DETERMINATION OF THE DYNAMIC CHARACTERISTICS OF MECHANISMS USING ENERGY METHODS

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DETERMINATION OF THE DYNAMIC CHARACTERISTICS OF

MECHANISMS USING ENERGY METHODS

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TABLE OF CONTENTS

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SYMBOLS

KE . . . Kinetic Energy. (ft-lbs)

E . . . Input or Output Energy. (ft-lbs)

T . . . Torque. (ft-lbs)

F . . . Force. (lbs)

I . . . Moment of Inertia. (slug-ft^2)

m . . . Mass. (slugs)

ω . . . Instantaneous Angular Velocity. (rad/sec)

α . . . Instantaneous Angular Acceleration. (rad/sec^2)

V . . . Instantaneous Linear Velocity. (ft/sec)

A . . . Instantaneous Linear Acceleration. (ft/sec^2)

θ, ϕ . . Angular Displacement. (radians)

s . . . Linear Displacement. (feet)

t . . . Time. (seconds)

G . . . Center of Gravity.

SIGN CONVENTION

Torques, angular displacements, angular velocities and angular accelerations are considered positive in a counterclockwise direction.

Forces, linear displacements, linear velocities and linear accelerations are considered positive if they act in the direction of the positive X or Y axis.

INTRODUCTION

The study of dynamic characteristics of mechanisms and machines has become more important with the advent of increased speeds in high-speed machinery. The designer of these mechanisms requires a method for accurately analyzing their dynamic characteristics. The purpose of this thesis is to develop and apply energy methods to the analysis of the dynamic characteristics of mechanisms.

Two general types of problems which are involved in the design of a mechanism are:

A. Determination of the dynamic forces resulting in a mechanism which has specified velocities and accelerations.

B. Determination of the velocities and accelerations resulting from the application of known forces.

The problem of determining the forces required to produce specified dynamic characteristics in a mechanism can be solved using vector polygon methods.[1] This method, however, does not provide a direct approach to the determination of velocities and accelerations resulting from the application of known forces. The dynamic characteristics usually are analyzed for one motion cycle of the mechanism and the vector polygon method becomes lengthy and tedious. The required forces are determined for a sufficient number of successive phases of the

[1] Ham, C. W. and E. J. Crane. Mechanics of Machinery, 3rd ed., McGraw-Hill Book Company, Inc., New York, 1948, 538 pp.

mechanism to yield a curve of the driving force versus the mechanism phase. The mechanism phase is either designated by an angular displacement of one of the links of the mechanism or by a linear displacement of some point in the mechanism which has translation. A driving force, which will vary in the manner determined by the vector polygon method, may be impractical to produce. A designer has at his disposal certain devices which can be adapted to driving mechanisms. Springs, air and hydraulic cylinders, and solenoids are three of these devices. All of these devices have known force-displacement relations. Placing these limitations on the available driving sources, it becomes necessary to analyze the dynamic characteristics resulting from the application of known forces. This indicates that a method for solving type B problems would be desirable. Several attempts have been made toward this goal.[2,3] However, the methods developed were not general in their application.

Studying a mechanism from an energy viewpoint limits the analysis to scalar quantities and allows for easier analytical solutions. The energy in a mechanism is a function of the velocities. Therefore, a dynamic analysis of a mechanism would only require a velocity analysis to determine certain characteristics which can be applied in the energy method equations. The energy method solutions indicate two possible methods of representing and analyzing a mechanism. They are an equivalent moment of inertia or an equivalent mass system. Either of these

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[2]Quinn, B. E. Energy Method for Determining Dynamic Characteristics of Mechanisms, Journal of Applied Mechanics (ASME Trans.) Vol. 71, 1949, 283-288

[3]VanSickle, R. C. and T. P. Goodman. Spring Actuated Linkage Analysis to Increase Speed, Product Engineering, Vol. 24, No. 7, 1953, 152-157

systems is adaptable to the solution of either type of problem previously mentioned.

DEVELOPMENT OF THE ENERGY METHOD EQUATIONS

In the analysis that follows, strain energy, potential energy and bearing friction will be neglected. Therefore, the only energy that will be considered is the input, output and kinetic energy of the mechanism.

Equivalent Moment of Inertia

The kinetic energy of a rigid body having plane motion is equal to the kinetic energy due to rotation about its center of gravity plus the kinetic energy due to translation of its center of gravity. Referring to Figure 1, the kinetic energy of each of the links is:

$$\left.\begin{array}{ll} \text{link 2} & KE_2 = 1/2\ I_2 \omega_2^2 \\ \text{link 3} & KE_3 = 1/2\ I_{g3} \omega_3^2 + 1/2\ M_3\ V_{g3}^2 \\ \text{link 4} & KE_4 = 1/2\ I_4 \omega_4^2 \end{array}\right] \qquad (1)$$

I_2 and I_4 are the moments of inertia of links 2 and 4 about their respective centers of rotation. I_{g3} is the moment of inertia of link 3 about its center of gravity (G_3). The subscript on the angular velocity symbols (ω) indicate the respective links. V_{g3} represents the instantaneous linear velocity of the center of gravity of link 3.

The total kinetic energy of a system of rigid bodies is equal to the algebraic sum of the kinetic energy of each body in the system. Therefore, the total kinetic energy (K_t) of the mechanism for the configuration shown is:

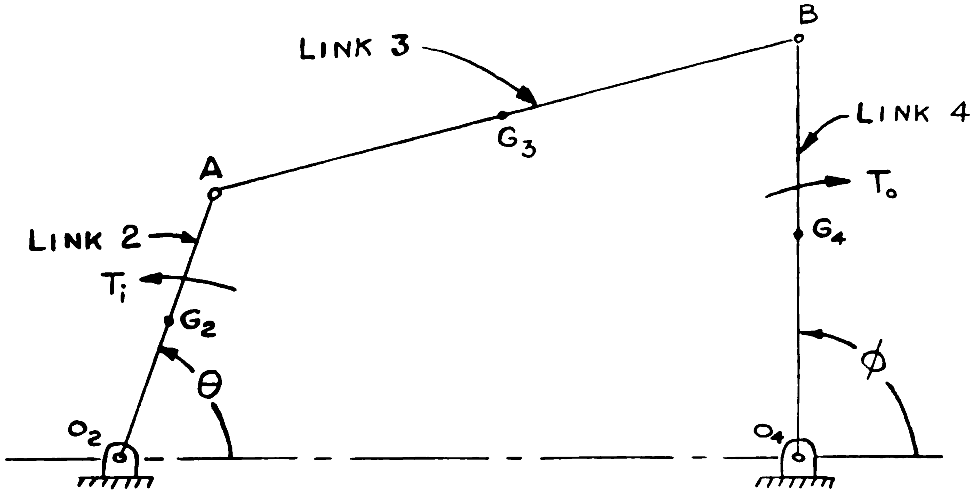

FIGURE 1. A FOUR-BAR MECHANISM.

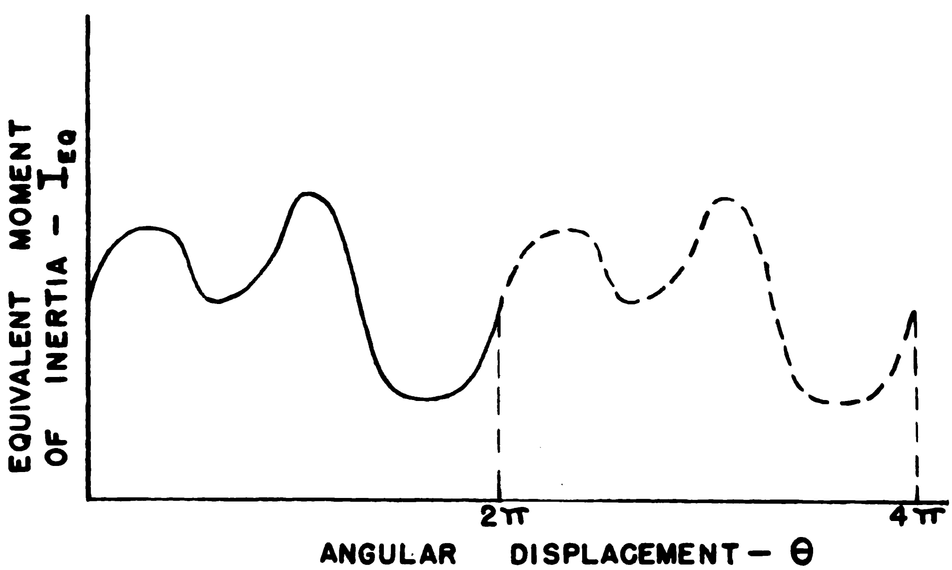

FIGURE 2. AN EQUIVALENT MOMENT OF INERTIA CURVE.

$$KE_t = KE_2 + KE_3 + KE_4$$

or,

$$KE_t = 1/2\ I_2 \omega_2^2 + 1/2\ I_{g3} \omega_3^2 + 1/2\ M_3\ V_{g3}^2 + 1/2\ I_4 \omega_4^2 \qquad (2)$$

This instantaneous kinetic energy is attributed to a variable mass which is rotating about the fixed point O_2 at an instantaneous angular velocity equal to that of link 2; link 2 is the input link of the mechanism. It is apparent that the kinetic energy of this variable mass will be:

$$KE_{eq} = 1/2\ I_{eq} \omega_2^2 \qquad (3)$$

I_{eq} is the instantaneous moment of inertia of the variable mass. Equating the two kinetic energy expressions and solving for the equivalent moment of inertia, the following expressions result:

$$KE_{eq} = KE_t$$

$$I_{eq} = [I_2 + I_4 \left(\frac{\omega_4}{\omega_2}\right)^2 + I_{g3} \left(\frac{\omega_3}{\omega_2}\right)^2 + M_3 \left(\frac{V_{g3}}{\omega_2}\right)^2] \qquad (4)$$

Equation 4 gives a relation between the moments of inertia of each link in the mechanism and the equivalent moment of inertia, when the kinetic energy of the mechanism is referred to link 2. A similar expression will result if a different reference is chosen.

A more general expression for the equivalent moment of inertia for any mechanism, regardless of its complexity is:

$$I_{eq} = \frac{2}{\omega_r^2} \sum_{K=2}^{K=n} (KE)_K \qquad (4a)$$

where (ω_r) is the angular velocity of the reference link and (n) is the number of links in the mechanism.

In Equation 4 there are ratios of the angular velocities of the links of the mechanism which can be determined by a velocity analysis. The angular velocity ratios are a function of the lengths of the links and the angular positions of the links. Therefore, it is not necessary to know a specific value for the angular velocity of a link. The analysis is usually carried out assuming a constant angular velocity for the input or reference link. After this velocity ratio analysis has been completed it is possible to construct a curve of the equivalent moment of inertia versus the angular position of the reference link, using Equation 4. A typical equivalent moment of inertia curve is shown in Figure 2.

Some of the simple mechanisms will be adaptable to complete analytical solution by writing the angular velocity ratios as functions of the crank angle (θ), then substituting in Equation 4 to obtain an analytical expression for the equivalent moment of inertia.

The mechanism is supplied with energy in the form of an input torque (T_i), at link 2, and energy is removed in the form of an output torque (T_o) at link 4. The instantaneous angular velocity of link 2, after link 2 has moved through an angular displacement ($\Delta\theta$), can be found by applying the principle of dynamics that is stated below:

> "The work done on a system of particles by all of the external and internal forces in any displacement of the system is equal to the change in the kinetic energy of the system in the same displacement."[4]

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[4]Seely, M. S. and N. E. Ensign. Analytical Mechanics for Engineers, 3rd ed., John Wiley and Sons, Inc., New York, 1948, p 299

The total work done, on the mechanism, will be equal to the difference between the input and output energies. Referring to Figure 3, the input energy (E_i) will be:

$$E_i = \int_{\theta_a}^{\theta_b} T_i \, d\theta \tag{5}$$

From Figure 4, the output energy (E_o) will be:

$$E_o = \int_{\phi_a}^{\phi_b} T_o \, d\phi \tag{6}$$

The limits used to evaluate the output energy from Equation 6 must be compatible with the limits used to evaluate the input energy from Equation 5. Each set of limits should represent the same change in position of the links of the mechanism. To avoid errors in the choice of limits it is possible to reconstruct the output torque curve of Figure 4 to a curve of output torque versus the reference crank angle (θ), since the angle (ϕ) is a function of the angle (θ). However, in many cases the values of the limits, ϕ_a and ϕ_b, will be determined which are compatible with the limits, θ_a and θ_b. To evaluate the energy quantities given by the Equations 5 and 6, it will be necessary to know the relation between the torques and their respective angles. If the torque relations are not adaptable to analytical expressions it will be necessary to construct the two torque curves and by graphical or numerical means evaluate the energy. The difference between the input and output energy, or the net input energy, can be determined by constructing curves similar to those in Figure 5. The area, cdef, between the two

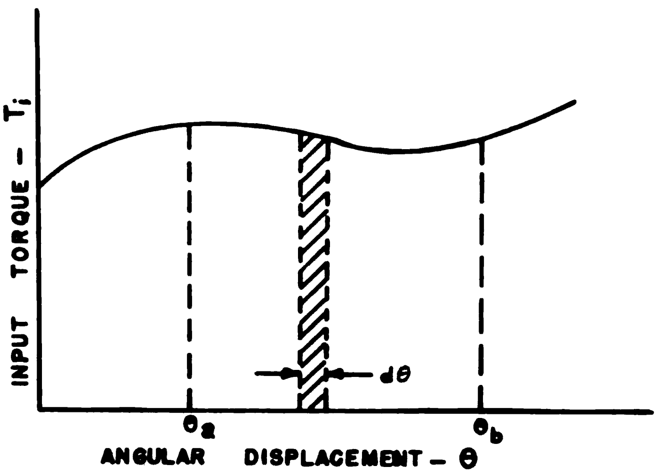


FIGURE 3. AN INPUT TORQUE CURVE.

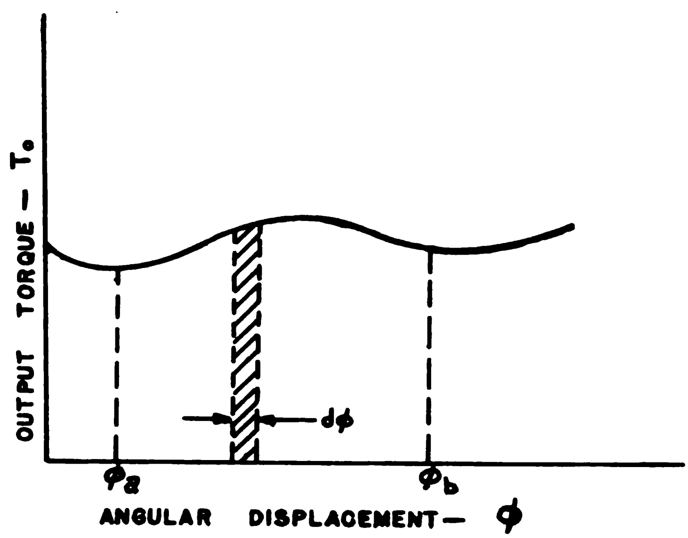


FIGURE 4. AN OUTPUT TORQUE CURVE.

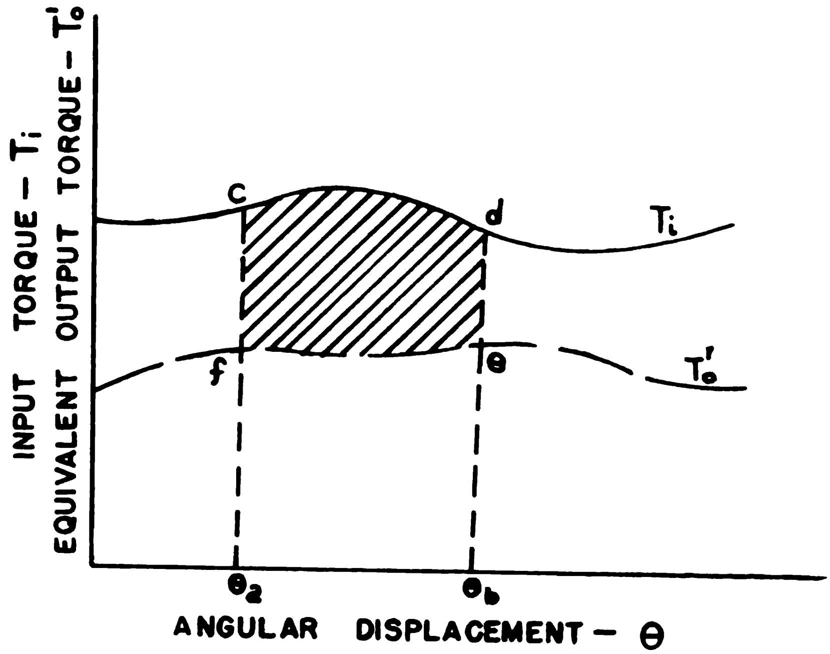


FIGURE 5. INPUT AND EQUIVALENT OUTPUT TORQUE CURVES.

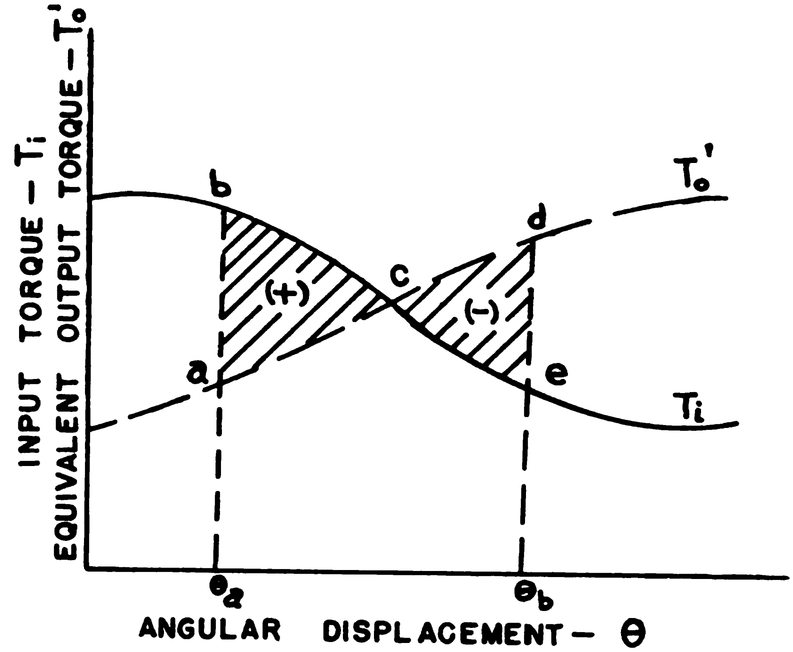


FIGURE 6. INPUT AND EQUIVALENT OUTPUT TORQUE CURVES.

torque curves within the displacement interval will represent the net input energy during that interval. This area can represent a negative input or loss in net input energy if more than one half of the total area indicated lies above the input torque curve. This is illustrated in Figure 6. The area, abc, represents a net gain of input energy while the area, cde, represents a net loss in input energy to the mechanism. To construct the curve in Figure 5 for an equivalent output torque (T_o') versus angular displacement (θ) it is necessary to use the following relations:

$$\int_{\theta_a}^{\theta_b} T_o' \, d\theta = \int_{\phi_a}^{\phi_b} T_o \, d\phi$$

Differentiation with respect to θ yields:

$$T_o' = T_o \frac{d\phi}{d\theta}$$

but,
$$\frac{d\phi}{d\theta} = \frac{\omega_4}{\omega_2}$$

therefore:
$$T_o' = T_o \frac{\omega_4}{\omega_2} \qquad (7)$$

Equation 7 indicates that the value of the output torque (T_o) at any position can be converted to an equivalent output torque with reference to the input link, if it is multiplied by the angular velocity ratio of the output to the input link at that position.

Subtraction of Equation 6 from Equation 5 and equating this result to the change in the kinetic energy of the equivalent system during the same displacement interval yields the following expressions:

$$E_i - E_o = KE_b - KE_a$$

$$\int_{\theta_a}^{\theta_b} T_i \, d\theta - \int_{\phi_a}^{\phi_b} T_o \, d\phi = 1/2(I_{eq})_b \omega_b^2 - 1/2(I_{eq})_a \omega_a^2 \qquad (8)$$

The subscripts on I_{eq} and ω indicate the angular positions of the equivalent system and the reference link of the actual mechanism. The equivalent system is rotating at the same angular velocity as the reference link, link 2. Therefore, ω_b will equal the instantaneous angular velocity of link 2 at the angular position θ_b. Solving Equation 8 for the angular velocity yields the following equation:

$$\omega_b^2 = \frac{2}{(I_{eq})_b} \left[\int_{\theta_a}^{\theta_b} T_i \, d\theta - \int_{\phi_a}^{\phi_b} T_o \, d\phi + 1/2(I_{eq})_a \, \omega_a^2 \right] \qquad (8a)$$

If the net input energy supplied to the mechanism during a particular angular displacement of the reference link is equal to zero, ($E_i = E_o$), then the instantaneous angular velocity after this displacement will be given by the following relation:

$$\omega_b = \omega_a \sqrt{\frac{(I_{eq})_a}{(I_{eq})_b}} \qquad (8b)$$

The terms $(I_{eq})_a$ and $(I_{eq})_b$ denote the value of the equivalent moment of inertia of the mechanism at positions θ_a and θ_b, respectively. If the reference link of the mechanism is rotating at an instantaneous angular velocity (ω_a), at the angular position θ_a, and is allowed to rotate to position θ_b without a change in the total energy of the mechanism, the instantaneous angular velocity of the reference link will change in accordance with Equation 8b. This change in angular velocity of the reference link would indicate a transfer of energy between the links of the mechanism.

To determine the angular acceleration (α_b) of the reference link at the angular position θ_b, Equation 8a is differentiated once with respect to θ, and the relation, $\alpha = \omega \frac{d\omega}{d\theta}$, is used to obtain:

$$\alpha_b = \frac{[(T_i)_b - (T_o)_b (\frac{\omega_4}{\omega_2})_b]}{(I_{eq})_b} - \frac{(\frac{dI_{eq}}{d\theta})_b \left[\int_{\theta_a}^{\theta_b} T_i d\theta - \int_{\phi_a}^{\phi_b} T_o d\phi + 1/2 (I_{eq})_a \omega_a^2\right]}{(I_{eq})_b^2} \tag{9}$$

Substitution of Equation 8a in Equation 9 yields:

$$\alpha_b = \frac{[(T_i)_b - (T_o)_b (\frac{\omega_4}{\omega_2})_b]}{(I_{eq})_b} - \frac{(\omega^2)_b}{2(I_{eq})_b} (\frac{dI_{eq}}{d\theta})_b \tag{9a}$$

When performing the differentiation, it must be remembered that the terms $(I_{eq})_b$, T_i, T_o and ω_b are functions of the angular position θ, while $(I_{eq})_a$ and ω_a have specific values as determined at position θ_a.

In Equation 9a, the terms $(T_i)_b$, $(T_o)_b$, $(I_{eq})_b$, $(\frac{dI_{eq}}{d\theta})_b$ and $(\frac{\omega_4}{\omega_2})_b$ are determined for the angular position, θ_b, of the reference link. The torques $(T_i)_b$ and $(T_o)_b$ are obtained from the input and output torque curves or analytical expressions. The angular velocity ratio $(\frac{\omega_4}{\omega_2})_b$

can be determined by graphical methods or from the data used to construct the equivalent moment of inertia curve. The value of the equivalent moment of inertia $(I_{eq})_b$ and the slope $(\frac{dI_{eq}}{d\theta})_b$ are obtained from the equivalent moment of inertia curve. The slope $(\frac{dI_{eq}}{d\theta})$ can be determined graphically for each particular position being analyzed or the equivalent moment of inertia curve can be graphically or numerically differentiated and a slope curve constructed to permit a complete analysis of the mechanism.

The angular acceleration can also be obtained by graphical differentiation of the angular velocity versus the angular displacement curve, which is obtained from Equation 8a, and using the relationship, $\alpha = \omega \frac{d\omega}{d\theta}$. Both methods require one graphical differentiation and would introduce approximately the same error in the analysis. However, the equivalent moment of inertia curve has been constructed for use in obtaining the angular velocity from Equation 8a. It would therefore seem more desirable to differentiate this curve and use Equation 9a to obtain the angular acceleration.

Rewriting Equation 9a yields:

$$[(T_i)_b - (T_o)_b(\frac{\omega_4}{\omega_2})_b] = \frac{(\omega^2)_b}{2}(\frac{dI_{eq}}{d\theta})_b + (I_{eq})_b\alpha_b \qquad (9b)$$

This equation is more convenient for finding the torques required to meet specified angular velocities and accelerations.

To determine the time required for the linkage to move through a particular angular displacement, it is necessary to construct a curve of the reciprocal of the angular velocity $(\frac{1}{\omega})$ versus the angular displacement. Equation 8a is used to compute the angular velocity. The area under this curve, within the displacement limits, will then represent the time. This can be shown by the following equations:

$$\omega = \frac{d\theta}{dt}$$

or,

$$dt = \frac{1}{\omega} d\theta$$

$$t_b - t_a = \int_{\theta_a}^{\theta_b} \frac{1}{\omega} d\theta \qquad (10)$$

The curve of $(\frac{1}{\omega})$ versus θ can be graphically integrated to obtain a time versus angular displacement curve. In some cases it would only be necessary to measure the total area under the curve to determine the time for a complete motion cycle of the reference link.

Equivalent Mass

The preceding outline offers a method for the complete dynamic analysis of a mechanism when the mechanism is reduced to a single rotating mass with a variable moment of inertia. A similar analysis results for an equivalent mass system if a reference point which has translation is chosen instead of a reference link which is rotating about a fixed point. Referring to the mechanism in Figure 7, the kinetic energy for any configuration of the mechanism can be determined from the following expression:

$$KE_t = 1/2\ I_2 \omega_2^2 + 1/2\ I_{g3} \omega_3^2 + 1/2\ M_3 V_{g3}^2 + 1/2\ M_4 V_p^2 \qquad (11)$$

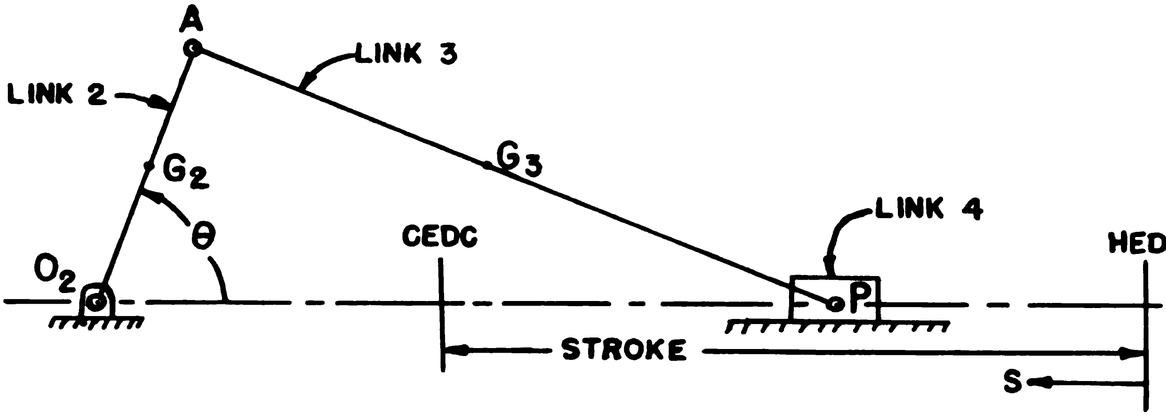


FIGURE 7. A SLIDER CRANK MECHANISM.

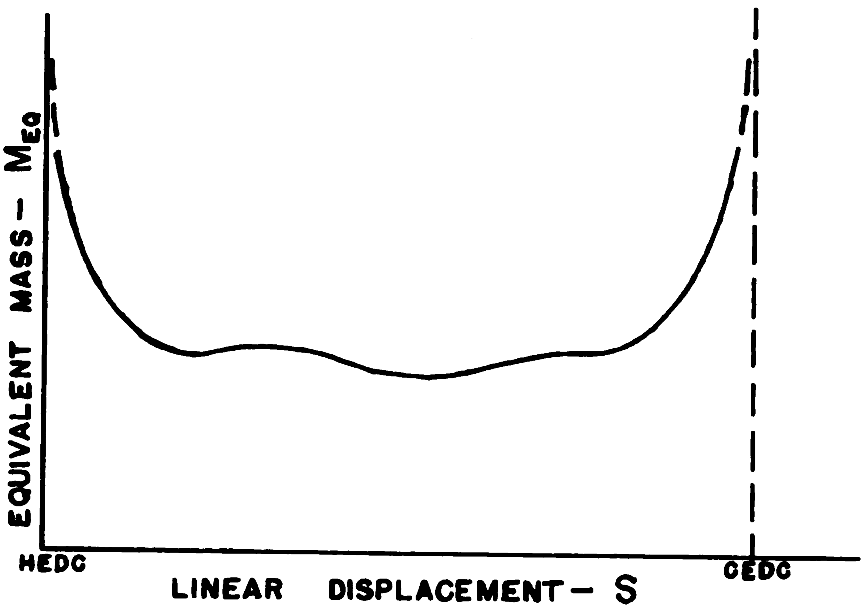


FIGURE 8. AN EQUIVALENT MASS CURVE.

The instantaneous kinetic energy of a variable mass, which is translating at an instantaneous linear velocity equal to that of the point "P" in the actual mechanism, will be:

$$KE_{eq} = 1/2\ M_{eq}\ V_p^2 \tag{12}$$

If the mechanism and the variable mass are assumed to possess the same kinetic energy at all times, then the mass of the equivalent system must vary in order to maintain this energy balance. The expression used to find the value of the equivalent mass of the variable mass body, for any configuration of the mechanism, can be determined by equating the two kinetic energy expressions and solving for M_{eq}.

$$M_{eq} = \left[I_2\left(\frac{\omega_2}{V_p}\right)^2 + I_{g3}\left(\frac{\omega_3}{V_p}\right)^2 + M_3\left(\frac{V_{g3}}{V_p}\right)^2 + M_4 \right] \tag{13}$$

For a mechanism with "n" links and a point "P" chosen as the translation reference point, the equivalent mass equation becomes:

$$M_{eq} = \frac{2}{V_p^2} \sum_{K=2}^{K=n} (KE)_K \tag{13a}$$

The velocity ratios in Equation 13 vary with each particular configuration of the mechanism. Therefore, the ratios are determined for several configurations which will completely represent one motion cycle of the reference point. Then a curve of the equivalent mass versus the linear displacement of the reference point is constructed. Figure 8 indicates a method for representing the data obtained from Equation 13 or 13a.

Because the reference point in this analysis has reciprocating motion, the velocity becomes zero at each end of its path. Similarly, the angular velocity for an oscillating crank would become zero at the end of its path. Referring to Equation 12, this would indicate that the kinetic energy of the equivalent system becomes equal to zero. However, the kinetic energy of the actual mechanism is not necessarily equal to zero. Therefore, the equivalent mass must become infinitely large. The value of this equivalent mass will be indeterminate from Equation 13. It will therefore be necessary to construct the equivalent mass curve as accurately as possible up to the head end and crank end dead center positions, but not including these positions. As the curve in Figure 8 indicates, there would be considerable error introduced at the two end positions. It will be shown later that the slope $\left(\frac{dM_{eq}}{ds}\right)$ of the equivalent mass curve will be required for the analysis. This slope, in the region at the ends of the path of motion, will be very susceptible to error and this error may be objectionable in the analysis. However, some mechanisms of this type will only be analyzed through a portion of their entire motion cycle and if this motion is restricted to the accurate portion of the equivalent mass curve, the equivalent mass analysis can be used. If a complete analysis of this mechanism is required it will be necessary to use the equivalent moment of inertia method; in which case, the input or output torque curve can be replaced by a force-displacement relation similar to the one shown in Figure 9. This will not introduce any new problems to the analysis because the energy, either input or output, will be determinate from the force-displacement curve.

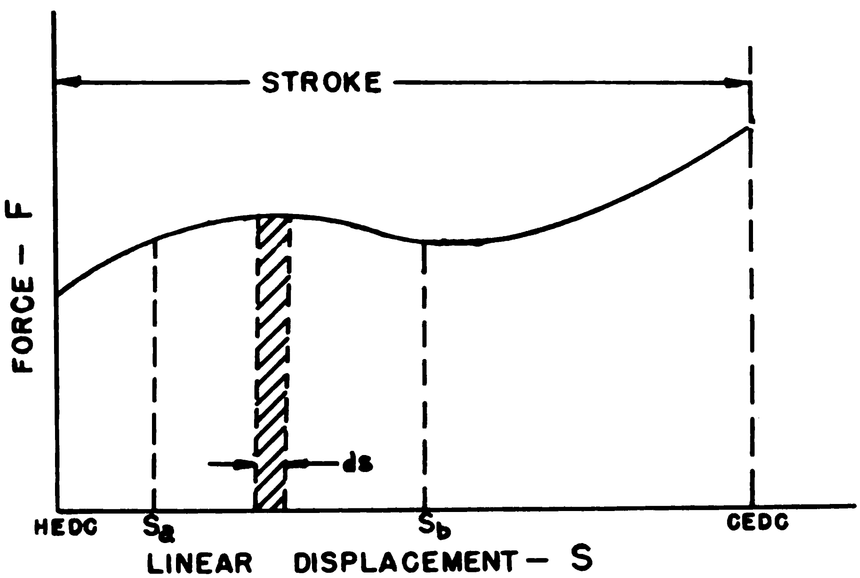


FIGURE 9. A FORCE CURVE.

The linear velocity (V_p) of the reference point in the mechanism of Figure 7 can be determined in a manner similar to that previously outlined for finding the angular velocity of the reference link in the mechanism of Figure 1. The input energy supplied to the mechanism during a change in the configuration of the mechanism, assuming an input torque versus angular displacement relation as shown in Figure 3 is known, can be determined by the following equation:

$$E_i = \int_{\theta_a}^{\theta_b} T_i \, d\theta \qquad (14)$$

The output energy during the same configuration change can be determined if the output force-displacement relation is known. The curve in Figure 9 represents a general relation between the force acting on the piston and the linear displacement of the piston. The output energy is obtained from the following equation:

$$E_o = \int_{s_a}^{s_b} F \, ds \qquad (15)$$

where the limits, s_a and s_b, are compatible with the limits, θ_a and θ_b, respectively.

The force-displacement curve of Figure 9 can be changed to an equivalent output torque (T_o') versus angular displacement curve if the value of the force at a particular configuration is multiplied by the ratio of the linear velocity of the piston to the angular velocity of the input link at that position. This relation can be shown by the following equations:

$$\int_{\theta_a}^{\theta_b} T_o' \, d\theta = \int_{s_a}^{s_b} F \, ds$$

where (T_o') is the equivalent output torque function acting on link 2. Differentiation with respect to θ yields:

$$T_o' = F \left(\frac{ds}{d\theta}\right)$$

but,

$$\frac{ds}{d\theta} = \frac{V_p}{\omega_2}$$

therefore,

$$T_o' = F \left(\frac{V_p}{\omega_2}\right) \tag{16}$$

The net input energy supplied to the mechanism during a particular displacement interval can be determined by subtracting Equation 15 from Equation 14. The following expressions result if the net input energy supplied to the mechanism is equated to the change in the kinetic energy of the equivalent system during the same displacement interval.

$$E_i - E_o = KE_b - KE_a$$

$$\int_{\theta_a}^{\theta_b} T_i \, d\theta - \int_{s_a}^{s_b} F \, ds = 1/2(M_{eq})_b(V_p)_b^2 - 1/2(M_{eq})_a(V_p)_a^2 \tag{17}$$

Solving Equation 17 for the instantaneous linear velocity of the piston yields the following equation:

$$(V_p)_b^2 = \frac{2}{(M_{eq})_b}\left[\int_{\theta_a}^{\theta_b} T_i \, d\theta - \int_{s_a}^{s_b} F \, ds + 1/2(M_{eq})_a(V_p)_a^2\right] \tag{17a}$$

Equation 17a can be used to compute the instantaneous linear velocity of the reference point in the mechanism after a particular change in the configuration of the mechanism.

The instantaneous linear velocity of the reference point, after a particular displacement during which the net input energy supplied to the mechanism is equal to zero ($E_i = E_o$), will be obtained from the following equation:

$$(V_p)_b = (V_p)_a \sqrt{\frac{(M_{eq})_a}{(M_{eq})_b}} \qquad (17b)$$

Equation 17b indicates that if the mechanism was supplied with a certain amount of energy then allowed to coast, without a change in the total energy of the mechanism, the velocity of the reference point will vary. This indicates that the kinetic energy of each link varies in order to maintain the overall energy level and consequently energy must be transferred between the links of the mechanism.

Differentiation of Equation 17a with respect to s and use of the relation, $A_p = V_p(\frac{dV_p}{ds})$, yields the following expression for the instantaneous linear acceleration of the reference point:

$$(A_p)_b = \frac{[(T_i)_b(\frac{\omega_2}{V_p})_b - (F)_b]}{(M_{eq})_b} - \frac{(\frac{dM_{eq}}{ds})_b\left[\int_{\theta_a}^{\theta_b} T_i \, d\theta - \int_{s_a}^{s_b} F \, ds + 1/2(M_{eq})_a(V_p)_a^2\right]}{(M_{eq})_b^2} \qquad (18)$$

Substitution of Equation 17a in Equation 18 yields:

$$(A_p)_b = \frac{[(T_i)_b \ (\frac{\omega_2}{V_p})_b - (F)_b]}{(M_{eq})_b} - \frac{(V_p)_b^2}{2(M_{eq})_b} (\frac{dM_{eq}}{ds})_b \qquad (18a)$$

The terms $(M_{eq})_a$ and $(V_p)_a^2$, in Equation 17a, are not a function of the linear displacement but have specific values as determined at (s_a). All other terms in Equation 17a are a function of the displacement. The subscript, outside the parentheses, on the terms in Equation 18a indicate the position at which the functions are evaluated. The force (F) is a function of the linear displacement (s). The torque (T_i) is a function of the crank angle (θ), and the equivalent mass (M_{eq}) is a function of the linear displacement. All of these functions are represented by their respective graphs or curves.

The instantaneous linear acceleration of the reference point can also be obtained by graphical differentiation of the linear velocity versus linear displacement curve. The velocity-displacement curve can be constructed using Equation 17a.

The time required for the reference point to move through a particular linear displacement can be determined by constructing a curve of the reciprocal of the linear velocity $(\frac{1}{V_p})$ versus the linear displacement. The area under this curve, within the displacement limits, will represent the time for that displacement. The equations for obtaining the time are as follows:

$$V = \frac{ds}{dt}$$

or,

$$dt = \int_{s_a}^{s_b} \frac{1}{V}\, ds$$

Integration yields:

$$t_b - t_a = \int_{s_a}^{s_b} \frac{1}{V}\, ds \qquad (19)$$

The two equivalent systems derived from energy methods are very general, but the equivalent moment of inertia equations will apply to all mechanisms and is considered the more general of the two systems.

APPLICATIONS OF THE ENERGY METHOD EQUATIONS

To better understand the potentialities and to evaluate the merits of the energy method, four problems are presented which illustrate the application of the derived equations to actual mechanisms. All of the problems are solved using the equations and methods derived for an equivalent moment of inertia system.

The first problem uses a parallel crank mechanism subjected to constant input and output torques. The mechanism represents a special case of the equations.

The second problem uses a four-bar mechanism subjected to a constant input torque.

The third problem uses the same mechanism that is used in problem two with a more general condition on the applied forces. The mechanism is considered to be an opening electrical switch.

The fourth problem uses a slider crank mechanism subjected to an input force on the piston.

Parallel Crank Mechanism

The parallel crank mechanism shown in Figure 10 has the following physical properties:

Weight of link 2	10 lbs
Weight of link 3	20 lbs
Weight of link 4	10 lbs
Moments of inertia of links 2 and 4 about their respective centers of gravity	0.012 slug-ft^2
Moment of inertia of link 3 about its center of gravity	0.020 slug-ft^2

The mechanism starts from rest at a position corresponding to $\theta = 0^\circ$ and is subjected to a constant input torque of 10 ft-lbs and a constant output torque of 5 ft-lbs. The torques act in the directions indicated on the mechanism.

The equations and methods derived from the energy equations are used to construct the angular velocity and acceleration curves. Also the time required for one motion cycle is to be determined. The solution of this problem is accomplished in the following four steps:

(1.) Equation 4 is used to construct the equivalent moment of inertia curve.

$$I_{eq} = [I_2 + I_4 \left(\frac{\omega_4}{\omega_2}\right)^2 + I_{g3}\left(\frac{\omega_3}{\omega_2}\right)^2 + M_3\left(\frac{V_{g3}}{\omega_2}\right)^2]$$

The parallel crank mechanism is unique in that the following conditions apply:

$$\omega_2 = \omega_4$$

$$\omega_3 = 0$$

$$V_{g3} = V_a = (\overline{O_2A})\,\omega_2$$

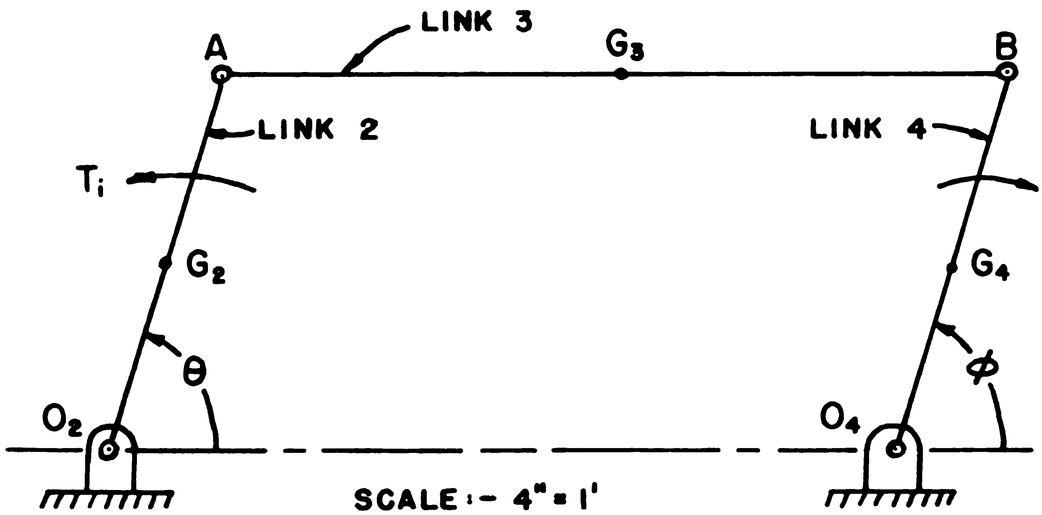


FIGURE 10. PARALLEL CRANK MECHANISM.

Substitution of these conditions in Equation 4 yields:

$$I_{eq} = I_2 + I_4 + M_3 (\overline{O_2A})^2$$

Therefore, the equivalent moment of inertia is constant for all phases of the mechanism. The values of the moments of inertia given in the statement of the problem are about the centers of gravity of the links and the values in the above equation are about the centers of rotation. The parallel axis theorem is applied to find I_2 and I_4.

$$I_2 = I_{g2} + M_2(\overline{O_2G_2})^2$$

$$I_2 = 0.0314 \text{ slug-ft}^2$$

$$I_4 = 0.0314 \text{ slug-ft}^2$$

Therefore,

$$I_{eq} = 0.2178 \text{ slug-ft}^2$$

(2.) Equation 8a is used to construct the angular velocity curve. For this problem, the torques are constant, the initial angular velocity (ω_a) is equal to zero and the angle θ is equal to the angle ϕ.

$$T_i = 10 \text{ ft-lbs} \quad \ldots\ldots\ldots \quad \int_0^\theta T_i \, d\theta = 10\,\theta$$

$$T_o = 5 \text{ ft-lbs} \quad \ldots\ldots\ldots \quad \int_0^\theta T_o \, d\theta = 5\,\theta$$

Substitution of the above relations in Equation 8a yields the following general equation for the instantaneous angular velocity of the reference link as a function of the angular displacement.

$$\omega = 6.78\sqrt{\theta}$$

The angular displacement (θ) is measured in radians. The curve of the angular velocity versus the angular displacement is shown in Figure 11 for one revolution of the reference link. The above equation will apply for more than one revolution.

(3.) The angular acceleration curve is constructed next using Equation 9. The equation is simplified by the condition that I_{eq} is constant. Therefore:

$$\frac{dI_{eq}}{d\theta} = 0$$

$$\alpha = \frac{T_i - T_o\left(\frac{\omega_4}{\omega_2}\right)}{I_{eq}}$$

$$\alpha = \frac{10 - 5}{0.2178} = 23 \text{ rad/sec}^2$$

The angular acceleration for this problem is constant. This can also be determined by differentiation of the equation for the angular velocity.

$$\omega = 6.78\sqrt{\theta}$$

$$\omega^2 = 46\ \theta$$

$$2\omega\frac{d\omega}{d\theta} = 46$$

$$\alpha = 46/2 = 23 \text{ rad/sec}^2$$

(4.) The time required for the mechanism to complete one motion cycle can be determined in two ways.

a. The angular velocity equation determined for this problem is substituted in Equation 10.

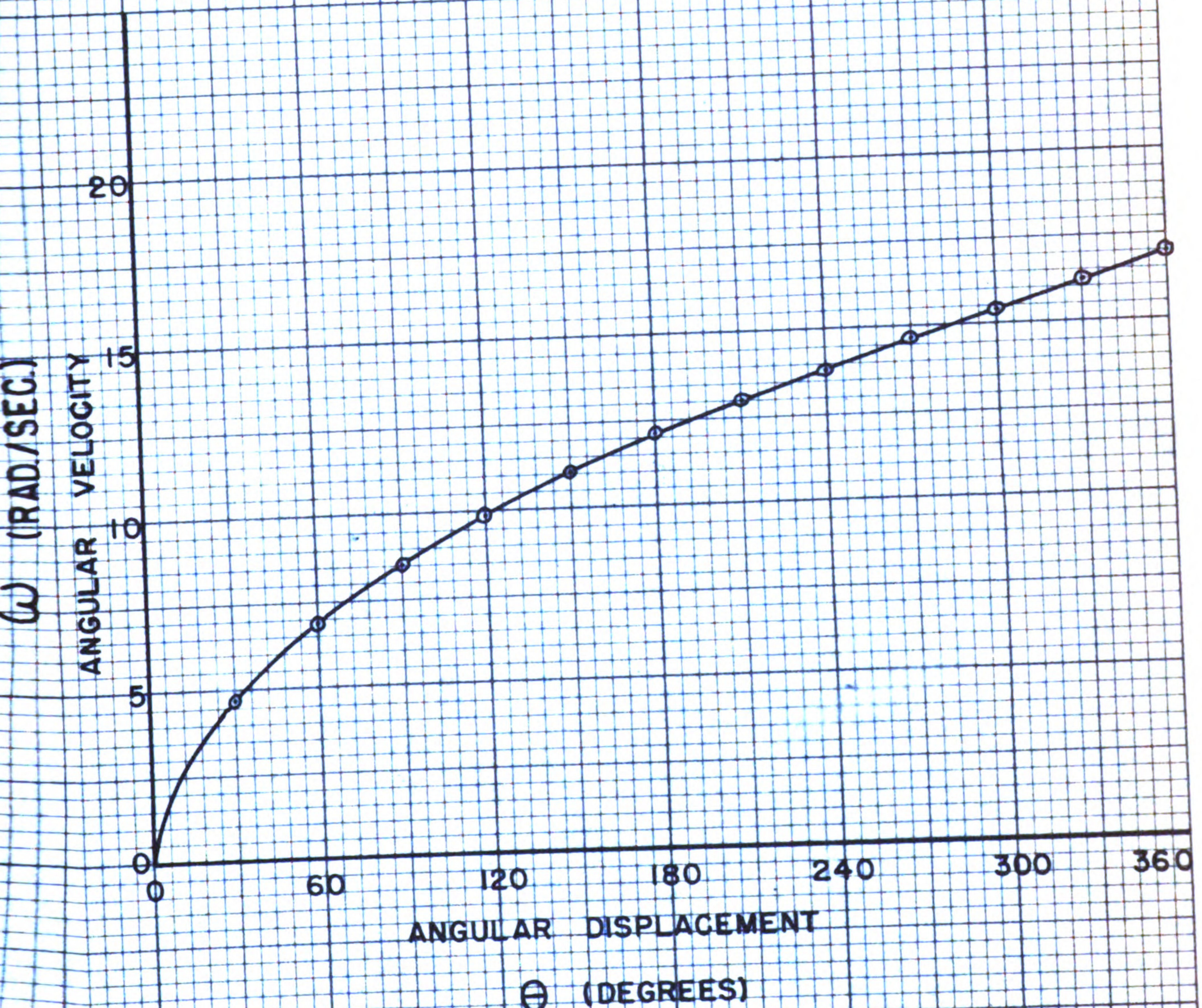


FIGURE 11. ANGULAR VELOCITY CURVE FOR PARALLEL CRANK MECHANISM.

$$\omega = 6.78 \sqrt{\theta}$$

$$t = \int_{0}^{2\pi} \frac{1}{\omega} d\theta$$

$$t = \frac{2\sqrt{\theta}}{6.78} = 0.74 \text{ seconds}$$

b. Since I_{eq} is constant, the mechanism can be treated as a rigid body rotating about a fixed point. Therefore, the equations of motion for a rigid body apply. The rigid body starts from rest and is subjected to a constant acceleration.

$$\omega_f = \omega_o + \alpha t$$

$$\omega_f = 23 \text{ t}$$

The angular velocity after one motion cycle is equal to 17 rad/sec (from Figure 11). Therefore,

$$t = 17/23 = 0.74 \text{ seconds}$$

A check on the results of this problem can be made using the vector polygon methods.

Four-Bar Mechanism

The four-bar mechanism shown in Figure 12a has the following physical properties:

Weight of link 2	10 lbs
Weight of link 3	3.22 lbs
Weight of link 4	10 lbs
Moments of inertia of links 2 and 4 about their respective centers of gravity	0.012 slug-ft^2
Moment of inertia of link 3 about its center of gravity	0.020 slug-ft^2

The mechanism starts from rest at a position corresponding to $\theta = 0^\circ$ and is subjected to a constant input torque of 10 ft-lbs. The torque acts in the direction indicated on the mechanism.

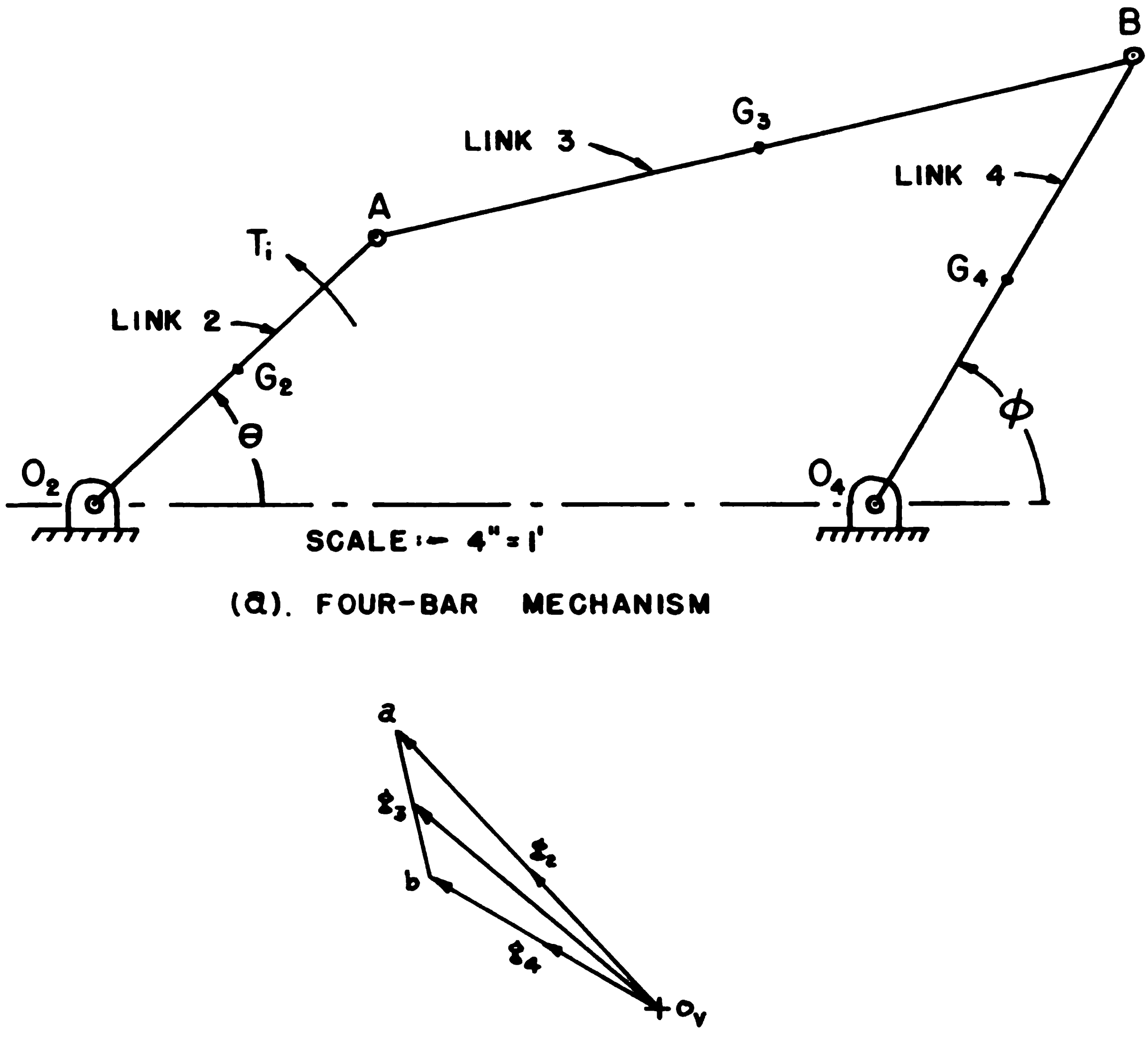
The angular velocity and the angular acceleration curves, for link 2, are constructed in the following manner:

(1.) Equation 4 is used to construct the equivalent moment of inertia curve.

$$I_{eq} = \left[I_2 + I_4\left(\frac{\omega_4}{\omega_2}\right)^2 + I_{g3}\left(\frac{\omega_3}{\omega_2}\right)^2 + M_3\left(\frac{V_{g3}}{\omega_2}\right)^2 \right]$$

The velocity ratios, in Equation 4, are determined from a velocity vector polygon. The velocity polygon is drawn for a sufficient number of phases to construct the equivalent moment of inertia curve. A sample velocity polygon is shown in Figure 12b. The data obtained from the polygons and the information required to construct the curve are tabulated in Table I. The equivalent moment of inertia versus the angular displacement curve is shown in Figure 13.

(2.) The angular velocity curve is constructed using Equation 8a. Since the mechanism starts from rest at $\theta = 0^\circ$ and is subjected to a constant input torque of 10 ft-lbs, the following conditions apply:



(a). FOUR-BAR MECHANISM

(b). VELOCITY POLYGON

FIGURE 12. FOUR-BAR MECHANISM AND VELOCITY POLYGON.

TABLE I

EQUIVALENT MOMENT OF INERTIA DATA FOR
FOUR-BAR MECHANISM

θ degrees	$\frac{\omega_4}{\omega_2}$	$\frac{\omega_3}{\omega_2}$	$\frac{V_{g3}}{\omega_2}$	I_{eq} * slug-ft^2
0	1.000	1.000	0.308	0.0923
7 1/2	0.614	0.825	0.199	0.0608
15	0.238	0.633	0.192	0.0449
22 1/2	0.050	0.470	0.263	0.0429
30	0.281	0.340	0.333	0.0474
37 1/2	0.434	0.250	0.389	0.0538
45	0.532	0.187	0.424	0.0590
60	0.664	0.100	0.467	0.0672
75	0.725	0.050	0.492	0.0722
90	0.750	0.008	0.500	0.0740
105	0.750	0.021	0.500	0.0740
120	0.725	0.059	0.490	0.0720
135	0.682	0.100	0.475	0.0687
150	0.600	0.162	0.445	0.0631
165	0.482	0.241	0.400	0.0559
180	0.338	0.330	0.349	0.0494
195	0.185	0.420	0.299	0.0449
210	0.050	0.484	0.258	0.0428
225	0.055	0.510	0.246	0.0428
240	0.150	0.513	0.250	0.0437
255	0.249	0.478	0.283	0.0460
270	0.350	0.408	0.333	0.0495
285	0.475	0.292	0.391	0.0560
300	0.662	0.100	0.468	0.0674
307 1/2	0.782	0.038	0.508	0.0764
315	0.913	0.200	0.550	0.0888
322 1/2	1.075	0.396	0.588	0.1054
330	1.220	0.600	0.600	0.1213
337 1/2	1.355	0.830	0.600	0.1338
345	1.390	1.000	0.550	0.1424
352 1/2	1.265	1.060	0.445	0.1241
360	1.000	1.000	0.308	0.0923

$$* \; I_{eq} = \left[.0314 + .0314\left(\frac{\omega_4}{\omega_2}\right)^2 + .020\left(\frac{\omega_3}{\omega_2}\right)^2 + .1\left(\frac{V_{g3}}{\omega_2}\right)^2\right]$$

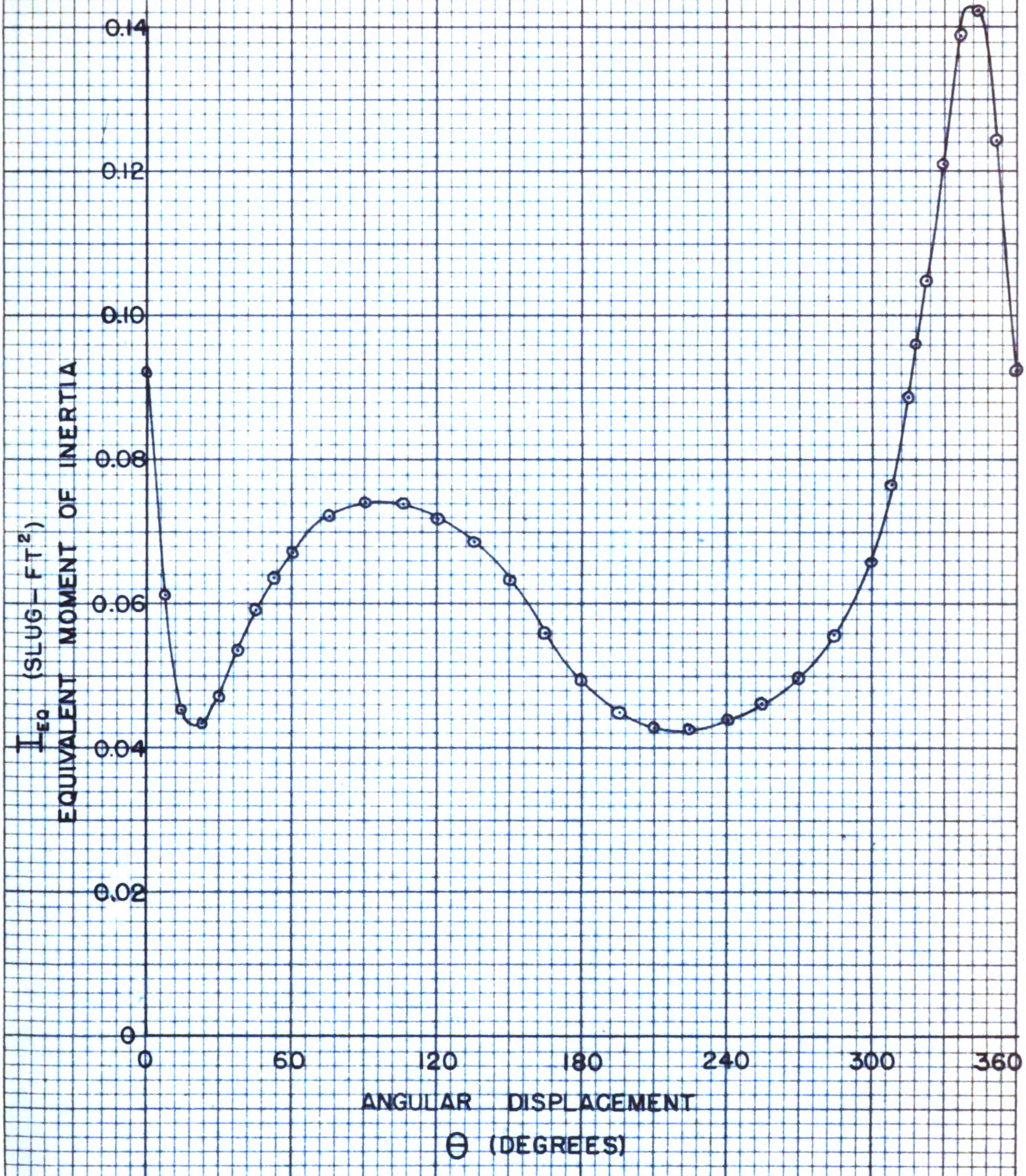


FIGURE 13. EQUIVALENT MOMENT OF INERTIA CURVE FOR FOUR-BAR MECHANISM.

$$T_i = 10 \text{ ft-lbs} \quad \ldots\ldots\ldots \quad \int_0^{\theta} T_i \, d\theta = 10\,\theta$$

$$T_o = 0$$

$$\omega_a = 0$$

Therefore,

$$\omega = \sqrt{\frac{20\,\theta}{I_{eq}}}$$

Table II contains the results of computations based on the above equation. These data are plotted in Figure 14 to graphically represent the angular velocity of link 2 versus the angular displacement.

(3.) The angular acceleration curve is constructed using Equation 9a. This equation is simplified by the conditions of the problem to obtain:

$$\alpha = \frac{2\,T_i - \omega^2\left(\frac{dI_{eq}}{d\theta}\right)}{2\,I_{eq}}$$

The construction of the angular acceleration curve, using the above equation, requires three intermediate steps. First, the value of the angular velocity must be read from the angular velocity curve for the number of positions required to adequately represent a complete motion cycle. Secondly, the equivalent moment of inertia must be tabulated for the same positions. Thirdly, the slope of the equivalent moment of inertia curve must be determined at each of these positions. Graphical methods for the determination of the slope are sufficiently accurate for use in most problems. The values of the slope in Table III were determined for the required positions. Table III also contains the information necessary to compute the angular acceleration. The angular

TABLE II

ANGULAR VELOCITY DATA FOR FOUR-BAR MECHANISM

θ degrees	θ radians	$20\ \theta$ ft-lb-rad	I_{eq} slug-ft^2	ω^2 (rad/sec)^2	ω rad/sec
0	0	0	0.0923	0	0
15	0.262	5.24	0.0449	116.8	10.8
30	0.524	10.48	0.0474	221.0	14.9
60	1.045	20.90	0.0672	311.0	17.6
90	1.571	31.40	0.0740	425.0	20.6
120	2.095	41.90	0.0720	584.0	24.2
150	2.620	52.40	0.0631	830.0	28.8
180	3.142	62.84	0.0494	1273.0	35.6
210	3.670	73.40	0.0423	1718.0	41.5
240	4.190	83.80	0.0437	1918.0	43.7
270	4.710	94.20	0.0495	1910.0	43.6
300	5.240	104.80	0.0674	1554.0	39.4
315	5.500	110.00	0.0888	1240.0	35.2
330	5.760	115.20	0.1214	950.0	30.8
345	6.020	120.40	0.1424	845.0	29.0
360	6.283	125.60	0.0923	1360.0	36.9
390	6.800	136.00	0.0474	2875.0	53.6

TABLE III

ANGULAR ACCELERATION DATA FOR FOUR-BAR MECHANISM

θ degrees	ω^2 (rad/sec)^2	$\frac{dI}{d\theta}$ $\frac{\text{slug-ft}^2}{\text{radians}}$	I_{eq} slug-ft^2	α rad/sec^2
0	0	-0.242	0.0923	108
15	117	-0.053	0.0449	290
30	221	0.043	0.0474	100
60	311	0.024	0.0672	94
90	425	0.000	0.0740	135
120	584	-0.010	0.0720	179
150	830	-0.028	0.0631	342
180	1273	-0.024	0.0494	514
210	1718	-0.005	0.0428	334
240	1913	0.005	0.0437	120
270	1910	0.020	0.0495	-134
300	1554	0.060	0.0674	-545
315	1240	0.114	0.0888	-685
330	950	0.134	0.1214	-440
345	845	-0.019	0.1424	127
360	1360	-0.242	0.0923	1890

curve is plotted in Figure 15. The angular acceleration is equal to zero at two different values of the angular displacement. They are 250 degrees and 345 degrees. Referring to the angular velocity curve, Figure 14, these angular displacement values correspond to points on the curve where the slope is equal to zero, which is expected.

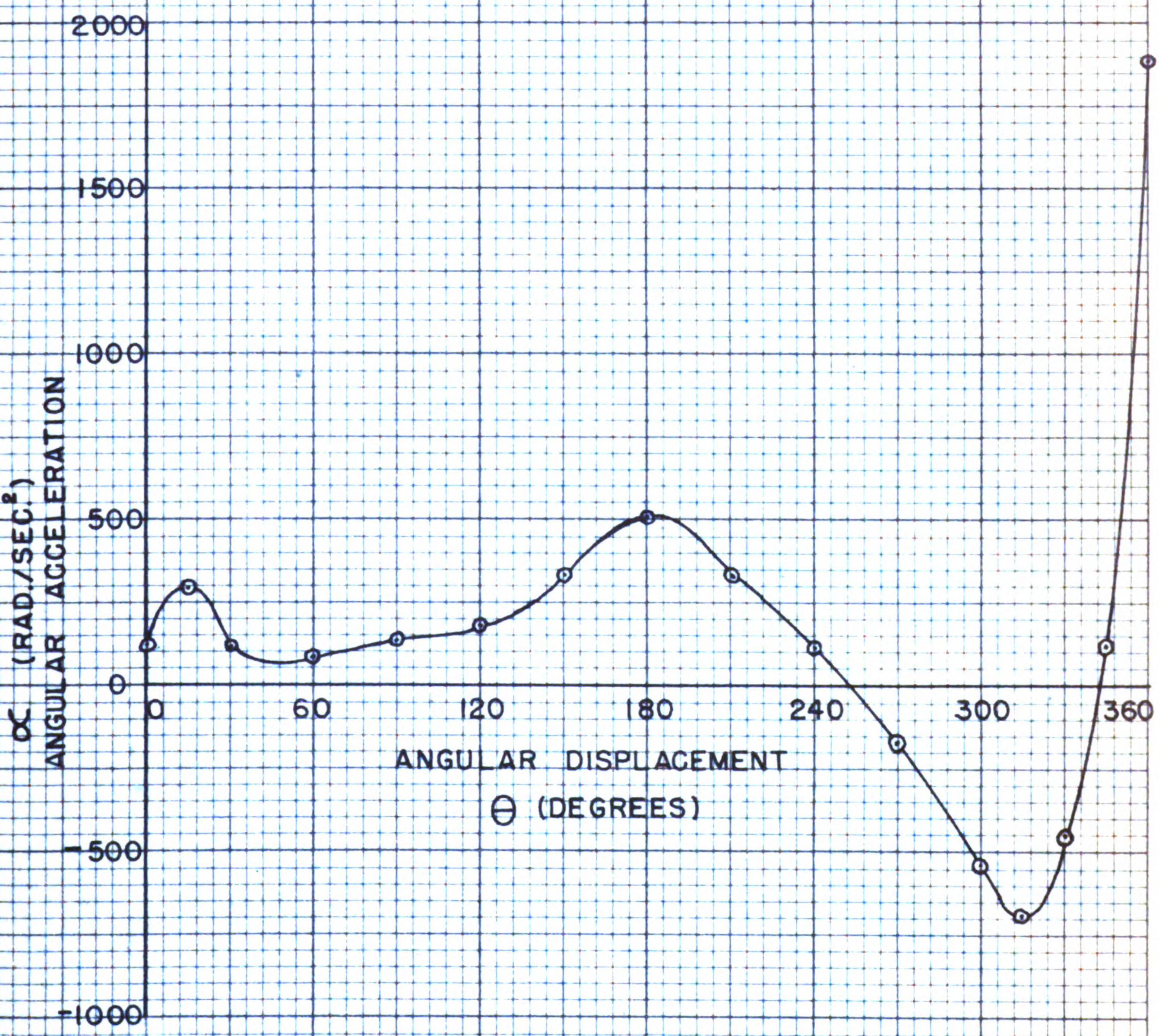


FIGURE 15. ANGULAR ACCELERATION CURVE FOR FOUR-BAR MECHANISM.

Four-Bar Mechanism Used as a Switch

The four-bar mechanism shown in Figure 16 has the same physical properties as the mechanism in the preceding problem. The mechanism is considered to be an opening electrical switch. The mechanism starts from rest at a position where θ = 22 1/2°. It is subjected to an input torque that is represented graphically in Figure 17 and an output torque that is represented in Figure 18. The equivalent moment of inertia curve is the same curve as constructed for the preceding problem and is shown in Figure 13.

The angular velocity curve is constructed for the displacement interval represented by θ = 22 1/2° to θ = 180°. Equation 8a is used to compute the angular velocity of link 2 at a series of angular displacements. Table IV contains the computed angular velocities. It also contains the data obtained from the equivalent moment of inertia curve and that data obtained from the two torque curves. These data are plotted in Figure 19.

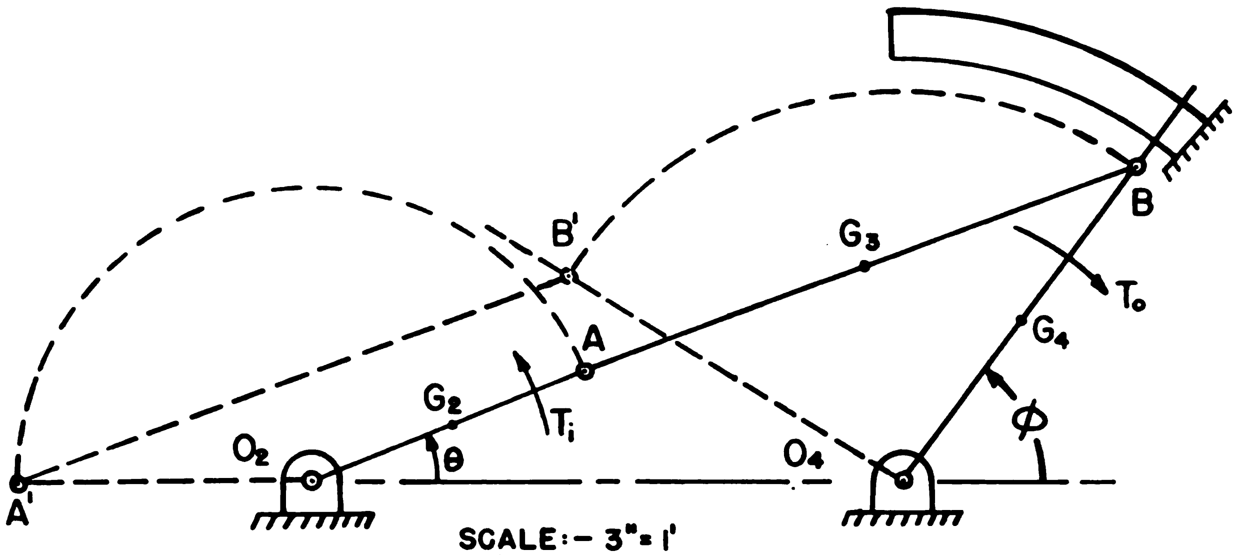


FIGURE 16. FOUR-BAR MECHANISM USED AS A SWITCH.

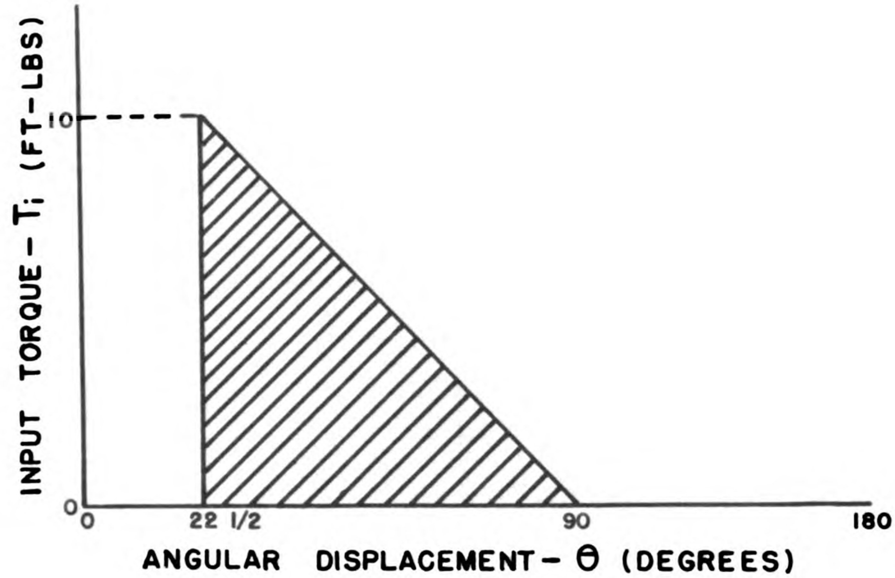


FIGURE 17. INPUT TORQUE CURVE.

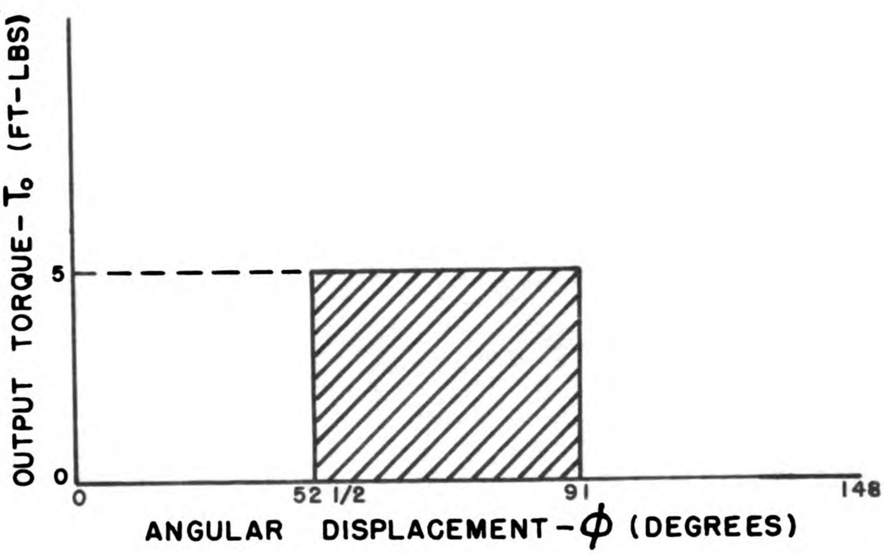


FIGURE 18. OUTPUT TORQUE CURVE.

TABLE IV

ANGULAR VELOCITY DATA FOR FOUR-BAR MECHANISM USED AS A SWITCH

θ degrees	$\int T_i \, d\theta$ ft-lbs	$\int T_o \, d\phi$ ft-lbs	I_{eq} slug-ft^2	ω^2 (rad/sec)^2	ω rad/sec
0	0	0	--	--	--
22 1/2	0	0	0.0429	0	0
30	1.24	0.175	0.0474	45.0	6.70
37 1/2	2.33	0.393	0.0538	71.8	8.46
45	3.26	0.707	0.0590	86.7	9.30
52 1/2	4.07	1.090	0.0632	94.5	9.70
60	4.72	1.484	0.0672	96.4	9.80
67 1/2	5.25	1.965	0.0700	94.0	9.69
75	5.62	2.400	0.0722	89.3	9.45
82 1/2	5.83	2.880	0.0738	80.0	8.95
90	5.90	3.360	0.0740	68.3	8.39
105	5.90	3.360	0.0740	68.8	8.39
120	5.90	3.360	0.0720	70.5	8.40
135	5.90	3.360	0.0687	74.0	8.60
150	5.90	3.360	0.0631	80.5	8.95
165	5.90	3.360	0.0559	91.0	9.55
180	5.90	3.360	0.0494	103.0	10.10

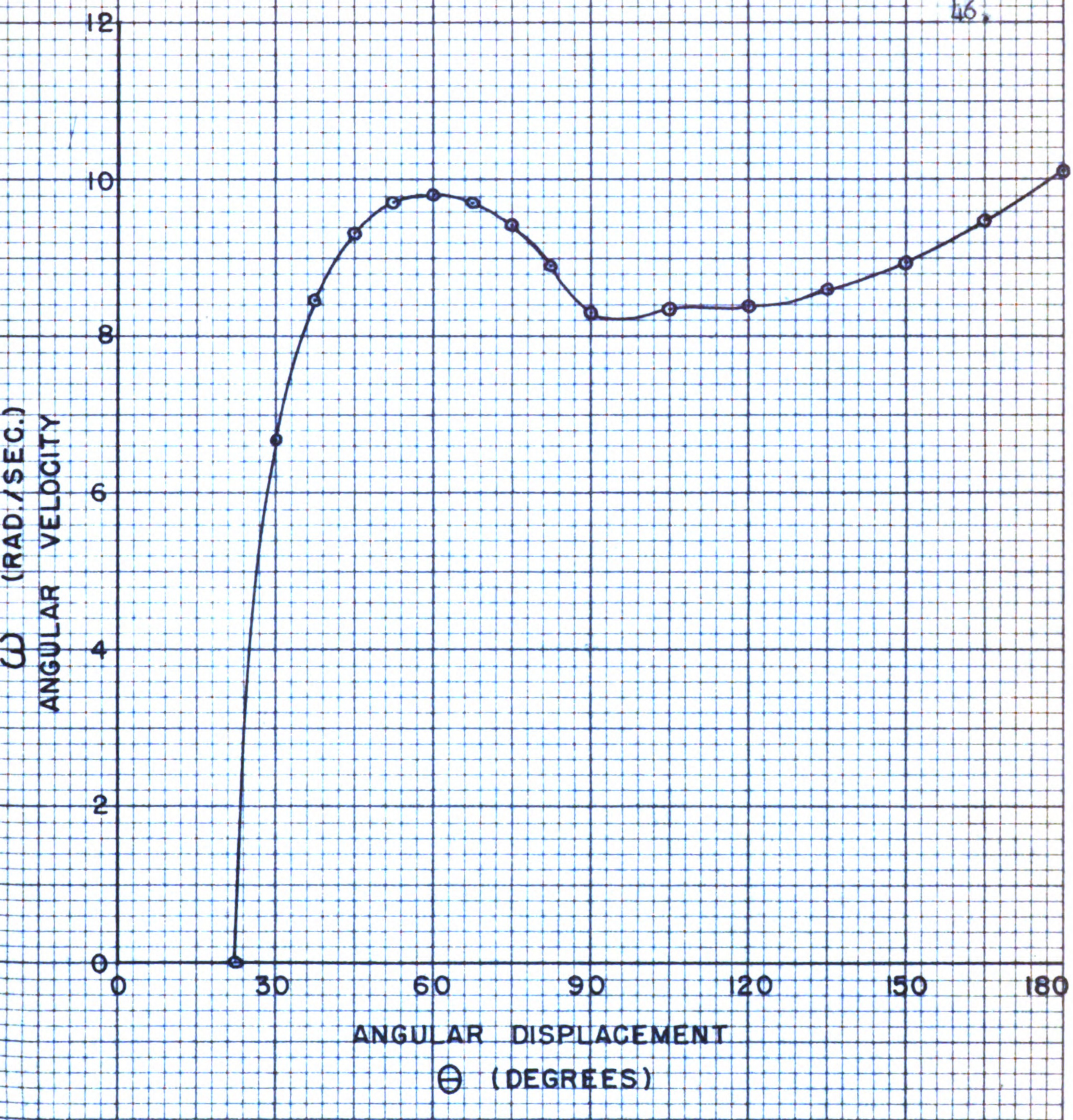


FIGURE 19. ANGULAR VELOCITY CURVE FOR FOUR-BAR MECHANISM USED AS A SWITCH.

Slider Crank Mechanism

The slider crank mechanism shown in Figure 20 has the following physical properties:

Weight of link 2	3.22 lbs
Weight of link 3	3.22 lbs
Weight of the piston	3.22 lbs
Moment of inertia of link 2 about its center of gravity	0.0234 slug-ft^2
Moment of inertia of link 3 about its center of gravity	0.0200 slug-ft^2

The mechanism starts from rest at a position corresponding to $\theta = 0^{o}$. The piston is subjected to an input force which decreases linearly from the head-end dead center position. The force-displacement relation is expressed analytically as:

$$F_i = 100 - 200\ s$$

where (s) is the linear displacement, in feet, of the piston. It is measured positively from the head-end dead center position. The input force is equal to zero at the crank-end dead center position. The mechanism completes the remaining one-half cycle without energy being supplied or removed.

The angular velocity curve for link 2 is constructed in the following manner:

(1.) Equation 4 is used to construct the equivalent moment of inertia curve. For this mechanism, the equation is rewritten in the following form:

$$I_{eq} = [I_2 + I_{g3}\left(\frac{\omega_3}{\omega_2}\right)^2 + M_3\left(\frac{V_{g3}}{\omega_2}\right)^2 + M_4\left(\frac{V_p}{\omega_2}\right)^2]$$

Table V contains the data obtained from a velocity analysis and also the computed data necessary to construct the equivalent moment of

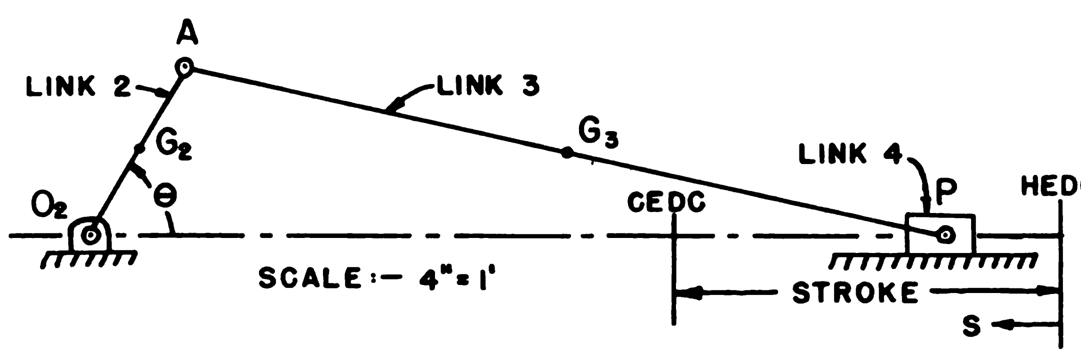


FIGURE 20. SLIDER CRANK MECHANISM.

TABLE V

EQUIVALENT MOMENT OF INERTIA DATA FOR SLIDER CRANK MECHANISM

θ degrees	$\frac{V_p}{\omega_2}$	$\frac{V_{g3}}{\omega_2}$	$\frac{\omega_3}{\omega_2}$	I_{eq}* slug-ft^2
0	0	0.125	0.250	0.0328
15	0.080	0.141	0.242	0.0338
30	0.150	0.175	0.217	0.0362
45	0.208	0.213	0.179	0.0392
60	0.246	0.238	0.127	0.0418
75	0.258	0.252	0.067	0.0431
90	0.250	0.250	0	0.0425
105	0.225	0.238	0.067	0.0408
120	0.190	0.213	0.127	0.0383
135	0.144	0.184	0.179	0.0357
150	0.098	0.155	0.217	0.0343
165	0.050	0.134	0.242	0.0332
180	0	0.125	0.250	0.0328
195	0.050	0.134	0.242	0.0332
210	0.098	0.155	0.217	0.0343
225	0.144	0.184	0.179	0.0357
240	0.190	0.213	0.127	0.0383
255	0.225	0.238	0.067	0.0408
270	0.250	0.250	0	0.0425
285	0.258	0.252	0.067	0.0431
300	0.246	0.238	0.127	0.0418
315	0.208	0.213	0.179	0.0392
330	0.150	0.175	0.217	0.0362
345	0.080	0.141	0.242	0.0338
360	0	0.125	0.250	0.0328

$$^*I_{eq} = [.030 + .020\left(\frac{\omega_3}{\omega_2}\right)^2 + .1\left(\frac{V_{g3}}{\omega_2}\right)^2 + .1\left(\frac{V_p}{\omega_2}\right)^2]$$

inertia curve. These data are plotted in Figure 21. For slider crank mechanisms, the equivalent moment of inertia curve is very nearly sinusoidal in nature and the portion of the curve from 180 degrees to 360 degrees is a mirror image of the portion from 0 degrees to 180 degrees.

(2.) For this mechanism, Equation 8a is rewritten and simplified to obtain:

$$\omega^2 = \frac{2\int_0^s F\,ds}{I_{eq}}$$

where,

$$\int_0^s F\,ds = E_i$$

and,

$$E_i = 100\,s\,(1 - s)$$

The above equation is used to compile the data in Table VI. These data are plotted in Figure 22 to represent the angular velocity of link 2 for one motion cycle. In this analysis, it is assumed that the input force on the piston can cause turning of the crank when the mechanism is at the head-end dead center phase.

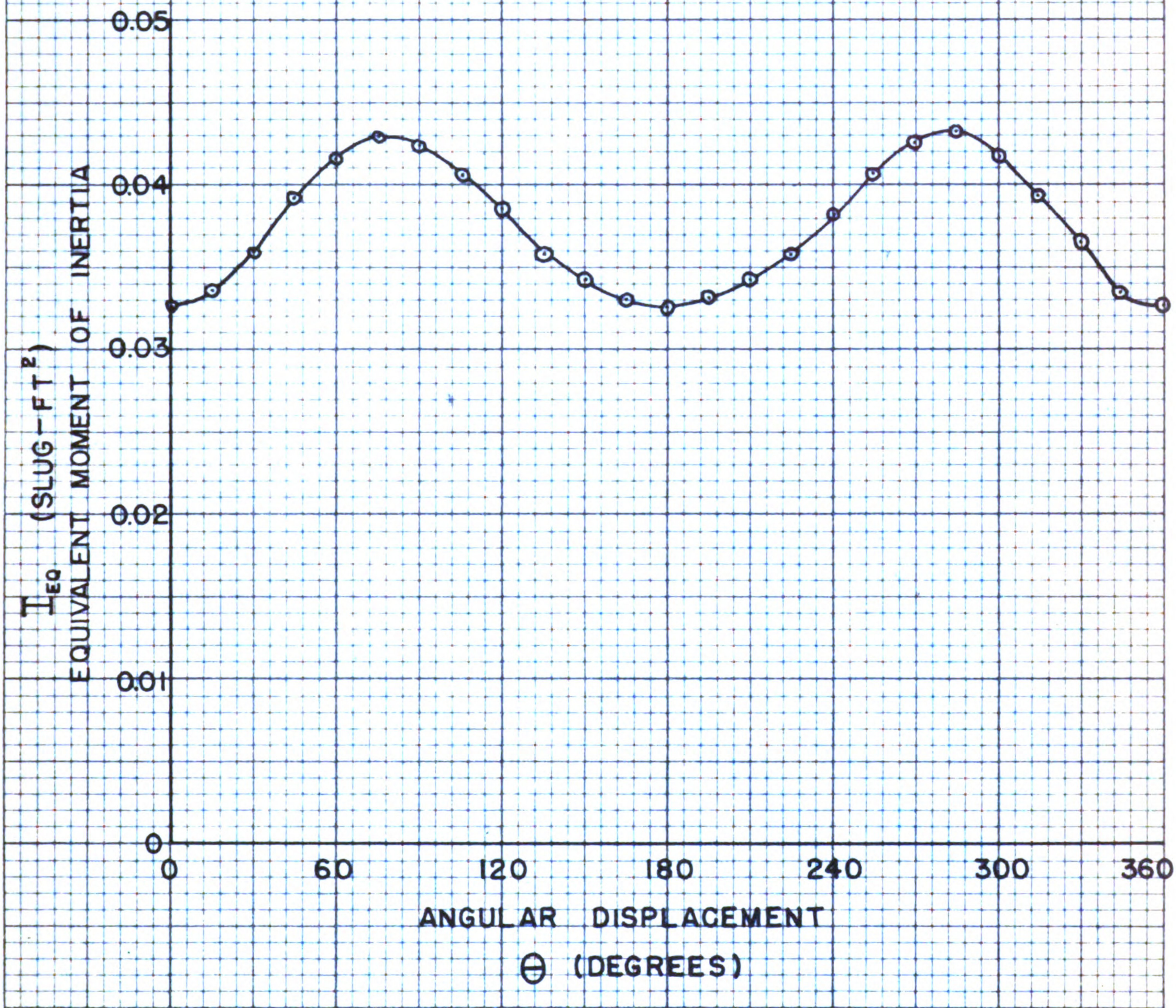


FIGURE 21. EQUIVALENT MOMENT OF INERTIA CURVE FOR SLIDER CRANK MECHANISM.

TABLE VI

ANGULAR VELOCITY DATA FOR SLIDER CRANK MECHANISM

θ degrees	E_i ft lbs	I_{eq} slug-ft^2	ω^2 (rad/sec)^2	ω rad/sec
0	0	0.0328	0	0
15	0.99	0.0338	59	7.7
30	3.92	0.0362	217	14.7
45	8.06	0.0392	412	20.3
60	12.65	0.0413	606	24.6
75	16.70	0.0431	775	27.8
90	20.18	0.0425	952	30.9
105	22.50	0.0408	1100	33.2
120	23.90	0.0383	1250	35.3
135	24.60	0.0357	1380	37.2
150	24.89	0.0343	1450	38.1
165	24.91	0.0332	1500	38.7
180	25.00	0.0328	1525	39.0
210	25.00	0.0343	1460	38.2
240	25.00	0.0383	1305	36.1
270	25.00	0.0425	1175	34.2
300	25.00	0.0418	1195	34.6
330	25.00	0.0362	1380	37.2
360	25.00	0.0328	1525	39.0

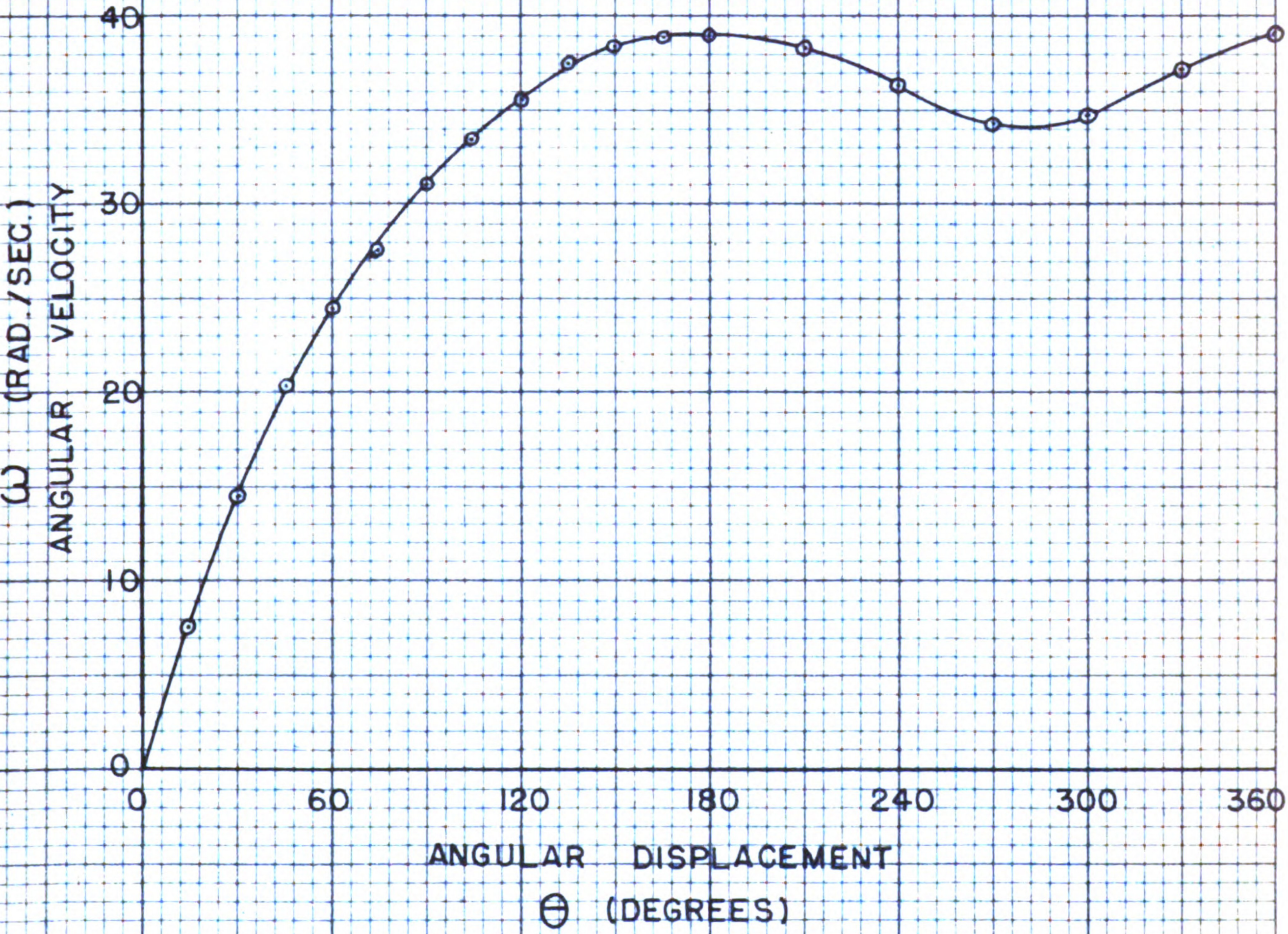


FIGURE 22. ANGULAR VELOCITY CURVE FOR SLIDER CRANK MECHANISM.

SUMMARY

The equations and methods derived from the energy equations are general. They can be used for determining the dynamic characteristics of any mechanism. This is evident from the fact that either of the two general types of problems can be solved using these equations. The velocity polygon will supply the information necessary to perform a complete analysis of a mechanism. It will not be necessary to determine accelerations which may be difficult in the case of complex mechanisms.

The proposed methods consist of the following steps which lead to the complete analysis of a mechanism subjected to known forces.

A. Construction of an equivalent moment of inertia or equivalent mass curve.

B. Construction of an angular velocity curve.

C. Construction of an angular acceleration curve.

D. Construction of a time curve.

Step A requires a preliminary operation involving a velocity-vector polygon. The remaining steps are performed using the equations and methods proposed.

The energy method solutions indicate two methods of representing a mechanism for analysis. They are the equivalent moment of inertia and the equivalent mass systems. The equivalent moment of inertia is the more general of the two systems. However, both systems are applicable for the solution of the two types of general problems.

The energy method as applied to mechanisms has certain disadvantages which limit the accuracy of the analysis. The effect of friction is cumulative, and the error may increase during the motion cycle. The determination of the velocity ratios from the velocity polygon and the determination of the slope of the equivalent moment of inertia or angular velocity curves will introduce errors. Such errors, however, are difficult to determine analytically. Some of these errors can be minimized by careful application of the graphical methods required.

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