STUDY OF A CLASS OF LANDAU-LIFSHITZ EQUATIONS OF FERROMAGNETISM WITHOUT EXCHANGE ENERGY

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ABSTRACT

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Landau-Lifshitz equations of ferromagnetism, which are based on several competing energy contributions, are important mathematical models for the evolution of magnetization field $\mathbf{m}$ of a ferromagnetic material. Many problems, such as existence, stability, regularity, asymptotic behavior, thin-film limit and numerical computation, have been well studied for the Landau-Lifshitz equations that include the so-called exchange energy. However, these problems turn out to be quite challenging for equations without the exchange energy. The main reason is that when the exchange energy is included, one automatically has the magnetization vector $\mathbf{m} \in L^\infty((0,\infty); H^1(\Omega))$ from energy estimates, which gives some compactness and stability that are needed for using the standard methods; however, in the cases without the exchange energy, one only has $\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega))$, which is too rough to get the needed compactness and stability. In this thesis, we investigate some problems for models of reduced Landau-Lifshitz equations with no-exchange energy.

In Chapter 1, we introduce the Landau-Lifshitz theory of ferromagnetism and summarize the main results of the thesis. The readers can check out the main results quickly in this chapter and then go to the corresponding chapters for details of proof, more discussions and further references.

In Chapter 2, we study the *quasi-stationary limit* of a simple Landau-Lifshitz-Maxwell system with the permittivity parameter ϵ approaching zero and, using this quasi-stationary limit, establish the existence of global weak solutions to the reduced Landau-Lifshitz equa-

tions with initial value $\mathbf{m}_0 \in L^\infty(\Omega)$.

In Chapter 3, we establish a local L^2-stability theorem for the global weak solutions in finite time. The key in the proof of stability theorem is that we split the nonlocal term $H_\mathbf{m}$ into two parts: one is bounded in $L^\infty(\Omega)$ and the other bounded in $L^2(\Omega)$. Using this stability theorem, we also provide another proof for the existence of global weak solutions for a full expression of the no-exchange energy with applied field $\mathbf{a}(x) \in L^\infty(\Omega)$.

In Chapter 4, we prove a higher time regularity for the regular solutions, using mainly induction method, together with several interpolation results. In this chapter, we also study the weak ω-limit sets for the so-called *soft-case* and study the asymptotic behaviors for the special case when Ω is ellipsoid and initial values $\mathbf{m}_0$ are constant.

In Chapter 5, we investigate a different model called the fractional Landau-Lifshitz equations and establish the existence of global weak solutions with initial value $\mathbf{m}_0 \in H^\alpha(\Omega)$, where $0 < \alpha < 1$. In this new model, in contrast to the case when only the nonlocal term $H_\mathbf{m}$ is included, we have some compactness in $H^\alpha(\Omega)$, which enables us to apply the Galerkin method to establish the existence of global weak solution.

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Chapter 1

Introduction and Main Results

The well-known Landau-Lifshitz theory in ferromagnetism models the state of magnetization vector $\mathbf{m}$ of a ferromagnetic material occupying a domain Ω in $\mathbf{R}^3$ based on a formulation of the total energy $\mathcal{E}(\mathbf{m})$ consisting of several competing terms:

$$\mathcal{E}(\mathbf{m}) = \frac{\kappa}{2}\int_\Omega |\nabla \mathbf{m}|^2\,dx + \int_\Omega \varphi(\mathbf{m})\,dx - \int_\Omega \mathbf{a}(x)\cdot \mathbf{m}\,dx + \frac{1}{2}\int_{\mathbf{R}^3} |H_{\mathbf{m}}|^2\,dx. \tag{1.1}$$

We refer to [4, 34, 37, 38] for more backgrounds on such a model. The first term of $\mathcal{E}(\mathbf{m})$ is the *exchange energy*, penalizing the spacial change of $\mathbf{m}$; this term could also be given in terms of a positive-definite quadratic form of $\nabla\mathbf{m}$. The second term is the *anisotropy energy* due to crystallographic properties of the material. The third term is the *interaction energy* due to a given applied magnetic field $\mathbf{a}(x)$. The last term is the *magnetostatic energy* of the stray field $H_{\mathbf{m}}$ induced by $\mathbf{m}$ through a simplified Maxwell equations:

$$\operatorname{curl} H_{\mathbf{m}} = 0, \quad \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \quad \text{in } \mathbf{R}^3, \tag{1.2}$$

where χ_Ω is the characteristic function of domain Ω. In this theory, the (rescaled) *saturation condition*: $|\mathbf{m}| = 1$ is usually assumed over Ω.

The dynamic Landau-Lifshitz equation governing the evolution of magnetization $\mathbf{m} =$

$\mathbf{m}(x,t)$ is given by

$$\partial_t \mathbf{m} = \gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} + \alpha \frac{\gamma}{|\mathbf{m}|} \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) \tag{1.3}$$

on $\Omega \times [0, \infty)$, where $\gamma < 0$ is the electron gyromagnetic ratio, $\alpha \geq 0$ is the Landau-Lifshitz phenomenological damping parameter, and $\mathbf{H}_{\text{eff}}$ is the total *effective magnetic field* defined by the functional derivative of $\mathcal{E}(\mathbf{m})$ as

$$\mathbf{H}_{\text{eff}} = -\frac{\partial \mathcal{E}}{\partial \mathbf{m}} = \kappa \Delta \mathbf{m} - \varphi'(\mathbf{m}) + \mathbf{a}(x) + H_{\mathbf{m}}. \tag{1.4}$$

Note that the Landau-Lifshitz equation (1.3) can be written as a Landau-Lifshitz-Gilbert equation:

$$\partial_t \mathbf{m} = \gamma(1+\alpha^2) \mathbf{m} \times \mathbf{H}_{\text{eff}} + \frac{\alpha}{|\mathbf{m}|} \mathbf{m} \times \partial_t \mathbf{m}. \tag{1.5}$$

Most existing studies, including existence, stability, asymptotic behavior and regularity, on (1.3) or (1.5) are for models with the exchange energy, that is, when $\mathbf{H}_{\text{eff}} = \kappa \Delta \mathbf{m} - \varphi'(\mathbf{m}) + \mathbf{a}(x) + H_{\mathbf{m}}$ with $\kappa > 0$; however, few results have been established for the case when the exchange energy is excluded, because of the lack of compactness and stability in such a case.

We consider the reduced Landau-Lifshitz equation (1.3) without exchange energy and study properties of the corresponding solutions. In following sections, we briefly introduce some background and our results on existence, stability, asymptotic behavior and regularity problems for reduced Landau-Lifshitz equations.

1.1 Existence

There are many results for existence of solutions to (1.3) or (1.5) with the exchange energy, see [1, 3, 7, 8, 10, 17, 22, 27, 49]. In these cases, the initial data $\mathbf{m}_0$ usually need to be smooth enough (e.g. in $H^1(\Omega)$ or $H^2(\Omega)$) in order to use the Galerkin method and elliptic estimates to establish the existence of solutions. Later, we will see that the most important key in the proof of existence is that we need strong convergence for some certain sequence of $\mathbf{m}$. When we include exchange energy, the strong convergence condition is much easier to obtain than the case without exchange energy. Specifically, with exchange energy, most of previous research work can easily get following bound for $\mathbf{m}$ by applying energy estimation.

$$\mathbf{m} \in L^1((0,\infty); H^1(\Omega)) \tag{1.6}$$

The bound (1.6) is also very important to many other problems, such as asymptotic behavior of solutions. Actually, most of results about Landau-Lifshitz equation with exchange energy based on (1.6). Later we will see that in our case (without exchange energy), from energy estimation, the best bound we can get for $\mathbf{m}$ is,

$$\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega))$$

That is why our problem is more challenging.

Existence of weak solution for rough initial data

As introduced above, we will prove existence of solutions to the dynamic Landau-Lifshitz equation (1.3) for a total energy $\mathcal{E}(\mathbf{m})$ without the exchange energy or simply called reduced dynamic Landau-Lifshitz equation (1.3). Therefore, we have

$$\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a}(x) + H_{\mathbf{m}}$$

This Cauchy problem can be written as a *quasi-stationary* system:

$$\begin{cases} \partial_t \mathbf{m} = F(x, \mathbf{m}, H_{\mathbf{m}}) \quad \text{ in } \Omega \times (0, \infty), \\ \operatorname{curl} H_{\mathbf{m}} = 0, \quad \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0 \;\text{ in } \mathbf{R}^3, \\ \mathbf{m}(x, 0) = \mathbf{m}_0(x) \quad \text{on } \Omega, \end{cases} \tag{1.7}$$

with a given initial datum $\mathbf{m}_0 \in L^\infty(\Omega)$, where $F : \Omega \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is an appropriate function (the *Landau-Lifshitz interaction* function) specifically given in Chapter 2.

In order to prove the existence of global solution, we wish to have tight bound like (1.6) for solutions or approximate solutions of system (1.7) and then easily extract strong convergent subsequence in L^2, which would handle the nonlinear term $F(x, \mathbf{m}, H_{\mathbf{m}})$.

For smooth initial data $\mathbf{m}_0 \in H^2(\Omega)$ with $\frac{\partial \mathbf{m}_0}{\partial \nu}|_{\partial\Omega} = 0$, previous research work of Carbou and Fabrie [8] has established the existence to the similar equation without exchange energy using the singular perturbation method: include $\kappa\Delta\mathbf{m}$ in $\mathbf{H}_{\text{eff}}$ and let $\kappa \to 0$. But unfortunately, for initial data only in $L^\infty(\Omega)$, their method does not directly work. We need to find other ways to obtain the compactness for approximate solutions to (1.7) with rough initial data only in L^∞.

We study the system (1.7) as a *quasi-stationary limit* of the following Landau-Lifshitz-Maxwell system of electro-magnetism when the permittivity parameter ϵ is constant and approaches zero; the more general case of Maxwell equations with variable permittivity has been studied in Jochmann [29]. However, the system (1.8) studied below is much simpler and the method used is quite different, but more direct.

$$\begin{cases} \epsilon\partial_t E - \operatorname{curl} H = 0, \\ \partial_t(H + M\chi_\Omega) + \operatorname{curl} E = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \\ \partial_t M = F(x, M, H) \quad \text{in } \Omega \times (0,\infty), \\ (E,H)|_{t=0} = (E_0, H_0) \;\text{ on } \mathbf{R}^3, \quad M|_{t=0} = \mathbf{m}_0 \;\text{ on } \Omega, \end{cases} \tag{1.8}$$

where the initial data E_0, H_0 for electric and magnetic fields E, H are any vector-fields satisfying

$$E_0, H_0 \in L^2(\mathbf{R}^3; \mathbf{R}^3), \quad \operatorname{div} E_0 = \operatorname{div}(H_0 + \mathbf{m}_0\chi_\Omega) = 0. \tag{1.9}$$

For $\epsilon > 0$, the existence of certain weak solutions to the Cauchy problem (1.8) can be obtained in the same way as studied in Joly, Metivier and Rauch [31] with $\epsilon = 1$. Therefore, for any $\epsilon > 0$, there exists a global weak solution M^ϵ. We have following theorem for the convergence of sequence M^ϵ as $\epsilon \to 0$.

Theorem 1.1.1 (Chapter 2, section 2.1). *We have $M^\epsilon \to \mathbf{m}$ strongly in both $C^0([0,T]; L^2(\Omega))$ and $L^2(\Omega_T)$ for all $0 < T < \infty$.*

The strong convergence result in Theorem 1.1.1 is exactly what we need to handle the nonlinear term in the Cauchy problem (1.7). We use quite different methods to get such compactness comparing to the case with exchange energy. From details in Chapter 2, Section 2.1 for proof of Theorem 1.1.1, one can notice that we never have the same bound as (1.6). To the best of our knowledge, in the reduced Landau-Lifshitz model, one should not be able to get (1.6).

With Theorem 1.1.1, we establish the existence of global weak solution to the Cauchy problem (1.7).

Theorem 1.1.2 (Chapter 2, section 2.2)**.** *Let* $\mathbf{m}_0 \in L^\infty(\Omega)$ *and let* E_0, H_0 *be any functions satisfying* (1.9)*. Then any function* $\mathbf{m}$ *determined by the convergence in Theorem 1.1.1 is a weak solution to the Cauchy problem* (1.7)*.*

1.2 Stability

Definition 1.2.1. Given $\mathbf{m}_0 \in L^\infty(\Omega)$, a *global weak solution* to the Cauchy problem (1.7) is a function

$$\mathbf{m} \in W^{1,\infty}_{loc}([0,\infty); L^2(\Omega; \mathbf{R}^3)) \cap L^\infty((0,\infty); L^\infty(\Omega; \mathbf{R}^3)) \tag{1.10}$$

such that $\mathbf{m}(0) = \mathbf{m}_0$ in $L^2(\Omega)$ with $\mathbf{m} = \mathbf{m}(\cdot, t)$, equation

$$\partial_t \mathbf{m} = F(x, \mathbf{m}, H_{\mathbf{m}}), \tag{1.11}$$

holds both in $L^\infty((0,\infty); L^2(\Omega))$ and in the sense of distribution on $\Omega \times (0,T)$, for all $0 < T < \infty$.

In this section we will investigate the stability of global weak solutions to *quasi-stationary* system (1.7) including function $\mathbf{a}(x)$; for a similar result on the Maxwell system, see [31, Theorem 6.1] and [13, Theorem 5.1].

Theorem 1.2.1 (Chapter 3, Section 3.1)**.** *Let* $0 < R, T < \infty$ *be given. Then there exist constants* $C = C(R,T) > 0$, $c = c(R,T) > 0$ *and* $\rho = \rho(R,T) > 0$ *such that, for any weak solution* $\mathbf{m}^k$ *to the system* (1.7) *with applied field* $\mathbf{a}^k$ *and initial datum* $\mathbf{m}^k(0) = \mathbf{m}^k_0$

satisfying $\|\mathbf{a}^k\|_{L^\infty}+\|\mathbf{m}_0^k\|_{L^\infty} \le R$ *for* $k=1,2$*, if* $\mu = \max\{\|\mathbf{m}_0^1-\mathbf{m}_0^2\|_{L^2}, \|\mathbf{a}^1-\mathbf{a}^2\|_{L^2}\} \le c$*, then one has, for all* $t \in [0,T]$*,*

$$\|\mathbf{m}^1(t) - \mathbf{m}^2(t)\|_{L^2(\Omega)} \le C\mu^\rho. \tag{1.12}$$

This stability result also implies the uniqueness of weak solution to system (1.7).

1.3 Existence of global solutions

Based on the previous stability theorem, we present a new method for the existence of global weak solution to (1.7) with general applied fields $\mathbf{a}$ and initial data $\mathbf{m}_0$.

First, we show the existence of global solution to (1.7) for smooth fields $\mathbf{a}$ and initial data $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. Define $\mathbf{f}(\mathbf{m}) = F_{\mathbf{a}}(x,\mathbf{m},H_{\mathbf{m}})$. We show $\mathbf{f}\colon H^2(\Omega;\mathbf{R}^3) \to H^2(\Omega;\mathbf{R}^3)$ and is *locally Lipschitz*; the proof uses a critical estimate that $H_{\mathbf{m}} \in H^2(\Omega;\mathbf{R}^3)$ for all $\mathbf{m} \in H^2(\Omega;\mathbf{R}^3)$ (see, e.g., [8, 31]). By the abstract ODE theory in Banach spaces, problem (1.7) has a local solution belongs to $H^2(\Omega;\mathbf{R}^3)$ if $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. Then a no-blowup result (Theorem 3.2.4) shows that the local solution is in fact global on $t \in [0,\infty)$.

We remark that in the special case when $\varphi = 0$ and $\mathbf{a} = 0$ (thus $\mathbf{H}_{\text{eff}} = H_{\mathbf{m}}$), for smooth initial data $\mathbf{m}_0 \in H^2(\Omega)$ with $\frac{\partial \mathbf{m}_0}{\partial \nu}|_{\partial\Omega} = 0$, Carbou and Fabrie [8] also established the global existence through a singular perturbation method, by including $\kappa\Delta\mathbf{m}$ in $\mathbf{H}_{\text{eff}}$ and letting $\kappa \to 0$.

Once we have obtained the global existence for smooth data $\mathbf{a}$ and $\mathbf{m}_0$, we use approximation and the stability result Theorem 1.2.1 to establish the existence for general data.

Theorem 1.3.1 (Chapter 3, section 3.2)**.** *Let* $\mathbf{a} \in L^\infty(\Omega;\mathbf{R}^3)$*. Given any initial datum*

$\mathbf{m}_0 \in L^\infty(\Omega;\mathbf{R}^3)$, *the problem* (1.7) *has an unique global weak solution.*

1.4 Time regularity and special asymptotics

Higher regularity in time

In this section, we introduce the higher time-regularity result of solution to LLG equation (1.13).

$$\begin{cases} \mathbf{m}_t = \gamma \mathbf{m} \times H_{\mathbf{m}} + \gamma\alpha \mathbf{m} \times (\mathbf{m} \times H_{\mathbf{m}}) \ \text{ in } \Omega \times (0,\infty), \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{1.13}$$

where Ω is a bounded smooth domain in $\mathbf{R}^3$ and $\mathbf{m}_0 \in H^2(\Omega;\mathbf{R}^3)$. The existence of such regular solutions has been proved in Section 3.2 by applying abstract ODE theory.

Similar regularity problem has been studied by Cimrak and Keer [6] for LLG with exchange energy. In their proof, they highly take advantage of energy estimation from exchange energy and then apply induction method. Now even without exchange energy, we also get similar time regularity. Our method is actually also inspired by Cimrak and Keer, using induction method; however, because we drop off exchange energy, we do not have any a priori estimates as being used in [6]. One can refer to Section 4.1 for details of proof of Theorem 1.4.1.

Theorem 1.4.1 (Chapter 4, Section 4.1)**.** *For any time $T > 0$ and initial unit vector $\mathbf{m}_0 \in H^2(\Omega)$, the regular solution, i.e. $\mathbf{m} \in H^1([0,T];H^2(\Omega))$, satisfies*

$$\|\partial_t^{p+1}\mathbf{m}\|_{H^2(\Omega)} \le C$$

where C is one constant only depending on $T, p, \|\mathbf{m}_0\|_{H^2(\Omega)}$.

Asymptotic behavior for constant initial data on ellipsoid domains

We now consider asymptotic behavior of regular solutions to Landau-Lifshitz equation (1.13). One of difficulties is that without exchange energy $\nabla \mathbf{m}$, we do not have much control for weak sequence with respect to time t. Thus, methods used in Carbou and Fabrie [7] can not be used in our case. Recently, research work about asymptotic behavior of global weak solutions to Landau-Lifshitz equation (1.13) has been done by Yan ([51],[52]). In [52], without exchange energy, Yan studied asymptotic behaviors of equation (1.13) in the weak-star convergence of $L^\infty(\Omega, \mathbf{R}^3)$. The equilibrium set of equation (1.13) is

$$\mathbf{m} \times H_{\mathbf{m}} = 0 \quad \text{ on } \Omega.$$

Yan [51] has investigated the equilibrium set under the general framework of partial differential inclusions and the vectorial calculus of variations based on the notation quasiconvexity in Ball [2], Dacorogna [11] and Morrey [40]. He has proved the following theorem.

Theorem 1.4.2 (Yan [51])**.** *Let $\mathbf{m}_j \rightharpoonup \mathbf{m}$ weak-star in $L^\infty(\Omega; \mathbf{R}^3)$ as $j \to \infty$. If*

$$\lim_{j\to\infty} \int_\Omega (|\mathbf{m}_j|^2 + 2|\mathbf{m}_j \times H_{\mathbf{m}_j}| - 1)^+ dx = 0$$

then the weak-star limit $\mathbf{m}$ satisfies $|\mathbf{m}|^2 + 2|\mathbf{m} \times H_{\mathbf{m}}| \le 1$ a.e. on Ω.

It is very obvious that all points in equilibrium set satisfies condition in Theorem 1.4.2, thus Yan [52] described the behavior of equilibrium set in some sense.

In Section 4.2, we devote to prove a similar result as Theorem 1.4.2 in a different way. We first derive an energy identity for the global weak solutions to the Landau-Lifshitz equation (1.3).

Theorem 1.4.3. *The global weak solution* $\mathbf{m}$ *to* (1.7) *with bounded initial data satisfies the energy identity*

$$\mathcal{E}(\mathbf{m}(t)) - \mathcal{E}(\mathbf{m}(s)) = \gamma\alpha \int_s^t \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx d\tau \quad \forall\; 0 \le s \le t < \infty. \tag{1.14}$$

Furthermore, if $\gamma\alpha < 0$, *then* $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$.

Therefore, the global-in-time regularity for weak solutions (even for regular solutions) is that

$$\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega; \mathbf{R}^3)) \quad \text{with} \quad \mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3)).$$

But this regularity is not enough to have strong convergence as $t \to \infty$; it would be enough if one has $\mathbf{m}_t \in L^1((0,\infty); L^2(\Omega; \mathbf{R}^3))$ (see [32]). Therefore, it is quite challenging to study the asymptotic behaviors for even the regular solutions. The solution orbits for general initial data may not have strong ω-limit points; we thus study the *weak ω-limit set*:

$$\omega^*(\mathbf{m}_0) = \{\tilde{\mathbf{m}} \mid \exists\; t_j \uparrow \infty \text{ such that } \mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}} \text{ weakly in } L^2(\Omega; \mathbf{R}^3)\}. \tag{1.15}$$

We give an estimate of $\omega^*(\mathbf{m}_0)$ for the so-called *soft-case*, where there is no anisotropy energy ($\varphi = 0$).

Theorem 1.4.4. *Let* $\gamma\alpha < 0$, $\varphi = 0$ *and* $\mathbf{a} \in L^\infty(\Omega; \mathbf{R}^3)$. *Then, for any* $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$ *with* $|\mathbf{m}_0(x)| = 1$ *a.e. on* Ω, *it follows that*

$$\omega^*(\mathbf{m}_0) \subseteq \{\tilde{\mathbf{m}} \in L^\infty(\Omega; \mathbf{R}^3) \mid |\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \; a.e. \; on \; \Omega\}. \tag{1.16}$$

This theorem generalizes some results in [52]. For more results on further special case when $\mathbf{a} = 0$, see [51, 52].

Instead of considering the asymptotic behavior of global weak solutions, in this section, we only try to determine asymptotic behavior of solutions to Landau-Lifshitz equation (1.13) under a special case when initial value $\mathbf{m}_0$ is constant over Ω and Ω set is ellipsoid. It mainly involves two step.

The first step is that we need to write explicit form for nonlocal term $H_{\mathbf{m}}$ under our special geometry with ellipsoid domains. After some work has been done in Chapter 4, Section 4.2.1, we can write

$$H_{\mathbf{m}} = -\Lambda \mathbf{m} \quad \text{on } \Omega,$$

where Λ is a positive definite matrix; the details of how we determine matrix Λ can be found in Chapter 4, Section 4.2.1.

Then, we will use Lyapunov Theorem to determine global stable equilibrium point as follows,

Theorem 1.4.5 (Chapter 4, Section 4.2)**.** *Let us assume that $\mathbf{m}_0$ is constant and Ω is ellipsoid,*

$$\Omega = \{x \in \mathbf{R}^3 | \sum_{i=1}^{3} \frac{x_i^2}{a_i} < 1\}$$

b_1, b_2, b_3 are positive numbers determined by (4.23). If $b_k = \min\{b_1, b_2, b_3\}$, then $\pm e^k$ are asymptotically stable critical points to Landau-Lifshitz equation (1.13)*, where $\{e^1, e^2, e^3\}$ are the standard basis vectors of $\mathbf{R}^3$.*

1.5 Existence of fractional Landau-Lifshitz equations

Motivation

Before we go in details about the existence result for our new model called fractional Landau-Lifshitz equations, let us first see motivation behind it. The micromagnetic energy is given by,

$$\mathcal{E}(\mathbf{m}) = \frac{\kappa}{2} \int_{\Omega} |\nabla \mathbf{m}|^2 + \int_{\Omega} \varphi(\mathbf{m}) - \int_{\Omega} \mathbf{a}(x) \cdot \mathbf{m} + \frac{1}{2} \int_{\mathbf{R}^3} |H_{\mathbf{m}}|^2. \tag{1.17}$$

One popular research topic for functional energy (1.17) is to explore global minimizers under non-convexity constraints $|\mathbf{m}| = 1$. Another interesting problem is to investigate its limiting behavior in different asymptotic regimes. Those regimes are different with relations among following terms,

$$\begin{aligned} t &= \text{thickness of the film} \\ l &= \text{length scale of the cross section} \\ d &= \text{characteristic length scale of the magnetic material} \end{aligned} \tag{1.18}$$

There are two well studied regimes: (a) The large-body limit, in which $\frac{d}{l} \to 0$ while $\frac{t}{l}$ is fixed. See [28], [18], [48] and [42]. (b) The small-aspect-ratio limit, in which $\frac{t}{l} \to 0$ while $\frac{d}{l}$ is fixed. In this case, when the external filed is constant, the asymptotic variational problem predicts a uniform magnetization; see [21].

DeSimone et al [15] considered different regimes: (a) Ω is a cylindrical domain of thickness t with cross section Ω', such as $\Omega = \Omega' \times (0, t)$; (b) $\mathbf{m}$ does not depend on the thickness direction x_3; (c) $t \ll l$. Under such regime, DeSimone et al [15] derived following convergent magnetostatic energy through Fourier transform:

$$\int_{\mathbf{R}^3} |H_{\mathbf{m}}|^2 dx = \int_{\mathbf{R}^2} | \wedge^{-\frac{1}{2}} (\nabla' \mathbf{m}')|^2 dx \tag{1.19}$$

where $\mathbf{m'} = (\mathbf{m}_1, \mathbf{m}_2)$ and $\wedge$ denotes the square root of Laplacian $(-\triangle)^{\frac{1}{2}}$. It is pretty interesting that the exchange energy is dissipative under this regime and magnetostatic energy reduced to (1.19). This result motivated some research work associated with energy (1.19) applied to Landau-Lifshitz equation; see [23], [24], [25], [26].

Fractional Landau-Lifshitz equation

To extend the models considered in [23], [24], [25], [26], we would like to prove the existence of global weak solution to fractional Landau-Lifshitz equation (1.20) including magnetostatic term with periodic boundary condition:

$$\mathbf{m}_t = \gamma \mathbf{m} \times \mathcal{F}_{\mathbf{m}} + \gamma \mathbf{m} \times (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \qquad \text{in } \mathbf{R}^n, \tag{1.20}$$

where

$$\mathcal{F}_{\mathbf{m}} = \wedge^{2\alpha} \mathbf{m} + H_{\mathbf{m}} \qquad 0 < \alpha < 1. \tag{1.21}$$

We focus on the existence of global weak solution(defined in Chapter 5, see definition (5.0.1)) in special domain $\Omega = [0, 2\pi]^n$ with periodic boundary conditions.

Now, we define $H_{\mathbf{m}}$ in energy term (1.21) as,

$$\widehat{H_{\mathbf{m}}}(\xi) = \begin{cases} \dfrac{(\xi \cdot \widehat{\mathbf{m}}) \cdot \xi}{|\xi|^2}, \quad \xi \neq 0. \\ \\ 0, \quad \xi = 0. \end{cases} \tag{1.22}$$

The operator $\wedge$ denotes the square root of Laplacian $(-\triangle)^{\frac{1}{2}}$, so $\wedge^{\beta}\mathbf{m}$ can be understood in terms of Fourier transform:

$$\mathfrak{F}(\wedge^{\beta}\mathbf{m}(x)) = |\xi|^{\beta}\widehat{\mathbf{m}}(\xi) \tag{1.23}$$

A priori estimates

We mainly use Galerkin method to prove existence. Therefore, we first project our problem into finite space in $L^2(\Omega)$ and the solution can be easily established by standard ODE method. Then we are going to prove a priori estimation in order to extract strong convergence subsequence.

Lemma 1.5.1 (Chapter 5, Section 5.2). *Let $\mathbf{m}_0 \in H^{\alpha}(\Omega)$, then for any $0 < T < \infty$, the approximate solutions $\mathbf{m}_N$ to systems (5.8) in Chapter 5, satisfy,*

$$\max_{0\le t\le T} \|\mathbf{m}_N\|^2_{H^{\alpha}(\Omega)} \le C_1$$

where C_1 only depends on initial data $\|\mathbf{m}_0\|^2_{H^{\alpha}(\Omega)}$. Moreover, for $1 \le r \le r^* = \frac{d}{d-\alpha}$, where d is dimension.

$$\|\frac{\partial \mathbf{m}_N}{\partial t}\|_{L^r(Q_T)} \le C_2$$

and for $1 < r \le r^* = \frac{d}{d-\alpha}$

$$\|\mathbf{m}_N(t_1) - \mathbf{m}_N(t_2)\|_{L^r(\Omega)} \le C_2|t_2 - t_1|^{\frac{r-1}{r}}$$

where C_2 only depends on initial data $\|\mathbf{m}_0\|^2_{H^{\alpha}(\Omega)}$ and time T, $Q_T = (0,T)\times\Omega$.

It is not enough to extract strong convergence subsequence; we still need following compactness lemma,

Lemma 1.5.2 ([39]). *Let B_0, B, B_1 be three Banach space such that,*

$$B_0 \subset B \subset B_1$$

where the injections are continuous and B_0, B_1 are reflexive and $B_0 \hookrightarrow B$ is compact. Denote

$$W = \{v | v \in L^{p_0}(0, T; B_0), \frac{dv}{dt} \in L^{p_1}(0, T; B_1)\}$$

for $T < \infty$ and $1 < p_0, p_1 < \infty$. Then W equipped with the norm

$$\|v\|_{L^{p_0}(0,T;B_0)} + \|\frac{dv}{dt}\|_{L^{p_1}(0,T;B_1)}$$

is a Banach space and the embedding $W \hookrightarrow L^{p_0}(0, T; B)$ is compact.

Now, we can proved that there exists some $\mathbf{m} \in L^\infty(0, T; H^\alpha(\Omega))$ such that,

$$\begin{aligned}
&\mathbf{m}_N \rightharpoonup \mathbf{m} \quad \text{weakly in } L^p(0, T; H^\alpha(\Omega)) \text{ for } 1 < p < \infty \\
&\mathbf{m}_N \to \mathbf{m} \quad \text{strongly in } L^p(0, T; H^\beta(\Omega)) \text{ for } 1 < p < \infty, 0 \le \beta \le \alpha \\
&\frac{\partial \mathbf{m}_N}{\partial t} \rightharpoonup \frac{\partial \mathbf{m}}{\partial t} \quad \text{weakly in } L^r(Q_T) \text{ for } 1 < r
\end{aligned} \tag{1.24}$$

Existence

With compactness result (1.24), it is easy to establish existence.

Theorem 1.5.3 (Chapter 5, section 5.4). *Let $0 < \alpha < 1$ and $\mathbf{m}_0 \in H^\alpha(\Omega)$, then for any $0 < T < \infty$, then there exists at least one global weak solution to fractional Landau-Lifshitz* *equation (1.20) such that*

$$\mathbf{m} \in L^\infty(0,T;H^\alpha(\Omega)) \bigcap C^{0,\frac{r-1}{r}}(0,T;L^r(\Omega))$$

for $1 < r \leq r^* = \frac{d}{d-\alpha}$, *where* d *is dimension.*

Chapter 2

Existence for Reduced Landau-Lifshitz Equations

In this chapter, we study the existence for reduced Landau-Lifshitz equations given by

$$\partial_t \mathbf{m} = \gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} + \alpha \frac{\gamma}{|\mathbf{m}|} \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \tag{2.1}$$

where

$$\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a}(x) + H_{\mathbf{m}} \tag{2.2}$$

with

$$\operatorname{curl} H_{\mathbf{m}} = 0, \quad \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0 \quad \text{in } \mathbf{R}^3. \tag{2.3}$$

We establish the existence of global weak solution using two methods for different initial data. First, we study the *quasi-stationary limit* of a simple Landau-Lifshitz-Maxwell system with the permittivity parameter ϵ approaching zero and, using this quasi-stationary limit, establish the existence of global weak solutions to the reduced Landau-Lifshitz equations with initial value $\mathbf{m}_0 \in L^\infty(\Omega)$. We then give a different method for the existence when the initial datum is more regular (e.g., in $H^2(\Omega)$) by considering the problem as an abstract ordinary differential equation in an appropriate Banach space.

2.1 Quasi-stationary limit of Landau-Lifshitz-Maxwell systems

We study the Cauchy problem for equations (2.1)-(2.3) with a given bounded initial datum $\mathbf{m}(x,0) = \mathbf{m}_0(x)$. This Cauchy problem can be written as a *quasi-stationary* system:

$$\begin{cases} \partial_t \mathbf{m} = F(x, \mathbf{m}, H_{\mathbf{m}}) \quad \text{in } \Omega \times (0,\infty), \\ \operatorname{curl} H_{\mathbf{m}} = 0, \quad \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_\Omega) = 0 \quad \text{in } \mathbf{R}^3, \\ \mathbf{m}(x,0) = \mathbf{m}_0(x) \quad \text{on } \Omega, \end{cases} \tag{2.4}$$

with a given initial datum $\mathbf{m}_0 \in L^\infty(\Omega)$, where $F : \Omega \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is an appropriate function (the *Landau-Lifshitz interaction* function), which can be written in a general form:

$$F(x, M, H) = \mathbb{F}(M)H + \mathbf{a}(x, M) \tag{2.5}$$

with certain specific functions $\mathbb{F}(M)$ and $\mathbf{a}(x, M)$; our results are valid for more general interaction functions $F(x, M, H)$ of this form satisfying some conditions to be specified later (see (2.13)-(2.14) below).

Definition 2.1.1. Given $\mathbf{m}_0 \in L^\infty(\Omega)$, a *weak solution* to the Cauchy problem (2.4) is a function

$$\mathbf{m} \in W^{1,\infty}((0,\infty); L^2(\Omega)) \cap L^\infty((0,\infty); L^\infty(\Omega)) \tag{2.6}$$

such that $\mathbf{m}(0) = \mathbf{m}_0$ in $L^2(\Omega)$ and, with $H_{\mathbf{m}}$ defined by (2.3) with $\mathbf{m} = \mathbf{m}(\cdot, t)$, equation

$$\partial_t \mathbf{m} = F(x, \mathbf{m}, H_{\mathbf{m}}), \tag{2.7}$$

holds both in $L^\infty((0,\infty);L^2(\Omega))$ and in the sense of distribution on $\Omega\times[0,\infty)$.

Remark 1. We remark that, for the Landau-Lifshitz equation (2.1) with $\mathbf{H}_{\text{eff}}$ given by (2.2), since $F(x,\mathbf{m},H_{\mathbf{m}})\cdot\mathbf{m}=0$, the length $|\mathbf{m}(x,t)|$ is preserved for weak solutions; therefore, if the initial datum $\mathbf{m}_0$ satisfies the saturation condition $|\mathbf{m}_0(x)|=1$ then any weak solution $\mathbf{m}(x,t)$ to the Cauchy problem (2.4) will also satisfy the saturation condition: $|\mathbf{m}(x,t)|=1$ for all $t>0$. In general, our assumptions on interaction function F will guarantee the L^∞-norm of $\mathbf{m}(\cdot,t)$ be non-increasing for $t>0$ (see Section 2.1.1).

In the special case of $\mathbf{H}_{\text{eff}}=H_{\mathbf{m}}$, for the regular initial data $\mathbf{m}_0\in H^2(\Omega)$ with $\frac{\partial\mathbf{m}_0}{\partial\nu}|_{\partial\Omega}=0$, certain weak solution of (2.4) has been obtained in [8] as limit of the regular solution $\mathbf{m}_\kappa$ to the Landau-Lifshitz equation (2.1) with $\mathbf{H}_{\text{eff}}=\kappa\Delta\mathbf{m}+H_{\mathbf{m}}$ when $\kappa\to0^+$. However, this singular perturbation method does not work for rough initial data $\mathbf{m}_0$ in $L^\infty(\Omega)$.

We study the Cauchy problem (2.4) as a quasi-stationary limit of the following Landau-Lifshitz-Maxwell system when the permittivity parameter ϵ approaches zero:

$$\begin{cases}\epsilon\partial_t E-\operatorname{curl}H=0,\\ \partial_t(H+M\chi_\Omega)+\operatorname{curl}E=0\quad\text{in }\mathbf{R}^3\times(0,\infty),\\ \partial_t M=F(x,M,H)\quad\text{in }\Omega\times(0,\infty),\\ (E,H)|_{t=0}=(E_0,H_0)\ \ \text{on }\mathbf{R}^3,\quad M|_{t=0}=\mathbf{m}_0\ \ \text{on }\Omega,\end{cases}\tag{2.8}$$

where the initial data E_0,H_0 for electric and magnetic fields E,H are any vector-fields satisfying

$$E_0,H_0\in L^2(\mathbf{R}^3;\mathbf{R}^3),\quad \operatorname{div}E_0=\operatorname{div}(H_0+\mathbf{m}_0\chi_\Omega)=0.\tag{2.9}$$

For $\epsilon>0$, the existence of certain weak solutions to the Cauchy problem (2.8) can be obtained in the same way as studied in [31] for $\epsilon = 1$. More general Landau-Lifshitz-Maxwell systems with variable dielectric permittivity $\epsilon(x)$ and magnetic permeability $\mu(x)$ have been studied in [29], together with the quasi-stationary limit as the variable $\epsilon(x) \to 0$. Long-time asymptotic problems for such systems have been addressed in [7, 30, 32].

One of the main purposes is to present a more direct way to study the asymptotic behavior of weak solution $(E^\epsilon, H^\epsilon, M^\epsilon)$ to the Cauchy problem (2.8) as $\epsilon \to 0^+$. We prove that, for all $0 < T < \infty$, a subsequence M^ϵ converges strongly to a function $\mathbf{m}$ in $C^0([0,T]; L^2(\Omega))$ and $L^2(\Omega \times (0,T))$ (see Theorem 2.1.3 below) and, furthermore, the limit $\mathbf{m} = \mathbf{m}(x,t)$ is a weak solution to Cauchy problem (2.4) (see Theorem 2.2.1 below). Our direct proof is motivated by the methods of [31] with, however, a completely new parameter asymptotics, and is much different in nature from the studies based on semi-group theory in [29].

The rest of section is organized as follows. We will first give the assumptions on the interaction functions $F(x,M,H)$ appearing in the general Cauchy problem (2.4) above and present some preliminaries, including two useful compensated compactness results. Then, we prove the strong convergence of M^ϵ as $\epsilon \to 0^+$ using a direct approach motivated by the work [31], which is much different from the abstract semi-group techniques used in [29]. Finally, we prove the existence of weak solution to (2.4) directly from the strong convergence theorem.

2.1.1 General assumptions and preliminaries

We assume Ω is a bounded domain in $\mathbf{R}^3$; the boundedness of Ω will simplify many technical assumptions otherwise needed as in [29, 31].

Let $F(x,M,H) = \mathbb{F}(M)H + \mathbf{a}(x,M)$ be defined as in (2.5), where we assume $\mathbb{F}\colon \mathbf{R}^3 \to$

$\mathbf{R}^{3\times 3}$ and $\mathbf{a}\colon \Omega \times \mathbf{R}^3 \to \mathbf{R}^3$ satisfy the following condition (see also [29]):

$$
\begin{aligned}
&(a) \quad \mathbb{F}(M)^T M = 0, \quad \mathbf{a}(x, M) \cdot M \leq 0 \quad \forall\, M \in \mathbf{R}^3,\ x \in \Omega;\\
&(b) \quad \mathbf{a}(x, 0) \in L^2(\Omega);\\
&(c) \quad \begin{aligned}[t] &|\mathbf{a}(x, M') - \mathbf{a}(x, M)| + |\mathbb{F}(M') - \mathbb{F}(M)| \leq C_1(R)\,|M' - M|\\ &\forall\, R > 0,\ x \in \Omega,\ |M'|, |M| \leq R, \text{ where } C_1(R) \text{ is constant.}\end{aligned}
\end{aligned}
\tag{2.10}
$$

In particular, we have

$$
F(x, M, H) \cdot M \leq 0 \quad \forall\, x \in \Omega,\ M,\ H \in \mathbf{R}^3 \tag{2.11}
$$

$$
|F(x, M, H)| \leq C_2(R)(|H| + 1) + |\mathbf{a}(x, 0)| \quad \forall\, |M| \leq R \tag{2.12}
$$

for all $x \in \Omega$, $H \in \mathbf{R}^3$, $R \geq 0$, where $C_2(R)$ is a constant depending on R.

Consider the Landau-Lifshitz function defined by $L(0, H) = 0$ and

$$
L(M, H) = \gamma M \times H + \alpha \frac{\gamma}{|M|} M \times (M \times H), \quad M \neq 0, \tag{2.13}
$$

where α, γ are constants. Then the function $F(x, M, H)$ appearing in (2.4) is given by $F(x, M, H) = \mathbb{F}(M)H + \mathbf{a}(x, M)$ with

$$
\mathbb{F}(M)H = L(M, H), \quad \mathbf{a}(x, M) = L(M, \mathbf{a}(x)) - L(M, \varphi'(M)). \tag{2.14}
$$

It is easily seen that this particular function F satisfies all assumptions in condition (2.10); in fact, one has $F(x, M, H) \cdot M = 0$ for all $x \in \Omega$ and $M,\ H \in \mathbf{R}^3$. This property in particular yields the equality in the length estimate (2.18) below.

The orthogonal decomposition $L^2(\mathbf{R}^3;\mathbf{R}^3) = L^2_{\|}(\mathbf{R}^3;\mathbf{R}^3) \oplus L^2_{\perp}(\mathbf{R}^3;\mathbf{R}^3)$ is standard, where $L^2_{\|}(\mathbf{R}^3;\mathbf{R}^3), L^2_{\perp}(\mathbf{R}^3;\mathbf{R}^3)$ are the subspaces of curl-free or divergence-free functions in the sense of distributions, respectively. This decomposition can be explicitly given in terms of the Fourier transform $\hat{f}$ of $f \in L^2(\mathbf{R}^3;\mathbf{R}^3)$: $f = f_{\|} + f_{\perp}$, where

$$\hat{f}_{\|} = (\xi \cdot \hat{f})\xi/|\xi|^2, \quad \hat{f}_{\perp} = \hat{f} - (\xi \cdot \hat{f})\xi/|\xi|^2 = -\xi \times (\xi \times \hat{f})/|\xi|^2.$$

The projection operator $P_{\|}(f) = f_{\|}$ also extends to a bounded linear operator on $L^p(\mathbf{R}^3;\mathbf{R}^3)$ for all $1 < p < \infty$, with operator norm bounded by $C_0 p$ when $p \geq 2$, where C_0 is independent of $p \geq 2$; see [45].

2.1.2 Two Compensated Compactness Lemmas

We give two compensated compactness results in a suitable form to be used later; both are special case of the more general results [20, 46, 47].

Let $b \in C_c^{\infty}(\mathbf{R}^3)$ and $[b, P_{\|}] = bP_{\|} - P_{\|}b \colon L^2(\mathbf{R}^3;\mathbf{R}^3) \to L^2(\mathbf{R}^3;\mathbf{R}^3)$ be the commutator. We have the following special compactness result from the well-known div-curl lemma [46].

Lemma 2.1.1. *Let $f^k \rightharpoonup 0$ weakly in $L^2(\mathbf{R}^3;\mathbf{R}^3)$ and $L^p(\mathbf{R}^3;\mathbf{R}^3)$ for some $p > 2$. Then $g^k = [b, P_{\|}]f^k \to 0$ strongly in $L^2(\Omega;\mathbf{R}^3)$ for all bounded sets Ω.*

Proof. From

$$g^k = bf^k_{\|} - (bf^k)_{\|} = (bf^k)_{\perp} - bf^k_{\perp},$$

we have $\operatorname{div} g^k = -\nabla b \cdot f^k_{\perp}$, $\operatorname{curl} g^k = \nabla b \times f^k_{\|}$ in the sense of distributions on $\mathbf{R}^3$, and so both $\{\operatorname{div} g^k\}$ and $\{\operatorname{curl} g^k\}$ are compact in $H^{-1}(\mathbf{R}^3)$. By the div-curl lemma, $|g^k|^2 = g^k \cdot g^k \to 0$

in the sense of distributions on $\mathbf{R}^3$. Since $\{g^k\}$ is bounded in $L^p(\mathbf{R}^3)$ with $p > 2$, we have $g^k \to 0$ strongly in $L^2(\Omega)$ for all bounded sets Ω. $\square$

Let U be any domain in $\mathbf{R}^3$ and, for $0 < T \leq \infty$, let $U_T = U \times (0, T)$. Let $c \in \mathbf{R}$ be any constant. Denote $\Box_c u = \Delta u - c u_{tt}$, where Δ is the Laplacian with respect to $x \in \mathbf{R}^3$. Note that $\tilde{\Delta} = \Box_{-1}$ is the Laplacian operator with respect to $(x,t) \in \mathbf{R}^3 \times \mathbf{R}$ and $-\Box_1$ is the usual wave operator on $\mathbf{R}^3 \times \mathbf{R}$. The following lemma, with $c = 0$, will be used later; the result is a special case of more general analyses on micro-local defect measures or H-measures and orthogonality of sequences [20, 47].

Lemma 2.1.2. *Let $u^k \rightharpoonup u$ and $v^k \rightharpoonup v$ weakly in $L^2(U_T)$ as $k \to \infty$. Suppose $\{\Box_c u^k\}$ is compact in $H^{-2}(U_T)$ and $\{\partial_t v^k\}$ is bounded in $L^2(U_T)$. Then $u^k v^k \to uv$ in the sense of distributions on U_T.*

Proof. By considering $u^k - u$ and $v^k - v$, we can assume $u = v = 0$. Given any $\zeta \in C_c^\infty(U_T)$, we need to show

$$\lim_{k\to\infty} \iint_{U_T} \zeta(x,t) u^k(x,t) v^k(x,t)\, dxdt = 0. \tag{2.15}$$

Let A, B be two regular domains in U_T such that $\operatorname{supp}\zeta \subset\subset A \subset\subset B \subset\subset U_T$. We solve the following Dirichlet problem:

$$\begin{cases} \tilde{\Delta} w^k = \Delta w^k + w^k_{tt} = v^k \ \text{ in } B \\ w^k = 0 \ \text{ on } \partial B. \end{cases} \tag{2.16}$$

By the standard elliptic theory, this problem has a unique solution $w^k \in H_0^1(B) \cap H^2(B)$, which also satisfies the estimate $\|w^k\|_{H^2(B)} \leq C_1 \|v^k\|_{L^2(B)}$. Hence $\{w^k\}$ is bounded in $H^2(B)$. By the Rellich compactness theorem, there exists a subsequence, still denoted by

k, such that $\{w^k\}$ converges strongly in $H^1(B)$ as $k \to \infty$. From (2.16), it follows that $\tilde{\Delta}(w_t^k) = v_t^k \in L^2(B)$ and hence by the (interior) L^2-estimates,

$$\|w_t^k\|_{H^2(A)} \leq C_2\,(\|v_t^k\|_{L^2(B)} + \|w_t^k\|_{L^2(B)}).$$

This shows that $\{w_t^k\}$ is bounded in $H^2(A)$ and hence for a subsequence, denoted again by k, $\{w_t^k\}$ converges strongly in $H^1(A)$. This implies the strong convergence of $\{w_{tt}^k\}$ in $L^2(A)$ as $k \to \infty$. Extending w^k by zero onto U_T, we write

$$\iint_{U_T} \zeta u^k v^k \, dxdt = \iint_{U_T} \zeta u^k (\Delta w^k + w_{tt}^k)\, dxdt$$

$$= \iint_{U_T} (1+c)\zeta u^k w_{tt}^k \, dxdt + \iint_{U_T} \zeta u^k \Box_c w^k \, dxdt := I_k + II_k,$$

where the first term $I_k \to 0$ as $k \to \infty$ since $u^k \rightharpoonup 0$ and $\{w_{tt}^k\}$ converges strongly in $L^2(A)$. The second term can be written as

$$II_k = \iint_{U_T} u^k(\Box_c(\zeta w^k) - w^k \Box_c \zeta - 2\nabla\zeta \cdot \nabla w^k + 2c\zeta_t w_t^k)\, dxdt$$

$$= \iint_{U_T} u^k \Box_c(\zeta w^k)\, dxdt - \iint_{U_T} u^k\,(w^k \Box_c \zeta + 2\nabla\zeta \cdot \nabla w^k - 2c\zeta_t w_t^k)\, dxdt.$$

The second integral on the right-hand side approaches zero as $k \to \infty$ because $u^k \rightharpoonup 0$ and $\{w^k\}$ converges strongly in $H^1(B)$. Since $\{\zeta w^k\}$ is bounded in $H^2(U_T)$, we can estimate the first integral as follows:

$$\left| \iint_{U_T} u^k \, \Box_c(\zeta w^k)\, dxdt \right| = \left| \langle \Box_c u^k, \zeta w^k \rangle \right|$$

$$\leq \|\Box_c u^k\|_{H^{-2}(U_T)} \|\zeta w^k\|_{H^2(U_T)} \to 0$$

as $k \to \infty$, where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^{-2}(U_T)$ and $H_0^2(U_T)$. This completes the proof of (2.15). Finally, the lemma is proved. $\Box$

2.1.3 Strong convergence of M^ϵ

Let $0 < \epsilon < 1$. Let $(E^\epsilon, H^\epsilon, M^\epsilon)$ be the weak solution to the Cauchy problem (2.8) with the property $(E^\epsilon, H^\epsilon) \in C^0([0,\infty); L^2(\mathbf{R}^3))$ and $M^\epsilon \in W^{1,\infty}_{loc}([0,\infty); L^2(\Omega)) \cap L^\infty_{loc}([0,\infty); L^\infty(\Omega))$; the existence of such a weak solution has been established in [29, 31].

Estimates and weak convergence of M^ϵ and H^ϵ

Standard estimates using (2.9), (2.11) and (2.12) and Gronwall's inequality, yield that, for all $t > 0$,

$$\operatorname{div} E^\epsilon = \operatorname{div}(H^\epsilon + M^\epsilon \chi_\Omega) = 0 \quad \text{in } \mathbf{R}^3, \tag{2.17}$$

$$|M^\epsilon(x,t)| \leq |\mathbf{m}_0(x)| \quad a.e.\ x \in \Omega, \tag{2.18}$$

$$\int_{\mathbf{R}^3} (\epsilon |E^\epsilon(t)|^2 + |H^\epsilon(t)|^2) dx \leq C_3 + e^{C_3 t} \int_{\mathbf{R}^3} (\epsilon |E_0|^2 + |H_0|^2) dx, \tag{2.19}$$

where C_3 is a constant depending only on $\|\mathbf{a}(\cdot, 0)\|_{L^2}$ and $\|\mathbf{m}_0\|_{L^\infty}$.

Since $\{M^\epsilon\}$ is bounded in $L^\infty(\Omega \times (0,\infty))$ and $L^\infty((0,\infty); L^2(\Omega))$, there exist a function $\mathbf{m} \in L^\infty(\Omega \times (0,\infty)) \cap L^\infty((0,\infty); L^2(\Omega))$ and a subsequence of $\epsilon \to 0$, still denoted by ϵ, such that

$$M^\epsilon \rightharpoonup \mathbf{m} \quad \text{weakly* in } L^\infty(\Omega \times (0,\infty)) \cap L^\infty((0,\infty); L^2(\Omega)). \tag{2.20}$$

Let $0 < T < \infty$ and $\Omega_T = \Omega \times (0,T)$ and $U_T = \mathbf{R}^3 \times (0,T)$. By energy estimate (2.19), third equation of (2.8) and condition (2.12), it follows that the sequences $\{\sqrt{\epsilon}E^\epsilon\}, \{H^\epsilon\}$ are bounded in $L^\infty((0,T); L^2(\mathbf{R}^3))$ and $L^2(U_T)$ and that the sequence $\{\partial_t M^\epsilon\}$ is bounded in $L^\infty((0,T); L^2(\Omega))$ and $L^2(\Omega_T)$. Let $\tilde{E}$, $\tilde{H}$ and $\tilde{S}$ be the limit of any weakly convergent subsequence of $\{\sqrt{\epsilon}E^\epsilon\}, \{H^\epsilon\}$ and $\{\partial_t M^\epsilon\}$ along the chosen $\epsilon \to 0$ in the respective Banach spaces. Clearly $\tilde{S} = \partial_t \mathbf{m}$. Note that $\epsilon E_t^\epsilon \to 0$ and $H^\epsilon \to \tilde{H}$ also in the sense of distributions on U_T. Hence, from the first equation of system (2.8), using (2.17), we have

$$\operatorname{curl} \tilde{H}(t) = 0, \quad \operatorname{div}(\tilde{H}(t) + \mathbf{m}(t)\chi_\Omega) = 0$$

in the sense of distributions on $\mathbf{R}^3$ for *almost every* $t \in (0,T)$; therefore, $\tilde{H}(t) = -P_{\|}(\mathbf{m}(t)\chi_\Omega)$ for almost every $t \in (0,\infty)$. This implies that any weak limit $\tilde{H}$ is uniquely determined by $\mathbf{m}$. This uniqueness also shows that the whole sequence $\{H^\epsilon\}$ (along the chosen $\epsilon \to 0$) converges weakly in both $L^\infty((0,T); L^2(\mathbf{R}^3))$ and $L^2(U_T)$ for all $0 < T < \infty$; the limit is given by

$$H^\infty(t) = -P_{\|}(\mathbf{m}(t)\chi_\Omega) = H_{\mathbf{m}(t)}, \tag{2.21}$$

where $H_{\mathbf{m}(t)}$ is defined by $\mathbf{m}(t)$ through (1.2) above. Moreover, since $M^\epsilon \in W^{1,\infty}((0,T); L^2(\Omega)) \subset C^0([0,T]; L^2(\Omega))$, we have

$$\mathbf{m} \in W^{1,\infty}((0,T); L^2(\Omega)) \subset C^0([0,T]; L^2(\Omega)) \quad \forall\ 0 < T < \infty; \tag{2.22}$$

hence, we can also assume (2.21) holds for *all* $t > 0$.

Strong convergence of M^ϵ

The main result of this section is the following.

Theorem 2.1.3. *We have $M^\epsilon \to \mathbf{m}$ strongly in both $C^0([0,T];L^2(\Omega))$ and $L^2(\Omega_T)$ for all $0 < T < \infty$.*

A closer look at the proof of [29, Theorem 1.1] shows that our theorem follows from that proof since the conductivity σ is zero here. However the proof in [29] involves other complicated techniques aimed for handling the variable permittivity $\epsilon(x)$ and permeability $\mu(x)$. We present a different and direct proof of this result in our setting, which is motivated by the methods in [31].

The rest of this section is devoted to the proof of Theorem 2.1.3.

Weighted energy estimates

Assume $T > 0$ and $\epsilon, \delta \to 0$ are the chosen subsequence. We want to estimate some weighted norm of $M^\epsilon(t) - M^\delta(t)$ in $L^2(\Omega)$ for $0 \le t \le T$.

In what follows, we use C_j to denote various constants depending on $\|\mathbf{m}_0\|_{L^\infty}$ and perhaps T or other quantities to be specified as needed. We write

$$
\begin{aligned}
\partial_t(M^\epsilon - M^\delta) &= F(x, M^\epsilon, H^\epsilon) - F(x, M^\delta, H^\delta) \\
&= \mathbb{F}(M^\epsilon)H^\epsilon - \mathbb{F}(M^\delta)H^\delta + \mathbf{a}(x, M^\epsilon) - \mathbf{a}(x, M^\delta) \\
&= (\mathbb{F}(M^\epsilon) - \mathbb{F}(M^\delta))H^\infty + \mathbf{a}(x, M^\epsilon) - \mathbf{a}(x, M^\delta) \\
&\quad + \mathbb{F}(M^\epsilon)(H^\epsilon - H^\infty) - \mathbb{F}(M^\delta)(H^\delta - H^\infty).
\end{aligned} \tag{2.23}
$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}|M^\epsilon - M^\delta|^2 \le & |(\mathbb{F}(M^\epsilon) - \mathbb{F}(M^\delta))H^\infty| \cdot |M^\epsilon - M^\delta| \\ & + |\mathbf{a}(x, M^\epsilon) - \mathbf{a}(x, M^\delta)| \cdot |M^\epsilon - M^\delta| \\ & + (\mathbb{F}(M^\epsilon)(H^\epsilon - H^\infty) - \mathbb{F}(M^\delta)(H^\delta - H^\infty)) \cdot (M^\epsilon - M^\delta) \\ \le & C_1|M^\epsilon - M^\delta|^2(|H^\infty| + 1) \\ & + (\mathbb{F}(M^\epsilon)(H^\epsilon - H^\infty) - \mathbb{F}(M^\delta)(H^\delta - H^\infty)) \cdot (M^\epsilon - M^\delta),\end{aligned} \tag{2.24}$$

where $C_1 = C_1(R)$ is the constant in (2.10-c) with $R = \|\mathbf{m}_0\|_{L^\infty}$. Let $a(x,t)$ be the function defined by

$$a(x,t) = |x|^2 + C_1 \int_0^t (1 + |H^\infty(x,s)|)\, ds \quad (x \in \mathbf{R}^3,\ t \ge 0). \tag{2.25}$$

For all $t \ge 0$, since $e^{-a(\cdot,t)} \in L^2(\mathbf{R}^3)$ and $M^\epsilon(\cdot,t) \in L^\infty(\Omega)$, it follows that

$$e^{-a(t)} M^\epsilon(t) \rightharpoonup e^{-a(t)}\mathbf{m}(t) \ \text{ weakly in } L^2(\Omega) \quad \forall\, t \ge 0. \tag{2.26}$$

Furthermore, by (2.24),

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}(e^{-2a}|M^\epsilon - M^\delta|^2) \\ &\le e^{-2a}(\mathbb{F}(M^\epsilon)(H^\epsilon - H^\infty) - \mathbb{F}(M^\delta)(H^\delta - H^\infty)) \cdot (M^\epsilon - M^\delta).\end{aligned} \tag{2.27}$$

Integrating (2.27) with respect to $x \in \Omega$ and on time-interval $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\|e^{-a(t)}(M^\epsilon(t)-M^\delta(t))\|^2_{L^2(\Omega)} \\
&\qquad\qquad \leq \int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\epsilon)(H^\epsilon - H^\infty)\cdot(M^\epsilon - M^\delta)\,dxds \\
&\qquad\qquad\quad - \int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\delta)(H^\delta - H^\infty)\cdot(M^\epsilon - M^\delta)\,dxds \\
&\qquad\qquad := f^\epsilon_{\epsilon,\delta}(t) - f^\delta_{\epsilon,\delta}(t),
\end{aligned} \tag{2.28}$$

where functions $f^\rho_{\epsilon,\delta}(t)$ (with $\rho = \epsilon,\ \delta$) are defined by

$$f^\rho_{\epsilon,\delta}(t) = \int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\rho)(H^\rho - H^\infty)\cdot(M^\epsilon - M^\delta)dxds. \tag{2.29}$$

To analyze $f^\rho_{\epsilon,\delta}(t)$, we split it into two terms:

$$\begin{aligned}
f^\rho_{\epsilon,\delta}(t) =& \int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\rho)(H^\rho_\perp - H^\infty_\perp)\cdot(M^\epsilon - M^\delta)dxds \\
&+ \int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\rho)(H^\rho_\| - H^\infty_\|)\cdot(M^\epsilon - M^\delta)dxds \\
:=& g^\rho_{\epsilon,\delta}(t) + h^\rho_{\epsilon,\delta}(t).
\end{aligned} \tag{2.30}$$

By (2.17) and (2.21), $H^\rho_\| - H^\infty_\| = -P_\|(\tilde{M}^\rho - \tilde{\mathbf{m}})$, where $\tilde{M}^\rho = M^\rho\chi_\Omega$ and $\tilde{\mathbf{m}} = \mathbf{m}\chi_\Omega$. So, the function $h^\rho_{\epsilon,\delta}(t)$ in (2.30) can be rewritten as

$$\begin{aligned}
h^\rho_{\epsilon,\delta}(t) =& -\int_0^t \int_\Omega e^{-2a}\mathbb{F}(M^\rho)(P_\|(\tilde{M}^\rho - \tilde{\mathbf{m}}))\cdot(M^\epsilon - M^\delta)dxds \\
=& -\int_0^t \int_\Omega \mathbb{F}(M^\rho)(P_\|(e^{-a}(\tilde{M}^\rho - \tilde{\mathbf{m}})))\cdot e^{-a}(M^\epsilon - M^\delta)dxds \\
& - \int_0^t \int_\Omega \mathbb{F}(M^\rho)([e^{-a}, P_\|](\tilde{M}^\rho - \tilde{\mathbf{m}}))\cdot e^{-a}(M^\epsilon - M^\delta)dxds \\
& := -I_3 - I_4,
\end{aligned} \tag{2.31}$$

where $[e^{-a}, P_{\|}]f = e^{-a}f_{\|} - (e^{-a}f)_{\|}$ denotes the commutator operator. For I_3, we have:

$$\begin{aligned}|I_3| &= \left|\int_0^t \int_\Omega \mathbb{F}(M^\rho)(P_{\|}(e^{-a}(\tilde{M}^\rho - \tilde{\mathbf{m}}))) \cdot e^{-a}(M^\epsilon - M^\delta)dxds\right| \\ &\leq C_3 \int_0^t \left(\|e^{-a}(M^\rho - \mathbf{m})\|_{L^2(\Omega)}^2 + \|e^{-a}(M^\epsilon - M^\delta)\|_{L^2(\Omega)}^2\right) ds \\ &\leq 4C_3 \int_0^t \left(\|e^{-a}(M^\epsilon - \mathbf{m})\|_{L^2(\Omega)}^2 + \|e^{-a}(M^\epsilon - M^\delta)\|_{L^2(\Omega)}^2\right) ds.\end{aligned} \tag{2.32}$$

For I_4, we have the following result.

Lemma 2.1.4. *For any $\eta > 0$, there exists $\xi = \xi(\eta, T, R) > 0$ such that*

$$|I_4| < \eta/4 \quad \forall\ 0 < \epsilon, \delta < \xi, \quad \forall\ t \in [0, T]. \tag{2.33}$$

Proof. Note that

$$|I_4| \leq C_3 \int_0^T \|[e^{-a(s)}, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)} ds. \tag{2.34}$$

We claim

$$\sup_{0\leq t\leq T} \|[e^{-a(s)}, P_{\|}](\tilde{M}^\rho(s) - \tilde{\mathbf{m}}(s))\|_{L^2(\Omega)} < \infty \tag{2.35}$$

$$\lim_{\rho\to 0} \|[e^{-a(s)}, P_{\|}](\tilde{M}^\rho(s) - \tilde{\mathbf{m}}(s))\|_{L^2(\Omega)} = 0 \quad \forall\ s \in (0, T). \tag{2.36}$$

Once (2.35) and (2.36) are proved, by Lebesgue dominated convergence theorem, we have

$$\lim_{\rho\to 0} \int_0^T \|[e^{-a(s)}, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)} ds = 0$$

and hence (2.33) follows by (2.34). To prove (2.35), note that, by the boundedness of $P_{\|}$ on

$L^4(\mathbf{R}^3)$,

$$\|[e^{-a}, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)} \leq C_0 \|e^{-a}\|_{L^4(\Omega)} \|M^\rho - \mathbf{m}\|_{L^4(\Omega)}.$$

To prove the point-wise convergence (2.36), fix $s \in [0, T]$ and note that

$$\|[e^{-a(s)}, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)} \leq I_5 + I_6$$

$$:= \|[e^{-a(s)} - b, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)} + \|[b, P_{\|}](\tilde{M}^\rho - \tilde{\mathbf{m}})\|_{L^2(\Omega)},$$

where $b \in C_c^\infty(\Omega)$. Again, by the boundedness of $P_{\|}$ on $L^4(\mathbf{R}^3)$, we have

$$I_5 \leq C_0 \|e^{-a(s)} - b\|_{L^4(\Omega)} \|M^\rho - \mathbf{m}\|_{L^4(\Omega)} \leq C_4 \|e^{-a(s)} - b\|_{L^4(\Omega)}.$$

Choose $b \in C_c^\infty(\Omega)$ so that $\|e^{-a(s)} - b\|_{L^4(\Omega)}$ and hence I_5 are arbitrarily small. Once b is chosen, since $\tilde{M}^\rho(s) \rightharpoonup \tilde{\mathbf{m}}(s)$ weakly in $L^p(\mathbf{R}^3)$ for all $p \geq 2$, by Lemma 2.1.1, we have $I_6 \to 0$ as $\rho \to 0$. Hence (2.36) is proved. $\square$

Combining (2.28)–(2.33) above, we have, for all $t \in [0, T]$ and $0 < \epsilon, \delta < \xi$, with $\xi = \xi(\eta, T, R)$ determined in Lemma 2.1.4,

$$\begin{aligned} &\|e^{-a(t)}(M^\epsilon(t) - M^\delta(t))\|_{L^2}^2 \leq \eta + 2(g_{\epsilon,\delta}^\epsilon(t) - g_{\epsilon,\delta}^\delta(t)) \\ &+C_5 \int_0^t \left(\|e^{-a(s)}(M^\epsilon - M^\delta)\|_{L^2}^2 + \|e^{-a(s)}(M^\epsilon - \mathbf{m})\|_{L^2}^2 \right) ds, \end{aligned} \tag{2.37}$$

where $g_{\epsilon,\delta}^\rho(t)$ is the function defined in (2.30) by

$$g_{\epsilon,\delta}^\rho(t) = \int_0^t \int_\Omega e^{-2a} \mathbb{F}(M^\rho)(H_\perp^\rho - H_\perp^\infty) \cdot (M^\epsilon - M^\delta) dx ds. \tag{2.38}$$

From (2.37), using Gronwall's inequality, we have

$$\int_0^t \|e^{-a(s)}(M^\epsilon - M^\delta)\|_{L^2}^2 \, ds \le \int_0^t 2e^{(t-s)C_5}(g_{\epsilon,\delta}^\epsilon(s) - g_{\epsilon,\delta}^\epsilon(s)) \, ds$$
$$+ Te^{TC_5}\left(\eta + C_5 \int_0^t \|e^{-a(s)}(M^\epsilon - \mathbf{m})\|_{L^2}^2 \, ds\right).$$

Plugging this inequality into the right-hand of (2.37), we have obtained the following result.

Proposition 2.1.5. *For each $\eta > 0$, there exists $\xi = \xi(\eta, T, R) > 0$ such that, for all $0 < \epsilon, \delta < \xi$ and $t \in [0, T]$,*

$$\begin{aligned} \|e^{-a(t)}(M^\epsilon(t) - M^\delta(t))\|_{L^2}^2 &\le G_{\epsilon,\delta}^\epsilon(t) - G_{\epsilon,\delta}^\delta(t) \\ &+ C_6\left(\eta + \int_0^t \|e^{-a(s)}(M^\epsilon - \mathbf{m})\|_{L^2}^2 \, ds\right), \end{aligned} \tag{2.39}$$

where $C_6 = C_6(R, T)$ is a constant depending only on R, T and, for $\rho = \epsilon, \delta$,

$$G_{\epsilon,\delta}^\rho(t) = 2g_{\epsilon,\delta}^\rho(t) + \int_0^t 2e^{(t-s)C_5} g_{\epsilon,\delta}^\rho(s) \, ds. \tag{2.40}$$

Estimates of $g_{\epsilon,\delta}^\rho(t)$

We now study the function $g_{\epsilon,\delta}^\rho(t)$ defined by (2.38). Since $\operatorname{curl} H^\infty = 0$ and so $H_\perp^\infty = 0$, we have thus

$$g_{\epsilon,\delta}^\rho(t) = \int_0^t \int_\Omega e^{-2a(s)} \mathbb{F}(M^\rho)(H_\perp^\rho) \cdot (M^\epsilon - M^\delta) dx ds. \tag{2.41}$$

Let

$$g^\epsilon(t) = \int_0^t \int_\Omega e^{-2a(s)} \mathbb{F}(M^\epsilon)(H_\perp^\epsilon) \cdot (M^\epsilon - \mathbf{m}) dx ds. \tag{2.42}$$

We have the following result.

Proposition 2.1.6. *There exists a constant $C_7 = C_7(R,T)$ such that*

$$|g^{\epsilon}_{\epsilon,\delta}(t)| + |g^{\delta}_{\epsilon,\delta}(t)| \le C_7 \quad \forall\ 0 < \epsilon, \delta < 1,\ t \in [0,T] \tag{2.43}$$

$$\lim_{\delta \to 0} g^{\delta}_{\epsilon,\delta}(t) = 0 \quad \forall\ 0 < \epsilon < 1,\ t \in [0,T] \tag{2.44}$$

$$\lim_{\delta \to 0} g^{\epsilon}_{\epsilon,\delta}(t) = g^{\epsilon}(t) \quad \forall\ 0 < \epsilon < 1,\ t \in [0,T] \tag{2.45}$$

$$\lim_{\epsilon \to 0} g^{\epsilon}(t) = 0 \quad \forall\ t \in [0,T]. \tag{2.46}$$

Proof. It is easy to see that

$$|g^{\rho}_{\epsilon,\delta}(t)| \le C_8 \int_0^T \|H^{\rho}(s)\|_{L^2(\mathbf{R}^3)} ds \le C_7 \quad \text{by (2.19)}.$$

The convergence (2.45) follows easily from the weak convergence $M^{\delta} \rightharpoonup M^{\infty}$. The proofs of (2.44) and (2.46) are similar; so we only give the proof of (2.44). To this end, we write

$$\begin{aligned} g^{\delta}_{\epsilon,\delta}(t) &= \int_0^t \int_{\Omega} e^{-2a} \mathbb{F}(M^{\delta})(H^{\delta}_{\perp}) \cdot (M^{\epsilon} - M^{\delta})\, dxds \\ &= \int_0^t \int_{\Omega} e^{-2a} (\mathbb{F}(M^{\delta})^T (M^{\epsilon} - M^{\delta})) \cdot H^{\delta}_{\perp}\, dxds \\ &= \int_0^t \int_{\Omega} e^{-2a} (\mathbb{F}(M^{\delta})^T M^{\epsilon}) \cdot H^{\delta}_{\perp}\, dxds. \end{aligned} \tag{2.47}$$

Let $\Omega_t = \Omega \times (0,t)$. Since $e^{-2a} \in L^2(\Omega_t)$ for all $0 < t \le T$, the limit (2.44) will be proved if we show that

$$(\mathbb{F}(M^{\delta})^T M^{\epsilon}) \cdot H^{\delta}_{\perp} \rightharpoonup 0 \text{ weakly in } L^2(\Omega_t) \text{ as } \delta \to 0. \tag{2.48}$$

Since the sequence $\{(\mathbb{F}(M^{\delta})^T M^{\epsilon}) \cdot H^{\delta}_{\perp}\}$ is bounded in $L^2(\Omega_t)$, to show its weak convergence to 0, we only need to show $(\mathbb{F}(M^{\delta})^T M^{\epsilon}) \cdot H^{\delta}_{\perp} \to 0$ in the sense of distributions on Ω_t as $\delta \to 0$. We prove this by using Lemma 2.1.2 above. It is easy to see function $g(\mathbf{m},\mathbf{n}) = \mathbb{F}(\mathbf{m})^T \mathbf{n}$

restricted to the set $\mathbf{B}_R = \{(\mathbf{m}, \mathbf{n}) \,|\, |\mathbf{m}| \le R,\ |\mathbf{n}| \le R\}$, where $R = \|\mathbf{m}_0\|_{L^\infty(\Omega)}$, is Lipschitz continuous with Lipschitz constant $\le C_9(R)$. We can thus extend this function $g(\mathbf{m}, \mathbf{n})$ to a Lipschitz function $G(\mathbf{m}, \mathbf{n})$ on whole $(\mathbf{m}, \mathbf{n}) \in \mathbf{R}^3 \times \mathbf{R}^3$ with Lipschitz constant $\le C_9(R)$. Note that $(\mathbb{F}(M^\delta)^T M^\epsilon) \cdot H_\perp^\delta = G(M^\epsilon, M^\delta) \cdot H_\perp^\delta$. Hence

$$\|G(M^\epsilon, M^\delta)\|_{L^2(\Omega_t)} \le C_{10} < \infty. \tag{2.49}$$

Since G is Lipschitz and $\partial_s M^\epsilon$ and $\partial_s M^\delta$ both exist as integrable functions on Ω_t, we have $\partial_s(G(M^\epsilon, M^\delta))$ exists as an integrable function on Ω_t and

$$\begin{aligned} |\partial_s(G(M^\epsilon, M^\delta))| &\le C_9(R)\,(|\partial_s M^\epsilon| + |\partial_s M^\delta|) \\ &\le C_9(R)\,(|F(x, M^\epsilon, H^\epsilon)| + |F(x, M^\delta, H^\delta)|) \end{aligned}$$

and hence

$$\|\partial_s(G(M^\epsilon, M^\delta))\|_{L^2(\Omega_t)} \le C_{11} < \infty. \tag{2.50}$$

From first two equations of the Maxwell system (2.8) for $\delta > 0$, with s as time-variable, we deduce

$$\Delta(H_\perp^\delta) = \delta \partial_{ss}^2 (\tilde{M}_\perp^\delta + H_\perp^\delta) = \delta(\tilde{M}_\perp^\delta + H_\perp^\delta)_{ss} \tag{2.51}$$

in the sense of distributions on $\mathbf{R}^3 \times (0, \infty)$, where again $\tilde{M}^\delta = M^\delta \chi_\Omega$. Since the sequences $\{H_\perp^\delta\}$ and $\{\tilde{M}_\perp^\delta\}$ are bounded in $L^2(\Omega_t)$, we have

$$\Box_0 H_\perp^\delta = \Delta H_\perp^\delta \to 0 \quad \text{strongly in } H^{-2}(\Omega_t) \quad \text{as } \delta \to 0. \tag{2.52}$$

Clearly, $H_\perp^\delta \rightharpoonup H_\perp^\infty = 0$ weakly in $L^2(\Omega_t)$ as $\delta \to 0$, by (2.50), (2.52) and Lemma 2.1.2

above, we have that $G(M^\epsilon, M^\delta) \cdot H_\perp^\delta \to 0$ in the sense of distributions on Ω_t, as $\delta \to 0$; hence, (2.48) is proved. This completes the proof of Proposition 2.1.6. $\square$

Proof of Theorem 2.1.3

We now complete the proof of Theorem 2.1.3. Taking the limit as $\delta \to 0$ in (2.39), using (2.26), Fatou's lemma and (2.43)–(2.45), we have

$$\begin{aligned} &\|e^{-a(t)}(M^\epsilon(t) - \mathbf{m}(t))\|^2_{L^2(\Omega)} \\ &\leq G^\epsilon(t) + C_6\left(\eta + \int_0^t \|e^{-a(s)}(M^\epsilon - \mathbf{m})\|^2_{L^2(\Omega)}\,ds\right) \end{aligned} \tag{2.53}$$

for all $0 < \epsilon < \xi$ and all $t \in [0, T]$, where

$$G^\epsilon(t) = 2g^\epsilon(t) + \int_0^t 2e^{(t-s)C_5} g^\epsilon(s)\,ds.$$

From (2.53), another use of Gronwall's inequality yields

$$\int_0^T \|e^{-a(t)}(M^\epsilon - \mathbf{m})\|^2_{L^2(\Omega)}\,dt \leq C_{12}\,\eta + \int_0^T e^{(T-t)C_6} G^\epsilon(t)\,dt \tag{2.54}$$

for all $0 < \epsilon < \xi$. Since $\eta > 0$ is arbitrary, using (2.46), (2.40), we have

$$\lim_{\epsilon \to 0} \int_0^T \|e^{-a(t)}(M^\epsilon - \mathbf{m})\|^2_{L^2(\Omega)}\,dt = 0.$$

Hence the sequence $\{e^{-a} M^\epsilon\}$ converges to $e^{-a}\mathbf{m}$ strongly in $L^2(\Omega_T)$ and $L^2((0, T); L^2(\Omega))$. Since $a(x, t)$ is finite almost everywhere and e^{-a} is not zero almost everywhere, for any sequence of $\{M^\epsilon\}$ we can extract a subsequence converging point-wise almost everywhere on

Ω; the limit must be $\mathbf{m}$ from the strong convergence of $\{e^{-a}M^\epsilon\}$. By Lebesgue's dominated convergence theorem, we have that the whole sequence $\{M^\epsilon\}$ converges to $\mathbf{m}$ strongly in $L^2(\Omega_T)$. This and (2.53) imply the convergence: $\|M^\epsilon(t) - \mathbf{m}(t)\|_{L^2} \to 0$ for all $t \in [0,T]$.

On the other hand, the third equation of system (2.8) and (2.12) imply that:

$$\|M^\epsilon(t_1) - M^\epsilon(t_2)\|_{L^2(\Omega)} \leq C_{13}\,|t_1 - t_2| \quad \forall\, t_1,\, t_2 \in [0,T].$$

Thus $\{M^\epsilon\}$ is equi-continuous in $C^0([0,T];L^2(\Omega))$. This, combined with the point-wise convergence of $\{M^\epsilon(t)\}$ to $\mathbf{m}(t)$ for all $t \in [0,T]$, implies that $M^\epsilon \to \mathbf{m}$ in $C^0([0,T];L^2(\Omega))$. The proof is now completed.

2.2 Existence for rough initial data

We finish proving the existence of solutions to the Cauchy problem (2.4).

Theorem 2.2.1. *Let $\mathbf{m}_0 \in L^\infty(\Omega)$ and let E_0, H_0 be any functions satisfying* (2.9). *Then any function $\mathbf{m}$ determined by the convergence* (2.20) *is a weak solution to the Cauchy problem* (2.4).

Proof. By (2.22), $\mathbf{m} \in W^{1,\infty}((0,\infty);L^2(\Omega))$. By Theorem 2.1.3, $M^\epsilon \to \mathbf{m}$ in $C^0([0,T];L^2(\Omega))$; hence $\mathbf{m}(0) = \mathbf{m}_0$ because $M^\epsilon(0) = \mathbf{m}_0$. Also, by the strong convergence of $M^\epsilon \to \mathbf{m}$ and weak convergence of $H^\epsilon \rightharpoonup H^\infty$ in $L^2(\Omega_T)$, we have the weak convergence of $F(x, M^\epsilon, H^\epsilon) \rightharpoonup F(x, \mathbf{m}, H^\infty)$ in $L^2(\Omega_T)$ along the chosen sequence $\epsilon \to 0$. From the equation $\partial_t M^\epsilon = F(x, M^\epsilon, H^\epsilon)$, it follows that $\partial_t \mathbf{m} = F(x, \mathbf{m}, H^\infty)$ in $L^\infty((0,\infty);L^2(\Omega))$ and in the sense of distribution on $\Omega \times (0,\infty)$. Finally, by (2.21), $H^\infty = -P_{\|}(\mathbf{m}\chi_\Omega) = H_{\mathbf{m}}$ and hence $\mathbf{m}$ is a weak solution to (2.4). $\square$

Chapter 3

Local L^2-Stability of Solutions in Finite Time

From Chapter 2, we have proved the existence of solutions with very rough initial value condition. In this chapter, we will continue to prove the stability result for global weak solutions. We also consider initial value problem (2.1)-(2.3) in Chapter 2 as a *quasi-stationary* system:

$$\begin{cases} \partial_t \mathbf{m} = F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}}) \quad \text{in } \Omega \times (0, \infty), \\ \operatorname{curl} H_{\mathbf{m}} = 0, \ \operatorname{div}(H_{\mathbf{m}} + \mathbf{m}\chi_{\Omega}) = 0 \quad \text{in } \mathbf{R}^3 \text{ for all } t \in [0, \infty), \\ \mathbf{m}(x, 0) = \mathbf{m}_0(x) \quad \text{on } \Omega, \end{cases} \tag{3.1}$$

where $F_{\mathbf{a}}(x, \mathbf{m}, H)$, specifying the dependence on applied field $\mathbf{a}$, is the *Landau-Lifshitz interaction* function given by

$$F_{\mathbf{a}}(x, \mathbf{m}, H) = \mathbb{L}(\mathbf{m}, -\varphi'(\mathbf{m}) + \mathbf{a}(x) + H),$$

with $\mathbb{L}(\mathbf{m}, \mathbf{n})$ linear in $\mathbf{n}$ and defined by

$$\mathbb{L}(\mathbf{m}, \mathbf{n}) = \gamma \mathbf{m} \times \mathbf{n} + \gamma\alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{n}), \quad \mathbf{m}, \mathbf{n} \in \mathbf{R}^3. \tag{3.2}$$

In this chapter, we aim to prove the stability of global weak solutions(see Definition in Section 1.2) to *quasi-stationary* system (3.1); we first introduce the important Lemma about how we split nonlocal term $H_{\mathbf{m}}$, then show the main stability result. The last section will be one of applications of stability theorem, which proves the existence of global weak solution to reduced Landau-Lifshitz equation with $\mathbf{a}(x) \in L^{\infty}(\Omega)$ for rough initial data.

3.1 Stability of weak solutions

3.1.1 Decomposition of $H_{\mathbf{m}}$

The following lemma aims to handle the nonlocal term $H_{\mathbf{m}}$. Actually, the stability result will be trivial if we have $H_{\mathbf{m}} \in L^{\infty}(\Omega)$. However, such bound is valid only when $\mathbf{m} \in H^2(\Omega)$ (see Lemma 3.2.1). In our case, we only require initial value $\mathbf{m} \in L^{\infty}(\Omega)$; therefore, $H_{\mathbf{m}}$ can not be bounded in L^{∞}. Following lemma enables us to split $H_{\mathbf{m}}$ into two parts: one is bounded in L^{∞}; the other is bounded in $L^2(\Omega)$ (also see [13, Lemma 5.2] and [31, Lemma 6.2]).

The projection operator used below is the same as Helmholtz Decomposition introduced in Chapter 2, Section 2.1 and it is pretty easy to see that $H = -P_{\|}(\mathbf{m}\chi_{\Omega})$ with this operator.

Lemma 3.1.1. *Let* $\mathbf{m} \in L^{\infty}(\Omega; \mathbf{R}^3)$ *and* $H = -P_{\|}(\mathbf{m}\chi_{\Omega}) = -\tilde{\mathbf{m}}_{\|}$. *Then, for all* $\lambda \geq e$, $H = H^{\lambda} + (H - H^{\lambda})$ *on* $\mathbf{R}^3$, *where* H^{λ} *is a function such that*

$$\|H^{\lambda}\|_{L^{\infty}} \leq C \ln \lambda, \quad \|H - H^{\lambda}\|_{L^2} \leq \frac{C|\Omega|^{1/2}}{\lambda}, \tag{3.3}$$

with constant $C = C_0' \|\mathbf{m}\|_{L^{\infty}}$ *for an absolute constant* C_0'.

Proof. Define H^λ by

$$H^\lambda(x) = \begin{cases} H(x) & |H(x)| \le C \ln \lambda \\ \\ 0 & \text{otherwise} \end{cases}$$

Since $H = -\tilde{\mathbf{m}}_{\|}$, where $\tilde{\mathbf{m}} = \mathbf{m}\chi_\Omega$, we have, for all $p \ge 2$,

$$\begin{aligned} \|H - H^\lambda\|_{L^2}^2 &= \int_{|\tilde{\mathbf{m}}_{\|}| > C \ln \lambda} |\tilde{\mathbf{m}}_{\|}|^2 \, dx \\ &\le \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^2 |\{x : |\tilde{\mathbf{m}}_{\|}| > C \ln \lambda\}|^{\frac{p-2}{p}} \\ &\le \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^2 \frac{1}{(C \ln \lambda)^{p-2}} \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^{p-2} \\ &= \|\tilde{\mathbf{m}}_{\|}\|_{L^p}^p \frac{1}{(C \ln \lambda)^{p-2}}. \end{aligned}$$

However, the boundedness of $P_{\|}$ on L^p yields that, for all $p \ge 2$,

$$\|\tilde{\mathbf{m}}_{\|}\|_{L^p} \le C_0 p \|\tilde{\mathbf{m}}\|_{L^p} \le C_0 p \|\mathbf{m}\|_{L^\infty(\Omega)} |\Omega|^{1/p},$$

where C_0 is an abstract constant independent of p for all $p \ge 2$; see [45]. Hence

$$\|H - H^\lambda\|_{L^2}^2 \le \frac{|\Omega|(C_1 p)^p}{(C \ln \lambda)^{p-2}},$$

where $C_1 = C_0 \|\mathbf{m}\|_{L^\infty}$. We choose $C = 4eC_1$ and $p = 4 \ln \lambda \ge 4$ to obtain

$$\|H - H^\lambda\|_{L^2}^2 \le \frac{|\Omega|(C_1 p)^p}{(C \ln \lambda)^{p-2}} = |\Omega| \frac{(C \ln \lambda)^2}{\lambda^4}$$

and hence

$$\|H - H^\lambda\|_{L^2} \le \frac{C|\Omega|^{1/2} \ln \lambda}{\lambda^2} \le \frac{C|\Omega|^{1/2}}{\lambda},$$

using $\ln\lambda \le \lambda$ for $\lambda \ge e$. This proves (3.3). □

3.1.2 Stability Theorem

We now prove our main stability result, Theorem 1.2.1, which generalizes our previous result [13, Theorem 5.1] to the case of different applied fields $\mathbf{a}(x)$. A similar stability result including the different anisotropy functions $\varphi(\mathbf{m})$ can also be proved; for a similar result on Maxwell systems, see [31, Theorem 6.1].

Assume $\mathbf{m}^k$ $(k = 1, 2)$ is any weak solution to the problem (3.1) with given applied field $\mathbf{a}^k$ and initial datum $\mathbf{m}_0^k$ satisfying

$$\|\mathbf{a}^k\|_{L^\infty} + \|\mathbf{m}_0^k\|_{L^\infty} \le R \quad \text{for } k = 1, 2, \tag{3.4}$$

where $R > 0$ is a given constant. Then, Theorem 1.2.1 will be proved once we prove the following result.

Theorem 3.1.2. *Given any $0 < T < \infty$, there exist constants $C = C(R,T) > 0$, $c = c(R,T) > 0$ and $\rho = \rho(R,T) > 0$ such that, if $\mu = \max\{\|\mathbf{m}_0^1 - \mathbf{m}_0^2\|_{L^2}, \|\mathbf{a}^1 - \mathbf{a}^2\|_{L^2}\} \le c$, then one has, for all $t \in [0,T]$,*

$$\|\mathbf{m}^1(t) - \mathbf{m}^2(t)\|_{L^2(\Omega)} \le C\,\mu^\rho. \tag{3.5}$$

Proof. Step 1. Let $\delta\mathbf{m} = \mathbf{m}^1(t) - \mathbf{m}^2(t)$ and $\delta F = F_{\mathbf{a}^1}(x, \mathbf{m}^1, H_1) - F_{\mathbf{a}^2}(x, \mathbf{m}^2, H_2)$, where $H_k = H_{\mathbf{m}^k}$ for $k = 1, 2$. Then $\partial_t(\delta\mathbf{m}) = \delta F$ and hence

$$\partial_t(\|\delta\mathbf{m}(t)\|_{L^2}) \le \|\partial_t(\delta\mathbf{m}(t))\|_{L^2} = \|\delta F(t)\|_{L^2}.$$

So we have

$$\|\delta\mathbf{m}(t)\|_{L^2} - \|\delta\mathbf{m}_0\|_{L^2} \leq \int_0^t \|\delta F(s)\|_{L^2}\, ds. \tag{3.6}$$

Step 2. The function $\mathbb{L}(\mathbf{m}, \mathbf{n})$ defined by (3.2) above can be written as

$$\mathbb{L}(\mathbf{m}, \mathbf{n}) = \mathbb{B}(\mathbf{m}) \cdot \mathbf{n}, \tag{3.7}$$

where $\mathbb{B}(\mathbf{m})$ is a 3×3-matrix for each $\mathbf{m} \in \mathbf{R}^3$; note that each element of $\mathbb{B}(\mathbf{m})$ is a quadratic function of $\mathbf{m}$. Given any $\mathbf{m}^k, \mathbf{n}^k \in \mathbf{R}^3$ $(k = 1, 2)$, letting $\delta\mathbf{m} = \mathbf{m}^1 - \mathbf{m}^2, \delta\mathbf{n} = \mathbf{n}^1 - \mathbf{n}^2$, by virtue of $\mathbb{L}(\mathbf{m}^1, \mathbf{n}^1) - \mathbb{L}(\mathbf{m}^2, \mathbf{n}^2) = [\mathbb{L}(\mathbf{m}^1, \mathbf{n}^1) - \mathbb{L}(\mathbf{m}^2, \mathbf{n}^1)] + \mathbb{L}(\mathbf{m}^2, \mathbf{n}^1 - \mathbf{n}^2)$, one can write

$$\mathbb{L}(\mathbf{m}^1, \mathbf{n}^1) - \mathbb{L}(\mathbf{m}^2, \mathbf{n}^2) = \mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n}^1) \cdot \delta\mathbf{m} + \mathbb{B}(\mathbf{m}^2) \cdot \delta\mathbf{n}, \tag{3.8}$$

where $\mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n}^1)$ is a matrix function given by

$$\mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{n}^1) = \int_0^1 \frac{\partial \mathbb{L}}{\partial t}(t\mathbf{m}^1 + (1-t)\mathbf{m}^2, \mathbf{n}^1)\, dt. \tag{3.9}$$

Step 3. By Remark 1 in Section 2.1, it follows that $\|\mathbf{m}^k(t)\|_{L^\infty} \leq R$ $(k = 1, 2)$ for all $t \geq 0$. From $F_{\mathbf{a}^k}(x, \mathbf{m}^k, H_k) = -\mathbb{L}(\mathbf{m}^k, \varphi'(\mathbf{m}^k)) + \mathbb{L}(\mathbf{m}^k, \mathbf{a}^k(x)) + \mathbb{L}(\mathbf{m}^k, H_k)$, by (3.4) and (3.8), we obtain the following point-wise estimate for δF:

$$|\delta F| \leq A|\delta H| + B(|H_1| + 1)|\delta\mathbf{m}| + D|\delta\mathbf{a}|, \tag{3.10}$$

where $\delta H = H_1 - H_2 = H_{\delta\mathbf{m}}$, $\delta\mathbf{a} = \mathbf{a}^1(x) - \mathbf{a}^2(x)$, and $A = A(R), B = B(R), D = D(R)$ are constants depending only on R. We apply Lemma 3.1.1 to function $H_1(t) = -P_{\|}(\mathbf{m}^1(t)\chi_\Omega)$. For any $\lambda \geq e$, let $H_1 = H_1^\lambda + (H - H_1^\lambda)$, where H_1^λ is given in Lemma 3.1.1 with constant

$C = C_0' \|\mathbf{m}^1(t)\|_{L^\infty} \le C_0' R$. So, by (3.10), we have the $L^2(\Omega)$-norm estimate:

$$
\begin{aligned}
\|\delta F\|_{L^2} \le & A\|\delta H\|_{L^2} + B(C\ln\lambda + 1)\|\delta\mathbf{m}\|_{L^2} \\
& + B\frac{C|\Omega|^{\frac{1}{2}}}{\lambda}\|\delta\mathbf{m}\|_{L^\infty} + D\|\delta\mathbf{a}\|_{L^2} \\
\le & (A' + B'\ln\lambda)\|\delta\mathbf{m}\|_{L^2} + \frac{C'}{\lambda} + D\|\delta\mathbf{a}\|_{L^2},
\end{aligned} \tag{3.11}
$$

using $\|\delta H\|_{L^2} \le \|H_{\delta\mathbf{m}}\|_{L^2(\mathbf{R}^3)} \le \|\delta\mathbf{m}\|_{L^2}$, where constants A', B', C' depend on R.

Step 4. From (3.6) and (3.11), it follows that

$$
\begin{aligned}
\|\delta\mathbf{m}(t)\|_{L^2} - \|\delta\mathbf{m}_0\|_{L^2} & \le \int_0^t \|\delta F(s)\|_{L^2}\, ds \\
& \le \int_0^t \left((A' + B'\ln\lambda)\|\delta\mathbf{m}(s)\|_{L^2} + \frac{C'}{\lambda} + D\|\mathbf{a}\|_{L^2} \right) ds \\
& = \frac{C't}{\lambda} + \|\delta\mathbf{a}\|_{L^2} Dt + (A' + B'\ln\lambda)\int_0^t \|\delta\mathbf{m}(s)\|_{L^2}\, ds.
\end{aligned}
$$

From this, a Gronwall inequality yields

$$
\begin{aligned}
\|\delta\mathbf{m}(t)\|_{L^2} & \le \left(\|\delta\mathbf{m}_0\|_{L^2} + \frac{C't}{\lambda} + \|\delta\mathbf{a}\|_{L^2} Dt \right) e^{A't + B't\ln\lambda} \\
& \le \left(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} + \frac{C't}{\lambda} \right) e^{A't}\lambda^{B't} \quad \forall\ 0 \le t \le T.
\end{aligned} \tag{3.12}
$$

Step 5. We consider two cases.

Case 1. Assume both $\delta\mathbf{m}_0 = 0$ and $\delta\mathbf{a} = 0$. Then, by (3.12),

$$
\|\delta\mathbf{m}(t)\|_{L^2} \le C' t e^{A't}\lambda^{B't - 1}. \tag{3.13}
$$

Let $t_0 = \frac{1}{B'+1}$. If $0 \le t \le t_0$, then $B't - 1 < 0$ and hence, by (3.13) with $\lambda \to \infty$, we have

$\delta\mathbf{m}(t) = 0$ for all $t \in [0, t_0]$. With $\mathbf{m}^k(t_0)$ as initial datum at time t_0, we obtain $\delta\mathbf{m}(t) = 0$ on $[t_0, 2t_0]$; eventually, we have $\delta\mathbf{m}(t) = 0$ for all $t \geq 0$; hence (3.5) holds. This also shows the uniqueness of the weak solution to the system.

Case 2. Assume $0 < \|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq 1/e < 1$. In this case, setting $\lambda = (\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{-1} \geq e$ in (3.12), we obtain

$$\|\delta\mathbf{m}(t)\|_{L^2} \leq (1 + C't)e^{A't}(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{1-B't}.$$

Let $t_1 = \frac{1}{2(B'+1)}$ and $C_1 = (1 + C't_1)e^{A't_1} > 1$. Then $1 - B't \geq \frac{1}{2}$ for all $0 \leq t \leq t_1$; hence

$$\|\delta\mathbf{m}(t)\|_{L^2} \leq C_1(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}} \quad \forall\, 0 \leq t \leq t_1.$$

Adding $DT\|\delta\mathbf{a}\|_{L^2}$ to both sides, we obtain

$$\|\delta\mathbf{m}(t)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq C_2\,(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}} \quad \forall\, 0 \leq t \leq t_1, \tag{3.14}$$

where $C_2 = C_1 + 1$ depends only on R.

Step 6. Combining Cases 1 and 2 in Step 5 above, with the constants $t_1 = t_1(R)$ and $C_2 = C_2(R) > 1$ determined above, we have that, if $\|\delta\mathbf{m}_0\|_{L^2} + DT\,\|\delta\mathbf{a}\|_{L^2} \leq 1/e$, then

$$\|\delta\mathbf{m}(t)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq C_2\,(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}} \quad \forall\, 0 \leq t \leq t_1. \tag{3.15}$$

Assume

$$C_2(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}} \leq 1/e. \tag{3.16}$$

Then, by (3.15), $\|\delta\mathbf{m}(t_1)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq 1/e$. With $\mathbf{m}^k(t_1)$ as initial datum at time t_1, we apply (3.15) again to obtain

$$\begin{aligned}\|\delta\mathbf{m}(t_1+t)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq& C_2(\|\delta\mathbf{m}(t_1)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2}} \\ \leq& C_2^{1+\frac{1}{2}}(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{4}} \quad \forall\ 0 \leq t \leq t_1.\end{aligned}$$

We have thus proved that, if (3.16) holds then

$$\|\delta\mathbf{m}(t)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq C_2^{1+\frac{1}{2}}(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{4}} \quad \forall\ 0 \leq t \leq 2t_1.$$

By induction, we obtain that, for $k = 1, 2, \cdots$, if

$$C_2^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}}(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2^k}} \leq 1/e, \tag{3.17}$$

then

$$\begin{aligned}\|\delta\mathbf{m}(t)\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} &\leq C_2^{1+\frac{1}{2}+\cdots+\frac{1}{2^k}}(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2^{k+1}}} \\ &\leq C_2^2(\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2})^{\frac{1}{2^{k+1}}} \quad \forall\ 0 \leq t \leq 2^k t_1. \end{aligned} \tag{3.18}$$

Step 7. In this step, we complete the proof of the theorem. Let k be the integer such that $2^{k-1}t_1 < T \leq 2^k t_1$. Define

$$\rho = \rho(R, T) = 1/(2^{k+1}), \quad c = c(R, T) = (C_2^2 e)^{-2^k}/(1 + DT).$$

Assume $\mu = \max\{\|\delta\mathbf{m}_0\|_{L^2}, \|\delta\mathbf{a}\|_{L^2}\} \leq c$. Then

$$\|\delta\mathbf{m}_0\|_{L^2} + DT\|\delta\mathbf{a}\|_{L^2} \leq (1 + DT)\mu \leq (C_2^2 e)^{-2^k},$$

from which it is easily seen that (3.17) holds; so, by (3.18),

$$\|\delta \mathbf{m}(t)\|_{L^2} + DT\|\delta \mathbf{a}\|_{L^2} \le C_2^2(\|\delta \mathbf{m}_0\|_{L^2} + DT\|\delta \mathbf{a}\|_{L^2})^{\rho} \quad \forall\, 0 \le t \le T.$$

Therefore,

$$\|\delta \mathbf{m}(t)\|_{L^2} \le C_2^2\,(1+DT)^{\rho}\,\mu^{\rho} \quad \forall\, t \in [0,T];$$

this proves (3.5) with constant $C = C_2^2(1+DT)^{\rho}$. □

3.2 Existence of global weak solutions

In this section, we present a proof for the existence of global weak solution to (3.1) based on the stability theorem proved above. To this end, we introduce a nonlinear function

$$\mathbf{f}(\mathbf{m}) = F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}}) = -\mathbb{L}(\mathbf{m}, \varphi'(\mathbf{m})) + \mathbb{L}(\mathbf{m}, \mathbf{a}(x)) + \mathbb{L}(\mathbf{m}, H_{\mathbf{m}}) \tag{3.19}$$

for $\mathbf{m} \in L^{\infty}(\Omega; \mathbf{R}^3)$, where $H_{\mathbf{m}}$ is defined by Maxwell equation in (3.1) and $\mathbb{L}$ is defined by (3.2). As before, we always assume the anisotropy function $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is smooth.

3.2.1 Properties of f for smooth applied fields

In this subsection, we assume the applied field $\mathbf{a}$ belongs to $C^{\infty}(\bar{\Omega}; \mathbf{R}^3)$ and show that, in this case, $\mathbf{f}\colon H^2(\Omega; \mathbf{R}^3) \to H^2(\Omega; \mathbf{R}^3)$ and is locally Lipschitz. We need some estimates.

Lemma 3.2.1. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Then the following* *estimates hold on* $H^2(\Omega;\mathbf{R}^3)$:

$$\|\mathbf{m}\|_{L^\infty(\Omega)} + \|\mathbf{m}\|_{W^{1,p}(\Omega)} \le C_0 \|\mathbf{m}\|_{H^2(\Omega)} \quad \forall\ 1 \le p \le 6, \tag{3.20}$$

$$\|H_{\mathbf{m}}\|_{H^2(\Omega)} \le C_1 \|\mathbf{m}\|_{H^2(\Omega)}. \tag{3.21}$$

Proof. We omit the proof, but only mention that (3.20) is a simple consequence of the well-known embeddings: $H^2(\Omega) \subset W^{1,6}(\Omega) \subset C^{\frac{1}{2}}(\bar{\Omega}) \subset L^\infty(\Omega)$ for bounded smooth domain $\Omega \subset \mathbf{R}^3$, and that estimate (3.21) has been, e.g., proved in [8]. Finally, we remark that, from (3.20) and (3.21), it follows that, with constant $C_2 = C_0 C_1$,

$$\|H_{\mathbf{m}}\|_{L^\infty(\Omega)} \le C_2 \|\mathbf{m}\|_{H^2(\Omega)} \quad \forall\ \mathbf{m} \in H^2(\Omega;\mathbf{R}^3). \tag{3.22}$$

□

The main result of the subsection is the following local Lipschitz property of $\mathbf{f}$ on $H^2(\Omega;\mathbf{R}^3)$.

Proposition 3.2.2. *Function* $\mathbf{f}$ *maps space* $H^2(\Omega;\mathbf{R}^3)$ *into itself and is locally Lipschitz on* $H^2(\Omega;\mathbf{R}^3)$.

Proof. Since $\mathbf{f}(0) = 0$, the self-mapping property of $\mathbf{f}$ will follow from the local Lipschitz property of $\mathbf{f}$ on $H^2(\Omega;\mathbf{R}^3)$.

To prove the local Lipschitz property of $\mathbf{f}$, given any two functions $\mathbf{m}^1, \mathbf{m}^2 \in H^2(\Omega;\mathbf{R}^3)$ satisfying

$$\max\{\|\mathbf{m}^1\|_{H^2(\Omega)}, \|\mathbf{m}^2\|_{H^2(\Omega)}\} \le R, \tag{3.23}$$

where $R < \infty$ is a constant, we need to show that

$$\|\mathbf{f}(\mathbf{m}^1) - \mathbf{f}(\mathbf{m}^2)\|_{H^2(\Omega)} \leq L \,\|\mathbf{m}^1 - \mathbf{m}^2\|_{H^2(\Omega)} \tag{3.24}$$

for a (local Lipschitz) constant $L = L(R) < \infty$ depending on R.

By (3.19), we write $\mathbf{f}(\mathbf{m}^1) - \mathbf{f}(\mathbf{m}^2) = I_1 + I_2$, where

$$I_1 = \mathbb{L}(\mathbf{m}^1, \mathbf{a} - \varphi'(\mathbf{m}^1)) - \mathbb{L}(\mathbf{m}^2, \mathbf{a} - \varphi'(\mathbf{m}^2))$$

and $I_2 = \mathbb{L}(\mathbf{m}^1, H_{\mathbf{m}^1}) - \mathbb{L}(\mathbf{m}^2, H_{\mathbf{m}^2})$. Let $\delta\mathbf{m} = \mathbf{m}^1 - \mathbf{m}^2$. Then, by (3.8),

$$I_2 = \mathbb{A}(\mathbf{m}^1, \mathbf{m}^2, H_{\mathbf{m}^1}) \cdot \delta\mathbf{m} + \mathbb{B}(\mathbf{m}^2) \cdot H_{\delta\mathbf{m}}, \tag{3.25}$$

where $\mathbb{A}, \mathbb{B}$ are functions defined in *Step 2* of the proof of Theorem 3.1.2 above. We also write I_1 as

$$I_1 = \int_0^1 \frac{d}{dt}\mathbb{L}(\mathbf{m}^2 + t\delta\mathbf{m}, \mathbf{a} - \varphi'(\mathbf{m}^2 + t\delta\mathbf{m}))\,dt = \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot \delta\mathbf{m}, \tag{3.26}$$

where $\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})$ is certain smooth function of $(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$. Note that $\mathbb{C}$ is linear in $\mathbf{a}$. We aim to show

$$\|I_k\|_{H^2(\Omega)} \leq L(R)\,\|\delta\mathbf{m}\|_{H^2(\Omega)} \quad (k = 1, 2)$$

for some constant $L(R)$ depending on R. By (3.23), (3.22) and Lemma 3.2.1, it follows that, for $k = 1, 2$,

$$\|\mathbf{m}^k\|_{L^\infty(\Omega)} + \|H_{\mathbf{m}^k}\|_{L^\infty(\Omega)} + \|H_{\mathbf{m}^k}\|_{H^2(\Omega)} + \|\nabla \mathbf{m}^k\|_{L^4(\Omega)} \leq C_3 R. \tag{3.27}$$

We proceed with several steps.

Step 1: **Estimation of I_1.** Clearly, by (3.26) and (3.27),

$$\|I_1\|_{L^2(\Omega)} \leq \|\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})\|_{L^\infty} \|\delta \mathbf{m}\|_{L^2} \leq L(R) \, \|\delta \mathbf{m}\|_{L^2(\Omega)}.$$

We estimate the H^2-norm. Denote by ∂_j the first partial derivative with respect to x_j and by ∂^2_{ij} the second partial derivative with respect to x_j and x_i $(i, j = 1, 2, 3)$. Note that

$$\partial_j(I_1) = \partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot \delta \mathbf{m} + \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot (\delta \mathbf{m})_{x_j}$$

and

$$\begin{aligned}\partial^2_{ij}(I_1) = \partial^2_{ij}(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot \delta \mathbf{m} + \partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta \mathbf{m})_{x_i} \\ + \partial_i(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) \cdot (\delta \mathbf{m})_{x_j} + \mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}) \cdot (\delta \mathbf{m})_{x_i x_j}.\end{aligned}$$

Since $\partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})) = (\partial_{\mathbf{m}^1}\mathbb{C}) \cdot \mathbf{m}^1_{x_j} + (\partial_{\mathbf{m}^2}\mathbb{C}) \cdot \mathbf{m}^2_{x_j} + (\partial_{\mathbf{a}}\mathbb{C}) \cdot \mathbf{a}_{x_j}$ has L^2-norm controlled by R, we have

$$\begin{aligned}\|\partial_j(I_1)\|_{L^2} &\leq \|\partial_j(\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a}))\|_{L^2} \|\delta \mathbf{m}\|_{L^\infty} + \|\mathbb{C}(\mathbf{m}^1, \mathbf{m}^2, \mathbf{a})\|_{L^\infty} \|(\delta \mathbf{m})_{x_j}\|_{L^2} \\ &\leq L(R) \, \|\delta \mathbf{m}\|_{H^2(\Omega)}.\end{aligned}$$

Similarly, $\partial^2_{ij}(\mathbb{C}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{a}))$ contains terms up to second derivatives of $\mathbf{a}$ and terms like $(\partial^2_{\mathbf{m}^p\mathbf{m}^q}\mathbb{C})\cdot\mathbf{m}^k_{x_i}\cdot\mathbf{m}^l_{x_j}$ and $(\partial_{\mathbf{m}^p}\mathbb{C})\cdot\mathbf{m}^q_{x_{i'}x_{j'}}$, with certain choices of $p,q,k,l\in\{1,2\}$ and $i',j'\in\{i,j\}$. Hence $\|\partial^2_{ij}(\mathbb{C}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{a}))\|_{L^2}$ is bounded by the quantity

$$C(R)\left(\||\nabla\mathbf{m}^1|^2\|_{L^2}+\||\nabla\mathbf{m}^2|^2\|_{L^2}+\|\nabla^2\mathbf{m}^1\|_{L^2}+\|\nabla^2\mathbf{m}^2\|_{L^2}+\|\mathbf{a}\|_{H^2}\right),$$

which, due to $\||\nabla\mathbf{m}|^2\|_{L^2}=\|\nabla\mathbf{m}\|^2_{L^4}\le C\,\|\mathbf{m}\|^2_{H^2}$, is in fact bounded by another constant $C(R)$. From this, similar to the term $\partial_j(I_1)$, the L^2-norm of the first or fourth term of $\partial^2_{ij}(I_1)$ is bounded by $L(R)\|\delta\mathbf{m}\|_{H^2(\Omega)}$. The second and third terms of $\partial^2_{ij}(I_1)$ can be estimated as follows:

$$\begin{aligned}
&\|\partial_j(\mathbb{C}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{a}))\cdot(\delta\mathbf{m})_{x_i}+\partial_i(\mathbb{C}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{a}))\cdot(\delta\mathbf{m})_{x_j}\|_{L^2}\\
&\le 2\|\nabla(\mathbb{C}(\mathbf{m}^1,\mathbf{m}^2,\mathbf{a}))\|_{L^4}\cdot\|\nabla(\delta\mathbf{m})\|_{L^4}\\
&\le C(R)\left(\|\mathbf{m}^1\|_{W^{1,4}}+\|\mathbf{m}^2\|_{W^{1,4}}+\|\nabla\mathbf{a}\|_{L^4}\right)\cdot\|\delta\mathbf{m}\|_{W^{1,4}}\\
&\le L(R)\,\|\delta\mathbf{m}\|_{H^2(\Omega)}.
\end{aligned}$$

This proves $\|I_1\|_{H^2(\Omega)}\le L(R)\|\delta\mathbf{m}\|_{H^2(\Omega)}$.

Step 2: **Estimation of I_2.** We write $I_2=I_{21}+I_{22}$ with

$$I_{21}=\mathbb{A}(\mathbf{m}^1,\mathbf{m}^2,H_{\mathbf{m}^1})\cdot\delta\mathbf{m},\quad I_{22}=\mathbb{B}(\mathbf{m}^2)\cdot H_{\delta\mathbf{m}}.$$

The term I_{21} is more like term I_1, except the constant field $\mathbf{a}$ is replaced by the fields $H_{\mathbf{m}^1}$. Since $H_{\mathbf{m}^1}\in H^2(\Omega)$ and $\|H_{\mathbf{m}^1}\|_{L^\infty}+\|H_{\mathbf{m}^1}\|_{H^2(\Omega)}\le C_1\|\mathbf{m}^1\|_{H^2(\Omega)}\le C_1R$, estimation resulting from $H_{\mathbf{m}^1}$ in $\mathbb{A}$ can be handled in a much similar way as the term $\mathbf{a}$ in $\mathbb{C}$ of I_1.

The term I_{22} is simpler but slightly different than I_1 in that $H_{\delta\mathbf{m}}$ is in place of $\delta\mathbf{m}$. Nevertheless this term can also be estimated in a similar fashion as I_1, using the following estimate on $H_{\delta\mathbf{m}}$:

$$\|H_{\delta\mathbf{m}}\|_{L^\infty(\Omega)} + \|\nabla H_{\delta\mathbf{m}}\|_{L^4(\Omega)} + \|H_{\delta\mathbf{m}}\|_{H^2(\Omega)} \leq C_5 \|\delta\mathbf{m}\|_{H^2(\Omega)}.$$

We eventually obtain $\|I_2\|_{H^2(\Omega)} \leq L(R)\|\mathbf{m}\|_{H^2(\Omega)}$. This completes the proof. $\square$

3.2.2 Existence of global solution for smooth data

We continue to assume $\mathbf{a} \in C^\infty(\bar{\Omega}; \mathbf{R}^3)$ in this subsection. Let $X = H^2(\Omega; \mathbf{R}^3)$. With function $\mathbf{f}\colon X \to X$ defined above, we formulate the problem (1.7) as an abstract ODE on X by

$$\begin{cases} \frac{d\mathbf{m}}{dt} = \mathbf{f}(\mathbf{m}), \\ \mathbf{m}(0) = \mathbf{m}_0. \end{cases} \tag{3.28}$$

A solution $\mathbf{m}$ to (3.28) on $[0,T]$ is a function $\mathbf{m} \in C([0,T]; X)$ that satisfies

$$\mathbf{m}(t) = \mathbf{m}_0 + \int_0^t \mathbf{f}(\mathbf{m}(s))\, ds \quad \forall\ 0 \leq t \leq T.$$

We say $\mathbf{m}$ is a solution to (3.28) on $[0,T)$ if $\mathbf{m}$ is a solution on $[0,T']$ for all $0 < T' < T$ (in this case T could be ∞).

Theorem 3.2.3. *Given any $\mathbf{m}_0 \in X$,* (3.28) *has a unique solution $\mathbf{m}$ on $[0,\infty)$. This solution is also a global weak solution to problem* (3.1)*.*

Proof. Given $\mathbf{m}_0 \in X$, since $\mathbf{f}$ is locally Lipschitz on X, from the abstract theory, there exists $T > 0$ such that (3.28) has a unique solution $\mathbf{m}$ on $[0, T]$. Let

$$T_* = \sup \left\{ T > 0 \;\middle|\; \text{(3.28) has a unique solution on } [0, T] \right\}.$$

We claim that $T_* = \infty$, which implies that (3.28) has a unique global solution $\mathbf{m}$ defined on $[0, \infty)$. Clearly, this solution is also a global weak solution to the Cauchy problem (1.7) above.

Suppose $T_* < \infty$. Then, by the elementary ODE theory, a solution $\mathbf{m}$ to (3.28) would exist on $[0, T_*)$ and satisfy

$$\lim_{t \to T_*^-} \|\mathbf{m}(t)\|_X = \infty.$$

The following asserts that this finite-time blowup is impossible; this completes the proof of Theorem 3.2.3. □

Theorem 3.2.4. *Given any $T > 0$, if $\mathbf{m}$ is a solution to (3.28) on $[0, T)$, then*

$$\sup_{t \in [0, T)} \|\mathbf{m}(t)\|_X \leq C_{T, \|\mathbf{m}_0\|_X} < \infty. \tag{3.29}$$

The proof of this theorem involves lots of technical estimates and will be delayed to the next individual section.

3.2.3 Existence of global weak solution for rough data

In this subsection, we assume both applied field $\mathbf{a}$ and initial datum $\mathbf{m}_0$ are in $L^\infty(\Omega; \mathbf{R}^3)$.

Let $\mathbf{a}^\epsilon, \mathbf{m}_0^\epsilon \in C^\infty(\bar{\Omega}; \mathbf{R}^3)$ be such that

$$\|\mathbf{a}^\epsilon\|_{L^\infty} + \|\mathbf{m}_0^\epsilon\|_{L^\infty} \le R \quad \forall\, \epsilon > 0, \tag{3.30}$$

$$\lim_{\epsilon \to 0^+} (\|\mathbf{a}^\epsilon - \mathbf{a}\|_{L^2} + \|\mathbf{m}_0^\epsilon - \mathbf{m}_0\|_{L^2}) = 0, \tag{3.31}$$

$$\mathbf{a}^\epsilon \to \mathbf{a},\ \mathbf{m}_0^\epsilon \to \mathbf{m}_0 \quad \text{pointwise in } \Omega. \tag{3.32}$$

Consider the Cauchy problem (3.1) with applied field $\mathbf{a}^\epsilon$ and initial datum $\mathbf{m}_0^\epsilon$. Then, by Theorem 3.2.3, for each $\epsilon > 0$, (3.1) has a global weak solution $\mathbf{m}^\epsilon$. Since $\mathbf{m}^\epsilon \cdot \mathbf{f}(\mathbf{m}^\epsilon) = 0$, it follows that $\partial_t(|\mathbf{m}^\epsilon(x,t)|^2) = 0$ and hence $|\mathbf{m}^\epsilon(x,t)| = |\mathbf{m}_0^\epsilon(x)|$ for a.e. $x \in \Omega$ and all $t > 0$. This implies

$$\|\mathbf{m}^\epsilon(t)\|_{L^\infty} = \|\mathbf{m}_0^\epsilon\|_{L^\infty} \le R. \tag{3.33}$$

For each $n \in \{1, 2, 3, \cdots\}$, our stability result (Theorems 1.2.1 and 3.1.2) implies that sequence $\{\mathbf{m}^\epsilon\}$ is Cauchy in Banach space $C([0,n]; L^2(\Omega; \mathbf{R}^3))$ as $\epsilon \to 0^+$. Therefore, $\mathbf{m}^\epsilon \to \mathbf{m}$ in $C([0,n]; L^2(\Omega; \mathbf{R}^3))$ as $\epsilon \to 0^+$ for some $\mathbf{m} \in C([0,n]; L^2(\Omega; \mathbf{R}^3))$. (Presumably, $\mathbf{m} = \mathbf{m}_n$ depends on n.) Hence, by (3.31),

$$\mathbf{m}(0) = \mathbf{m}_0. \tag{3.34}$$

We also have $H_{\mathbf{m}^\epsilon} \to H_{\mathbf{m}}$ in $C([0,n]; L^2(\Omega; \mathbf{R}^3))$. It follows that $\mathbf{m}^\epsilon \to \mathbf{m}$ and $H_{\mathbf{m}^\epsilon} \to H_{\mathbf{m}}$ also in $L^2(\Omega \times (0,n))$ as $\epsilon \to 0^+$. Using a subsequence, we can assume

$$\mathbf{m}^\epsilon(x,t) \to \mathbf{m}(x,t), \quad H_{\mathbf{m}^\epsilon}(x,t) \to H_{\mathbf{m}}(x,t) \quad \text{pointwise in } \Omega \times (0,n).$$

Therefore, $F_{\mathbf{a}^\epsilon}(x, \mathbf{m}^\epsilon, H_{\mathbf{m}^\epsilon}) \to F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}})$ pointwise in $\Omega \times (0,n)$. This shows $\partial_t \mathbf{m} =$

$F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}})$ in the sense of distribution on $\Omega \times (0, n)$.

Note also that $F_{\mathbf{a}^\epsilon}(x, \mathbf{m}^\epsilon, H_{\mathbf{m}^\epsilon}) \in L^2(\Omega; \mathbf{R}^3)$ uniformly on ϵ and $t \in (0, n)$; this implies that $\partial_t \mathbf{m} = F_{\mathbf{a}}(x, \mathbf{m}, H_{\mathbf{m}})$ holds in $L^\infty((0, n); L^2(\Omega))$ and that $\mathbf{m} \in W^{1,\infty}([0, n); L^2(\Omega; \mathbf{R}^3))$. Combining with (3.34), we have proved that $\mathbf{m} = \mathbf{m}_n$ is a weak solution to (3.1) on $\Omega \times (0, n)$. By the uniqueness of weak solutions, we have $\mathbf{m}_{n+1} = \mathbf{m}_n$ on $\Omega \times (0, n)$; therefore, the sequence $\{\mathbf{m}_n\}_1^\infty$ defines a unique function $\mathbf{m}$ by setting $\mathbf{m}(x, t) = \mathbf{m}_n(x, t)$ with $n = [t] + 1$. It is easy to see that $\mathbf{m}$ is a global weak solution to (3.1).

Finally, we have proved the following theorem.

Theorem 3.2.5. *Let* $\mathbf{a} \in L^\infty(\Omega; \mathbf{R}^3)$. *Given any initial datum* $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$, *the problem* (3.1) *has a unique global weak solution.*

3.2.4 Proof of Theorem 3.2.4

In this separate section, we give the proof of Theorem 3.2.4. This involves the special form of function $\mathbb{L}(\mathbf{m}, \mathbf{n})$ and several estimates.

In what follows, assume $\mathbf{a} \in C^\infty(\bar{\Omega}; \mathbf{R}^3)$, $0 < T < \infty$ and $\mathbf{m}$ is a solution to (3.28) on $[0, T)$ with initial datum $\mathbf{m}_0 \in H^2(\Omega; \mathbf{R}^3)$. Assume

$$\|\mathbf{m}_0\|_{L^\infty(\Omega)} = R > 0.$$

Then, similar to (3.33) above, we have

$$\|\mathbf{m}(t)\|_{L^\infty} = \|\mathbf{m}_0\|_{L^\infty} = R, \ \|\mathbf{m}(t)\|_{L^2} = \|\mathbf{m}_0\|_{L^2} \leq R|\Omega|^{\frac{1}{2}} \quad \forall\, 0 \leq t < T. \tag{3.35}$$

We would like to show

$$\sup_{t\in[0,T)} \|\mathbf{m}(t)\|_{H^2(\Omega)} \leq C_{T,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{3.36}$$

Let

$$y(t) = 1 + \|\mathbf{m}(t)\|^2_{H^2(\Omega)} = 1 + \|\mathbf{m}(t)\|^2_{L^2} + \|\nabla\mathbf{m}(t)\|^2_{L^2} + \|\nabla^2\mathbf{m}(t)\|^2_{L^2}.$$

The goal is to show

$$y'(t) \leq Cy(t)(1 + \ln y(t)) \quad \forall\ 0 < t < T, \tag{3.37}$$

where $C = C(R)$ is a constant depending on R. Once (3.37) is proved, one easily obtains that

$$\ln(y(t)) \leq (\ln(y(0)) + 1)\, e^{CT} < \infty \quad \forall\ t \in [0, T),$$

from which (3.36) follows.

The rest of the section is devoted to proving (3.37).

Energy estimates

It is convenient to use the special structure of function $\mathbb{L}$ to write function $\mathbf{f}(\mathbf{m})$ as follows:

$$\mathbf{f}(\mathbf{m}) = \mathbb{B}(\mathbf{m}) \cdot \mathbf{a} - \mathbb{B}(\mathbf{m}) \cdot \varphi'(\mathbf{m}) + \mathbb{B}(\mathbf{m}) \cdot H_{\mathbf{m}},$$

where $\mathbb{B}(\mathbf{m})$ is a 3×3 matrix defined in (3.7) above, whose elements are quadratic functions of $\mathbf{m}$; hence $\mathbb{B}''(\mathbf{m}) = \mathbb{D}$ is a constant tensor. However, this special structure of $\mathbb{B}$ is not used; in fact, the following arguments are valid for arbitrary smooth functions $\mathbb{B}$.

Differentiating equation in (3.28) with respect to x_i yields

$$\begin{aligned}\frac{d\mathbf{m}_{x_i}}{dt} &= \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot \mathbf{a} + \mathbb{B}(\mathbf{m}) \cdot \mathbf{a}_{x_i} \\ &\quad - \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot \varphi'(\mathbf{m}) - \mathbb{B}(\mathbf{m}) \cdot \varphi''(\mathbf{m}) \cdot \mathbf{m}_{x_i} \\ &\quad + \mathbb{B}'(\mathbf{m}) \cdot \mathbf{m}_{x_i} \cdot H_{\mathbf{m}} + \mathbb{B}(\mathbf{m}) \cdot (H_{\mathbf{m}})_{x_i}.\end{aligned} \tag{3.38}$$

Again, differentiating equation (3.38) with respect to x_j yields

$$\begin{aligned}\frac{d\mathbf{m}_{x_i x_j}}{dt} &= \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot \mathbf{a} + \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot \mathbf{a} + \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot \mathbf{a}_{x_j} \\ &\quad + \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot \mathbf{a}_{x_i} + \mathbb{B} \cdot \mathbf{a}_{x_i x_j} \\ &\quad - \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot \varphi' - \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot \varphi' + \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot \varphi'' \cdot \mathbf{m}_{x_j} \\ &\quad - \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot \varphi'' \cdot \mathbf{m}_{x_i} - \mathbb{B} \cdot \varphi''' \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} - \mathbb{B} \cdot \varphi'' \cdot \mathbf{m}_{x_i x_j} \\ &\quad + \mathbb{D} \cdot \mathbf{m}_{x_j} \cdot \mathbf{m}_{x_i} \cdot H_{\mathbf{m}} + \mathbb{B}' \cdot \mathbf{m}_{x_i x_j} \cdot H_{\mathbf{m}} + \mathbb{B}' \cdot \mathbf{m}_{x_i} \cdot (H_{\mathbf{m}})_{x_j} \\ &\quad + \mathbb{B}' \cdot \mathbf{m}_{x_j} \cdot (H_{\mathbf{m}})_{x_i} + \mathbb{B} \cdot (H_{\mathbf{m}})_{x_i x_j}.\end{aligned} \tag{3.39}$$

Dot-product of (3.38) with $\mathbf{m}_{x_i}$ and of (3.39) with $\mathbf{m}_{x_i x_j}$ yields the following identities:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{m}_{x_i}\|_{L^2}^2\right) = \int_\Omega \left(\mathbf{m}_{x_i} \cdot \frac{d\mathbf{m}_{x_i}}{dt}\right) dx, \tag{3.40}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{m}_{x_i x_j}\|_{L^2}^2\right) = \int_\Omega \left(\mathbf{m}_{x_i x_j} \cdot \frac{d\mathbf{m}_{x_i x_j}}{dt}\right) dx. \tag{3.41}$$

The energy estimates involve estimating the right-hand sides of (3.40) and (3.41) with terms $\frac{d\mathbf{m}_{x_i}}{dt}$, $\frac{d\mathbf{m}_{x_i x_j}}{dt}$ given by the right-hand sides of (3.38) and (3.39).

More subtle inequalities

To handle the terms involved in the integrals on the right-hand sides of (3.40) and (3.41), more subtle inequalities are needed.

Lemma 3.2.6. *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary. Then*

$$\begin{aligned} &\|\nabla \mathbf{n}\|_{L^4} \leq C_6 \, \|\mathbf{n}\|_{L^\infty}^{\frac{1}{2}} \|\mathbf{n}\|_{H^2}^{\frac{1}{2}}, \\ &\|H_{\mathbf{n}}\|_{L^\infty(\Omega)} \leq C_{\|\mathbf{n}\|_{L^\infty}} \, (1 + \ln^+(\|\mathbf{n}\|_{H^2})), \end{aligned} \qquad \forall \ \mathbf{n} \in H^2(\Omega; \mathbf{R}^3), \tag{3.42}$$

where $\ln^+ t = \max\{\ln t, 0\}$ *for* $t > 0$ *and* $C_{\|\mathbf{n}\|_{L^\infty}} < \infty$ *depends on* $\|\mathbf{n}\|_{L^\infty(\Omega)}$.

Proof. The first inequality of (3.42) is a consequence of the well-known Gagliardo-Nirenberg inequality:

$$\|\nabla^j f\|_{L^q(\mathbf{R}^n)} \leq C \, \|f\|_{L^r(\mathbf{R}^n)}^{1-\theta} \|\nabla^l f\|_{L^p(\mathbf{R}^n)}^{\theta},$$

where $\theta = j/l \in (0, 1)$ and $1/q = \theta/p + (1-\theta)/r$, $1 \leq p, r \leq \infty$. Here $j = 1, l = 2, p = 2, q = 4, r = \infty$ and $\theta = 1/2$. While the second inequality of (3.42) is a Judovic-type inequality proved, e.g., in [31, Lemma 7.2]. □

The following result is an immediate consequence of this lemma and (3.35).

Proposition 3.2.7. *For the solution* $\mathbf{m}(t)$*, with* $y(t)$ *defined above, it follows that*

$$\begin{aligned} &\|\nabla \mathbf{m}(t)\|_{L^4(\Omega)}^4 \leq C_7 \, y(t), \\ &\|H_{\mathbf{m}}(t)\|_{L^\infty(\Omega)} \leq C_8 \, (1 + \ln y(t)), \end{aligned} \qquad \forall \ 0 \leq t < T, \tag{3.43}$$

where C_7, C_8 *are constants depending on* $R = \|\mathbf{m}_0\|_{L^\infty}$.

Energy estimates (continued) and proof of (3.37)

First of all, the integral on right-hand side of (3.40) is bounded by

$$C(R)\int_\Omega \Big(|\nabla \mathbf{m}|^2 + |\nabla \mathbf{m}| + |\nabla \mathbf{m}|^2|H_\mathbf{m}| + |\nabla \mathbf{m}||\nabla(H_\mathbf{m})|\Big)\, dx.$$

The third term is bounded by $C(R)\|H_\mathbf{m}\|_{L^\infty(\Omega)}\|\nabla \mathbf{m}\|_{L^2}^2$ and hence, by (3.43b), is bounded by $C(R)y(t)(1+\ln y(t))$, while all the other terms are bounded by $C(R)\|\mathbf{m}\|_{H^2}^2$ and hence by $C(R)y(t)$. Therefore,

$$\frac{d}{dt}\Big(\|\mathbf{m}_{x_i}\|_{L^2}^2\Big) \le C(R)y(t)(1+\ln y(t)), \quad \forall\, 0<t<T. \tag{3.44}$$

Similarly, the integrand of the right-hand side of (3.41) is bounded by a constant $C(R)$ times

$$|\nabla \mathbf{m}|^2|\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}|^2 + |\nabla \mathbf{m}||\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}| + |\nabla^2 \mathbf{m}||\nabla^2(H_\mathbf{m})|$$

$$+|\nabla \mathbf{m}|^2|H_\mathbf{m}||\nabla^2 \mathbf{m}| + |H_\mathbf{m}||\nabla^2 \mathbf{m}|^2 + |\nabla \mathbf{m}||\nabla^2 \mathbf{m}||\nabla(H_\mathbf{m})|.$$

Integrals of terms in the first group can all be bounded by $Cy(t)$. Integrals of the first two terms in the second group can be bounded by constant times

$$\|H_\mathbf{m}\|_{L^\infty(\Omega)}(\|\nabla \mathbf{m}\|_{L^4}^4 + \|\nabla^2 \mathbf{m}\|_{L^2}^2),$$

which, by (3.43a-b), is bounded by $Cy(t)(1+\ln y(t))$. Finally, the integral of the last term in the second group can be estimated as follows:

$$\int_\Omega |\nabla \mathbf{m}||\nabla^2 \mathbf{m}||\nabla(H_\mathbf{m})| dx \le \||\nabla \mathbf{m}|\cdot|\nabla(H_\mathbf{m})|\|_{L^2(\Omega)}\||\nabla^2 \mathbf{m}|\|_{L^2(\Omega)}$$

$$\leq \|\nabla \mathbf{m}\|_{L^4(\Omega)} \|\nabla (H_{\mathbf{m}})\|_{L^4(\Omega)} \||\nabla^2 \mathbf{m}|\|_{L^2(\Omega)},$$

which, by using Lemma 3.2.6, is bounded by

$$\leq C\|\mathbf{m}\|_{H^2(\Omega)}^{\frac{1}{2}} \|H_{\mathbf{m}}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|H_{\mathbf{m}}\|_{H^2(\Omega)}^{\frac{1}{2}} \|\mathbf{m}\|_{H^2(\Omega)}$$

$$\leq C\|\mathbf{m}\|_{H^2(\Omega)}^{\frac{1}{2}} \|H_{\mathbf{m}}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\mathbf{m}\|_{H^2(\Omega)}^{\frac{1}{2}} \|\mathbf{m}\|_{H^2(\Omega)} = C\|\mathbf{m}\|_{H^2(\Omega)}^2 \|H_{\mathbf{m}}\|_{L^\infty(\Omega)}^{\frac{1}{2}}$$

$$\leq C\, y(t) \cdot (1 + \ln y(t))^{\frac{1}{2}} \leq C\, y(t)(1 + \ln y(t)).$$

Therefore, by (3.41), we have obtained that

$$\frac{d}{dt}\left(\|\mathbf{m}_{x_i x_j}\|_{L^2}^2\right) \leq C(R) y(t)(1 + \ln y(t)), \quad \forall\ 0 < t < T. \tag{3.45}$$

Summing up $i, j = 1, 2, 3$ in (3.44) and (3.45) and using (3.33), we obtain (3.37).

Remark 2. By the local Lipschitz property of $\mathbf{f}(\mathbf{m})$, from (3.36), one easily obtains

$$\sup_{t \in [0,T)} \|\mathbf{m}_t\|_{H^2(\Omega)} \leq C_{T, \|\mathbf{m}_0\|_{H^2}} < \infty. \tag{3.46}$$

In next chapter, we prove higher time-regularity for solutions.

Chapter 4

Higher Time Regularity and Special Asymptotics

4.1 Higher time regularity

The higher time regularity has been studied for Landau-Lifshitz equation with exchange energy by Cimrak and Keer [6]. We study a higher time-regularity of weak solutions for simple Landau-Lifshitz equation

$$\begin{cases} \mathbf{m}_t = \gamma \mathbf{m} \times H_{\mathbf{m}} + \gamma\alpha \mathbf{m} \times (\mathbf{m} \times H_{\mathbf{m}}) \ \text{ in } \Omega \times (0, \infty), \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{4.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^3$ and $\mathbf{m}_0 \in H^2(\Omega; \mathbf{R}^3)$.

Theorem 4.1.1. *For any time $T > 0$, the solution $\mathbf{m}$ to* (4.1) *satisfies, for $p = 0, 1, 2, \cdots$,*

$$\sup_{t \in [0,T]} \|\partial_t^{p+1} \mathbf{m}\|_{H^2(\Omega)} \leq C < \infty, \tag{4.2}$$

where C is constant depending on T, p and $\|\mathbf{m}_0\|_{H^2(\Omega)}$.

Proof. We use induction on p. The case for $p = 0$ is already mentioned in Remark 2 in Section 3.2.4. Let us assume (4.2) holds for all powers up to $p - 1$. We consider the case for

p. Note that $\partial_t^i(H_{\mathbf{m}}) = H_{\partial_t^i \mathbf{m}}$ and hence, by (3.22),

$$\|H_{\partial_t^i \mathbf{m}}\|_{H^2(\Omega)} \leq C\|\partial_t^i \mathbf{m}\|_{H^2(\Omega)}.$$

Therefore, by the induction assumption, it follows that, for all $t \in [0, T]$,

$$\|\partial_t^i(H_{\mathbf{m}})\|_{H^2(\Omega)} \leq C\|\partial_t^i \mathbf{m}\|_{H^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty \quad \forall\ 0 \leq i \leq p. \tag{4.3}$$

Taking p^{th}-derivatives with respect to t to equation (4.1) yields

$$\partial_t^{p+1}\mathbf{m} = \gamma \sum_{i+j=p} \partial_t^i \mathbf{m} \times \partial_t^j H_{\mathbf{m}} + \gamma\alpha \sum_{i+j+k=p} \partial_t^i \mathbf{m} \times (\partial_t^j \mathbf{m} \times \partial_t^k H_{\mathbf{m}}). \tag{4.4}$$

We need to prove $\|\partial_t^{p+1}\mathbf{m}\|_{H^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty$.

Estimation of $\|\partial_t^{p+1}\mathbf{m}\|_{L^2(\Omega)}$

Since, $\forall\ 0 \leq i \leq p$,

$$\|\partial_t^i(H_{\mathbf{m}})\|_{L^\infty(\Omega)} \leq C\|\partial_t^i(H_{\mathbf{m}})\|_{H^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty,$$

the L^2-norm of each term on the right-hand side of (4.4) can be bounded by the L^∞-norms of its factors, which are in turn bounded by constant $C_{T,p,\|\mathbf{m}_0\|_{H^2}}$. Hence we have

$$\|\partial_t^{p+1}\mathbf{m}\|_{L^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}}. \tag{4.5}$$

Estimation of $\|\partial_t^{p+1}\nabla \mathbf{m}\|_{L^2(\Omega)}$

Taking $\partial_l = \partial_{x_l}$ on equation (4.1) yields

$$\begin{aligned}\partial_l \mathbf{m}_t &= \gamma\partial_l(\mathbf{m}\times H_{\mathbf{m}}) + \gamma\alpha\partial_l(\mathbf{m}\times(\mathbf{m}\times H_{\mathbf{m}}))\\ &= \gamma\mathbf{m}_{x_l}\times H_{\mathbf{m}} + \gamma\mathbf{m}\times(H_{\mathbf{m}})_{x_l} + \gamma\alpha\mathbf{m}\times(\mathbf{m}_{x_l}\times H_{\mathbf{m}})\\ &+ \gamma\alpha\mathbf{m}\times(\mathbf{m}\times(H_{\mathbf{m}})_{x_l}) + \gamma\alpha\mathbf{m}_{x_l}\times(\mathbf{m}\times H_{\mathbf{m}})\end{aligned} \tag{4.6}$$

Taking p^{th} derivative with respect to t on Eq. (4.6) yields

$$\begin{aligned}\partial_t^{p+1}\mathbf{m}_{x_l} =& \gamma\sum_{i+j=p}\partial_t^i\mathbf{m}_{x_l}\times\partial_t^j H_{\mathbf{m}} + \gamma\sum_{i+j=p}\partial_t^i\mathbf{m}\times\partial_t^j(H_{\mathbf{m}})_{x_l}\\ &\gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k(H_{\mathbf{m}})_{x_l}) + \gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}_{x_l}\times\partial_t^k H_{\mathbf{m}})\\ &+\gamma\alpha\sum_{i+j+k=p}\partial_t^i\mathbf{m}_{x_l}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_{\mathbf{m}})\end{aligned}$$

In order to estimate $\|\partial_t^{p+1}\mathbf{m}_{x_l}\|_{L^2(\Omega)}$, it is sufficient to estimate the following L^2-norms.

$$\begin{aligned}&\|\sum_{i+j=p}\partial_t^i\mathbf{m}_{x_l}\times\partial_t^j H_{\mathbf{m}}\|_{L^2(\Omega)}.\\ &\|\sum_{i+j=p}\partial_t^i\mathbf{m}\times\partial_t^j(H_{\mathbf{m}})_{x_l}\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k(H_{\mathbf{m}})_{x_l})\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}\times(\partial_t^j\mathbf{m}_{x_l}\times\partial_t^k H_{\mathbf{m}})\|_{L^2(\Omega)}.\\ &\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}_{x_l}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_{\mathbf{m}})\|_{L^2(\Omega)}.\end{aligned}$$

All these norms can be estimated in the same way: For each of the individual cross-product integrands, use the L^2-norm of a sole factor with x_l-derivative and use the L^∞-norms for the other factor or factors. All these norms can be bounded by constant $C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty$. Finally, summing up $l = 1, 2, 3$, we have proved

$$\|\partial_t^{p+1}\nabla\mathbf{m}\|_{L^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{4.7}$$

Estimation of $\|\partial_t^{p+1}\triangle\mathbf{m}\|_{L^2(\Omega)}$

Differentiating (4.6) with respect to x_l and summing up over $l = 1, 2, 3$ yields that

$$\begin{aligned}\triangle\mathbf{m}_t &= \gamma\triangle\mathbf{m}\times H_\mathbf{m} + \gamma\mathbf{m}\times\triangle H_\mathbf{m} + \gamma\sum_l \mathbf{m}_{x_l}\times(H_\mathbf{m})_{x_l} \\ &\quad + \gamma\alpha[\triangle\mathbf{m}\times(\mathbf{m}\times H_\mathbf{m}) + \mathbf{m}\times(\triangle\mathbf{m}\times H_\mathbf{m}) + \mathbf{m}\times(\mathbf{m}\times\triangle H_\mathbf{m})] \\ &\quad + \gamma\alpha\sum_l[\mathbf{m}_{x_l}\times(\mathbf{m}_{x_l}\times H_\mathbf{m}) + \mathbf{m}_{x_l}\times(\mathbf{m}\times(H_\mathbf{m})_{x_l}) + \mathbf{m}\times(\mathbf{m}_{x_l}\times(H_\mathbf{m})_{x_l})].\end{aligned} \tag{4.8}$$

Differentiating equation (4.8) p times with respect to t will yield a formula for $\partial_t^{p+1}\triangle\mathbf{m}$. To estimate $\|\partial_t^{p+1}\triangle\mathbf{m}\|_{L^2(\Omega)}$, we do not need to estimate every single term because lots of them are similar; it is sufficient to estimate the following 4 L^2-norms:

$$\|\sum_{i+j=p}\partial_t^i\triangle\mathbf{m}\times\partial_t^j H_\mathbf{m}\|_{L^2(\Omega)}. \tag{4.9}$$

$$\|\sum_{i+j=p}\partial_t^i\mathbf{m}_{x_l}\times\partial_t^j(H_\mathbf{m})_{x_l}\|_{L^2(\Omega)}. \tag{4.10}$$

$$\|\sum_{i+j+k=p}\partial_t^i\triangle\mathbf{m}\times(\partial_t^j\mathbf{m}\times\partial_t^k H_\mathbf{m})\|_{L^2(\Omega)}. \tag{4.11}$$

$$\|\sum_{i+j+k=p}\partial_t^i\mathbf{m}_{x_l}\times(\partial_t^j\mathbf{m}_{x_l}\times\partial_t^k H_\mathbf{m})\|_{L^2(\Omega)}. \tag{4.12}$$

For (4.9), we use

$$\|\partial_t^i \triangle \mathbf{m} \times \partial_t^j H_{\mathbf{m}}\|_{L^2(\Omega)} \leq \|\partial_t^j H_{\mathbf{m}}\|_{L^\infty} \|\partial_t^i \triangle \mathbf{m}\|_{L^2(\Omega)}.$$

For (4.10), we use

$$\|\partial_t^i \mathbf{m}_{x_l} \times \partial_t^j (H_{\mathbf{m}})_{x_l}\|_{L^2(\Omega)} \leq \|\partial_t^i \nabla \mathbf{m}\|_{L^4(\Omega)} \|\partial_t^j \nabla H_{\mathbf{m}}\|_{L^4(\Omega)}.$$

For (4.11), we use

$$\|\partial_t^i \triangle \mathbf{m} \times (\partial_t^j \mathbf{m} \times \partial_t^k H_{\mathbf{m}})\|_{L^2(\Omega)} \leq \|\partial_t^j \mathbf{m}\|_{L^\infty} \|\partial_t^k H_{\mathbf{m}}\|_{L^\infty} \|\partial_t^i \triangle \mathbf{m}\|_{L^2(\Omega)}.$$

For (4.12), we use

$$\|\partial_t^i \mathbf{m}_{x_l} \times (\partial_t^j \mathbf{m}_{x_l} \times \partial_t^k H_{\mathbf{m}})\|_{L^2} \leq \|\partial_t^k H_{\mathbf{m}}\|_{L^\infty} \|\partial_t^i \nabla \mathbf{m}\|_{L^4} \|\partial_t^j \nabla \mathbf{m}\|_{L^4}.$$

Finally, from these estimates, we obtain

$$\|\partial_t^{p+1} \triangle \mathbf{m}\|_{L^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty. \tag{4.13}$$

Combining (4.5), (4.7) and (4.13), we have shown that

$$\|\partial_t^{p+1} \mathbf{m}\|_{H^2(\Omega)} \leq C_{T,p,\|\mathbf{m}_0\|_{H^2}} < \infty.$$

This completes the induction process and hence the proof. □

Remark 3. Theorem 4.1.1 is also valid for the general equation (3.28) with smooth functions

$\varphi(\mathbf{m})$ and $\mathbf{a}(x)$; the proof should be similar.

4.2 Weak ω-limit sets

We first study the energy decay for global weak solutions to the Landau-Lifshitz equation (4.1). We write the initial value problem as

$$\begin{cases} \mathbf{m}_t = \mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}}) \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{4.14}$$

in terms of the Landau-Lifshitz interaction function $\mathbb{L}$ defined by (3.2), where the effective magnetic field $\mathbf{H}_{\text{eff}}$ is given by (2.2).

4.2.1 The energy identity

Let $\mathcal{E}(\mathbf{m})$ be defined by (1.1). Assume $\mathbf{a}$, $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$.

Theorem 4.2.1. *The global weak solution* $\mathbf{m}$ *to* (4.14) *satisfies the energy identity*

$$\mathcal{E}(\mathbf{m}(t)) - \mathcal{E}(\mathbf{m}(s)) = \gamma\alpha \int_s^t \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dx d\tau \quad \forall\ 0 \le s \le t < \infty. \tag{4.15}$$

Furthermore, if $\gamma\alpha < 0$, *then* $\mathbf{m}_t \in L^2((0, \infty); L^2(\Omega; \mathbf{R}^3))$.

Proof. By the definition of $\mathbf{H}_{\text{eff}}$, (4.15) follows from the identity $L(\mathbf{m}, \mathbf{n}) \cdot \mathbf{n} = -\alpha\gamma|\mathbf{m} \times \mathbf{n}|^2$ for all $\mathbf{m}, \mathbf{n} \in \mathbf{R}^3$ and the property

$$\frac{d}{dt}(\mathcal{E}(\mathbf{m}(t))) = -\int_\Omega \mathbf{H}_{\text{eff}} \cdot \mathbf{m}_t \, dx \quad a.e.\ (0, \infty).$$

If $\gamma\alpha < 0$, by (4.15), it follows that $\int_0^\infty \int_\Omega |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \, dxdt \leq \mathcal{E}(\mathbf{m}_0)/|\alpha\gamma| < \infty$. Also from the equation (4.14),

$$
\begin{aligned}
|\mathbf{m}_t|^2 =& |\mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}})|^2 \\
=& \gamma^2 |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 + (\gamma\alpha)^2 |\mathbf{m}|^2 |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \\
\leq& C \, |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2,
\end{aligned}
$$

where constant C depends on $\|\mathbf{m}_0\|_{L^\infty}$. Hence $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$. $\square$

Remark 4. The stability result and all the regularity estimates previously proved are for *finite time*; the only global space for the solutions (even for the regular solutions) is

$$
\mathbf{m} \in L^\infty((0,\infty); L^\infty(\Omega; \mathbf{R}^3)) \quad \text{with} \quad \mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3)).
$$

But this space is not enough to have strong convergence as $t \to \infty$; it would be enough if one has $\mathbf{m}_t \in L^1((0,\infty); L^2(\Omega; \mathbf{R}^3))$ (see [32]). Therefore, it is quite challenging to study the asymptotic behaviors of even regular solutions. The solution orbits for general initial data may not have strong ω-limit points. See [52] for some recent results on the estimation of *weak* ω-limit set.

4.2.2 Weak ω-limit sets and the estimate for soft-case

Given $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$, let $\mathbf{m}$ be the global weak solution to the initial value problem (4.14) and define the weak ω-limit set for $\mathbf{m}$ to be

$$
\omega^*(\mathbf{m}_0) = \{\tilde{\mathbf{m}} \mid \exists \, t_j \uparrow \infty \text{ such that } \mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}} \text{ weakly in } L^2(\Omega; \mathbf{R}^3)\}. \tag{4.16}
$$

We give an estimate of $\omega^*(\mathbf{m}_0)$ for the so-called *soft-case*, where there is no anisotropy energy ($\varphi = 0$). For more results on further special case when $\mathbf{a} = 0$, see [51, 52].

Theorem 4.2.2. *Let $\gamma\alpha < 0$, $\varphi = 0$ and $\mathbf{a} \in L^\infty(\Omega; \mathbf{R}^3)$. Then, for any $\mathbf{m}_0 \in L^\infty(\Omega; \mathbf{R}^3)$ with $|\mathbf{m}_0(x)| = 1$ a.e. on Ω, it follows that*

$$\omega^*(\mathbf{m}_0) \subseteq \{\tilde{\mathbf{m}} \in L^\infty(\Omega; \mathbf{R}^3) \mid |\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \text{ a.e. on } \Omega\}. \tag{4.17}$$

Proof. Let $\mathbf{m}$ be the global weak solution to (4.14) with the given initial datum $\mathbf{m}_0$. Then $|\mathbf{m}(t)| = 1$ a.e. on Ω for all $t \ge 0$. Assume $\mathbf{m}(t_j) \rightharpoonup \tilde{\mathbf{m}}$ weakly in $L^2(\Omega; \mathbf{R}^3)$ for a sequence $t_j \uparrow \infty$. In the following, we show that

$$|\tilde{\mathbf{m}}|^2 + 2|\tilde{\mathbf{m}} \times (\mathbf{a} + H_{\tilde{\mathbf{m}}})| \le 1 \text{ a.e. on } \Omega. \tag{4.18}$$

Let $e(t) = \mathcal{E}(\mathbf{m}(t))$. Then, by (4.15), $e(t)$ is non-increasing and bounded and hence $e(t)$ has limit as $t \to \infty$; this again by (4.15) implies

$$e(t_j + 1) - e(t_j) = \gamma\alpha \int_{t_j}^{t_j+1} \|\mathbf{m}(t) \times (\mathbf{a} + H_{\mathbf{m}(t)})\|_{L^2}^2 \, dt \to 0.$$

Hence there exists some $s_j \in [t_j, t_j + 1]$ such that

$$\|\mathbf{m}(s_j) \times (\mathbf{a} + H_{\mathbf{m}(s_j)})\|_{L^2(\Omega)} \to 0. \tag{4.19}$$

By Theorem 4.2.1, $\mathbf{m}_t \in L^2((0,\infty); L^2(\Omega; \mathbf{R}^3))$; hence

$$\|\mathbf{m}(s_j) - \mathbf{m}(t_j)\|_{L^2} \le \int_{t_j}^{s_j} \|\mathbf{m}_t(t)\|_{L^2} dt \le (s_j - t_j)^{\frac{1}{2}} \left(\int_{t_j}^{s_j} \|\mathbf{m}_t\|_{L^2}^2 dt \right)^{\frac{1}{2}} \to 0,$$

which yields $\mathbf{m}(s_j) \rightharpoonup \tilde{\mathbf{m}}$ weakly in $L^2(\Omega; \mathbf{R}^3)$. Therefore, by (4.19), (4.18) follows from the following lemma with $\mathbf{m}_j = \mathbf{m}(s_j)$. This completes the proof. $\square$

Lemma 4.2.3. *Let* $\mathbf{m}_j \rightharpoonup \tilde{\mathbf{m}}$ *weakly in* $L^2(\Omega; \mathbf{R}^3)$ *and satisfy the following conditions:*

$$(a) \quad |\mathbf{m}_j| = 1 \ a.e.\, \Omega; \quad (b) \quad \|\mathbf{m}_j \times (\mathbf{a} + H_{\mathbf{m}_j})\|_{L^2(\Omega)} \to 0.$$

Then $\tilde{\mathbf{m}}$ *satisfies the condition* (4.18) *above.*

Proof. This result can be proved by a similar method as used for [51, Theorem 1.1]. However, we present a different but direct proof based on the div-curl lemma [46]. For any $\mathbf{m} \in L^\infty(\Omega; \mathbf{R}^3)$, let $G_{\mathbf{m}} = \mathbf{m}\chi_\Omega + H_{\mathbf{m}}$. Then $\operatorname{div} G_{\mathbf{m}} = 0$ on $\mathbf{R}^3$. Denote

$$G_j = \mathbf{a} + G_{\mathbf{m}_j}, \ H_j = \mathbf{a} + H_{\mathbf{m}_j}; \quad \tilde{G} = \mathbf{a} + G_{\tilde{\mathbf{m}}}, \ \tilde{H} = \mathbf{a} + H_{\tilde{\mathbf{m}}}.$$

Then $G_j \rightharpoonup \tilde{G}$, $H_j \rightharpoonup \tilde{H}$ weakly in $L^2(\Omega; \mathbf{R}^3)$ and, by the div-curl lemma [46],

$$\int_\Omega G_j \cdot H_j \phi \, dx \to \int_\Omega \tilde{G} \cdot \tilde{H} \phi \, dx \quad \forall \, \phi \in C_0^\infty(\Omega). \tag{4.20}$$

Since $\mathbf{m}_j = G_j - H_j$ on Ω, it follows that

$$\begin{aligned} |\mathbf{m}_j|^2 + 2|\mathbf{m}_j \times (\mathbf{a} + H_{\mathbf{m}_j})| =& |G_j - H_j|^2 + 2|G_j \times H_j| \\ =& |G_j|^2 + |H_j|^2 + 2|G_j \times H_j| - 2G_j \cdot H_j. \end{aligned} \tag{4.21}$$

Note that function $f(\mathbf{m}, \mathbf{n}) = |\mathbf{m}|^2 + |\mathbf{n}|^2 + 2|\mathbf{m} \times \mathbf{n}|$ is convex on $(\mathbf{m}, \mathbf{n}) \in \mathbf{R}^3 \times \mathbf{R}^3$. Hence, for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$, one has

$$\liminf_{j\to\infty} \int_\Omega (|G_j|^2 + |H_j|^2 + 2|G_j \times H_j|)\phi \geq \int_\Omega (|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}|)\phi. \tag{4.22}$$

By assumptions (a), (b), from (4.20)–(4.22), it follows that

$$\int_\Omega (|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}| - 2\tilde{G} \cdot \tilde{H})\phi \, dx \leq \int_\Omega \phi \, dx$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$. This implies

$$|\tilde{G}|^2 + |\tilde{H}|^2 + 2|\tilde{G} \times \tilde{H}| - 2\tilde{G} \cdot \tilde{H} \leq 1 \quad a.e. \ \Omega,$$

which, exactly, is equivalent to (4.18). This completes the proof. □

4.3 Special dynamics for constant initial data on ellipsoid domains

We now study a special case of (4.14) where applied field $\mathbf{a}(x) = \mathbf{a}$ is constant, domain Ω is an ellipsoid, and initial datum $\mathbf{m}_0$ is a constant unit vector. Therefore, in (4.14), the effective magnetic field $\mathbf{H}_{\text{eff}}$ is now given by

$$\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a} + H_{\mathbf{m}}$$

as above, but with constant vector **a**. In what follows, we assume ellipsoid domain Ω is given by

$$\Omega = \{x \in \mathbf{R}^3 \mid \sum_{i=1}^{3} x_i^2/a_i < 1\},$$

where $a_i > 0$ are constants.

4.3.1 Associated ODE system on $\mathbf{R}^3$

Let b_i $(i = 1, 2, 3)$ be defined as below:

$$b_i = \frac{1}{2}\int_0^\infty \frac{\sqrt{a_1 a_2 a_3}\, dt}{(a_i + t)\sqrt{(a_1 + t)(a_2 + t)(a_3 + t)}}. \tag{4.23}$$

Note that, if Ω is a ball in $\mathbf{R}^3$, all b_i's are equal to 1/3.

Theorem 4.3.1 (Pedragal and Yan [44])**.** *For each $k = 1, 2, 3$, the Dirichlet-Neumann problem*

$$\triangle\omega = 0 \quad \text{in } \Omega^c \tag{4.24}$$

with boundary condition

$$\omega|_{\partial\Omega} = x_k, \qquad \frac{\partial\omega}{\partial\nu}|_{\partial\Omega} = (1 - \frac{1}{b_k})\nu_k$$

ν is the unit normal on $\partial\Omega$ pointing outward of Ω, has an unique solution ω satisfies $\omega \in H^1_{loc}(\Omega^c)$ and $|\nabla\omega| \in L^2(\Omega^c)$.

With Theorem 4.3.1, one can prove following lemma; the proof is in [44].

Lemma 4.3.2. *Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the functions determined in Theorem 4.3.1. For each*

$k = 1, 2, 3,$ *let*

$$u_k(x) = \begin{cases} b_k x_k & x \in \Omega, \\ b_k \omega_k(x) & x \in \Omega^c \end{cases} \tag{4.25}$$

Then $u_k \in H^1_{loc}(\mathbf{R}^n)$, $F^k = \nabla u_k \in L^2(\mathbf{R}^n; \mathbf{R}^n)$, *and*

$$\operatorname{curl} F^k = 0, \quad \operatorname{div}(-F^k + e^k \chi_\Omega) = 0, \quad \text{in } \mathbf{R}^3, \tag{4.26}$$

where e^1, e^2, e^3 *are the standard basis vectors of* $\mathbf{R}^3$.

From this lemma, the magneto-static stray field $H_{\mathbf{m}}$ induced by any *constant* field $\mathbf{m}$ has constant value on Ω given by

$$H_{\mathbf{m}}|_\Omega = -\Lambda \mathbf{m} \quad (\forall\ \mathbf{m} \in \mathbf{R}^3),$$

where Λ is a diagonal matrix of positive numbers. In fact, $\Lambda = \operatorname{diag}(b_1, b_2, b_3)$.

Let $\mathbf{m}_0$ be a constant unit vector. Then, problem (4.14) reduces to the following ODE system on $\mathbf{m} \in \mathbf{R}^3$:

$$\begin{cases} \dot{\mathbf{m}} = \Phi(\mathbf{m}), & t > 0, \\ \mathbf{m}(0) = \mathbf{m}_0, \end{cases} \tag{4.27}$$

where function $\Phi \colon \mathbf{R}^3 \to \mathbf{R}^3$ is defined by

$$\Phi(\mathbf{m}) = \mathbb{L}(\mathbf{m}, -\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda \mathbf{m}), \quad \forall\ \mathbf{m} \in \mathbf{R}^3. \tag{4.28}$$

Since $\mathbf{m} \cdot \Phi(\mathbf{m}) = 0$, system (4.27) also preserves the length of $\mathbf{m}(t)$. Thus we have $|\mathbf{m}(t)| = 1$ for all $t \geq 0$. Moreover, $\mathbb{L}(\mathbf{m}, \mathbf{n}) = 0$ if and only if $\mathbf{m} \times \mathbf{n} = 0$; hence, the equilibrium points of (4.27), that is, the solutions of $\Phi(\mathbf{m}) = 0$ on unit sphere $|\mathbf{m}| = 1$, are characterized by vectors $\mathbf{m} \in \mathbf{R}^3$ for which there is a real number $\lambda \in \mathbf{R}$ such that

$$-\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda\mathbf{m} = \lambda\mathbf{m}, \quad |\mathbf{m}| = 1. \tag{4.29}$$

This condition is equivalent to $\mathbf{m}$ being a critical point of the function

$$P(\mathbf{m}) = \frac{1}{2}\Lambda\mathbf{m} \cdot \mathbf{m} - \mathbf{a} \cdot \mathbf{m} + \varphi(\mathbf{m}) \tag{4.30}$$

on unit sphere $|\mathbf{m}| = 1$.

In most cases, there will be at least two distinct equilibrium points for system (4.27); for example, all maximum or minimum points of P on $|\mathbf{m}| = 1$ (always exist) are such points.

4.3.2 Lyapunov function and the special dynamics

The dynamics of system (4.27) can be studied by the classical ODE theory. For example, we have the following result.

Theorem 4.3.3. *Function P is a Lyapunov function for system* (4.27)*. The ω-limit set of* (4.27) *is contained in the set of all critical points of P on unit sphere.*

Proof. Since $\mathbf{H}_{\text{eff}} = -\varphi'(\mathbf{m}) + \mathbf{a} - \Lambda\mathbf{m} = -\nabla P(\mathbf{m})$, $\forall\ \mathbf{m} \in \mathbf{R}^3$, it follows that for solution $\mathbf{m} = \mathbf{m}(t)$ of (4.27)

$$\frac{d}{dt}P(\mathbf{m}(t)) = \nabla P(\mathbf{m}) \cdot \mathbf{m}_t = -\mathbf{H}_{\text{eff}} \cdot \mathbb{L}(\mathbf{m}, \mathbf{H}_{\text{eff}}) = \gamma\alpha|\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 \le 0.$$

Hence P is a Lyapunov function for system (4.27).

To show the second part of the theorem, assume $\mathbf{m}(t_j) \to \bar{\mathbf{m}}$ for a sequence $t_j \uparrow \infty$. Let $p(t) = P(\mathbf{m}(t))$. Then $p(t)$ is smooth and has limit as $t \to \infty$. Hence $p(t_j + 1) - p(t_j) = p'(s_j) \to 0$ for some $s_j \in (t_j, t_j + 1)$. Since $\gamma\alpha < 0$, this implies

$$|\mathbf{m}(s_j) \times \mathbf{H}_{\text{eff}}(s_j)| = |\mathbf{m}(s_j) \times \nabla P(\mathbf{m}(s_j))| \to 0. \tag{4.31}$$

As above, since $\mathbf{m}_t \in L^2(0, \infty)$, one has

$$|\mathbf{m}(s_j) - \mathbf{m}(t_j)| \leq \int_{t_j}^{s_j} |\mathbf{m}_t| dt \leq (s_j - t_j)^{\frac{1}{2}} \left(\int_{t_j}^{s_j} |\mathbf{m}_t|^2 dt \right)^{\frac{1}{2}} \to 0.$$

This implies $\mathbf{m}(s_j) \to \bar{\mathbf{m}}$; hence, by (4.31), $|\bar{\mathbf{m}} \times \nabla P(\bar{\mathbf{m}})| = 0$, which proves $\bar{\mathbf{m}}$ is a critical point of P on unit sphere. This completes the proof. $\square$

Finally, we prove a special result for $\mathbf{a} = 0$ and $\varphi = 0$.

Proposition 4.3.4. *Let b_1, b_2, b_3 be positive numbers determined by* (4.23)*. If $b_k = \min\{b_1, b_2, b_3\}$, then $\pm e^k$ are asymptotically stable equilibrium points to the system* (4.27)*, where e^1, e^2, e^3 are the standard basis vectors of $\mathbf{R}^3$.*

Proof. Without loss of generality, let us assume $b_1 = \min\{b_1, b_2, b_3\}$. It is trivial to see that P has a strict relative minimum at $(\pm 1, 0, 0)$. According to the Lyapunov stability theorem, $(\pm 1, 0, 0)$ are the asymptotically stable equilibrium points. $\square$

Chapter 5

Existence for Fractional Landau-Lifshitz Equations

In this chapter, we prove the existence of global weak solution to fractional Landau-Lifshitz equation including magneto-static term with periodic boundary condition:

$$\mathbf{m}_t = \gamma \mathbf{m} \times \mathcal{F}_{\mathbf{m}} + \gamma \mathbf{m} \times (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \qquad \text{in } \mathbf{R}^n, \tag{5.1}$$

where

$$\mathcal{F}_{\mathbf{m}} = \wedge^{2\alpha} \mathbf{m} + H_{\mathbf{m}} \qquad 0 < \alpha < 1. \tag{5.2}$$

Let $\Omega = [0, 2\pi]^n$ denote the periodic box in all directions and $\mathbb{Z}$ denote the dual lattice associated to Ω in $\mathbf{R}^n$. The two terms in energy form (5.2), for $x \in \Omega$, are defined as:

$$\widehat{H_{\mathbf{m}}}(\xi) = \begin{cases} \dfrac{(\xi \cdot \widehat{\mathbf{m}}) \cdot \xi}{|\xi|^2}, & \xi \neq 0. \\ \\ 0, & \xi = 0. \end{cases} \tag{5.3}$$

The operator $\wedge$ denotes the square root of Laplacian $(-\triangle)^{\frac{1}{2}}$, so $\wedge^{\beta}\mathbf{m}$ can be understood in terms of Fourier transform:

$$\mathfrak{F}(\wedge^{\beta} \mathbf{m}(x)) = |\xi|^{\beta} \widehat{\mathbf{m}}(\xi) \tag{5.4}$$

Definition 5.0.1. Let $\mathbf{m}_0 \in H^\alpha$, $|\mathbf{m}_0| = 1$, $\mathbf{m}$ is a global weak solution to equation (5.1) if

$$\mathbf{m} \in L^\infty((0,T); H^\alpha(\Omega))$$

$$\int_{Q_T} \frac{\partial \mathbf{m}}{\partial t} \cdot \varphi = \gamma \int_{Q_T} (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \cdot \varphi + \gamma \int_{Q_T} \mathbf{m} \times (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \cdot \varphi$$

for all $\varphi \in C^\infty(Q_T)$. Here $Q_T = (0,T) \times \Omega$.

Remark 5. We remark that, as being the same as weak solution in Chapter 2, the length $|\mathbf{m}(x,t)|$ is preserved for weak solutions; therefore, if the initial datum $\mathbf{m}_0$ satisfies the saturation condition $|\mathbf{m}_0(x)| = 1$ then any weak solution $\mathbf{m}(x,t)$ to the fractional equation (5.1) will also satisfy the saturation condition: $|\mathbf{m}(x,t)| = 1$ for all $t > 0$.

5.1 Notations and preliminaries

Let L^p denote the space of all the p-th integrable function f in Ω with associated norm given by:

$$\|f(x)\|_{L^p} = (\int_\Omega |f(x)|^p)^{\frac{1}{p}}$$

For any $r \in \mathbf{R}$, we define the homogeneous Sobolev space H^r for all tempered distribution function $f(x)$ such that $\|f\|_{H^r}$ is finite, where $\|\cdot\|_{H^r}$ is defined through Fourier transform:

$$\|f\|_{H^r} = \| \wedge^r f(x)\|_{L^2} = (\sum_{\xi \in \mathbb{Z}} |\xi|^{2r} |\widehat{f}(\xi)|^2)^{\frac{1}{2}}$$

For $1 \le p \le \infty$ and $r \in \mathbf{R}$, the space $H^{r,p}(\Omega)$ consists of all $f(x)$ such that $f = \wedge^{-r} g$, where $g \in L^p(\Omega)$. The corresponding norm is given by:

$$\|f\|_{H^{r,p}} = \|\mathfrak{F}^{-1}(|\xi|^r \widehat{f})\|_{L^p}$$

The following commutator estimate comes from a general result of Coifman and Meyer [9]; see also [35, 36].

Lemma 5.1.1. *Suppose that $s > 0$ and $p \in (1, \infty)$. Then*

$$\| \wedge^s (fg) - f \wedge^s g\|_{L^p} \le C \left(\|\nabla f\|_{L^{p_1}} \|g\|_{H^{s-1,p_2}} + \|f\|_{H^{s,p_3}} \|g\|_{L^{p_4}} \right),$$

where $p_1, p_2, p_3, p_4 \in (1, \infty)$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$$

and f, g are such that the right hand side terms make sense.

For fractional derivative, we also have Hardy-Littlewood-Sobolev theorem; the proof can be found in Stein [45].

Lemma 5.1.2. *Suppose that $q > 1$, $p \in [q, \infty]$ and*

$$\frac{1}{p} + \frac{s}{d} = \frac{1}{q}$$

Assume that $\wedge^s f \in L^q$, then $f \in L^p$ and there is a constant $C > 0$ such that

$$\|f\|_{L^p} \le C \| \wedge^s f\|_{L^q}$$

As a corollary, if $f = \wedge^{-s} g$ *for* $g \in L^q$*, then*

$$\|g\|_{H^{-s,p}} = \| \wedge^{-s} g\|_{L^p} \leq C\|g\|_{L^q}$$

Later we are going to apply the Galerkin method; therefore, we need to consider following eigenvalue problem,

$$\begin{cases} \wedge^{2\alpha} u = \mu u \\ \text{with periodic boundary conditions.} \end{cases} \tag{5.5}$$

Because $\wedge^{-2\alpha}$ is a compact self-adjoint operator in $L^2(\Omega)$, there exists a completed orthogonal family of $L^2(\Omega)$, say $\{\omega_j\}_{j\in N}$, being eigenvectors of $\wedge^{-2\alpha}$,

$$\wedge^{-2\alpha}\omega_j = \mu_j \omega_j$$

where sequence μ_j is decreasing and tends to 0. By setting $\nu_j = \mu_j^{-1}$, we obtain,

$$\wedge^{2\alpha}\omega_j = \nu_j \omega_j \tag{5.6}$$

where $0 < \nu_1 \leq \nu_2 \leq ...$ The family $\{\omega_j\}$ satisfies

$$(\omega_j, \omega_k) = \delta_{jk}, \qquad \text{Kronecker symbol}$$

5.2 A priori estimates

We let $V_n = \bigcup_{k=1}^{n} \omega_k \in L^2(\Omega)$, where ω_k is eigenvectors in (5.6). We look for solutions to approximate fractional Landau-Lifshitz equation (5.1) in the form of

$$\mathbf{m}_N = \sum_{k=1}^{N} \varphi_{k,N}(t)\omega_k(x) \tag{5.7}$$

In order to show such $\{\varphi_{k,N}(t)\}$ exist, we take inner product of equation (5.1) with each $\omega_k(x)$, $k = 1, 2, ..., N$,

$$\int_\Omega \frac{\partial \mathbf{m}_N}{\partial t} \cdot \omega_k(x) = \gamma \int_\Omega \mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N} \cdot \omega_k(x) + \gamma \int_\Omega \mathbf{m}_N \times (\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}) \cdot \omega_k(x) \tag{5.8}$$

where

$$\mathcal{F}_{\mathbf{m}_N} = \wedge^{2\alpha}\mathbf{m}_N + P_N(H_{\mathbf{m}_N})$$

We define P_N as the orthogonal projection,

$$P_N : \dot{H}^r(\Omega) \to V_N = \{\omega_1, \omega_2, ..., \omega_N\}$$

The local existence of solutions $\{\varphi_{k,N}(t)\}$ is justified by standard Picard's theorem. In order to extend the local weak solution to $[0, T]$ and take $N \to \infty$, we need to establish following a priori estimate,

Lemma 5.2.1. *Let $\mathbf{m}_0 \in H^\alpha(\Omega)$, then for any $0 < T < \infty$, the approximate solutions $\mathbf{m}_N$ to systems (5.8) satisfy,*

$$\max_{0 \le t \le T} \|\mathbf{m}_N\|^2_{H^\alpha(\Omega)} \le C_1$$

where C_1 only depends on initial data $\|\mathbf{m}_0\|^2_{H^\alpha(\Omega)}$. Moreover, for $1 \le r \le r^* = \frac{d}{d-\alpha}$, where d is dimension,

$$\|\frac{\partial \mathbf{m}_N}{\partial t}\|_{L^r(Q_T)} \le C_2$$

and for $1 < r \le r^* = \frac{d}{d-\alpha}$

$$\|\mathbf{m}_N(t_1) - \mathbf{m}_N(t_2)\|_{L^r(\Omega)} \le C_2|t_2 - t_1|^{\frac{r-1}{r}}$$

where C_2 only depends on initial data $\|\mathbf{m}_0\|^2_{H^\alpha(\Omega)}$ and time T.

Proof. Multiply systems (5.8) with $\varphi_{k,N}(t)$, $k = 1, 2, .., N$ and sum up together, then we have,

$$\frac{d}{dt}\int_\Omega |\mathbf{m}_N|^2 dx = 0$$

So,

$$\|\mathbf{m}_N\|_{L^2(\Omega)} \le \|\mathbf{m}_0\|_{L^2(\Omega)} \tag{5.9}$$

Multiply systems (5.8) with $\varphi'_{k,N}(t)$, $k = 1, 2, .., N$, where $\varphi'_{k,N}(t)$ satisfies,

$$\mathcal{F}_{\mathbf{m}_N} = \sum_{k=1}^{N} \varphi'_{k,N}(t)\omega_k(x)$$

and sum up together,

$$\int_\Omega \frac{\partial \mathbf{m}_N}{\partial t} \cdot \wedge^{2\alpha}\mathbf{m}_N + \int_\Omega \frac{\partial \mathbf{m}_N}{\partial t} \cdot P_N(H_{\mathbf{m}_N}) + \int_\Omega |\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}|^2 = 0$$

By using periodic boundary condition,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\wedge^\alpha \mathbf{m}_N|^2 + \int_\Omega \frac{\partial \mathbf{m}_N}{\partial t}\cdot P_N(H_{\mathbf{m}_N}) + \int_\Omega |\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}|^2 = 0$$

For mapping operator P_N,

$$\int_\Omega \frac{\partial \mathbf{m}_N}{\partial t}\cdot P_N(H_{\mathbf{m}_N}) = \int_\Omega \frac{\partial \mathbf{m}_N}{\partial t}\cdot H_{\mathbf{m}_N}$$

Also we have,

$$\int_\Omega \mathbf{m}_N \cdot H_{\mathbf{m}_N} = \sum \widehat{\mathbf{m}_N}\cdot \widehat{H_{\mathbf{m}_N}} = \int_\Omega |H_{\mathbf{m}_N}|^2$$

So,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\wedge^\alpha \mathbf{m}_N|^2 + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |H_{\mathbf{m}_N}|^2 + \int_\Omega |\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}|^2 = 0 \tag{5.10}$$

Integrating (5.10) with respect to time from $[0,T]$,

$$\max_{0\le t\le T} \|\wedge^\alpha \mathbf{m}_N\|^2_{L^2(\Omega)} + \max_{0\le t\le T}\|H_{\mathbf{m}_N}\|^2_{L^2(\Omega)} + \|\mathbf{m}_N\times\mathcal{F}_{\mathbf{m}_N}\|^2_{L^2(0,T;L^2(\Omega))} \le C_1 \tag{5.11}$$

where C_1 only depends on initial data $\|\wedge^\alpha \mathbf{m}_0\|^2_{L^2(\Omega)}$. By Sobolev embedding, for any $1 \le p \le p^* = \frac{2d}{d-2\alpha}$, we have,

$$\max_{0\le t\le T}\|\mathbf{m}_N(t)\|_{L^p(\Omega)} \le k_1 C_1$$

By Holder inequality, for $1 \le r \le r^* = \frac{d}{d-\alpha}$,

$$\left(\int_\Omega |\mathbf{m}_N\times\mathcal{F}_{\mathbf{m}_N}|^r dx\right)^{\frac{1}{r}} \le \|\mathcal{F}_{\mathbf{m}_N}\|_{L^2(\Omega)}\left(\int_\Omega |\mathbf{m}_N|^{\frac{2r}{2-r}}dx\right)^{\frac{2-r}{2r}} \le k_2 C_1$$

since $\frac{2r}{2-r} \leq p^*$ when $r < r^*$. Moreover, we have for $1 \leq r \leq r^* = \frac{d}{d-\alpha}$,

$$
\begin{aligned}
(\int_{Q_T} |\frac{\partial \mathbf{m}_N}{\partial t}|^r) &\leq \int_{Q_T} |\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}|^r + \int_{Q_T} |\mathbf{m}_N \times (\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N})|^r \\
&\leq k_2 C_1 + \int_0^T \|\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}\|_{L^2(\Omega)} (\int_\Omega |\mathbf{m}_N|^{\frac{2r}{2-r}} dx)^{\frac{2-r}{2r}} dt \\
&\leq k_3 C_2
\end{aligned}
\tag{5.12}
$$

where C_2 only depends on initial data $\| \wedge^\alpha \mathbf{m}_0\|^2_{L^2(\Omega)}$ and time T. With the same $r > 1$ as defined above, we can further prove the approximation solution $\mathbf{m}_N$ is continuous in time,

$$
\begin{aligned}
\|\mathbf{m}_N(t_1) - \mathbf{m}_N(t_2)\|_{L^r(\Omega)} &\leq \| \int_{t_1}^{t_2} \frac{\partial \mathbf{m}_N}{\partial t} dt \|_{L^r(\Omega)} \\
&\leq \int_{t_1}^{t_2} \|\frac{\partial \mathbf{m}_N}{\partial t}\|_{L^r(\Omega)} dt \\
&\leq |t_2 - t_1|^{\frac{r-1}{r}} (\int_{Q_T} |\frac{\partial \mathbf{m}_N}{\partial t}|^r)^{\frac{1}{r}} \\
&\leq C_2 |t_2 - t_1|^{\frac{r-1}{r}}
\end{aligned}
\tag{5.13}
$$

□

Based on Lemma 5.2.1, we have following conclusions for approximate solution $\mathbf{m}_N$,

Lemma 5.2.2. *For arbitrage N and time T, under the condition of Lemma 5.2.1, the initial value problem for approximation systems (5.8) has at least one time continuous and global weak solution.*

5.3 Compactness

In order to take $N \to \infty$ with the approximation solutions, we still need strong convergence based on Lemma 5.2.1. The compactness lemma below can be found in [39],

Lemma 5.3.1. *Let B_0, B, B_1 be three Banach space such that,*

$$B_0 \subset B \subset B_1$$

where the injections are continuous and B_0, B_1 are reflexive and $B_0 \hookrightarrow B$ is compact. Denote

$$W = \{v | v \in L^{p_0}(0, T; B_0), \frac{dv}{dt} \in L^{p_1}(0, T; B_1)\}$$

for $T < \infty$ and $1 < p_0, p_1 < \infty$. Then W equipped with the norm

$$\|v\|_{L^{p_0}(0,T;B_0)} + \|\frac{dv}{dt}\|_{L^{p_1}(0,T;B_1)}$$

is a Banach space and the embedding $W \hookrightarrow L^{p_0}(0, T; B)$ is compact.

From Lemma 5.2.1, we know $\mathbf{m}_N$ is uniformly bounded in

$$L^\infty(0, T; H^\alpha(\Omega)) \bigcap W^{1,r}(0, T; L^r(\Omega))$$

With compactness lemma 5.3.1, there exists some $\mathbf{m} \in L^\infty(0, T; H^\alpha(\Omega))$ such that,

$$\begin{aligned}
&\mathbf{m}_N \rightharpoonup \mathbf{m} \quad \text{weakly in } L^p(0, T; H^\alpha(\Omega)) \text{ for } 1 < p < \infty \\
&\mathbf{m}_N \to \mathbf{m} \quad \text{strongly in } L^p(0, T; H^\beta(\Omega)) \text{ for } 1 < p < \infty, 0 \le \beta \le \alpha \\
&\frac{\partial \mathbf{m}_N}{\partial t} \rightharpoonup \frac{\partial \mathbf{m}}{\partial t} \quad \text{weakly in } L^r(Q_T) \text{ for } 1 < r
\end{aligned} \tag{5.14}$$

5.4 Existence

We are going to prove the global existence of weak solution to fractional LLG (5.1) by determining weak convergence limit of approximate solution $\mathbf{m}_N$ when $N \to \infty$. From compactness result(5.14), it is obvious that we have,

$$\int_{Q_T} \frac{\partial \mathbf{m}_N}{\partial t} \cdot \varphi \to \int_{Q_T} \frac{\partial \mathbf{m}}{\partial t} \cdot \varphi \quad , \forall \varphi \in C^\infty(Q_T)$$

For the nonlocal nonlinear term, we first observe that,

$$\|\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}\|^2_{L^2(0,T;L^2(\Omega))} \leq C_1$$

which has been proved in (5.11). So there exists $\eta \in L^2(0,T;L^2(\Omega))$, such that,

$$\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N} \rightharpoonup \eta \quad \text{weakly in } L^2(0,T;L^2(\Omega))$$

We would like to show that $\eta = \mathbf{m} \times \mathcal{F}_{\mathbf{m}}$ in $L^2(0,T;L^2(\Omega))$. Let $\varphi \in H^s(\Omega)$ for $s > \alpha + d/2$,

$$\begin{aligned} \int_{Q_T} \mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N} \cdot \varphi &= \int_{Q_T} \mathbf{m}_N \times P_N(H_{\mathbf{m}_N}) \cdot \varphi + \int_{Q_T} \mathbf{m}_N \times \wedge^{2\alpha} \mathbf{m}_N \cdot \varphi \\ &= A_1 + A_2 \end{aligned} \tag{5.15}$$

From compactness result(5.14),

$$A_1 = \int_{Q_T} \mathbf{m}_N \times P_N(H_{\mathbf{m}_N}) \cdot \varphi \to \int_{Q_T} \mathbf{m} \times H_{\mathbf{m}} \cdot \varphi$$

For term A_2,

$$
\begin{aligned}
A_2 &= \int_{Q_T} \mathbf{m}_N \times \wedge^{2\alpha} \mathbf{m}_N \cdot \varphi \\
&= -\int_{Q_T} \wedge^{\alpha} \mathbf{m}_N \cdot \wedge^{\alpha}(\mathbf{m}_N \times \varphi)
\end{aligned}
\tag{5.16}
$$

In order to show that $\eta = \mathbf{m} \times \mathcal{F}_{\mathbf{m}}$ in $L^2(0,T;L^2(\Omega))$, it is enough to prove

$$
\int_{Q_T} \wedge^{\alpha} \mathbf{m}_N \cdot \wedge^{\alpha}(\mathbf{m}_N \times \varphi) \rightharpoonup \int_{Q_T} \wedge^{\alpha} \mathbf{m} \cdot \wedge^{\alpha}(\mathbf{m} \times \varphi) \quad \text{weakly in } L^2(0,T;L^2(\Omega))
$$

We set commutator $\Phi_\varphi(\mathbf{m}) = [\wedge^\alpha, \varphi]\mathbf{m} = \wedge^\alpha(\mathbf{m} \times \varphi) - \wedge^\alpha \mathbf{m} \times \varphi$, then it is equivalent to prove

$$
\int_{Q_T} \wedge^{\alpha} \mathbf{m}_N \cdot \Phi_\varphi(\mathbf{m}_N - \mathbf{m}) + \int_{Q_T} \wedge^{\alpha}(\mathbf{m}_N - \mathbf{m}) \cdot \Phi_\varphi(\mathbf{m}) \to 0
$$

From the compactness result (5.14), it is obvious,

$$
\int_{Q_T} \wedge^{\alpha}(\mathbf{m}_N - \mathbf{m}) \cdot \Phi_\varphi(\mathbf{m}) \to 0
$$

From Lemma 5.1.1 and Lemma 5.1.2,

$$
\begin{aligned}
\|\Phi_\varphi(\mathbf{m}_N - \mathbf{m})\|_{L^2(\Omega)} &\leq C(\|\nabla \varphi\|_{L^{p_1}} \|\mathbf{m}_N - \mathbf{m}\|_{H^{\alpha-1,p_2}} + \|\varphi\|_{H^{\alpha,p_3}} \|\mathbf{m}_N - \mathbf{m}\|_{L^{p_4}}) \\
&\leq C(\|\nabla \varphi\|_{L^{p_1}} \|\mathbf{m}_N - \mathbf{m}\|_{L^2(\Omega)} + \|\varphi\|_{H^{\alpha,p_3}} \|\mathbf{m}_N - \mathbf{m}\|_{H^\beta(\Omega)}) \\
&\leq C\|\varphi\|_{H^s(\Omega)} \|\mathbf{m}_N - \mathbf{m}\|_{H^\beta(\Omega)}
\end{aligned}
\tag{5.17}
$$

where $p_2, p_3 \in (1, \infty)$

$$
\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}
$$

$$\frac{1-\alpha}{d} + \frac{1}{p_2} = \frac{1}{2}$$

$$\frac{\beta}{d} + \frac{1}{p_4} = \frac{1}{2} \quad 0 < \beta < \alpha$$

$$s > \frac{d}{2} - 1$$

Therefore,

$$|\int_{Q_T} \wedge^\alpha \mathbf{m}_N \cdot \Phi_\varphi(\mathbf{m}_N - \mathbf{m})| \leq C\|\varphi\|_{H^s(\Omega)}\| \wedge^\alpha \mathbf{m}_N\|_{L^2(\Omega_T)}\|\mathbf{m}_N - \mathbf{m}\|_{L^2(0,T;H^\beta(\Omega))} \to 0$$

So, $\forall \varphi \in C^\infty(Q_T)$

$$\int_{Q_T} (\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}) \cdot \varphi \to \int_{Q_T} (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \cdot \varphi$$

Furthermore, since $\mathbf{m}_N \to \mathbf{m}$ strongly in $L^2(Q_T)$,

$$\int_{Q_T} (\mathbf{m}_N \times \mathcal{F}_{\mathbf{m}_N}) \times \mathbf{m}_N \cdot \varphi \to \int_{Q_T} (\mathbf{m} \times \mathcal{F}_{\mathbf{m}}) \times \mathbf{m} \cdot \varphi$$

The theorem is summarized as below

Theorem 5.4.1. *Let $0 < \alpha < 1$ and $\mathbf{m}_0 \in H^\alpha(\Omega)$, then for any $0 < T < \infty$, then there exists at least one global weak solution to fractional Landau-Lifshitz equation (5.1) such that*

$$\mathbf{m} \in L^\infty(0,T;H^\alpha(\Omega)) \bigcap C^{0,\frac{r-1}{r}}(0,T;L^r(\Omega))$$

for $1 < r \leq r^* = \frac{d}{d-\alpha}$.

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