EQUIVARIANT ALGEBRAIC COBORDISM AND DOUBLE POINT RELATIONS

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ABSTRACT

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For a reductive connected group or a finite group over a field of characteristic zero, we define an equivariant algebraic cobordism theory by a generalized version of the double point relation of Levine-Pandharipande. We prove basic properties and the well-definedness of a canonical fixed point map. We also find explicit generators of the algebraic cobordism ring of the point when the group is finite abelian.

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1. Introduction

Cobordism is a deep and well-developed theory in topology. According to Thom's definition, two dimension d smooth oriented manifolds M, N are said to be cobordant if there exists a dimension $d+1$ smooth oriented manifold with boundary $M \amalg (-N)$ (Negative sign means opposite orientation). By definition, the set of all cobordism classes, with addition given by disjoint union and multiplication given by product, is called the oriented bordism ring U_* (grading given by dimension). This ring was well-studied. For instance, Thom showed that the torsion free part can be described by $U_* \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[x_{4k} \mid k \geq 1]$. In addition, Milnor and Wall showed that all torsion has order 2. The main technique involved was the use of the Thom spectrum which we will briefly explain below.

Consider a $SO(n)$-bundle E over a manifold X. Let D be the set of all vectors (fiberwise) with length ≤ 1 and S be the set of all vectors with length 1. Then, the Thom space is defined as the quotient space D/S and denoted by $T(E)$. Now consider the classifying space $BSO(n)$ with universal $SO(n)$-bundle E_n. Denote the Thom space $T(E_n)$ by $MSO(n)$. Notice that $E_n \times \mathbb{R}^1$ becomes a $SO(n+1)$-bundle over $BSO(n)$ and hence induces the classifying map $BSO(n) \to BSO(n+1)$ and $E_n \times \mathbb{R}^1 \to E_{n+1}$. Apply the Thom space construction on both sides of the second map, we get

$$MSO(n) \wedge S^1 \cong T(E_n \times \mathbb{R}^1) \to T(E_{n+1}) = MSO(n+1).$$

That defines the Thom spectrum MSO. We can then consider the homotopy groups of MSO, namely $\pi_k(MSO) \overset{def}{=} \varinjlim_n \pi_{n+k}(MSO(n))$. The importance of the Thom spectrum comes from the isomorphism $U_k \tilde{\to} \pi_k(MO)$ which is given by the Pontrjagin-Thom construction (see [St]).

More generally, for a smooth oriented manifold X, we say two maps $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$, where Y_1, Y_2 are both dimension d smooth oriented manifolds, are cobordant if there exists a map $F : Z \to X$ such that Z is a dimension $d+1$ smooth oriented manifold with boundary and $F|_{\partial Z} = f_1 \amalg (-f_2)$ (Negative sign means opposite orientation on domain). The set of all cobordism classes with addition given by disjoint union is denoted by $U_*(X)$ (grading given by dimension of the domain of the map). That is the oriented bordism group.

Other than oriented bordism theory, there are other bordism (or cobordism) theories. For example, for a stably complex manifold X, we define

$$MU_k(X) \overset{def}{=} \varinjlim_n [S^{2n+k}, MU(n) \wedge X]$$

and

$$MU^k(X) \overset{def}{=} \varinjlim_n [S^{2n-k} \wedge X, MU(n)]$$

where $MU(n)$ is the Thom space of the universal $U(n)$-bundle over the classifying space $BU(n)$. This way, one defines the complex bordism theory (given by $MU_*(X)$) and the complex cobordism theory (given by $MU^*(X)$). Milnor showed [Mil] that the complex bordism ring MU_* is just a polynomial ring $\mathbb{Z}[x_{2k} \mid k \geq 1]$ and $MU^* \cong MU_{-*}$.

Moreover, this complex cobordism theory is equipped with Chern classes and it leads to what is called the formal group law. More precisely, for each complex vector bundle E over X of rank r, there are Chern classes $c_i(E) \in MU^{2i}(X)$ for $1 \leq i \leq r$ associated to it (see [CoF]). It turns out the complex cobordism group $MU^*(\mathbb{CP}^\infty)$ is given by the power series ring $MU^*[[s]]$ and the tensor product map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ will define a Hopf-algebra structure on $MU^*(\mathbb{CP}^\infty)$. Thus, we obtain a map

$$MU^*[[s]] \cong MU^*(\mathbb{CP}^\infty) \to MU^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong MU^*(\mathbb{CP}^\infty) \hat{\otimes} MU^*(\mathbb{CP}^\infty) \cong MU^*[[u, v]].$$

Denote the image of s by $F \in MU^*[[u, v]]$. Since $\mathbb{CP}^\infty$ is the classifying space for $U(1)$ and the elements s, u and v correspond to $c_1(\mathcal{O}(1))$, $c_1(\mathcal{O}(1, 0))$ and $c_1(\mathcal{O}(0, 1))$ respectively, we obtain the following relation for any pairs of complex line bundles L_1, L_2 over X :

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

as elements inside $MU^*(X)$. This power series F is called a formal group law over MU^* (see [Q]).

Unfortunately, because of the lack of the notion of boundary in the category of algebraic varieties, an algebraic version of cobordism theory can not be defined in a similar manner. There is a naive approach which turns out to be unsuccessful. We may define two dimension d smooth projective varieties X, X' to be cobordant if there exists a morphism $Y \to \mathbb{P}^1$

where Y is a dimension $d+1$ smooth projective variety such that X, X' are the fibers over 0, 1 respectively. This approach was also addressed by M. Levine and F. Morel (see remark 1.2.9 in [LeMo] for more detail). Consider the case when $d = 1$. Since the genus and the number of connected components are invariant under this concept of cobordism, we can not decompose a smooth genus g curve. Hence, the cobordism group of curves will be much bigger than $\mathbb{Z}$, which is what we expect from the theory of complex cobordism.

Nevertheless, in [LeMo], Levine and Morel managed to define an algebraic cobordism theory Ω, which is an analog of the complex cobordism theory, in spite of the absence of notion of boundary. However, the definition is relatively complicated. Roughly speaking, if X is a separated scheme of finite type over the ground field k, then we consider elements of the form $(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r)$ where f is projective, Y is an irreducible smooth variety over k and the sheaves $\mathcal{L}_i$ are line bundles over Y (the order of $\mathcal{L}_i$ does not matter and the number r of line bundles can be zero). The dimension of $(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r)$ is defined to be $\dim Y - r$. There is a natural notion of isomorphism on elements of this form. Denote the free abelian group generated by isomorphism classes of elements of this form by $\mathcal{Z}(X)$. Let $\underline{\Omega}(X)$ be the quotient of $\mathcal{Z}(X)$ by the subgroup corresponding to imposing the axioms **(Dim)** and **(Sect)** (following the notations in [LeMo]). The algebraic cobordism group $\Omega(X)$ is defined to be the quotient of $\underline{\Omega}(X) \otimes_{\mathbb{Z}} \mathbb{L}$, where $\mathbb{L}$ is the Lazard ring, by the $\mathbb{L}$-submodule corresponding to imposing the formal group law **(FGL)**.

This cobordism theory satisfies a number of basic properties, **(D1)-(D4)**, **(A1)-(A8)**, **(Dim), (Sect)** and **(FGL)** (following the notation in [LeMo]). It also satisfies some more advanced properties, for example, the localization property and the homotopy invariance property. Moreover, the cobordism ring $\Omega(\operatorname{Spec} k)$ will be isomorphic to the Lazard ring $\mathbb{L}$ when the characteristic of k is 0, which is what we expect from the complex cobordism theory (see Corollary 1.2.11 and Theorem 4.3.7 in [LeMo]).

One may wonder if it is possible to construct an algebraic cobordism theory via a more geometric approach. Suppose X is a smooth variety over k. We may consider the abelian group $M(X)^+$ generated by isomorphism classes over X of projective morphisms $f : Y \to X$ where Y is a smooth variety over k. The hope is that by imposing some reasonable relations, we will obtain an algebraic cobordism theory that also satisfies some previously mentioned properties. Such a construction was introduced by M. Levine and R. Pandharipande in [LeP]. A relation called "double point relation" was introduced and it was shown that the theory ω obtained by imposing this relation is canonically isomorphic to the theory Ω under the assumption that the characteristic of k is 0 (see Theorem 1 of [LeP]).

More precisely, let $\phi : Y \to X \times \mathbb{P}^1$ be a projective morphism where Y is an equidimensional smooth variety over k. Consider the fibers for the composition $Y \to X \times \mathbb{P}^1 \to \mathbb{P}^1$. Suppose the fiber Y_ξ is a smooth divisor on Y and the fiber Y_0 can be expressed as the union of two smooth divisors A, B such that A intersects B transversely. Then, the double point relation is

$$\begin{aligned}[Y_\xi \hookrightarrow Y \to X] \;&=\; [A \hookrightarrow Y \to X] + [B \hookrightarrow Y \to X] \\ &\quad - [\mathbb{P}(\mathcal{O} \oplus \mathcal{N}_{A\cap B \hookrightarrow A}) \to A \cap B \hookrightarrow Y \to X].\end{aligned}$$

The objective of the current paper is to develop an algebraic cobordism theory of varieties with group action that assembles the theories of Levine-Morel and Levine-Pandharipande. For this, we go back to topology for inspiration. In topology, for a compact Lie group G, the concept of G-equivariant bordism was first studied by Conner and Floyd (see [Co] or [H]). In their approach, for a G-space X, we consider the set of maps $Y \to X$ where Y is a stable almost complex G-manifold. Define the notion of G-bordism similarly to form the geometric unitary bordism group of X, denoted by $U^G_*(X)$. Another approach was pursued by Tom Dieck [T]. Let $\xi^G_n \to BU(n,G)$ be the universal unitary n-dimensional G-bundle and $MU(n,G)$ be its Thom space. Then, the homotopy theoretic unitary G-bordism group of X is defined by

$$MU^G_{2k}(X) \overset{def}{=} \varinjlim_{V} [\, S^V,\, MU(\dim_{\mathbb{C}} V - k, G) \wedge X\,]^G$$

and

$$MU^G_{2k+1}(X) \overset{def}{=} \varinjlim_{V} [\, S^V \wedge S^1,\, MU(\dim_{\mathbb{C}} V - k, G) \wedge X\,]^G$$

where V runs through all unitary G-representations (see [T]). Inspired by the isomorphism between the principal G-bordism group over a point and the oriented bordism group $MSO(BG)$ (where $EG \to BG$ is the universal G-bundle) when G is finite (see [Co]), there is also a third G-equivariant bordism theory defined by the following equation :

$$MU_*^{G,h}(X) \stackrel{def}{=} MU_*((X \times EG)/G).$$

In the case when X is a point, there are some maps relating the three theories.

$$U_*^G \xrightarrow{a} MU_*^G \xrightarrow{b} MU_*^{G,h}$$

The map a is given by the same Pontrjagin-Thom construction, but an inverse can not be constructed in the same manner due to the lack of transversality when there is group action. Indeed, the map a is never surjective (unless the group G is trivial) because there are non-trivial elements in the negative degrees of MU_*^G. However, the injectivity of the map a was shown by Loffler and Comezana when G is abelian (see [Lo] and [Ma]). On the other hand, when G is abelian, the map b identifies the I-adic completion of MU_*^G, where I is the augmentation ideal, to $MU_*^{G,h}$ (see [GrMa]).

There are some computational results on different versions of equivariant bordism ring. In [Ko], Kosniowski gave a list of G-spaces which multiplicatively generate the geometric unitary bordism ring U_*^G over MU_* when G is a cyclic group of prime order. When G is an abelian compact Lie group, Sinha gave a list of elements and relations that describe the structure of the homotopy theoretic unitary bordism ring MU_*^G as a MU_*-algebra (see [Si]). Since $MU_*^{G,h}$ can be identified with the I-adic completion of MU_*^G when G is abelian, we also obtain the structure of $MU_*^{G,h}$.

Following this pattern, we can expect to also have several different approaches to equivariant algebraic cobordism theory. In order to define an analog of the homotopy theoretic bordism theory MU^G in the algebraic geometry setup, one possible way is through Voevodsky's machinery of $\mathbb{A}^1$-homotopy theory (see [MoV]). A (non-equivariant) algebraic cobordism theory defined this way is discussed in [V], but, to our knowledge, an equivariant version of this has not yet been considered.

To define an analog of the theory $MU^{G,h}$, one can employ Totaro's approximation of EG. In [EG], Edidin and Graham successfully defined an equivariant Chow ring following this line of thought. For a given dimension n algebraic space X with G-action and for a fixed integer

i, pick a G-representation V and an invariant open set U inside such that G acts freely on U and the codimension of $V - U$ is larger than $n - i$. Then, $X \times U \to (X \times U)/G$ will be a principle G-bundle. Moreover, the Chow group $A_{i+\dim V - \dim G}((X \times U)/G)$ is indeed independent of the choice of the pair (V, U). Hence, the equivariant Chow group $A_i^G(X)$ is defined to be $A_{i+\dim V - \dim G}((X \times U)/G)$.

Unfortunately, since the independence of choice relies on the fact that the negative (cohomological) degrees of Chow groups are always zero, i.e. $A^i = 0$ whenever $i < 0$, equivariant algebraic cobordism theory can not be defined in the exact same manner. One approach is by considering a whole system of good pairs $\{(V, U)\}$ and define the equivariant algebraic cobordism group $\Omega_G^i(X)$ to be the inverse limit of $\Omega^i((X \times U)/G)$ (see [HeLop] for more details). Another, possibly equivalent, approach was pursued by Krishna [Kr].

Aside from these two homotopical approaches, one can also define an equivariant algebraic cobordism theory analogously to the geometric bordism theory U^G, namely by considering varieties with G-action and imposing the G-action also on the double point relation. Suppose G is an algebraic group over k and X is a smooth G-variety over k. This is what we do in this paper. We can consider the abelian group $M_G(X)^+$ generated by isomorphism classes of G-equivariant projective morphism $f : Y \to X$ where Y is also a smooth G-variety. For a morphism $\phi : Y \to X \times \mathbb{P}^1$ where Y is a smooth G-variety, $\mathbb{P}^1$ is equipped with the trivial action and ϕ is projective and G-equivariant satisfying the same conditions on the fibers Y_ξ and Y_0 as before, we impose the exact same equation with all objects involved equipped with their naturally inherited G-actions. Then, all morphisms involved will also be naturally equivariant.

For technical reasons, we focus on the case when the characteristic of k is zero and G is either a finite group or a connected reductive group. Observe that if there is a projective, G-equivariant morphism $Y \to X$ and smooth G-invariant divisors Y_ξ, A, B on Y satisfying the conditions in the double point relation, then Y_ξ is equivariantly linearly equivalent to $A + B$ and $Y_\xi + A + B$ is a reduced strict normal crossing divisor. Suppose we are given a smooth, G-invariant, very ample divisor C on Y. Due to the lack of transversality in the equivariant setting, the choice of the pairs of smooth G-invariant divisors A, B such that $C \sim A + B$ and $A + B + C$ is a reduced strict normal crossing divisor may become seriously limited, if not impossible. To remedy this, it is preferable to impose a more general relation which we call generalized double point relation.

More precisely, suppose X, Y are both smooth varieties with G-action and $\phi : Y \to X$ is an equivariant projective morphism. Assume there are smooth invariant divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on Y such that $A_1 + \cdots + A_n$ is equivariantly linearly equivalent to $B_1 + \cdots + B_m$ and $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor. Then, the generalized double point relation $GDPR(n, m)$ we will impose is of the form

$$\begin{aligned} & [A_1 \hookrightarrow Y \to X] + [A_2 \hookrightarrow Y \to X] + \cdots + [A_n \hookrightarrow Y \to X] + \text{extra terms} \\ = \; & [B_1 \hookrightarrow Y \to X] + [B_2 \hookrightarrow Y \to X] + \cdots + [B_m \hookrightarrow Y \to X] + \text{extra terms} \end{aligned}$$

where the extra terms are of the form $[\mathbb{P} \to C \hookrightarrow Y \to X]$ such that C is the intersection of some of the divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ and $\mathbb{P} \to C$ is an admissible tower (see subsection 6.3 for the definition). Denote the left hand side of the above equation by $L(\phi, A_1, \ldots, A_n, B_1, \ldots, B_m)$ and the right hand side by $R(\phi, A_1, \ldots, A_n, B_1, \ldots, B_m)$. Hence, we define the (geometric) equivariant algebraic cobordism group, denoted by $\mathcal{U}_G(X)$, to be the quotient of $M_G(X)^+$ by the abelian subgroup generated by

$$L(\phi, A_1, \ldots, A_n, B_1, \ldots, B_m) - R(\phi, A_1, \ldots, A_n, B_1, \ldots, B_m)$$

for all equivariant projective morphisms $\phi : Y \to X$ and all possible set of invariant divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ satisfying the conditions described above. We conjecture that the generalized double point relation is indeed stronger than the double point relation (See Remark 6.23 in the text).

An important observation is that the generalized double point relation actually holds in the non-equivariant theory ω. In other words, our equivariant algebraic cobordism theory in the case when G is trivial coincides with the non-equivariant algebraic cobordism theory, i.e . $\mathcal{U}_{\{1\}}(X) \cong \omega(X)$ for all smooth varieties X. That means this theory $\mathcal{U}_G$ can be thought as a generalization of ω. In addition, although the generalized double point relation may look tedious, it is actually easier to use because of the freedom of the number of divisors involved.

Using this theory, we are able to define a "fixed-point map" which is similar to a well-known construction in topology. Recall the definition of the fixed point map in topology (see [T]). For simplicity, suppose G is a finite group of prime order p. Then, there are exactly p non-isomorphic irreducible complex G-representations. Denote them by $V_1, \ldots, V_p$. For a unitary G-manifold M, let F be a component of the fixed point set M^G. The normal bundle of F inside M can be written as $\oplus_{i=1}^p V_i \otimes N_i$ for some complex vector bundles N_i over F with no G-action. Compose the classifying map of N_i with the natural map $BU(\text{rank of } N_i) \to BU$. We get a map $F \to BU$ which we will denote by f_i. Thus, the fixed point map

$$\phi : U_*^G \to \bigoplus_{i=1}^p MU_*(BU)$$

is given by sending $[M]$ to the sum of

$$([f_1 : F \to BU], \ldots, [f_p : F \to BU])$$

over all components F. If we add up the elements $[f_i]$ and push them down to the bordism ring, we obtain a map

$$U_*^G \to \bigoplus_{i=1}^p MU_*(BU) \to MU_*(BU) \to MU_*$$

given by

$$[M] \mapsto \sum_F ([f_1], \ldots, [f_p]) \mapsto \sum_{F,i} [f_i] \mapsto \sum_{F,i} [F] = p\,[M^G].$$

Assume the ground field k has characteristic 0 as before. If the group G is finite, then the fixed point locus of any smooth variety over k is again smooth (Proposition 3.4 in [Ed]). The same statement also holds when G is reductive (by Proposition 7.1). So, for a smooth variety X, we have an abelian group homomorphism from $M_G(X)^+$ to $M(X^G)^+$ defined by sending $[Y \to X]$ to $[Y^G \to X^G]$, which we will also call fixed point map. One of our main results is the following Theorem (Corollary 7.3 in the text) which can be considered as an analog of the topological fixed point map.

Theorem 1. *For any smooth G-variety X, sending $[Y \to X]$ to $[Y^G \to X^G]$ defines an abelian group homomorphism*

$$\mathcal{F} : \mathcal{U}_G(X) \to \omega(X^G).$$

We also managed to find a set of generators for the equivariant algebraic cobordism ring of the point $\operatorname{Spec} k$ when G is a finite abelian group with exponent e and k contains a primitive e-th root of unity. We can naturally embed the non-equivariant algebraic cobordism ring

$$\mathbb{L} \cong \omega(\operatorname{Spec} k) \cong \mathcal{U}_{\{1\}}(\operatorname{Spec} k)$$

inside the equivariant algebraic cobordism ring $\mathcal{U}_G(\operatorname{Spec} k)$ (by assigning trivial G-action) (see Corollary 7.4). This construction provides $\mathcal{U}_G(\operatorname{Spec} k)$ with a $\mathbb{L}$-algebra structure. Then, the following Theorem describes a set of generators of $\mathcal{U}_G(\operatorname{Spec} k)$ (see Theorem 6.22 for more detail).

Theorem 2. *Suppose G is a finite abelian group with exponent e and k contains a primitive e-th root of unity. Then, the equivariant algebraic cobordims ring $\mathcal{U}_G(\operatorname{Spec} k)$ is generated by the set of exceptional objects*

$$\{E_{n,H,H'} \mid n \geq 0 \text{ and } G \supseteq H \supseteq H'\}$$

and the set of admissible towers over $\operatorname{Spec} k$ *as a $\mathbb{L}$-algebra.*

Here is the definition of the exceptional objects. For an integer $n \geq 0$ and a pair of subgroups $G \supseteq H \supseteq H'$, since G is abelian, we can write

$$H/H' \cong H_1 \times \cdots \times H_a$$

where H_i is a cyclic group of order $p_i^{m_i}$ for a prime p_i. Define an (H/H')-action on

$$\text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a]$$

by assigning H_i to act faithfully on $k-\text{span}\{v_i\}$ and trivially on other generators, for all $1 \leq i \leq a$. Then, the exceptional object is defined as, with the natural G-action,

$$E_{n,H,H'} \overset{def}{=} G/H \times \text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \,/\, (v_1^{p_1^{m_1}} - g_1, \ldots, v_a^{p_a^{m_a}} - g_a)$$

where $g_i \in k[x_0, \ldots, x_n]$ is homogeneous with degree $p_i^{m_i}$ such that $E_{n,H,H'}$ is smooth with dimension n ($[E_{n,H,H'}] \in \mathcal{U}_G(\operatorname{Spec} k)$ is independent of the choice of $\{g_i\}$).

Let us now give a brief outline of this paper. In section 2, we state some basic notions and assumptions we will be using throughout the paper. In section 3, we give the precise definition of generalized double point relation and also the definition of our equivariant algebraic cobordism theory $\mathcal{U}_G$. We also show that the generalized double point relation holds in the non-equivariant theory ω. Then, a number of basic properties, namely **(D1)-(D4)** and **(A1)-(A8)** that does not involve the first Chern class operator, will be stated and proved. The last subsection will be devoted to the investigation of the case when the action is free. In this case, we show an isomorphism $\omega(X/G) \cong \mathcal{U}_G(X)$.

In section 4, we handle the (first) Chern class operator. We first define the notion of "nice" G-linearized invertible sheaves. Then, we define the Chern class operator $c(\mathcal{L})$ for all such sheaves and prove the most important property of this operator : formal group law **(FGL)**. Next, we extend the definition to arbitrary G-linearized invertible sheaves with stronger assumptions on G and k (in particular, G is a finite abelian group).

In section 5, we will first prove the rest of the list of basic properties, i.e. **(D1)-(D4)** and **(A1)-(A8)** that involve the Chern class operator. Then, we will show the properties **(Dim)** and **(FGL)**.

The whole section 6 will be devoted to proving the Theorem about the set of generators of the equivariant cobordism ring $\mathcal{U}_G(\operatorname{Spec} k)$ as a $\mathbb{L}$-algebra. The first subsection in section 6 will be dedicated to an interesting general technique which we will call splitting principle by blowing up along invariant smooth centers. Finally, in the last section, we will prove the well-definedness of the fixed point map, i.e. Theorem 7.2.

2. Notations and assumptions

Throughout this paper, we work over a field k with characteristic 0. We will denote by Sm the category of smooth quasi-projective schemes over k. We will often refer to this as varieties even though they do not have to be irreducible. The identity morphism will be denoted by $\mathbb{I}_X : X \to X$. The groups which act on varieties are either reductive connected groups or finite groups over k. So, they are always affine over $\operatorname{Spec} k$. We will often use the symbol π_k to denote the structure morphism $X \to \operatorname{Spec} k$ and the symbol π_i to denote the projection of $X_1 \times \cdots \times X_n$ onto its i-th component X_i.

As in [MuFoKi], an action of a group scheme G on a variety X is by definition a morphism $\sigma : G \times X \to X$ such that

1. The two morphisms $\sigma \circ (\mathbb{I}_G \times \sigma)$ and $\sigma \circ (\mu \times \mathbb{I}_X)$ from $G \times G \times X$ to X agree, where $\mu : G \times G \to G$ is the group law of G.
2. The composition
$$X \tilde{\to} \operatorname{Spec} k \times X \xrightarrow{e \times \mathbb{I}_X} G \times X \xrightarrow{\sigma} X$$
is equal to $\mathbb{I}_X$, where e is the identity morphism.

For any $\alpha \in G$ and $x \in X$, we will denote $\sigma(\alpha, x)$ by $\alpha \cdot x$, or simply αx if there is no confusion. We will say that the action is proper if the morphism $G \times X \to X \times X$ given by $(\alpha, x) \mapsto (\alpha \cdot x, x)$ is proper. Similarly, we will say the action is free if the above map is a closed immersion. This notion is stronger than the notion "set-theoretically free". According to Lemma 8 of [EG], set-theoretically free and proper implies free. In the case when G is a finite group scheme, the two morphisms $\sigma, \pi_2 : G \times X \to X$ are both proper. That means the morphism $G \times X \to X \times X$ above is proper. Hence, in this case, "set-theoretically free" is equivalent to free. Morphisms between G-varieties are always assumed to be G-equivariant unless specified otherwise. We will denote by G-Sm the category with objects in Sm with G action and

$$\operatorname{Mor}_{G\text{-}Sm}(X, Y) = \{f : X \to Y \mid f \text{ is } G\text{-equivariant}\}.$$

If X is in G-Sm and $\mathcal{E}$ is a locally free coherent sheaf on X with rank r, then a G-linearization of $\mathcal{E}$ is a collection of isomorphisms $\{\phi_\alpha : \alpha^* \mathcal{E} \tilde{\to} \mathcal{E} \mid \alpha \in G\}$ that satisfies the cocycle condition :

$$\phi_{\alpha\beta} = \phi_\beta \circ (\beta^* \phi_\alpha),$$

as isomorphisms from $(\alpha\beta)^*\mathcal{E}$ to $\mathcal{E}$, for all $\alpha, \beta \in G$. There is a natural definition of isomorphism associated to it. The set of isomorphism classes of invertible sheaves on X with a G-linearization will be denoted by $\mathrm{Pic}^G(X)$.

If X, Y are two objects in G-Sm, then $X \times Y$ is considered to be in G-Sm with G acting diagonally. An object $Y \in G$-Sm is called G-irreducible if there exists an irreducible component Y' of Y such that $G \cdot Y' = Y$. The set of isomorphism classes of invertible sheaves on X with a G-linearization will be denoted by $\mathrm{Pic}^G(X)$. For a locally free sheaf $\mathcal{E}$ of rank r over a k-scheme X, the corresponding vector bundle E over X will be given by

$$E \stackrel{def}{=} \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{E}^\vee.$$

The same applies to the case that X is a G-scheme over k and $\mathcal{E}$ is G-linearized.

Recall the definition of transversality from [LeP]. For objects $A, B, C \in Sm$ and morphisms $f : A \to C$ and $g : B \to C$, we say f, g are transverse if $A \times_C B$ is smooth and for all irreducible components $A' \subseteq A$ and $B' \subseteq B$ such that $f(A'), g(B')$ are both contained in the same irreducible component $C' \subseteq C$, we have either

$$\dim A' \times_{C'} B' = \dim A' + \dim B' - \dim C'$$

or $A' \times_{C'} B' = \emptyset$. If A, B are both subschemes of C, we say that A, B are transverse if the inclusion morphisms are transverse. If $f : A \to C$ and x is point in C, we say that x is a regular value of f if the inclusion morphism $x \hookrightarrow C$ and f are transverse.

Also recall the definition of principal G-bundle from [EG]. A morphism $f : X \to Y$ is called a principal G-bundle if G acts on X, the morphism f is flat, surjective, G-equivariant for the trivial G-action on Y and the morphism

$$G \times X \to X \times_Y X,$$

defined by $(\alpha, x) \mapsto (\alpha \cdot x, x)$, is an isomorphism.

For a morphism $f : X \to Y$ and a point $y \in Y$, we denote the fiber product $\operatorname{Spec} k(y) \times_Y X$ by $f^{-1}(y)$ where $k(y)$ is the residue field of y and $\operatorname{Spec} k(y) \to Y$ is the morphism corresponding to y. Similarly, if Z is a subscheme of Y, then we denote $Z \times_Y X$ by $f^{-1}(Z)$. If A, B are both subschemes of X, then we denote $A \times_X B$ by $A \cap B$.

In this paper, for a G-irreducible object $X \in G\text{-}Sm$, a G-invariant divisor D on X is a Weil divisor of the form $\sum_i m_i D_i$ where D_i are distinct, G-invariant, G-irreducible, reduced, codimension 1, closed subscheme of X. We call such a divisor smooth if all the multiplicities m_i are 1 and D_i are smooth and disjoint. We call a G-invariant divisor $A_1 + \cdots + A_n$ reduced strict normal crossing divisor if each A_i is a smooth G-invariant divisor and, for each $I \subseteq \{1, \ldots, n\}$, the closed subscheme $\cap_{i \in I} A_i$ is smooth with codimension $|I|$ in X.

3. Geometric equivariant algebraic cobordism $\mathcal{U}_G$

3.1. **Preliminaries.** Before digging into the equivariant algebraic cobordism theory, we need to understand more about G-invariant divisors and G-linearized invertible sheaves.

Weil Divisors :

Let X be a G-irreducible object in G-Sm. A G-invariant, G-irreducible reduced closed subscheme $D \subseteq X$ with codimension 1 will be called a prime G-invariant Weil divisor. A G-invariant Weil divisor is a finite $\mathbb{Z}$-linear combination of prime divisors, i.e. $D = \sum n_i D_i$. A G-invariant Weil divisor D is called effective if n_i are all non-negative. Let $\mathcal{K}$ be the sheaf of total quotient rings of $\mathcal{O}_X$, which has its natural G-action. We say that two G-invariant Weil divisors D, D' are G-equivariantly linearly equivalent, denoted by $D \sim D'$, if there is an element $f \in \mathrm{H}^0(X, \mathcal{K}^*)^G$ such that $D - D' = \operatorname{div} f$ where $\operatorname{div} f$ is defined in the usual way.

Cartier Divisors :

Similar to the definition of Cartier divisors in Ch II, section 6 in [Ha], a G-invariant Cartier divisor is an element in $\mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*)^G$. We say two G-invariant Cartier divisors D, D' are G-equivariantly linearly equivalent if $D - D'$ is in the image of

$$\mathrm{H}^0(X, \mathcal{K}^*)^G \to \mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*)^G.$$

As usual, we will represent a G-invariant Cartier divisor by $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open cover of X and $f_i \in \mathrm{H}^0(U_i, \mathcal{K}^*)$. The (left) G-action on the sheaf $\mathcal{K}$ (or the sheaf $\mathcal{O}$) is given explicitly by

$$(\alpha \cdot f)(x) = f(\alpha^{-1} \cdot x)$$

for any $f \in \mathcal{K}$ (or in $\mathcal{O}$) and $\alpha \in G$. Then, the Cartier divisor D being G-invariant implies

$$\{(U_i, f_i)\} = \{(\alpha \cdot U_i, \alpha \cdot f_i)\}$$

as elements in $\mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*)$ for all $\alpha \in G$. In other words, $(\alpha \cdot f_i)/f_j$ is a unit in $\mathcal{O}_{(\alpha \cdot U_i) \cap U_j}$ for all i, j. Since X is smooth, we have a one-to-one correspondence between the set of G-invariant Weil divisors and the set of G-invariant Cartier divisors by the same construction as in [Ha]. Moreover, the notion of G-equivariantly linearly equivalent is also preserved.

Hence, from now on, we will use the two notions interchangeably. Furthermore, divisors are always assumed to be G-invariant unless specified otherwise and linear equivalence means G-equivariant linear equivalence.

G-linearized invertible sheaves :

For a given G-invariant divisor D on a smooth G-variety X, we can construct a G-linearized invertible sheaf naturally. We will denote it by $\mathcal{O}_X(D)$. Here is the construction.

The underlying invertible sheaf structure is given by the natural definition as in Ch II, section 6 in [Ha] : if D is represented by $\{(U_i, f_i)\}$, then we define $\mathcal{O}_X(D)$ by the following equation :

$$\mathcal{O}_X(D)|_{U_i} \overset{def}{=} \mathcal{O}_{U_i} f_i^{-1}$$

for all i.

The G-linearization of $\mathcal{O}_X(D)$ can be defined as the following. Consider the case when D is a prime G-invariant divisor. Then, it defines an ideal sheaf $\mathcal{I}$ which is naturally G-linearized. Then, the natural isomorphism $\mathcal{O}_X(-D) \cong \mathcal{I}$ induces a G-linearization on $\mathcal{O}_X(-D)$. Hence, we can define the G-linearization by taking the dual, namely, $\mathcal{O}_X(D) \overset{def}{=} \mathcal{O}_X(-D)^\vee$. In general, if $D = \sum n_i D_i$ for some prime G-invariant divisors D_i, then we define $\mathcal{O}_X(D) \overset{def}{=} \otimes O_X(D_i)^{\otimes n_i}$.

The G-linearization structure on $\mathcal{O}_X(D)$ can be explicitly given as the following. For a given $\alpha \in G$, we will define an isomorphism $\phi_\alpha : \alpha^*\mathcal{O}(D) \to \mathcal{O}(D)$. Let us consider the restriction on U_i, the domain becomes

$$\begin{aligned}
(\alpha^*\mathcal{O}(D))|_{U_i} &= \alpha^*(\mathcal{O}(D)|_{\alpha U_i}) = \alpha^*(\mathcal{O}_{U_j \cap \alpha U_i} f_j^{-1}) \\
&\quad \text{(further restricted on } U_j \cap \alpha U_i) \\
&= \mathcal{O}_{U_i \cap \alpha^{-1} U_j}\, \alpha^{-1} \cdot f_j^{-1}.
\end{aligned}$$

On the other hand, the codomain becomes $\mathcal{O}_{U_i \cap \alpha^{-1} U_j}\, f_i^{-1}$ when restricted on $U_i \cap \alpha^{-1} U_j$. Then, we define

$$\phi_\alpha|_{U_i \cap \alpha^{-1} U_j} : \mathcal{O}\, \alpha^{-1} \cdot f_j^{-1} \to \mathcal{O} f_i^{-1}$$

by sending $\alpha^{-1} \cdot f_j^{-1}$ to $(f_i \,/\, (\alpha^{-1} \cdot f_j))\, f_i^{-1}$. Since $\phi_\alpha|_{U_i \cap \alpha^{-1} U_j}$ is an identity map, ϕ_α is well-defined and is an isomorphism.

We need to check the cocycle condition

$$\phi_{\alpha\beta} = \phi_\beta \circ (\beta^* \phi_\alpha) : (\alpha\beta)^* \mathcal{O}(D) \to \mathcal{O}(D).$$

For simplicity, we will denote $\mathcal{O}_X$ (or other base) by simply $\mathcal{O}$. Notice that, by the above definition, $\phi_\beta : \beta^*\mathcal{O}(D) \to \mathcal{O}(D)$ corresponds to $\mathcal{O}\, \beta^{-1} \cdot f_j^{-1} \to \mathcal{O} f_i^{-1}$ and $\phi_\alpha : \alpha^*\mathcal{O}(D) \to \mathcal{O}(D)$ corresponds to $\mathcal{O}\, \alpha^{-1} \cdot f_k^{-1} \to \mathcal{O} f_j^{-1}$. So, the morphism $\beta^*\phi_\alpha : \beta^*\alpha^*\mathcal{O}(D) \to \beta^*\mathcal{O}(D)$ corresponds to $\mathcal{O}\, \beta^{-1}\alpha^{-1} \cdot f_k^{-1} \to \mathcal{O}\, \beta^{-1} \cdot f_j^{-1}$. On the other hand, $\phi_{\alpha\beta} : (\alpha\beta)^*\mathcal{O}(D) \to \mathcal{O}(D)$ corresponds to $\mathcal{O}\, \beta^{-1}\alpha^{-1} \cdot f_k^{-1} \to \mathcal{O} f_i^{-1}$. Thus, the domains and codomains of $\phi_{\alpha\beta}$ and $\phi_\beta \circ (\beta^*\phi_\alpha)$ are represented by the same generators and all the morphisms are identities. Hence, they commute.

It remains to check its independence of the choice of representations $\{(U_i, f_i)\}$ of the Cartier divisor. In other words, if D is represented by $\{(U_i, f_i)\}$ where $f_i \in \mathrm{H}^0(U_i, \mathcal{O}^*)$, then the G-linearized invertible sheaf it defined will be G-equivariantly isomorphic to the structure sheaf. To see this, we define a morphism from $\mathcal{O}(D)$ to $\mathcal{O}$ by patching the morphisms $\mathcal{O} f_i^{-1} \to \mathcal{O}$ in which we send f_i^{-1} to f_i^{-1}. Then, it is a well-defined isomorphism. The commutativity of the following diagram implies that this morphism is G-equivariant.

$$\begin{array}{ccc} \mathcal{O}\, \alpha^{-1} \cdot f_j^{-1} & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow \\ \mathcal{O} f_i^{-1} & \longrightarrow & \mathcal{O} \end{array}$$

This natural construction also takes G-equivariantly linearly equivalent divisors to isomorphic G-linearized invertible sheaves, i.e. if f is in $\mathrm{H}^0(X, \mathcal{K}^*)^G$, then $\mathcal{O} f^{-1} \tilde{\to} \mathcal{O}$ by sending f^{-1} to 1.

Unfortunately, we do not have the one-to-one correspondence between the set of G-invariant divisor classes and the set of isomorphism classes of G-linearized invertible sheaves. Here is a simple reason. If the G-action on X is trivial, then the G-action on any G-invariant divisor will be trivial too. Hence, the G-action on the line bundle corresponding to $\mathcal{O}(D)$ must be trivial. But, there are certainly G-equivariant line bundles over X with non-trivial fiberwise G-actions.

The following are some basic properties of G-invariant divisors.

Proposition 3.1. *Suppose X, Y are objects in G-Sm.*

(1) *If $f : X \to Y$ is a morphism in G-Sm and D is a G-invariant divisor on Y such that f^*D is a G-invariant divisor on X, then $f^*\mathcal{O}(D) \cong \mathcal{O}(f^*D)$.*

(2) *If D is a G-invariant divisor on X and Z is a closed subscheme of X such that $Z \cap \mathrm{Supp}D$ is empty, then $\mathcal{O}_X(D)|_Z \cong \mathcal{O}_Z$.*

(3) *If D is a G-invariant divisor on X, then $\mathcal{O}_X(D) \cong \mathcal{O}_X$ if and only if $D \sim 0$.*

Proof. (1) Suppose D is represented by $\{(U_i, g_i)\}$. Then, the G-invariant divisor f^*D can be represented by $\{(f^{-1}(U_i), f^*g_i)\}$. Thus,

$$(f^*\mathcal{O}(D))|_{f^{-1}(U_i)} = f^*(\mathcal{O}_{U_i} g_i^{-1}) = \mathcal{O}_{f^{-1}(U_i)}\, f^*g_i^{-1}.$$

On the other hand, $\mathcal{O}(f^*D)|_{f^{-1}(U_i)} = \mathcal{O}_{f^{-1}(U_i)}\, f^*g_i^{-1}$. So they are isomorphic. The compatibility of the G-action is easy to check.

(2) Suppose D is represented by $\{(U_i, g_i)\}$ and $i : Z \hookrightarrow X$ is the closed immersion. Since $Z \cap \mathrm{Supp}D = \emptyset$, by refinement, we can assume U_i either has empty intersection with Z or, otherwise, g_i is a unit in $\mathcal{O}_{U_i}$. Thus, i^*D is a G-invariant divisor on Z and can be represented by $\{(U_i \cap Z, g_i|_Z)\}$, or simply $\{(Z, 1)\}$ by the independence of representation. That means

$$\mathcal{O}_X(D)|_Z \cong \mathcal{O}_Z(i^*D) \cong \mathcal{O}_Z.$$

(3) As mentioned before, if D and D' are G-equivariantly linearly equivalent, then they define the same G-linearized invertible sheaf, i.e. $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$. So, it is enough to show if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, then $D \sim 0$. Suppose D is represented by $\{(U_i, g_i)\}$. Then, the isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ over U_i is given by sending 1 to $a_i g_i^{-1}$ for some $a_i \in \mathcal{O}_{U_i}^*$. The fact that the isomorphism is globally defined implies that $a_i(g_j/g_i) = a_j$. Thus,

$$h \overset{def}{=} \frac{a_i}{g_i} = \frac{a_j}{g_j} \in \mathrm{H}^0(X, \mathcal{K}^*).$$

Since $a_i g_i^{-1}$ corresponds to 1, the G-action on h is trivial. Hence, the two G-invariant divisors $\{(U_i, g_i)\}$ and $\{(U_i, a_i)\}$ are G-equivariantly linearly equivalent. The result then follows from the fact that $a_i \in \mathcal{O}^*_{U_i}$. □

Remark 3.2. By property (3), we can consider the set of G-invariant divisor classes on X as a natural subgroup of $\operatorname{Pic}^G(X)$.

We will also use the following fact about projective bundles from time to time.

Proposition 3.3. *For an object $X \in G$-Sm, suppose $\mathcal{L}$ is in $\operatorname{Pic}^G(X)$ and $\mathcal{E}$ is a G-linearized locally free sheaf of rank r over X. Then $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ are naturally isomorphic as G-equivariant projective bundles over X.*

Proof. First of all, we define a morphism from $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ to $\mathbb{P}(\mathcal{E})$ without considering the group action. Let $\{U_i\}$ be an open cover of X such that $\mathcal{E}|_{U_i}$ is trivial and $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} l_i$. Then, we define a morphism

$$\gamma : \mathcal{E}|_{U_i} \to \mathcal{E} \otimes \mathcal{L}|_{U_i}$$

by $e \mapsto e \otimes l_i$. This induces a morphism

$$f|_{U_i} : \mathbb{P}(\mathcal{E} \otimes \mathcal{L}|_{U_i}) = \text{Proj Sym } \mathcal{E} \otimes \mathcal{L}|_{U_i} \to \text{Proj Sym } \mathcal{E}|_{U_i} = \mathbb{P}(\mathcal{E}|_{U_i}).$$

We claim that $\{f|_{U_i}\}$ will patch together to form a morphism from $\mathbb{P}(\mathcal{E} \otimes \mathcal{L})$ to $\mathbb{P}(\mathcal{E})$ and it will be an isomorphism of projective bundles over X.

Let $\sigma_{\mathcal{E}}$, $\sigma_{\mathcal{L}}$ be the transition functions of $\mathcal{E}$, $\mathcal{L}$ respectively from U_i to U_j. Then, we have $\sigma_{\mathcal{L}}(l_i) = a\, l_j$ for some $a \in \mathcal{O}^*_{U_j}$ and the transition function for $\mathcal{E} \otimes \mathcal{L}$ will be $\sigma_{\mathcal{E}} \otimes \sigma_{\mathcal{L}}$. Then,

$$\begin{aligned} (\sigma_{\mathcal{E}} \otimes \sigma_{\mathcal{L}}) \circ \gamma(e) &= (\sigma_{\mathcal{E}} \otimes \sigma_{\mathcal{L}})(e \otimes l_i) \\ &= \sigma_{\mathcal{E}}(e) \otimes \sigma_{\mathcal{L}}(l_i) \\ &= \sigma_{\mathcal{E}}(e) \otimes a\, l_j \\ &= a\, (\sigma_{\mathcal{E}}(e) \otimes l_j). \end{aligned}$$

On the other hand,

$$\gamma \circ \sigma_{\mathcal{E}}(e) = \sigma_{\mathcal{E}}(e) \otimes l_j.$$

If we consider $\sigma_{\mathcal{E}}(e) \otimes l_j$ and $a\left(\sigma_{\mathcal{E}}(e) \otimes l_j\right)$ as elements in Sym $\mathcal{E} \otimes \mathcal{L}$, then they agree, in homogeneous coordinates. Hence, $\{f|_{U_i}\}$ patch together to form a morphism f. Moreover, it is obviously an isomorphism and a projective bundle morphism.

It remains to check if f is G-equivariant. The G-linearization on $\mathcal{L}$ is described by a set of isomorphisms $\{\phi_{\mathcal{L},\alpha} : \alpha^*\mathcal{L} \tilde{\to} \mathcal{L}\}$. When restricted on $U_i \cap \alpha^{-1}U_j$, $\phi_{\mathcal{L},\alpha}$ defines an isomorphism from $\mathcal{O}l_j$ to $\mathcal{O}l_i$. So, $\phi_{\mathcal{L},\alpha}(l_j) = b_\alpha l_i$ for some $b_\alpha \in \mathcal{O}^*_{U_i \cap \alpha^{-1}U_j}$. Similarly, if $\{\phi_{\mathcal{E},\alpha}\}$ and $\{\phi_{\mathcal{E}\otimes\mathcal{L},\alpha}\}$ defines the G-linearizations on $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}$ respectively, then

$$\gamma \circ \phi_{\mathcal{E},\alpha}(e) = \phi_{\mathcal{E},\alpha}(e) \otimes l_i.$$

On the other hand,

$$\begin{aligned} \phi_{\mathcal{E}\otimes\mathcal{L},\alpha} \circ \gamma(e) &= \phi_{\mathcal{E}\otimes\mathcal{L},\alpha}(e \otimes l_j) \\ &= \phi_{\mathcal{E},\alpha}(e) \otimes \phi_{\mathcal{L},\alpha}(l_j) \\ &= \phi_{\mathcal{E},\alpha}(e) \otimes b_\alpha l_i \\ &= b_\alpha(\phi_{\mathcal{E},\alpha}(e) \otimes l_i). \end{aligned}$$

For the same reason, they agree in homogeneous coordinates and hence, f is G-equivariant.

□

3.2. **Generalized double point relation.** In [LeP] (Definition 0.2), the graded cobordism group $\omega_*(X)$ is defined as the quotient of the free abelian group generated by symbols $[f : Y \to X]$ where Y is an object in *Sm* and f is a projective morphism, by an equivalence relation called double point relation. More precisely, suppose $Y \in$ *Sm* is equidimensional and there is a projective morphism $\phi : Y \to X \times \mathbb{P}^1$ such that a closed point $0 \neq \xi \in \mathbb{P}^1$ is a regular value of $\pi_2 \circ \phi$ (in other words, $Y_\xi \stackrel{def}{=} (\pi_2 \circ \phi)^{-1}(\xi)$ is a smooth divisor on Y), while the fiber $Y_0 = A \cup B$ for some smooth divisors A, B and $A + B$ is a reduced strict normal crossing divisor. Then, the double point relation is

$$\begin{aligned} [Y_\xi \hookrightarrow Y \to X] &= [A \hookrightarrow Y \to X] + [B \hookrightarrow Y \to X] \\ &\quad - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to A \cap B \hookrightarrow Y \to X]. \end{aligned}$$

We refer the reader to section 0 in [LeP] for more details.

In addition, in section 5.2 in [LeP], a relation called extended double point relation is also introduced. Suppose $Y \in Sm$ is equidimensional and there is a projective morphism $\phi : Y \to X$. In addition, suppose we have divisors A, B, C on Y such that $A + B + C$ is a reduced strict normal crossing divisor and $C \sim A + B$. Then, the extended double point relation is

$$
\begin{aligned}
[C \hookrightarrow Y \to X] &= [A \hookrightarrow Y \to X] + [B \hookrightarrow Y \to X] \\
&- [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to A \cap B \hookrightarrow Y \to X] \\
&+ [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to A \cap B \cap C \hookrightarrow Y \to X] \\
&- [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to A \cap B \cap C \hookrightarrow Y \to X].
\end{aligned}
$$

On one hand, if we assume C does not intersect $A \cup B$, then this is the same as the double point relation. On the other hand, Lemma 5.2 in [LeP] shows that the extended double point relation holds in the theory ω defined by the double point relation. One may then expect the existence of similar formulas when $Y_0 = A_1 \cup A_2 \cup A_3$ in the double point relation setup, or when $B \sim A_1 + A_2 + A_3$ in the extended double point relation setup. Indeed, it is possible to write a formula for arbitrary number of divisors. For induction purposes, we will consider the extended double point relation setup.

More precisely, suppose X is a separated scheme of finite type over k and $\phi : Y \to X$ is a projective morphism with $Y \in Sm$ such that Y is equidimensional. Moreover, suppose there are divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on Y such that

$$A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$$

and $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor. Then, we expect a formula of the form

$$[A_1 \to X] + \cdots + [A_n \to X] + \text{extra terms} = [B_1 \to X] + \cdots + [B_m \to X] + \text{extra terms}.$$

We will give such a formula inductively. For this purpose, we will consider the following.

Definition 3.4. Define a polynomial ring over $\mathbb{Z}$ with commuting variables :

$$\mathcal{R} \stackrel{def}{=} \mathbb{Z}[\{X_i, Y_j, U_k^p, V_l^q\}]$$

where $i, j, k, l \geq 1$ and $1 \leq p, q \leq 3$.

Then, we define some elements in $\mathcal{R}$ inductively.

Definition 3.5. Let $E_1^X,\ F_1^X \stackrel{def}{=} 0$. For $n \geq 2$, define

$$\begin{aligned} E_n^X &\stackrel{def}{=} E_{n-1}^X - (X_1 + \cdots + X_{n-1} + E_{n-1}^X) X_n U_{n-1}^1 - X_n F_{n-1}^X \\ F_n^X &\stackrel{def}{=} F_{n-1}^X + (X_1 + \cdots + X_{n-1} + E_{n-1}^X) X_n (U_n^2 - U_n^3). \end{aligned}$$

Similarly, define E_n^Y, F_n^Y by replacing X by Y and U by V in E_n^X, F_n^X respectively, namely,

$$\begin{aligned} E_1^Y,\ F_1^Y &\stackrel{def}{=} 0 \\ E_n^Y &\stackrel{def}{=} E_{n-1}^Y - (Y_1 + \cdots + Y_{n-1} + E_{n-1}^Y) Y_n V_{n-1}^1 - Y_n F_{n-1}^Y \\ F_n^Y &\stackrel{def}{=} F_{n-1}^Y + (Y_1 + \cdots + Y_{n-1} + E_{n-1}^Y) Y_n (V_n^2 - V_n^3) \end{aligned}$$

for $n \geq 2$. Also, for $n, m \geq 1$, define the elements $G_{n,m}^X$ as the following :

$$G_{n,m}^X \stackrel{def}{=} X_1 + \cdots + X_n + E_n^X + (Y_1 + \cdots + Y_m) F_n^X + E_m^Y F_n^X.$$

Finally, define $G_{n,m}^Y$ by interchanging X and Y in $G_{n,m}^X$, namely,

$$G_{n,m}^Y \stackrel{def}{=} Y_1 + \cdots + Y_n + E_n^Y + (X_1 + \cdots + X_m) F_n^Y + E_m^X F_n^Y.$$

For a projective morphism $\phi : Y \to X$, such that Y is equidimensional, and divisors $A_1, \ldots, A_n,\ B_1, \ldots, B_m$ on Y such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$ and $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor, we define an abelian group homomorphism

$$\mathcal{G} : \mathcal{R} \to \omega(X)$$

as the following.

First of all, any term with X_i such that $i > n$, or Y_j such that $j > m$, or U_k^p such that $k > n$, or V_l^q such that $l > m$ is sent to 0. Then, we send

$$\begin{aligned}
1 &\mapsto [Y \to X] \\
X_i &\mapsto [A_i \to Y \to X] \\
Y_j &\mapsto [B_j \to Y \to X] \\
U_k^1 &\mapsto [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D)) \to Y \to X] \\
&\quad \text{where } D \overset{def}{=} A_1 + \cdots + A_k. \text{ Denote it by } [P_k^1 \to X] \text{ for simplicity.} \\
U_k^2 &\mapsto [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-A_k) \oplus \mathcal{O}(-D)) \to Y \to X] \\
&\quad \text{Denote it by } [P_k^2 \to X]. \\
U_k^3 &\mapsto [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-A_k) \oplus \mathcal{O}(-D)) \to Y \to X] \\
&\quad \text{Denote it by } [P_k^3 \to X]. \\
V_l^q &\mapsto [Q_l^q \to Y \to X] \\
&\quad \text{where } Q_l^q \text{ is defined in the same manner as } P_l^q \text{ with divisors } B_l \text{ and} \\
&\quad D = B_1 + \cdots + B_l \text{ instead.}
\end{aligned}$$

Finally, we send the general term $X_i \cdots Y_j \cdots U_k^p \cdots V_l^q \cdots$ to

$$[A_i \times_Y \cdots \times_Y B_j \times_Y \cdots \times_Y P_k^p \times_Y \cdots \times_Y Q_l^q \times_Y \cdots \to Y \to X].$$

In order for the homomorphism $\mathcal{G}$ to be well-defined, we need to check that, in general, the morphism

$$A_i \times_Y \cdots \times_Y B_j \times_Y \cdots \times_Y P_k^p \times_Y \cdots \times_Y Q_l^q \times_Y \cdots \to Y \to X$$

is projective and its domain is smooth. Notice that

$$\mathcal{G}(X_i^n) = [A_i \times_Y \cdots \times_Y A_i \to X] = [A_i \to X],$$

which is projective and A_i is smooth. Since $A_1+\cdots+A_n+B_1+\cdots+B_m$ is a reduced strict normal crossing divisor, the same is true for the value of $\mathcal{G}$ at any term with X_i, Y_j only. In addition, the morphisms $P_k^p \to Y$ and $Q_l^q \to Y$ are both projective and smooth. That means $\mathcal{G}:\mathcal{R}\to\omega(X)$ is well-defined. Then, the generalized double point relation $GDPR(n,m)$ is the equality :

$$\mathcal{G}(G_{n,m}^X)=\mathcal{G}(G_{m,n}^Y).$$

Remark 3.6. Observe that for any $n,m\geq 1$, the terms in $G_{n,m}^X$ or $G_{m,n}^Y$ are always of the form

$$X_i\cdots Y_j\cdots U_k^p\cdots V_l^q\cdots$$

where the powers for X_i, Y_j are either 0 or 1. In other words, self intersection will never happen in any $GDPR(n,m)$. Moreover, $1\leq i,k\leq n$ and $1\leq j,l\leq m$. In addition, $\mathcal{G}(G_{n,m}^X)$, $\mathcal{G}(G_{m,n}^Y)$ are both in $\omega_{\dim Y-1}(X)$.

The generalized double point relation is indeed a generalization of the double point relation and the extended double point relation. For example, if we apply the definition on the setup $\mathbb{I}_X : Y=X\to X$ with $A_1+A_2\sim B_1$, we get

$$\begin{aligned}
E_2^X &= -\,X_1X_2U_1^1\\
F_2^X &= X_1X_2(U_2^2-U_2^3)\\
G_{2,1}^X &= X_1+X_2+E_2^X+Y_1F_2^X\\
&= X_1+X_2-X_1X_2U_1^1+Y_1X_1X_2(U_2^2-U_2^3)\\
G_{1,2}^Y &= Y_1
\end{aligned}$$

Hence, the $GDPR(2,1)$ is the equality

$$\begin{aligned}
&[A_1\hookrightarrow X]+[A_2\hookrightarrow X]-\ [\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(A_1))\to A_1\cap A_2\hookrightarrow X]\\
&+[\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbb{P}(\mathcal{O}(-A_2)\oplus\mathcal{O}(-A_1-A_2))\to B_1\cap A_1\cap A_2\hookrightarrow X]\\
&-[\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-A_2)\oplus\mathcal{O}(-A_1-A_2))\to B_1\cap A_1\cap A_2\hookrightarrow X]\\
=\ &[B_1\hookrightarrow X],
\end{aligned}$$

which is exactly the extended double point relation as Lemma 5.2 in [LeP]. If we further assume that B_1 is disjoint from A_1, A_2, then we get

$$[A_1 \hookrightarrow X] + [A_2 \hookrightarrow X] - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A_1)) \to A_1 \cap A_2 \hookrightarrow X] = [B_1 \hookrightarrow X],$$

which is the double point relation in [LeP] (because $\mathcal{N}_{A_1 \cap A_2 \hookrightarrow A_2} \cong \mathcal{O}_{A_1 \cap A_2}(A_1)$).

Our first goal is to prove $GDPR(n, m)$ holds in the theory ω. In other words, we will show that imposing the generalized double point relation is equivalent to imposing the double point relation.

To be more precise, suppose X is a separated scheme of finite type over k and $\phi : Y \to X$ is a projective morphism such that $Y \in \mathit{Sm}$ is equidimensional. Moreover, suppose there are divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on Y such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$ and $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor. We want to show

$$\mathcal{G}(G^X_{n,m}) = \mathcal{G}(G^Y_{m,n})$$

where $\mathcal{G} : \mathcal{R} \to \omega(X)$ is the corresponding group homomorphism.

First of all, observe that $\mathcal{G}(G^X_{n,m}) = \phi_* \circ \mathcal{G}'(G^X_{n,m})$ where $\mathcal{G}'$ is the map defined by the setup $\mathbb{I} : Y \to Y$ with the same set of divisors on Y. Similarly, $\mathcal{G}(G^Y_{m,n}) = \phi_* \circ \mathcal{G}'(G^Y_{m,n})$. Hence, we reduce to the case when $\phi = \mathbb{I}_X$. In particular, we may assume X is in Sm and is equidimensional.

Suppose X is a separated scheme of finite type over k and $\mathcal{L}$ is an invertible sheaf over X. Recall that in [LeP], there is a corresponding operator $\tilde{c}_1(\mathcal{L}) \in \mathrm{End}\,(\omega(X))$ which is called the first Chern class operator. For simplicity, we will denote it by $c(\mathcal{L})$ and call it Chern class operator for the rest of this paper.

We are going to prove $GDPR(n, m)$ by induction. For this purpose, we need to modify the definition of $\mathcal{G}$. Suppose $X \in \mathit{Sm}$ is equidimensional and there are divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on X such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$. Then, we define a ring homomorphism $\mathcal{G} : \mathcal{R} \to \mathrm{End}\,(\omega(X))$ by

$$
\begin{aligned}
X_i &\mapsto c(\mathcal{O}(A_i)) \\
Y_j &\mapsto c(\mathcal{O}(B_j)) \\
U_k^a &\mapsto p_{a*}p_a^* \\
&\quad \text{where } p_a : P_k^a \to X \\
V_l^b &\mapsto q_{b*}q_b^* \\
&\quad \text{where } q_b : Q_l^b \to X
\end{aligned}
$$

if $1 \leq i, k \leq n$ and $1 \leq j, l \leq m$ (The morphisms p_{a*}, p_a^*, q_{b*}, q_b^* are all well-defined because p_a, q_b are both smooth and projective.). Otherwise, send them to zero.

For well-definedness of $\mathcal{G}$, we need to check the commutativity of some endomorphisms. Axiom **(A5)** in ω implies that $c(\mathcal{L})c(\mathcal{L}') = c(\mathcal{L}')c(\mathcal{L})$. In addition, for $p : P_k^a \to X$, we have

$$
c(\mathcal{L})p_*p^* = p_*c(p^*\mathcal{L})p^* = p_*p^*c(\mathcal{L})
$$

and same for q. For the commutativity between p_*p^* and $(p')_*(p')^*$ where $p : P \overset{def}{=} P_k^a \to X$ and $p' : P' \overset{def}{=} P_{k'}^{a'} \to X$, consider the following commutative diagram :

$$
\begin{array}{ccc}
P \times_X P' & \xrightarrow{\overline{p'}} & P \\
\overline{p}\downarrow & & \downarrow p \\
P' & \xrightarrow{p'} & X
\end{array}
$$

By axiom **(A2)** in ω,

$$
p_*p^*(p')_*(p')^* = p_*\overline{p'}_*\overline{p}^*(p')^* = (p')_*\overline{p}_*\overline{p'}^*p^* = (p')_*(p')^*p_*p^*.
$$

The commutativity between q and q', p and q follow from similar arguments. Hence, the ring homomorphism $\mathcal{G} : \mathcal{R} \to \mathrm{End}\,(\omega(X))$ is well-defined.

The statement we are going to prove is $\mathcal{G}(G_{n,m}^X) = \mathcal{G}(G_{m,n}^Y)$ as elements in $\mathrm{End}\,(\omega(X))$. Notice that we do not assume $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ to be a reduced strict normal crossing divisor in the setup anymore. Moreover, if A_i is a smooth divisor, then

$$
\mathcal{G}(X_i)[\mathbb{I}_X] = c(\mathcal{O}(A_i))[\mathbb{I}_X] = [A_i \hookrightarrow X]
$$

by the **(Sect)** axiom in [LeP]. So, the statement corresponding to this modified $\mathcal{G}$ is actually stronger than what we aimed to prove at the beginning (we will make this more precise later). For simplicity, we will still call this statement $GDPR(n, m)$ within this proof.

Here is the outline of the proof. We will prove that $GDPR(n, m)$ holds by assuming $GDPR(n, 1)$. Then, for $n \geq 3$, we will prove $GDPR(n, 1)$ by assuming $GDPR(n-1, 1)$ and $GDPR(2, 1)$. Thus, we reduce the proof of $GDPR(n, m)$ to the proof of $GDPR(2, 1)$, which is essentially the extended double point relation in [LeP]. But since the definition of $\mathcal{G}$ is modified, $GDPR(2, 1)$ becomes a stronger statement. Hence, there is still some works needed to be done.

Suppose $GDPR(n, 1)$ holds. Then, for a given equidimensional $X \in Sm$ and divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on X, let $C \stackrel{def}{=} A_1 + \cdots + A_n$. Consider the setup corresponding to $A_1 + \cdots + A_n \sim C$. From $GDPR(n, 1)$, we get $\mathcal{G}(G^X_{n,1}) = \mathcal{G}(G^Y_{1,n})$. This means that, as elements in $\mathrm{End}\,(\omega(X))$,

$$c(\mathcal{O}(C)) = \mathcal{G}(X_1 + \cdots + X_n + E^X_n) + \mathcal{G}(F^X_n)c(\mathcal{O}(C)).$$

Similarly, by considering the setup $C \sim B_1 + \cdots + B_m$, we get

$$c(\mathcal{O}(C)) = \mathcal{G}'(Y_1 + \cdots + Y_m + E^Y_m) + \mathcal{G}'(F^Y_m)c(\mathcal{O}(C))$$

with corresponding $\mathcal{G}'$.

Now, consider the map $\mathcal{G}''$ corresponding to the setup $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$. Then, by observing that $\mathcal{G} = \mathcal{G}''$ on terms without Y or V and $\mathcal{G}' = \mathcal{G}''$ on terms without X or U, we have

$$\begin{aligned}
& c(\mathcal{O}(C)) \\
= \; & \mathcal{G}(X_1 + \cdots + X_n + E^X_n) + \mathcal{G}(F^X_n)c(\mathcal{O}(C)) \\
= \; & \mathcal{G}''(X_1 + \cdots + X_n + E^X_n) + \mathcal{G}''(F^X_n)\;(\mathcal{G}'(Y_1 + \cdots + Y_m + E^Y_m) + \mathcal{G}'(F^Y_m)c(\mathcal{O}(C))) \\
= \; & \mathcal{G}''(X_1 + \cdots + X_n + E^X_n) + \mathcal{G}''(F^X_n)\;(\mathcal{G}''(Y_1 + \cdots + Y_m + E^Y_m) + \mathcal{G}''(F^Y_m)c(\mathcal{O}(C))) \\
= \; & \mathcal{G}''(G^X_{n,m}) + \mathcal{G}''(F^X_n F^Y_m)c(\mathcal{O}(C)).
\end{aligned}$$

On the other hand,

$$\begin{aligned} & c(\mathcal{O}(C)) \\ = \ & \mathcal{G}'(Y_1 + \cdots + Y_m + E_m^Y) + \mathcal{G}'(F_m^Y)c(\mathcal{O}(C)) \\ = \ & \mathcal{G}''(G_{m,n}^Y) + \mathcal{G}''(F_n^X F_m^Y)c(\mathcal{O}(C)). \end{aligned}$$

Then, the result follows from cancelling $\mathcal{G}''(F_n^X F_m^Y)c(\mathcal{O}(C))$ on both sides. That means it is enough to show $GDPR(n,1)$.

Assume $GDPR(n-1,1)$ and $GDPR(2,1)$ are true. Now, we start with a setup $A_1 + \cdots + A_n \sim B$. Let $C \stackrel{def}{=} A_1 + \cdots + A_{n-1}$. Consider the setup $C + A_n \sim B$. Define $\sigma \stackrel{def}{=} p_{1*}p_1^*$ and $\sigma' \stackrel{def}{=} p_{2*}p_2^* - p_{3*}p_3^*$ where

$$\begin{aligned} p_1 &: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(C)) \to X \\ p_2 &: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-A_n) \oplus \mathcal{O}(-B)) \to X \\ p_3 &: \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-A_n) \oplus \mathcal{O}(-B)) \to X. \end{aligned}$$

Then, by $GDPR(2,1)$, we get

$$\begin{aligned} c(\mathcal{O}(B)) = \ & c(\mathcal{O}(C)) + c(\mathcal{O}(A_n)) \\ & - c(\mathcal{O}(C))c(\mathcal{O}(A_n))\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))c(\mathcal{O}(A_n))\sigma'. \end{aligned} \tag{1}$$

By $GDPR(n-1,1)$ corresponding to the setup $A_1 + \cdots + A_{n-1} \sim C$, we have $\mathcal{G}'(G_{n-1,1}^X) = \mathcal{G}'(G_{1,n-1}^Y)$ where $\mathcal{G}'$ is the corresponding ring homomorphism. That implies

$$c(\mathcal{O}(C)) = \mathcal{G}'(X_1 + \cdots + X_{n-1} + E_{n-1}^X) + c(\mathcal{O}(C))\mathcal{G}'(F_{n-1}^X). \tag{2}$$

Now, consider the setup $A_1 + \cdots + A_n \sim B$ and call the corresponding ring homomorphism $\mathcal{G}$. Then, $\mathcal{G} = \mathcal{G}'$ on terms involving only X_i, U_k^p, if $1 \le i,k \le n-1$. Also, we have $\mathcal{G}(X_n) = c(\mathcal{O}(A_n))$.

For simplicity, we will drop the notation $\mathcal{G}$. Hence, as elements in $\mathrm{End}\,(\omega(X))$,

$$
\begin{aligned}
& c(\mathcal{O}(B)) \\
= \quad & c(\mathcal{O}(C)) + X_n - c(\mathcal{O}(C))X_n\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))X_n\sigma' \\
& \text{(by equation (1))} \\
= \quad & (X_1 + \cdots + X_{n-1} + E^X_{n-1} + c(\mathcal{O}(C))F^X_{n-1}) + X_n \\
& - X_n\sigma \; (X_1 + \cdots + X_{n-1} + E^X_{n-1} + c(\mathcal{O}(C))F^X_{n-1}) \\
& + c(\mathcal{O}(B))X_n\sigma' \; (X_1 + \cdots + X_{n-1} + E^X_{n-1} + c(\mathcal{O}(C))F^X_{n-1}) \\
& \text{(by equation (2))} \\
= \quad & X_1 + \cdots + X_{n-1} + X_n \\
& + E^X_{n-1} - (X_1 + \cdots + X_{n-1} + E^X_{n-1})X_n\sigma \\
& + c(\mathcal{O}(B))\sigma' X_n(X_1 + \cdots + X_{n-1} + E^X_{n-1}) \\
& + c(\mathcal{O}(C))F^X_{n-1} - c(\mathcal{O}(C))F^X_{n-1}X_n\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))\sigma' X_n F^X_{n-1},
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& X_1 + \cdots + X_n + (E^X_n + X_n F^X_{n-1}) + c(\mathcal{O}(B))(F^X_n - F^X_{n-1}) \\
+ \quad & c(\mathcal{O}(C))F^X_{n-1} - c(\mathcal{O}(C))F^X_{n-1}X_n\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))\sigma' X_n F^X_{n-1}
\end{aligned}
$$

by definition of E^X_n and F^X_n and the fact that $\sigma = \mathcal{G}(U^1_{n-1})$ and $\sigma' = \mathcal{G}(U^2_n - U^3_n)$. Notice that the last three terms are

$$
\begin{aligned}
& c(\mathcal{O}(C))F^X_{n-1} - c(\mathcal{O}(C))F^X_{n-1}X_n\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))\sigma' X_n F^X_{n-1} \\
= \quad & (c(\mathcal{O}(B)) - X_n + c(\mathcal{O}(C))X_n\sigma - c(\mathcal{O}(B))c(\mathcal{O}(C))X_n\sigma') \; F^X_{n-1} \\
& - c(\mathcal{O}(C))F^X_{n-1}X_n\sigma + c(\mathcal{O}(B))c(\mathcal{O}(C))\sigma' X_n F^X_{n-1} \\
& \text{(by equation (1))} \\
= \quad & (c(\mathcal{O}(B)) - X_n)F^X_{n-1}.
\end{aligned}
$$

Hence,

$$\begin{aligned} c(\mathcal{O}(B)) &= X_1 + \cdots + X_n + E_n^X + X_n F_{n-1}^X + c(\mathcal{O}(B))(F_n^X - F_{n-1}^X) \\ &\quad + (c(\mathcal{O}(B)) - X_n) F_{n-1}^X \\ &= X_1 + \cdots + X_n + E_n^X + c(\mathcal{O}(B)) F_n^X, \end{aligned}$$

which is exactly $\mathcal{G}(G_{1,n}^Y) = \mathcal{G}(G_{n,1}^X)$. That means it is enough to show $GDPR(2,1)$.

Suppose $X \in \mathit{Sm}$ is equidimensional and $\mathcal{L}$, $\mathcal{M}$ are two invertible sheaves over X. Define an element $H(\mathcal{L},\mathcal{M}) \in \mathrm{End}\,(\omega(X))$ by :

$$\begin{aligned} H(\mathcal{L},\mathcal{M}) &\stackrel{def}{=} c(\mathcal{L}) + c(\mathcal{M}) - c(\mathcal{L})c(\mathcal{M}) p_{1*} {p_1}^* \\ &\quad + c(\mathcal{L})c(\mathcal{M})c(\mathcal{L}\otimes\mathcal{M})(p_{2*}{p_2}^* - p_{3*}{p_3}^*) - c(\mathcal{L}\otimes\mathcal{M}) \\ &\quad \text{where } p_1 : \mathbb{P}(\mathcal{O}\oplus\mathcal{L}) \to X \\ &\quad p_2 : \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1)) \to \mathbb{P}(\mathcal{M}^\vee \oplus (\mathcal{L}\otimes\mathcal{M})^\vee) \to X \\ &\quad p_3 : \mathbb{P}(\mathcal{O}\oplus\mathcal{M}^\vee \oplus (\mathcal{L}\otimes\mathcal{M})^\vee) \to X. \end{aligned}$$

Observe that if A, B, C are divisors on X such that $A + B \sim C$, then

$$H(\mathcal{O}_X(A), \mathcal{O}_X(B)) \stackrel{def}{=} \mathcal{G}(G_{2,1}^X) - \mathcal{G}(G_{1,2}^Y)$$

where $\mathcal{G}$ is the ring homomorphism corresponding to the setup $A + B \sim C$. In other words, it is enough to show $H(\mathcal{L},\mathcal{M}) = 0$ for any equidimensional $X \in \mathit{Sm}$ and invertible sheaves $\mathcal{L}$, $\mathcal{M}$ over X. For this purpose, we need the following two Lemmas.

Lemma 3.7. *Suppose $f : X' \to X$ is a morphism in Sm such that X, X' are both equidimensional and $\mathcal{L}$, $\mathcal{M}$ are two invertible sheaves over X.*

1. *If f is projective, then $H(\mathcal{L},\mathcal{M}) f_* = f_* H(f^*\mathcal{L}, f^*\mathcal{M})$.*
2. *If f is smooth, then $H(f^*\mathcal{L}, f^*\mathcal{M}) f^* = f^* H(\mathcal{L},\mathcal{M})$.*

Proof. 1. Axiom **(A3)** in ω implies that $c(\mathcal{L}) f_* = f_* c(f^*\mathcal{L})$. For p_i, consider the commutative diagram

$$\begin{array}{ccc} P^i \times_X X' & \xrightarrow{p_i'} & X' \\ f' \downarrow & & \downarrow f \\ P^i & \xrightarrow{p_i} & X \end{array}$$

By axiom **(A2)**, we have $p_{i*}{p_i}^* f_* = p_{i*} f'_* {p_i'}^* = f_* p'_{i*} {p_i'}^*$ and the morphisms p_i' are

$$\begin{aligned} P^1 \times_X X' &= \mathbb{P}(\mathcal{O} \oplus f^*\mathcal{L}) \to X' \\ P^2 \times_X X' &= \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(f^*\mathcal{M}^\vee \oplus f^*(\mathcal{L} \otimes \mathcal{M})^\vee) \to X' \\ P^3 \times_X X' &= \mathbb{P}(\mathcal{O} \oplus f^*\mathcal{M}^\vee \oplus f^*(\mathcal{L} \otimes \mathcal{M})^\vee) \to X'. \end{aligned}$$

2. Similarly, axiom **(A4)** implies that $c(f^*\mathcal{L})f^* = f^*c(\mathcal{L})$. For p_i, we can consider the same diagram above and we get $p'_{i*}{p_i'}^* f^* = p'_{i*} f'^* {p_i}^* = f^* p_{i*} {p_i}^*$. ☐

Lemma 3.8. *Suppose X is a smooth k-scheme, $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ are invertible sheaves over X and $L_1, L_2, \ldots, L_n$ are the corresponding line bundles over X. Let $\tilde{X} \stackrel{def}{=} L_1 \times_X L_2 \times_X \cdots \times_X L_n$ and $\pi : \tilde{X} \to X$ be the projection. Then, there are canonically defined global sections $s_i \in \mathrm{H}^0(\tilde{X}, \pi^*\mathcal{L}_i)$ such that, for each i, the section s_i will cut out a smooth divisor D_i on $\tilde{X}$ and $D_1 + \cdots + D_n$ is a reduced strict normal crossing divisor.*

Proof. Define $s_i : \tilde{X} \to \tilde{X} \times_X L_i$ by $(x, v_1, \ldots, v_n) \mapsto (x, v_1, \ldots, v_n, v_i)$. This is a canonically defined global section of $\pi^*\mathcal{L}_i$. It cuts out the divisor $D_i \stackrel{def}{=} \{(x, v_1, \ldots, v_n) \mid v_i = 0\}$. Moreover, the intersection of $D_{i_1}, \ldots, D_{i_j}$ is just $\{(x, v_1, \ldots, v_n) \mid v_{i_1} = v_{i_2} = \cdots = v_{i_j} = 0\}$, which is smooth and has codimension j in $\tilde{X}$. ☐

We are now ready to prove $H(\mathcal{L}, \mathcal{M}) = 0$. First of all,

$$H(\mathcal{L}, \mathcal{M})[f] = H(\mathcal{L}, \mathcal{M}) f_*[\mathbb{I}] = f_* H(f^*\mathcal{L}, f^*\mathcal{M})[\mathbb{I}].$$

So, it is enough to consider the element $[\mathbb{I}]$. Let $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ be the invertible sheaves $\mathcal{L}$, $\mathcal{M}$, $\mathcal{L} \otimes \mathcal{M}$ respectively and $\pi : \tilde{X} \to X$ as in Lemma 3.8. Then, we have

$$\pi^* H(\mathcal{L}, \mathcal{M})[\mathbb{I}] = H(\pi^*\mathcal{L}, \pi^*\mathcal{M})\pi^*[\mathbb{I}] = H(\pi^*\mathcal{L}, \pi^*\mathcal{M})[\mathbb{I}].$$

By the extended homotopy property in ω, $\pi^* : \omega(X) \to \omega(\tilde{X})$ is an isomorphism. That means it is enough to prove $H(\mathcal{L}, \mathcal{M})[\mathbb{I}] = 0$ when there are divisors A, B, C on X such that

$\mathcal{L} \cong \mathcal{O}_X(A)$, $\mathcal{M} \cong \mathcal{O}_X(B)$, $C \sim A + B$ and $A + B + C$ is a reduced strict normal crossing divisor. In this case,

$$\begin{aligned}
& H(\mathcal{L}, \mathcal{M})[\mathbb{I}] \\
= \; & c(\mathcal{O}(A))[\mathbb{I}] + c(\mathcal{O}(B))[\mathbb{I}] - c(\mathcal{O}(A))c(\mathcal{O}(B))p_{1*}{p_1}^*[\mathbb{I}] \\
& + c(\mathcal{O}(A))\, c(\mathcal{O}(B))\, c(\mathcal{O}(C))\, (p_{2*}{p_2}^* - p_{3*}{p_3}^*)[\mathbb{I}] - c(\mathcal{O}(C))[\mathbb{I}] \\
= \; & [A \hookrightarrow X] + [B \hookrightarrow X] - p_{1*}{p_1}^* c(\mathcal{O}(A))c(\mathcal{O}(B))[\mathbb{I}] \\
& + (p_{2*}{p_2}^* - p_{3*}{p_3}^*)\, c(\mathcal{O}(A))\, c(\mathcal{O}(B))\, c(\mathcal{O}(C))\, [\mathbb{I}] - [C \hookrightarrow X] \\
& \text{(by } \textbf{(Sect)} \text{ axiom in } \omega) \\
= \; & [A \hookrightarrow X] + [B \hookrightarrow X] - p_{1*}{p_1}^*[A \cap B \hookrightarrow X] \\
& + (p_{2*}{p_2}^* - p_{3*}{p_3}^*)[A \cap B \cap C \hookrightarrow X] - [C \hookrightarrow X] \\
& \text{(by } \textbf{(Sect)} \text{ axiom)} \\
= \; & [A \hookrightarrow X] + [B \hookrightarrow X] - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to A \cap B \hookrightarrow X] \\
& + [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to A \cap B \cap C \hookrightarrow X] \\
& - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to A \cap B \cap C \hookrightarrow X] - [C \hookrightarrow X] \\
= \; & 0
\end{aligned}$$

by the extended double point relation in [LeP] (Lemma 5.2). Hence, we proved the following Proposition.

Proposition 3.9. *Suppose $X \in Sm$ is equidimensional and $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ are divisors on X such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$. Let $\mathcal{G} : \mathcal{R} \to \operatorname{End}(\omega(X))$ be the corresponding map constructed before. Then, $\mathcal{G}(G^X_{n,m}) = \mathcal{G}(G^Y_{m,n})$.*

We can now apply this statement to prove that the generalized double point relation holds in ω.

Corollary 3.10. *Suppose X is a separated scheme of finite type over k and there is a projective morphism $\phi : Y \to X$ such that Y is in Sm and is equidimensional. Moreover, suppose $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ are divisors on Y such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$ and*

$A_1+\cdots+A_n+B_1+\cdots+B_m$ is a reduced strict normal crossing divisor. Let $\mathcal{G}:\mathcal{R}\to\omega(X)$ be the corresponding map constructed before. Then, $\mathcal{G}(G^X_{n,m})=\mathcal{G}(G^Y_{m,n})$.

Proof. By definition, $\mathcal{G}(G^X_{n,m})=\phi_*\circ\mathcal{G}'(G^X_{n,m})$ and $\mathcal{G}(G^Y_{m,n})=\phi_*\circ\mathcal{G}'(G^Y_{m,n})$ where $\mathcal{G}'$ is the map corresponding to the setup $\mathbb{I}_Y:Y\to Y$ with the same set of divisors. So, we may assume $\phi=\mathbb{I}_X$. Then, it follows from the fact that

$$\begin{aligned} & \mathcal{G}(G^X_{n,m})[\mathbb{I}_X] \\ & \text{(the modified definition } \mathcal{G}:\mathcal{R}\to\operatorname{End}(\omega(X))) \\ = \; & \mathcal{G}(G^X_{n,m}) \\ & \text{(the original definition } \mathcal{G}:\mathcal{R}\to\omega(X)) \end{aligned}$$

and similarly for $G^Y_{m,n}$. □

Remark 3.11. Notice that in the generalized double point relation setup $\phi:Y\to X$ with $A_1+\cdots+A_n\sim B_1+\cdots+B_m$ on Y, we do not assume A_i or B_j to be nonempty. If $\mathcal{G}$ is the map corresponding to $A_1+\cdots+A_n\sim B_1+\cdots+B_m$ and $\mathcal{G}'$ is the map corresponding to

$$A_1+\cdots+A_n+\sum_{i=n+1}^{N}C_i\sim B_1+\cdots+B_m+\sum_{j=m+1}^{M}D_j$$

where $\{C_i,D_j\}$ are zero divisors, then

$$\mathcal{G}(G^X_{n,m})=\mathcal{G}'(G^X_{n,m})=\mathcal{G}'(G^X_{N,M}) \quad\text{and}\quad \mathcal{G}(G^Y_{m,n})=\mathcal{G}'(G^Y_{m,n})=\mathcal{G}'(G^Y_{M,N}).$$

Indeed, notice that if a general term $X_i\cdots U^p_k\cdots$ in $\mathcal{R}$ contains X_i or Y_j with $n+1\le i\le N$ or $m+1\le j\le M$, then $\mathcal{G}'(X_i\cdots U^p_k\cdots)=0$. By definition,

$$E^X_{n+1}=E^X_n+\sum \text{ terms with } X_{n+1}.$$

Inductively,

$$E^X_N=E^X_n+\sum \text{ terms with } X_i \text{ where } n+1\le i\le N.$$

Similar facts hold for F_N^X, E_M^Y and F_M^Y. Hence,

$$\begin{aligned} G_{N,M}^X &= X_1 + \cdots + X_N + E_N^X + (Y_1 + \cdots + Y_M)F_N^X + E_M^Y F_N^X \\ &= X_1 + \cdots + X_n + E_n^X + (Y_1 + \cdots + Y_m)F_n^X + E_m^Y F_n^X \\ &\quad + \sum \text{ terms with } X_i \text{ where } n+1 \le i \le N \\ &\quad + \sum \text{ terms with } Y_j \text{ where } m+1 \le j \le M. \end{aligned}$$

That means $\mathcal{G}'(G_{N,M}^X) = \mathcal{G}'(G_{n,m}^X) = \mathcal{G}(G_{n,m}^X)$. Similarly, $\mathcal{G}'(G_{M,N}^Y) = \mathcal{G}'(G_{m,n}^Y) = \mathcal{G}(G_{m,n}^Y)$.

3.3. **Definition and basic properties.** Now we will define our equivariant algebraic cobordism theory using the generalized double point relation.

Definition 3.12. For an object X in *G-Sm*, let $M_G(X)$ be the set of isomorphism classes over X of projective morphisms $f : Y \to X$ in *G-Sm*. Then, $M_G(X)$ is a monoid under disjoint union of domains, i.e.

$$[Y \to X] + [Y' \to X] \stackrel{def}{=} [Y \amalg Y' \to X].$$

We define the abelian group $M_G(X)^+$ as the group completion of $M_G(X)$.

The i-th graded piece (cohomological grading) : $(M_G(X)^+)^i$, when X is equidimensional, is given by $[Y \to X]$ where Y is equidimensional and $i = \dim X - \dim Y$. We also have homological grading $M_G(X)_i^+$ where i denotes the dimension of Y, if Y is equidimensional.

Remark 3.13. The main reason for focusing on quasi-projective X instead of just separated scheme of finite type over k as in [LeP] is because we will sometimes consider the quotient X/G and the operation of taking quotient works better in the quasi-projective category.

Next, we will define the notion of equivariant generalized double point relation which is the equivariant analog of the generalized double point relation we just defined in section 3.2. To be more precise, we will consider the following setup.

Let $\phi : Y \to X$ be a projective morphism in *G-Sm* such that Y is equidimensional. In addition, $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ are G-invariant divisors on Y such that $A_1 + \cdots + A_n \sim$

$B_1 + \cdots + B_m$ (G-equivariantly linearly equivalent) and $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor. In this setup, we construct a corresponding abelian group homomorphism $\mathcal{G} : \mathcal{R} \to M_G(X)^+$ by the exact same definition as in section 3.2. Notice that all objects involved are smooth varieties with natural G-action and all morphisms involved are naturally G-equivariant. We will call the collection of $\phi : Y \to X$ together with the divisors as above a generalized double point relation setup over X, or GDPR setup.

Definition 3.14. The equivariant algebraic cobordism group $\mathcal{U}_G(X)$ is defined as the quotient of $M_G(X)^+$ by the subgroup generated by all expressions $\mathcal{G}(G^X_{n,m}) - \mathcal{G}(G^Y_{m,n})$ where $\mathcal{G}$ corresponds to some GDPR setup over X.

Remark 3.15. As pointed out in remarks 3.6, if $\phi : Y \to X$ is the morphism defining $\mathcal{G}$, then $\mathcal{G}(G^X_{n,m})$, $\mathcal{G}(G^Y_{m,n})$ both lie in $M_G(X)^+_{\dim Y - 1}$. Hence, if X is equidimensional, we can define a homological (cohomological) grading on $\mathcal{U}_G(X)$, namely

$$\mathcal{U}_G(X) = \bigoplus_i \mathcal{U}^i_G(X) = \bigoplus_i \mathcal{U}^G_i(X)$$

where $\mathcal{U}^G_i(X)$ is defined as the quotient of $M_G(X)^+_i$ by the subgroup generated by all expressions $\mathcal{G}(G^X_{n,m}) - \mathcal{G}(G^Y_{m,n})$ such that $\mathcal{G}$ corresponds to some GDPR setup over X where the dimension of the domain of ϕ is $i + 1$. Similarly, the group $\mathcal{U}^i_G(X)$ is the quotient of $(M_G(X)^+)^i$ with GDPR setups over X when the dimension of the domain of ϕ is $\dim X - i + 1$.

Generalized double point relation is a generalization of the double point relation in the equivariant configuration.

Proposition 3.16. *Suppose $\phi : Y \to X \times \mathbb{P}^1$ is a projective morphism in G-Sm (with trivial G-action on $\mathbb{P}^1$) such that Y is equidimensional. Let $\xi \in \mathbb{P}^1$ be a closed point. Assume that the fiber $Y_\xi \overset{def}{=} (\pi_2 \circ \phi)^{-1}(\xi)$ is a smooth G-invariant divisor on Y and there exist smooth G-invariant divisors A, B on Y such that $Y_0 = A \cup B$ and A, B intersect transversely, then*

$$[Y_\xi \to X] = [A \to X] + [B \to X] - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to A \cap B \to X]$$

as elements in $\mathcal{U}_G(X)$.

Proof. Since Y_ξ is disjoint from A, B and A, B intersect transversely, $Y_\xi + A + B$ is a reduced strict normal crossing divisor on Y. In addition, since $\mathbb{P}^1$ has trivial G-action, $Y_\xi \sim A + B$. That defines a generalized double point relation setup $\pi_1 \circ \phi : Y \to X$ with $Y_\xi \sim A + B$. Thus, we obtain the equality $\mathcal{G}(G_{1,2}^X) = \mathcal{G}(G_{2,1}^Y)$ in $\mathcal{U}_G(X)$ which is exactly

$$[Y_\xi \to X] = [A \to X] + [B \to X] - [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to A \cap B \to X].$$

□

In [LeP], M. Levine and R. Pandharipande listed several natural axioms and properties that an algebraic cobordism theory should satisfy. Here, we will show the equivariant version of some of them.

(D1) If $f : X \to X'$ in G-*Sm* is projective, then there is an abelian group homomorphism

$$f_* : \mathcal{U}_*^G(X) \to \mathcal{U}_*^G(X').$$

Moreover, if f, g are both projective, then $(g \circ f)_* = g_* \circ f_*$.

Proof. As in the ω_* theory of [LeP], the push-forward f_* is given by sending $[h : Y \to X]$ to $[f \circ h : Y \to X']$. We need to check that it preserves the generalized double point relation.

Suppose a generalized double point relation on X is defined by a projective morphism $\phi : Y \to X$ in G-*Sm* with $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$. It defines a homomorphism $\mathcal{G} : \mathcal{R} \to M_G(X)^+$. We can then consider the generalized double point relation on X' given by $f \circ \phi : Y \to X'$ with the same set of divisors. This will also define a homomorphism $\mathcal{G}' : \mathcal{R} \to M_G(X')^+$. Thus, for a general term $X_i \cdots Y_j \cdots U_k^p \cdots V_l^q \cdots$ in $\mathcal{R}$,

$$\begin{aligned}
& f_* \circ \mathcal{G}(X_i \cdots Y_j \cdots U_k^p \cdots V_l^q \cdots) \\
= \; & f_*[A_i \times_Y \cdots \times_Y B_j \times_Y \cdots \times_Y P_k^p \times_Y \cdots \times_Y Q_l^q \times_Y \cdots \to X] \\
= \; & [A_i \times_Y \cdots \times_Y B_j \times_Y \cdots \times_Y P_k^p \times_Y \cdots \times_Y Q_l^q \times_Y \cdots \to X \to X'].
\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathcal{G}'(X_i\cdots Y_j\cdots U_k^p\cdots V_l^q\cdots)\\ =\;&[A_i\times_Y\cdots\times_Y B_j\times_Y\cdots\times_Y P_k^p\times_Y\cdots\times_Y Q_l^q\times_Y\cdots\to X\to X'].\end{aligned}$$

That implies $f_*\circ\mathcal{G}=\mathcal{G}'$. In particular, $f_*\circ\mathcal{G}(G^X_{n,m})=\mathcal{G}'(G^X_{n,m})$ and $f_*\circ\mathcal{G}(G^Y_{m,n})=\mathcal{G}'(G^Y_{m,n})$, which means $f_*\circ\mathcal{G}(G^X_{n,m})=f_*\circ\mathcal{G}(G^Y_{m,n})$ in $\mathcal{U}_G(X')$. So, the group homomorphism $f_*:\mathcal{U}^G(X)\to\mathcal{U}^G(X')$ is well-defined. Clearly, it preserves the homological grading and $(g\circ f)_*=g_*\circ f_*$. $\square$

(D2) If $f:X'\to X$ in *G-Sm* is smooth such that X,X' are both equidimensional, then there is an abelian group homomorphism

$$f^*:\mathcal{U}^*_G(X)\to\mathcal{U}^*_G(X').$$

Proof. Let $[Y\to X]$ be an element $\mathcal{U}_G(X)$, then we define the pull-back $f^*[Y\to X]$ as $[Y\times_X X'\to X']$. First of all, $Y\times_X X'$ is a smooth variety with natural diagonal G-action and the morphism $Y\times_X X'\to X'$ is projective and G-equivariant.

Consider a GDPR setup over X given by $\phi:Y\to X$ with divisors $A_1,\ldots,A_n$, $B_1,\ldots,B_m$ on Y and $\mathcal{G}$ be the corresponding map. We have the following commutative diagram :

$$\begin{array}{ccc} Y'\overset{def}{=}Y\times_X X' & \xrightarrow{f'} & Y\\ \phi'\big\downarrow & & \big\downarrow\phi\\ X' & \xrightarrow{f} & X\end{array}$$

We obtain a generalized double point relation setup over X' given by $\phi':Y'\to X'$ with divisors $f'^*A_1,\ldots,f'^*A_n,f'^*B_1,\ldots,f'^*B_m$ on Y'. Let $\mathcal{G}'$ be the corresponding homomorphism. The smoothness of f' implies that $f'^*A_1+\cdots+f'^*A_n+f'^*B_1+\cdots+f'^*B_m$ is still a reduced strict normal crossing divisor. Observe that if $P^1_k=\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(D))$ is a G-equivariant projective bundle over Y, then $P^1_k\times_Y Y'\cong\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(f'^*D))$, as G-equivariant projective bundles over Y'. So,

$$\mathcal{G}'(U^1_k)=[P^1_k\times_Y Y'\to Y']=f^*[P^1_k\to Y]=f^*\circ\mathcal{G}(U^1_k).$$

Similar statements with respect to U_k^p and V_l^q also hold. For a general term,

$$\begin{aligned} & f^* \circ \mathcal{G}(X_i \cdots U_k^p \cdots) \\ = \; & f^*[A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots \to X] \\ = \; & [(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots) \times_X X' \to X']. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathcal{G}'(X_i \cdots U_k^p \cdots) \\ = \; & [(A_i \times_Y Y') \times_{Y'} \cdots \times_{Y'} (P_k^p \times_Y Y') \times_{Y'} \cdots \to X']. \\ = \; & [(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots) \times_Y Y' \to X']. \\ = \; & [(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots) \times_X X' \to X']. \end{aligned}$$

That shows the well-definedness of $f^* : \mathcal{U}_G(X) \to \mathcal{U}_G(X')$. Since f is smooth, taking fiber product with $f : X' \to X$ preserves codimension. Thus, f^* preserves the cohomological grading. □

(D3) In [LeP], there is a discussion of the first Chern class operator. This will be addressed in the next section.

(D4) For each pair (X, X') of objects in *G-Sm*, there is a bilinear, graded pairing

$$\times : \mathcal{U}_i^G(X) \times \mathcal{U}_j^G(X') \to \mathcal{U}_{i+j}^G(X \times X')$$

which is commutative, associative and admits a distinguished element $1 \in \mathcal{U}_0^G(\operatorname{Spec} k)$ as a unit.

Proof. The definition is standard. We define

$$[f : Y \to X] \times [f' : Y' \to X'] \stackrel{def}{=} [f \times f' : Y \times Y' \to X \times X'].$$

Suppose a GDPR setup over X is given by $\phi : Z \to X$ with divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on Z and $\mathcal{G}$ be the corresponding homomorphism. We need to show

$$\mathcal{G}(G_{n,m}^X) \times [f' : Y' \to X'] = \mathcal{G}(G_{m,n}^Y) \times [f' : Y' \to X'].$$

Without loss of generality, we may assume Y' is equidimensional. Consider the GDPR setup over $X \times X'$ given by $\phi \times f' : Z \times Y' \to X \times X'$ with divisors $\pi_1^* A_1, \ldots, \pi_1^* A_n, \pi_1^* B_1, \ldots, \pi_1^* B_m$ on $Z \times Y'$. Let $\mathcal{G}'$ be the corresponding homomorphism.

Observe that if $P_k^1 = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))$, then $P_k^1 \times Y' = \mathbb{P}(\mathcal{O}_{Z \times Y'} \oplus \mathcal{O}_{Z \times Y'}(\pi_1^* D))$. So,

$$\mathcal{G}'(U_k^1) = [P_k^1 \times Y' \to X \times X'] = [P_k^1 \to X] \times [Y' \to X'] = \mathcal{G}(U_k^1) \times [Y' \to X'].$$

Similar statements with respect to U_k^p and V_l^q also hold. For a general term,

$$\begin{aligned} & [f'] \times \mathcal{G}(X_i \cdots U_k^p \cdots) \\ = \; & [f'] \times [A_i \times_Z \cdots \times_Z P_k^p \times_Z \cdots \to X] \\ = \; & [(A_i \times_Z \cdots \times_Z P_k^p \times_Z \cdots) \times Y' \to X \times X']. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathcal{G}'(X_i \cdots U_k^p \cdots) \\ = \; & [(A_i \times Y') \times_{Z \times Y'} \cdots \times_{Z \times Y'} (P_k^p \times Y') \times_{Z \times Y'} \cdots \to X \times X']. \\ = \; & [(A_i \times_Z \cdots \times_Z P_k^p \times_Z \cdots) \times Y' \to X \times X']. \end{aligned}$$

That shows the well-definedness of $\times$. It is not hard to see that this product is graded, associative and commutative. The unit in $\mathcal{U}_0^G(\operatorname{Spec} k)$ is simply $[\mathbb{I} : \operatorname{Spec} k \to \operatorname{Spec} k]$. □

Remark 3.17. We will refer to

$$\times : \mathcal{U}_i^G(X) \times \mathcal{U}_j^G(X') \to \mathcal{U}_{i+j}^G(X \times X')$$

as the external product. This external product gives $\mathcal{U}_*^G(\operatorname{Spec} k)$ a graded ring structure and $\mathcal{U}_*^G(X)$ a graded $\mathcal{U}_*^G(\operatorname{Spec} k)$-module structure. In addition, if $f : X \to X'$ is a projective morphism in *G-Sm*, then the push-forward $f_* : \mathcal{U}_*^G(X) \to \mathcal{U}_*^G(X')$ will be a graded $\mathcal{U}_*^G(\operatorname{Spec} k)$-module homomorphism. Similarly, if $f : X \to X'$ in *G-Sm* is smooth such that X, X' are equidimensional, then the pull-back $f^* : \mathcal{U}^*_G(X') \to \mathcal{U}^*_G(X)$ will be a graded $\mathcal{U}^*_G(\operatorname{Spec} k)$-module homomorphism.

The following two properties can be easily derived from the definitions, similarly to [LeP].

(A1) If $f : X \to X'$ and $g : X' \to X''$ are both smooth and X, X', X'' are all equidimensional, then

$$(g \circ f)^* = f^* \circ g^*.$$

Moreover, $\mathbb{I}^*$ is the identity homomorphism.

□

(A2) If $f : X \to Z$ is projective and $g : Y \to Z$ is smooth such that X, Y, Z are all equidimensional, then we have $g^* f_* = f'_* g'^*$ in the pull-back square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

□

(A3), (A4), (A5) in [LeP] are properties involving the Chern class operator. Hence, they will be addressed in the next section.

(A6) If f, g are projective, then

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times.$$

Proof. Let $f : X \to X'$ and $g : Z \to Z'$. The statement follows from the commutativity of the following diagram, which is easy to check.

$$\begin{array}{ccc} \mathcal{U}_G(X) \times \mathcal{U}_G(Z) & \xrightarrow{\times} & \mathcal{U}_G(X \times Z) \\ f_* \times g_* \downarrow & & \downarrow (f \times g)_* \\ \mathcal{U}_G(X') \times \mathcal{U}_G(Z') & \xrightarrow{\times} & \mathcal{U}_G(X' \times Z') \end{array}$$

□

(A7) If f, g are smooth with equidimensional domains and codomains, then

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

Proof. It follows from the commutativity of the previous diagram with vertical arrows reversed. □

3.4. **Results for free action.** Consider the set of objects $Y \in G\text{-}Sm$ such that the geometric quotient (definition 0.6 in [MuFoKi]) Y/G exists as scheme over k, lies in Sm and the map $Y \to Y/G$ is a principal G-bundle. Denote this set of objects by $\mathcal{D}$. We will consider $\mathcal{D}$ as a full subcategory of $G\text{-}Sm$. Suppose X is a variety in $\mathcal{D}$, it turns out that there is a one-to-one correspondence between morphisms $Z \to X/G$ in the category Sm and G-equivariant morphisms $Y \to X$ in the category $G\text{-}Sm$. This important observation will lead us to the proof of the isomorphism

$$\omega(X/G) \tilde{\to} \mathcal{U}_G(X)$$

for any $X \in \mathcal{D}$.

Throughout this paper, we will call going from X to X/G "descent" and going from X/G to X "ascent".

Proposition 3.18. *If $f : Y \to X$ is a morphism in $G\text{-}Sm$ and X is in $\mathcal{D}$, then Y is also in $\mathcal{D}$.*

Proof. Recall that the group scheme G we are working with is either a reductive connected group over k or a finite group.

Consider the case when G is connected and reductive. Since Y is quasi-projective, the map $Y \to X$ is quasi-projective. Then, there exists an invertible sheaf $\mathcal{L}$ over Y (may not be G-linearized) which is very ample relative to X. By Theorem 1.6 in [Su], since Y is normal, there exists a positive integer m such that $\mathcal{L}^m$ ($\overset{def}{=} \mathcal{L}^{\otimes m}$) admits a G-linearization. Then, by Proposition 7.1 in [MuFoKi], we have the following commutative diagram in which Y/G is quasi-projective and $Y \to Y/G$ is a principal G-bundle.

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y/G & \longrightarrow & X/G \end{array}$$

Since $Y \to Y/G$ is a principal G-bundle, the morphism $Y \to Y/G$ is locally trivial in the étale topology. That means that Y/G can be covered by étale neighborhoods W for which we have the following commutative diagram :

$$\begin{array}{ccc} W \times G & \xrightarrow{\text{étale}} & Y \\ \downarrow & & \downarrow \\ W & \xrightarrow{\text{étale}} & Y/G \end{array}$$

Hence, Y is smooth if and only if Y/G is smooth.

For the case when G is finite, just replace $\mathcal{L}^m$ by $\otimes_{\alpha \in G} \alpha^* \mathcal{L}$. □

The following is mostly a standard application of descent theory, but we need to make sure we preserve the smoothness and quasi-projectiveness assumptions.

Proposition 3.19. *For any object $X \in \mathcal{D}$,*

(1) *There is a one-to-one correspondence between the set of morphisms $f : Z \to X/G$ in Sm and the set of morphisms $g : Y \to X$ in G-Sm, given by sending $Z \to X/G$ to its fiber product with $X \to X/G$. Moreover, its inverse is given by sending $Y \to X$ to $Y/G \to X/G$.*

(2) *The above map defines a one-to-one correspondence between the set of projective morphisms $f : Z \to X/G$ in Sm and the set of projective morphisms $g : Y \to X$ in G-Sm.*

(3) *The above map defines a one-to-one correspondence between the set of vector bundles $E' \to X/G$ and the set of G-equivariant vector bundles $E \to X$.*

Proof. (1) For ascent, consider the following commutative diagram :

$$\begin{array}{ccc} Z \times_{X/G} X & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Z & \xrightarrow{f} & X/G \end{array}$$

There is a natural G-action on $Z \times_{X/G} X$ and g is G-equivariant. Since X, Z are quasi-projective, $Z \times_{X/G} X$ is quasi-projective.

Claim 1 : If X is an object in $\mathcal{D}$, then the morphism $X \to X/G$ is smooth.

Since $X \to X/G$ is a principal G-bundle, it is flat and locally trivial in the étale topology. Thus, we have the following commutative diagram :

$$\begin{array}{ccc} W \times G & \xrightarrow{\text{étale}} & X \\ \downarrow & & \downarrow \\ W & \xrightarrow{\text{étale}} & X/G \end{array}$$

Let x be a point in X/G and K be the algebraic closure of $k(x)$. Then, by taking fiber product with $\operatorname{Spec} K \to \operatorname{Spec} k(x)$, we have the following commutative diagram :

$$\begin{array}{ccc} W_K \times G & \xrightarrow{\text{étale}} & X_K \\ \downarrow & & \downarrow \\ W_K & \xrightarrow{\text{étale}} & \operatorname{Spec} K \end{array}$$

Clearly, $\dim X_K = \dim W_K \times G = \dim G$ and X_K is regular. The claim then follows from Theorem 10.2 in Ch III in [Ha]. $\triangle$

Since the morphism $X \to X/G$ is smooth and Z is smooth, $Z \times_{X/G} X$ is smooth. That shows the well-definedness of ascent.

For descent, consider the following commutative diagram :

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Y/G & \xrightarrow{f \overset{def}{=} g/G} & X/G \end{array}$$

By Proposition 3.18, Y is in $\mathcal{D}$. So, Y/G is in *Sm*. The fact that these two constructions are inverse to each other is standard and follows from descent theory.

(2) Ascent clearly preserves projectiveness. For descent, it follows from the descent of properness (Proposition 2 of [EG]) and the fact that Y/G is quasi-projective.

(3) Ascent clearly takes vector bundles to G-equivariant vector bundles. For descent, it follows from Lemma 1 of [EG]. $\square$

We are now ready to prove the following Theorem.

Theorem 3.20. *Suppose X is an object in $\mathcal{D}$. Sending $[Z \to X/G]$ to $[Z \times_{X/G} X \to X]$ defines an abelian group isomorphism*

$$\Psi : \omega^*(X/G) \to \mathcal{U}^*_G(X).$$

Proof. Define the inverse homomorphism Ψ^{-1} by sending $[Y \to X]$ to $[Y/G \to X/G]$. We will call Ψ "ascent" and Ψ^{-1} "descent".

First of all, we need to prove that Ψ is well-defined. By Proposition 3.19, Ψ is well-defined at the level of $M(X/G)^+$. In this proof, we will denote the fiber product with $X \to X/G$ by a star, i.e. $W^* \overset{def}{=} W \times_{X/G} X$. We also denote by $\pi : X \to X/G$ the projection. Consider the following commutative diagram :

$$\begin{array}{ccc} Y^* & \xrightarrow{\phi^*} & X \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\phi} & X/G \times \mathbb{P}^1 \end{array}$$

where ϕ corresponds to a double point relation setup over X/G (the fiber Y_ξ is a smooth divisor, $Y_0 = A \cup B$ for some smooth divisors A, B and A, B intersect transversely).

We want to show that ϕ^* gives an equivariant double point relation setup over X. Notice that Y^* is in $\mathcal{D}$ because X is in $\mathcal{D}$ (Proposition 3.18). So, Y^* is smooth and the projection $Y^* \to Y$ is smooth (claim 1 in the proof of Proposition 3.19). Then, Y^* is equidimensional, $(Y_\xi)^* = (Y^*)_\xi$, A^* and B^* are G-invariant divisors on Y^*,

$$A^* \cup B^* = (A \cup B)^* = (Y_0)^* = (Y^*)_0$$

and A^*, B^* intersect transversely. Clearly, ϕ^* is projective. Hence, that gives us an equivariant double point relation setup over X. By Proposition 3.16, we obtain the following equation in $\mathcal{U}_G(X)$:

$$[Y^*_\xi \to X] = [A^* \to X] + [B^* \to X] - [\mathbb{P}(\mathcal{O}_{D^*} \oplus \mathcal{O}_{D^*}(A^*)) \to X] \tag{3}$$

where $D \overset{def}{=} A \cap B$.

On the other hand, the double point relation on X/G corresponding to ϕ is

$$[Y_\xi \to X/G] = [A \to X/G] + [B \to X/G] - [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(A)) \to X/G].$$

If we apply Ψ on this equation, we will get

$$[Y^*_\xi \to X] = [A^* \to X] + [B^* \to X] - [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(A)) \times_{X/G} X \to X]. \tag{4}$$

Since

$$\mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(A)) \times_{X/G} X \cong \mathbb{P}(\,\pi^*(\mathcal{O}_D \oplus \mathcal{O}_D(A))\,) \cong \mathbb{P}(\mathcal{O}_{D^*} \oplus \mathcal{O}_{D^*}(A^*)),$$

equations (3) and (4) are equivalent. This finishes the first half of the proof : well-definedness of Ψ.

It remains to show the well-definedness of the inverse Ψ^{-1}. By Proposition 3.19, it is well-defined at the level of $M_G(X)^+$. It remains to show that for a given GDPR setup $\phi : Y \to X$ with divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on Y and corresponding homomorphism $\mathcal{G}$,

$$\Psi^{-1} \circ \mathcal{G}(G^X_{n,m}) = \Psi^{-1} \circ \mathcal{G}(G^Y_{m,n})$$

as elements in $\omega(X/G)$.

First of all, Y is in $\mathcal{D}$ (by Proposition 3.18) implies that Y/G is in Sm and is equidimensional. In addition, for all i, the G-invariant divisor A_i is in $\mathcal{D}$. So, A_i/G is in Sm. Moreover, $\dim A_i/G = \dim A_i - \dim G$ implies that A_i/G is a smooth divisor on Y/G. By similar arguments,

$$A_1/G + \cdots + A_n/G + B_1/G + \cdots + B_m/G$$

is a reduced strict normal crossing divisor on Y/G. On the other hand, by definition, there exists $f \in \mathrm{H}^0(Y, \mathcal{K}^*)^G$ such that

$$A_1 + \cdots + A_n - B_1 - \cdots - B_m = \operatorname{div} f.$$

By the fact that $\mathrm{H}^0(Y, \mathcal{K}^*)^G \cong \mathrm{H}^0(Y/G, \mathcal{K}^*)$, we can consider f as an element in $\mathrm{H}^0(Y/G, \mathcal{K}^*)$ and deduce that

$$A_1/G + \cdots + A_n/G - B_1/G - \cdots - B_m/G = \operatorname{div} f.$$

By Proposition 3.19, $\phi/G : Y/G \to X/G$ is projective. Hence, we obtain a GDPR setup over X/G given by $\phi/G : Y/G \to X/G$ with divisors $A_1/G, \ldots, A_n/G$, $B_1/G, \ldots, B_m/G$ on Y/G. Let $\mathcal{G}'$ be the corresponding homomorphism. By Corollary 3.10,

$$\mathcal{G}'(G^X_{n,m}) = \mathcal{G}'(G^Y_{m,n})$$

in $\omega(X/G)$. So, it will be enough to show $\mathcal{G}' = \Psi^{-1} \circ \mathcal{G}$. We will need the following claim first.

Claim : For morphisms $Z \to X$ and $Z' \to X$ with $X,\ Z,\ Z' \in \mathcal{D}$, we have the following isomorphism :

$$(Z \times_X Z')/G \cong Z/G \times_{X/G} Z'/G.$$

Notice that

$$\begin{aligned}
(Z/G \times_{X/G} Z'/G) \times_{X/G} X &\cong Z/G \times_{X/G} (Z'/G \times_{X/G} X) \\
&\cong Z/G \times_{X/G} Z' \\
&\quad \text{(by Proposition 3.19)} \\
&\cong Z/G \times_{X/G} X \times_X Z' \\
&\cong Z \times_X Z' \\
&\quad \text{(by Proposition 3.19).}
\end{aligned}$$

Again, by Proposition 3.19, we get

$$Z/G \times_{X/G} Z'/G \cong ((Z/G \times_{X/G} Z'/G) \times_{X/G} X)\,/G \cong (Z \times_X Z')/G.$$

The proves the claim. $\triangle$

Consider a general term $X_i \cdots U_k^p \cdots$ in $\mathcal{R}$. On one hand,

$$\begin{aligned}
\mathcal{G}'(X_i \cdots U_k^p \cdots) &= [A_i/G \times_{Y/G} \cdots \times_{Y/G} (P_k^p)' \times_{Y/G} \cdots \to X/G] \\
&\quad \text{where } (P_k^p)' \text{ is the corresponding tower defined by } \{A_i/G\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Psi^{-1} \circ \mathcal{G}(X_i \cdots U_k^p \cdots) &= \Psi^{-1}[A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots \to X] \\
&\quad \text{where } P_k^p \text{ is the corresponding tower defined by } \{A_i\} \\
&= [(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots)/G \to X/G] \\
&= [A_i/G \times_{Y/G} \cdots \times_{Y/G} P_k^p/G \times_{Y/G} \cdots \to X/G] \\
&\quad \text{(by the claim).}
\end{aligned}$$

Thus, it remains to show $(P_k^p)' \cong P_k^p/G$. Consider the case when $p = 1$. Let D be the divisor $A_1 + \cdots + A_k$. Then, we have

$$\begin{aligned} (P_k^1)' \times_{Y/G} Y &\cong \mathbb{P}(\,\pi^*(\mathcal{O}_{Y/G} \oplus \mathcal{O}_{Y/G}(D/G))\,) \\ &\cong \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(D)) \\ &= P_k^1. \end{aligned}$$

By Proposition 3.19, we have $(P_k^1)' \cong P_k^1/G$. Similarly, $(P_k^p)' \cong P_k^p/G$ for $p = 2, 3$. □

When X is an object in $\mathcal{D}$, there are some natural formulas relating the push-forward, pull-back and external product with their non-equivariant versions.

Proposition 3.21. *Suppose $f : X' \to X$ is a morphism in $\mathcal{D}$.*

(1) *If f is projective, then f/G is projective, we have push-forward*

$$(f/G)_* : \omega(X'/G) \to \omega(X/G)$$

and

$$f_* = \Psi \circ (f/G)_* \circ \Psi^{-1}$$

as morphisms from $\mathcal{U}_G(X')$ to $\mathcal{U}_G(X)$.

(2) *If f is smooth and X, X' are both equidimensional, then f/G is smooth, we have pull-back*

$$(f/G)^* : \omega(X/G) \to \omega(X'/G)$$

and

$$f^* = \Psi \circ (f/G)^* \circ \Psi^{-1}$$

as morphisms from $\mathcal{U}_G(X)$ to $\mathcal{U}_G(X')$.

Proof. (1) First of all, f/G is projective by Proposition 3.19. Also, X/G, X'/G are both in *Sm*. Hence, the push-forward $(f/G)_* : \omega(X'/G) \to \omega(X/G)$ is well-defined. Moreover, by definition,

$$\begin{aligned}
\Psi \circ (f/G)_* \circ \Psi^{-1}\,[Y \to X'] &= \Psi \circ (f/G)_*\,[Y/G \to X'/G] \\
&= \Psi\,[Y/G \to X/G] \\
&= [Y/G \times_{X/G} X \to X] \\
&= [Y \to X].
\end{aligned}$$

(2) By the descent of smoothness (Proposition 2 of [EG]), the morphism f/G is smooth. Also, X/G, $X'/G \in \mathit{Sm}$ are both equidimensional. Hence, the pull-back $(f/G)^* : \omega(X/G) \to \omega(X'/G)$ is well-defined. Moreover,

$$\begin{aligned}
\Psi \circ (f/G)^* \circ \Psi^{-1}\,[Y \to X] &= \Psi \circ (f/G)^*\,[Y/G \to X/G] \\
&= \Psi\,[Y/G \times_{X/G} X'/G \to X'/G] \\
&= [Y/G \times_{X/G} X'/G \times_{X'/G} X' \to X'] \\
&= [Y/G \times_{X/G} X' \to X'] \\
&= [Y/G \times_{X/G} X \times_X X' \to X'] \\
&= [Y \times_X X' \to X'] \\
&\quad \text{by Proposition 3.19.}
\end{aligned}$$

□

There is a also similar formula for the external product, which is somewhat harder to state. We need some trivial facts first.

Let $\gamma : G \to H$ be a group scheme homomorphism between the group schemes G, H. Then, for all $X \in H\text{-}\mathit{Sm}$, it induces a natural abelian group homomorphism

$$\Phi_\gamma : \mathcal{U}_H(X) \to \mathcal{U}_G(X)$$

by sending $[Y \to X]$ with H-actions to $[Y \to X]$ with G-actions via γ. This homomorphism obviously respects GDPR, so Φ_γ is well-defined.

Denote the ascending homomorphism corresponding to G-action as $\Psi_G : \omega(X/G) \to \mathcal{U}_G(X)$.

Proposition 3.22. *Suppose X, X' are two objects in $\mathcal{D}$. Then, the external product*

$$\times : \mathcal{U}_G(X) \times \mathcal{U}_G(X') \to \mathcal{U}_G(X \times X')$$

of the element $(a, b) \in \mathcal{U}_G(X) \times \mathcal{U}_G(X')$ can be given by

$$a \times b = \Phi_\Delta \circ \Psi_{G\times G}(\Psi_G^{-1} a \times \Psi_G^{-1} b)$$

where $\Delta : G \to G \times G$ is the diagonal morphism.

Proof. Follows from the definition. □

4. The Chern class operator $c(\mathcal{L})$

Suppose X is an object in G-Sm and $\mathcal{L}$ is a G-linearized invertible sheaf over X. Our goal in this section is to define an abelian group homomorphism

$$c(\mathcal{L}) : \mathcal{U}_*^G(X) \to \mathcal{U}_{*-1}^G(X)$$

which satisfies some natural properties.

Recall that in section 4 of [LeP], when $\mathcal{L}$ is a globally generated invertible sheaf over a k-scheme $X \in Sm$, $c(\mathcal{L}) : \omega_*(X) \to \omega_{*-1}(X)$ is defined as follow. Let $[f : Y \to X]$ be an element in $\omega(X)$ such that Y is irreducible. Since $f^*\mathcal{L}$ is a globally generated invertible sheaf over Y, there is a smooth divisor H on Y such that $\mathcal{O}_Y(H) \cong f^*\mathcal{L}$. Then, we define $c(\mathcal{L})[f : Y \to X] \overset{def}{=} [H \hookrightarrow Y \to X]$.

It is natural to try to give a similar version in our equivariant setting. However, since there is no assumption on how the group G acts on the scheme X, there is no guarantee that even a single non-zero invariant global section of $\mathcal{L}$ can be found. For example, if the action on X is transitive, then no matter how nice a G-linearized invertible sheaf $\mathcal{L}$ over X is, there is no invariant global section that cuts out an invariant divisor. Hence, $c(\mathcal{L})[\mathbb{I} : X \to X]$ can not be defined in a similar manner.

Moreover, even if there is an invariant section cutting out a smooth invariant divisor, it may not be generic. For example, take $G \overset{def}{=} GL(2)$ and $X \overset{def}{=} \mathbb{P}^2$ with action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{def}{=} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Consider the case when $\mathcal{L} = \mathcal{O}(1)$, which is naturally G-linearized. Then, there is only one invariant section $s \in \mathrm{H}^0(X, \mathcal{L})^G$ that cuts out an invariant divisor, namely $s = z$. In this case, for a projective map $f : Y \to X$, we can not define $c(\mathcal{L})[f : Y \to X]$ by $f^*(s)$ because there is no reason to believe that H_{f^*s} (the subscheme cut out by f^*s) will be smooth, or even a divisor. So, it is important that the choice of section is generic. Indeed, we will see later that this freedom of choice is essential for the well-definedness of our Chern class operator.

4.1. **First approach.** As pointed out in the subsection 3.4, the theory $\mathcal{U}_G$ works nicely in the subcategory $\mathcal{D}$. Hence, our first approach is to restrict to this subcategory and define the Chern class operator. We first need a little lemma to ensure we stay inside the quasi-projective setup.

Lemma 4.1. *If X is quasi-projective over k and $\pi : E \to X$ is a vector bundle, then E is quasi-projective over k.*

Proof. Consider $\mathbb{P}(\mathcal{E}^\vee \oplus \mathcal{O}_X)$ where $\mathcal{E}$ is the locally free sheaf over X corresponding to E. Since $\mathbb{P}(\mathcal{E}^\vee \oplus \mathcal{O}_X) \to X$ is projective, the scheme $\mathbb{P}(\mathcal{E}^\vee \oplus \mathcal{O}_X)$ is quasi-projective. Then, E can be considered as an open set inside $\mathbb{P}(\mathcal{E}^\vee \oplus \mathcal{O}_X)$, hence is quasi-projective. □

Here is the natural definition of $c(\mathcal{L})$ when X is in $\mathcal{D}$.

Definition 4.2. Suppose X is an object in $\mathcal{D}$ and $\mathcal{L}$ is a sheaf in $\mathrm{Pic}^G(X)$. We define the Chern class operator $c(\mathcal{L}) : \mathcal{U}_*^G(X) \to \mathcal{U}_{*-1}^G(X)$ by

$$c(\mathcal{L}) \stackrel{def}{=} \Psi \circ c(\pi_* \mathcal{L}^G) \circ \Psi^{-1}$$

where $\pi : X \to X/G$ is the quotient map and $\Psi : \omega(X/G) \tilde{\to} \mathcal{U}_G(X)$ is the ascent isomorphism defined in subsection 3.4.

Since X is in $\mathcal{D}$, the sheaf $\pi_*\mathcal{L}^G$ over X/G is invertible. Hence, the abelian group homomorphism $c(\pi_*\mathcal{L}^G) : \omega_*(X/G) \to \omega_{*-1}(X/G)$ is well-defined (see sections 4 and 9 in [LeP] for more detail).

Remark 4.3. For $X \in \mathcal{D}$ and $\mathcal{L} \in \mathrm{Pic}^G(X)$ such that $\mathcal{L}$ is globally generated by invariant sections, we can construct $c(\mathcal{L})[f : Y \to X]$ by following the definitions of Ψ and $c(\pi_*\mathcal{L}^G)$.

First, descend $Y \to X$ to get $Y/G \to X/G$. Then, $(f/G)^*(\pi_*\mathcal{L}^G)$ will be a globally generated invertible sheaf over Y/G (A G-linearized invertible sheaf $\mathcal{L}$ being globally generated by invariant sections is equivalent to $\pi_*\mathcal{L}^G$ being globally generated). Pick a global section

$s \in \mathrm{H}^0(Y/G, (f/G)^*(\pi_*\mathcal{L}^G))$ that cuts out a smooth divisor H_s on Y/G. Then, ascend $H_s \to Y/G \to X/G$ to obtain $[H_s \times_{X/G} X \to X]$. Thus,

$$c(\mathcal{L})[f : Y \to X] = [H_s \times_{X/G} X \to X].$$

It can be seen that $c(\mathcal{L})[f : Y \to X]$ can also be obtained in the following way. Since $\mathcal{L}$ is globally generated by invariant sections, $f^*\mathcal{L}$ is also globally generated by invariant sections. Pick a section $s' \in \mathrm{H}^0(Y, f^*\mathcal{L})^G$ that cuts out an invariant smooth divisor $H_{s'}$ on Y. Then,

$$c(\mathcal{L})[f : Y \to X] = [H_{s'} \to Y \to X].$$

Because of the natural isomorphism between $\mathcal{U}_G(X)$ and $\omega(X/G)$ when X is in $\mathcal{D}$, we can now easily show the equivariant versions of some properties of the Chern class operator listed in [LeP], namely **(A3)-(A5)**, **(A8)**, **(Dim)**, etc.

4.2. **Second approach.** Instead of imposing a restriction on X, we may impose a restriction on $\mathcal{L}$. Our second approach is to first define the notion of a "nice" G-linearized invertible sheaf. Then, we define the Chern class operator for "nice" sheaves $\mathcal{L}$ and extend this definition to more general G-linearized invertible sheaves through the formal group law.

Before proceeding to describe this second approach, let us recall the definition of the formal group law and some basic properties.

We denote the Lazard ring by $\mathbb{L}$ (see section 1.1 in [LeMo]). Let $\{a_{ij}\}$ with $i, j \geq 0$ and $(i, j) \neq (0, 0)$ be the standard set of generators of the Lazard ring, i.e. $\mathbb{L} = \mathbb{Z}[a_{ij}]$. Then, the formal group law F is the power series in $\mathbb{L}[[u, v]]$:

$$F(u, v) = \sum_{i,j \geq 0} a_{ij} u^i v^j = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j$$

(see section 2.4.3 in [LeMo]). To help our intuition, we will think of the formal group law as giving "addition". By definition, we have

$$\begin{aligned} F(u, 0) &= u. \\ F(u, v) &= F(v, u). \\ F(u, F(v, w)) &= F(F(u, v), w) \end{aligned}$$

and the relations on a_{ij} are the ones imposed by these equalities.

Moreover, there is a power series $\chi(u) \in \mathbb{L}[[u]]$ that satisfies

$$F(u, \chi(u)) = 0.$$

The power series $\chi(u)$ can be regarded as giving the "inverse" of u. Hence, we can define "subtraction" by

$$F^{-}(u,v) \overset{def}{=} F(u, \chi(v)).$$

For our purpose, we also need the notion of "multiplication by a positive integer" :

$$F^{n}(u) \overset{def}{=} F(u, F(u, \cdots F(u,u) \cdots))$$

$$(n-1 \text{ times application of } F)$$

Finally, we will need the notion "division by a positive integer". For simplicity, denote $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{n}]$ by $\mathbb{L}_n$. The Lazard's Theorem states that $\mathbb{L}$ is a polynomial algebra over integers with infinitely many generators (see [L]). In particular, $\mathbb{L}$ has no torsion and $\mathbb{L} \hookrightarrow \mathbb{L}_n$.

Lemma 4.4. *For all $n \geq 1$, there exists a power series in $\mathbb{L}_n[[u]]$, denoted by $F^{1/n}(u)$, such that*

$$F^{1/n}(F^{n}(u)) = F^{n}(F^{1/n}(u)) = u.$$

Proof. Let $F^{n}(u) \overset{def}{=} \sum_{i \geq 1} a_i u^i$ for some $a_i \in \mathbb{L}$.

Claim : $a_1 = n$.

We proceed by induction on n. Obviously, the claim is true for $n = 1$. Suppose the claim is true for $n-1$. Notice that we can always ignore terms with degree of u greater than 1. Hence,

$$\begin{aligned} F^{n}(u) &= F(u, F^{n-1}(u)) \\ &= u + F^{n-1}(u) + \text{higher degree terms} \\ &= u + (n-1)(u) + \cdots \\ &= nu + \cdots . \end{aligned}$$

That proves the claim. $\triangle$

Let $F^{1/n}(u) \stackrel{def}{=} \sum_{i\geq 1} b_i u^i \in \mathbb{L}_n[[u]]$ with coefficients $\{b_i\}$ yet to be determined. The equality we want is

$$u = F^{1/n}(F^n(u)) = b_1(a_1u + a_2u^2 + \cdots) + b_2(a_1u + a_2u^2 + \cdots)^2 + \cdots .$$

That gives us the following set of equations :

$$\begin{aligned} 1 &= b_1a_1 \\ 0 &= b_1a_2 + b_2a_1^2 \\ 0 &= b_1a_3 + b_2 2a_1a_2 + b_3a_1^3, \text{ etc.} \end{aligned}$$

Thus, we have $b_1 = 1/a_1 = 1/n \in \mathbb{L}_n$. After $b_1, \ldots, b_{i-1}$ are determined, we can define $b_i \in \mathbb{L}_n$ by the equation with respect to u^i and the fact that the term corresponding to b_i is just $b_i a_1^i = n^i b_i$. That gives us a power series $F^{1/n}(u) \in \mathbb{L}_n[[u]]$ such that $u = F^{1/n}(F^n(u))$.

To show the second equality $F^n(F^{1/n}(u)) = u$, let $F^n(F^{1/n}(u)) \stackrel{def}{=} \sum_{i\geq 1} c_i u^i$. Then,

$$\begin{aligned} b_1u + b_2u^2 \cdots &= F^{1/n}(u) \\ &= F^{1/n}(F^n(F^{1/n}(u))) \\ &= F^{1/n}(\sum_{i\geq 1} c_i u^i) \\ &= b_1(c_1u + c_2u^2 + \cdots) + b_2(c_1u + c_2u^2 + \cdots)^2 + \cdots . \end{aligned}$$

By comparing the coefficients, we obtain the following set of equations :

$$\begin{aligned} b_1 &= b_1c_1 \\ b_2 &= b_1c_2 + b_2c_1^2 \\ b_3 &= b_1c_3 + b_2 2c_1c_2 + b_3c_1^3, \text{ etc.} \end{aligned}$$

Since $b_1 = \frac{1}{n}$, the first equation implies $c_1 = 1$. Substituting $c_1 = 1$ into the second equation implies that $c_2 = 0$. Inductively, $c_i = 0$ for all $i \geq 2$. Hence, $F^n(F^{1/n}(u)) = u$. $\square$

Remark 4.5. By examining the proof carefully, it can be shown that if $F^{1/n}(u) = \sum_{i\geq 1} b_i u^i$, then $n^{i(i+1)/2} b_i \in \mathbb{L}$.

As mentioned at the beginning of this subsection, we will start by defining the notion of a nice G-equivariant invertible sheaf.

Definition 4.6. Suppose X is an object in G-Sm and $\mathcal{L}$ is a sheaf in $\mathrm{Pic}^G(X)$. We say that $\mathcal{L}$ is nice if there exists a morphism in G-Sm, $\psi : X \to \mathbb{P}^n$ (with trivial G-action on $\mathbb{P}^n$) such that $\mathcal{L} \cong \psi^*\mathcal{O}(1)$.

Here are some basic properties.

Lemma 4.7. *Suppose X is an object in G-Sm.*

1. *The structure sheaf $\mathcal{O}_X$ is nice.*
2. *If the sheaves $\mathcal{L}$, $\mathcal{L}' \in \mathrm{Pic}^G(X)$ are both nice, then $\mathcal{L} \otimes \mathcal{L}'$ is also nice.*
3. *If $f : X \to Y$ is a morphism in G-Sm and $\mathcal{L} \in \mathrm{Pic}^G(Y)$ is nice, then $f^*\mathcal{L}$ is nice.*

Proof. 1. By considering the map $\psi : X \to \mathbb{P}^0 \cong \operatorname{Spec} k$.

2. Suppose we have two morphisms $\psi : X \to \mathbb{P}^n$ and $\psi' : X \to \mathbb{P}^m$ such that $\psi^*\mathcal{O}(1) \cong \mathcal{L}$ and $\psi'^*\mathcal{O}(1) \cong \mathcal{L}'$. Let ψ'' be the following composition :

$$X \xrightarrow{\psi\times\psi'} \mathbb{P}^n \times \mathbb{P}^m \xrightarrow{Segre} \mathbb{P}^N.$$

Then, $\psi''^*\mathcal{O}(1) \cong \mathcal{L} \otimes \mathcal{L}'$.

3. By definition. □

We will start with a definition of the Chern class operator which depends on ψ. Suppose that $\mathcal{L}$ is a sheaf in $\mathrm{Pic}^G(X)$ and there is a map $\psi : X \to \mathbb{P}^n$ such that $\psi^*\mathcal{O}(1) \cong \mathcal{L}$. We would like to define $c_\psi(\mathcal{L})[f : Y \to X]$ as $[Y \times_{\mathbb{P}^n} H \to Y \to X]$ where H is a hyperplane in $\mathbb{P}^n$ such that $Y \times_{\mathbb{P}^n} H$ is a smooth invariant divisor on Y. Clearly, it is enough to consider the case when Y is G-irreducible. In what follows, we will show that this is well-defined, i.e. that such an H exists, that this element is independent of the choice of H and that the construction respects GDPR.

Lemma 4.8. *Denote the dual projective space $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)))$ by $(\mathbb{P}^n)^*$. Then, there is a non-empty open set U in $(\mathbb{P}^n)^*$ such that for any section s in U, the closed subscheme*

$Y \times_{\mathbb{P}^n} H \subseteq Y$, where H is the hyperplane in $\mathbb{P}^n$ cut out by the section s, is a smooth invariant divisor on Y .

Proof. This is a variation of the Bertini's Theorem when $\operatorname{char} k = 0$. We have $f : Y \to X$ and $\psi : X \to \mathbb{P}^n$ as above. Let $\mathcal{H}$ be the analog of the universal Cartier divisor, i.e.

$$\mathcal{H} \overset{def}{=} \{ (y, s) \mid s(\psi \circ f(y)) = 0 \} \subseteq Y \times (\mathbb{P}^n)^*.$$

Claim : $\mathcal{H}$ is smooth and of dimension $\dim Y + n - 1$.

Let $\mathbb{P}^n = \operatorname{Proj} k[x_0, \ldots, x_n]$ and $(\mathbb{P}^n)^* = \operatorname{Proj} k[c_0, \ldots, c_n]$. Let $D(x_i)$ be the affine open subscheme of $\mathbb{P}^n$ given by $x_i \neq 0$ and similarly for $D(c_i)$. Also, let $\operatorname{Spec} A$ be an affine open subscheme of $(\psi \circ f)^{-1}(D(x_i))$. Then, $\psi \circ f$ is locally given by a map $\operatorname{Spec} A \to \operatorname{Spec} k[x_0/x_i, \ldots, x_n/x_i]$, which corresponds to sending the elements x_j/x_i to some elements $a_j \in A$. So, the universal Cartier divisor $\mathcal{H}$ is locally given by the equation $\sum_{j \neq i}(c_j/c_i)a_j = 0$ inside $\operatorname{Spec} A \times D(c_i)$. Hence, the claim is true. $\triangle$

Consider the projection $\mathcal{H} \to (\mathbb{P}^n)^*$. For a section $s \in (\mathbb{P}^n)^*$, the fiber is exactly $Y \times_{\mathbb{P}^n} H$ where H is the hyperplane cut out by s. Hence, the open set we want will be the set of regular values of this projection map. $\square$

Lemma 4.9. *Let s, s' be two sections in $(\mathbb{P}^n)^*$, cutting out H, H' respectively, such that $Y \times_{\mathbb{P}^n} H$ and $Y \times_{\mathbb{P}^n} H'$ are both smooth invariant divisors on Y. Then we have*

$$[Y \times_{\mathbb{P}^n} H \to X] = [Y \times_{\mathbb{P}^n} H' \to X]$$

as elements in $\mathcal{U}_G(X)$.

Proof. Observe that H, H' are equivariantly linearly equivalent divisors on $\mathbb{P}^n$. Thus,

$$Y \times_{\mathbb{P}^n} H = (\psi \circ f)^* H \sim (\psi \circ f)^* H' = Y \times_{\mathbb{P}^n} H'$$

as invariant divisors on Y. The result then follows from $GDPR(1, 1)$. $\square$

Lemma 4.10. *Sending $[Y \to X]$ to $[Y \times_{\mathbb{P}^n} H \to X]$ defines an abelian group homomorphism from $\mathcal{U}^G_*(X)$ to $\mathcal{U}^G_{*-1}(X)$.*

Proof. As before, let $\mathcal{G}$ be the map corresponding to a GDPR setup $Y \to X$ with divisors $A_1, \dots, A_n$, $B_1, \dots, B_m$ on Y. We need to show

$$c_\psi(\mathcal{L}) \circ \mathcal{G}(G^X_{n,m}) = c_\psi(\mathcal{L}) \circ \mathcal{G}(G^Y_{m,n}).$$

For simplicity, we will denote $X \times_{\mathbb{P}^n} H$ by X_H. Consider the projective morphism $Y_H \to X_H$. By the freedom of choice of H, we may assume X_H is a smooth invariant divisor on X and the same for Y_H. In particular, Y_H, X_H are both in *G-Sm* and Y_H is equidimensional. Similarly, we may assume the same property holds for A_{iH} and B_{jH} and also,

$$A_{1H} + \cdots + A_{nH} + B_{1H} + \cdots + B_{mH}$$

is a reduced strict normal crossing divisor on Y_H. Since the divisors are given by pull-back along $Y_H \to Y$, we have

$$A_{1H} + \cdots + A_{nH} \sim B_{1H} + \cdots + B_{mH}.$$

Thus, we can define a map $\mathcal{G}' : \mathcal{R} \to \mathcal{U}_G(X_H)$ by the GDPR setup $Y_H \to X_H$ with $A_{1H} + \cdots + A_{nH} \sim B_{1H} + \cdots + B_{mH}$. So, it is enough to show

$$c_\psi(\mathcal{L}) \circ \mathcal{G} = i_* \circ \mathcal{G}'$$

where $i : X_H \hookrightarrow X$.

For a general term $X_i \cdots U^p_k \cdots$,

$$\begin{aligned} c_\psi(\mathcal{L}) \circ \mathcal{G}(X_i \cdots U^p_k \cdots) &= c_\psi(\mathcal{L})[A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots \to X] \\ &= [(A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots)_H \to X] \\ &= [A_{iH} \times_{Y_H} \cdots \times_{Y_H} (P^p_k)_H \times_{Y_H} \cdots \to X_H \to X]. \end{aligned}$$

Hence, it is enough to show $(P^p_k)_H$ is the same as the corresponding tower given by invariant divisors $\{A_{iH}\}$. The $p = 1$ case follows from the fact that

$$\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(D))_H \cong \mathbb{P}(\mathcal{O}_{Y_H} \oplus \mathcal{O}_{Y_H}(D_H))$$

and the $p = 2,3$ cases can be proved similarly. That shows the well-definedness of the homomorphism. The fact that it sends $\mathcal{U}^G_*(X)$ to $\mathcal{U}^G_{*-1}(X)$ is clear. $\square$

Hence, we have the following definition.

Definition 4.11. Suppose that $\mathcal{L}$ is a sheaf in $\mathrm{Pic}^G(X)$ such that there exists an equivariant morphism $\psi : X \to \mathbb{P}^n$ with $\psi^*\mathcal{O}(1) \cong \mathcal{L}$. We define the Chern class operator $c_\psi(\mathcal{L}) : \mathcal{U}^G_*(X) \to \mathcal{U}^G_{*-1}(X)$ by

$$c_\psi(\mathcal{L})[f : Y \to X] \overset{def}{=} [Y \times_{\mathbb{P}^n} H \to Y \to X]$$

where H is a hyperplane in $\mathbb{P}^n$ such that $Y \times_{\mathbb{P}^n} H$ is an invariant smooth divisor on Y.

We definitely do not want the definition of the Chern class operator to depend on the particular morphism $\psi : X \to \mathbb{P}^n$.

Lemma 4.12. *$c_\psi(\mathcal{L})$ is independent of ψ.*

Proof. Suppose we have two equivariant morphisms $\psi_1 : X \to \mathbb{P}^n$ and $\psi_2 : X \to \mathbb{P}^m$ such that $\psi_1^*\mathcal{O}(1) \cong \mathcal{L} \cong \psi_2^*\mathcal{O}(1)$. Consider the pull-back of sections

$$\psi_1^* : \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \to \mathrm{H}^0(X, \mathcal{L}).$$

Then, the image of ψ_1^* will lie in $\mathrm{H}^0(X,\mathcal{L})^G$ and the same for ψ_2. Let $\{s_{1i}\}$ be a k-basis for $\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$ and $\{s_{2j}\}$ be a k-basis for $\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}(1))$. Then, $k-\mathrm{span}\{\psi_1^* s_{1i},\ \psi_2^* s_{2j}\}$ will be a finite dimensional vector space in $\mathrm{H}^0(X,\mathcal{L})^G$. In addition, it is base-point free. This defines an equivariant morphism $\psi_3 : X \to \mathbb{P}^N$ which can be factored as $X \to \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ or $X \to \mathbb{P}^m \hookrightarrow \mathbb{P}^N$. Also, $\psi_3^*\mathcal{O}(1) \cong \mathcal{L}$. Thus, it is enough to show $c_{\psi_1}(\mathcal{L}) = c_{\psi_3}(\mathcal{L})$.

Consider an element $[Y \to X]$ in $\mathcal{U}_G(X)$. Pick a hyperplane $H \subseteq \mathbb{P}^N$ such that $\mathbb{P}^n \cap H$ is a hyperplane in $\mathbb{P}^n$ (this is equivalent to $\mathbb{P}^n \times_{\mathbb{P}^N} H$ being a smooth divisor on $\mathbb{P}^n$) and $Y \times_{\mathbb{P}^N} H$ is a smooth divisor on Y. Then,

$$\begin{aligned} c_{\psi_1}(\mathcal{L})[Y \to X] &= [Y \times_{\mathbb{P}^n} (\mathbb{P}^n \cap H)] \\ &= [Y \times_{\mathbb{P}^N} H] \\ &= c_{\psi_3}(\mathcal{L})[Y \to X]. \end{aligned}$$

□

Hence, for a nice G-linearized invertible sheaf $\mathcal{L}$ over $X \in G\text{-}Sm$, we have a natural definition of the Chern class operator

$$c(\mathcal{L}) : \mathcal{U}^G_*(X) \to \mathcal{U}^G_{*-1}(X).$$

4.3. **Special pull-back and the formal group law.** Recall that in the ω_* theory in [LeP], we have the following property (Proposition 9.4 in [LeP]).

For any $X \in Sm$ and invertible sheaves $\mathcal{L}$, $\mathcal{M}$ over X, we have

$$c(\mathcal{L} \otimes \mathcal{M}) = F(c(\mathcal{L}), c(\mathcal{M}))$$

as abelian group endomorphisms on $\omega(X)$ where $F \in \mathbb{L}[[u,v]] \cong \omega(\operatorname{Spec} k)[[u,v]]$ is the formal group law with $\omega(X)$ considered as a $\omega(\operatorname{Spec} k)$-module by the external product. Since the Chern class operator always cuts down the dimension of the domain by one, $F(c(\mathcal{L}), c(\mathcal{M}))$ indeed acts as a finite sum on any given element in $\omega(X)$.

We will follow the notation in [LeP] and denote this property by **(FGL)**. Our objective in this subsection is to prove it holds in our equivariant setting, when all G-linearized invertible sheaves involved are nice. First of all, we will need some basic facts.

Proposition 4.13. *Suppose $f : Y \to X$ is a morphism in G-Sm. Then, there exists a G-representation V and an equivariant immersion $i : Y \hookrightarrow \mathbb{P}(V) \times X$ such that $f = \pi_2 \circ i$. If we further assume f to be projective, then i will be a closed immersion.*

Proof. First, assume that G is reductive and connected. Since Y is quasi-projective, there exists an (not necessarily equivariant) immersion $i_0 : Y \hookrightarrow \mathbb{P}^n$. Define $\mathcal{L} \overset{def}{=} i_0^*\mathcal{O}(1)$ as an (not necessarily G-linearized) invertible sheaf over Y. By Theorem 1.6 in [Su], there exists an integer m such that $\mathcal{L}^m$ is G-linearizable. Fix a G-linearization of $\mathcal{L}^m$. Since we have a G-linearized very ample invertible sheaf $\mathcal{L}$ over Y, by Proposition 1.7 in [MuFoKi], there exists an equivariant immersion $i_1 : Y \hookrightarrow \mathbb{P}(V)$ for some G-representation V such that $i_1^*O(1) \cong \mathcal{L}^m$. Then, the map $i_1 \times f : Y \to \mathbb{P}(V) \times X$ will be the equivariant immersion we want.

Now assume that G is finite. As above, $\mathcal{L} = i_0^*\mathcal{O}(1)$ is a very ample invertible sheaf over Y. Then, $\otimes_{\alpha\in G}\ \alpha^*\mathcal{L}$ will be a G-linearized very ample invertible sheaf over Y, which gives us the equivariant immersion i_1.

If f is projective, then the image of $i = i_1 \times f$ will be a closed subscheme of $\mathbb{P}(V) \times X$. □

Suppose X is a scheme over k and U is a subscheme of X. We will denote the closure of U in X by $\mathrm{clos}_X U$. Also denote the singular locus of X by $\mathrm{Sing}(X)$.

Proposition 4.14. *(Equivariant immersion with smooth closure)*

(1) *If Y is an object in G-Sm, then there exists a G-representation V where Y can be equivariantly embedded into $\mathbb{P}(V)$ such that its closure is smooth.*

(2) *Suppose X, Y are objects in G-Sm and $U \subseteq X$ is an invariant open subscheme. If a morphism $f : Y \to U$ in G-Sm is equivariant and projective, then there exists a G-representation V, an equivariant closed immersion $i : Y \hookrightarrow U \times \mathbb{P}(V)$ such that $f = \pi_1 \circ i$, and $\mathrm{clos}_{X\times\mathbb{P}(V)} Y$ is smooth.*

Proof. (1) By Proposition 4.13, we may assume there exists an equivariant immersion $Y \hookrightarrow \mathbb{P}(V')$ for some G-representation V'. By the canonical resolution of singularities (Theorem 1.6 in [BiMi]), for any variety Z over k ($\mathrm{char\,k} = 0$), there exists a smooth variety Z^{res} and a morphism $Z^{res} \to Z$ which is given by a series of blowups along canonically chosen smooth centers. As pointed out in Remarks 4-1-1 in [M], since the blowups are canonical, Z^{res} has a natural G-action and $Z^{res} \to Z$ will be G-equivariant. Apply this on our case by setting $Z \stackrel{def}{=} \mathrm{clos}_{\mathbb{P}(V')} Y$, then we have an equivariant morphism $\pi : Z^{res} \to Z$.

First of all, since Y is smooth, π is an isomorphism away from $\mathrm{Sing}(Z) \subseteq Z - Y$. That implies the equivariant immersion $Y \hookrightarrow Z$ lifts to an equivariant immersion $Y \hookrightarrow Z^{res}$ and $\mathrm{clos}_{Z^{res}} Y = Z^{res}$. Moreover, Z^{res} is projective because π is projective and Z is projective. By Proposition 4.13, Z^{res} can be equivariantly embedded into $\mathbb{P}(V)$ for some G-representation V. Hence, we have $Y \hookrightarrow Z^{res} \hookrightarrow \mathbb{P}(V)$ such that $\mathrm{clos}_{\mathbb{P}(V)} Y = Z^{res}$ is smooth.

(2) Since $f : Y \to U$ is projective, by Proposition 4.13, there exists an equivariant immersion $i' : Y \hookrightarrow U \times \mathbb{P}(V')$ for some G-representation V' such that $f = \pi_1 \circ i'$. Consider $U \times \mathbb{P}(V')$ as an invariant open subscheme in $X \times \mathbb{P}(V')$ and let $Z \stackrel{def}{=} \mathrm{clos}_{X\times\mathbb{P}(V')} Y$. By

canonical resolution of singularities as above, we have an equivariant projective morphism $Z^{res} \to Z$. By considering $Z^{res} \to Z \hookrightarrow X \times \mathbb{P}(V') \to X$, we know that $Z^{res} \to X$ is equivariant and projective. By Proposition 4.13, there exists an equivariant immersion $Z^{res} \hookrightarrow X \times \mathbb{P}(V)$ for some G-representation V. Again, the equivariant immersion $Y \hookrightarrow Z$ lifts to $Y \hookrightarrow Z^{res}$ and we have $Y \hookrightarrow Z^{res} \hookrightarrow X \times \mathbb{P}(V)$ where $\mathrm{clos}_{X \times \mathbb{P}(V)} Y = Z^{res}$ is smooth. Consider the following commutative diagram :

$$\begin{array}{ccccc} Z^{res} & \hookrightarrow & X \times \mathbb{P}(V) & & \\ \downarrow & & & \searrow & \\ Z & \hookrightarrow & X \times \mathbb{P}(V') & \to & X. \end{array}$$

Consider its restriction over U. Then, we obtain the following commutative diagram :

$$\begin{array}{ccccc} Z^{res}|_U \cong Y & \hookrightarrow & U \times \mathbb{P}(V) & & \\ \downarrow & & & \searrow & \\ Z|_U \cong Y & \hookrightarrow & U \times \mathbb{P}(V') & \to & U. \end{array}$$

That gives us an equivariant closed immersion $i : Y \hookrightarrow U \times \mathbb{P}(V)$ such that the closure $\mathrm{clos}_{X \times \mathbb{P}(V)} Y = Z^{res}$ is smooth. Moreover, the composition $\pi_1 \circ i$ is given by

$$Y \tilde{\to} Z^{res}|_U \tilde{\to} Z|_U \cong Y \hookrightarrow U \times \mathbb{P}(V') \to U,$$

which is $\pi_1 \circ i' = f$. □

In order to prove the **(FGL)** property , we need some reduction of arguments, which requires the following special type of pull-back.

Let $\psi : X \to \prod_i \mathbb{P}^{n_i}$ be a G-equivariant morphism where $X \in G\text{-}Sm$ is equidimensional and the G-action on $\prod_i \mathbb{P}^{n_i}$ is trivial. We are going to define $\psi^* : \mathcal{U}_G(\prod_i \mathbb{P}^{n_i}) \to \mathcal{U}_G(X)$. Our proof is basically the equivariant version of Lemma 6.1 in [LeP]. Let Q be the group scheme $\prod_i GL(n_i + 1)$ which acts on $\prod_i \mathbb{P}^{n_i}$ naturally. We consider Q as a variety with trivial G-action, so Q is in G-*Sm*.

Lemma 4.15. *Let $f : Y \to \prod_i \mathbb{P}^{n_i}$ be a projective morphism in G-Sm such that Y is G-irreducible.*

(1) *There exists a non-empty open subscheme $U(\psi, f) \subseteq Q$ such that, for all closed points $\beta \in U(\psi, f)$, the morphisms $\beta \cdot \psi$ and f are transverse.*

(2) *For any two closed points $\beta, \beta' \in U(\psi, f)$, we have*

$$[X \times_{\beta \cdot \psi} Y \to X] = [X \times_{\beta' \cdot \psi} Y \to X]$$

as elements in $\mathcal{U}_G(X)$.

Proof. (1) First of all, $\beta \cdot \psi$ is G-equivariant because $\beta : \prod_i \mathbb{P}^{n_i} \tilde{\to} \prod_i \mathbb{P}^{n_i}$ is trivially G-equivariant. Define a map $Q \times X \to \prod_i \mathbb{P}^{n_i}$ by $(\beta, x) \mapsto \beta \cdot \psi(x)$, which is clearly G-equivariant. In addition, since Q acts on $\prod_i \mathbb{P}^{n_i}$ transitively, the map

$$T_\beta Q \oplus T_x X = T_{(\beta,x)}(Q \times X) \to T_{\beta\psi(x)}(\prod_i \mathbb{P}^{n_i})$$

is surjective ($T_x X$ means the tangent space of X at x). Since the domain and codomain are both smooth, by Proposition 10.4 in Ch. III in [Ha] ($\operatorname{char} \mathrm{k} = 0$), the map $Q \times X \to \prod_i \mathbb{P}^{n_i}$ is smooth. That implies $(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y$ is smooth.

Let $(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y \to Q$ be the projection. If a closed point $\beta \in Q$ is a regular value, then $((Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y)_\beta = X \times_{\beta \cdot \psi} Y$ is smooth and

$$\begin{aligned}
\dim X \times_{\beta \cdot \psi} Y &= \dim((Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y)_\beta \\
&= \dim(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y - \dim Q \\
&= \dim Q \times X + \dim Y - \dim \prod_i \mathbb{P}^{n_i} - \dim Q \\
&= \dim X + \dim Y - \dim \prod_i \mathbb{P}^{n_i}.
\end{aligned}$$

In other words, f and $\beta \cdot \psi$ are transverse. Hence, the open set $U(\psi, f)$ we want is just the set of regular values of $(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y \to Q$.

(2) Consider the following commutative diagram :

$$\begin{array}{ccccccc}
(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y & \longrightarrow & Q \times X & \longrightarrow & Q & \longrightarrow & \mathbb{A}^N \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
(U \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y & \longrightarrow & U \times X & \longrightarrow & U \overset{def}{=} Q \cap \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \overset{def}{=} \text{line through } \beta, \beta'
\end{array}$$

where the group scheme $Q = \prod_i GL(n_i + 1)$ is considered as an open subscheme of $\mathbb{A}^N$ for some large N (trivial G-action on $\mathbb{A}^N$). Notice that U is a non-empty open subscheme of $\mathbb{A}^1$.

All maps in the diagram are trivially G-equivariant. The morphism $(Q \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y \to Q \times X$ is projective because it is an extension from f. By using a smaller U (as long as $U \subseteq U(\psi, f)$), we can assume the projection map

$$(U \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y \to U$$

to be smooth. Hence, $(U \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y$ is smooth. Notice that the fibers are

$$((U \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y)_\beta = X \times_{\beta \cdot \psi} Y.$$

Denote the map

$$Z \stackrel{def}{=} (U \times X) \times_{\prod_i \mathbb{P}^{n_i}} Y \to U \times X$$

by g. Then, g is a projective morphism in G-Sm. In addition, Z is equidimensional because U is equidimensional and $Z \to U$ is smooth. By Proposition 4.14, there exists a G-equivariant closed immersion $i : Z \hookrightarrow (U \times X) \times \mathbb{P}(V)$ for some G-representation V such that $g = \pi_1 \circ i$ and the closure of Z in $(\mathbb{P}^1 \times X) \times \mathbb{P}(V)$ is smooth. Let us denote this closure by $\overline{Z}$. Thus, we obtain a projective morphism $\overline{Z} \to \mathbb{P}^1 \times X \to X$ in G-Sm such that the fibers of $\overline{Z}$ over $\beta, \beta' \in \mathbb{P}^1$ agree with the fibers of Z over β, β', namely $\overline{Z}_\beta = Z_\beta$ and $\overline{Z}_{\beta'} = Z_{\beta'}$. Since β, β' can be considered as G-invariant divisors on $\mathbb{P}^1$ and they are G-equivariantly linearly equivalent, we have $\overline{Z}_\beta \sim \overline{Z}_{\beta'}$, as G-invariant divisors on $\overline{Z}$. Hence, by $GDPR(1,1)$,

$$[X \times_{\beta \cdot \psi} Y \to X] = [\overline{Z}_\beta \to X] = [\overline{Z}_{\beta'} \to X] = [X \times_{\beta' \cdot \psi} Y \to X].$$

□

We will define the special pull-back $\psi^* : \mathcal{U}_G^*(\prod_i \mathbb{P}^{n_i}) \to \mathcal{U}_G^*(X)$ by sending the element $[f : Y \to \prod_i \mathbb{P}^{n_i}]$ to $[X \times_{\beta \cdot \psi} Y \to X]$ with $\beta \in U(\psi, f)$. Its well-definedness is given by the following Lemma.

Lemma 4.16. *Sending $[f : Y \to \prod_i \mathbb{P}^{n_i}]$ to $[X \times_{\beta \cdot \psi} Y \to X]$ defines an abelian group homomorphism from $\mathcal{U}_G^*(\prod_i \mathbb{P}^{n_i})$ to $\mathcal{U}_G^*(X)$.*

Proof. This proof is roughly the same as the proof of the well-definedness of $c_\psi(\mathcal{L})$. We need to show it respects GDPR. This can be achieved by using the fact that the choice of β in the group Q is generic which is similar to the generic choice of H in $\mathbb{P}^n$ in the other proof.

As before, let $\mathcal{G}$ be the map corresponding to a GDPR setup $\phi : Y \to \prod_i \mathbb{P}^{n_i}$ with G-invariant divisors $A_1, \ldots, A_n,\ B_1, \ldots, B_m$ on Y. Consider the following commutative diagram :

$$\begin{array}{ccc} Y' \stackrel{def}{=} Y \times_{\prod_i \mathbb{P}^{n_i}} X & \xrightarrow{(\beta\cdot\psi)'} & Y \\ \phi' \downarrow & & \downarrow \phi \\ X & \xrightarrow{\beta\cdot\psi} & \prod_i \mathbb{P}^{n_i} \end{array}$$

By picking $\beta \in U(\psi, \phi)$, we may assume that Y' is smooth and of dimension

$$\dim X + \dim Y - \dim \prod_i \mathbb{P}^{n_i}.$$

Similarly, there is a non-empty open subscheme $U \subseteq Q$ such that $A'_i \stackrel{def}{=} (\beta \cdot \psi)'^{-1}(A_i)$ is a smooth invariant divisor on Y' for all $\beta \in U$. By taking intersection with some more open subschemes, we may assume $A'_1 + \cdots + A'_n + B'_1 + \cdots + B'_m$ is a reduced strict normal crossing divisor on Y' for all β in some non-empty open subscheme $U' \subseteq Q$. The divisors are given by pull-back, so $A'_1 + \cdots + A'_n \sim B'_1 + \cdots + B'_m$. Thus, $\phi' : Y' \to X$ together with $A'_1, \ldots, A'_n, B'_1, \ldots, B'_m$ defines a GDPR setup over X. Denote its corresponding map by $\mathcal{G}'$.

For a general term $X_i \cdots U^p_k \cdots$,

$$\begin{aligned} \psi^* \circ \mathcal{G}(X_i \cdots U^p_k \cdots) &= \psi^*[A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots \to \prod_i \mathbb{P}^{n_i}] \\ &= [X \times_{\beta\cdot\psi} (A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots) \to X] \\ &= [(X \times_{\beta\cdot\psi} A_i) \times_{Y'} \cdots \times_{Y'} (X \times_{\beta\cdot\psi} P^p_k) \times_{Y'} \cdots \to X]. \\ &= [A'_i \times_{Y'} \cdots \times_{Y'} (X \times_{\beta\cdot\psi} P^p_k) \times_{Y'} \cdots \to X]. \end{aligned}$$

On the other hand,

$$\mathcal{G}'(X_i \cdots U^p_k \cdots) = [A'_i \times_{Y'} \cdots \times_{Y'} (P^p_k)' \times_{Y'} \cdots \to X].$$

Observe that $X \times_{\beta\cdot\psi} P^p_k = Y' \times_Y P^p_k \cong (P^p_k)'$. Hence, $\psi^* \circ \mathcal{G} = \mathcal{G}'$. □

Hence, for any G-equivariant morphism $\psi : X \to \prod_i \mathbb{P}^{n_i}$ such that X is equidimensional, we obtain a special pull-back

$$\psi^* : \mathcal{U}_G^*(\prod_i \mathbb{P}^{n_i}) \to \mathcal{U}_G^*(X)$$

which sends $[f : Y \to \prod_i \mathbb{P}^{n_i}]$ to $[X \times_{\beta\cdot\psi} Y \to X]$ where β is a closed point in Q such that $\beta \cdot \psi$ and f are transverse.

Now we can proceed to the proof of **(FGL)**. Here are a few simple properties we will need.

Lemma 4.17. *Suppose $\psi : X \to \mathbb{P}^n \times \mathbb{P}^m$ is a morphism in G-Sm such that X is equidimensional. Denote the sheaves $\pi_1^*\mathcal{O}_{\mathbb{P}^n}(1)$, $\pi_2^*\mathcal{O}_{\mathbb{P}^m}(1)$ and $\pi_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^m}(1)$ by $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$ and $\mathcal{O}(1,1)$ respectively.*

(1) *If $\mathcal{L}$ is either $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$ or $\mathcal{O}(1,1)$, then $\mathcal{L}$ is nice and*

$$\psi^* \circ c(\mathcal{L}) = c(\psi^*\mathcal{L}) \circ \psi^*$$

as morphisms from $\mathcal{U}_G(\mathbb{P}^n \times \mathbb{P}^m)$ to $\mathcal{U}_G(X)$.

(2) *The special pull-back ψ^* is a $\mathcal{U}_G(\mathrm{Spec}\, k)$-module homomorphism.*

Proof. (1) The sheaves $\mathcal{O}(1,0), \mathcal{O}(0,1)$ and $\mathcal{O}(1,1)$ are nice by definition. The equalities follow immediately from our construction.

(2) Same reason as the usual smooth pull-back. □

Lemma 4.18. *Suppose $f : X \to X'$ is a projective morphism in G-Sm and $\mathcal{L} \in \mathrm{Pic}^G(X')$ is a nice invertible sheaf, then*

$$f_* \circ c(f^*\mathcal{L}) = c(\mathcal{L}) \circ f_*$$

as morphisms from $\mathcal{U}_G(X)$ to $\mathcal{U}_G(X')$.

Proof. Let $[Y \to X]$ be an element in $\mathcal{U}_G(X)$ and $\psi : X' \to \mathbb{P}^n$ be a morphism in *G-Sm* such that $\psi^*\mathcal{O}(1) \cong \mathcal{L}$. Then,

$$\begin{aligned} c(\mathcal{L}) \circ f_*[Y \to X] &= c(\mathcal{L})[Y \to X'] \\ &= [Y \times_{\mathbb{P}^n} H \to X'] \end{aligned}$$

(fiber product via the map $Y \to X \to X' \to \mathbb{P}^n$).

On the other hand,

$$\begin{aligned} f_* \circ c(f^*\mathcal{L})[Y \to X] &= f_*[Y \times_{\mathbb{P}^n} H \to X] \\ &\quad \text{(fiber product via the map } Y \to X \to X' \to \mathbb{P}^n) \\ &= [Y \times_{\mathbb{P}^n} H \to X']. \end{aligned}$$

□

We are now ready to prove the formal group law property **(FGL)** of the Chern class operator for nice G-linearized invertible sheaves. As mentioned before, the formal group law is the power series

$$F(u, v) = \sum_{i,j \geq 0} a_{ij} u^i v^j \in \mathbb{L}[[u, v]].$$

For nice sheaves $\mathcal{L}, \mathcal{M} \in \mathrm{Pic}^G(X)$, we consider $F(c(\mathcal{L}), c(\mathcal{M}))$ as a morphism from $\mathcal{U}_*^G(X)$ to $\mathcal{U}_{*-1}^G(X)$ given by

$$\sum_{i,j \geq 0} a_{ij} c(\mathcal{L})^i \circ c(\mathcal{M})^j$$

where a_{ij} are considered as elements in $\mathcal{U}_G(\mathrm{Spec}\, k)$ via the maps

$$\mathbb{L} \cong \omega(\mathrm{Spec}\, k) \cong \mathcal{U}_{\{1\}}(\mathrm{Spec}\, k) \xrightarrow{\Phi_\gamma} \mathcal{U}_G(\mathrm{Spec}\, k)$$

where Φ_γ is induced by the group scheme homomorphism $\gamma : G \to \{1\}$ (See definition of Φ_γ in subsection 3.4. We will see that this is a ring embedding in Corollary 7.4). As in the non-equivariant theory, the Chern class operator decreases the homological grading by one. Since we have $\mathcal{U}_i^G(X) = 0$ when $i < 0$, the power series $\sum_{i,j\geq 0} a_{ij} c(\mathcal{L})^i \circ c(\mathcal{M})^j$ indeed acts as a finite sum for any given element in $\mathcal{U}^G(X)$.

Proposition 4.19. *If X is an object in G-Sm and $\mathcal{L}$, $\mathcal{M} \in \mathrm{Pic}^G(X)$ are both nice, then*

$$c(\mathcal{L} \otimes \mathcal{M}) = F(c(\mathcal{L}), c(\mathcal{M}))$$

as morphisms from $\mathcal{U}_*^G(X)$ to $\mathcal{U}_{*-1}^G(X)$.

Proof. Since $[f : Y \to X] = f_*[\mathbb{I}_Y]$, by Lemma 4.18, it is enough to prove the statement on the element $[\mathbb{I}_X]$ such that $X \in$ *G-Sm* is equidimensional.

Let $\psi_1 : X \to \mathbb{P}^n$ and $\psi_2 : X \to \mathbb{P}^m$ be the maps such that $\psi_1^*\mathcal{O}(1) \cong \mathcal{L}$ and $\psi_2^*\mathcal{O}(1) \cong \mathcal{M}$. Let $\psi : X \to \mathbb{P}^n \times \mathbb{P}^m$ be the map defined by ψ_1 and ψ_2. Then,

$$\begin{aligned} c(\mathcal{L})[\mathbb{I}_X] &= c(\psi^*\mathcal{O}(1,0)) \circ \psi^*[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}] \\ &= \psi^* \circ c(\mathcal{O}(1,0))[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}] \\ &\quad \text{(by Lemma 4.17).} \end{aligned}$$

The same holds for $\mathcal{M}$ and $\mathcal{L}\otimes\mathcal{M}$. Hence, we have

$$c(\mathcal{L}\otimes\mathcal{M})[\mathbb{I}_X] = \psi^* \circ c(\mathcal{O}(1,1))[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}]$$

and

$$F(c(\mathcal{L}), c(\mathcal{M}))[\mathbb{I}_X] = \psi^* \circ F(c(\mathcal{O}(1,0)), c(\mathcal{O}(0,1)))[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}].$$

Thus, without loss of generality, we can assume $X = \mathbb{P}^n \times \mathbb{P}^m$, $\mathcal{L} = \mathcal{O}(1,0)$ and $\mathcal{M} = \mathcal{O}(0,1)$. Notice that the G-actions on X, $\mathcal{L}$, $\mathcal{M}$ and $\mathbb{I}_X$ are all trivial now. Let

$$\Phi_\gamma : \omega(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathcal{U}_{\{1\}}(\mathbb{P}^n \times \mathbb{P}^m) \to \mathcal{U}_G(\mathbb{P}^n \times \mathbb{P}^m)$$

be the abelian groups homomorphism induced by the group scheme homomorphism $\gamma : G \to \{1\}$. By Proposition 9.4 in [LeP], **(FGL)** holds in the non-equivariant theory ω_*. In particular,

$$c(\mathcal{O}(1,1))[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}] = F(c(\mathcal{O}(1,0)), c(\mathcal{O}(0,1)))[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}] \tag{5}$$

as elements in $\omega(\mathbb{P}^n \times \mathbb{P}^m)$. Observe that, for $\mathcal{L} = \mathcal{O}(1,0)$, $\mathcal{O}(0,1)$ or $\mathcal{O}(1,1)$, we have

$$\Phi_\gamma \circ c(\mathcal{L})[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}] = [H_s \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m] = c(\mathcal{L})[\mathbb{I}_{\mathbb{P}^n\times\mathbb{P}^m}]$$

where $s \in \mathrm{H}^0(\mathbb{P}^n \times \mathbb{P}^m, \mathcal{L})$ is a global section such that H_s is a smooth divisor on $\mathbb{P}^n \times \mathbb{P}^m$. By applying Φ_γ on equation (5), the same equality holds in $\mathcal{U}_G(\mathbb{P}^n \times \mathbb{P}^m)$. □

4.4. **Extending the definition.** In order to extend our definition to arbitrary G-linearized invertible sheaves, we need to first consider the sheaf $\mathcal{O}(1) \in \mathrm{Pic}^G(\mathbb{P}(V))$ for arbitrary G-representation V. In the case when G is a finite abelian group with exponent e, it turns out the only way to define $c(\mathcal{O}(1))$, so that the property **(FGL)** still holds, will force us to invert the element $e \in \mathbb{Z}$. Hence, we introduce the notation

$$\mathcal{U}_G(X)[1/e] \stackrel{def}{=} \mathcal{U}_G(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/e].$$

Remarks 4.20. We will explain why we cannot expect a more general definition of $c(\mathcal{L})$ that satisfies the **(FGL)** without inverting the exponent of the group. Let us consider the following example. Suppose G is a cyclic group of order p (prime) and the ground field k contains a primitive p-th root of unity ξ. Let $V \stackrel{def}{=} k-\mathrm{span}\{x, y\}$ with action $\alpha \cdot x = \xi x$ and $\alpha \cdot y = y$ where α is a generator of G. Let $X \stackrel{def}{=} \mathbb{P}(V)$.

Suppose we have defined $c(\mathcal{O}(1)) : \mathcal{U}_G(X) \to \mathcal{U}_G(X)$ such that **(FGL)** holds. Then, we will have

$$c(\mathcal{O}(p))[\mathbb{I}_X] = c(\mathcal{O}(1)^{\otimes p})[\mathbb{I}_X] = F^p(c(\mathcal{O}(1)))[\mathbb{I}_X].$$

Notice that

$$F(c(\mathcal{O}(1)), c(\mathcal{O}(1)))[\mathbb{I}_X] = 2\, c(\mathcal{O}(1))[\mathbb{I}_X] + a_{11} c(\mathcal{O}(1))^2[\mathbb{I}_X] + \cdots .$$

For any $i \geq 2$, the element $c(\mathcal{O}(1))^i[\mathbb{I}_X]$ lies in $\mathcal{U}^G_{1-i}(X)$, which is zero because the dimension of X is one. So, we have $F(c(\mathcal{O}(1)), c(\mathcal{O}(1)))[\mathbb{I}_X] = 2\, c(\mathcal{O}(1))[\mathbb{I}_X]$. Inductively, we get $F^p(c(\mathcal{O}(1)))[\mathbb{I}_X] = p\, c(\mathcal{O}(1))[\mathbb{I}_X]$.

On the other hand, consider the G-equivariant map $\psi : X \to \mathbb{P}^1$ (with trivial action on $\mathbb{P}^1$) given by $(x; y) \mapsto (x^p; y^p)$. Then, $\mathcal{O}_X(p) \cong \psi^* \mathcal{O}_{\mathbb{P}^1}(1)$. Hence, $\mathcal{O}_X(p)$ is nice. By the definition of the Chern class operator for nice G-linearized invertible sheaves,

$$c(\mathcal{O}(p))[\mathbb{I}_X] = [H_p \hookrightarrow \mathbb{P}(V)] = [G \hookrightarrow \mathbb{P}(V)]$$

where $H_p \cong G$ (the k-scheme of p points with free G-action). Hence, by pushing down both equalities to $\mathcal{U}_G(\mathrm{Spec}\, k)$, we obtain

$$[G] = p\, a \tag{6}$$

where $a \stackrel{def}{=} \pi_{k*}(c(\mathcal{O}(1))[\mathbb{I}_X])$ and $\pi_k : X \to \mathrm{Spec}\, k$.

Let $[Z_1] - [Z_2]$ be a representative of $a \in \mathcal{U}_G(\operatorname{Spec} k)$. Consider the group scheme homomorphism $\{1\} \to G$. It induces an abelian groups homomorphism

$$\Phi : \mathcal{U}_0^G(\operatorname{Spec} k) \to \mathcal{U}_0^{\{1\}}(\operatorname{Spec} k) \cong \omega_0(\operatorname{Spec} k) \cong \mathbb{Z}.$$

That implies

$$p\,(\Phi[Z_1] - \Phi[Z_2]) = \Phi(p\,a) = \Phi[G] = p$$

as elements in $\omega_0(\operatorname{Spec} k)$. Since there is no torsion in $\omega_0(\operatorname{Spec} k) \cong \mathbb{Z}$, we conclude that $\Phi[Z_1] - \Phi[Z_2] = 1$. On the other hand, since the order of the group G is a prime and the dimension of Z_1 is zero, $Z_1 \cong \operatorname{Spec} A_t \amalg \operatorname{Spec} A_f$ where the action on A_t is trivial and the action on A_f is free. Moreover, A_t can be written as the product of $K_{t,i}$, where $K_{t,i}$ are finite field extensions of k. Similarly, $Z_2 \cong \operatorname{Spec} B_t \amalg \operatorname{Spec} B_f$ and $B_t = \prod_j L_{t,j}$. By Lemma 2.3.4 in [LeMo], we have $[\operatorname{Spec} K] = [K : k][\mathbb{I}_{\operatorname{Spec} k}]$ as elements in $\omega(\operatorname{Spec} k)$, where $[K : k]$ denotes the degree of the field extension. Hence,

$$(7) \quad 1 = \Phi[Z_1] - \Phi[Z_2] = \sum_i [K_{t,i} : k] + \Phi[\operatorname{Spec} A_f] - \sum_j [L_{t,j} : k] - \Phi[\operatorname{Spec} B_f].$$

Let us consider an G-irreducible component W of $\operatorname{Spec} A_f$. It can either be $\operatorname{Spec} K$ with free action, or the disjoint union of p copies of $\operatorname{Spec} K$ with G permuting them. In the first case,

$$\Phi[W] = \Phi[\operatorname{Spec} K] = [K : k] = [K : K^G][K^G : k] = p[K^G : k].$$

In the second case,

$$\Phi[W] = \Phi[\operatorname{Spec} \textstyle\prod_{i=1}^{p} K] = p[K : k].$$

Either case, p divides $\Phi[W]$. Hence, p divides $\Phi[\operatorname{Spec} A_f]$. Similarly, p divides $\Phi[\operatorname{Spec} B_f]$.

Now, if we apply the fixed point map $\mathcal{F} : \mathcal{U}_G(\operatorname{Spec} k) \to \omega(\operatorname{Spec} k)$ on equation (6) (see section 7 for details), we obtain

$$
\begin{aligned}
0 = \mathcal{F}[G] &= \mathcal{F}(p\,([Z_1] - [Z_2])) \\
&= p\,(\mathcal{F}([Z_1]) - \mathcal{F}([Z_2])) \\
&= p\,([\mathrm{Spec}\, A_t] - [\mathrm{Spec}\, B_t]) \\
&= p\,(\sum_i [K_{t,i} : k] - \sum_j [L_{t,j} : k]).
\end{aligned}
$$

That implies

$$
0 = \sum_i [K_{t,i} : k] - \sum_j [L_{t,j} : k]. \tag{8}
$$

Combining equations (7) and (8) and the fact that p divides $\Phi[\mathrm{Spec}\, A_f]$ and $\Phi[\mathrm{Spec}\, B_f]$, we get a contradiction.

Hence, it is impossible to define $c(\mathcal{O}(1))$ as an operator on $\mathcal{U}_G(X)$ such that **(FGL)** holds. It can also be seen in this example that the natural definition of $c(\mathcal{O}(1))[\mathbb{I}_X]$ should be $(1/p)[H_p \hookrightarrow X]$, as an element in $\mathcal{U}_G(X)[1/p]$.

In order to simplify the calculation, we need a condition on G and k such that any irreducible G-representation will be of dimension 1.

Definition 4.21. We will say that the pair (G, k) is split, if the group G is finite abelian with exponent e and the field k contains a primitive e-th root of unity.

Lemma 4.22. *If the pair (G, k) is split, then any irreducible G-representation has dimension one.*

Proof. Recall that we are assuming $\mathrm{char}\, \mathrm{k} = 0$. We can easily see that when (G, k) is split, we have $k[G] \cong \prod k$. The result then follows. □

For the rest of this subsection, we assume that the pair (G, k) is split. In this case, we can extend our definition of the Chern class operator to arbitrary G-linearized invertible sheaves. In order to preserve the **(FGL)** property, we would like to define $c(\mathcal{L})$ by the following formula :

$$
F^{1/e}(F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M})))
$$

where $\mathcal{M}$ is in $\text{Pic}^G(X)$ such that $\mathcal{L}^e \otimes \mathcal{M}$ and $\mathcal{M}$ are both nice (recall that $\mathcal{L}^e$ means $\mathcal{L}^{\otimes e}$ and $F^{1/e}(u)$ is the operation "division by e" in formal group law, see subsection 4.2 for details). We need the following two Lemmas for its well-definedness.

Lemma 4.23. *For any $\mathcal{L} \in \text{Pic}^G(X)$, there exists an invertible sheaf $\mathcal{M} \in \text{Pic}^G(X)$ such that $\mathcal{L}^e \otimes \mathcal{M}$ and $\mathcal{M}$ are both nice.*

Proof. Let us first consider the case when $X = \mathbb{P}(V)$ where V is a G-representation and $\mathcal{L} = \mathcal{O}(1)$. By Lemma 4.22, $X \cong \text{Proj } k[x_0, \ldots, x_n]$ such that, for all i, $k-\text{span}\{x_i\}$ is a 1-dimensional G-representation. Let $Y \overset{def}{=} \text{Proj } k[y_0, \ldots, y_n]$ with trivial action and

$$\psi : X = \text{Proj } k[x_0, \ldots, x_n] \to \text{Proj } k[y_0, \ldots, y_n] = Y$$

be the morphism corresponding to the k-algebra homomorphism $k[y_0, \ldots, y_n] \to k[x_0, \ldots, x_n]$ defined by $y_i \mapsto x_i^e$. Since e is the exponent of G, the map ψ is G-equivariant. Observe that this map can also be considered as an e-uple embedding followed by a linear projection on some G-invariant open subscheme. Hence, we have $\psi^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(e)$. In other words, the sheaf $\mathcal{O}_X(e)$ is nice.

For general $X \in G\text{-}Sm$ and $\mathcal{L} \in \text{Pic}^G(X)$, by Proposition 4.14, there exists an equivariant immersion $\psi : X \hookrightarrow \mathbb{P}(V)$. For large enough m, the sheaf $\mathcal{L} \otimes \psi^*\mathcal{O}(m)$ will be very ample. By embedding $\mathbb{P}(V)$ into some larger $\mathbb{P}(V')$, we can assume $m = 1$. Since $\mathcal{L} \otimes \psi^*\mathcal{O}(1)$ is very ample and G-linearized, by Proposition 1.7 in [MuFoKi], there exists an equivariant immersion $\psi' : X \hookrightarrow \mathbb{P}(V'')$ such that $\psi'^*\mathcal{O}(1) \cong \mathcal{L} \otimes \psi^*\mathcal{O}(1)$. Hence, we have $\psi'^*\mathcal{O}(e) \cong \mathcal{L}^e \otimes \psi^*\mathcal{O}(e)$. Then, the result follows because $\psi^*\mathcal{O}(e)$ and $\psi'^*\mathcal{O}(e)$ are both nice. □

Lemma 4.24. *For any two sheaves $\mathcal{M}$, $\mathcal{M}' \in \text{Pic}^G(X)$ such that $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{L}^e \otimes \mathcal{M}$ and $\mathcal{L}^e \otimes \mathcal{M}'$ are all nice, we have*

$$F^{1/e}(F^-(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M}))) = F^{1/e}(F^-(c(\mathcal{L}^e \otimes \mathcal{M}'), c(\mathcal{M}')))$$

as homomorphisms from $\mathcal{U}_G(X)[1/e]$ to $\mathcal{U}_G(X)[1/e]$.

Proof. By the fact that all sheaves involved are nice and Proposition 4.19, we have

$$F(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M}')) = c(\mathcal{L}^e \otimes \mathcal{M} \otimes \mathcal{M}') = F(c(\mathcal{L}^e \otimes \mathcal{M}'), c(\mathcal{M})).$$

That implies

$$\begin{aligned} F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M})) &= F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}'), c(\mathcal{M}')) \\ F^{1/e}(F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M}))) &= F^{1/e}(F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}'), c(\mathcal{M}'))). \end{aligned}$$

□

Definition 4.25. Assume the pair (G, k) is split. Suppose X is in G-Sm and $\mathcal{L}$ is in $\mathrm{Pic}^G(X)$. We define the abelian group homomorphism $c(\mathcal{L}) : \mathcal{U}_*^G(X)[1/e] \to \mathcal{U}_{*-1}^G(X)[1/e]$ by the following formula :

$$c(\mathcal{L}) \stackrel{def}{=} F^{1/e}(F^{-}(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M})))$$

where $\mathcal{M}$ is in $\mathrm{Pic}^G(X)$ such that $\mathcal{L}^e \otimes \mathcal{M}$, $\mathcal{M}$ are both nice.

Remark 4.26. Suppose $\mathcal{L} \in \mathrm{Pic}^G(X)$ is nice. In this new definition, we can pick $\mathcal{M}$ to be $\mathcal{L}$. Then,

$$F^{1/e}(F^{-}(c(\mathcal{L}^e \otimes \mathcal{L}), c(\mathcal{L}))) = F^{1/e}(c(\mathcal{L}^e)) = c(\mathcal{L}).$$

That means the new definition is indeed a generalization of the definition of the Chern class operator for nice G-linearized invertible sheaves.

Suppose X is an object in $\mathcal{D}$ and $\mathcal{L} \in \mathrm{Pic}^G(X)$. Then we have two definitions of the Chern class operator (as operators on $\mathcal{U}_G(X)[1/e]$), given by the first and second approach. The last part of this section is to show that they agree.

Lemma 4.27. *Suppose X is an object in $\mathcal{D}$ and $\mathcal{L}$, $\mathcal{M}$ are sheaves in $\mathrm{Pic}^G(X)$. Let $\pi : X \to X/G$ be the quotient map. Then, we have*

$$\pi_*(\mathcal{L} \otimes \mathcal{M})^G \cong \pi_*\mathcal{L}^G \otimes \pi_*\mathcal{M}^G.$$

For any two sheaves $\mathcal{L}$, $\mathcal{M} \in \mathrm{Pic}(X/G)$, we have

$$\pi^*(\mathcal{L} \otimes \mathcal{M}) \cong (\pi^*\mathcal{L}) \otimes (\pi^*\mathcal{M}).$$

In other words, descent and ascent both commutes with tensor product.

Proof. The second statement follows from a basic property of pull-back. For descent, since $X \to X/G$ is a principle G-bundle, there is a one-to-one correspondence between $\mathrm{Pic}^G(X)$ and $\mathrm{Pic}(X/G)$ given by π^* and $\pi_*(-)^G$. Therefore,

$$\begin{aligned} \pi_*\mathcal{L}^G \otimes \pi_*\mathcal{M}^G &\cong \pi_*(\pi^*(\pi_*\mathcal{L}^G \otimes \pi_*\mathcal{M}^G))^G \\ &\cong \pi_*((\pi^*\pi_*\mathcal{L}^G) \otimes (\pi^*\pi_*\mathcal{M}^G))^G \\ &\cong \pi_*(\mathcal{L} \otimes \mathcal{M})^G. \end{aligned}$$

□

Suppose the pair (G, k) is split, X is an object in $\mathcal{D}$ and $\mathcal{L} \in \mathrm{Pic}^G(X)$. Denote the corresponding Chern class operator defined by the first approach by $c'(\mathcal{L})$, i.e.

$$c'(\mathcal{L}) = \Psi \circ c(\pi_*\mathcal{L}^G) \circ \Psi^{-1}$$

from $\mathcal{U}_G(X)[1/e]$ to $\mathcal{U}_G(X)[1/e]$. Also denote the corresponding Chern class operator defined by the second approach by $c''(\mathcal{L})$, i.e.

$$c''(\mathcal{L})[Y \to X] = [Y \times_{\mathbb{P}^n} H \to X]$$

when $\mathcal{L}$ is nice (see subsection 4.2 for details), and for general $\mathcal{L} \in \mathrm{Pic}^G(X)$,

$$c''(\mathcal{L}) = F^{1/e}(F^-(c''(\mathcal{L}^e \otimes \mathcal{M}), c''(\mathcal{M})))$$

from $\mathcal{U}_G(X)[1/e]$ to $\mathcal{U}_G(X)[1/e]$ where $\mathcal{M}$ is in $\mathrm{Pic}^G(X)$ such that $\mathcal{L}^e \otimes \mathcal{M}$, $\mathcal{M}$ are both nice.

Proposition 4.28. *For any $X \in \mathcal{D}$ and $\mathcal{L} \in \mathrm{Pic}^G(X)$, we have*

$$c'(\mathcal{L}) = c''(\mathcal{L})$$

as group homomorphisms from $\mathcal{U}_G(X)[1/e]$ to $\mathcal{U}_G(X)[1/e]$.

Proof. If $\mathcal{L} \in \mathrm{Pic}^G(X)$ is nice, then there is an equivariant morphism $\psi : X \to \mathbb{P}^n$ such that $\psi^*\mathcal{O}(1) \cong \mathcal{L}$. By definition,

$$c''(\mathcal{L})[f : Y \to X] = [Y \times_{\mathbb{P}^n} H \to X]$$

where H is a hyperplance in $\mathbb{P}^n$ such that $Y \times_{\mathbb{P}^n} H$ is an invariant smooth divisor on Y. Let $s \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$ be the global section that cuts out H. Then, $Y \times_{\mathbb{P}^n} H$ is cut out by the invariant global section $(\psi \circ f)^* s \in \mathrm{H}^0(Y, f^*\mathcal{L})^G$. On the other hand, by remark 4.3, $c'(\mathcal{L})[Y \to X]$ can also be given by the divisor cut out by any invariant global section $s' \in \mathrm{H}^0(Y, f^*\mathcal{L})^G$ as long as the divisor is smooth. Hence, $c'(\mathcal{L}) = c''(\mathcal{L})$ when $\mathcal{L}$ is nice.

For general $\mathcal{L} \in \mathrm{Pic}^G(X)$, let $F^{1/e}(u) \stackrel{def}{=} \sum_{i \geq 1} b_i\, u^i$ and $F^-(u, v) \stackrel{def}{=} \sum_{j,k \geq 0} c_{jk}\, u^j v^k$. Then, we have

$$\begin{aligned}
c''(\mathcal{L}) &= F^{1/e}(F^-(c''(\mathcal{L}^e \otimes \mathcal{M}), c''(\mathcal{M}))) \\
&= \sum_i b_i \, (\sum_{j,k} c_{jk} c''(\mathcal{L}^e \otimes \mathcal{M})^j c''(\mathcal{M})^k)^i \\
&= \sum_i b_i \, (\sum_{j,k} c_{jk} \Psi \circ c(\pi_*(\mathcal{L}^e \otimes \mathcal{M})^G)^j \circ c(\pi_* \mathcal{M}^G)^k \circ \Psi^{-1})^i \\
&\quad \text{(the two definitions agree for nice sheaves)}
\end{aligned}$$

$$\begin{aligned}
&= \Psi \circ (\sum_i b_i \, (\sum_{j,k} \; c_{jk} \; c(\pi_*(\mathcal{L}^e \otimes \mathcal{M})^G)^j \; c(\pi_* \mathcal{M}^G)^k \,)^i) \circ \Psi^{-1} \\
&= \Psi \circ F^{1/e}(F^-(c(\pi_*(\mathcal{L}^e \otimes \mathcal{M})^G), c(\pi_* \mathcal{M}^G))) \circ \Psi^{-1} \\
&= \Psi \circ F^{1/e}(F^-(c((\pi_* \mathcal{L}^G)^e \otimes \pi_* \mathcal{M}^G), c(\pi_* \mathcal{M}^G))) \circ \Psi^{-1} \\
&\quad \text{(by Lemma 4.27)} \\
&= \Psi \circ c(\pi_* \mathcal{L}^G) \circ \Psi^{-1} \\
&\quad (\text{ because } \textbf{(FGL)} \text{ holds in } \omega(X/G)\,) \\
&= c'(\mathcal{L}).
\end{aligned}$$

□

5. More properties for $\mathcal{U}_G$

In this section, we will state and prove some more basic properties in our equivariant algebraic cobordism theory $\mathcal{U}_G$, equipped with the Chern class operator for nice G-linearized invertible sheaves. Some properties are related to the Chern class operator. In that case, we will also prove them in the theory $\mathcal{U}_G[1/e] \overset{def}{=} \mathcal{U}_G \otimes_{\mathbb{Z}} \mathbb{Z}[1/e]$ for arbitrary G-linearized invertible sheaves assuming that the pair (G, k) is split (recall that e is the exponent of G). The non-equivariant version of these properties can be found in [LeP].

At this stage, we have established projective push-forward **(D1)**, smooth pull-back **(D2)**, Chern class operator **(D3)** and external product **(D4)**. For convenience, we will briefly recall here some of the properties already shown in section 3.

(A1) If $f : X \to X'$ and $g : X' \to X''$ are both smooth and X, X', X'' are all equidimensional, then

$$(g \circ f)^* = f^* \circ g^*.$$

Moreover, $\mathbb{I}^*$ is the identity homomorphism.

(A2) If $f : X \to Z$ is projective and $g : Y \to Z$ is smooth such that X, Y, Z are all equidimensional, then we have $g^* f_* = f'_* g'^*$ in the pull-back square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

(A3) If $f : X \to X'$ is projective and $\mathcal{L} \in \mathrm{Pic}^G(X')$ is nice, then

$$f_* \circ c(f^* \mathcal{L}) = c(\mathcal{L}) \circ f_*$$

in the theory $\mathcal{U}_G$. Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L} \in \mathrm{Pic}^G(X')$.

Proof. The first part of the statement follows from Lemma 4.18. For the second part,

$$
\begin{aligned}
f_* \circ c(f^*\mathcal{L}) &= f_* \circ F^{1/e}(F^-(c(f^*\mathcal{L}^e \otimes f^*\mathcal{M}), c(f^*\mathcal{M}))) \\
&\quad \text{(for some } \mathcal{M} \in \mathrm{Pic}^G(X') \text{ such that } \mathcal{L}^e \otimes \mathcal{M},\ \mathcal{M} \text{ are both nice)} \\
&= f_* \circ \sum_i b_i \,(\sum_{j,k} c_{jk}\, c(f^*\mathcal{L}^e \otimes f^*\mathcal{M})^j\, c(f^*\mathcal{M})^k)^i \\
&\quad \text{where } b_i,\ c_{jk} \text{ are coefficients for } F^{1/e}(u),\ F^-(u,v) \text{ respectively} \\
&= (\sum_i b_i \,(\sum_{j,k} c_{jk}\, c(\mathcal{L}^e \otimes \mathcal{M})^j\, c(\mathcal{M})^k)^i) \circ f_* \\
&\quad \text{(by Lemma 4.18 and the fact that } \mathcal{L}^e \otimes \mathcal{M},\ \mathcal{M} \text{ are nice).}
\end{aligned}
$$

Hence,

$$f_* \circ c(f^*\mathcal{L}) = F^{1/e}(F^-(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M}))) \circ f_* = c(\mathcal{L}) \circ f_*.$$

□

(A4) If $f : X \to X'$ is smooth, X, X' are both equidimensional and $\mathcal{L} \in \mathrm{Pic}^G(X')$ is nice, then

$$f^* \circ c(\mathcal{L}) = c(f^*\mathcal{L}) \circ f^*$$

in the theory $\mathcal{U}_G$. Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L} \in \mathrm{Pic}^G(X')$.

Proof. Suppose that $\psi : X' \to \mathbb{P}^n$ is a morphism in G-*Sm* such that $\mathcal{L} \cong \psi^*\mathcal{O}(1)$. Let $[Y \to X']$ be an element in $\mathcal{U}_G(X')$ and H be a hyperplane in $\mathbb{P}^n$ such that $Y \times_{\mathbb{P}^n} H$ is a smooth invariant divisor on Y. Then,

$$
\begin{aligned}
f^* \circ c(\mathcal{L})[Y \to X'] &= f^*[Y \times_{\mathbb{P}^n} H \to X'] \\
&= [X \times_{X'} (Y \times_{\mathbb{P}^n} H) \to X].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
c(f^*\mathcal{L}) \circ f^*[Y \to X'] &= c(f^*\mathcal{L})[X \times_{X'} Y \to X] \\
&= [(X \times_{X'} Y) \times_{\mathbb{P}^n} H \to X].
\end{aligned}
$$

Hence, they agree. The proof for arbitrary $\mathcal{L}$ is similar to the proof of the similar statement of **(A3)**. ☐

(A5) If $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^G(X)$ are both nice, then

$$c(\mathcal{L}) \circ c(\mathcal{L}') = c(\mathcal{L}') \circ c(\mathcal{L})$$

in the theory $\mathcal{U}_G$. Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^G(X)$.

Proof. Suppose that $\mathcal{L}$, $\mathcal{L}'$ are nice and let $\psi : X \to \mathbb{P}^n$ and $\psi' : X \to \mathbb{P}^m$ be the corresponding maps for $\mathcal{L}$ and $\mathcal{L}'$ respectively. Then, for some appropriately chosen hyperplanes $H \subseteq \mathbb{P}^n$ and $H' \subseteq \mathbb{P}^m$,

$$\begin{aligned} c(\mathcal{L}) \circ c(\mathcal{L}')\ [Y \to X] &= c(\mathcal{L})\ [Y \times_{\mathbb{P}^m} H' \to X] \\ &= [(Y \times_{\mathbb{P}^m} H') \times_{\mathbb{P}^n} H \to X] \\ &= [(Y \times_{\mathbb{P}^n} H) \times_{\mathbb{P}^m} H' \to X] \\ &= c(\mathcal{L}') \circ c(\mathcal{L})\ [Y \to X]. \end{aligned}$$

The statement for arbitrary $\mathcal{L}$, $\mathcal{L}' \in \mathrm{Pic}^G(X)$ can be shown by a similar argument as before. ☐

(A6) If f, g are projective, then

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times.$$

(A7) If f, g are smooth with equidimensional domains and codomains, then

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

(A8) Let a, b be elements in $\mathcal{U}_G(X), \mathcal{U}_G(X')$ respectively and let $\mathcal{L} \in \mathrm{Pic}^G(X)$ be a nice invertible sheaf. Then we have

$$c(\mathcal{L})(a) \times b = c(\pi_1^* \mathcal{L})(a \times b).$$

Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L} \in \mathrm{Pic}^G(X)$.

Proof. Suppose that $\mathcal{L}$ is nice. Without loss of generality, we can assume $a = [Y \to X]$ and $b = [Y' \to X']$. Let $\psi : X \to \mathbb{P}^n$ be the map corresponding to $\mathcal{L}$. Then, for some $H \subseteq \mathbb{P}^n$,

$$\begin{aligned}
(c(\mathcal{L})[Y \to X]) \times [Y' \to X'] &= [Y \times_{\mathbb{P}^n} H \to X] \times [Y' \to X'] \\
&= [(Y \times_{\mathbb{P}^n} H) \times Y' \to X \times X'] \\
&= [(Y \times Y') \times_{\mathbb{P}^n} H \to X \times X'] \\
&\quad \text{(via the map } Y \times Y' \to X \times X' \to X \to \mathbb{P}^n) \\
&= c(\pi_1^*\mathcal{L})[Y \times Y' \to X \times X'].
\end{aligned}$$

For arbitrary $\mathcal{L} \in \mathrm{Pic}^G(X)$,

$$\begin{aligned}
c(\mathcal{L})(a) \times b &= F^{1/e}(F^-(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M})))(a) \times b \\
&= (\sum_i b_i (\sum_{j,k} c_{jk}\, c(\mathcal{L}^e \otimes \mathcal{M})^j\, c(\mathcal{M})^k)^i(a)) \times b \\
&= (\sum_{j,k} d_{jk}\, c(\mathcal{L}^e \otimes \mathcal{M})^j\, c(\mathcal{M})^k(a)) \times b \\
&\quad \text{(expand the series out and denote the coefficients by } d_{jk}) \\
&= \sum_{j,k} d_{jk}\, c(\pi_1^*\mathcal{L}^e \otimes \pi_1^*\mathcal{M})^j\, c(\pi_1^*\mathcal{M})^k\, (a \times b) \\
&= \sum_i b_i (\sum_{j,k} c_{jk}\, c(\pi_1^*\mathcal{L}^e \otimes \pi_1^*\mathcal{M})^j\, c(\pi_1^*\mathcal{M})^k)^i\, (a \times b) \\
&= F^{1/e}(F^-(c(\pi_1^*\mathcal{L}^e \otimes \pi_1^*\mathcal{M}), c(\pi_1^*\mathcal{M})))\, (a \times b) \\
&= c(\pi_1^*\mathcal{L})(a \times b).
\end{aligned}$$

□

(Dim) If $\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_r \in \mathrm{Pic}^G(X)$ are nice invertible sheaves and $r > \dim X$, then

$$c(\mathcal{L}_1) \circ c(\mathcal{L}_2) \circ \cdots \circ c(\mathcal{L}_r)[\mathbb{I}_X] = 0.$$

Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L}_1,\ \mathcal{L}_2, \ldots, \mathcal{L}_r \in \mathrm{Pic}^G(X)$.

Proof. It follows from the fact that $c(\mathcal{L}) : \mathcal{U}^G_*(X) \to \mathcal{U}^G_{*-1}(X)$ and $\mathcal{U}^G_{<0}(X) = 0$. $\square$

(FGL) If $\mathcal{L}$, $\mathcal{L}' \in \mathrm{Pic}^G(X)$ are nice invertible sheaves, then

$$c(\mathcal{L} \otimes \mathcal{L}') = F(c(\mathcal{L}), c(\mathcal{L}'))$$

in the theory $\mathcal{U}_G$. Moreover, if the pair (G, k) is split, then the same statement holds in the theory $\mathcal{U}_G[1/e]$ for arbitrary $\mathcal{L}$, $\mathcal{L}' \in \mathrm{Pic}^G(X)$.

Proof. The statement for nice $\mathcal{L}$, $\mathcal{L}'$ was proved in section 4. For arbitrary $\mathcal{L}$, $\mathcal{L}'$,

$$\begin{aligned}
F(c(\mathcal{L}), c(\mathcal{L}')) &= F(\ F^{1/e}(F^-(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M})))\ ,\ F^{1/e}(F^-(c(\mathcal{L}'^{\,e} \otimes \mathcal{M}'), c(\mathcal{M}')))\ ) \\
&= F^{1/e}(F(\ F^-(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{M}))\ ,\ F^-(c(\mathcal{L}'^{\,e} \otimes \mathcal{M}'), c(\mathcal{M}'))\ )) \\
&= F^{1/e}(F^-(\ F(c(\mathcal{L}^e \otimes \mathcal{M}), c(\mathcal{L}'^{\,e} \otimes \mathcal{M}'))\ ,\ F(c(\mathcal{M}), c(\mathcal{M}'))\ )) \\
&= F^{1/e}(F^-(\ c(\mathcal{L}^e \otimes \mathcal{M} \otimes \mathcal{L}'^{\,e} \otimes \mathcal{M}')\ ,\ c(\mathcal{M} \otimes \mathcal{M}')\ )) \\
&= c(\mathcal{L} \otimes \mathcal{L}')
\end{aligned}$$

because $(\mathcal{L} \otimes \mathcal{L}')^e \otimes (\mathcal{M} \otimes \mathcal{M}')$ and $\mathcal{M} \otimes \mathcal{M}'$ are both nice.

$\square$

6. Generators for the equivariant algebraic cobordism ring

The main objective of this section is to prove Theorem 6.22, which gives a set of generators of the equivariant algebraic cobordism ring $\mathcal{U}_G(\operatorname{Spec} k)$. To achieve this, we need to use a different version of splitting principle. We will assume the pair (G, k) is split in this section.

6.1. **Splitting principle by blowing up along invariant smooth centers.** In this subsection, for a sheaf $\mathcal{E}$ over Y and a map $f : X \to Y$, we will denote $f^*\mathcal{E}$ by $\mathcal{E}_X$ if there is no confusion. Suppose X is a scheme over k and Z is a closed subscheme of X. We will denote the blow up of X along Z by $\mathrm{Blow}_Z X$.

The main result in this subsection is similar to the equivariant analog of Theorem 4.7 in [Kl].

Let $S \in G\text{-}Sm$ be a ground scheme. Suppose $\mathcal{N}$ is a G-linearized locally free sheaf of rank N over S and $\mathcal{A} \hookrightarrow \mathcal{N}$ is a rank 1 G-linearized locally free subsheaf. Recall the definition in section 2.1 [Kl].

The scheme $\sigma_{1,n}(\mathcal{A}, \mathcal{N})$ is defined as the closed subscheme of Grassmannian $Gr_n(\mathcal{N})$ satisfying the following. A point $(s, \mathcal{H}) \in Gr_n(\mathcal{N})$ (i.e. $s \in S$ and $\mathcal{H}$ is a n-quotient of $\mathcal{N}|_s$) is inside $\sigma_{1,n}(\mathcal{A}, \mathcal{N})$ if the composition $\mathcal{A}|_s \to \mathcal{N}|_s \to \mathcal{H}$ is zero.

Also recall the following definition in section 3.1 in [Kl].

Suppose X is in $G\text{-}Sm$ and $\mathcal{N}$ is a G-linearized locally free sheaf of rank N over $\operatorname{Spec} k$. An equivariant immersion $X \hookrightarrow Gr_r(\mathcal{N})$ is called twisted if it is the Segre product of an equivariant map $X \to Gr_r(\mathcal{N}_1)$ and an equivariant immersion $X \hookrightarrow \mathbb{P}(\mathcal{N}_2)$ for some G-linearized locally free sheaves $\mathcal{N}_1, \mathcal{N}_2$ over $\operatorname{Spec} k$.

Proposition 6.1. *Suppose $X \in G\text{-}Sm$ is G-irreducible with dimension d and there is a twisted equivariant immersion*

$$X \hookrightarrow Gr_r(\mathcal{N}) \overset{def}{=} Y$$

for some G-linearized locally free sheaf $\mathcal{N}$ of rank N over $\operatorname{Spec} k$ *($1 \leq r < N$). Moreover, there is a 1-dimensional character ψ such that the dimension of the ψ component* $\mathrm{H}^0(\operatorname{Spec} k, \mathcal{N})_\psi$ *is greater than r. Let $Z \overset{def}{=} Gr_{N-1}(\mathcal{N})$ and $\mathcal{A}$ be the universal subbundle over Z ($\mathcal{A} \hookrightarrow \mathcal{N}_Z$ with rank 1). Then, there exists a closed point z of the fixed point locus*

Z^G, with residue field $k(z) \cong k$, such that the closed subscheme $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \subseteq Y$ is smooth with codimension r and the dimension of $X \cap \sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ is $d - r$.

Proof. This statement is similar to Theorem 3.3 in [Kl]. First of all, notice that

$$X \times Z \hookrightarrow Y \times Z = Gr_r(\mathcal{N}) \times Z \cong Gr_r(\mathcal{N}_Z).$$

On the other hand, the subsheaf $\mathcal{A} \hookrightarrow \mathcal{N}_Z$ induces $\sigma_{1,r}(\mathcal{A}, \mathcal{N}_Z)$, which is a closed subscheme of $Gr_r(\mathcal{N}_Z)$. So, we will consider $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ and $X \cap \sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ as fibers of

$$\sigma_{1,r}(\mathcal{A}, \mathcal{N}_Z) \hookrightarrow Y \times Z \to Z$$

and

$$(X \times Z) \cap \sigma_{1,r}(\mathcal{A}, \mathcal{N}_Z) \hookrightarrow Y \times Z \to Z$$

respectively.

Suppose the G-representation corresponding to $\mathcal{N}$ is given by a k-basis $\{e_1, e_2, \ldots, e_N\}$ such that each e_i defines a 1-dimensional G-representation. Let U_N be the invariant affine open subscheme of Z corresponding to $e_1, \ldots, e_{N-1}$. Then, $U_N = \operatorname{Spec} k[s_1, \ldots, s_{N-1}]$. Since $Z = Gr_{N-1}(\mathcal{N}) \cong \mathbb{P}(\mathcal{N}^\vee)$ and $\dim \mathrm{H}^0(\operatorname{Spec} k, \mathcal{N})_\psi \geq r+1$, without loss of generality, we may assume G acts on the coordinates $s_1, \ldots, s_r$ trivially. In addition, it can be shown that $\mathcal{A} \hookrightarrow \mathcal{N}_Z$ is defined by

$$f \stackrel{def}{=} \left(\sum_{i=1}^{N-1} s_i e_i \right) - e_N$$

over U_N.

Let $U_{1,2,\ldots,r}$ be the affine open subscheme of Y corresponding to $e_1, \ldots, e_r$. Then, we have $U_{1,2,\ldots,r} = \operatorname{Spec} k[t_{i,j}]$ where $1 \leq i \leq r$ and $1 \leq j \leq N - r$. Let $(\mathcal{N}/\mathcal{G}, z) = (t_{i,j}, s_k)$ be a closed point in

$$\operatorname{Spec} k[t_{i,j}, s_k] = U_{1,2,\ldots,r} \times U_N \subseteq Y \times Z = Gr_r \mathcal{N}_Z.$$

Then, the map $\mathcal{A}|_z \to \mathcal{N} \to \mathcal{N}/\mathcal{G}$ at this point corresponds to

$$k\text{−span}\{f\} \hookrightarrow \oplus_{i=1}^{N} k\text{−span}\{e_i\} \to (\oplus_{i=1}^{N} k\text{−span}\{e_i\}) \,/\, k\text{−span}\{g_1, \ldots, g_{N-r}\}$$

where

$$g_i \stackrel{def}{=} \left(\sum_{j=1}^{r} t_{j,i} e_j \right) - e_{r+i}$$

for $1 \leq i \leq N-r$. The composition being zero is equivalent to $f \in k-\text{span}\{g_1, \ldots, g_{N-r}\}$, which is equivalent to

$$h_i \stackrel{def}{=} s_i + \left(\sum_{j=1}^{N-r-1} s_{j+r} t_{i,j} \right) - t_{i,(N-r)} = 0$$

for $1 \leq i \leq r$. So, $\sigma_{1,r}(\mathcal{A}, \mathcal{N}_Z)$ is cut out by the equations $h_1, \ldots, h_r$ inside $U_{1,2,\ldots,r} \times U_N$.

Let $z = (q_1, \ldots, q_{N-1})$ be a closed point in U_N. Then, when restricted on the fiber of $U_{1,2,\ldots,r} \times U_N \to U_N$ over z, the closed subscheme $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,2,\ldots,r}$ is cut out by r linear equations :

$$h_i = q_i + \left(\sum_{j=1}^{N-r-1} q_{j+r} t_{i,j} \right) - t_{i,(N-r)} = 0,$$

where $1 \leq i \leq r$. So, $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,2,\ldots,r}$ is smooth and of codimension r. Moreover, since $X \hookrightarrow Gr_r(\mathcal{N})$ is a twisted immersion and $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,2,\ldots,r}$ is given by r linear equations $\{h_i = 0\}$, the scheme $X \cap \sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,2,\ldots,r}$ is of dimension $d-r$ (See the proof of Theorem 3.3 in [Kl] for details).

Because of the symmetry of f, the only other affine open subscheme of Y we need to consider is $U_{1,\ldots,r-1,N}$. In this case, the map $\mathcal{A}|_z \to \mathcal{N} \to \mathcal{N}/\mathcal{G}$ corresponds to

$$k-\text{span}\{f\} \hookrightarrow \oplus_{i=1}^{N} k-\text{span}\{e_i\} \to \left(\oplus_{i=1}^{N} k-\text{span}\{e_i\}\right) / \, k-\text{span}\{g_1, \ldots, g_{N-r}\}$$

where

$$g_i \stackrel{def}{=} \left(\sum_{j=1}^{r-1} t_{j,i} e_j \right) + t_{r,i} e_N - e_{r+i}$$

for $1 \leq i \leq N-r-1$ and

$$g_{N-r} \stackrel{def}{=} \left(\sum_{j=1}^{r-1} t_{j,N-r} e_j \right) + t_{r,N-r} e_N - e_r.$$

Hence, the equations that cut $\sigma_{1,r}(\mathcal{A}, \mathcal{N}_Z)$ out are

$$h_i \stackrel{def}{=} s_i + \left(\sum_{j=1}^{N-r-1} s_{j+r} t_{i,j} \right) + t_{i,N-r} s_r = 0$$

for $1 \le i \le r-1$ and

$$h_r \stackrel{def}{=} -1 + \left(\sum_{j=1}^{N-r-1} s_{j+r} t_{r,j} \right) + t_{r,N-r} s_r = 0.$$

Let B be the closed subscheme of U_N defined by the equations $s_r = s_{r+1} = \cdots = s_{N-1} = 0$ and $z = (q_1, \ldots, q_{N-1})$ be a closed point in $U_N - B$. Then, in the fiber of $U_{1,\ldots,r-1,N} \times U_N \to U_N$ over z, the closed subscheme $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ is cut out by r linear equations

$$h_i = q_i + \left(\sum_{j=1}^{N-r-1} q_{j+r} t_{i,j} \right) + t_{i,N-r} q_r = 0$$

for $1 \le i \le r-1$ and

$$h_r = -1 + \left(\sum_{j=1}^{N-r-1} q_{j+r} t_{r,j} \right) + t_{r,N-r} q_r = 0.$$

Since at least one of $q_r, \ldots, q_{N-1}$ is non-zero, the linear equations $\{h_i \mid 1 \le i \le r\}$ are linearly independent. Hence, by the same reason, $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,\ldots,r-1,N}$ is smooth with codimension r and $X \cap \sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \cap U_{1,\ldots,r-1,N}$ is of dimension $d - r$.

For a different affine open subscheme $U_{i_1,\ldots,i_{r-1},N}$ of Y, there is a corresponding "bad" closed subscheme B of U_N defined by the set of equations $\{s_j = 0\}$ where $j \notin \{i_1, \ldots, i_{r-1}\}$. Hence, the result follows by picking $z = (q_1, \ldots, q_r, 0, \ldots, 0)$ such that $q_1, \ldots, q_r$ are all non-zero. $\square$

Suppose $\mathcal{A} \hookrightarrow \mathcal{N}$ are G-linearized locally free sheaves of rank 1, N respectively, over $\operatorname{Spec} k$. Let $Y \stackrel{def}{=} Gr_{r-1}(\mathcal{N}/\mathcal{A})$ and $\mathcal{Q}^Y$ be its universal quotient. Let $\mathcal{K}$ be the kernel of the composition $\mathcal{N}_Y \to (\mathcal{N}/\mathcal{A})_Y \to \mathcal{Q}^Y$. Define a map $g : Gr_1(\mathcal{K}) \to Gr_r(\mathcal{N})$ as the following.

For a point $(y, \mathcal{H})$ in $Gr_1(\mathcal{K})$, we get an exact sequence

$$0 \to \mathcal{G} \to \mathcal{K}|_y \to \mathcal{H} \to 0$$

where the rank of $\mathcal{G}$ will be $N - r$. Since $\mathcal{K}|_y \hookrightarrow \mathcal{N}|_y = \mathcal{N}$, we can consider $\mathcal{N}/\mathcal{G}$, which is of rank r. Thus, we define

$$g(y, \mathcal{H}) \stackrel{def}{=} \mathcal{N}/\mathcal{G}.$$

Proposition 6.2. *The map $g : Gr_1(\mathcal{K}) \to Gr_r(\mathcal{N})$ constructed above is equivariantly isomorphic to the map corresponding to the blow up of $Gr_r(\mathcal{N})$ along $\sigma_{1,r}(\mathcal{A}, \mathcal{N})$.*

Proof. This is the analog of Theorem 4.4 in [Kl]. First of all, it is not hard to see that g is equivariant. Let $X \stackrel{def}{=} Gr_r(\mathcal{N})$, $Y \stackrel{def}{=} Gr_1(\mathcal{K})$ and $\tilde{X} \stackrel{def}{=} \mathrm{Blow}_{\sigma_{1,r}(\mathcal{A},\mathcal{N})} Gr_r(\mathcal{N})$. Also denote the blow up map from $\tilde{X}$ to X by π. By Theorem 4.4 in [Kl], there exists an isomorphism $\mu : \tilde{X} \to Y$ such that $g \circ \mu = \pi$. So, it is enough to show μ is equivariant. Take an invariant open subscheme $U \subseteq X$ such that $g|_U$ and $\pi|_U$ are both isomorphisms. Since $g|_U, \pi|_U$ are both equivariant, the map $\mu|_U = (g|_U)^{-1} \circ \pi|_U$ is also equivariant. Now, a map being equivariant is a closed condition. Hence, μ is equivariant. □

Theorem 6.3. *Suppose $X \in G\text{-}Sm$ is G-irreducible and $\mathcal{E}$ is a G-linearized locally free sheaf of rank r over X. Then, there exists an equivariant morphism $f : \tilde{X} \to X$, which is the composition of a series of blow ups along invariant smooth centers, and a G-linearized invertible subsheaf $\mathcal{L} \hookrightarrow f^*\mathcal{E}$ over $\tilde{X}$ such that the sequence*

$$0 \to \mathcal{L} \to f^*\mathcal{E} \to (f^*\mathcal{E})/\mathcal{L} \to 0$$

is exact and $(f^*\mathcal{E})/\mathcal{L}$ is locally free with rank $r - 1$.

Proof. Let d be the dimension of X. The result is trivially true if $d = 0$, so we may assume $d \geq 1$. By Proposition 4.13, we can embed X into $\mathbb{P}(\mathcal{N}_2)$ for some G-linearized locally free sheaf $\mathcal{N}_2$ over $\operatorname{Spec} k$. Denote $\mathcal{E} \otimes \mathcal{O}_X(m)$ by $\mathcal{E}(m)$ for simplicity. Assume X is projective first. Let $\mathcal{N}_1$ be the G-linearized locally free sheaf over $\operatorname{Spec} k$ corresponding to $\mathrm{H}^0(X, \mathcal{E}(m))$. For a sufficiently large m, we can assume the induced map $(\mathcal{N}_1)_X \to \mathcal{E}(m)$ is surjective and defines an equivariant immersion $X \hookrightarrow Gr_r(\mathcal{N}_1)$, which sends x to $\mathcal{E}(m)|_x$. Then, we define a twisted equivariant immersion $i : X \hookrightarrow Gr_r(\mathcal{N}) \overset{def}{=} Y$ as the Segre product of $X \hookrightarrow Gr_r(\mathcal{N}_1)$ and $X \hookrightarrow \mathbb{P}(\mathcal{N}_2)$. In particular, $\mathcal{N} \cong \mathcal{N}_1 \otimes \mathcal{N}_2$.

By construction, $i^*\mathcal{Q}^Y \cong \mathcal{E}(m+1)$ where $\mathcal{Q}^Y$ is the universal quotient of Y. Since dim $\mathrm{H}^0(X, \mathcal{E}(m))$ is a polynomial of m with degree d, we may assume there is a 1-dimensional character ψ such that the ψ component $\mathrm{H}^0(X, \mathcal{E}(m))_\psi$ has dimension much larger than r.

If X is not projective, we can pick $\mathcal{N}_1$ to be a sheaf corresponding to some finite dimensional G-representation inside $\mathrm{H}^0(X, \mathcal{E}(m))$ and construct $i : X \hookrightarrow Y$ in the same manner.

Let $\mathcal{A}$ be the universal subbundle of $Gr_{N-1}(\mathcal{N})$. Let V_1, V_2 and V be the G-representations corresponding to $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}$ respectively. Then, the dimension of the ψ component of V_1 is much larger than r by construction. Thus, there is a 1-dimensional character ψ' such that the dimension of the ψ' component of V is much larger than r. Hence, by Proposition 6.1, there exists a closed point z of the fixed point locus of $Gr_{N-1}(\mathcal{N})$, with residue field $k(z) \cong k$, such that $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N}) \subseteq Y$ is smooth with codimension r and $X \cap \sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ has dimension $d - r$.

For such z, denote $\sigma_{1,r}(\mathcal{A}|_z, \mathcal{N})$ by σ for simplicity. Then, we have smooth invariant closed subschemes X, σ of Y with dimension d and $\dim Y - r$ respectively. Moreover, $X \cap \sigma$ has dimension $d - r$. By applying the embedded desingularization theorem in [BiMi] on $X \cup \sigma \hookrightarrow Y$, we obtain the following commutative diagram :

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{i'} & Y' \\ f\downarrow & & \downarrow p \\ X & \xrightarrow{i} & Y \end{array}$$

where $p : Y' \to Y$ is the composition of a series of blow ups along smooth invariant centers and $f : \tilde{X} \to X$ is the map corresponding to the strict transform of X. In addition, $\tilde{X} \cup \langle\sigma\rangle$ (denote the strict transform by $\langle\ \rangle$) is smooth and if E is the sum of the exceptional divisors on Y', then $\tilde{X}$, $\langle\sigma\rangle$ and E will intersect transversely. Since X and σ are both smooth and do not contain each other, according to the Theorem 1.6 in [BiMi], it is not hard to see that each smooth invariant center is either a proper closed subscheme of the strict transform of X, or a subscheme away from it. Hence, f is the composition of a series of blow ups along smooth invariant centers.

Observe that $\tilde{X}$ and $\langle\sigma\rangle$ are disjoint because $\tilde{X} \cup \langle\sigma\rangle$ is smooth. In addition,

$$i'^{-1} \circ p^{-1}(\sigma) = i'^{-1}(\langle\sigma\rangle \cup E) = \tilde{X} \cap (\langle\sigma\rangle \cup E) = \tilde{X} \cap E,$$

which is an invariant divisor on $\tilde{X}$. By the universal property of blow up, there is a unique map $j : \tilde{X} \to \mathrm{Blow}_\sigma Y$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \mathrm{Blow}_\sigma Y \\ i' \downarrow & & \downarrow q \\ Y' & \xrightarrow{p} & Y \end{array}$$

Since z is a fixed point with $k(z) \cong k$, the sheaf $\mathcal{A}|_z$ is a G-linearized locally free sheaf of rank 1 over $\mathrm{Spec}\, k$ and it is naturally embedded inside $\mathcal{N}$. Following the construction before. Let $Y_1 \stackrel{def}{=} Gr_{r-1}(\mathcal{N}/\mathcal{A}|_z)$, $\mathcal{Q}^{Y_1}$ be its universal quotient, $\mathcal{K}$ be the kernel of $\mathcal{N}_{Y_1} \to \mathcal{Q}^{Y_1}$ and $\tilde{Y} \stackrel{def}{=} Gr_1(\mathcal{K})$. By Proposition 6.2, the equivariant map $g : \tilde{Y} \to Y$ is equivariantly isomorphic to $q : \mathrm{Blow}_\sigma Y \to Y$. Moreover, as pointed out in (4.1) in [Kl], there is an exact sequence

$$0 \to \mathcal{L}' \stackrel{def}{=} \mathcal{Q}^{\tilde{Y}} \to g^*\mathcal{Q}^Y \to (\mathcal{Q}^{Y_1})_{\tilde{Y}} \to 0 \tag{9}$$

of G-linearized locally free sheaves over $\tilde{Y}$ where $\mathcal{L}'$ is of rank 1.

Consider the following commutative diagram :

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{j} & \mathrm{Blow}_\sigma Y & = & \mathrm{Blow}_\sigma Y \\ f\downarrow & & q\downarrow & & \mu\downarrow \\ X & \xrightarrow{i} & Y & \xleftarrow{g} & \tilde{Y} \end{array}$$

On one hand, $f^*i^*Q^Y \cong f^*\mathcal{E}(m+1)$. On the other hand, if we pull back the exact sequence (9) by μ and then j. We got an exact sequence of G-linearized locally free sheaves over $\tilde{X}$

$$0 \to j^*\mu^*\mathcal{L}' \to f^*\mathcal{E}(m+1) \cong j^*\mu^*g^*\mathcal{Q}^Y \to j^*\mu^*(\mathcal{Q}^{Y_1})_{\tilde{Y}} \to 0.$$

The result then follows by twisting the whole sequence by $f^*\mathcal{O}_X(-m-1)$.

□

6.2. **Basic structure of G-linearized invertible sheaves.** In this subsection, we will state and prove some results about the structure of G-linearized invertible sheaves over some $X \in G\text{-}Sm$.

Lemma 6.4. *For any $X \in G\text{-}Sm$, we have*

$$\operatorname{kernel}\{\operatorname{Pic}^G(X) \to \operatorname{Pic}(X)\} = \pi_k^* \operatorname{Pic}^G(\operatorname{Spec} k)$$

where $\operatorname{Pic}^G(X) \to \operatorname{Pic}(X)$ is the forgetful map.

Proof. Finding the kernel of the forgetful map is the same as asking how many G-linearizations can $\mathcal{O}_X$ have. A G-linearization of $\mathcal{O}_X$ can be described by a set of isomorphisms

$$\{\alpha^* : \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X \mid \alpha \in G\}.$$

Each isomorphism α^* induces an isomorphism

$$\alpha^* : \mathrm{H}^0(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{H}^0(X, \mathcal{O}_X)$$

which sends 1 to some element $a_\alpha \in \mathrm{H}^0(X, \mathcal{O}_X)$. Since $a_\alpha^e = 1$ (e is the exponent of G) and the pair (G, k) is split, a_α is in k^*. In other words, there exists a 1-dimensional character χ such that $\alpha^*(1) = \chi(\alpha)$ for all $\alpha \in G$. Then, the result follows from the one to one correspondence between the set of 1-dimensional characters and $\operatorname{Pic}^G(\operatorname{Spec} k)$. $\square$

Proposition 6.5. *Suppose $X \in G\text{-}Sm$ is G-irreducible and $\mathcal{L}$ is a G-linearized invertible sheaf over X. Then, there exist an invariant divisor D on X and a sheaf $\mathcal{N} \in \operatorname{Pic}^G(\operatorname{Spec} k)$ such that*

$$\mathcal{L} \cong \mathcal{O}_X(D) \otimes \pi_k^* \mathcal{N}.$$

Proof. Without loss of generality, we may assume the action on X is faithful. Let U be a non-empty, invariant open subscheme of X such that the action on U is free. By Theorem 1 in section 7 of [Mu], the geometric quotient U/G exists as a variety over k and $\pi : U \to U/G$ is an étale morphism. By picking a smaller U, we may further assume U/G to be smooth. Let $D_1, \ldots, D_n$ be some invariant divisors on X such that $D_i \subseteq X - U$ for all i and the codimension of $X - U - \cup_i D_i$ in X is at least 2.

Claim 1 : The kernel of the restriction map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(U)$ is generated by $\{\mathcal{O}_X(D_i)\}$ and $\mathrm{Pic}^G(\mathrm{Spec}\, k)$.

Consider the following commutative diagram :

$$\begin{array}{ccccccc}
 & & \mathrm{Pic}^G(\mathrm{Spec}\, k) & & & & \\
 & & \pi_k^* \downarrow & & & & \\
\mathbb{Z}^n & \xrightarrow{a} & \mathrm{Pic}^G(X) & \xrightarrow{c} & \mathrm{Pic}^G(U) & & \\
\mathbb{I}\downarrow & & b\downarrow & & b\downarrow & & \\
\mathbb{Z}^n & \xrightarrow{a} & \mathrm{Pic}(X) & \xrightarrow{c} & \mathrm{Pic}(U) & \longrightarrow & 0
\end{array}$$

where a sends "1" in the i-th position to $\mathcal{O}_X(D_i)$, b is the forgetful map and c is the restriction map.

Clearly, the third row is exact. Moreover, by Lemma 6.4, the second column is also exact. Then, the result follows from some diagram chasing. $\triangle$

Since the action on U is free, according to Proposition 2 in section 7 in [Mu], there is a one-to-one correspondence between $\mathrm{Pic}^G(U)$ and $\mathrm{Pic}(U/G)$. In particular, $\pi_*(\mathcal{L}|_U)^G$ is an invertible sheaf over U/G. Since U/G is smooth, there is a divisor D' on U/G such that $\pi_*(\mathcal{L}|_U)^G \cong \mathcal{O}_{U/G}(D')$. Thus, we have

$$\begin{aligned}
\mathcal{L}|_U &\cong \pi^*(\pi_*(\mathcal{L}|_U)^G) \\
&\cong \pi^*\mathcal{O}_{U/G}(D') \\
&\cong \mathcal{O}_U(\pi^* D') \\
&\quad (\pi : U \to U/G \text{ is étale}).
\end{aligned}$$

Consider the sheaf $\mathcal{O}_X(D'') \in \mathrm{Pic}^G(X)$ where D'' is the invariant divisor on X given by the closure of $\pi^* D'$ in X. Hence, $\mathcal{L} \otimes \mathcal{O}_X(-D'')$ will be in the kernel of the restriction map $\mathrm{Pic}^G(X) \to \mathrm{Pic}^G(U)$. By claim 1, there are integers $\{m_i\}$ and a sheaf $\mathcal{N} \in \mathrm{Pic}^G(\mathrm{Spec}\, k)$ such that

$$\mathcal{L} \otimes \mathcal{O}_X(-D'') \cong \mathcal{O}_X(\sum_i m_i D_i) \otimes \pi_k^* \mathcal{N}.$$

The result then follows by defining $D \overset{def}{=} D'' + \sum_i m_i D_i$. $\square$

6.3. **Reduction of towers.** Next, we will define the notion of quasi-admissible tower and admissible tower and prove we can reduce an quasi-admissible tower into something much simplier. This subsection is an analog of section 7 in [LeP].

Definition 6.6. Suppose Y is an object in G-Sm. A morphism $\mathbb{P} \to Y$ in G-Sm is called a quasi-admissible tower over Y with length n if it can be factored into

$$\mathbb{P} = \mathbb{P}_n \to \mathbb{P}_{n-1} \to \cdots \to \mathbb{P}_1 \to \mathbb{P}_0 = Y$$

such that, for all $0 \leq i \leq n-1$, $\mathbb{P}_{i+1} = \mathbb{P}(\mathcal{E}_i)$ where $\mathcal{E}_i$ is the direct sum of sheaves which is either the pull-back of a G-linearized locally free sheaves over Y, or the pull back of $\mathcal{O}_{\mathbb{P}_j}(m)$ for some integer m and $1 \leq j \leq i$.

In this subsection, for an object $Y \in G$-Sm, an invariant divisor D on Y and a G-linearized locally free sheaf $\mathcal{E}$ over Y, we will denote $\mathcal{E} \otimes \mathcal{O}_Y(D)$ by $\mathcal{E}(D)$ for simplicity. Moreover, if $\mathbb{P} \to Y$ is a quasi-admissible tower, then we will denote the pull-back of $\mathcal{E}$ as a sheaf over $\mathbb{P}_i$ by $\mathcal{E}$ if there is no confusion.

Definition 6.7. Suppose Y is an object in G-Sm. We will call a sheaf $\mathcal{L} \in \mathrm{Pic}^G(Y)$ admissible if there exist invariant smooth divisors $D_1, \ldots, D_k$ on Y and a sheaf $\mathcal{N} \in \mathrm{Pic}^G(\mathrm{Spec}\, k)$ such that

$$\mathcal{L} \cong \mathcal{O}_Y(\textstyle\sum_{i=1}^k m_i D_i) \otimes \pi_k^* \mathcal{N}$$

for some integers $\{m_i\}$. Denote the subgroup of $\mathrm{Pic}^G(Y)$ generated by admissible invertible sheaves by $\mathrm{APic}^G(Y)$. Also, define the group of admissible invertible sheaves over $\mathbb{P}_i$ by

$$\mathrm{APic}^G(\mathbb{P}_i) \overset{def}{=} \mathrm{APic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1).$$

Then, a quasi-admissible tower $\mathbb{P} \to Y$ is called admissible if all sheaves involved in the construction are admissible invertible sheaves.

Remark 6.8. If all the G-linearized locally free sheaves involved in the construction of a tower $\mathbb{P} \to Y$ are invertible, then it is a quasi-admissible tower.

Proof. Since

$$\mathrm{Pic}(\mathbb{P}_i) = \mathrm{Pic}(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1)$$

and, by Lemma 6.4, the kernel of the forgetful map $\mathrm{Pic}^G(\mathbb{P}_i) \to \mathrm{Pic}(\mathbb{P}_i)$ is given by $\mathrm{Pic}^G(\mathrm{Spec}\, k)$, we have

$$\mathrm{Pic}^G(\mathbb{P}_i) = \mathrm{Pic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1).$$

Then, $\mathbb{P} \to Y$ is a quasi-admissible tower by definition. □

Lemma 6.9. *Suppose $Y \in G$-Sm is G-irreducible and $\mathcal{L}$ is a sheaf in $\mathrm{Pic}^G(Y)$. Moreover, $\mathcal{E}$ is the direct sum of a finite number of invertible sheaves in $\mathrm{Pic}^G(Y)$ and D is an invariant smooth divisor on Y. Let*

$$\begin{aligned} A &\stackrel{def}{=} \mathbb{P}(\mathcal{E} \oplus \mathcal{L} \oplus \mathcal{L}(D))|_D, \\ B &\stackrel{def}{=} \mathbb{P}(\mathcal{E} \oplus \mathcal{L}), \\ C &\stackrel{def}{=} \mathbb{P}(\mathcal{E} \oplus \mathcal{L}(D)), \\ \mathbb{P} &\stackrel{def}{=} \mathbb{P}(\mathcal{E} \oplus \mathcal{L} \oplus \mathcal{L}(D)). \end{aligned}$$

Then A, B, C are invariant smooth divisors on $\mathbb{P}$, the sum of them is a reduced strict normal crossing divisor, $A + B \sim C$ and

$$\begin{aligned} \mathcal{O}_{\mathbb{P}}(A) &\cong \mathcal{O}_{\mathbb{P}}(\pi^* D) \\ \mathcal{O}_{\mathbb{P}}(B) &\cong (\pi^*\mathcal{L}(D))^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \mathcal{O}_{\mathbb{P}}(C) &\cong (\pi^*\mathcal{L})^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

where π is the projection $\mathbb{P} \to Y$.

Proof. The fact that A, B, C are smooth divisors on $\mathbb{P}$ and the sum of them is a reduced strict normal crossing divisor was stated in section 7.2 in [LeP]. They are obviously invariant. Since π is smooth, $O_{\mathbb{P}}(A) \cong \mathcal{O}_{\mathbb{P}}(\pi^* D)$. Moreover, as in the proof of Lemma 7.1 in [LeP], $\mathbb{P}(\mathcal{E} \oplus \mathcal{L}) \subseteq \mathbb{P}(\mathcal{E} \oplus \mathcal{L} \oplus \mathcal{L}(D))$ is given by the vanishing of the composition of equivariant morphisms

$$\pi^*\mathcal{L}(D) \to \pi^*(\mathcal{E} \oplus \mathcal{L} \oplus \mathcal{L}(D)) \to \mathcal{O}_{\mathbb{P}}(1).$$

Hence,

$$O_{\mathbb{P}}(B) = O_{\mathbb{P}}(\mathbb{P}(\mathcal{E} \oplus \mathcal{L})) \cong (\pi^*\mathcal{L}(D))^\vee \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Similarly,

$$O_{\mathbb{P}}(C) = O_{\mathbb{P}}(\mathbb{P}(\mathcal{E} \oplus \mathcal{L}(D))) \cong (\pi^*\mathcal{L})^\vee \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Then, we have

$$\begin{aligned}
\mathcal{O}_{\mathbb{P}}(A) \otimes \mathcal{O}_{\mathbb{P}}(B) &\cong \mathcal{O}_{\mathbb{P}}(\pi^*D) \otimes (\pi^*\mathcal{L}(D))^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \\
&\cong \pi^*\mathcal{O}_Y(D) \otimes \pi^*\mathcal{L}(D)^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \\
&\cong \pi^*\mathcal{L}^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \\
&\cong \mathcal{O}_{\mathbb{P}}(C)
\end{aligned}$$

By remark 3.2, that implies $A + B \sim C$. □

Lemma 6.10. *Suppose Y is G-irreducible and D is an invariant smooth divisor on Y. If $\mathbb{P} \to Y$ is an admissible tower with length n and $\mathbb{P}_{i+1} = \mathbb{P}(\oplus_{j=1}^r \mathcal{L}_j)$, then there exist an admissible tower $\mathbb{P}' \to Y$ of length n and quasi-admissible towers $Q_0, Q_1, Q_2, Q_3 \to D$ such that*

$$\mathbb{P}' = \mathbb{P}'_n \to \cdots \to \mathbb{P}'_{i+1} \to \mathbb{P}_i \to \cdots \to \mathbb{P}_0 = Y$$

where $\mathbb{P}'_{i+1} = \mathbb{P}((\oplus_{j=1}^{r-1} \mathcal{L}_j) \oplus \mathcal{L}_r(D))$ and we have the following equality in $\mathcal{U}_G(Y)$:

$$[\mathbb{P}' \to Y] - [\mathbb{P} \to Y] = [Q_0 \to D \to Y] - [Q_1 \to D \to Y] + [Q_2 \to D \to Y] - [Q_3 \to D \to Y].$$

Proof. Let $\hat{\mathbb{P}}_{i+1} \overset{def}{=} \mathbb{P}((\oplus_{j=1}^r \mathcal{L}_j) \oplus \mathcal{L}_r(D))$. Then, we have $\hat{\mathbb{P}}_{i+1} \to \mathbb{P}_i$ and $\mathbb{P}_{i+1} \hookrightarrow \hat{\mathbb{P}}_{i+1}$.

We will first construct an admissible tower

$$\hat{\mathbb{P}} = \hat{\mathbb{P}}_n \to \cdots \to \hat{\mathbb{P}}_{i+1} \to \mathbb{P}_i \to \cdots \to \mathbb{P}_0 = Y$$

such that $\mathbb{P}_k = \mathbb{P}_{i+1} \times_{\hat{\mathbb{P}}_{i+1}} \hat{\mathbb{P}}_k$ for all $k > i$. Since

$$\mathrm{APic}^G(\mathbb{P}_{i+1}) = \mathrm{APic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_{i+1}}(1)$$

$$\mathrm{APic}^G(\hat{\mathbb{P}}_{i+1}) = \mathrm{APic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1) + \mathbb{Z}\mathcal{O}_{\hat{\mathbb{P}}_{i+1}}(1)$$

and the restriction map $\mathrm{Pic}^G(\hat{\mathbb{P}}_{i+1}) \to \mathrm{Pic}^G(\mathbb{P}_{i+1})$ sends $\mathcal{O}_{\hat{\mathbb{P}}_{i+1}}(1)$ to $\mathcal{O}_{\mathbb{P}_{i+1}}(1)$, if we write $\mathbb{P}_{i+2} = \mathbb{P}(\oplus \mathcal{L}'_{j'})$ for some $\mathcal{L}'_{j'} \in \mathrm{APic}^G(\mathbb{P}_{i+1})$, then we can define $\hat{\mathbb{P}}_{i+2} \overset{def}{=} \mathbb{P}(\oplus \mathcal{L}'_{j'})$ by considering $\mathcal{L}'_{j'}$ as in $\mathrm{APic}^G(\hat{\mathbb{P}}_{i+1})$. Similarly, for higher levels, $\mathrm{APic}^G(\hat{\mathbb{P}}_k) \to \mathrm{APic}^G(\mathbb{P}_k)$ is surjective and $\hat{\mathbb{P}}_{k+1}$ can be constructed.

Next, we will construct the admissible tower $\mathbb{P}' \to Y$ and quasi-admissible tower $Q_0 \to Y$. As in the statement, $\mathbb{P}'_{i+1} \overset{def}{=} \mathbb{P}((\oplus_{j=1}^{r-1} \mathcal{L}_j) \oplus \mathcal{L}_r(D))$, which can be naturally embedded inside $\hat{\mathbb{P}}_{i+1}$. Then, we define $\mathbb{P}'_k \overset{def}{=} \mathbb{P}'_{i+1} \times_{\hat{\mathbb{P}}_{i+1}} \hat{\mathbb{P}}_k$ for all $k > i+1$, which is clearly admissible. The quasi-admissible tower Q_0 are defined by pull-back, i.e. $(Q_0)_j \overset{def}{=} D \times_Y \hat{\mathbb{P}}_j$ for all $0 \leq j \leq n$.

By Lemma 6.9, $\mathbb{P}_{i+1}$, $\mathbb{P}'_{i+1}$ and $(Q_0)_{i+1}$ are all invariant smooth divisors on $\hat{\mathbb{P}}_{i+1}$, the sum of them is a reduced strict normal crossing divisor and $(Q_0)_{i+1} + \mathbb{P}_{i+1} \sim \mathbb{P}'_{i+1}$. Pull them back to the top level, we have $Q_0 + \mathbb{P} \sim \mathbb{P}'$ as invariant smooth divisors on $\hat{\mathbb{P}}$. By $GDPR(2,1)$, we have

$$
\begin{aligned}
[\mathbb{P}' \hookrightarrow \hat{\mathbb{P}}] &= [Q_0 \hookrightarrow \hat{\mathbb{P}}] + [\mathbb{P} \hookrightarrow \hat{\mathbb{P}}] \\
&- [(Q_0 \cap \mathbb{P}) \times_{\hat{\mathbb{P}}} P^1 \to \hat{\mathbb{P}}] \\
&+ [(Q_0 \cap \mathbb{P} \cap \mathbb{P}') \times_{\hat{\mathbb{P}}} P^2 \to \hat{\mathbb{P}}] \\
&- [(Q_0 \cap \mathbb{P} \cap \mathbb{P}') \times_{\hat{\mathbb{P}}} P^3 \to \hat{\mathbb{P}}]
\end{aligned} \tag{10}
$$

as elements in $\mathcal{U}_G(\hat{\mathbb{P}})$, where

$$
\begin{aligned}
P^1 &\overset{def}{=} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(Q_0)) \\
P^2 &\overset{def}{=} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-\mathbb{P}) \oplus \mathcal{O}(-\mathbb{P}')) \\
P^3 &\overset{def}{=} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-\mathbb{P}) \oplus \mathcal{O}(-\mathbb{P}')).
\end{aligned}
$$

We then denote $(Q_0 \cap \mathbb{P}) \times_{\hat{\mathbb{P}}} P^1$, $(Q_0 \cap \mathbb{P} \cap \mathbb{P}') \times_{\hat{\mathbb{P}}} P^2$ and $(Q_0 \cap \mathbb{P} \cap \mathbb{P}') \times_{\hat{\mathbb{P}}} P^3$ by Q_1, Q_2 and Q_3 respectively. They all clearly lie over D. Since the towers Q_1, Q_2, $Q_3 \to D$ are all constructed by G-linearized invertible sheaves, by Remark 6.8, they are all quasi-admissible towers. Hence, the result follows by pushing down equality (10) to $\mathcal{U}_G(Y)$. ∎

Remark 6.11. Notice that $\mathbb{P}'_j = \mathbb{P}_j$ for all $j < i+1$. For $j > i+1$, if we identify the admissible invertible sheaves over $\mathbb{P}_{j-1}$ that comes from Y to those over $\mathbb{P}'_{j-1}$ and also the sheaves of the form $\mathcal{O}(m)$ for some integer m, then $\mathbb{P}'_j$ is defined by the exact same set of admissible invertible sheaves as $\mathbb{P}_j$.

Lemma 6.12. *Suppose Y is an object in G-Sm and $\mathcal{E}$ is a G-linearized locally free sheaf of rank r over Y. Furthermore, there exists an exact sequence of G-linearized sheaves over Y*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0$$

such that $\mathcal{L}$ and $\mathcal{E}/L$ are locally free of rank 1, $r-1$ respectively. Then,

$$\mathbb{P}(\mathcal{E}) \sim \mathbb{P}((\mathcal{E}/\mathcal{L}) \oplus \mathcal{L})$$

as invariant smooth divisors on $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ and they intersect transversely.

Proof. Without loss of generality, we may assume Y is G-irreducible. $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}((\mathcal{E}/\mathcal{L}) \oplus \mathcal{L})$ are obviously invariant smooth divisors on $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ and their intersection is $\mathbb{P}(\mathcal{E}/\mathcal{L})$. So, we only need to prove they are equivariantly linearly equivalent.

Ignore the G-action first. Locally, over an affine open subscheme U_i, we have $\mathcal{E} \cong Re_{i1} \oplus \cdots \oplus Re_{ir}$ where $R \overset{def}{=} \mathcal{O}_Y(U_i)$. Similarly, $\mathcal{L} \cong Rf_i$. Let $\phi : \mathcal{L} \hookrightarrow \mathcal{E}$ be the embedding of sheaves as in the statement. For simplicity, denote $\mathbb{P}(\mathcal{E} \oplus \mathcal{L})$ by $\mathbb{P}$, $\mathbb{P}(\mathcal{E})$ by A and $\mathbb{P}((\mathcal{E}/\mathcal{L}) \oplus \mathcal{L})$ by B. Locally, $\mathbb{P} = \text{Proj } R[e_{i1}, \ldots, e_{ir}, f_i]$, A is defined by $f_i = 0$ and B is defined by $\phi(f_i) = 0$. So, it is enough to show $g \overset{def}{=} f_i/\phi(f_i) \in \mathcal{K}(\mathbb{P})^*$ is independent of i, namely, $f_i/\phi(f_i) = f_j/\phi(f_j)$.

On the intersection $U_i \cap U_j$, we can consider the ratio $f_i/f_j \overset{def}{=} \sigma_{ij} \in \mathcal{O}(U_i \cap U_j)^*$, which defines the transition function of $\mathcal{L}$. On the other hand, since $\phi : \mathcal{L} \hookrightarrow \mathcal{E}$ is a morphism between sheaves, we can also consider the ratio $\phi(f_i)/\phi(f_j)$ and it should be σ_{ij} too. That means

$$\frac{f_i}{f_j} = \sigma_{ij} = \frac{\phi(f_i)}{\phi(f_j)}.$$

Hence, g is independent of i. Finally,

$$\alpha \cdot g = \alpha \cdot \frac{f_i}{\phi(f_i)} = \frac{\alpha \cdot f_i}{\alpha \cdot \phi(f_i)} = \frac{\alpha \cdot f_i}{\phi(\alpha \cdot f_i)} = g.$$

□

The following result is an analog of Lemma 5.1 in [LeP].

Lemma 6.13. *If X is in G-Sm and Z is an invariant smooth closed subscheme of X, then, as elements in $\mathcal{U}_G(X)$,*

$$[\mathrm{Blow}_Z X \to X] - [\mathbb{I}_X] = -[\mathbb{P}_1 \to Z \hookrightarrow X] + [\mathbb{P}_2 \to Z \hookrightarrow X]$$

for some projective morphisms $\mathbb{P}_1$, $\mathbb{P}_2 \to Z$ in G-Sm.

Proof. Without loss of generality, X is G-irreducible. Let $Y \overset{def}{=} \mathrm{Blow}_{Z\times 0}(X \times \mathbb{P}^1)$ (trivial action on $\mathbb{P}^1$). Consider the projective map $Y \to X \times \mathbb{P}^1$. For any closed point $\xi \neq 0$ in $\mathbb{P}^1$, we have $[Y_\xi \to X] = [\mathbb{I}_X]$, where Y_ξ denotes the fiber of Y over ξ as before. Consider the fiber of Y over 0, we have $Y_0 = A \cup B$ where $A \overset{def}{=} \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{N}^\vee_{Z\hookrightarrow X})$ (the exceptional divisor) and $B \overset{def}{=} \mathrm{Blow}_Z X$ (the strict transform of X). In addition, $A \cap B = \mathbb{P}(\mathcal{N}^\vee_{Z\hookrightarrow X})$. Hence, Y_ξ, A, B are all invariant smooth divisors on Y and A, B intersect transversely. In other words, $Y \to X \times \mathbb{P}^1$ defines an equivariant DPR. By Proposition 3.16, we have

$$\begin{aligned}
&[\mathbb{I}_X] \\
=\ & [\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{N}^\vee_{Z\hookrightarrow X}) \to X] + [\mathrm{Blow}_Z X \to X] - [\mathbb{P}(\mathcal{O}_{A\cap B} \oplus \mathcal{O}_{A\cap B}(A)) \to A \cap B \to X] \\
=\ & [\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{N}^\vee_{Z\hookrightarrow X}) \to Z \hookrightarrow X] + [\mathrm{Blow}_Z X \to X] - [\mathbb{P}(\mathcal{O}_{A\cap B} \oplus \mathcal{O}_{A\cap B}(A)) \to Z \hookrightarrow X].
\end{aligned}$$

Then, the result follows from defining $\mathbb{P}_1 \overset{def}{=} \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{N}^\vee_{Z\hookrightarrow X})$ and $\mathbb{P}_2 \overset{def}{=} \mathbb{P}(\mathcal{O}_{A\cap B} \oplus \mathcal{O}_{A\cap B}(A))$ and the fact that $A \cap B = \mathbb{P}(\mathcal{N}^\vee_{Z\hookrightarrow X})$ is projective over Z. □

Remark 6.14. We can express $\mathbb{P}_2$ in a different way. Consider the following commutative diagram :

$$\begin{array}{ccc}
\mathbb{P}(\mathcal{O}_{A\cap B} \oplus \mathcal{O}_{A\cap B}(A)) & \longrightarrow & \mathbb{P}(\mathcal{N}^\vee_{Z\hookrightarrow X}) = A \cap B \\
\downarrow & & \downarrow \\
\mathbb{P}(\mathcal{O}_A \oplus \mathcal{O}_A(A)) & \longrightarrow & \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{N}^\vee_{Z\hookrightarrow X}) = A
\end{array}$$

Since A is the exceptional divisor of the blowup $Y \to X \times \mathbb{P}^1$, we have $\mathcal{O}_A(A) \cong \mathcal{O}_A(-1)$. Thus, $\mathcal{O}_{A\cap B}(A) \cong \mathcal{O}_{A\cap B}(-1)$. Hence,

$$\mathbb{P}_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{N}^\vee_{Z\hookrightarrow X}) \to Z$$

$$\mathbb{P}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}(\mathcal{N}^{\vee}_{Z \hookrightarrow X}) \to Z.$$

Definition 6.15. Define $\mathcal{U}_G(\operatorname{Spec} k)'$ to be the abelian subgroup of $\mathcal{U}_G(\operatorname{Spec} k)$ generated by admissible towers over $\operatorname{Spec} k$.

Remarks 6.16. If $\mathbb{P} \to \operatorname{Spec} k$ and $\mathbb{P}' \to \operatorname{Spec} k$ are two admissible towers, then the product $\mathbb{P} \times \mathbb{P}' \to \mathbb{P} \to \operatorname{Spec} k$ is also an admissible tower over $\operatorname{Spec} k$. In other words, $\mathcal{U}_G(\operatorname{Spec} k)'$ is a subring of $\mathcal{U}_G(\operatorname{Spec} k)$.

Proposition 6.17. *For any quasi-admissible tower $\mathbb{P} \to Y$ where Y is G-irreducible, there exist elements $a_i \in \mathcal{U}_G(\operatorname{Spec} k)'$ and maps $Y_i' \to Y$ in G-Sm with $\dim Y_i' \leq \dim Y$ such that*

$$[\mathbb{P} \to Y] = \sum_i a_i \, [Y_i' \to Y]$$

as elements in $\mathcal{U}_G(Y)$.

Proof. We will prove the statement by induction on dimension of Y. We will handle the induction step first. Suppose $\dim Y \geq 1$. Let $\mathcal{U}_G(Y)'$ be the subgroup of $\mathcal{U}_G(Y)$ generated by elements of the form $[\mathbb{P} \to Y' \to Y]$ where $Y' \in G\text{-}Sm$ is G-irreducible with dimension less than $\dim Y$ and $\mathbb{P} \to Y'$ is a quasi-admissible tower. So, elements in $\mathcal{U}_G(Y)'$ will be handled by the induction assumption. Let $\mathbb{P} \to Y$ be a quasi-admissible tower. If the length of the tower n is 0, then we are done. Suppose $n \geq 1$.

Step 1 : Reduction to a quasi-admissible tower constructed only by G-linearized invertible sheaves.

Define the integer "total rank" as the sum of ranks of all sheaves involved in all levels. Also, define the integer "number of sheaves" as the number of sheaves in all levels. For example, the tower $\mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_2 \oplus \mathcal{E}_3) \to Y$ has total rank = rank $\mathcal{E}_1$ + rank $\mathcal{E}_2$ + rank $\mathcal{E}_3$ and number of sheaves 3.

Assume that, for the tower $\mathbb{P} \to Y$, number of sheaves is less than total rank. Then, there exists a sheaf $\mathcal{E}$, which is used in the construction of some level $\mathbb{P}_i$, has rank greater than 1. Notice that $\mathcal{E}$ has to come from Y because the tower is quasi-admissible. Let $\mathbb{P}_i \overset{def}{=} \mathbb{P}((\oplus_j \mathcal{E}_j) \oplus \mathcal{E})$. By Theorem 6.3, there exists a map $\pi : \tilde{Y} \to Y$, which is the composition of a series of blow ups along invariant smooth centers with dimensions less than

$\dim Y$, and a G-linearized invertible sheaf $\mathcal{L}$ over $\tilde{Y}$ such that the sequence of G-linearized sheaves

$$0 \to \mathcal{L} \to \pi^*\mathcal{E} \to (\pi^*\mathcal{E})/\mathcal{L} \to 0$$

is exact and $(\pi^*\mathcal{E})/\mathcal{L}$ is locally free with rank $r-1$.

Define the tower $\tilde{\mathbb{P}} \to \tilde{Y}$ by pulling back each level, namely $\tilde{\mathbb{P}}_i = \mathbb{P}_i \times_Y \tilde{Y}$. Then, the sheaves in the construction at each level of $\tilde{\mathbb{P}}$ is exactly the same as $\mathbb{P}$ if we identify $\pi^*\mathcal{M}$ and $\mathcal{M}$. Thus, $\tilde{\mathbb{P}} \to \tilde{Y}$ is a quasi-admissible tower with the same total rank and number of sheaves.

Claim 1 : $\pi_*[\tilde{\mathbb{P}} \to \tilde{Y}] - [\mathbb{P} \to Y]$ lies in $\mathcal{U}_G(Y)'$.

First, assume π is given by a single blow up along some invariant smooth center $Z \subseteq Y$. Observe that $\tilde{\mathbb{P}}$ can be considered as the blow up of $\mathbb{P}$ along $\mathbb{P}|_Z$. By Lemma 6.13, we obtain the equality

$$[\tilde{\mathbb{P}} \to \mathbb{P}] - [\mathbb{I}_{\mathbb{P}}] = -[Q_1 \to \mathbb{P}|_Z \to \mathbb{P}] + [Q_2 \to \mathbb{P}|_Z \to \mathbb{P}].$$

Pushing them down to Y, we get

$$\pi_*[\tilde{\mathbb{P}} \to \tilde{Y}] - [\mathbb{P} \to Y] = -[Q_1 \to \mathbb{P}|_Z \to Z \to Y] + [Q_2 \to \mathbb{P}|_Z \to Z \to Y].$$

Notice that the tower $\mathbb{P}|_Z \to Z$ is trivially quasi-admissible and, by Remark 6.14, the sheaves involved in the construction of $Q_1,\ Q_2 \to \mathbb{P}|_Z$ are either of the form $\mathcal{O}(m)$ or $\mathcal{N}^\vee_{\mathbb{P}|_Z \hookrightarrow \mathbb{P}} \cong \mathcal{N}^\vee_{Z \hookrightarrow Y}$ in our notation. That implies $Q_1 \to Z$ and $Q_2 \to Z$ are both quasi-admissible towers. The result then follows from the fact that $\dim Z < \dim Y$. The general case with more blow ups follows easily from the fact that $\pi_*\mathcal{U}_G(\tilde{Y})' \subseteq \mathcal{U}_G(Y)'$. $\triangle$

Hence, without loss of generality, we may assume the splitting

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0$$

happens in the original tower $\mathbb{P} \to Y$. Next, we will construct towers $\hat{\mathbb{P}}, \mathbb{P}' \to Y$ in a similar manner as in the proof of Lemma 6.10. Define

$$\hat{\mathbb{P}}_i \overset{def}{=} \mathbb{P}((\oplus_j \mathcal{E}_j) \oplus \mathcal{E} \oplus \mathcal{L}) \quad \text{and} \quad \mathbb{P}'_i \overset{def}{=} \mathbb{P}((\oplus_j \mathcal{E}_j) \oplus (\mathcal{E}/\mathcal{L}) \oplus \mathcal{L}).$$

Then, by Lemma 6.12, $\mathbb{P}_i$ and $\mathbb{P}'_i$ are equivariantly linearly equivalent invariant smooth divisors on $\hat{\mathbb{P}}_i$ and they intersect transversely. For each level $k > i$, we construct $\hat{\mathbb{P}}_k$ by the same set of sheaves used in $\mathbb{P}_k$ to form a tower

$$\hat{\mathbb{P}} \stackrel{def}{=} \hat{\mathbb{P}}_n \to \cdots \to \hat{\mathbb{P}}_i \to \mathbb{P}_{i-1} \to \cdots \to Y.$$

Also, for each level $k > i$, we construct $\mathbb{P}'_k$ by fiber product, namely, $\mathbb{P}'_k \stackrel{def}{=} \hat{\mathbb{P}}_k \times_{\hat{\mathbb{P}}_i} \mathbb{P}'_i$ to form another tower

$$\mathbb{P}' \stackrel{def}{=} \mathbb{P}'_n \to \cdots \to \mathbb{P}'_i \to \mathbb{P}_{i-1} \to \cdots \to Y.$$

In this case, $\mathbb{P} \sim \mathbb{P}'$ as invariant smooth divisors on $\hat{\mathbb{P}}$ and they intersect transversely. By $GDPR(1,1)$, we have $[\mathbb{P} \hookrightarrow \hat{\mathbb{P}}] = [\mathbb{P}' \hookrightarrow \hat{\mathbb{P}}]$ and hence,

$$[\mathbb{P} \to Y] = [\mathbb{P}' \to Y]$$

as elements in $\mathcal{U}_G(Y)$. Observe that, for each level $k \neq i$, the set of sheaves involved in the construction of $\mathbb{P}'_k$ is exactly the same as those of $\mathbb{P}_k$ in our notation. For level i, by definition, $\mathbb{P}'_i = \mathbb{P}((\oplus_j \mathcal{E}_j) \oplus (\mathcal{E}/\mathcal{L}) \oplus \mathcal{L})$. Hence, $\mathbb{P}' \to Y$ is a quasi-admissible tower with the same total rank as $\mathbb{P} \to Y$ and one higher number of sheaves. By repeating this procedure, we will obtain the highest number of sheaves possible : the number of sheaves is equal to the total rank. That means all sheaves involved in the construction of the quasi-admissible tower are G-linearized invertible sheaves.

Step 2 : Reduction to an admissible tower.

By step 1, we may assume $\mathbb{P} \to Y$ is a quasi-admissible tower constructed by G-linearized invertible sheaves only. For each $\mathcal{L} \in \mathrm{Pic}^G(Y)$ used in the construction, there is an invariant divisor $D_{\mathcal{L}}$ on Y and a sheaf $\mathcal{N}_{\mathcal{L}} \in \mathrm{Pic}^G(\mathrm{Spec}\, k)$ such that

$$\mathcal{L} \cong \mathcal{O}_Y(D_{\mathcal{L}}) \otimes \pi_k^* \mathcal{N}_{\mathcal{L}}$$

by Proposition 6.5. We can then represent such a (Weil) divisor as a linear combination of prime divisors $\{D_{\mathcal{L},k}\}$ on Y. Let

$$\{D_1, \ldots, D_N\} \stackrel{def}{=} \{D_{\mathcal{L},k} \text{ where } \mathcal{L} \text{ is used in the construction of } \mathbb{P} \to Y\}.$$

Consider $\cup_{k=1}^N D_k$ as a reduced closed subscheme of Y. Apply the embedded desingularization Theorem in [BiMi] on $\cup_{k=1}^N D_k \hookrightarrow Y$, we obtain a map $\pi : \tilde{Y} \to Y$, which is the composition of a series of blow ups along invariant smooth centers such that $\left\langle \cup_{k=1}^N D_k \right\rangle$ is smooth. Let $\{E_l\}$ be the set of exceptional divisors. Since $\left\langle \cup_{k=1}^N D_k \right\rangle = \cup_{k=1}^N \langle D_k \rangle$ is smooth, the strict transforms $\{\langle D_k \rangle\}$ are disjoint invariant smooth divisors on $\tilde{Y}$. Moreover, we have

$$\pi^* \mathcal{O}_Y(D_k) \cong \mathcal{O}_{\tilde{Y}}(\langle D_k \rangle + \sum_l m_l E_l)$$

for some integers m_l and all invariant divisors involved are smooth. Hence, $\pi^* \mathcal{L}$ are all admissible and the tower $\tilde{\mathbb{P}} \to \tilde{Y}$ defined by $\tilde{\mathbb{P}}_i \stackrel{def}{=} \mathbb{P}_i \times_Y \tilde{Y}$ becomes admissible. By claim 1, we reduce to the case when $\mathbb{P} \to Y$ is an admissible tower.

Step 3 : Reduction to an admissible tower with $\mathbb{P}_1 = \mathbb{P}(\pi_k^* \mathcal{E}_1)$ where $\mathcal{E}_1$ is a G-linearized locally free sheaf over $\operatorname{Spec} k$.

By step 2, we may assume $\mathbb{P} \to Y$ is an admissible tower. Consider the first level $\mathbb{P}_1 = \mathbb{P}(\oplus_{j=1}^r \mathcal{L}_j)$. Since the sheaves $\mathcal{L}_j$ are admissible, we have $\mathcal{L}_j \cong \mathcal{O}_Y(\sum_k \pm D_{jk}) \otimes \pi_k^* \mathcal{N}_j$ for some invariant smooth divisors D_{jk} on Y and some $\mathcal{N}_j \in \operatorname{Pic}^G(\operatorname{Spec} k)$. By lemma 6.10, we can twist $\mathbb{P} \to Y$ to $\mathbb{P}' \to Y$ so that $\mathbb{P}'_1 = \mathbb{P}((\oplus_{j \neq p} \mathcal{L}_j) \oplus \mathcal{L}_p(D))$ and the difference will be given by quasi-admissible towers $Q \to D$. Notice that

$$[Q \to D \hookrightarrow Y] = \sum_i [Q_i \to D_i \hookrightarrow Y]$$

where $\{D_i\}$ are the G-components of D and $Q_i \stackrel{def}{=} Q \times_D D_i$ defines a quasi-admissible tower over D_i. So, $[\mathbb{P} \to Y] - [\mathbb{P}' \to Y]$ lie in $\mathcal{U}_G(Y)'$. Hence, by twisting each $\mathcal{L}_j$ by suitable choices of D, we may assume there exists a sheaf $\mathcal{L}' \in \operatorname{APic}^G(Y)$ such that

$$\mathcal{L}_j \cong \mathcal{L}' \otimes \pi_k^* \mathcal{N}_j$$

for all j. In other words,

$$\mathbb{P}_1 = \mathbb{P}(\mathcal{L}' \otimes \pi_k^* \mathcal{E}_1)$$

where $\mathcal{E}_1 \stackrel{def}{=} \oplus_j \mathcal{N}_j$ is a G-linearized locally free sheaf over $\operatorname{Spec} k$. Notice that $\mathbb{P}(\mathcal{L}' \otimes \pi_k^* \mathcal{E}_1)$ is isomorphic to $\mathbb{P}(\pi_k^* \mathcal{E}_1)$ as equivariant projective bundles over Y. If we define $\mathbb{P}'_1 \stackrel{def}{=} \mathbb{P}(\pi_k^* \mathcal{E}_1)$

and $\mathbb{P}'_i \overset{def}{=} \mathbb{P}_i \times_{\mathbb{P}_1} \mathbb{P}'_1$ for all $2 \leq i \leq n$, then we obtain a tower $\mathbb{P}' \to Y$ which is isomorphic to $\mathbb{P} \to Y$. Since all the sheaves involved in the construction of $\mathbb{P}'$ are invertible, by Remark 6.8, $\mathbb{P}' \to Y$ is a quasi-admissible tower. By applying step 2 on $\mathbb{P}' \to Y$, we obtain an admissible tower $\tilde{\mathbb{P}} \to \tilde{Y}$. Then, the result follows from claim 1 and the fact that

$$\tilde{\mathbb{P}}_1 = \mathbb{P}'_1 \times_Y \tilde{Y} \cong \mathbb{P}(\pi_k^* \mathcal{E}_1).$$

Step 4 : Finish the induction step.

By step 3, it is enough to prove the statement in the case when $\mathbb{P} \to Y$ is an admissible tower with $\mathbb{P}_1 = \mathbb{P}(\pi_k^* \mathcal{E}_1)$. Consider the second level $\mathbb{P}_2 = \mathbb{P}(\oplus_{j=1}^r \mathcal{L}_j)$. Since the sheaves $\mathcal{L}_j$ are admissible and

$$\mathrm{APic}^G(\mathbb{P}_1) = \mathrm{APic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_1}(1),$$

by the same trick as in step 3, we can twist $\mathbb{P} \to Y$ until there exists a sheaf $\mathcal{L}' \in \mathrm{APic}^G(Y)$ such that

$$\mathcal{L}_j \cong \mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}_1}(m_j) \otimes \pi_k^* \mathcal{N}_j$$

for all j. By Remark 6.11, the twisting will not affect $\mathbb{P}_1$. By defining

$$\mathcal{E}_2 \overset{def}{=} \oplus_j (\mathcal{O}_{\mathbb{P}(\mathcal{E}_1)}(m_j) \otimes \mathcal{N}_j)$$

and $p_1 : \mathbb{P}_1 = \mathbb{P}(\pi_k^* \mathcal{E}_1) \to \mathbb{P}(\mathcal{E}_1)$, we obtain an isomorphism

$$\mathbb{P}_2 = \mathbb{P}(\mathcal{L}' \otimes p_1^* \mathcal{E}_2) \cong \mathbb{P}(p_1^* \mathcal{E}_2).$$

Simiarly, we get an isomorphic quasi-admissible tower $\mathbb{P}' \to Y$ and then, an admissible tower $\tilde{P} \to \tilde{Y}$ by blow ups. Thus, we have the following commutative diagram :

$$\begin{array}{ccccc}
\mathbb{P}(\mathcal{E}_2) & \longleftarrow & \mathbb{P}(p_1^*\mathcal{E}_2) = \mathbb{P}'_2 & \longleftarrow & \mathbb{P}(q_1^* p_1^* \mathcal{E}_2) = \tilde{\mathbb{P}}_2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(\mathcal{E}_1) & \overset{p_1}{\longleftarrow} & \mathbb{P}(\pi_k^*\mathcal{E}_1) = \mathbb{P}'_1 & \overset{q_1}{\longleftarrow} & \mathbb{P}(\pi_k^*\mathcal{E}_1) = \tilde{\mathbb{P}}_1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}\, k & \longleftarrow & Y & \longleftarrow & \tilde{Y}
\end{array}$$

That handles the second level. By repeating the process until level n, we obtain an admissible tower

$$Q = Q_n = \mathbb{P}(\mathcal{E}_n) \to \cdots \to \mathbb{P}(\mathcal{E}_1) \to Q_0 = \operatorname{Spec} k$$

such that

$$[\mathbb{P} \to Y] = [Y \times Q \to Y] = [Q \to \operatorname{Spec} k][Y \to Y].$$

Step 5 : $\dim Y = 0$ case.

In this case, any G-linearized locally free sheaf $\mathcal{E}$ over Y splits into the direct sum of G-linearized invertible sheaves (by direct calculation or Theorem 6.3). Moreover, if $\mathcal{L}$ is a sheaf in $\operatorname{Pic}^G(Y)$, then, by Proposition 6.5, we have $\mathcal{L} \cong \mathcal{O}_Y(D) \otimes \pi_k^* \mathcal{N} \cong \pi_k^* \mathcal{N}$. That means

$$\mathbb{P}_1 = \mathbb{P}(\oplus \mathcal{L}_j) = \mathbb{P}(\oplus \pi_k^* \mathcal{N}_j) = Q_1 \times Y$$

where $Q_1 \overset{def}{=} \mathbb{P}(\oplus \mathcal{N}_j)$. The same argument applies to higher levels. Hence,

$$[\mathbb{P} \to Y] = [Q \to \operatorname{Spec} k][Y \to Y]$$

with admissible tower $Q \to \operatorname{Spec} k$. □

6.4. **Generators for $\mathcal{U}_G(\operatorname{Spec} k)$.** We are now in position to prove the generators Theorem. First of all, we will prove that any two birational objects Y, $Y' \in$ *G-Sm* agree in some truncated theory.

Definition 6.18. For any $X \in$ *G-Sm*, we define the abelian group $\overline{\mathcal{U}^G(X)}$ as the quotient of $\mathcal{U}^G(X)$ by the subgroup generated by elements of the form $[Z][Y \to X]$ where $[Z]$ is in $\mathcal{U}^G_{\geq 1}(\operatorname{Spec} k)'$ and $[Y \to X]$ is in $\mathcal{U}^G(X)$, i.e.

$$\overline{\mathcal{U}^G(X)} \overset{def}{=} \mathcal{U}^G(X) \,/\, \mathcal{U}^G_{\geq 1}(\operatorname{Spec} k)' \,\mathcal{U}^G(X).$$

Remark 6.19. $\overline{\mathcal{U}_G}$ can be considered as a theory on *G-Sm* with projective push-forward, smooth pull-back, Chern class operator (for nice invertible sheaves) and external product. In this truncated theory, the formal group law becomes additive, i.e.

$$c(\mathcal{L} \otimes \mathcal{M}) = c(\mathcal{L}) + c(\mathcal{M}).$$

Proof. In section 7.3 in [LeP], the abelian group $\omega(\operatorname{Spec} k)'$ is defined as the subgroup of $\omega(\operatorname{Spec} k)$ generated by admissible towers (Without group action, the notions of "admissible tower" in [LeP] and in our paper are equivalent). By Corollary 7.5 and equation 8.1 in [LeP], the coefficients a_{ij} used in the formal group law in the theory ω are all inside $\omega_{\geq 1}(\operatorname{Spec} k)'$. Then, the result follows from the fact that the formal group law in the theory ω and the formal group law in our theory $\mathcal{U}_G$ share the same set of coefficients a_{ij} if we consider $\omega(\operatorname{Spec} k) \hookrightarrow \mathcal{U}_G(\operatorname{Spec} k)$. □

Proposition 6.20. *Suppose Y, $Y' \in G$-Sm are both projective and G-irreducible. If they are equivariantly birational, then $[Y] = [Y']$ as elements in $\overline{\mathcal{U}_G(\operatorname{Spec} k)}$.*

Proof. By the equivariant weak factorization theorem (Theorem 0.3.1) in [AKMW], there exists a sequence of blowups and blowdowns along smooth invariant centers to go from Y to Y'. So, it is enough to consider a single blowup. By Lemma 6.13,

$$[Bl_Z\, Y \to Y] - [\mathbb{I}_Y] = -[\mathbb{P}_1 \to Z \to Y] + [\mathbb{P}_2 \to Z \to Y]$$

as elements in $\mathcal{U}_G(Y)$. Pushing them down to $\mathcal{U}_G(\operatorname{Spec} k)$ gives

$$[Bl_Z\, Y] - [Y] = -[\mathbb{P}_1 \to Z \to \operatorname{Spec} k] + [\mathbb{P}_2 \to Z \to \operatorname{Spec} k]$$

as elements in $\mathcal{U}_G(\operatorname{Spec} k)$. For simplicity, assume Z is G-irreducible. By Remark 6.14, $\mathbb{P}_1$, $\mathbb{P}_2 \to Z$ are both quasi-admissible towers. By Proposition 6.17, $[\mathbb{P}_i \to Z] = \sum a\,[Z' \to Z]$ for some $a \in \mathcal{U}_G(\operatorname{Spec} k)'$ and $Z' \in G$-Sm such that $\dim Z' \leq \dim Z$. Since $\dim \mathbb{P}_i = \dim Y > \dim Z$, the elements $\{a\}$ are all in $\mathcal{U}^G_{\geq 1}(\operatorname{Spec} k)'$. Hence, the element $[\mathbb{P}_i \to Z \to \operatorname{Spec} k]$ vanishes in $\overline{\mathcal{U}_G(\operatorname{Spec} k)}$. □

Finally, we are ready to prove our main Theorem. The generators of our equivariant algebraic cobordims ring $\mathcal{U}_G(\operatorname{Spec} k)$, as a $\mathbb{L}$-algebra, will be admissible towers over $\operatorname{Spec} k$ and some "exceptional objects". For an integer $n \geq 0$ and a pair of subgroups $G \supseteq H \supseteq H'$, since G is abelian, we can write

$$H/H' \cong H_1 \times \cdots \times H_a$$

where H_i is a cyclic group of order $M_i \overset{def}{=} p_i^{m_i}$ for some prime p_i. Let α_i be a generator of H_i. Define a (H/H')-action on Proj $k[x_0, \ldots, x_n, v_1, \ldots, v_a]$ as the following. First, H/H' acts on $x_0, \ldots, x_n$ trivially. Then, for all i, the subgroup H_i acts on v_i by $\alpha_i \cdot v_i = \xi_i v_i$ for some primitive M_i-th root of unity ξ_i. For all $j \neq i$, the subgroup H_j acts on v_i trivially.

Lemma 6.21. *There exist homogeneous polynomials $g_1, \ldots, g_a \in k[x_0, \ldots, x_n]$ with degrees $M_1, \ldots, M_a$ respectively, such that the projective variety*

$$\text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \,/\, (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a),$$

is smooth and has dimension n.

Proof. Let U be the open subscheme $\cup_{i=0}^{n} D(x_i)$ of Proj $k[x_0, \ldots, x_n, v_1, \ldots, v_a]$. For $1 \leq i \leq a$, let $\psi_i : U \to \mathbb{P}^{N_i}$ be the (H/H')-equivariant map sending $(x_0; \ldots; x_n; v_1; \ldots; v_a)$ to

$$(x_0^{M_i}; x_0^{M_i-1} x_1; \ldots; x_n^{M_i}; v_i^{M_i})$$

(the first $N_i - 1$ coordinates run though all degree M_i monomials given by $x_0, \ldots, x_n$). By Lemma 4.8, there exist hyperplanes $H_i \subseteq \mathbb{P}^{N_i}$ such that $U \times_{\mathbb{P}^{N_1}} H_1 \times_{\mathbb{P}^{N_2}} H_2 \times_{\mathbb{P}^{N_3}} \cdots \times_{\mathbb{P}^{N_a}} H_a$ is smooth and has dimension n. The result then follows by observing each H_i defines a homogeneous polynomial g_i with degree M_i and

$$U \times_{\mathbb{P}^{N_1}} H_1 \times_{\mathbb{P}^{N_2}} H_2 \times_{\mathbb{P}^{N_3}} \cdots \times_{\mathbb{P}^{N_a}} H_a = \text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \,/\, (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a).$$

□

Pick $g_1, \ldots, g_a$ as in Lemma 6.21. Let X be the projective variety

$$\text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \,/\, (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a).$$

Then, X is in (H/H')-*Sm*. Fix a set of representatives $\{\beta_j\}$ of G/H. The exceptional object $E_{n,H,H'}$ is defined as $G/H \times X$ such that for all $\alpha \in G$ and $(\beta_j, y) \in E_{n,H,H'}$,

$$\alpha \cdot (\beta_j, y) \overset{def}{=} (\beta_k,\ \gamma \cdot y)$$

where $\beta_k \in G/H$ and $\gamma \in H$ are uniquely determined by the equality $\alpha\beta_j = \beta_k\gamma$. We will see that the element $[E_{n,H,H'}] \in \mathcal{U}_G(\operatorname{Spec} k)$ is independent of the choice of $\{g_i\}$.

Theorem 6.22. *If the pair (G, k) is split, then $\mathcal{U}_G(\operatorname{Spec} k)$ is generated by the set of exceptional objects $\{E_{n,H,H'} \mid n \geq 0$ and $G \supseteq H \supseteq H'\}$ and the set of admissible towers over $\operatorname{Spec} k$ as a $\mathbb{L}$-algebra.*

Proof. Let S be the set of generators mentioned in the statement, i.e. $S \overset{def}{=} \{[E_{n,H,H'}],\ [\mathbb{P}]\}$. Consider the following diagram of abelian groups :

$$\begin{array}{ccc} \mathcal{U}^G_n(\operatorname{Spec} k) & & \\ \downarrow & \overset{P_n}{\searrow} & \\ \overline{\mathcal{U}^G_n(\operatorname{Spec} k)} & \overset{\overline{P_n}}{\dashrightarrow} & \mathcal{U}^G(\operatorname{Spec} k) \,/\, \mathbb{L}[S] \end{array}$$

Our goal is to prove S gives a set of generator of $\mathcal{U}_G(\operatorname{Spec} k)$ as $\mathbb{L}$-algebra. It is obviously enough to show that $P_n = 0$ for all n. Suppose we have shown that $P_0 = P_1 = \cdots = P_{n-1} = 0$. Then, since $\mathcal{U}^G(\operatorname{Spec} k)'$ is a subgroup of $\mathbb{L}[S]$ and

$$(\mathcal{U}^G_{\geq 1}(\operatorname{Spec} k)'\,\mathcal{U}^G(\operatorname{Spec} k)) \cap \mathcal{U}^G_n(\operatorname{Spec} k) = \sum_{i=1}^{n} \mathcal{U}^G_i(\operatorname{Spec} k)'\,\mathcal{U}^G_{n-i}(\operatorname{Spec} k),$$

the homomorphism $\overline{P_n}$ is well-defined and the diagram is commutative. In addition, $\overline{P_n} = 0$ will imply $P_n = 0$ and P_0, $\overline{P_0}$ agree. So, it is enough to show that $\overline{P_n} = 0$ for all n.

Suppose $n \geq 0$ and $[Y] \in \overline{\mathcal{U}^G_n(\operatorname{Spec} k)}$ is G-irreducible. Assume Y is irreducible and the G-action is faithful first. Let $G = G_1 \times \cdots \times G_a$ where G_i is a cyclic group of order $M_i \overset{def}{=} p_i^{m_i}$ for some prime p_i and α_i be a generator of G_i.

Claim 1 :

$$k(Y) \cong k(x_1, \ldots, x_n)[x_{n+1}, v_1, \ldots, v_a] \,/\, (f, v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a)$$

for some f, $g_i \in k[x_1, \ldots, x_{n+1}]$ such that the G-action on x_i is trivial, G_j acts on v_i trivially if $i \neq j$ and $\alpha_i \cdot v_i = \xi_i v_i$ where $\xi \in k$ is a primitive M_i-th root of unity.

Denote the function field $k(Y)$ by K. Since the G-action on Y is faithful, the degree of the extension K/K^G is equal to the order of G and it is a Galois extension (separable because $\operatorname{char} \mathrm{k} = 0$). Let K_i be the subfield of K consists of elements fixed by $\prod_{j \neq i} G_j$. Then,

$K = K_1 \cdots K_a$, the intersection of any $K_i \neq K_j$ is K^G and the extension K_i/K^G is Galois with $\mathrm{Gal}(K_i/K^G) \cong G_i$.

Since the dimension of the scheme Y/G is n, the field $k(Y/G) \cong K^G$ has transcendence degree n over k. So, there is an element $f \in k[x_1, \ldots, x_{n+1}]$ such that $K^G \cong k(x_1, \ldots, x_n)[x_{n+1}]/(f)$. Consider K_i as a G_i-representation over K^G. Since the pair (G_i, K^G) is split, K_i can be written as direct sum of 1-dimensional G_i-representations over K^G. Then, the action of α_i on at least one of the G_i-representations is given by ξ_i. Let b_i be a generator of such representation. Since $b_i^{M_i}$ is fixed by G_i, it is in K^G. Denote it by g_i. Without loss of generality, $g_i \in k[x_1, \ldots, x_{n+1}]$. Consider the polynomial

$$v^{M_i} - g_i = \prod_{j=0}^{M_i-1} (v - \xi^j b_i) \in K^G[v].$$

It is irreducible because if $j < M_i$, then α does not fix b_i^j, hence $b_i^j \notin K^G$. Since $v^{M_i} - g_i$ has degree M_i, the field K_i has to be generated by b_i. In other words,

$$K_i \cong k(x_1, \ldots, x_n)[x_{n+1}, v_i] \,/\, (f, v_i^{M_i} - g_i).$$

Also, the G-action on v_i corresponds to the G-action on b_i, which is exactly as the one described in the statement. $\triangle$

Let

$$Y' \overset{def}{=} \mathrm{Proj}\; k[x_0, \ldots, x_{n+1}, v_1, \ldots, v_a] \,/\, (f, v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a)$$

and

$$P' \overset{def}{=} \mathrm{Proj}\; k[x_0, \ldots, x_{n+1}, v_1, \ldots, v_a]$$

where f, $g_i \in k[x_0, \ldots, x_{n+1}]$ are homogeneous polynomials with degree d and M_i respectively, the G-action on x_i is trivial and the G-action on v_j are the one described in claim 1. For simplicity, we will denote P' simply as Proj $k[x, v]$. By claim 1, Y' is equivariantly birational to Y. By applying the embedded desingularization theorem [BiMi] on $Y' \hookrightarrow P'$, there is a commutative diagram

$$\begin{array}{ccc} \langle Y' \rangle & \longrightarrow & P \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & P' \end{array}$$

where $\langle Y' \rangle$, P are both in G-Sm. Since $\langle Y' \rangle$ is smooth and equivariantly birational to Y', by Proposition 6.20, we may assume $\langle Y' \rangle = Y$. Moreover, since $P \to P'$ is projective, by Proposition 4.13, there exist free variables $y_0, \dots, y_m$ with G-actions and a set of polynomials $\{h\} \subseteq k[x, v, y_0, \dots, y_m]$ which are bihomogeneous with respect to (x, v) and y such that

$$P \cong \text{BiProj } k[x, v][y]/(h)$$

and

$$Y = \text{BiProj } k[x, v][y] \,/\, (h) + (f, v_1^{M_1} - g_1, \dots, v_a^{M_a} - g_a).$$

Define a set of indices

$$J \overset{def}{=} \{\text{monomial in } k[x] \text{ with degree } d\} \amalg \coprod_i \{\text{monomial in } k[x] \text{ with degree } M_i\}.$$

Let $C \overset{def}{=} k[\{c_j \mid j \in J\}]$ be the polynomial ring generated by free variables indexed by J. Then, $f(x)$ can be considered as $f(c_0, x)$ for some $c_0 \in \operatorname{Spec} C$ and similarly for g_i. Let

$$T' \overset{def}{=} \text{Proj } C[x, v] \,/\, (f(c, x), v_1^{M_1} - g_1(c, x), \dots, v_a^{M_a} - g_a(c, x)).$$

If we assign a trivial G-action to $\operatorname{Spec} C$, then there is an equivariant, projective, surjective map $\phi' : T' \to \operatorname{Spec} C$ with fiber $T'_{c_0} \cong Y'$ and T' is a closed subscheme of $\operatorname{Spec} C \times P' \cong \text{Proj } C[x, v]$. Also let

$$T \overset{def}{=} \text{BiProj } C[x, v][y] \,/\, (h) + (f(c, x), v_1^{M_1} - g_1(c, x), \dots, v_a^{M_a} - g_a(c, x)).$$

Similarly, there is an equivariant, projective, surjective map $\phi : T \to \operatorname{Spec} C$ with fiber $T_{c_0} \cong Y$ and T is a closed subscheme of $\operatorname{Spec} C \times P \cong \text{BiProj } C[x, v][y]/(h)$.

Claim 2 : T is in G-Sm and has dimension $\dim \operatorname{Spec} C + n$.

Without loss of generality, k is algebraically closed. Notice that T is cut out from $\operatorname{Spec} C \times P$, which is smooth and has relative dimension $n+a+1$ over $\operatorname{Spec} C$, by the equations $f(c, x)$ and $v_i^{M_i} - g_i(c, x)$. We will show that the gradients $\{\nabla f(c, x), \nabla(v_i^{M_i} - g_i(c, x))\}$ are linearly independent and they are also linearly independent to any ∇h at any closed point in T. Since

h is in $k[x,v][y]$, $\phi_* \nabla h = 0$. So, it will be enough to show that the vectors

$$\{\phi_* \nabla f(c,x),\ \phi_* \nabla(v_i^{M_i} - g_i(c,x))\}$$

are linearly independent. Over $D(x_j)$, if we denote x_k/x_j by t_k, then we have

$$\phi_* \nabla f(c,x) = (t_0^d, t_0^{d-1} t_1, \ldots, t_{n+1}^d, 0, \ldots, 0)$$

and

$$\phi_* \nabla(v_i^{M_i} - g_i(c,x)) = -(0, \ldots, 0, t_0^{M_i}, t_0^{M_i - 1} t_1, \ldots, t_{n+1}^{M_i}, 0, \ldots, 0)$$

(zero except coordinates corresponding to the coefficients of g_i). Moreover, over $D(v_j)$, if we denote x_k/v_j by t_k, then we obtain the same equations for the vectors $\phi_* \nabla f(c,x)$ and $\phi_* \nabla(v_i^{M_i} - g_i(c,x))$. Thus, they are linearly independent as long as $(x_0; \cdots; x_{n+1}) \neq 0$. Suppose $x = (x_0; \cdots; x_{n+1}) = 0$ for a certain closed point in T. Then, $v_1, \ldots, v_a$ are all zero too. So, the coordinate of this point is $(c, 0; 0, y) \in T \subseteq C \times P$. We then get a contradiction by realizing that the map $C \times P \to P \to P'$ will send $(c, 0; 0, y)$ to $(0; 0) \in P'$. $\triangle$

The same argument also shows that T' is in *G-Sm* and has dimension $\dim \operatorname{Spec} C + n$. Notice that T and $\operatorname{Spec} C$ are both smooth, the map ϕ is projective, surjective and has relative dimension n and the fiber T_{c_0} is smooth with dimension n. So, the map ϕ is smooth if restricted in an open neighborhood of c_0 (because the point c_0 is not in the image of $\{\text{critical point}\}$, which is closed). Call such a neighborhood U_0. Pick a point $c_1 = (c_{1j}) \in \operatorname{Spec} C$ such that the fiber T'_{c_1} is in *G-Sm* with dimension n (such point exists by Lemma 6.21). Similarly, the map $\phi' : T' \to \operatorname{Spec} C$ is smooth if restricted in an open neighborhood of c_1. Call such a neighborhood U_1.

Claim 3 : There exists an equivariant, projective, birational map $\mu : T \to T'$ of schemes over $\operatorname{Spec} C$.

The map μ is given by the restriction of the map $\operatorname{Spec} C \times P \to \operatorname{Spec} C \times P'$ which sends $(c, x; v, y)$ to $(c, x; v)$. So, it is clearly equivariant and projective. Notice that $P \to P'$ is birational. That means if η_1 is the generic point of P', then $P_{\eta_1} \to \operatorname{Spec} \eta_1$ is an isomorphism, i.e. $\operatorname{Proj} k(x_*, v_*)[y]/(h_*) \tilde{\to} \operatorname{Spec} k(x_*, v_*)$ where x_*, v_* and h_* are the dehomogenizations of x, v and h with respect to x_0, respectively. Let η_2 be the generic point of T', as scheme over Spec C. Then,

$$\begin{aligned} T_{\eta_2} &\cong \text{Proj } C(x_*, v_*)[y] \,/\, (h_*) + (f_*, {v_{1*}}^{M_1} - g_{1*}, \ldots, {v_{a*}}^{Ma} - g_{a*}) \\ &\cong \text{Spec}\, C(x_*, v_*) \,/\, (f_*, {v_{1*}}^{M_1} - g_{1*}, \ldots, {v_{a*}}^{Ma} - g_{a*}) \\ &\cong \text{Spec}\, \eta_2. \end{aligned}$$

That means μ is birational, as a morphism of schemes over Spec C. $\triangle$

Denote the open subscheme $U_0 \cap U_1 \subseteq \text{Spec}\, C$ by U. Then, $\phi : T|_U \to U$ and $\phi' : T'|_U \to U$ are both smooth and $\mu : T|_U \to T'|_U$ has birational fibers (over U). Also, denote the affine line in Spec C connecting c_0 and c_1 by L and pick a closed point $c_2 \in U \cap L$. Consider the equivariant, projective map $\phi : T|_L \to L$. It is smooth over $U_0 \cap L$. That means $\text{Sing}(T|_L)$ is disjoint from the fibers T_{c_0} and T_{c_2}. By resolution of singularities (Theorem 1.6 in [BiMi]), we can assume $T|_L$ is smooth (The blow ups will not affect the two fibers). Now, $T|_L$ has fibers T_{c_0} and T_{c_2} which are both smooth invariant divisors. By Proposition 4.14, we can extend $T|_L \to L$ to some equivariant, projective map $\overline{T} \to \mathbb{P}^1$ where $\overline{T}$ is in *G-Sm*. Then, $GDPR(1,1)$ will imply

$$[T_{c_0} \hookrightarrow \overline{T}] = [T_{c_2} \hookrightarrow \overline{T}]$$

as elements in $\mathcal{U}_G(\overline{T})$. Push them down to Spec k, we got $[T_{c_0}] = [T_{c_2}]$. By applying the same argument on $\phi' : T'|_L \to L$, we got $[T'_{c_1}] = [T'_{c_2}]$. Hence,

$$[Y] = [T_{c_0}] = [T_{c_2}] = [T'_{c_2}] = [T'_{c_1}]$$

as elements in $\overline{\mathcal{U}_G(\text{Spec}\, k)}$ by Proposition 6.20 and the fact that T_{c_2}, T'_{c_2} are birational and are both smooth.

Because of the freedom of choice of $c_1 = (c_{1j})$, we can assume

$$Y \cong \text{Proj } k[x, v] \,/\, (f, v_1^{M_1} - g_1, \ldots, v_a^{Ma} - g_a)$$

for any choice of $f(x)$, $g_i(x)$ as long as the degrees of f, g_i are d, M_i respectively and Y is smooth. Consider the equivariant map

$$\psi : W \overset{def}{=} \text{Proj } k[x,v] \,/\, (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a) \to \text{Proj } k[x] \cong \mathbb{P}^{n+1}.$$

Then, Y can be considered as the preimage of a generic degree d hypersurface. More precisely, as elements in $\overline{\mathcal{U}_G(W)}$,

$$[Y \hookrightarrow W] = c(\psi^*\mathcal{O}(d))[\mathbb{I}_W] = d\, c(\psi^*\mathcal{O}(1))[\mathbb{I}_W]$$

because $\psi^*\mathcal{O}(1)$ is nice and formal group law becomes additive by Remark 6.19. In other words, it is enough to consider the case when $d = 1$. Without loss of generality, we may assume $f(x) = x_{n+1}$. Hence, we have

$$Y \cong \text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \,/\, (v_1^{M_1} - g_1(x), \ldots, v_a^{M_a} - g_a(x)),$$

which is the exceptional object $E_{n,G,\{1\}}$. So, $\overline{P_n}[Y] = 0$. That proves the case when Y is irreducible with faithful G-action.

If the G-action on $Y \in G$-*Sm* is faithful, but Y is reducible, then $Y \cong G/H \times X$ for some subgroup $H \subseteq G$ and some irreducible $X \in H$-*Sm*. By applying claim 1 on X with H-action, we can define $G/H \times X'$, $G/H \times X$, $G/H \times P'$ and $G/H \times P$ in the same manner to obtain the following commutative diagram :

$$\begin{array}{ccc} G/H \times X & \longrightarrow & G/H \times P \\ \downarrow & & \downarrow \\ G/H \times X' & \longrightarrow & G/H \times P' \end{array}$$

We can also define the polynomial ring C and the C-schemes $G/H \times T'$ and $G/H \times T$. We will also have G-equivariant, projective maps $\phi : G/H \times T \to \text{Spec}\, C$ and $\phi' : G/H \times T' \to \text{Spec}\, C$ such that ϕ is smooth around c_0 and ϕ' is smooth around some $c_1 = (c_{1j})$. Similarly, the natural map $\mu : G/H \times T \to G/H \times T'$ will also be G-equivariant, projective and has birational fibers over $\text{Spec}\, C$. Hence, as before,

$$[Y] = [G/H \times X] = [(G/H \times T)_{c_0}] = [(G/H \times T)_{c_2}] = [(G/H \times T')_{c_2}] = [(G/H \times T')_{c_1}].$$

In other words, we may assume

$$X \cong \text{Proj } k[x,v] \,/\, (f, v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a).$$

Define $\psi : G/H \times W \to \mathbb{P}^{n+1}$ similarly to get the same reduction on f. We may further assume

$$X \cong \text{Proj } k[x,v] \, / \, (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a).$$

Hence, we have $Y \cong G/H \times X \cong E_{n,H,\{1\}}$.

In general, if we have subgroups $G \supseteq H \supseteq H'$ such that the (G/H')-action on Y is faithful and $Y \cong G/H \times X$ for some irreducible $X \in$ *(H/H')-Sm*, then we may assume

$$X \cong \text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \, / \, (v_1^{M_1} - g_1(x), \ldots, v_a^{M_a} - g_a(x))$$

for some generic $g_1, \ldots, g_a$ where $v_1, \ldots, v_a$ are given by H/H'. Hence, $Y \cong E_{n,H,H'}$. That finishes the proof. □

Remark 6.23. Notice that we did not use the full power of the generalized double point relation in our proof of Theorem 6.22. More precisely, if we define our equivariant algebraic cobordism theory by imposing the extended double point relation $GDPR(2,1)$ alone, the same set of generators will still generate the equivariant algebraic cobordism ring. But, with the aid of the generalized double point relation, we can actually simplify the exceptional objects further.

Suppose the dimension of an exceptional object $E_{n,H,H'}$ is greater than the order of the group H/H'. Let

$$W \stackrel{def}{=} G/H \times \text{Proj } k[x_0, \ldots, x_n, v_1, \ldots, v_a] \, / \, (v_1^{M_1} - g_1, \ldots, v_{a-1}^{M_{a-1}} - g_{a-1}).$$

Then, the invariant smooth divisor $G/H \times \{v_a^{M_a} = g_a\} = E_{n,H,H'}$ is equivariantly linearly equivalent to the sum of invariant smooth divisors $G/H \times \{x_i = 0\}$ where i runs from 0 to $M_a - 1$. Moreover, by the freedom of choice of $\{g_i\}$, we can assume

$$E_{n,H,H'} + \sum_{i=0}^{M_a - 1} G/H \times \{x_i = 0\}$$

is a reduced strict normal crossing divisor. Thus, by the generalized double point relation $GDPR(M_a, 1)$,

$$[E_{n,H,H'} \hookrightarrow W] = \sum_{i=0}^{M_a - 1} [G/H \times \{x_i = 0\} \hookrightarrow W]$$

as elements in $\overline{\mathcal{U}_G(W)}$ ("extra terms" are always of the form $[\mathbb{P} \to Z \hookrightarrow W]$ where $\mathbb{P} \to Z$ is a quasi-admissible tower and $\dim Z < \dim \mathbb{P} = n$). In other words, it is enough to consider objects of the form

$$G/H \times \text{Proj } k[x_0, \ldots, x_{n-1}, v_1, \ldots, v_a] \,/\, (v_1^{M_1} - g_1, \ldots, v_{a-1}^{M_{a-1}} - g_{a-1})$$

instead. Similarly, we can apply the same argument to reduce $E_{n,H,H'}$ into

$$G/H \times \text{Proj } k[x_0, \ldots, x_{n-a}, v_1, \ldots, v_a] = G/H \times \mathbb{P}(V)$$

for some (H/H')-representation V. In particular, if the group G is a cyclic group of prime order, then

$$[G/H \times \mathbb{P}(V)] = [G/H]\,[\mathbb{P}(V)]$$

where V is some G-representation and H can be either G or the trivial group. Notice that $E_{0,\{1\},\{1\}} \cong G$ and $\mathbb{P}(V)$ is an admissible tower over $\operatorname{Spec} k$. Hence, only finite number of exceptional objects are needed to generate $\mathcal{U}_G(\operatorname{Spec} k)$ in this case.

7. Fixed point map

In this section, we will prove the well-definedness of the canonical fixed point map

$$\mathcal{F} : \mathcal{U}_G(X) \to \omega(X^G),$$

which is an analogue of the fixed point map in topology. Recall the following definition of fixed point locus from [Fo]. If X is a scheme over a field k, let X_G be the G-scheme X equipped with trivial G-action. If Y is a G-scheme over k, let $h_Y^G(X)$ be the set of morphisms from X_G to Y in the category of G-schemes over k. Then, $h_X^G(-)$ is a cotravariant functor from the category of schemes over k to category of sets. By Theorem 2.3 (for schemes of finite type over a field k), $h_X^G(-)$ is represented by a closed subscheme of X with trivial G-action. We refer to this closed subscheme as the fixed point locus of X and denote it by X^G.

In order to show that the fixed point map is well-defined, we need to first make sure the fixed point locus of any object in the category *G-Sm* stays inside the category *Sm* in our basic setup ($\operatorname{char} \mathrm{k} = 0$ and G is either a reductive connected group or a finite group).

Proposition 7.1. *For any object $X \in$ G-Sm, the fixed point locus X^G is smooth. Moreover, if $x \in X^G$ is a closed point, then there is no non-zero conormal vector in $\mathcal{N}^\vee_{X^G \hookrightarrow X}|_x$ which is fixed by the natural G-action.*

Proof. By Proposition 3.4 in [Ed], the fixed point locus X^G is smooth if G is finite. In the case when G is linearly reductive, let $x \in X^G$ be a closed point and $C(X, x)$ be the tangent cone of X at x. Since X is smooth at x, the tangent cone $C(X, x)$ is isomorphic to $\operatorname{Spec} k(x)[t_1, \ldots, t_d]$ where d is the dimension of $\mathcal{O}_{X,x}$ and $t_1, \ldots, t_d$ are independent indeterminates corresponding to a system of parameters of $\mathcal{O}_{X,x}$. Moreover, G acts on $k(x)$ trivially and the G-action on $t_1, \ldots, t_d$ is linear. By Theorem 5.2 in [Fo], we have $C(X, x)^G = C(X^G, x)$. Therefore, $C(X^G, x)$ is a linear subspace of $C(X, x)$, i.e. $\mathbb{A}^{d'}_{k(x)}$ for some d'. But then $d' = \dim C(X^G, x) = \dim X^G$. Hence, the fixed point locus X^G is smooth at x. That shows the first part of the statement.

For the second part, when G is finite, we have $\mathcal{T}X^G|_x \cong (\mathcal{T}X|_x)^G$ by Proposition 3.2 in [Ed]. Moreover, the following exact sequence of G-representations splits :

$$0 \to \mathcal{T}X^G|_x \to \mathcal{T}X|_x \to \mathcal{N}_{X^G \hookrightarrow X}|_x \to 0.$$

Hence, there is no non-zero normal vector of X^G which is fixed by G, and the same holds for conormal.

When G is reductive,

$$\mathcal{T}X^G|_x \cong \mathcal{T}C(X^G, x)|_0 \cong \mathcal{T}C(X, x)^G|_0 \cong (\mathcal{T}C(X, x)|_0)^G \cong (\mathcal{T}X|_x)^G.$$

Then the result follows similarly. □

Theorem 7.2. *Suppose X is an object in G-Sm and $\{Z\}$ is the set of irreducible components of its fixed point locus X^G. Then, sending $[Y \to X]$ to $\sum_Z [Y^G \times_{X^G} Z \to Z]$ defines an abelian group homomorphism :*

$$\mathcal{F} : \mathcal{U}_G(X) \to \bigoplus_Z \omega(Z).$$

Before going into the proof, let us illustrate how this fixed point map respects the generalized double point relation by the following example. We would like to thank Professor P. Brosnan for inspiration.

Example : Suppose $\mathbb{C}$ is the ground field and G is a cyclic group of order 3. Let $X(3)$ be the fine moduli space for generalized elliptic curves with $\Gamma(3)$-structure and $E \to X(3)$ be its corresponding universal family (see [DR]). By the $\Gamma(3)$-structure, there are two sections s, $s' : X(3) \to E$ such that, for each closed point $\mu \in X(3)$, $s(\mu)$ and $s'(\mu)$ is a set of generators of the 3-torsion $E_\mu[3]$. As in section 1.2 in [DR], the universal family can be given explicitly by

$$E = \{\nu(x^3 + y^3 + z^3) = 3\mu xyz\} \subseteq \text{Proj } \mathbb{C}[x, y, z] \times \text{Proj } \mathbb{C}[\mu, \nu]$$

projecting down to

$$X(3) = \text{Proj } \mathbb{C}[\mu, \nu] = \mathbb{P}^1.$$

Notice that the fiber over ∞ :

$$E_\infty = \{0 = 3xyz\} \subseteq \text{Proj } \mathbb{C}[x, y, z]$$

is a Néron 3-gon. Denote $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ by A, B, C respectively and $A \cap B$, $A \cap C$, $B \cap C$ by P, Q, R respectively. Then,

$$E_\infty - \{P, Q, R\} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{G}_m$$

and, without loss of generality, the element $s(\infty) \in E_\infty$ corresponds to an element $(0, \xi_3) \in \mathbb{Z}/3\mathbb{Z} \times \mathbb{G}_m$, where ξ_3 is a primitive cubic root of unity. In other words, if we define a G-action on E by translation by s, then

(1) $\phi : E \to X(3) \cong \mathbb{P}^1$ is a projective morphism in G-Sm (trivial G-action on $X(3)$).

(2) The fiber E_0 is an elliptic curve with free G-action.

(3) $E_\infty = A \cup B \cup C$ and A, B, C are all G-invariant.

(4) The G-actions on $A \cong B \cong C \cong \mathbb{P}^1$ are non-trivial and their fixed point loci are $A^G = \{P, Q\}$, $B^G = \{P, R\}$ and $C^G = \{Q, R\}$.

Now, consider the $GDPR(3, 1)$ setup given by $\pi_{\mathbb{C}} : E \to \text{Spec}\,\mathbb{C}$ with G-invariant divisors E_0, A, B, C on E such that $E_0 \sim A+B+C$ and $E_0+A+B+C$ is a reduced strict normal crossing divisor. Then, as elements in $\mathcal{U}^G(\text{Spec}\,\mathbb{C})$, we have

$$[A] + [B] + [C] - [\mathbb{P}_1] - [\mathbb{P}_2] - [\mathbb{P}_3] = [E_0]$$

where $\mathbb{P}_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to P$, $\mathbb{P}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A+B)) \to Q$ and $\mathbb{P}_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A+B)) \to R$. But since

$$\begin{aligned} \mathcal{O}(A)|_P &\cong \mathcal{N}_{A \hookrightarrow E}|_P \cong \mathcal{T}B|_P, \\ \mathcal{O}(A+B)|_Q &\cong \mathcal{O}(A)|_Q \cong \mathcal{N}_{A \hookrightarrow E}|_Q \cong \mathcal{T}C|_Q, \\ \mathcal{O}(A+B)|_R &\cong \mathcal{O}(B)|_R \cong \mathcal{N}_{B \hookrightarrow E}|_R \cong \mathcal{T}C|_R, \end{aligned}$$

we have $\mathbb{P}_1 \cong \mathbb{P}_2 \cong \mathbb{P}_3 \cong A \cong B \cong C$. Hence, $[E_0] = 0$ in $\mathcal{U}^G(\text{Spec}\,\mathbb{C})$. In this case, the fixed point map will take both sides to zero because the G-action on E_0 is free.

Furthermore, if we consider P as an irreducible component of E^G, sending $[Y \to E]$ to $[Y^G|_P \to P]$ will define a map from $\mathcal{U}^G(E)$ to $\omega(P)$. So, if we consider the $GDPR(3,1)$ setup given by $\mathbb{I}_E : E \to E$ with the same set of divisors, we will have

$$[A \hookrightarrow E] + [B \hookrightarrow E] + [C \hookrightarrow E] - [\mathbb{P}_1 \to E] - [\mathbb{P}_2 \to E] - [\mathbb{P}_3 \to E] = [E_0 \hookrightarrow E],$$

as elements in $\mathcal{U}^G(E)$. In this case, the fixed point map (restricted over P) will send the right hand side to zero and the left hand side to

$$[\mathbb{I}_P] + [\mathbb{I}_P] + 0 - [\, 0 \cup \infty \to P\,] - 0 - 0,$$

which is also zero.

Proof of Theorem 7.2. By Proposition 7.1, Z is smooth and sending $[Y \to X]$ to $[Y^G \times_{X^G} Z \to Z]$ is well-defined at the level of $M_G(X)^+ \to M(Z)^+$. If X^G is the empty set, then $\oplus_Z\ \omega(Z) = 0$ and there is nothing to prove. So, we can assume X^G is non-empty. The strategy of this proof is very similar to that of the Proposition 3.9.

First of all, it is clearly enough to show the well-definedness of $\mathcal{F}$ with respect to one fixed component Z, i.e. $\mathcal{F}_Z[Y \to X] = [Y^G \times_{X^G} Z \to Z]$. Consider a generalized double point relation setup given by $\phi : Y \to X$ with $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$ on Y. Let $\mathcal{G} : \mathcal{R} \to M_G(X)^+$ be the corresponding map. What we need to show is

$$\mathcal{F}_Z \circ \mathcal{G}(G^X_{n,m}) = \mathcal{F}_Z \circ \mathcal{G}(G^Y_{m,n})$$

as elements in $\omega(Z)$.

For a general term $X_i \cdots U^p_k \cdots$ in $\mathcal{R}$,

$$\begin{aligned}\mathcal{F}_Z \circ \mathcal{G}(X_i \cdots U^p_k \cdots) &= \mathcal{F}_Z[A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots \to Y \to X] \\ &= [(A_i \times_Y \cdots \times_Y P^p_k \times_Y \cdots)^G \times_{X^G} Z \to Y^G \times_{X^G} Z \to Z].\end{aligned}$$

If $Y^G \times_{X^G} Z$ is empty, then $\mathcal{F}_Z \circ \mathcal{G}(G^X_{n,m}) = \mathcal{F}_Z \circ \mathcal{G}(G^Y_{m,n}) = 0$. So, we may assume $Y^G \times_{X^G} Z$ is non-empty. Let $\{W\}$ be the set of irreducible components of $Y^G \times_{X^G} Z$ and $\pi_W : W \to Z$ be the natural projective map. Let $\mathcal{G}' : \mathcal{R} \to M_G(Y)^+$ be the map corresponding to the GDPR setup given by $\mathbb{I} : Y \to Y$ with the same set of divisors on Y. Then,

$$\begin{aligned}
\pi_{W*} \circ \mathcal{F}_W \circ \mathcal{G}'(X_i \cdots U_k^p \cdots) &= \pi_{W*} \circ \mathcal{F}_W[A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots \to Y] \\
&= \pi_{W*}[(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots)^G \times_{Y^G} W \to W] \\
&= [(A_i \times_Y \cdots \times_Y P_k^p \times_Y \cdots)^G \times_{Y^G} W \to W \to Z].
\end{aligned}$$

Hence,

$$\mathcal{F}_Z \circ \mathcal{G} = \sum_W \pi_{W*} \circ \mathcal{F}_W \circ \mathcal{G}'.$$

That means it is enough to prove

$$\mathcal{F}_W \circ \mathcal{G}'(G^X_{n,m}) = \mathcal{F}_W \circ \mathcal{G}'(G^Y_{m,n})$$

as elements in $\omega(W)$. In other words, we may assume $\phi = \mathbb{I}_X$. In particular, X is equidimensional. For simplicity, we will denote $\mathcal{F}_Z$ by $\mathcal{F}$.

Within this proof, we will call a G-linearized invertible sheaf $\mathcal{L}$ over X "good" if $\mathcal{L}|_Z$ has trivial G-action. Otherwise, we will call it "bad". We will also call an invariant divisor D on X "good" ("bad") if the corresponding G-linearized invertible sheaf $\mathcal{O}_X(D)$ is "good"("bad").

For a set of invariant divisors $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ on X such that $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$, we define a ring homomorphism $\mathcal{F}'$ from $\mathcal{R}$ to $\mathrm{End}\,(\omega(Z))$ by the following rules :

$$X_i \mapsto \begin{cases} c(\mathcal{O}(A_i)) & \text{if } A_i \text{ is good} \\ 1 & \text{if } A_i \text{ is bad} \end{cases}$$

$$U_k^1 \mapsto \begin{cases} (p_D^1)_*(p_D^1)^* & \text{if } D \overset{def}{=} A_1 + \cdots + A_k \text{ is good} \\ 2 & \text{if } D \text{ is bad} \end{cases}$$

where $p_D^1 : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D)) \to Z$

$$U_k^2 \mapsto \begin{cases} (p_k^2)_*(p_k^2)^* & \text{if } D,\ A_k,\ D+A_k \text{ are all good} \\ & \text{where } D \overset{def}{=} A_1 + \cdots + A_{k-1} \\ 2(p_D^1)_*(p_D^1)^* & \text{if } D \text{ is good but } A_k,\ D+A_k \text{ are bad} \\ 2 + (p_{A_k}^1)_*(p_{A_k}^1)^* & \text{if } A_k \text{ is good but } D,\ D+A_k \text{ are bad} \\ 2 + (p_{D+A_k}^1)_*(p_{D+A_k}^1)^* & \text{if } D+A_k \text{ is good but } A_k,\ D \text{ are bad} \\ 4 & \text{if } D,\ A_k,\ D+A_k \text{ are all bad} \end{cases}$$

where $p_k^2 : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-A_k) \oplus \mathcal{O}(-D-A_k)) \to Z$

$$U_k^3 \mapsto \begin{cases} (p_k^3)_*(p_k^3)^* & \text{if } D,\ A_k,\ D+A_k \text{ are all good} \\ & \text{where } D \overset{def}{=} A_1 + \cdots + A_{k-1} \\ 1 + (p_D^1)_*(p_D^1)^* & \text{if } D \text{ is good but } A_k,\ D+A_k \text{ are bad} \\ 1 + (p_{A_k}^1)_*(p_{A_k}^1)^* & \text{if } A_k \text{ is good but } D,\ D+A_k \text{ are bad} \\ 1 + (p_{D+A_k}^1)_*(p_{D+A_k}^1)^* & \text{if } D+A_k \text{ is good but } A_k,\ D \text{ are bad} \\ 3 & \text{if } D,\ A_k,\ D+A_k \text{ are all bad} \end{cases}$$

where $p_k^3 : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-A_k) \oplus \mathcal{O}(-D-A_k)) \to Z$

if $1 \le i, k \le n$. Otherwise, send it to zero. Define $\mathcal{F}'(Y_j)$, $\mathcal{F}'(V_l^q)$ similarly by replacing "A" by "B". As shown in the proof of Proposition 3.9, $c(\mathcal{O}(D))$ and p_*p^* commutes with each other. Hence, $\mathcal{F}'$ is well-defined. Notice that since, in the U_k^2, U_k^3 cases, we have $D + A_k \sim (A_1 + \cdots + A_k)$, it is impossible to have only one of A_k, D, $D + A_k$ being bad. Thus, the definition covers all possibilities.

Claim 1 : $\mathcal{F}'(G_{n,m}^X) = \mathcal{F}'(G_{m,n}^Y)$ as elements in $\mathrm{End}\,(\omega(Z))$.

By a similar symbolic cancelation as in the proof of Proposition 3.9, it is enough to show the claim in the case when $A + B \sim C$. In this case,

$$G^X_{2,1} = X_1 + X_2 - X_1X_2U^1_1 + Y_1X_1X_2(U^2_2 - U^3_2)$$

and

$$G^Y_{1,2} = Y_1.$$

We will prove the claim case by case.

Case 1 : A, B, C are all good.

In this case,

$$\begin{aligned}
\mathcal{F}'(G^X_{2,1}) &= \mathcal{F}'(X_1 + X_2 - X_1X_2U^1_1 + Y_1X_1X_2(U^2_2 - U^3_2)) \\
&= c(\mathcal{O}(A)) + c(\mathcal{O}(B)) - c(\mathcal{O}(A))c(\mathcal{O}(B))p^1_{A*}p^{1\,*}_A \\
&\quad + c(\mathcal{O}(C))c(\mathcal{O}(A))c(\mathcal{O}(B))(\,p^2_{2*}p^{2^*}_2 - p^3_{2*}p^{3^*}_2\,) \\
&\quad \text{where } p^1_A : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(A)) \to Z \\
&\quad p^2_2 : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to Z \\
&\quad p^3_2 : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-B) \oplus \mathcal{O}(-C)) \to Z.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{F}'(G^Y_{1,2}) &= \mathcal{F}'(Y_1) \\
&= c(\mathcal{O}(C)).
\end{aligned}$$

Thus, the difference $\mathcal{F}'(G^X_{2,1}) - \mathcal{F}'(G^Y_{1,2})$ is exactly what we defined to be $H(\mathcal{O}(A), \mathcal{O}(B))$ in the proof of Proposition 3.9, which was proved to be zero.

Case 2 : A is good but B, C are bad.

$$\begin{aligned}\mathcal{F}'(G^X_{2,1}) &= c(\mathcal{O}(A)) + 1 - c(\mathcal{O}(A))p^1_{A*}{p^1_A}^* \\ &\quad + c(\mathcal{O}(A))(\, 2p^1_{A*}{p^1_A}^* - 1 - p^1_{A*}{p^1_A}^* \,) \\ &= 1 \\ &= \mathcal{F}'(G^Y_{1,2}).\end{aligned}$$

Case 3 : B is good but A, C are bad.

$$\begin{aligned}\mathcal{F}'(G^X_{2,1}) &= 1 + c(\mathcal{O}(B)) - c(\mathcal{O}(B))(2) \\ &\quad + c(\mathcal{O}(B))(\, 2 + p^1_{B*}{p^1_B}^* - 1 - p^1_{B*}{p^1_B}^* \,) \\ &= 1 \\ &= \mathcal{F}'(G^Y_{1,2}).\end{aligned}$$

Case 4 : C is good but A, B are bad.

$$\begin{aligned}\mathcal{F}'(G^X_{2,1}) &= 1 + 1 - (1)(2) + c(\mathcal{O}(C))(\, 2 + p^1_{C*}{p^1_C}^* - 1 - p^1_{C*}{p^1_C}^* \,) \\ &= c(\mathcal{O}(C)) \\ &= \mathcal{F}'(G^Y_{1,2}).\end{aligned}$$

Case 5 : A, B, C are all bad.

$$\begin{aligned}\mathcal{F}'(G^X_{2,1}) &= 1 + 1 - (1)(2) + (1)(4-3) \\ &= 1 \\ &= \mathcal{F}'(G^Y_{1,2}).\end{aligned}$$

That proves the claim. $\triangle$

The next step is to verify the correspondence between $\mathcal{F}$ and $\mathcal{F}'$. To be more precise, let $\mathcal{G} : \mathcal{R} \to M_G(X)^+$ be the map corresponding to a GDPR setup given by $A_1 + \cdots + A_n \sim B_1 + \cdots + B_m$ on X such that $A_1 + \cdots + A_n + B_1 + \cdots + B_m$ is a reduced strict normal crossing divisor and let $\mathcal{F}' : \mathcal{R} \to \mathrm{End}\,(\omega(Z))$ be the map we just defined corresponding to this setup. Consider the fixed point map $\mathcal{F}$ as a map from $M_G(X)^+$ to $\omega(Z)$. The equation we are going to prove is

$$\mathcal{F} \circ \mathcal{G}(s) = \mathcal{F}'(s)[\mathbb{I}_Z] \tag{11}$$

for any element $s \in \mathbb{Z}\{X_i \cdots Y_j \cdots U_k^p \cdots V_l^q \cdots \mid \text{power of any } X_i, Y_j \leq 1\}$.

Suppose equation (11) is true. Then,

$$\begin{aligned}
\mathcal{F} \circ \mathcal{G}(G_{n,m}^X) &= \mathcal{F}'(G_{n,m}^X)[\mathbb{I}_Z] \\
&= \mathcal{F}'(G_{m,n}^Y)[\mathbb{I}_Z] \\
&\quad \text{(by claim 1)} \\
&= \mathcal{F} \circ \mathcal{G}(G_{m,n}^Y),
\end{aligned}$$

which is what we want. That means it is enough to verify equation (11). First of all, we need to understand the meaning of an invariant divisor being "good".

Claim 2 : Suppose D is a smooth invariant divisor on X. Then, D is good if and only if $D \cap Z$ is a smooth divisor on Z. Also, D is bad if and only if $D \cap Z = Z$.

First of all, observe that

$$D \cap Z = D \times_X Z = D \times_X X^G \times_{X^G} Z = D^G \times_{X^G} Z,$$

which is always smooth. If $D \cap Z = \emptyset$, then $\mathcal{O}_Z(D) \cong \mathcal{O}_Z$. That means it is good and $D \cap Z$ is the zero divisor.

Suppose $D \cap Z$ is non-empty. Take a closed point $x \in D \cap Z$. Notice that since the action on Z is trivial and Z is irreducible, the action $\mathcal{O}_Z(D)$ is trivial if and only if the action on $\mathcal{O}_Z(D)|_x$ is trivial. Moreover, $\mathcal{O}_Z(D)|_x \cong \mathcal{O}_D(D)|_x \cong \mathcal{N}_{D \hookrightarrow X}|_x$. Hence, the action on $\mathcal{N}_{D \hookrightarrow X}|_x$ is trivial if and only if D is good.

Suppose the action on $\mathcal{N}_{D \hookrightarrow X}|_x$ is trivial and $D \cap Z$ is not a divisor on Z. That means $D \cap Z = Z$, i.e. $Z \subseteq D$. Thus, we have a natural injective map $\mathcal{N}^\vee_{D \hookrightarrow X}|_x \hookrightarrow \mathcal{N}^\vee_{Z \hookrightarrow X}|_x$. It contradicts with the fact that there is no non-zero vector in $\mathcal{N}^\vee_{Z \hookrightarrow X}|_x$ fixed by G (Proposition 7.1).

Suppose $D \cap Z$ is a divisor on Z. Then D and Z intersect transversely. That means $\mathcal{T}X|_x = \mathcal{T}D|_x + \mathcal{T}Z|_x$ and $\mathcal{T}D|_x \cap \mathcal{T}Z|_x = \mathcal{T}(D \cap Z)|_x$. Therefore, we have $\mathcal{N}_{D \hookrightarrow X}|_x \hookrightarrow \mathcal{T}Z|_x \,/\, \mathcal{T}(D \cap Z)|_x$ and hence, the G-action on $\mathcal{N}_{D \hookrightarrow X}|_x$ is trivial. $\triangle$

Suppose the smooth invariant divisor A_i is good. Then, we have

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(X_i) &= \mathcal{F}[A_i \to X] \\ &= [A_i^G \times_{X^G} Z \to Z] \\ &= [A_i \cap Z \to Z] \\ &= c(\mathcal{O}(A_i))[\mathbb{I}_Z] \\ &\quad \text{(by claim 2 and } \textbf{(Sect)} \text{ axiom in the theory } \omega) \\ &= \mathcal{F}'(X_i)[\mathbb{I}_Z]. \end{aligned}$$

On the other hand, if A_i is bad, then we have $A_i \cap Z = Z$ by claim 2. In this case,

$$\mathcal{F} \circ \mathcal{G}(X_i) = [A_i^G \times_{X^G} Z \to Z] = [Z \to Z] = \mathcal{F}'(X_i)[\mathbb{I}_Z].$$

Hence, equation (11) holds for X_i and Y_j.

For U_k^1, if $D \overset{def}{=} A_1 + \cdots + A_k$ is good, then $\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))$ has trivial action. Thus,

$$\begin{aligned} \mathcal{F} \circ \mathcal{G}(U_k^1) &= [\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D))^G \times_{X^G} Z \to Z] \\ &= [\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D)) \to Z] \\ &= p^1_{D*} {p^1_D}^* [\mathbb{I}_Z] \\ &\quad \text{where } p^1_D : \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D)) \to Z \\ &= \mathcal{F}'(U_k^1)[\mathbb{I}_Z]. \end{aligned}$$

If D is bad, then $\mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))$ has non-trivial fiberwise action. That implies

$$\mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(D))^G|_Z = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))^G = \mathbb{P}(\mathcal{O}_Z(D)) \amalg \mathbb{P}(\mathcal{O}_Z).$$

Thus,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^1) &= [\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(D))^G\times_{X^G} Z\to Z]\\ &= [\mathbb{P}(\mathcal{O}_Z(D))\amalg\mathbb{P}(\mathcal{O}_Z)\to Z]\\ &= 2[\mathbb{I}_Z]=\mathcal{F}'(U_k^1)[\mathbb{I}_Z].\end{aligned}$$

Hence, equation (11) holds for U_k^1 and V_l^1.

For U_k^2, let $D\stackrel{def}{=}A_1+\cdots+A_{k-1}$ as in the definition of $\mathcal{F}'$. There are five different cases to consider.

Case 1 (Divisors D, A_k, $D+A_k$ are all good) :

The action on the projective bundle $\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))$ will be trivial and so is the projective bundle $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))$ above it. Thus,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^2) &= [\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))^G\times_{X^G} Z\to Z]\\ &= [\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))\to Z]\\ &= p^2_{k*}p^{2*}_k[\mathbb{I}_Z]\\ &\quad (p_k^2 \text{ as in the definition of } \mathcal{F}')\\ &= \mathcal{F}'(U_k^2)[\mathbb{I}_Z].\end{aligned}$$

Case 2 (Divisor D is good but A_k, $D+A_k$ are bad) :

In this case,

$$\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))\cong\mathbb{P}(\mathcal{O}_Z(D)\oplus\mathcal{O}_Z),$$

which has trivial action. Moreover, this isomorphism takes $\mathcal{O}(1)$ to $\mathcal{O}(1)\otimes\mathcal{O}_Z(-D-A_k)$. Hence, the tower

$$\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))\to Z$$

is isomorphic to

$$\mathbb{P}(\mathcal{O}\oplus(\mathcal{O}(1)\otimes\mathcal{O}_Z(-D-A_k)))\to\mathbb{P}(\mathcal{O}_Z(D)\oplus\mathcal{O}_Z)\to Z.$$

Hence,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^2) &= 2\,[\mathbb{P}(\mathcal{O}(D)\oplus\mathcal{O})\to Z]\\ &= 2\,p^1_{D*}{p^1_D}^*[\mathbb{I}_Z]\\ &= \mathcal{F}'(U_k^2)[\mathbb{I}_Z].\end{aligned}$$

Case 3 (Divisor A_k is good but D, $D+A_k$ are bad) :

Since D is bad, $\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))\cong\mathbb{P}(\mathcal{O}_Z(D)\oplus\mathcal{O}_Z)$ has fixed point locus $\mathbb{P}(\mathcal{O}_Z(-A_k))\amalg\mathbb{P}(\mathcal{O}_Z(-D-A_k))$. Moreover, the tower

$$\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbb{P}(\mathcal{O}_Z(-A_k))\to Z$$

is isomorphic to

$$\mathbb{P}(\mathcal{O}\oplus(\mathcal{O}(1)\otimes\mathcal{O}_Z(-A_k)))\to\mathbb{P}(\mathcal{O}_Z)\to Z,$$

which is simply $\mathbb{P}(\mathcal{O}\oplus\mathcal{O}_Z(-A_k))\to Z$ and also, the tower

$$\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(1))\to\mathbb{P}(\mathcal{O}_Z(-D-A_k))\to Z$$

is isomorphic to

$$\mathbb{P}(\mathcal{O}\oplus(\mathcal{O}(1)\otimes\mathcal{O}_Z(-D-A_k)))\to\mathbb{P}(\mathcal{O}_Z)\to Z.$$

Hence,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^2) &= [\mathbb{P}(\mathcal{O}(A_k)\oplus\mathcal{O})\to Z]+2[\mathbb{I}_Z]\\ &= (p^1_{A_k*}{p^1_{A_k}}^*+2)[\mathbb{I}_Z]\\ &= \mathcal{F}'(U_k^2)[\mathbb{I}_Z].\end{aligned}$$

Case 4 (Divisor $D+A_k$ is good but D, A_k are bad) :

Similarly, the fixed point locus of $\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))$ is the disjoint union of $\mathbb{P}(\mathcal{O}_Z(-A_k))$ and $\mathbb{P}(\mathcal{O}_Z(-D-A_k))$, and the corresponding towers are the same as in case

3. Hence,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^2) &= 2[\mathbb{I}_Z]+[\mathbb{P}(\mathcal{O}(D+A_k)\oplus\mathcal{O})\to Z]\\ &= (2+p^1_{D+A_k*}p^1_{D+A_k}{}^*)[\mathbb{I}_Z]\\ &= \mathcal{F}'(U_k^2)[\mathbb{I}_Z].\end{aligned}$$

Case 5 (Divisors D, A_k, $D+A_k$ are all bad) :

The fixed point locus of $\mathbb{P}(\mathcal{O}_Z(-A_k)\oplus\mathcal{O}_Z(-D-A_k))$ is again the disjoint union of $\mathbb{P}(\mathcal{O}_Z(-A_k))$ and $\mathbb{P}(\mathcal{O}_Z(-D-A_k))$, and the corresponding towers are the same. Hence,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^2) &= 2[\mathbb{I}_Z]+2[\mathbb{I}_Z]\\ &= \mathcal{F}'(U_k^2)[\mathbb{I}_Z].\end{aligned}$$

That proves equation (11) holds for U_k^2 and similarly for V_l^2.

For U_k^3, similarly, let $D \overset{def}{=} A_1+\cdots+A_{k-1}$. In case 1,

$$\mathcal{F}\circ\mathcal{G}(U_k^3)=[\mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-A_k)\oplus\mathcal{O}(-D-A_k))\to Z]=p^3_{k*}p^{3*}_k[\mathbb{I}_Z]=\mathcal{F}'(U_k^3)[\mathbb{I}_Z].$$

In case 2,

$$\begin{aligned}\mathcal{F}\circ\mathcal{G}(U_k^3) &= [\mathbb{P}(\mathcal{O})\amalg\mathbb{P}(\mathcal{O}(-A_k)\oplus\mathcal{O}(-D-A_k))\to Z]\\ &= [\mathbb{I}_Z]+[\mathbb{P}(\mathcal{O}(D)\oplus\mathcal{O})\to Z]\\ &= (1+p^1_{D*}p^1_D{}^*)[\mathbb{I}_Z]\\ &= \mathcal{F}'(U_k^3)[\mathbb{I}_Z].\end{aligned}$$

In case 3,

$$\mathcal{F}\circ\mathcal{G}(U_k^3)=[\mathbb{I}_Z]+[\mathbb{P}(\mathcal{O}(A_k)\oplus\mathcal{O})\to Z]=(1+p^1_{A_k*}p^1_{A_k}{}^*)[\mathbb{I}_Z]=\mathcal{F}'(U_k^3)[\mathbb{I}_Z].$$

In case 4,

$$\mathcal{F}\circ\mathcal{G}(U_k^3)=[\mathbb{I}_Z]+[\mathbb{P}(\mathcal{O}(D+A_k)\oplus\mathcal{O})\to Z]=(1+p^1_{D+A_k*}p^1_{D+A_k}{}^*)[\mathbb{I}_Z]=\mathcal{F}'(U_k^3)[\mathbb{I}_Z].$$

In case 5,

$$\mathcal{F}\circ\mathcal{G}(U_k^3) = [\mathbb{P}(\mathcal{O}) \amalg \mathbb{P}(\mathcal{O}(-A_k)) \amalg \mathbb{P}(\mathcal{O}(-D-A_k)) \to Z] = 3[\mathbb{I}_Z] = \mathcal{F}'(U_k^3)[\mathbb{I}_Z].$$

That proves equation (11) holds for U_k^3 and similarly for V_l^3.

Let s, t be two terms in

$$\overline{\mathcal{R}} \stackrel{def}{=} \mathbb{Z}\{X_i \cdots Y_j \cdots U_k^p \cdots V_l^q \cdots \mid \text{power of any } X_i,\, Y_j \leq 1\}.$$

By definition, the domain of $\mathcal{G}(st)$ = the domain of $\mathcal{G}(s) \times_X$ the domain of $\mathcal{G}(t)$. For simplicity, we will focus on domains. By abuse of notation, we will still call it $\mathcal{G}$. Observe that

$$\begin{aligned}
\mathcal{F}[Y_1 \times_X Y_2 \to X] &= [(Y_1 \times_X Y_2)^G \times_{X^G} Z \to Z] \\
&= [Y_1^G \times_{X^G} Y_2^G \times_{X^G} Z \to Z] \\
&= [(Y_1^G \times_{X^G} Z) \times_Z (Y_2^G \times_{X^G} Z) \to Z].
\end{aligned}$$

Hence, $\mathcal{F}(Y_1 \times_X Y_2) = \mathcal{F}(Y_1) \times_Z \mathcal{F}(Y_2)$, by abuse of notation again. Suppose $s \stackrel{def}{=} X_i$, Y_j, U_k^p or V_l^q and t is a term in $\overline{\mathcal{R}}$ such that st is also in $\overline{\mathcal{R}}$. By induction, we assume equation (11) holds for s and t. In that case,

$$\begin{aligned}
\mathcal{F}\circ\mathcal{G}(st) &= \mathcal{F}[\mathcal{G}(st) \to X] \\
&= \mathcal{F}[\mathcal{G}(s) \times_X \mathcal{G}(t) \to X] \\
&= [\mathcal{F}(\mathcal{G}(s) \times_X \mathcal{G}(t)) \to Z] \\
&= [\mathcal{F}\circ\mathcal{G}(s) \times_Z \mathcal{F}\circ\mathcal{G}(t) \to Z].
\end{aligned}$$

On the other hand,

$$\mathcal{F}'(st)[\mathbb{I}_Z] = \mathcal{F}'(s)\circ\mathcal{F}'(t)[\mathbb{I}_Z] = \mathcal{F}'(s)[\mathcal{F}\circ\mathcal{G}(t) \to Z]$$

by induction assumption. Denote $\mathcal{F} \circ \mathcal{G}(t)$ by Y and $Y \to Z$ by f. By the above calculation,

$$[\mathcal{F} \circ \mathcal{G}(s) \to Z] = m_1[\mathbb{I}_Z] + m_2[\mathbb{P} \to Z] + m_3[D \cap Z \hookrightarrow Z]$$

for some non-negative integers m_1, m_2, m_3, tower $\mathbb{P}$ and good, smooth, invariant divisor D on X.

Claim 3 : The map $\mathcal{F} \circ \mathcal{G}(s) \to Z$ is transverse to $f : Y \to Z$.

The claim is clearly true for $[\mathbb{I}_Z]$ and $[\mathbb{P} \to Z]$. So, we only need to consider the map $[D \cap Z \hookrightarrow Z]$ where D is a good, smooth, invariant divisor on X. Recall that

$$Y = \mathcal{F} \circ \mathcal{G}(X_i \cdots U_k^p \cdots) = \mathcal{F}(A_i) \times_Z \cdots \times_Z \mathcal{F}(P_k^p) \times_Z \cdots .$$

Since $\mathcal{F}(P_k^p)$ is the sum of towers and $\mathcal{F}(A_i) = Z$ when A_i is bad, we may assume it only involves good divisors, i.e.

$$\mathcal{F}(A_{i_1}) \times_Z \cdots \times_Z \mathcal{F}(B_{j_1}) \times_Z \cdots = A_{i_1} \cap \cdots \cap B_{j_1} \cap \cdots \cap Z.$$

Notice that since st is in $\overline{\mathcal{R}}$, the divisor D and the set of divisors $\{A_{i_1}, \cdots, B_{j_1}, \cdots\}$ are all distinct. For simplicity, we will only show the transversality involving good divisors D, D'. More precisely, we will show if D, D' are good, smooth, invariant divisors on X such that $D + D'$ is a reduced strict normal crossing divisor, then $D \cap Z + D' \cap Z$ is a reduced strict normal crossing divisor on Z.

Since X is equidimensional, D is equidimensional. Let W be an irreducible component of $Z \cap D$. Then, $D \in G\text{-}Sm$ is equidimensional, $D \cap D'$ is an invariant smooth divisor on D and W is an irreducible component of the fixed point locus of D. Notice that

$$\mathcal{O}_W(D \cap D') \cong \mathcal{O}_X(D')|_W \cong \mathcal{O}_Z(D')|_W.$$

Thus, $D \cap D'$ is a good divisor on D with respect to W, for all W, because D' is a good divisor on X with respect to Z. By applying claim 2 with X, D, Z replaced by D, $D \cap D'$, W respectively, $D \cap D' \cap W$ is a smooth divisor on W. So, $D \cap D' \cap Z$ is a smooth divisor on $D \cap Z$. Hence, $D \cap Z$ and $D' \cap Z$ intersect transversely inside Z. $\triangle$

Let $\{Y_i\}$ be the irreducible components of Y. Notice that f is projective. So, the push-forward $f_* : \omega(Y) \to \omega(Z)$ is well-defined. Since Y, Z are both smooth and quasi-projective, the map f is a local complete intersection morphism (See section 5.1.1 in [LeMo]). In addition, the algebraic cobordism theories ω and Ω are canonically isomorphic (Theorem 1 in [LeP]) and, for any local complete intersection morphism $g : X \to X'$ with equidimensional domain and codomain, the pull-back $g^* : \Omega(X') \to \Omega(X)$ is well-defined (see definition 6.5.10 in [LeMo]). Hence, $f^* : \omega(Z) \to \oplus_i \omega(Y_i) \cong \omega(Y)$ is also well-defined.

Suppose we have shown that

$$\mathcal{F}'(s)[f : Y \to Z] = f_* f^* \mathcal{F}'(s)[\mathbb{I}_Z]. \tag{12}$$

Then, we have

$$\begin{aligned}
\mathcal{F}'(st)[\mathbb{I}_Z] &= \mathcal{F}'(s)[f : Y \to Z] \\
&= f_* f^* \mathcal{F}'(s)[\mathbb{I}_Z] \\
&= f_* f^* [\mathcal{F} \circ \mathcal{G}(s) \to Z] \\
&\qquad \text{(by induction assumption)} \\
&= [(\mathcal{F} \circ \mathcal{G}(s)) \times_Z (\mathcal{F} \circ \mathcal{G}(t)) \to Z] \\
&\qquad \text{(by claim 3 and Theorem 6.5.12 in [LeMo])} \\
&= \mathcal{F} \circ \mathcal{G}(st).
\end{aligned}$$

That means equation (11) holds for $st \in \overline{R}$. Hence, it remains to show equation (12).

By the previous calculation,

$$\mathcal{F}'(s) = m_1 + m_2\, p_* p^* + m_3\, c(\mathcal{O}_Z(D))$$

for some non-negative integers m_1, m_2, m_3, smooth, projective map $p : \mathbb{P} \to Z$ and good, smooth, invariant divisor D on X. The equation obviously holds for the identity operator.

For $c(\mathcal{O}_Z(D))$,

$$\begin{aligned} c(\mathcal{O}_Z(D))[Y \to Z] &= [(D \cap Z) \times_Z Y \to Z] \\ &\quad \text{(by claim 3 and \textbf{(Sect)} axiom in } \omega) \\ &= f_* f^* [D \cap Z \to Z] \\ &\quad \text{(by claim 3 and the Theorem 6.5.12 in [LeMo])} \\ &= f_* f^* c(\mathcal{O}_Z(D))\,[\mathbb{I}_Z]. \end{aligned}$$

For $p_* p^*$,

$$\begin{aligned} p_* p^* [Y \to Z] &= [\mathbb{P} \times_Z Y \to Z] \\ &= f_* f^* [\mathbb{P} \to Z] \\ &\quad \text{(by Theorem 6.5.12 in [LeMo])} \\ &= f_* f^* p_* p^* [\mathbb{I}_Z]. \end{aligned}$$

That proves equation (12) and hence finishes the proof of the Theorem. ☐

Corollary 7.3. *If X is an object in G-Sm, then sending $[Y \to X]$ to $[Y^G \to X^G]$ defines an abelian group homomorphism*

$$\mathcal{F} : \mathcal{U}_G(X) \to \omega(X^G).$$

Proof. Let $\{Z\}$ be the set of irreducible components of the fixed point locus X^G. By Theorem 7.2, sending $[Y \to X]$ to $\sum_Z [Y^G \times_{X^G} Z \to Z]$ defines an abelian group homomorphism $\mathcal{U}_G(X) \to \oplus_Z \omega(Z)$. Then, the map $\mathcal{F} : \mathcal{U}_G(X) \to \omega(X^G)$ can be considered as the composition

$$\mathcal{U}_G(X) \to \oplus_Z \omega(Z) \to \oplus_Z \omega(X^G) \to \omega(X^G)$$

defined by sending

$$
\begin{aligned}
[Y \to X] &\mapsto \sum_Z [Y^G \times_{X^G} Z \to Z] \\
&\mapsto \sum_Z [Y^G \times_{X^G} Z \to Z \hookrightarrow X^G] \\
&\mapsto \sum_Z [Y^G \times_{X^G} Z \to Z \hookrightarrow X^G] = [Y^G \to X^G].
\end{aligned}
$$

□

Corollary 7.4. *Suppose X is an object in G-Sm with trivial G-action. Then, the abelian group $\omega(X) \cong \mathcal{U}_{\{1\}}(X)$ is a direct summand of $\mathcal{U}_G(X)$ via the homomorphism*

$$\Phi_\gamma : \mathcal{U}_{\{1\}}(X) \to \mathcal{U}_G(X)$$

induced by the group homomorphism $\gamma : G \to \{1\}$. In particular, the Lazard ring $\mathbb{L}$ is naturally a subring of the equivariant algebraic cobordism ring $\mathcal{U}_G(\operatorname{Spec} k)$.

Proof. The fixed point map

$$\mathcal{F} : \mathcal{U}_G(X) \to \omega(X^G) = \omega(X) \cong \mathcal{U}_{\{1\}}(X)$$

is a left inverse of the homomorphism $\mathcal{U}_{\{1\}}(X) \to \mathcal{U}_G(X)$. Also,

$$\Phi_\gamma : \mathbb{L} \cong \mathcal{U}_{\{1\}}(\operatorname{Spec} k) \to \mathcal{U}_G(\operatorname{Spec} k)$$

is a ring homomorphism. □

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