)}{B(p)}$.

This result can lead to several continuations.

If only a steady-state solution is required p is replaced by $j\omega$ and the fraction is then evaluated.

If both transient and steady-state are required, Operational Calculus, Laplace Transform, or even classical Differential Equations can be used.

All this is the analysis problem. Now consider the synthesis problem.

Consider the case where the circuit is to behave in such a way that the driving point impedance can be written as $\frac{A(p)}{B(p)} = Z$.

With that as a starting point, the problem is to synthesize a circuit with the above driving point impedance.

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⁵ Doherty & Keller. Mathematics of Modern Engineering. Vol. 1 1936 p63 New York: John Wiley & Sons.

⁶ Pipes, Louis A. Applied Mathematics for Engineers and Physicists. p86, 152 New York: McGraw-Hill.

The problem might have arisen from a very commonplace engineering experience; a complex circuit might have been designed and it's driving point impedance found to be Z^1 .

The circuit designer wishes to find a circuit that has the same Z^1 , but is composed of different elements and possibly fewer elements. Now he is faced with a problem in synthesis.

5. The polynomial in 'p'.

A few interesting facts can be learned by observing the character of the polynomial in p. Assume the polynomial is expressing a current function. Then there can be written: $i = \frac{A(p)}{B(p)} = \frac{C(p-p') (p-p^s) (p-p^h)}{(p-p'^1)(p-p'^s)(p-p^{h-1})}$.

The primed p represent the poles of the current, and the unprimed p represent the zeros. Each pole and zero can be either real or complex, depending on the form of the polynomial.

If operational calculus were to be used to continue the solution of the current the result would be a series of increasing and decreasing exponentials with both real and complex powers.

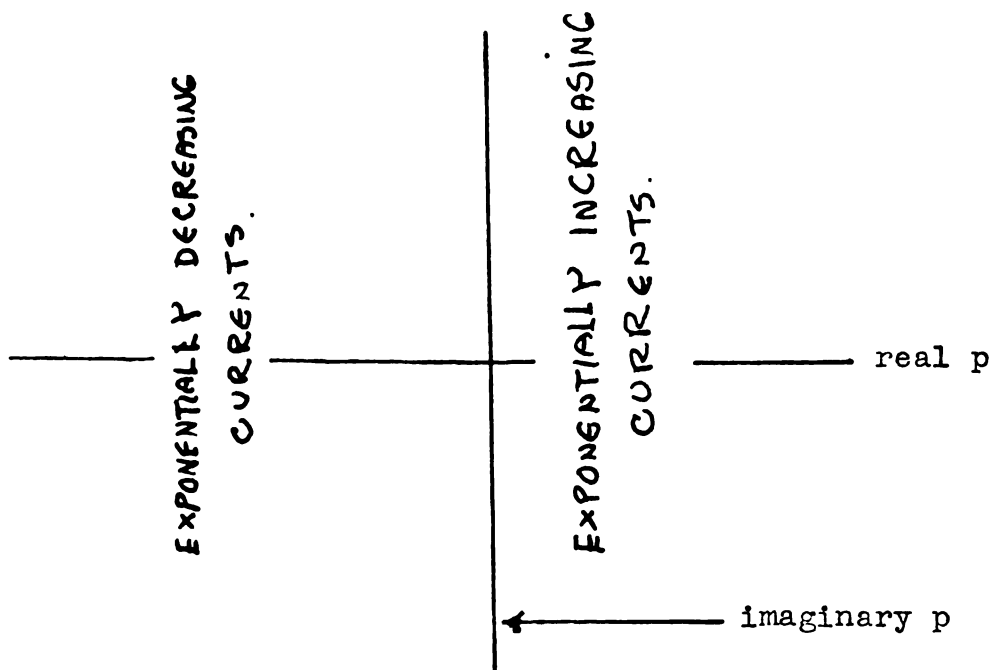
If the poles were not simple, the result might be an extremely complicated function of time.

A knowledge of the roots of the expression will give some indication of the form of the solution.

6. Plot of 'p'.

A small plot will give a picture of p . Assume p to be in the denominator. Note: since $p = j\omega$, an imaginary p will represent a negative real frequency.

fig. 1



7. Physically Realizable Networks.

This thesis is concerned with only physically realizable networks. They are defined as, 'networks of real elements, having no modes of free vibration that increase indefinitely with time.'^o Thus the forms e^{at} , $e^{j\omega t}$, $e^{j\omega t}t$ are not allowed.

From physical considerations, Dr. Bode has made a list of the requirements of the zeros of $B(p)$ which is the same as the poles of $\frac{A(p)}{B(p)}$.

These are the requirements for a physically realizable network.^{oo}

8. Network Criteria.

1. Zeros and poles are either real or conjugate complex pairs.
2. Real and imaginary components are respectively even and odd functions of frequency on the real frequency axis.
3. No zeros in the right half plane.
4. Zeros on the real frequency axis are all simple.
5. Real component of driving point impedance cannot be negative at real frequency.

6. Passive power must not be more than that consumed by generator and its conjugate.
7. A driving point impedance that meets the above list, when it has no zeros on the real frequency axis, is known as the ' minimum susceptance type. If the network has **no poles** on the real frequency axis, the network is called the ' minimum reactance type'. These two types are not mutually exclusive.

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° ibid 1 p. 120

°° ibid 1 p. 123

9. Multi-Resonant Circuits.

The circuit designer might have designed a circuit composed of many meshes. These meshes might have any combination of R, L, and C. He then looks at his network and asks if it can be reduced to a more simple form. Economy dictates that he uses the least number of elements.

He can easily evaluate the order of the denominator of the driving point impedance function. The procedure is as follows:

1. Number each mesh.
2. Follow the contour of each mesh and assign a value to that mesh.

Value is zero if there is neither L or C in the mesh.

Value is one if there is either L or C in the mesh.

Value is two if there is both L and C in the mesh.

3. Check off one L and C in the mesh if they appear there. Avoid checking off an element that appears as a mutual, if this is possible.

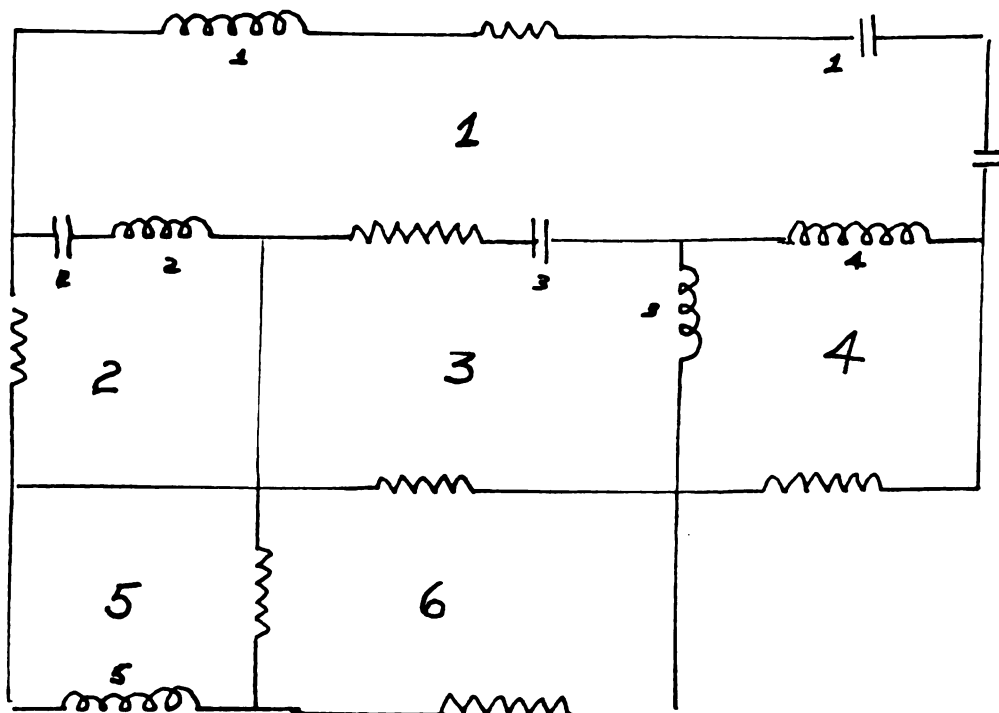
4. Add up the total number of mesh values.

This total is the order of the determinant of the driving point impedance denominator.

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° ibid 2 p. 187

Here is an example illustrating this technique:

fig. 2



Mesh #	Value
1	2
2	2
3	2
4	1
5	1
6	0
Total.....	8

10. Multi-Resonance; Non-Dissipative Cases.

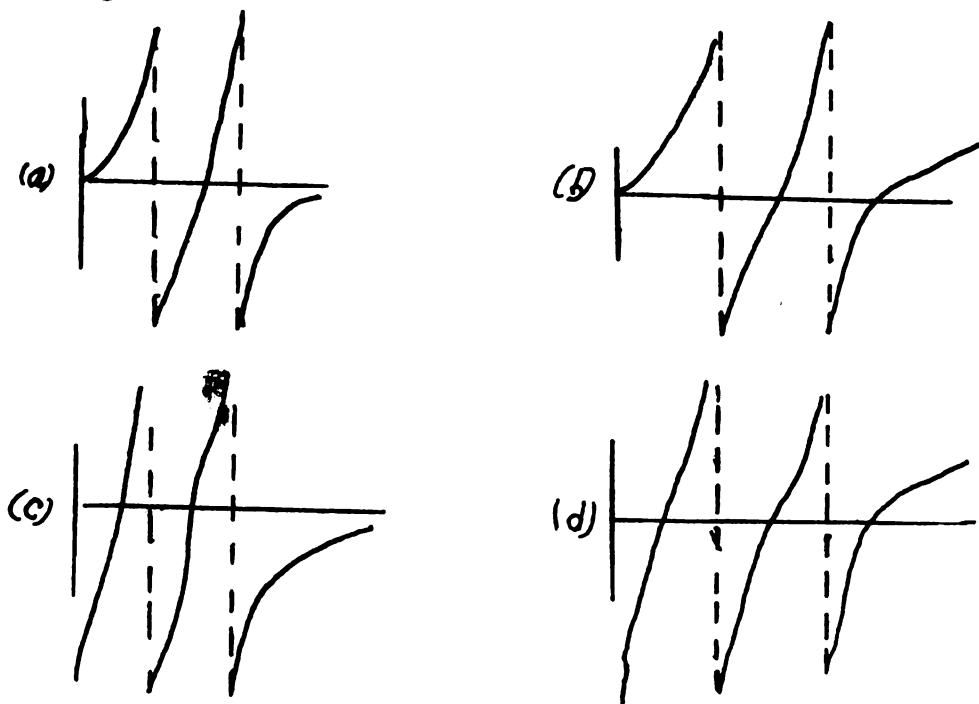
A circuit composed of only pure L and C is non-dissipative. A plot of its reactance versus time will show a series of alternating zeros and poles.

There are four basic cases. They are distinguished by their behavior at zero and infinite frequency.

- (a) Zero at 0 cps-----zero at infinity.
- (b) Zero at 0 cps-----pole at infinity.
- (c) Pole at 0 cps-----zero at infinity.
- (d) Pole at 0 cps-----pole at infinity.

Their plots appear below.

fig. 3



11. Canonic Forms.

The plot of reactance versus time gives the basic character of the circuit. The original circuit can be reduced to a canonic form. These forms will have the same reactance curve as the original function, and will contain the least number of elements. A given circuit can be represented by a very large number of canonic forms so economics will determine which one to use.

12. Preparation of the Problem.

Assume that a multi-resonant circuit has been designed. The driving point impedance can be written by elementary methods and is called $Z = \frac{A(p)}{B(p)}$.

Or perhaps the problem might have arisen that a circuit is required to have a given distribution of zeros and poles. These are problems in synthesis and these problems are solved by resorting to canonic forms.

The two most common methods will be shown.

13. Foster's Reactance Method.

Foster⁷ has devised a method to draw the canonic form.

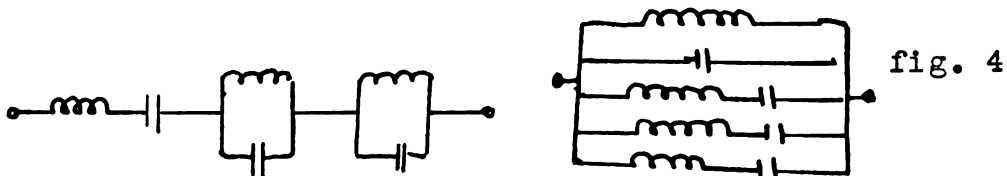
The required information is the location of the zeros and poles, and the reactance at any real frequency.

From $\frac{A(p)}{B(p)}$ it is easy to write the form

$$Z = \frac{H(p^2 - \omega_1^2)(p^2 - \omega_3^2)(p^2 - \omega_5^2)p}{(p^2 - \omega_2^2)(p^2 - \omega_4^2)(p^2 - \omega_6^2)}$$

The roots of the numerator are the zeros and the roots of the denominator are the poles.

The most general forms of the solution are as follows:



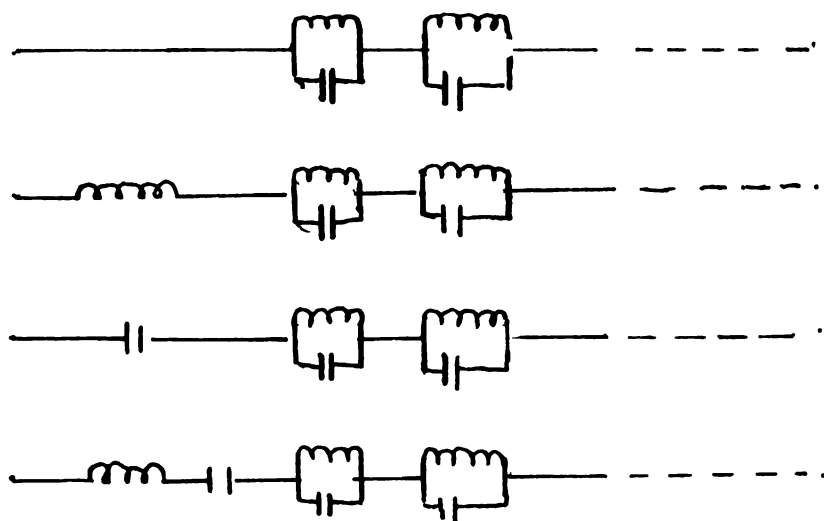
In the four basic cases mentioned before their solutions will appear as:

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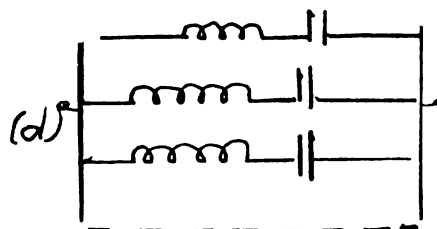
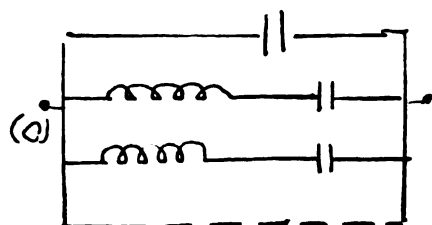
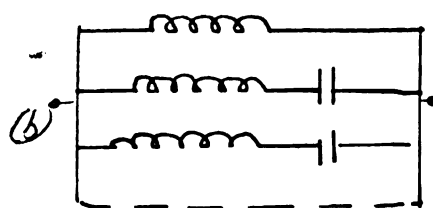
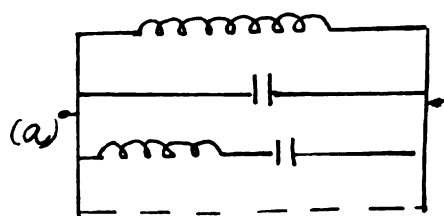
⁷ Foster, R.M. A Reactance Theorem. B.S.T.J.
April 1924. pp259-267.

First Type:

fig. 5



Second Type



An example of the use of Foster's Theorem is given:

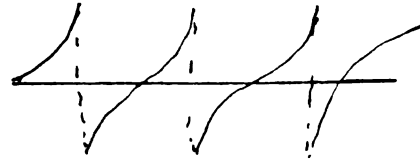
Problem: Design a circuit with this characteristic:

zeros at $\omega = 3, 5, 7$,

poles at $\omega = 2, 4, 6$,

$X = j12.5$ at $\omega = 0.5$

The reactance curve:



$$X = j\omega H \frac{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)}{(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)(\omega^2 - \omega_7^2)}$$

Evaluate H
 $\omega = 0.5$

$$j12.5 = \frac{j.5H(.25-9)(.25-25)(.25-49)}{(.25-4)(.25-16)(.25-36)} = 5$$

$$H = 5 = L_{2h} = L_3$$

$$C_k = \frac{-j\omega Y}{\omega^2 - \omega_k^2}$$

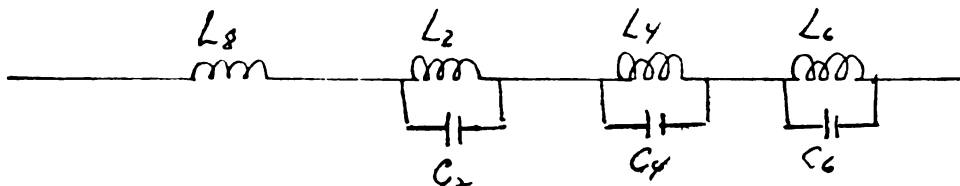
$$L_k = 1/\omega_k^2 C_k$$

$$C_2 = \frac{-(4-16)(4-36)}{5(4-9)(4-25)(4-49)} = .01625 \quad L^2 = 100/4(1.6) = 15.38$$

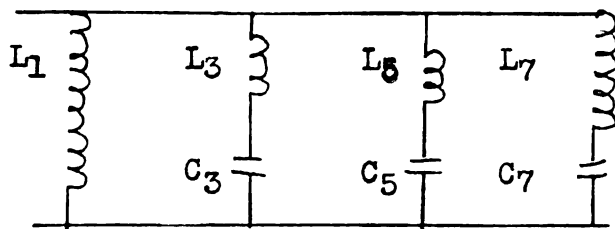
$$C_4 = \frac{-(16-4)(16-36)}{5(16-9)(16-25)(16-49)} = .02310 \quad L_4 = 100/16(2.3) = 2.704$$

$$C_6 = \frac{-(36-4)(36-16)}{5(36-9)(36-25)(36-49)} = .03315 \quad L_6 = 100/36(3.3) = .838$$

The final network will look like this:



The alternative Foster solution.



$$L_k = \frac{-j\omega Z}{\omega^2 - \omega_k^2}$$

$$C_k = 1/\omega_k^2 L$$

$$L_k = \frac{\omega_k^2(\omega_k^2-9)(\omega_k^2-25)(\omega_k^2-49)5}{(\omega_k^2-4)(\omega_k^2-16)(\omega_k^2-36)(\omega_k^2-\omega_k^2)}$$

$$L_1 = (\omega=0) \therefore = \frac{9(25)(49)(5)}{4(16)(36)} = 23.92$$

$$C_1 = 0$$

$$L_3 = \frac{5(9)(9-25)(9-49)}{9-4(9-16)(9-36)} = 30.5$$

$$C_3 = .00364$$

$$L_5 = \frac{5(25)(25-9)(25-49)}{(25-4)(25-16)(25-36)} = 23.1$$

$$C_5 = .00173$$

$$L_7 = \frac{5(49)(49-9)(49-25)}{(49-4)(49-16)(49-36)} = 12.18$$

$$C_7 = .00167$$

14. Cauer's Method.

Cauer^{8°} has a method similar to that of Foster^{°°}, to reduce a non-dissipative network to a canonic form.

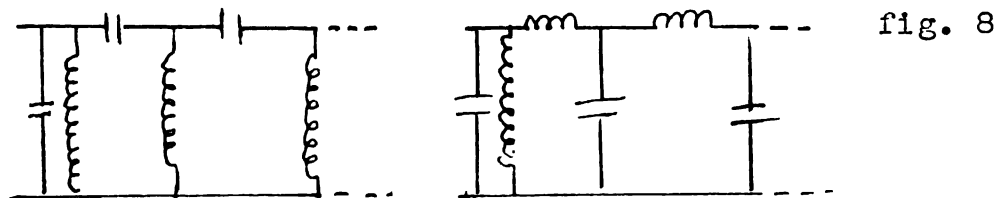
The loop equation of the network is written. The driving point impedance is then written as a fraction of two polynomials in p .

There are two possible solutions. The terms can be arranged in either ascending or descending powers of p .

This choice will determine the form of the solution. The two forms are equivalent, but one may be the wiser choice from the standpoint of economics.

After the terms are placed in order, the method of continued fractions is used to bring about a solution.

The two basic forms are:



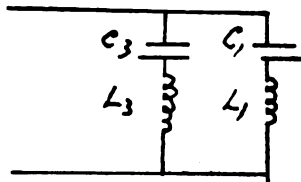
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⁸ Cauer, W. Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängigkeit. Archiv f. Elektrotechnik, vol. 17. p.355, 1927

° Ibid 3 p.198

°° Ibid 7

Cauer Method Example

Given:



$$1 \quad -L_3 p + 1/C_3 p$$

$$L_1 = 2 \times 10^{-3} \text{h} \quad C_1 = 2 \times 10^{-6} \text{fd.}$$

$$L_3 = .5 \times 10^{-3} \text{h} \quad C_3 = 4 \times 10^{-6} \text{fd.}$$

FIND: Canonic Forms.

$$I_1 = \frac{0 \quad (L_1 + L_3)p + (1/C_1 + 1/C_3)(1/p)}{L_3 p + 1/C_3 p \quad -L_3 p + 1/C_3 p}$$

$$-L_3 p + 1/C_3 p \quad (L_1 + L_3)p + (1/C_1 + 1/C_3)1/p$$

$$Z_1' = \frac{.5 \times 10^{-3} p + \frac{250k}{p} \quad (2.5 \times 10^{-3})p + \frac{1}{p(750k)} - .5 \times 10^{-3} p + \frac{250k}{p}}{2.5 \times 10^{-3} p + \frac{750k}{p}}$$

$$= \frac{1 \times 10^{-6} p^2 + 750 + 125,000 \times 10^6}{2.5 \times 10^{-3} p + \frac{750k}{p}} = \frac{10^{-6} p^4 + 750 p^2 + 125 \times 10^9}{2.5 \times 10^{-3} p^3 + 750 \times 10^3 p}$$

Arrange in ascending order.

$$Z = \frac{125 \times 10^9 + 750 p^2 + 1 \times 10^{-6} p^4}{750,000 p + 2.5 \times 10^{-3} p^3}$$

Continued Fraction Expansion

$$750,000 p = 2.5 \times 10^{-3} p^3 \quad \frac{(1/6 \times 10) \quad 1/p}{\frac{1.25 \times 10^9 + 750 p^2 + 10^{-6} p^4}{1.25 \times 10^9 + 416 p^2} + \dots}$$

$$\frac{167 \times 10^3}{p} + \frac{334 p^2 + 10^{-6} p^4}{750,000 p + 2.5 \times 10^{-3} p^3} \quad \frac{167 \times 10^3}{p}$$

Invert the remainder and continue the expansion.

$$334p^2 + 10p^4 \overline{) \begin{array}{r} 2.25 \times 10^3 / p \\ 750 \times 10^3 p + 2.5 \times 10^{-3} p^3 \\ \hline 750 \times 10^3 p + 2.25 \times 10^{-3} p^3 \\ \hline .25 \times 10^{-3} p^3 \end{array}} + \dots$$

$$\text{Result} \quad \frac{2.25 \times 10^3}{p} + \frac{.25 \times 10^{-3} p^3}{334p^2 + 10^{-6} p^4} \rightarrow \frac{2.25 \times 10^3}{p}$$

Invert remainder and continue division.

$$.25 \times 10^{-3} p^3 \overline{) \begin{array}{r} 1340 \times 10^3 / p \\ 334p^2 + 10^{-6} p^4 \\ \hline 334p^2 + 0 \\ \hline 10^{-6} p \end{array}} + \dots$$

$$\frac{1340 \times 10^3}{p} + \frac{10^{-6} p^4}{.25 \times 10^{-3} p^3} \rightarrow \frac{1340 \times 10^3}{p}$$

Invert remainder and divide.

$$10^{-6} p^4 \overline{) \begin{array}{r} 250 / p \\ .25 \times 10^{-3} p^3 \end{array}} \rightarrow \frac{250}{p}$$

The successive results represent a series of Z,Y, in order.

$$\frac{167 \times 10^3}{p} = C_1 = 6 \times 10 \text{ fd}$$

$$\frac{2.25 \times 10^3}{p} = L_1 = .00445 \times 10^{-3} \text{ h.}$$

$$\frac{1.34 \times 10^6}{p} = C_2 = .746 \times 10^{-6} \text{ fd.}$$

$$\frac{250}{p} = L_2 = .004 \text{ h.}$$

Thus the Cauer form is:

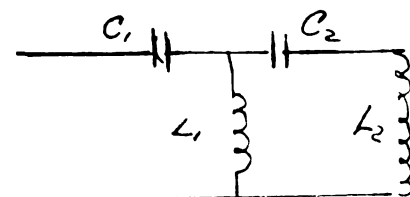


fig. 10

The other alternative

$$Z = \frac{10^{-6}p^4 + 750p^3 + 125 \times 10^9}{2.5 \times 10^{-3}p^3 + 750,000p}$$

$$\begin{array}{r} 2.5 \times 10^{-3}p^3 + 750,000p \overline{) \frac{10^{-6}p^4 + 750p^3 + 125 \times 10^9}{10^{-6}p^4 + 300p^3 + 0}} \\ \hline 450p^3 + 125 \times 10^9 \\ .4 \times 10^{-3}p = L_1p \end{array}$$

$$\begin{array}{r} 450p^3 + 125 \times 10^9 \overline{) \frac{5.55 \times 10^{-6}p + \dots}{2.5 \times 10^{-3}p^3 + 750,000p}} \\ \hline 2.5 \times 10^{-3}p^3 + 695,000p \\ 55,000p \\ 5.55 \times 10^{-6}p = C_1p \end{array}$$

$$\begin{array}{r} 55,000p \overline{) \frac{8.1 \times 10^{-3}p}{450p^3 + 125 \times 10^9}} \\ \hline 450p^3 \\ 125 \times 10^9 \\ 8.1 \times 10^{-3}p = L_2p \end{array}$$

$$\begin{array}{r} 125 \times 10^9 \overline{) \frac{.44 \times 10^{-6}p}{55 \times 10^3p}} \\ \hline 55 \times 10^3p \\ .44 \times 10^{-6}p = C_2p \end{array}$$

Network:

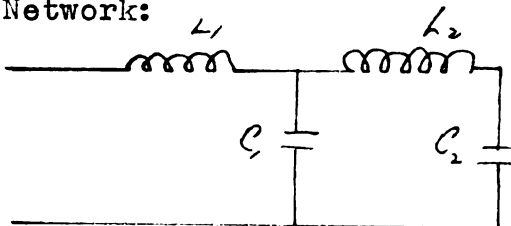


fig. 11

$$\begin{aligned} L_1 &= .4 \text{ mh} \\ L_2 &= 8.1 \text{ mh} \\ C_1 &= 5.55 \text{ uf} \\ C_2 &= .44 \text{ uf} \end{aligned}$$

Note in this particular problem, the second form gives results more easily constructed than the first form. This is not a general rule.

15. Minimum Resistance Function.

Theorem^o:

A passive immittance will continue to meet the requirements for physical realizability in passive networks, if it is diminished by any real constant, as long as the real component of the resultant expression does not become negative at any real frequency.

To use this theorem it is necessary to be able to convert a p expression into some expression containing ω .

Take the p expression, replace p by $j\omega$, and then separate the real and imaginary parts. Call the real part R' .

From R' can be subtracted any real number, as long as the remainder R'' is never negative for any real frequency. R' will reach a minimum value at some frequency, and as long as this value is subtracted from the original R' , the new expression R'' will never be negative.

Illustration:

Separate function of p into real and imaginary components. Differentiate the real part and find the minimum value. Subtract this value from the original expression.

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^o Ibid p. 172

Illustration: Minimum Resistance Function.

Given a passive network FIND
Minimum Resistance
Function.

$$Z = \frac{p^2 + p + 2}{5p^2 + 3p + 4}$$

$$p = j\omega \dots\dots\dots Z = \frac{-\omega^2 + j\omega + 2}{-5\omega^2 + 3j\omega + 4} \quad \text{rationalize}$$

$$Z = \frac{-\omega^2 + j\omega + 2(-5\omega^2 + 4 - 3j\omega)}{(-5\omega^2 + 4 + 3j\omega)(-5\omega^2 + 4 - 3j\omega)} \dots\dots\dots \frac{5\omega^4 - 11\omega^2 + 8 + j(-2\omega^3 - 2\omega)}{(4 - 5\omega^2)^2 + 9\omega^2}$$

Separate real from the imaginary. $Z = R + jX$

$$R = \frac{5\omega^4 - 11\omega^2 + 8}{(4 - 5\omega^2)^2 + 9\omega^2} \quad X = \frac{-2\omega^3 - 2\omega}{(4 - 5\omega^2)^2 + 9\omega^2}$$

Let $X = \omega^2$

$$R = \frac{5x^2 - 11x + 8}{(4 - 5x)^2 + 9x} \dots\dots\dots \frac{5x^2 - 11x + 8}{(4 - 5x)^2 + 9x} = \frac{5x^2 - 11x + 8}{25x^2 - 31x + 16}$$

$$\frac{dR}{dx} = 0 = (25x^2 - 31x + 16)(10x - 11) - (5x^2 - 11x + 8)(50x - 31)$$

$$0 = 120x^2 - 240x + 72 = 10x^2 - 20x + 6 \dots\dots\dots x = \frac{20 \pm \sqrt{400 - 240}}{20}$$

$$x = 1.63, \quad .3675$$

Choose the first root.

$$x = 1.63 \dots\dots R = \frac{13.3 - 17.9 + 8}{66.2 - 50.5 + 16} = \frac{3.4}{31.7} = 0.105$$

This is the minimum at $\omega^2 = 1.63$. By inspection we see that the alternate root gives a higher valued solution.

The minimum resistance function is then

$$Z' = Z - 0.105 = \frac{p^2(1 - .525) + p(1 - .315) + 2(1 - .420)}{5p^2 + 3p + 4}$$

$$= \frac{.48p^2 + 0.69p + 1.58}{5p^2 + 3p + 4}$$

Illustration: Minimum Conductance Function

The same procedure can be followed to find minimum conductance with an admittance function.

Given:

$$Y = \frac{5p^2 + 3p + 4}{p^2 + p + 2}$$

Find:

Minimum Conductance Function

$$Y = \frac{-5\omega^2 + 3j\omega + 4}{-\omega^2 + j\omega + 2} \quad \frac{-5\omega^2 + 3j\omega + 4}{\omega^4 - 3\omega^2 + 4} (2 - \omega^2 - j\omega)$$

$$= \frac{-10\omega^2 + 6j\omega + 8 + 5\omega^4 - 3j\omega^3 - 4\omega^2 + 5j\omega^3 + 3\omega^2 - 4j\omega}{\omega^4 - 3\omega^2 + 4} \quad \text{Let } x = \omega^2$$

$$Y = \frac{5\omega^4 - 11\omega^2 + 8 + j(2\omega^3 + 2\omega)}{\omega^4 - 3\omega^2 + 4}$$

$$G = \frac{5x^2 - 11x + 8}{x^2 - 3x + 4}$$

$$B = \frac{2\omega^3 + 2\omega}{\omega^4 - 3\omega^2 + 4}$$

$$\frac{dG}{dx} = 0 = (x^2 - 3x + 4)(10x - 11) - (5x^2 - 11x + 8)(2x - 3)$$

$$0 = -4x^2 + 24x - 20 \quad \text{Root } x = 1$$

$$x=1 \quad G = \frac{5 - 11 + 8}{1 - 3 + 4} = \frac{2}{2} = 1$$

$$Y' = Y - 1 = \frac{5p^2 + 3p + 4}{p^2 + p + 2} - 1 =$$

$$\text{Minimum Conductance Function} = \frac{4p^2 + 2p + 2}{p^2 + p + 2}$$

15. Minimum Reactance Function.

Theorem°

A passive will continue to meet the requirements of physical realizability in passive networks if it is diminished by the reactance or susceptance of one of its real frequency poles.

This is the same type problem as the minimum resistance problem.

Illustration:

Given:

FIND:

$$Z = \frac{-2p^2 + p + 1}{p^3 + p^2 + p + 1}$$

Minimum Reactance Expression.

$$Z = \frac{2p^2 + p + 1}{p^3 + p^2 + p + 1} \dots = \frac{2p^2 + p + 1}{(p+1)(p^2+1)} \quad \text{roots, } p=1, p=\pm j$$

$$A_0 = \lim_{p \rightarrow p_0} \frac{Z(p-p_0)}{p-p_0} = \lim_{p \rightarrow p_0} \frac{p^2 - p_0^2}{2p_0} Z \quad p_0^2 = -1$$

$$A_0 = \lim_{p \rightarrow j} \frac{(p^2 - 1)(2p^2 + p + 1)}{2j(p+1)(p^2+1)} = \frac{-2+j+1}{2j(1+j)} = \frac{1}{2}$$

$$\text{In an anti-resonant circuit } D = 2A_0 \quad L = \frac{-2A_0}{p_0^2}$$

$$D=1 \quad L=1$$

$$Z = \frac{p}{1+p^2} + (Z') = \frac{2p^2 + p + 1}{(1+p)(1+p^2)} \quad Z' = \frac{2p^2 + p + 1}{(p+1)(p^2+1)} - \frac{p}{1+p^2}$$

$$Z' = \frac{2p^2 + p + 1 - p^2 - p}{(p+1)(p^2+1)} = \frac{p^2 + 1}{(p^2+1)(p+1)} = \frac{1}{p+1}$$

$$Z = \frac{1}{p+1} + \frac{p}{p^2+1}$$



Fig. 12

.....
° Ibid 1 p. 175.

16. Brune's Method.

It may be required to find a circuit that is represented by a p expression. As long as the expression represents a passive network, this can be done by the method of Brune⁹. He has shown that any impedance satisfying the requirements to be passive can be represented by a physical network.

To do this, use is made of the technique developed to find minimum resistance, conductance and reactance functions.

The idea is to remove the poles of impedance until a simple resistance is reached.

The procedure is:
with a p expression given, change it to a minimum resistance function. Then remove the poles one by one. Remove a pole, subtract it from the minimum resistance function, and keep doing this till there remains only a pure resistance.

The process may be laborious.

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⁹ Brune, O. Synthesis of a Finite Two Terminal
 Network Whose Driving Point Impedance
 is a Prescribed Function of Frequency.
 Journal of Math. and Phys., vol. 10, pp.191-235.

Illustration: Brune Method.

Given:

FIND:

$$Z = \frac{p^2 + p + 2}{5p^2 + 3p + 4}$$

A Network to represent this expression.

From previous work the minimum resistance occurred when $\omega^2 = 1.63$. The minimum resistance function was found to be

$$\begin{aligned} Z &= \frac{.48p^2 + .69p + 1.58}{5p^2 + 3p + 4} \quad \omega^2 = 1.63 \quad \omega = 1.275 \\ &= \frac{-.48(1.63) + (.69j)(1.275) + 1.58}{-5(1.63) + j3(1.275) + 4} = \frac{.797 + j.882}{-4.15 + j3.83} \\ &= .211 / -90^\circ \quad \text{a pure negative L or a pure C.} \end{aligned}$$

$$\text{At } \omega = 1.275 \quad -L = .211 / 1.275 = .165h \quad \underline{L = -.165h}$$

Subtract from Z' .

$$Z'' = \frac{.48p^2 + .69p + 1.58}{5p^2 + 3p + 4} - .165h = \frac{.825p^3 + .975p^2 + 1.35 + 1.58}{5p^2 + 3p + 4}$$

$$\text{This can be factored.} = \frac{2.44(.338p + .397)(p^2 + 1.64)}{5p^2 + 3p + 4}$$

Invert and consider Y''

$$Y'' = \frac{5p^2 + 3p + 4}{2.44(.338p + .397)(p^2 + 1.64)} \quad \text{pole at } p = \pm 1.28$$

$$A_0 = \lim_{p \rightarrow p_0} \frac{p^2 - p_0^2}{2p_0} Y'' = \frac{1.64(5) + 3j(1.28) + 4}{2.56j(.397 + j)(1.28)(.338)(2.44)} = 1.55$$

$$1/L = 2A_0 \quad L = .323h \quad C = 2A_0/p_0^2 = .310/1.64 = 1.89$$

This is a series resonant circuit in shunt

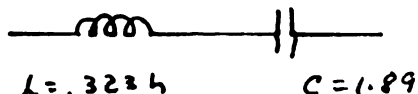


Fig. 13

The admittance of this shunt element is:

$\frac{3.10p}{1.64+p^2}$ Subtract this from Y'' and get Y'''

$$Y''' = \frac{5p^2+3p+4}{2.44(p^2+1.64)(3.38p+.397)} - \frac{3.10p}{1.64+p^2}$$

$$= \frac{5p^2+3p+4-2.56p^2-3p}{2.44(1.64+p^2)(.338p+.397)} = \frac{1}{-2.44 \quad .338p+.397}$$

$$Z''' = .338p+.397 \quad \dots\dots\dots Lp+R$$

$$L = .338 \quad R = .397$$

Now that all these computations are finished, the network can be drawn.

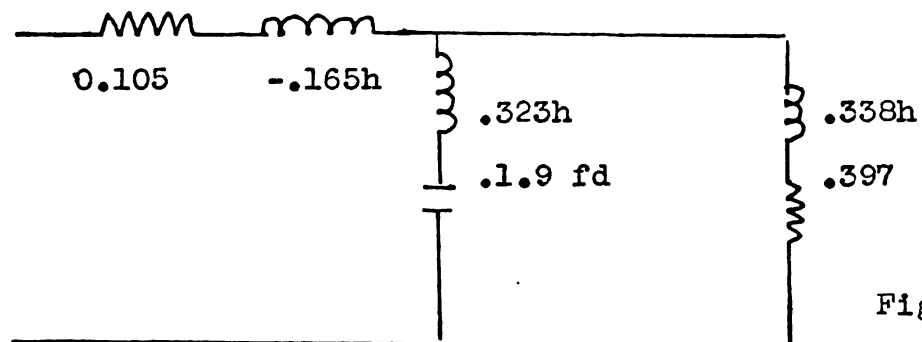


Fig. 14

The Brune treatment of an impedance function has been shown. The same treatment can be used to handle an admittance function.

Given:

$$Y = \frac{5p^2 + 3p + 2}{p^2 + p + 2}$$

FIND:

Brune network.

Minimum conductance when $\omega=1$ $G=1$

Minimum conductance expression

$$Y' = \frac{4p^2 + 2p + 2}{p^2 + p + 2} \quad \text{This was shown previously.}$$

$$\text{At } \omega=1 \quad \text{---- } Y' = \frac{-4+2j+2}{1+j+2} \quad \frac{-2-2j}{1+j} = 2 \angle -90^\circ$$

Treat as a negative L ... $\omega=1$... $1/L = -2$ $L = -.5h$

$$Z'' = \frac{p^2 + p + 2}{2p^2 + p + 1} - .5p = \frac{p^2 + p + 2 + 2p^3 + p^2 + p}{2p^2 + p + 1}$$

$$= \frac{p^3 + p^2 + p + 1}{2p^2 + p + 1} = \frac{(p^2 + 1)(p + 1)}{2p^2 + p + 1}$$

$$Y'' = \frac{2p^2 + p + 1}{(p^2 + 1)(p + 1)} \quad A_0 = \lim_{p \rightarrow p^0} \frac{p^2 - p_0^2}{2p_0} Z$$

$$A^0 = \frac{-2 + j + 1}{2(1+j)j} = \frac{1}{2}$$

In an admittance circuit series resonance

$$1/L = 2A_0 \quad C = \frac{2A_0}{-p_0^2}$$

$$= 1$$

$$= 1$$

Eliminate a series resonant circuit.

$$Y_{1111} = \frac{2p^2+p+1}{(p^2+1)(p+1)} - \frac{p}{(p^2+1)(p^2+1)(p+1)} \frac{1}{p+1} = \frac{1}{p+1}$$

This is a series R, L circuit in shunt with the rest of the circuit.

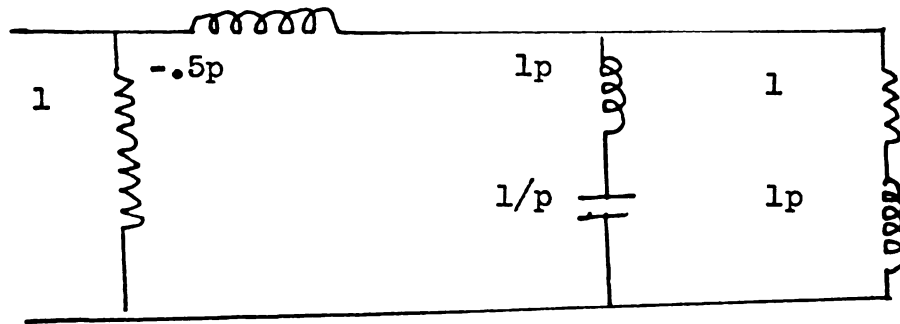
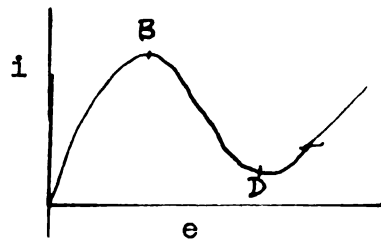


Fig. 15

The Brune method is more laborious in the impedance case. This is not a general rule, but is due to the choice of the original function. Another function might have been chosen which would tend to give a result most easily when the impedance approach were chosen.

17. Negative Resistance.

The concept of negative resistance^{10, 11} is introduced to facilitate the change from a passive network to an active network. The negative resistance can be considered as a source of power. One method used to realize this element is to use the negative slope found on a portion of the Tetrode Tube Characteristic.



From B to D the slope of the curve is negative.

Fig. 16

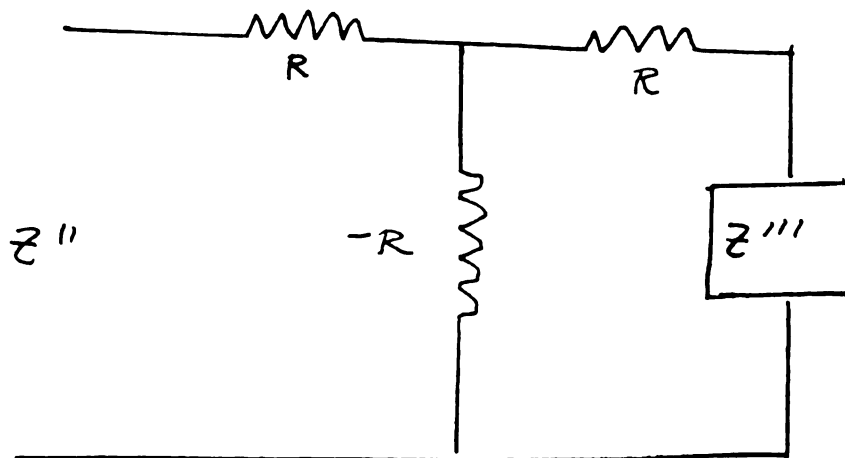
The use of negative resistance allows the use of two other circuit elements. They are negative inductance and negative capacitance.

In the general form, a T network is set up. The shunt element of the T is a negative resistance. The other two elements are equal in magnitude to the shunt element, but opposite in sign. The T is terminated with Z'' . The input impedance is $Z!$ --

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¹⁰ Everitt, W.L. Communication Engineering.
Second Edition, p. 509 McGraw-Hill Co. 1937.

¹¹ Emery, W.L. Ultra-High Frequency Engineering.
p. 162 Macmillan Co. 1942.

^o Ibid p. 188.



$$Z' = R + \frac{-R(R+Z)}{Z} = R - \frac{R^2 - RZ}{Z}$$

$$= \frac{RZ - R^2 - RZ}{Z} = \frac{-R^2}{Z}$$

This is the case where Z is a resistance.

In the case where Z is pure L.

$$Z' = R + \frac{-R(R+j\omega L)}{j\omega L} = \frac{j\omega LR - R^2 - j\omega LR}{j\omega L} = \frac{-R^2}{-j\omega L} = \frac{-R^2}{j\omega L}$$

Where Z'' is pure Capacitance

$$Z' = R + \frac{-R(R+1/j\omega C)}{1/j\omega C} = \frac{R}{j\omega C} - \frac{R^2}{1/j\omega C} - \frac{R}{j\omega C} = \frac{-R^2}{1/j\omega C}$$

Thus any negative impedance, including negative capacity and inductance, can be represented by terminating the T in the positive inverse of the required impedance.

An active network can be broken down into a combination of a passive network and a negative resistance network.

$$Z = \frac{p^2 + p + 4}{4p^2 + 2p + 2}$$

$$Y = \frac{4p^2 + 2p + 2}{p^2 + p + 4}$$

$$Y' - 1 = Y = \frac{4p^2 + 2p + 2}{p^2 + p + 2}$$

$$Y' = \frac{4p^2 + 2p + 2 + p^2 + p + 2}{p^2 + p + 2} = \frac{5p^2 + 3p + 4}{p^2 + p + 2}$$

This is an impedance in parallel with a negative resistance.

18. Inverse Network.

The inverse network^{12,°} is a useful tool. The inverse is usually taken with respect to a constant R . When this constant is set equal to 1, the structure found is called the reciprocal.

The dual is often confused with the reciprocal network. The dual is just a device used so that a network can be solved by means of the node equations. In this, the sources of voltage in the original network are replaced by sources of current. Instead of dealing with the impedances of the elements, the admittances are considered. However, there is no rearrangement of the circuit elements.

In the reciprocal circuit this is not true. Mathematically, the Matrix of the impedance is formed. Then the inverse of the matrix is found, by conventional Matrix Algebra. This inverse is then written as the matrix of the admittance of the circuit.

The inverse can be found in many cases without resorting to Matrix Algebra.

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¹² Zobel, O.J. B.S.T.J. January 1923, July 1928.

[°] Ibid 3 p. 203.

To find the inverse by geometric methods the following procedure is followed:

Locate a point in the center of each mesh, and one point outside the circuit.

Connect all the points by a series of lines. When a line passes through an element, that element will be replaced by its inverse.

An illustration follows:

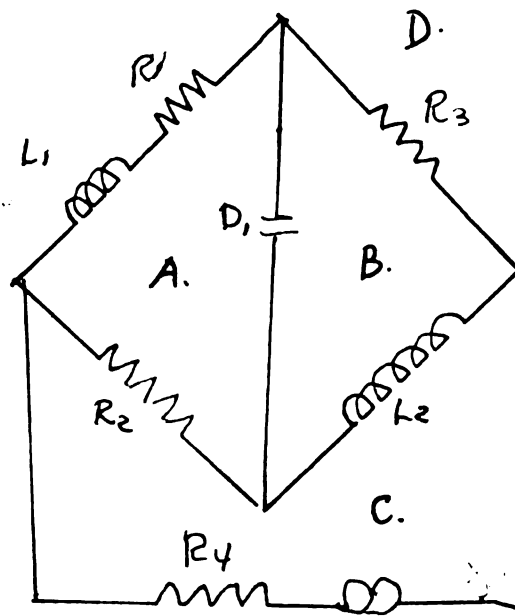


Fig. 18

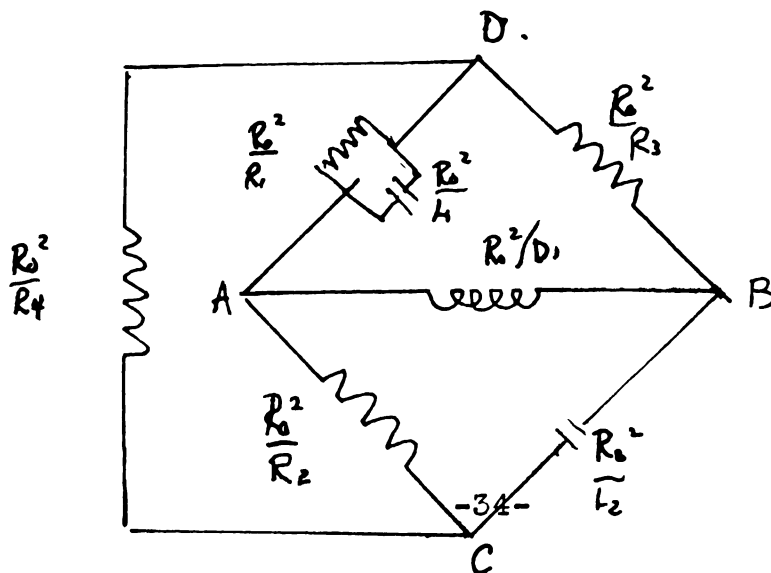
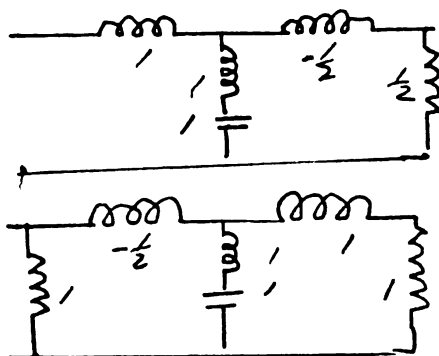


Fig. 19

This method, called 'Structural Inverse' will not work for all passive networks.¹³ If the required inverse is to be passive, the original network must not have any mutual inductance, for there is no inverse that is passive for mutual inductance.

However if active networks are allowed, the mathematical approach will always give a result.

The inverse is found, for example, for a Brune type network. Since the product of the two networks must equal a real number, it will be shown that the two following networks are inversely related to each other.



$$Z' = \frac{2p^2 + p + 1}{p^2 + p + 2}$$

Fig. 20

$$Z'' = \frac{(p^2 + p + 2)1}{2(2p^2 + p + 1)}$$

$$Z'' Z' = \frac{2p^2 + p + 1 (p^2 + p + 2) 1}{p^2 + p + 2 (2p^2 + p + 1) 2} = \frac{1}{2}$$

.....

¹³ Foster, R.M. Geometric Circuits of Electrical Networks. A.I.E.E. June 1932.

19. Complementary Networks.

A complementary network^o requires that the sum of the original network and its complementary network be a real constant.

Illustration.

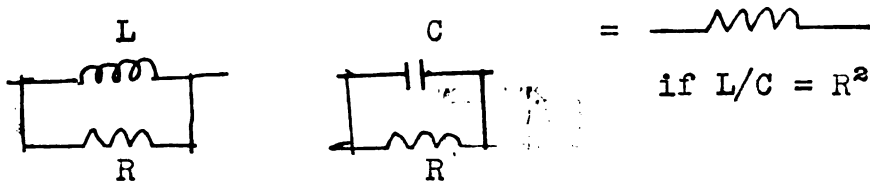


Fig. 21

$$Z = \frac{LpR}{Lp+R} + \frac{R/Cp}{R+1/Cp} = \frac{LpR^2+LR/C+LR/C+R^2/Cp}{LpR+R^2+L/C +R/Cp}$$

$$= \frac{R(LpR+L/C+L/C+R/Cp)}{(LpR+R^2+L/C+R/Cp)} = R$$

This shows the two networks to be complementary.

^o Ibid 13 p. 199.

20. Partial Fraction Expansion of a General Impedance.

Given an impedance function of p , by the method of partial fractions,¹⁴ a physical network can be drawn.

Given:

$$Z = \frac{3p^2 - 7p + 3}{p^2 - 3p + 2} = \frac{3(p - 1/756)(p - .564)}{(p - 1)(p - 2)}$$

pole at $p=1$ $3(-.756)(1-.564) = 1$

first fraction $Z' = 1/p-1$ This is $-R // C$

Subtract this fraction from the original.

$$Z'' = \frac{3p^2 - 7p + 3}{(p-1)(p-2)} - \frac{1}{p-1} = \frac{3p^2 - 8p + 5}{(p-2)(p-1)} = \frac{3p-5}{p-2}$$

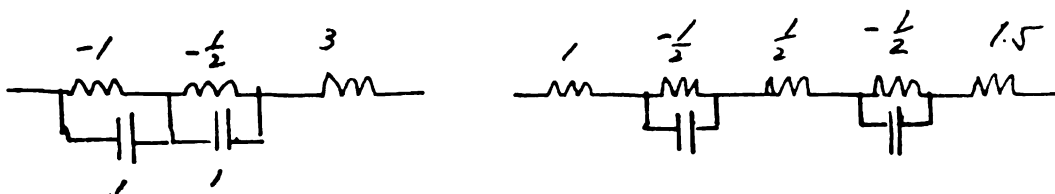
Residue at $p = 2$ Res. = 1

Second fraction $1/p-2$ This is a negative

R of $1/2$ in $//$ with $C = 1$.

The network will look like this. (See Fig. 22).

This form can be changed by distributing the 3 ohms along the circuit. Then the circuit takes the form:

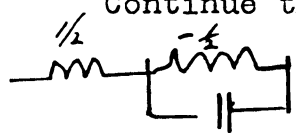


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¹⁴ Esbach, O.W. Handbook of Engineering Fundamentals
First Edition 1936 Ch. 2-08

° Ibid 3 pp. 192, 206.

Continue the analysis.



$$= \frac{1}{2} + \frac{1}{p-2} = \frac{p^2}{p-2}$$

Fig. 23

This expression can be represented as:

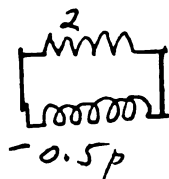
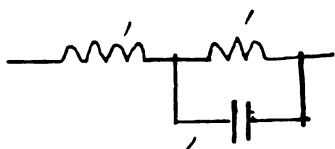


Fig. 24

Also analyze:



$$1 + \frac{1}{p-1} = \frac{p}{p-1} =$$



Fig. 25a

Fig. 25.

Then the entire network can be written as:

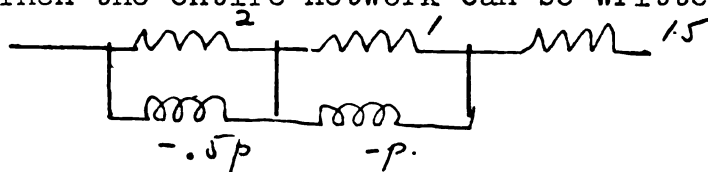


Fig. 27

This is only one out of many variations that can be constructed.

A more complicated network is solved as an example of partial fraction synthesis.

$$Z = \frac{p^2 + p + 2}{5p^2 + 3p + 4} \quad \text{roots } -.3 \pm j.844$$

$$= \frac{(p^2 + p + 2)}{(p + .3 - .844j)(p + .3 + .844j)} \quad \text{find residue}$$

$$p = -.3 - .844j$$

Res.

$$\frac{-.62 + j.506 - .3 - .844j + 2}{-1.688j} = \frac{1.08 - j3.38}{-1.688j}$$

$$\text{Residue} = .2 + j.643$$

$$\begin{aligned} Z' &= \frac{2(\text{Cap} - (\text{Capa} + \text{Cbpb}))}{p^2 - 2pap + (pa^2 + pb^2)} = \frac{2(.2p - (-.06 + .543j))}{p^2 + .6p + .09 + .71} \\ &= \frac{2(.2p - .483)}{p^2 + .6p + .80} = \frac{.4p - .966}{p^2 + .6p + .80} = \frac{.75(.534p - 1.28)}{p^2 + .6p + .05 + .75} \end{aligned}$$

Assume $R = .75$ in shunt, and remove it from the Z .

$$\begin{aligned} Z' &= \frac{.534p - 1.28}{p^2 + .6p + .05} \quad \text{roots } -.5, -.1 \\ &= \frac{.534p - 1.28}{(p + .5)(p + .1)} \quad \text{residue at } p = -.1 \quad -1.33/4 = -3.33 \end{aligned}$$

elements $-3.33/p + 1$ subtract from Z' and get:

$$Z'' = \frac{.534p - 1.28}{(p + .5)(p + .1)} - \frac{3.33}{p + .1} = \frac{1}{p + .5} \quad \text{This is } \begin{matrix} R=2 \\ C=1 \end{matrix}$$

The complete network will then appear as:

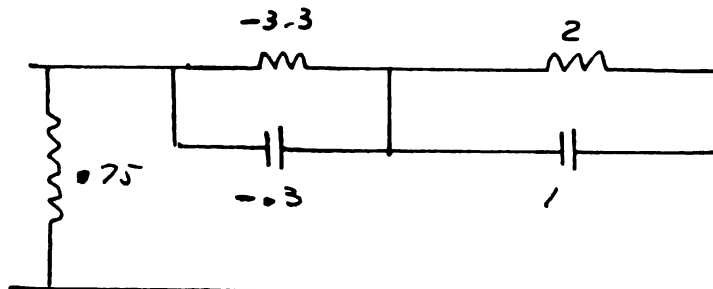
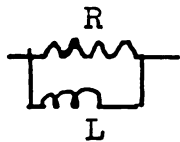


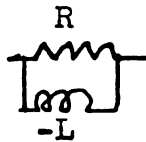
Fig. 28

21. 'p' Networks.

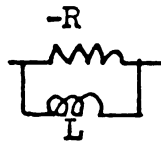
The partial fraction expansion results in expressions that represent two element networks. These expressions and their corresponding forms are shown below.



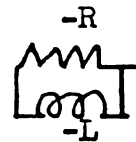
$$\frac{RLp}{Lp+R}$$



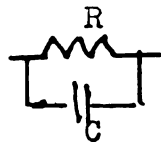
$$\frac{RLp}{Lp-R}$$



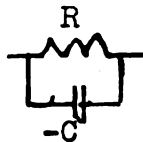
$$\frac{RLp}{R-Lp}$$



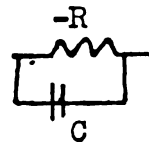
$$\frac{RLp}{-R-Lp}$$



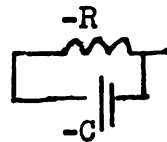
$$\frac{1/C}{p+1/CR}$$



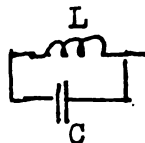
$$\frac{1/C}{1/CR-p}$$



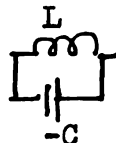
$$\frac{1/C}{p-1/CR}$$



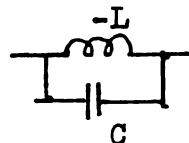
$$\frac{-1/C}{p+1/CR}$$



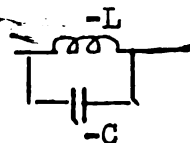
$$\frac{p/C}{p^2+1/CL}$$



$$\frac{p/C}{1/CL-p^2}$$



$$\frac{p/C}{p^2-1/CL}$$



$$\frac{-p/C}{p^2+1/CL}$$

22. Reconstruction of a Passive Impedance

From a Knowledge of Either Component.

Darlington^{15,°} has shown that, given a resistance function of time, or ω , the reactance function of the same impedance can be obtained, and visa versa. The sum of the two functions should be a minimum resistance function.

To illustrate:

A $R(\omega)$ is given. Then a $Z(\omega)$ is found.

Then the next step is to find the $X(\omega)$.

$$R(\omega) = \frac{1-\omega^2}{(\omega-1+j)(\omega-1-j)(\omega+1+j)(\omega+1-j)}$$

$$C'(-1-j) = \frac{1-(1-j)^2}{-8j+8} = \frac{1-2j}{-8j+8}$$

$$C''(1+j) = \frac{1-2j}{8j-8}$$

$$Z = \frac{2C'}{\omega-\omega'} + \frac{2C''}{\omega-\omega''} = \frac{1-2j}{4j-4(\omega-1+j)} + \frac{1-2j}{(4-4j)(\omega-1-j)}$$

$$= \frac{-4\omega+8-12j\omega+16j+4\omega+12j\omega+8-16j}{(\omega^2-2j)32j} = \frac{16-32j}{32(2+j\omega^2)}$$

$$= \frac{(16-32j)(2-j\omega^2)}{32(2+j\omega^2)(2-j\omega^2)} = \frac{32-16j\omega^2+64j-32\omega^2}{32(\omega^4+4)}$$

$$\text{Real } \frac{32(1-\omega^2)}{32(\omega^4+4)} \quad \text{Imaginary } \frac{j(-16\omega^2+64)}{32(\omega^4+4)}$$

$$\frac{1-\omega^2}{(\omega^4+4)} \quad \text{Checks with the original } R(\omega)$$

.....

¹⁵ Darlington Synthesis of Reactance Four Poles.
Journal of Mathematics and Physics. Sept. 1939.

° Ibid 1 p. 203.

23. Extension of Foster's Theorem to Dissipative Cases.

A network containing only two types of elements can be solved by an extension of Foster's ^o,^{oo} form. This has been shown by Cauer.

Theorem:

The expression for the impedance of a network made up of only two elements can be obtained from the expression of an impedance of a corresponding network of pure reactances, by replacing the multiplier p in the pure reactance expression by the impedance which corresponds to a pure inductance, and by replacing the p terms in the rest of the expression by the ratio of the impedance corresponding to a unit inductance and a unit capacity.

$$R, L \text{ Network } Z = \frac{kp(p-p_2^s)(p-p_4^s) \dots}{(p-p_1^s)(p-p_3^s)}$$

$$R, C \text{ Network } Z = \frac{k(p-p_2^s)(p-p_4^s) \dots}{(p-p_1^s)(p-p_3^s)}$$

.....

^o Ibid 1 p. 214.

^{oo} Ibid 3 p. 208.

Here is an example of the extension of Foster's form into the case of two element dissipative cases.

Given:

poles at $p=0, -2, -5$.

zeros at $p=-1, -3$.

Real part of the driving point impedance Z is 1 when $\omega=0$.

FIND: Two element network to have given characteristics.

$$Z = \frac{H(p+1)(p+3)}{p(p+2)(p+5)} = \frac{H(p^2+4p+3)}{p^3+7p^2+10p}$$

$$p = j\omega$$

$$Z = \frac{H(-\omega^2+4j\omega+3)}{(-j\omega-7\omega^2+10j\omega)} = \frac{H(-\omega^2+4j\omega+3)(-7\omega^2+j\omega^3-10j\omega)}{49\omega^4+\omega^6-20\omega^4+100\omega^2}$$

$$= \frac{H(7\omega^4-j\omega^5+10j\omega^3-28j\omega^3-4\omega^4+40\omega^2-21\omega^2+j\omega^2-30j\omega)}{\omega^2(\omega^4+29\omega^2+100)}$$

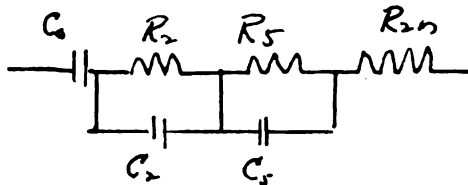
$$\text{Real part } \frac{H(3\omega^2+19)\omega^2}{(\omega^4+29\omega^2+100)\omega^2} \dots \text{ at } \omega=0 \text{ real part} = 1.$$

$$H = \frac{100}{19}$$

$$C_k = \frac{Y(p)}{p-p_k} \text{ as } p \text{ approaches } p_k, \quad p_k = -1/R_k C_k$$

$$R_{2n} = H$$

$$Y(p) = \frac{p(p+2)(p+5)}{H(p+1)(p+3)}$$



$$C_0 = \frac{19(2)(5)}{100(3)} = \frac{19}{30} \quad R_0 = 0$$

$$C_2 = \frac{19(-2)(3)}{100(-1)(1)} = \frac{114}{100} R_2 = \frac{100}{228}$$

$$C_5 = \frac{19(-3)(3)}{100(-4)(-2)} = \frac{152}{800} \quad R_5 = \frac{800}{760}$$

Continued Fraction Expansion may also be used to get an equivalent result.

$$Z(p) = \frac{100(p^3+4p+3)}{p^3+7p^2+10p} = \frac{15.8+21.03p+5.26p^2}{10p+7p^2+p^3}$$

$$10p+7p^2+p^3 \overline{) \begin{array}{r} 1.58/p \\ 15.8 + 21.03p + 5.26p^2 \\ \underline{15.8 + 12.27p + 1.58p^2} \\ 8.76p + 3.68p^2 \end{array}} \quad \begin{array}{l} 1.58/p \\ C = .633 \end{array}$$

$$8.76p+3.68p^2 \overline{) \begin{array}{r} 1.14 \\ 10p + 7p^2 + p^3 \\ \underline{10p + 4.2p^2} \\ + 2.8p^2 + p^3 \end{array}} \quad \begin{array}{l} 1.14 = R \end{array}$$

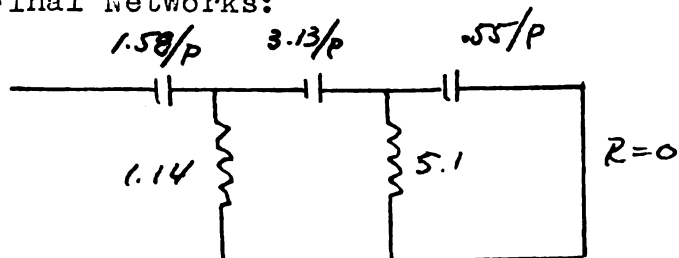
$$2.8p^2+p^3 \overline{) \begin{array}{r} 3.13/p \\ 8.76p + 3.68p^2 \\ \underline{8.76p + 3.13p^2} \\ .55p^2 \end{array}} \quad \begin{array}{l} 3.13/p \\ C = .32 \end{array}$$

$$.55p^2 \overline{) \begin{array}{r} 5.1 \\ 2.8p^2 + p^3 \\ \underline{2.8p^2} \\ p^3 \end{array}} \quad \begin{array}{l} 5.1 = R \end{array}$$

$$p^3 \overline{) \begin{array}{r} .55/p \\ .55p^2 \end{array}} \quad \begin{array}{l} .55/p \\ C = 1.82 \end{array}$$

A different network configuration could be obtained by arranging the original expression in **descending** powers of p.

Final Networks:



24. Transfer Impedance Functions.

The study, up to this stage, has been confined to two terminal networks. At this point, four terminal networks will be considered.

Here is the basic synthesis problem in four terminal networks:

Given a signal generator with its internal impedance r , find a network that will transmit a required characteristic of the signal to a load called R .

This problem requires the study of transfer impedance.

The transfer impedance will be noted as Z_t

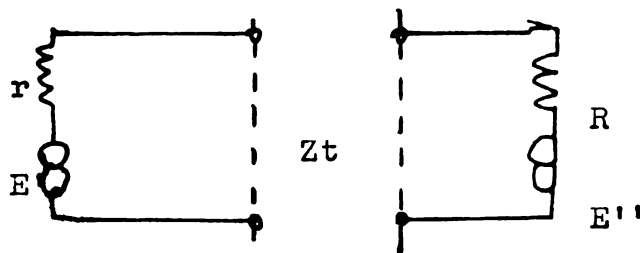


Fig. 30

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° Ibid 1 p. 226.

25. Construction of a General Transfer Impedance.

Mason¹⁶ has shown that the most efficient possible transmission between R' and R'' with passive networks occurs if the two were matched by an ideal transformer, and corresponds to

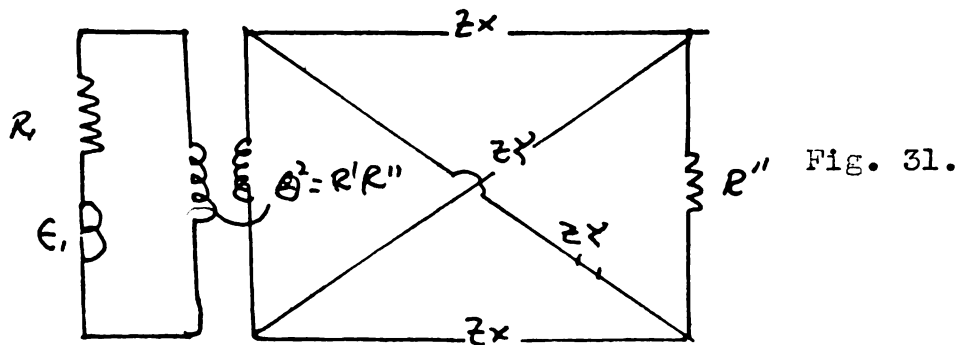
$$Z_t = 2 \sqrt{R' R''}$$

Everitt also shows this.

$$\text{Define } \theta = \lim \frac{Z_t}{2 \sqrt{R' R''}} = A + jB$$

$$Z_t = \frac{\Delta}{\Delta_{12}} = 2 \sqrt{R' R''} e^{\theta} = 2 \sqrt{R' R''} A + jB$$

The most advantageous way to represent the transfer impedance is to use the symmetrical lattice structure with an ideal transformer.



.....
¹⁶ Mason, W.P. Electromechanical Filters and Wave Transducers. 1942 New York: D. Van Nostrand. p. 24.

^o Ibid 9 p. 259.

The lattice is chosen because it has many general characteristics that are quite useful.^{17,°}

$$Z_t = \sqrt{Z_x Z_y} = \text{image impedance}$$

$$\tanh \theta/2 = \frac{\sqrt{Z_x}}{\sqrt{Z_y}}$$

Choose the image impedances to be equal to R''

by choosing $Z_x Z_y = (R'')^2$. Then the input impedance to the lattice is R'' .

$$\tanh \theta/2 = \sqrt{\frac{Z_x}{Z_y}} \quad \tanh \theta = \frac{1 - e^{-2\theta}}{1 + e^{2\theta}}$$

$$\tanh \theta/2 = \frac{1 - e^{-\theta}}{1 + e^{-\theta}} = \sqrt{\frac{Z_x}{Z_y}} = \frac{Z_x}{R}$$

$$\frac{e^{\theta} - 1}{e^{\theta} + 1} = \frac{Z_x}{R} \quad R e^{\frac{\theta}{2}} - R = Z_x e^{\frac{\theta}{2}} - Z_x$$

$$Z_x = (R) \frac{e^{\frac{\theta}{2}} - 1}{e^{\frac{\theta}{2}} + 1}$$

$$e^{\theta} = \frac{Z_x + R}{R - Z_x} = \frac{1 + Z_x/R}{1 - Z_x/R}$$

.....

¹⁷ Campbell Physical Theory of Electric Wave
Filters. B.S.T.J. Nov. 1922

[°] Ibid 3 Chapter X

26. Lattice Representations.

Theorem°:

The transfer impedance of any passive network can be represented by a symmetrical constant resistant lattice network, with resistance terminations. An elementary example follows:

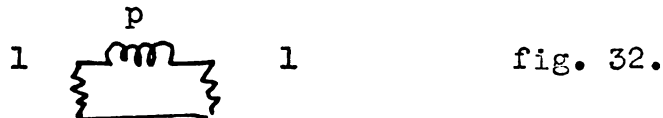


fig. 32.

$$Z_t = \frac{\Delta}{\Delta_{12}} = 2 + p \quad 2 \sqrt{R_1 R_2} = 2$$

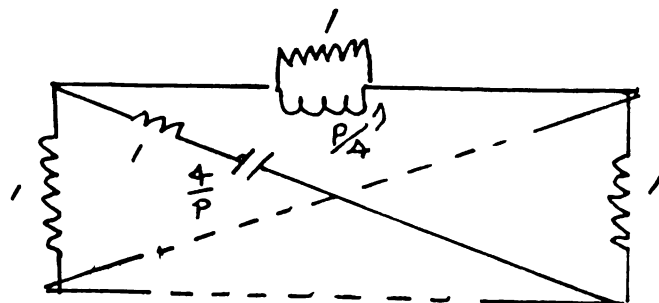
$$e^\theta = \frac{Z_t}{2 \sqrt{R_1 R_2}} = \frac{2+p}{2} = 1 + p/2$$

$$Z_x = R_1 \frac{(e^\theta - 1)}{e^\theta + 1} = \frac{1+p/2-1}{1+p/2+1} = \frac{p/4}{1+p/4}$$

$$Z_y = 1/Z_x = 4/p + 1$$

Lattice is then:

Fig. 33.



.....

° Ibid 1 p. 233.

A more complex network is presented.

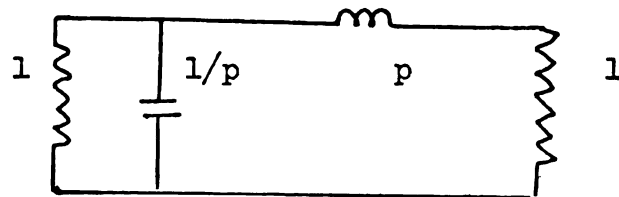


Fig. 34

$$Z_t = \frac{\begin{array}{cc} 1+1/p & -1/p \\ -1/p & p+1/p+1 \end{array}}{\begin{array}{cc} 1+1/p & 1 \\ -1/p & 0 \end{array}} = \frac{p+2/p+2}{1/p} = p^2+2p+2$$

$$e^\theta = \frac{Z_t}{2 \sqrt{R'R''}} = p^2/2+p+1$$

$$Z_x = \frac{e^\theta - 1}{e^\theta + 1} = \frac{p^2/2+p}{p^2/2+p+4} = \frac{p^2+2p}{p^2+2p+4} \quad \text{Reduce in Brune form.}$$

$$Y_x = \frac{p^2+2p+4}{p(p+2)} \quad p=0, \text{ residue} = 2 \quad \text{element } 2/p = Y$$

$$Y' = \frac{p^2+2p+4}{p(p+2)} - 2/p = \frac{p}{p+2}$$

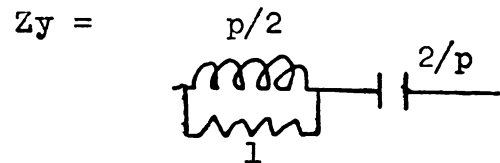
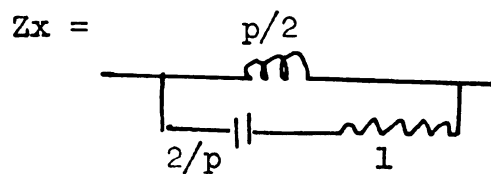
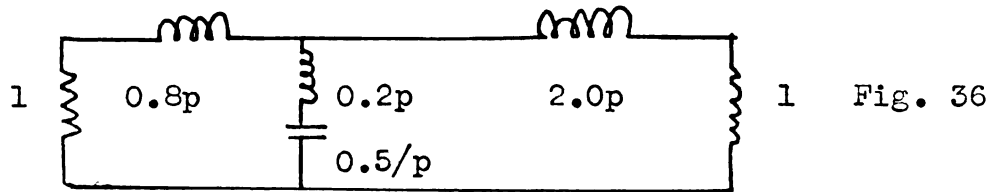


Fig. 35.

The fact that this method is laborious is not evident until a fairly complicated structure is given. This structure can be converted into a lattice, but it requires a lot of work by the present method.



$$Z_t = \frac{1 + .1p + .5/p}{-.2p - .5/p} \cdot \frac{-.2p - .5/p}{1 + 2.2p + .5/p}$$

$$Z_t = \frac{2.2p^3 + 3.2p^2 + 2.6 + 1/p + .25/p^2 - .04p^3 - .2 - .25p^3}{.2p + .5/p}$$

$$= \frac{2.16p^3 + 3.2p^2 + 2.4p + 1}{.2p^2 + .5}$$

$$e^\theta = \frac{1.08p^3 + 1.6p^2 + 1.2p + .5}{2p^2 + .5}$$

$$Z_x = \frac{e^\theta - 1}{e^\theta + 1} + \frac{1.08p^3 + 1.6p^2 + 1.2p + .5}{1.08p^3 + 1.6p^2 + 1.2p + .5} \cdot \frac{-2p^2 - .5}{.2p^2 + .5}$$

$$= \frac{1.08p^3 + 1.4p^2 + 1.2p}{1.08p^3 + 1.8p^2 + 1.2p + 1}$$

$$Y_{x+} = \frac{1.08p^3 + 1.8p^2 + 1.2p + 1}{1.08p^3 + 1.4p^2 + 1.2p} \quad \text{residue at } p=0 \quad \frac{1}{1.2}$$

This is $Y = 1/1.2p$

L of 1.2h

$$Y_X' = \frac{1.08p^3 + 1.8p^2 + 1.2p + 1}{(1.08p^2 + 1.4p + 1.2)p} - \frac{1}{1.2p}$$

$$= \frac{1.08p^3 + 1.8p^2 + 1.2p + 1 - .9p^2 - 1.165p - 1}{(1.08p^2 + 1.4p + 1.2)p}$$

$$Z_X' = \frac{1.08p^2 + 1.4p + 1.2}{(p + .791)(p + .0417)}$$

residue at $p = -.0417$

$$= \frac{.00189 - .0585 + 1.2}{.75} = \frac{1.14}{.75} = 1.52$$

Element is $\frac{1.52}{p + .0417}$

$$Z_X'' = \frac{1.08p^2 + 1.4p + 1.2}{(p + .791)(p + .0417)} - \frac{1.52}{(p + .0417)}$$

$$= \frac{1.08p^2 + 1.4p + 1.2 - 1.52p - 1.2}{(p + .0417)(p + .791)}$$

$$= \frac{1.08p^2 - .12p}{(p + .0417)(p + .791)}$$

$$Y_X''' = \frac{(p + .0417)(p + .791)}{(1.08p - 1.2)p} \quad \text{residue } p=0 \quad \frac{.033}{-.12} = -.275$$

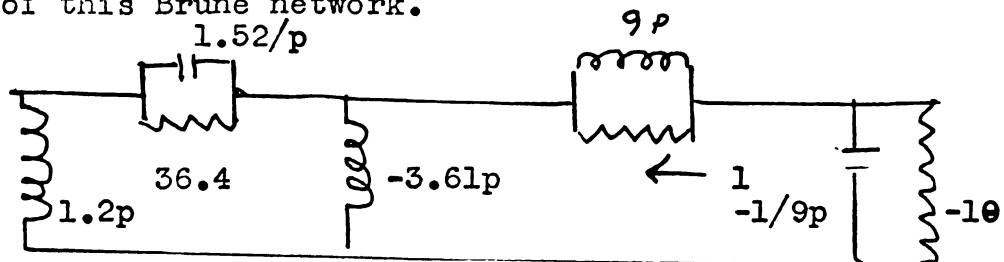
pole of $Y_X''' = -.275/p$ L of ---- -3.61

$$Y_X'''' = \frac{1.08p^2 + .9p + .035 + .3p - .035}{(1.08p - .12)} = \frac{(1.08p + 1.2)}{(1.08p - 1.2)}$$

$$Z''' = \frac{1.08p}{1.08p + 1.2} \quad \frac{-.12}{1.08p + 1.2}$$

$$R = 1, \quad L = .9 \quad R = -10, \quad C = -9$$

The x arm is the following. The y arm is the inverse of this Brune network.



27. Transfer Function Changes.

A passive transfer impedance function will continue to meet the requirements of physical realizability in a passive network if any of its real poles or any pair of its conjugate complex poles are replaced by its negative°. The change is equivalent to increasing or decreasing the transfer function by the phase shift of a corresponding all-pass section.

Given $e^\theta = -2 \frac{(p+1)}{p-1}$, write $e^\theta = \frac{-2(p+1)p+2}{p+2(p-2)}$

For the first part.

$$Z_x = \frac{\frac{-2(p+1)}{p+2} - 1}{\frac{-2(p+1)}{p+2} + 1} = \frac{-2p-2-p-2}{-2p+2+p+2} = \frac{-3p-4}{-p} = 3+4/p$$

This is a lattice

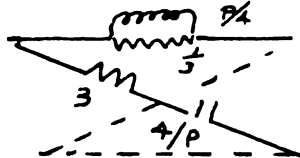


Fig. 38

in tandem with

$$\frac{\frac{p+2}{p-2} - 1}{\frac{p+2}{p-2} + 1} = \frac{p+2-p-2}{p+2+p-2} = 2/p$$

This is a lattice

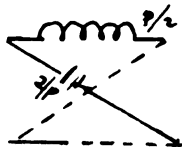


Fig. 39

Total Network.

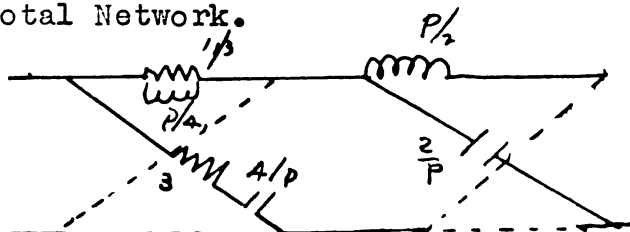


Fig. 40

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