LOAD DIFFUSION FROM AN ADHESIVE-BONDED STRINGER TO AN INFINITE SHEET

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THES

ABSTRACT

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The case considered here is that of a single stiffener of uniform cross-section and finite length, bonded to an infinite sheet by means of an elastic adhesive. The objective is to investigate the shear stress or shear flow in the adhesive layer, treating all members as linearly elastic. The interaction between the sheet and stiffener is idealized as a line loading. An integral equation in one variable is then formulated for the shear flow in the adhesive layer. This equation is solved in closed form for the case of a rigid sheet, but in general must be solved by numerical means. The solution reveals that the maximum stress concentration occurs at the loaded end of the stringer, with a smaller stress concentration appearing at the opposite end. These stress concentrations may be quite large when stiff adhesive layers are used. In this case the load transfer occurs mainly at the ends of the stringer. Thirty solutions have been tabulated together with instructions for interpolation.

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Lynn C. Lewis

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NOMENCLATURE

b	width of adhesive layer
q	shear flow in adhesive layer
t	thickness of sheet
u_1	displacement of stiffener
u_2	displacement of sheet
y	dimensionless axial coordinate
A	cross-sectional area of stiffener
E_1	Young's modulus for stiffener
E_2	Young's modulus for sheet
G	adhesive shear modulus
L	length of stiffener
P	external concentrated load
Q	dimensionless shear flow in the adhesive layer
T	axial force in stiffener
X, Y, Z	coordinate directions
β, γ	dimensionless parameters
η	thickness of adhesive layer
ν	Poisson's ratio for sheet
σ_X	sheet normal stress in X-direction
σ_Y	sheet normal stress in Y-direction
τ_{XY}	sheet shear stress
τ	shear stress in adhesive

I. INTRODUCTION

The use of sheet and stringer construction in airframe design has motivated considerable interest in a class of problems dealing with load diffusion from a stiffener into a sheet. This interest has been a result of the demands of the aerospace industry for lightweight, high-load-capacity structures.

Monolithic construction (sheet with integral stiffener) has been the subject of much investigation. The advent of chemical milling in the aerospace industry has made the monolithic case a very important and practical problem. Buell (1) investigated a semi-infinite sheet with a semi-infinite edge stiffener by means of a complex stress function for the sheet. Benscoter (2) considered an infinite sheet with a finite stiffener. He observed that the integral equation defining the shear flow between stiffener and sheet was formally identical with Prandtl's equation for the aerodynamic load distribution over a wing of finite span. This problem was solved approximately, using standard methods for the Prandtl equation. Koiter (3) investigated the case of an infinite sheet with a semi-infinite stiffener. He based his work on Benscoter's equation and presented a rigorous solution based on the application of Mellin transforms.

Riveting has also been a common means by which stiffeners have been attached to sheets. Bloom (4) investigated riveted sheet-and-stringer construction using the complex variable method of Muskhelishvili. He found that applying the method of compatible deformations to the sheet and stringer produces an infinite system of coupled algebraic equations in terms of the unknown rivet loads, solvable by truncation. Budiansky and Wu (5) have also contributed to the riveted-stringer load diffusion problem.

With the development of better adhesives, a large proportion of the structural connections in aerospace structures have been accomplished by adhesive bonding. Not only does the use of adhesives produce more uniform load diffusion in the joint and better fatigue life, but in the case of closed structural systems the adhesive provides adequate sealing for pressurization, fuel reservoirs, etc. The stress distribution in the adhesive layer of cemented lap joints has been investigated by Goland and Reissner (6). Solutions were obtained for two limiting cases, i.e., where the cement layer has negligible effect on the joint flexibility; and where joint flexibility is mainly due to the adhesive layer. This and other adhesive joint literature is reviewed by Benson (7).

In view of the foregoing, the problem which we consider here is of fundamental technical importance in aerospace construction. This is the case of a single stiffener, of constant cross-section and finite length, bonded by means of an adhesive to a sheet of infinite extent. The system is loaded by the application of an external force, acting at one end of the stiffener and in a direction parallel to the longitudinal axis of the stiffener. Our primary objective is to determine the shear flow or shear stress in the adhesive layer in the longitudinal direction. (Benthem (9) has examined the shear flow in a direction transverse to this longitudinal exis, for the case of a bonded hat-shaped stiffener.) In the present problem, the longitudinal shear flow is found to be governed by a one-dimensional integral equation, which appears to be analytically intractable and is therefore solved by a direct numerical procedure, for various combinations of the two dimensionless parameters governing the problem. Any bending moment introduced by the external force will be neglected, and the sheet, stringer, and adhesive layer are all assumed to be composed of linearly elastic materials. This neglect of bending implies that we are effectively splitting the stiffener into halves, one on each side of the sheet, with each side loaded by half the applied force. The sketches in the following section show the stiffener on one side of the sheet only, however.

Having made these assumptions we now proceed with the development of the governing equations.

II. MATHEMATICAL FORMULATION

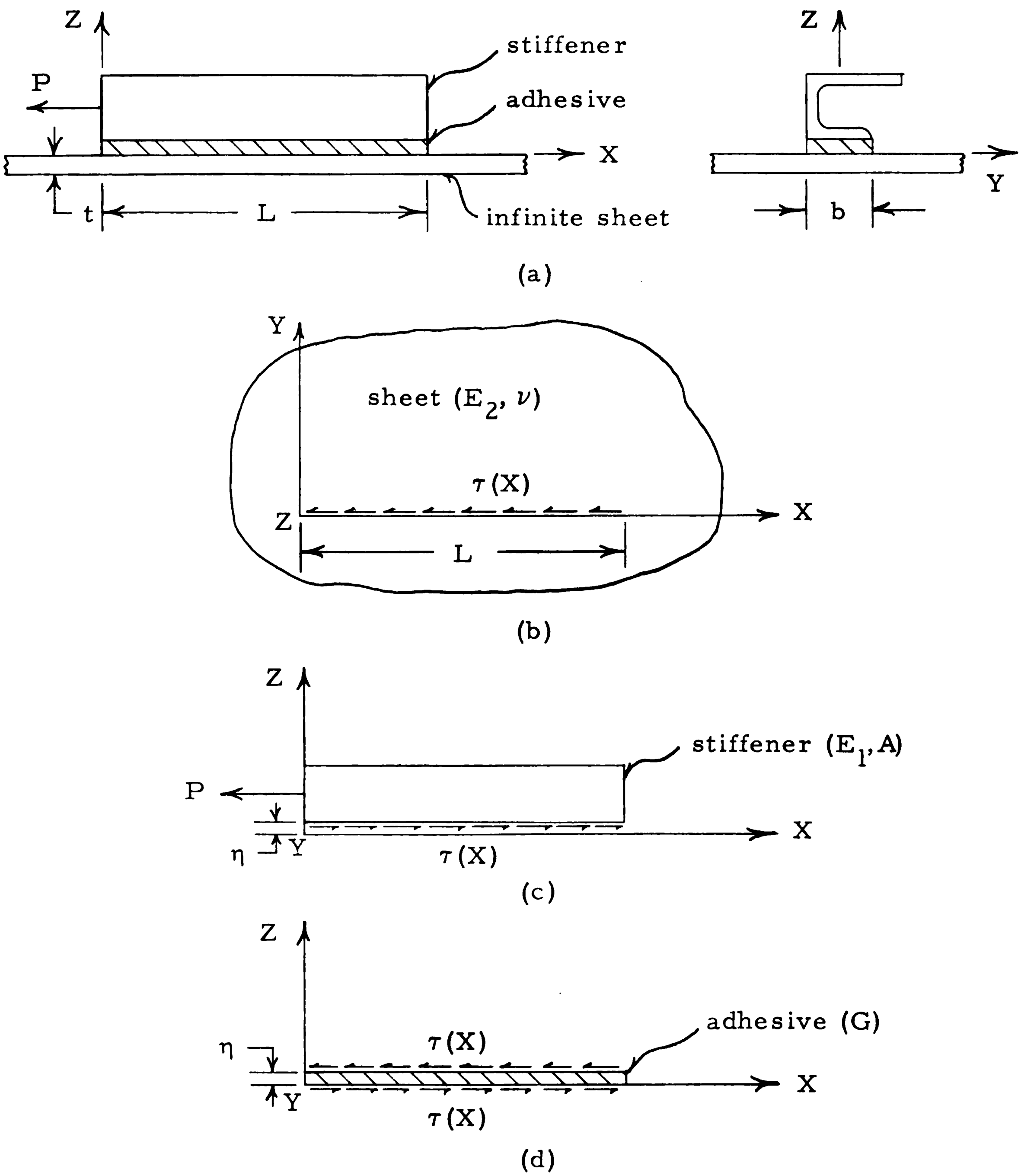

Figure 2-1. Infinite sheet with finite stiffener attached by means of an adhesive.

Referring to Figure 2-1, we consider an adhesive layer of small, constant thickness η, situated between a uniform reinforcing rib or stiffener of finite length L and an infinite elastic sheet. The adhesive layer is loaded by a system of continuous shear tractions τ, acting in the $Z = \eta$ plane and distributed from $X = 0$ to $X = L$. These stresses are applied by the stiffener, and their reactions act on the stiffener, both having been induced in the first instance by the external load **P**. The sheet is likewise loaded by a system of continuous tractions applied by the adhesive layer. These are distributed from $X = 0$ to $X = L$ in the plane $Z = 0$. The reactions to these stresses in turn act on the adhesive layer. It is now assumed that the adhesive layer is so thin (reflecting practice) that the adhesive stresses τ are uniform across the thickness of the adhesive layer. The complex three-dimensional stress disturbances at the ends $X = 0$ and $X = L$ of the adhesive layer are not examined in this study of the overall behavior of the joint.

Adopting a one-dimensional theory and neglecting bending, the adhesive shear strain may be expressed in the form

$$\gamma(X) = [u_2(X) - u_1(X)]/\eta \quad , \tag{2-1}$$

where $u_2(X)$ and $u_1(X)$ are the displacements of the sheet and stiffener respectively. The shear stress in the adhesive layer is given by

$$\tau(X) = G\,\gamma(X) \quad , \tag{2-2}$$

where G is the adhesive shear modulus. Axial equilibrium of a stiffener element of length dX requires that

$$dT(X)/dX + b\tau(X) = 0 \quad , \qquad (2\text{-}3)$$

where $T(X)$ is the axial force in the stiffener and b is the width of the adhesive layer in the Y-direction; see Fig. 2-1(a). The strain in the stiffener is given by the expression

$$du_1/dX = T(X)/E_1A \quad , \qquad (2\text{-}4)$$

which integrates to

$$u_1(X) = (1/E_1A)\int_0^X T(X)\,dX + u_{10} \quad , \qquad (2\text{-}5)$$

where u_{10} is the rigid-body displacement of the stiffener.

The sheet displacement u_2 along the line $X = 0$ may be obtained by means of a routine integration of the known strain distribution in an infinite sheet subjected to a concentrated load acting in the X-direction, together with the principle of superposition. The stress distribution for a concentrated force P_0 per unit sheet thickness, applied at the origin and acting in the positive X-direction, is given by Timoshenko (8).

$$\sigma_X = -(P_0 X/2\pi r^2)\,[\,(\nu + 3)/2 - (\nu + 1)\,Y^2/r^2] \qquad (2\text{-}6a)$$

$$\sigma_Y = -(P_0 X/2\pi r^2)\,[\,(\nu - 1)/2 + (\nu + 1)\,Y^2/r^2] \qquad (2\text{-}6b)$$

$$\tau_{XY} = -(P_0 Y/2\pi r^2)\,[\,(1 - \nu)/2 + (\nu + 1)\,X^2/r^2] \qquad (2\text{-}6c)$$

Here r is the radial coordinate with respect to the origin:

$$r = (X^2 + Y^2)^{1/2} \quad .$$

The normal strains for an arbitrary point are obtained by applying Hooke's law for plane stress, and these are readily integrated to obtain the sheet displacements. Specializing the result to the line X = 0, we get

$$u_2(X) = u_{20} - [\,(1 + \nu)\,(3 - \nu)/4\pi E_2]\;P_0 \ln |X| \quad , \qquad (2\text{-}7)$$

where u_{20} is a rigid-body displacement constant. Let the adhesive shear flow q be defined by

$$q(X) = \tau(X) \cdot b \quad . \qquad (2\text{-}8)$$

Replacing P_0 in Equation (2-7) by $-q(\xi)d\xi/t$ at $x = \xi$ and applying the principle of superposition, the sheet displacement for this problem becomes

$$u_2(X) = [\,(1 + \nu)\,(3 - \nu)/4\pi E_2 t\,] \int_0^L q(\xi)\,\ln|X - \xi|\,d\xi + u_{20}. \tag{2-9}$$

Integrating differential Equation (2-3), and using the boundary condition

$$T(0) = P \quad ,$$

we get

$$T(X) = P - \int_0^X \tau\, b d\xi \quad . \tag{2-10}$$

Since $q = \tau b$ Equation (2-10) becomes

$$T(X) = P - \int_0^X q(\xi)\,d\xi \quad . \tag{2-11}$$

Since

$$T(L) = 0,$$

we get

$$P = \int_0^L q(\xi)\, d\xi \quad . \tag{2-12}$$

Substituting, Equation (2-11) becomes

$$T(X) = \int_0^L q(\xi)\, d\xi - \int_0^X q(\xi)\, d\xi \quad ,$$

which may be written

$$T(X) = \int_X^L q(\xi)\, d\xi \quad . \tag{2-13}$$

Substituting in Equation (2-5) we get

$$u_1(X) = (1/E_1 A) \int_0^X dx_1 \int_{x_1}^L q(x_2) dx_2 + u_{10} \quad . \tag{2-14}$$

where x_1 and x_2 are dummy variables. We can arbitrarily dispose of one of the two rigid-body displacements. Taking $u_1(0) = 0$, Equation (2-9) and (2-14) become

$$u_1(X) = (1/E_1 A) \int_0^X dx_1 \int_{x_1}^L q(x_2)\, dx_2 \quad , \tag{2-15a}$$

$$u_2(X) = [(1 + \nu)(3 - \nu)/4\pi E_2 t] \int_0^L q(\xi) \ln|X - \xi| \, d\xi + u_{20} \quad , \tag{2-15b}$$

Substituting Equations (2-15) into (2-1) and combining this result with Equations (2-2) and (2-8) we obtain an integral equation for **q**

$$q(\mathbf{X}) = B_0 - (Gb/E_1 A\eta) \int_0^X dx_1 \int_{x_1}^L q(x_2) \, dx_2$$

$$+ [Gb(1 + \nu)(3 - \nu)/4\pi E_2 t\eta] \int_0^L q(\xi) \ln|X - \xi| \, d\xi \quad , \tag{2-16}$$

where

$$B_0 = u_{20} \, Gb/\eta \quad .$$

Making a change of variables to facilitate the use of dimensionless parameters, we define a dimensionless coordinate y and several dummy variables of integration by

$$y = X/L \; , \qquad y_i = x_i/L \;\; : \;\; (i = 1, 2) \; , \qquad u = \xi/L \quad .$$

A dimensionless shear flow Q and constant C_o are defined by

$$Q(y) = q(x)\, L/P \quad , \qquad\qquad C_0 = B_0 L/P \quad .$$

Equation (2-16) becomes

$$Q(y) = C_0 - (GbL^2/\eta E_1 A) \int_0^y dy_1 \int_{y_1}^1 Q(y_2)\, dy_2$$

$$+ [GbL(1 + \nu)(3 - \nu)/\eta 4\pi E_2 t] \int_0^1 Q(u)[\ln|y - u| + \ln L]\, du \quad . \tag{2-17}$$

Making the change of variables in Equation (2-12) we see that overall equilibrium requires

$$\int_0^1 Q(y) dy = 1 \quad . \tag{2-18}$$

Therefore (2-17) may be written

$$Q(y) = K - \beta^2 \int_0^y dy_1 \int_{y_1}^1 Q(y_2)\, dy_2 + \gamma^2 \int_0^1 Q(u) \ln|y - u|\, du \, , \tag{2-19}$$

where

$$K = C_0 + \gamma^2 \ln L \quad , \tag{2-20}$$

and

$$\beta^2 = GbL^2/\eta E_1 A \quad , \tag{2-21a}$$

$$\gamma^2 = GbL(1 + \nu)(3 - \nu)/4\pi\eta E_2 t \quad . \tag{2-21b}$$

The dimensionless parameters β and γ are the fundamental physical parameters of this problem.

At $y = 0$, the first integral vanishes and we see that K may also be expressed in terms of Q(0).

$$K = Q(0) - \gamma^2 \int_0^1 Q(u) \ln u du \quad . \tag{2-22}$$

Integrating the first integral in Equation (2-19) by parts with respect to y_1 , and applying Leibnitz's rule, the resulting equation, together with Equation (2-18), completely defines the function Q and the constant K.

$$Q(y) = K - \beta^2 y + \beta^2 \int_0^y (y - u) Q(u) du$$

$$+ \gamma^2 \int_0^1 Q(u) \ln |y - u| du \tag{2-23a}$$

$$1 = \int_0^1 Q(y) dy \tag{2-23b}$$

Equation (2-23a) would be a Volterra integral equation, except for the presence of the logarithmic integral. By substituting (2-23a) into (2-23b) we get

$$1 = K - \beta^2/2 + \beta^2 \int_0^1 \int_0^y (y - u)\, Q(u) du dy$$

$$+ \gamma^2 \int_0^1 \int_0^1 Q(u) \ln |y - u|\, du dy \quad . \tag{2-24}$$

This gives an explicit expression for eliminating K if desired. In the actual determination of Q, however, it proves to be simpler to avoid this procedure. The integral equation with K eliminated is therefore not given here.

Consider first the case where $\gamma = 0$. Equation (2-19) with $\gamma = 0$ can then be differentiated twice to produce

$$d^2Q/dy^2 - \beta^2 Q = 0 \quad , \tag{2-25}$$

which has the solution

$$Q = B_1 \sinh \beta y + B_2 \cosh \beta y \quad .$$

This suggests that we assume Equation (2-19) or (2-23a) to have a solution of the form

$$Q(y) = A_1 \sinh \beta y + A_2 \cosh \beta y + A_3 y + A_4 \quad . \tag{2-26}$$

Substituting into Equation (2-23a) with $\gamma = 0$, carrying out the integration, and equating coefficients of like powers of y, we find

$$A_1 = -K \tanh \beta \quad ,$$

$$A_2 = K \quad ,$$

$$A_3 = A_4 = 0 \quad .$$

It follows that

$$Q(y) = K (\cosh \beta y - \tanh \beta \sinh \beta y) \quad . \tag{2-27}$$

Substituting this result into Equation (2-23b) and integrating we get

$$K = \beta \coth \beta \quad ,$$

and thus we have the complete solution in closed form for the case $\gamma = 0$:

$$Q(y) = \beta \coth \beta (\cosh \beta y - \tanh \beta \sinh \beta y) \quad . \tag{2-28}$$

This solution could form the basis for a perturbation-type analysis, if γ is small. The sheet is effectively rigid when $\gamma = 0$, if we retain finite adhesive flexibility; see Equation (2-21b). The solution for $\gamma = 0$ thus reduces to the well-known Volkersen adhesive lap joint theory for the special case of one rigid member; this theory is documented in Benson (7).

The solution of Equations (2-23) for nonzero γ is obtained numerically here. This is accomplished by dividing the dimensionless adhesive length of unity into N equal increments of length h. The integrals are evaluated numerically, thus generating a system of linear algebraic equations in Q and K. The relation between N and h is given by

$$N = 1/h \quad .$$

The process of integration is carried out by a routine application of the trapezoidal rule, with the exception of the two increments on either side of the logarithmic singularity at $y = u$. In this region the integration is carried out with Q assumed constant and equal to its value at the singularity. Details of the numerical work are presented in Appendix B.

III. RESULTS AND DISCUSSION

3.1. Computational Results

The shear stress distribution in an adhesive layer bonding a finite stringer to an infinite sheet may be computed by a direct numerical process. The system of linear equations discussed in Chapter II and Appendix B may be solved by means of a Jordan pivotal-condensation technique. A solution must be obtained for each set of values of the parameters β and γ.

Considering again the two expressions (2-21) defining the nondimensional physical parameters β and γ we write

$$\beta = (k_a/k_1)^{1/2} \tag{3-1a}$$

$$\gamma = (k_a/k_2)^{1/2} \tag{3-1b}$$

where,

$$k_a = GLb/\eta \quad , \tag{3-2a}$$

$$k_1 = E_1 A/L \quad , \tag{3-2b}$$

$$k_2 = 4\pi E_2 t/\lambda \quad , \tag{3-2c}$$

and

$$\lambda = (1 + \nu)(3 - \nu) = -\nu^2 + 2\nu + 3 \quad . \tag{3-3}$$

The parameters k_a, k_1, and k_2 are the effective "spring constants" of the adhesive layer, stringer, and sheet respectively. The parameter β may then be interpreted as specifying the relative stiffnesses of the adhesive layer and the stringer, while γ represents the relative stiffnesses of the adhesive and the sheet.

The ratio of γ^2 to β^2 may be expressed by

$$\epsilon = \gamma^2/\beta^2 = k_1/k_2 \quad , \tag{3-4}$$

and may be interpreted as the relative stiffness of the stringer with respect to that of the sheet. Substituting Equations (3-2) into (3-4) we get

$$\epsilon = (\lambda/4\pi)(E_1/E_2)(A/tL) \quad . \tag{3-5}$$

Since the value of Poisson's ratio lies in the interval $0 < \gamma < 0.5$, it follows that the value of λ lies in the interval $3 < \lambda < 3.75$. For purposes of estimating the size of ϵ, we will assume that $\lambda/\pi \approx 1$. Therefore, Equation (3-5) becomes

$$\epsilon \approx (1/4)\ (E_1/E_2)\ (A/tL) \quad . \tag{3-6}$$

In most situations $E_1 \approx E_2$, so that the parameter ϵ depends primarily upon the geometry of the stringer and the adjacent sheet. Therefore Equation (3-6) becomes

$$\epsilon \approx A/4tL \quad . \tag{3-7}$$

In selecting a range for β, the constants which define β were given values typical for aerospace applications. While some practical problems may have been excluded, it seems likely that the principal phenomena of the problem have been exposed by the present choice. In order to facilitate interpolation in the calculated results, successive values of β increase by a factor of $\sqrt{10}$, so that log β increases by uniform increments. The five values chosen are $\beta = (10)^{-1/2}$, $(10)^0$, $(10)^{1/2}$, $(10)^{2/2}$, and $(10)^{3/2}$.

The parameter β is here regarded as the fundamental one, but only because the case $\gamma = 0$ ("pure-β problem") offers an exact solution. It appears that γ is likely to be comparable to (or smaller than) β in many physical problems, although the exceptional "pure-γ problem" ($\beta = 0$) is also conceivable in practice. Once again, the choice of the range of γ is thought to expose the characteristic behavior of this parameter sufficiently well for the present purposes. In the present calculations, γ appears indirectly: $\epsilon = \gamma^2/\beta^2$ is used as the primary parameter, and the uniform intervals $\sqrt{\epsilon} = 0(0.25)1.0$ are found to give a good spread of the curves Q(y) for any fixed value of β. Thus interpolation in the tabulated results with respect to $\sqrt{\epsilon}$ is facilitated. Note that the size of ϵ indicates the importance of the second integral of Equation (2-23a) in the solution.

The curves in Figures 3-1, 3-2, 3-3, and 3-4 show representative calculated distributions of dimensionless adhesive shear flow Q, as a function of the dimensionless coordinate y, for various values of β and ϵ. (Since the curves were plotted by the computer, only Y is available.) The rest of the calculations performed are tabulated in Appendix A. In each figure β is held constant and three curves are given, for the cases $\sqrt{\epsilon}$ = 0, 0.5, and 1.0. (The corresponding values of γ are listed at the top of each Figure.)

The function Q(y) may be interpreted as the stress concentration distribution in the adhesive since

$$Q = qL/P \quad , \tag{3-8}$$

where P/L is the average shear flow.

The shear flow Q may be interpreted in terms of two stressing mechanisms. Referring to the $\gamma = 0$ curves in Figures 3-1, 3-2, 3-3, and 3-4, we observe the asymmetric stress distribution characteristic of the pure-β mechanism. The curves show that the largest stress concentration is at the loaded end of the stringer (y = 0), and we observe a more-or-less exponential decline as y increases to one. For $\beta > 10$, the infinite-stringer case is effectively attained. When β is very large, all load is transferred in the immediate vicinity of the loaded end.

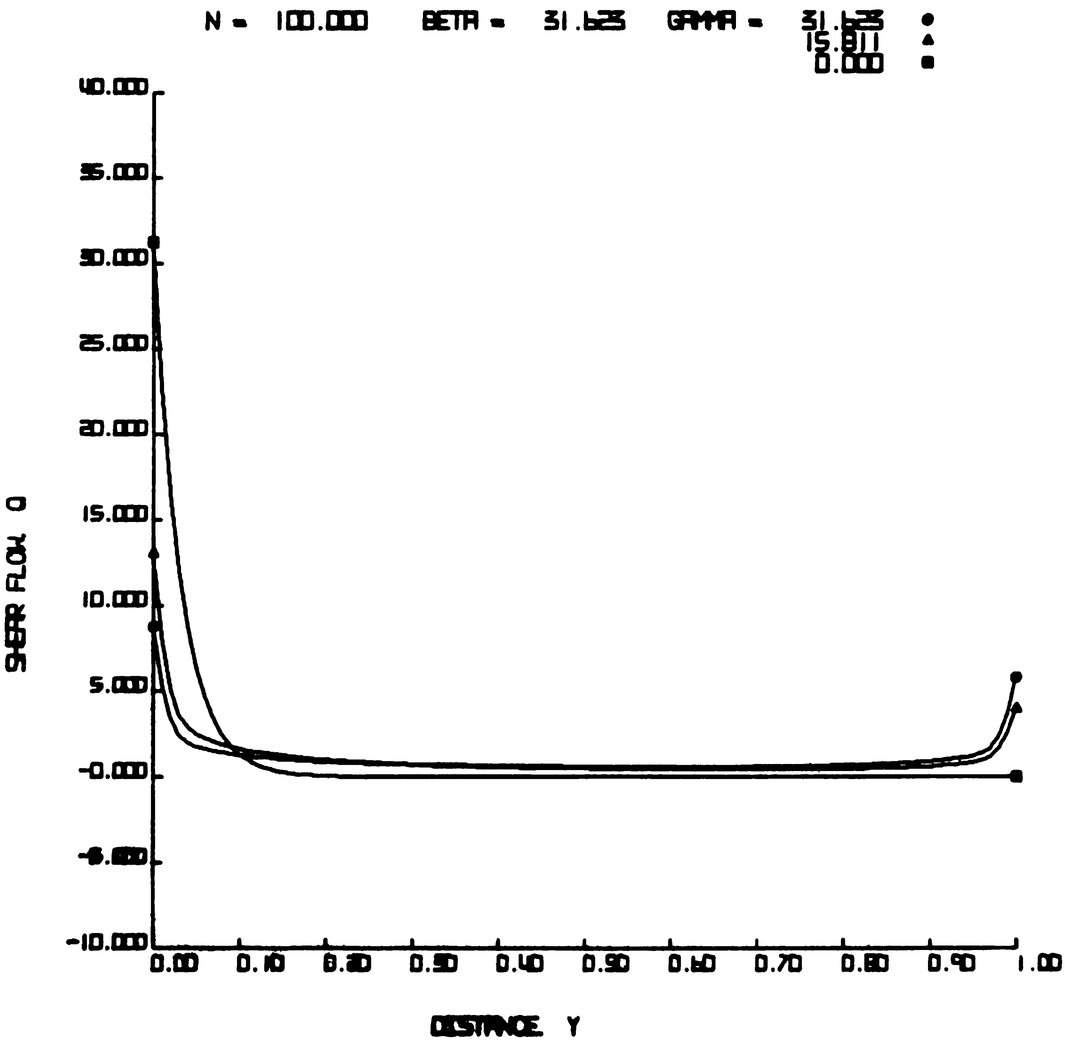

Figure 3-1. Shear flow Q for β = 31.623

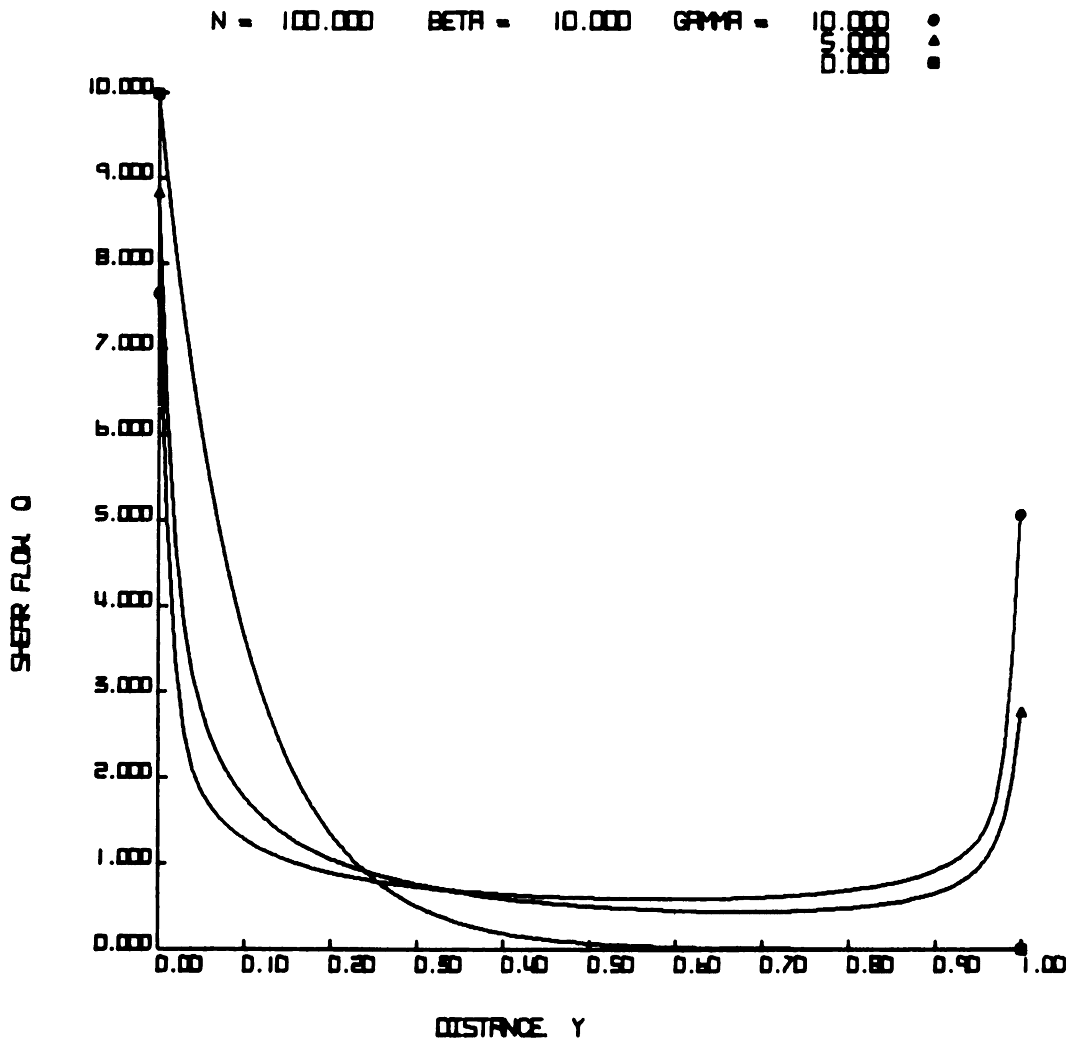

Figure 3-2. Shear flow Q for $\beta = 10.0$

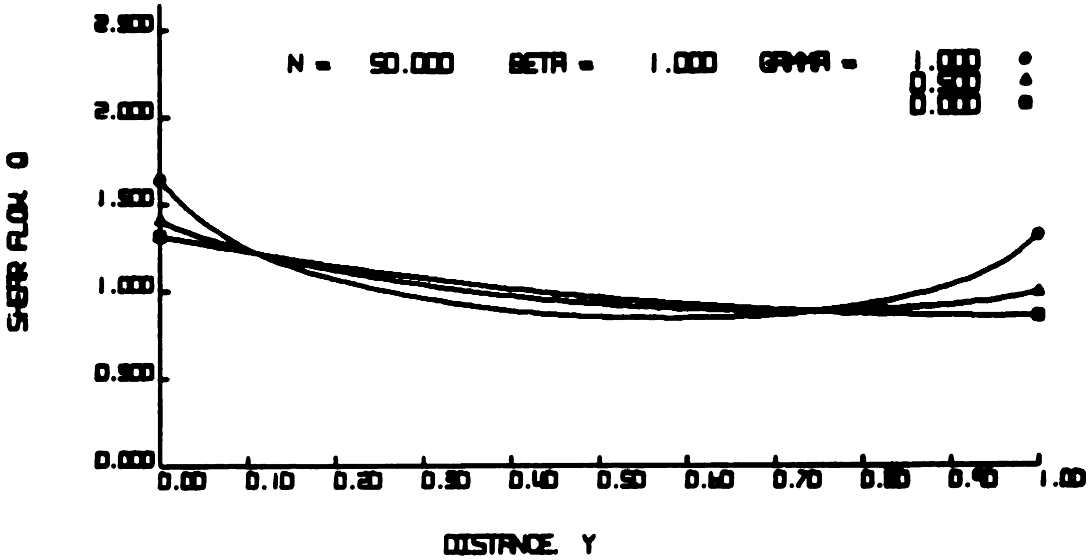

Figure 3-3. Shear flow Q for $\beta = 1.0$

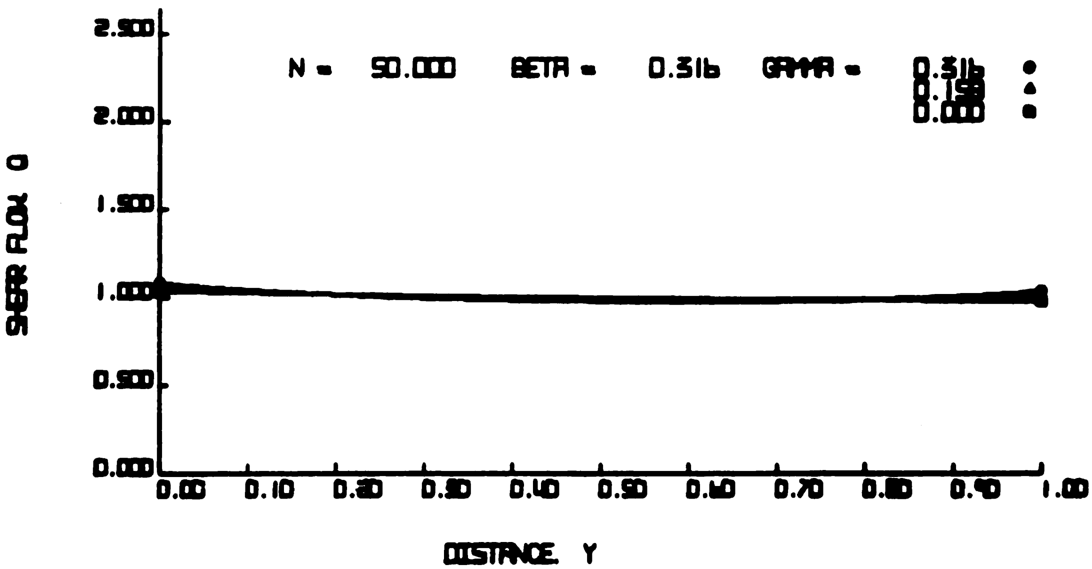

Figure 3-4. Shear flow Q for $\beta = 0.316$

The pure-β integral equation occurs when $\gamma = 0$ in Equation (2-23a). By direct differentiation, it may then be reduced to a Volkersen-type differential equation which may be solved exactly, as noted in Chapter II. From this solution, Equation (2-28), we see that the maximum shear flow occurs at $y = 0$ and may be expressed in terms of β.

$$Q_{MAX} = Q(0) = \beta \coth \beta \quad . \tag{3-9}$$

For $\beta > 3$

$$Q_{MAX} \approx \beta \quad . \tag{3-10}$$

Physically, the pure-β problem occurs when k_2 becomes large with respect to k_a; i.e., when a flexible stringer is attached to a rigid sheet by means of a flexible adhesive.

Representative solutions for the pure-γ problem are shown in Figure 3-5. The shear flow distribution here is a symmetric one, with stress concentrations at each end of the joint. This limiting case is achieved when k_1 becomes very large in comparison with k_a, as in the case of a rigid stringer attached to a flexible sheet by means of a flexible adhesive. The symmetry of $Q(y)$ in this case may be deduced from physical considerations, by imagining a rigid stringer with loads $P/2$ at each end, both acting in the negative y-direction. In the case of

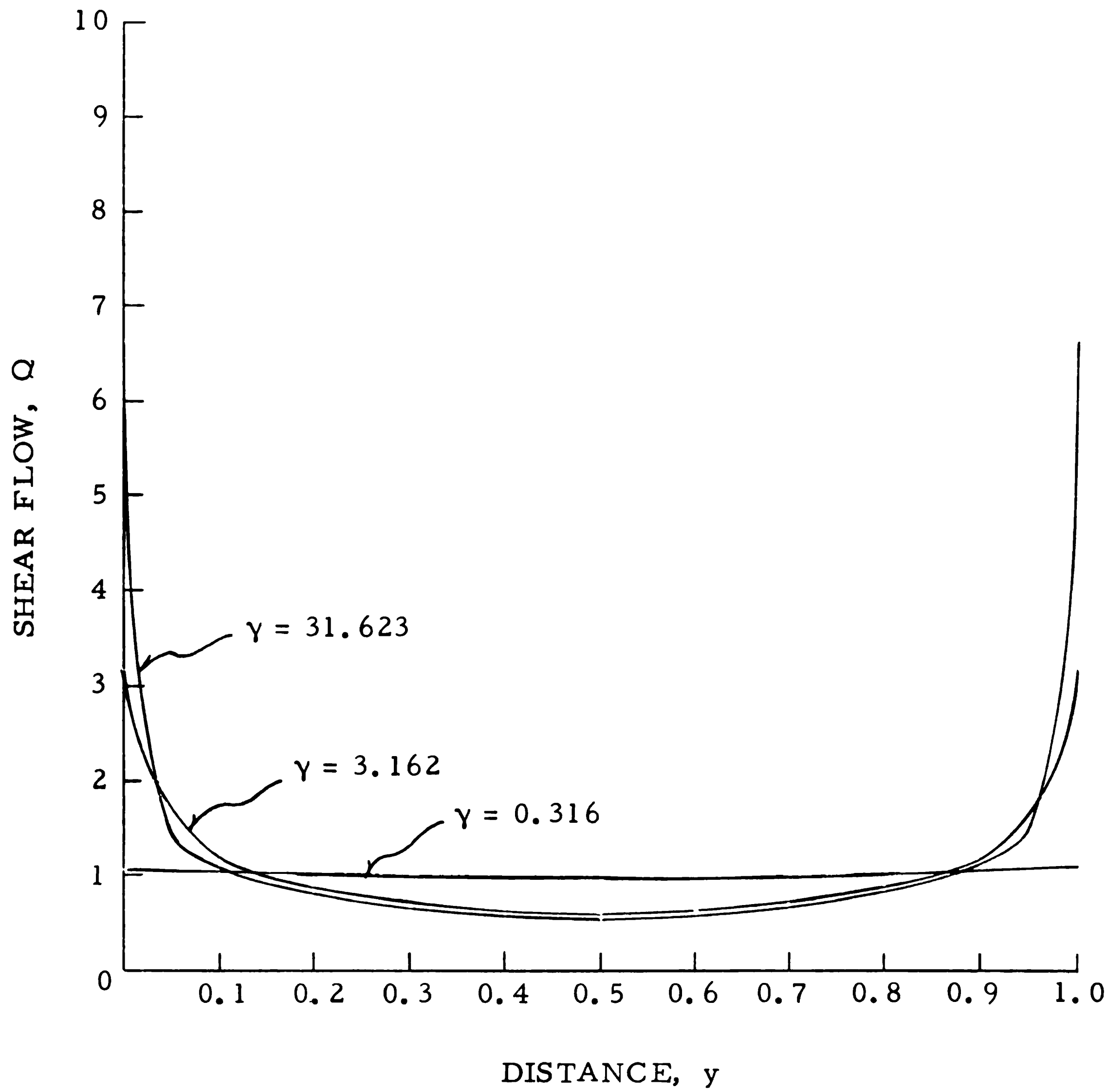

Figure 3-5. Shear flow Q for $\beta = 0$ and various values of γ.

large γ, the end stress concentrations indicate that load transfer is accomplished at the ends of the stiffener, for the most part.

Referring again to Figures 3-1, 3-2, 3-3, and 3-4, or to the integral Equation (2-23a), we see that the geometric parameter $\epsilon = \gamma^2/\beta^2$ controls the admixture of the two stressing mechanisms. As has been pointed out, ϵ specifies the relative stiffnesses of the stringer and sheet. On any one of these fixed-β plots, as ϵ increases an adhesive stress concentration appears at $y = 1$, accompanied by a significant change in the stress concentration at $y = 0$. This is the effect of adding the γ-mechanism. When β is large and the stresses are large, it is physically reasonable that an increase in ϵ (or γ), i.e., an increase in the flexibility of the sheet from the rigid state, should cause a decrease in stress concentration. This is observed for $\beta = 10^{3/2}$ and $\beta = 10$. For smaller values of β, the addition of the γ-mechanism causes an increase in the stress at $y = 0$. It is difficult to reason physically about the interplay of mechanisms involved in this phenomenon.

The maximum stress concentration occurs at $y = 0$ in all cases. As ϵ approaches unity and the γ-mechanism increases in importance, the stress distribution becomes more nearly symmetric with the interior of the joint experiencing a nearly uniform shear flow. As β decreases (more flexible adhesive or shorter stiffener), the magnitudes of the stress concentrations decrease, and in the limiting case when $\beta = \gamma = 0$, the shear flow becomes uniformly equal to unity: there is no stress concentration at all.

The monolithic case discussed in the Introduction is attained when the stringer becomes an integral part of the sheet, with the adhesive layer absent. We can achieve this condition in our present analysis if we multiply both sides of the integral equation (2-23a) by η/G. Now let the adhesive shear modulus G approach infinity and the thickness η of the adhesive layer approach zero. Recalling the definition of ϵ in Equation (3-4), we have

$$\int_0^y (y - u) Q(u)\, du + \epsilon \int_0^1 Q(u) \ln |y - u|\, du - y = 0 \quad . \quad (3\text{-}11)$$

By differentiating this with respect to y, redefining the variables suitably, and letting $L \to \infty$, the present equation can be reduced to that studied by Koiter (3). Since computation has been carried out here only for finite L and finite adhesive flexibility, the two problems are radically different and cannot be compared.

The tables in Appendix A present 25 sets of values of the dimensionless shear flow Q as a function of y, for uniform intervals of $\sqrt{\epsilon}$ and log β. This is done to facilitate two-way interpolation by the user. Five-point Lagrangian interpolation with respect to $\sqrt{\epsilon}$ and log β is thus possible for at least 51 values of y. Normally, only the values for $y = 0$ and $y = 1$ will be of interest in stress analysis. Appendix A also gives five values of the pure-γ solution, only three of which are plotted in Figure 3-5. Interpolation here is best done with respect to log γ.

While no exhaustive error analysis has been carried out, the results are believed to be reasonably accurate up to $\beta = 10$, for all values of ϵ. Basing any conclusions on the established accuracy of the $\gamma = 0$ family of solutions is admittedly doubtful unless ϵ is of moderate size, unfortunately. This is because the numerical procedure for the

γ-integral is the one most open to question. One possible way to test the solution is to fit calculated functions Q(y) accurately with polynomials in y, by sections. The γ-integral can then be evaluated analytically and the error in the integral equation for Q(y) assessed.

Another way is to increase the number of intervals and examine the results for changes. The sequence N = (number of intervals) = 75, 100, 125 would be illuminating, and would also offer the possibility of a Richardson-type extrapolation for the critical value of primary interest: Q(0). Improved accuracy, in principle, can be achieved for cases of large β and γ by using smaller intervals at the ends of the stringer, where the stress gradients are large. In the central regions where the variation of Q is small, a coarser interval is practicable to keep the total number of equations to be solved within reasonable limits. The available time has not permitted these refinements. The purpose of this thesis has been to offer a preliminary investigation of a very complex problem, and it is judged that the principal features of the physical problem have been adequately exposed.

3.2. Suggestions for Further Research

The principle of superposition allows us to use the solution of this fundamental problem for solving other problems of a more complex nature; for example, a compressive load on one end of the stringer and a tensile load on the other, two end tensile loads, etc.

Further research into the distribution of adhesive shear stresses in the transverse or Y-direction (throughout the width of the adhesive layer) should be considered. In the analysis presented here, the shear tractions applied to the sheet are assumed to be continuously distributed along a line coincident with the X-axis (dimensionless y-axis). Line loading is essentially singular in nature as far as sheet behavior is concerned and violates the real physical problem. It is a necessary recourse in a first study, to avoid the complexities of a truly two-dimensional problem. However, the more realistic problem warrants investigation.

The effect of bending and the elastic stability of the system should also be considered; bending in particular is of great practical importance. The consideration of the region of influence of the stiffener in the sheet would be useful for application to problems where several stringers are attached to the same sheet. For example, such a study would reveal when it is necessary to consider stringer interaction. Another obvious area for further analysis is the finite sheet problem, where the stiffener acts throughout the entire length of the sheet.

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Appendix A. TABULATED VALUES OF THE ADHESIVE SHEAR FLOW

The following tables give values for dimensionless shear flow, $Q = Q(y)$. Each table provides this information for a specific value of the parameter β and five values of the parameter $\epsilon = \gamma^2/\beta^2$. The values of ϵ are invariably taken as follows: $\sqrt{\epsilon} = 0,\ 0.25,\ 0.50,\ 0.75,\ 1.0$. The values of β are: $10^{-1/2}$, 10^0, $10^{1/2}$, $10^{2/2}$, $10^{3/2}$. In the actual tabulation, the values of γ corresponding to the given values of ϵ are shown as column headings. The variable y is the dimensionless axial coordinate, which ranges from 0 to 1, in some cases by intervals of 0.01 and in others of 0.02. (Since the tables are printed by computer, only a capital Y is available.)

The solution $Q(y)$ is also given when $\beta = 0$ ("pure-γ" case), for $\gamma = 10^{-1/2}$, 10^0, $10^{1/2}$, $10^{2/2}$, $10^{3/2}$.

To interpolate, calculate β and γ from the physical problem, then form $\sqrt{\epsilon} = \gamma/\beta$ and $p = 2\,(\log_{10}\beta) - 1$. For the tabulated values of β, $p = -2, -1, 0, 1, 2$, representing equal intervals. Interpolation with respect to p can then be carried out using Lagrangian interpolation coefficients, up to the five-point formula. Similarly, up to five-point interpolation is possible with respect to $\sqrt{\epsilon}$.

Interpolation in the pure-γ case may be carried out with respect to $\log_{10}\gamma$, by a scheme similar to that used for β.

TABLE 1. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 31.623

GAMMA =	31.623	23.717	15.811	7.906	0.000
Y					
0.00000	8.74969	10.06320	13.02620	20.72240	31.23480
0.01000	5.51104	6.34326	8.23971	13.39070	22.79650
0.02000	3.45091	3.97269	5.17851	8.64844	16.63790
0.03000	2.45299	2.81514	3.65494	6.11511	12.14310
0.04000	1.99644	2.27688	2.91951	4.73624	8.86253
0.05000	1.76383	1.99707	2.52064	3.90004	6.46826
0.06000	1.61612	1.81728	2.25880	3.32697	4.72082
0.07000	1.50307	1.67961	2.05882	2.89734	3.44546
0.08000	1.40868	1.56504	1.89418	2.55770	2.51465
0.09000	1.32770	1.46701	1.75459	2.28085	1.83530
0.10000	1.25767	1.38230	1.63468	2.05073	1.33948
0.11000	1.19673	1.30859	1.53070	1.85661	0.97761
0.12000	1.14333	1.24396	1.43974	1.69089	0.71350
0.13000	1.09616	1.18681	1.35947	1.54795	0.52075
0.14000	1.05417	1.13588	1.28805	1.42353	0.38006
0.15000	1.01652	1.09016	1.22406	1.31439	0.27739
0.16000	0.98255	1.04887	1.16634	1.21799	0.20245
0.17000	0.95173	1.01135	1.11399	1.13231	0.14776
0.18000	0.92362	0.97711	1.06628	1.05575	0.10784
0.19000	0.89788	0.94571	1.02259	0.98699	0.07871
0.20000	0.87423	0.91681	0.98244	0.92498	0.05744
0.21000	0.85242	0.89012	0.94540	0.86882	0.04192
0.22000	0.83224	0.86540	0.91112	0.81777	0.03060
0.23000	0.81353	0.84243	0.87931	0.77122	0.02233
0.24000	0.79613	0.82104	0.84971	0.72864	0.01630
0.25000	0.77993	0.80108	0.82211	0.68958	0.01190
0.26000	0.76481	0.78242	0.79631	0.65365	0.00868
0.27000	0.75068	0.76494	0.77215	0.62052	0.00634
0.28000	0.73746	0.74854	0.74949	0.58991	0.00462
0.29000	0.72507	0.73314	0.72819	0.56156	0.00338
0.30000	0.71345	0.71865	0.70816	0.53526	0.00246
0.31000	0.70254	0.70500	0.68928	0.51081	0.00180
0.32000	0.69229	0.69214	0.67147	0.48804	0.00131
0.33000	0.68266	0.68002	0.65465	0.46681	0.00096
0.34000	0.67362	0.66858	0.63875	0.44699	0.00070
0.35000	0.66511	0.65777	0.62372	0.42844	0.00051
0.36000	0.65712	0.64757	0.60948	0.41107	0.00037
0.37000	0.64961	0.63794	0.59599	0.39478	0.00027
0.38000	0.64255	0.62884	0.58321	0.37949	0.00020
0.39000	0.63594	0.62025	0.57110	0.36512	0.00014
0.40000	0.62974	0.61214	0.55960	0.35161	0.00011
0.41000	0.62393	0.60448	0.54870	0.33888	0.00008
0.42000	0.61851	0.59727	0.53836	0.32689	0.00006
0.43000	0.61345	0.59047	0.52855	0.31558	0.00004
0.44000	0.60875	0.58408	0.51924	0.30491	0.00003
0.45000	0.60438	0.57807	0.51042	0.29483	0.00002
0.46000	0.60036	0.57244	0.50206	0.28531	0.00002
0.47000	0.59665	0.56717	0.49414	0.27631	0.00001
0.48000	0.59326	0.56225	0.48665	0.26780	0.00001
0.49000	0.59018	0.55768	0.47957	0.25975	0.00001
0.50000	0.58740	0.55344	0.47288	0.25214	0.00000

TABLE 1. (CONT.)

GAMMA = Y	31.623	23.717	15.811	7.906	0.000
0.51000	0.58492	0.54953	0.46658	0.24493	0.00000
0.52000	0.58274	0.54594	0.46065	0.23812	0.00000
0.53000	0.58086	0.54267	0.45508	0.23167	0.00000
0.54000	0.57927	0.53972	0.44987	0.22558	0.00000
0.55000	0.57797	0.53708	0.44501	0.21982	0.00000
0.56000	0.57697	0.53476	0.44048	0.21438	0.00000
0.57000	0.57627	0.53275	0.43630	0.20924	0.00000
0.58000	0.57587	0.53105	0.43245	0.20440	0.00000
0.59000	0.57577	0.52966	0.42893	0.19984	0.00000
0.60000	0.57598	0.52860	0.42574	0.19556	0.00000
0.61000	0.57652	0.52787	0.42288	0.19154	0.00000
0.62000	0.57738	0.52746	0.42036	0.18779	0.00000
0.63000	0.57857	0.52739	0.41818	0.18428	0.00000
0.64000	0.58012	0.52768	0.41633	0.18103	0.00000
0.65000	0.58202	0.52832	0.41484	0.17802	0.00000
0.66000	0.58430	0.52934	0.41370	0.17526	0.00000
0.67000	0.58698	0.53074	0.41292	0.17274	0.00000
0.68000	0.59007	0.53255	0.41253	0.17047	0.00000
0.69000	0.59360	0.53479	0.41252	0.16844	0.00000
0.70000	0.59759	0.53747	0.41293	0.16667	0.00000
0.71000	0.60208	0.54063	0.41376	0.16516	-0.00000
0.72000	0.60709	0.54430	0.41505	0.16391	0.00000
0.73000	0.61267	0.54851	0.41681	0.16294	-0.00000
0.74000	0.61886	0.55330	0.41909	0.16226	0.00000
0.75000	0.62571	0.55872	0.42191	0.16189	0.00000
0.76000	0.63328	0.56482	0.42533	0.16184	-0.00000
0.77000	0.64164	0.57166	0.42938	0.16215	-0.00000
0.78000	0.65086	0.57932	0.43414	0.16284	-0.00000
0.79000	0.66105	0.58789	0.43966	0.16394	0.00000
0.80000	0.67230	0.59745	0.44603	0.16550	-0.00000
0.81000	0.68475	0.60813	0.45335	0.16757	0.00000
0.82000	0.69855	0.62007	0.46173	0.17022	-0.00000
0.83000	0.71389	0.63345	0.47132	0.17352	0.00000
0.84000	0.73098	0.64846	0.48227	0.17758	-0.00000
0.85000	0.75010	0.66536	0.49481	0.18250	0.00000
0.86000	0.77159	0.68446	0.50919	0.18843	-0.00000
0.87000	0.79588	0.70615	0.52575	0.19559	-0.00000
0.88000	0.82350	0.73095	0.54488	0.20420	0.00000
0.89000	0.85514	0.75946	0.56714	0.21459	-0.00000
0.90000	0.89163	0.79249	0.59318	0.22720	0.00000
0.91000	0.93398	0.83098	0.62385	0.24262	0.00000
0.92000	0.98337	0.87609	0.66024	0.26167	-0.00000
0.93000	1.04138	0.92941	0.70389	0.28560	0.00000
0.94000	1.11141	0.99420	0.75783	0.31655	-0.00000
0.95000	1.20386	1.08007	0.82990	0.35873	0.00000
0.96000	1.35159	1.21639	0.94254	0.42165	-0.00000
0.97000	1.64577	1.48391	1.15567	0.52768	-0.00000
0.98000	2.29490	2.06698	1.60542	0.72605	-0.00000
0.99000	3.64182	3.26987	2.51878	1.10219	0.00000
1.00000	5.76524	5.16529	3.95587	1.68883	-0.00000

TABLE 2. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 10.000

GAMMA = Y	10.000	7.500	5.000	2.500	0.000
0.00000	7.65098	8.09888	8.82164	9.68315	9.98752
0.01000	5.04775	5.49772	6.34954	7.84993	9.03746
0.02000	3.39134	3.83613	4.74805	6.56691	8.17777
0.03000	2.54212	2.94647	3.80785	5.64236	7.39986
0.04000	2.10601	2.45476	3.21701	4.93080	6.69595
0.05000	1.85383	2.14991	2.80928	4.35986	6.05900
0.06000	1.68357	1.93639	2.50588	3.88944	5.48264
0.07000	1.55437	1.77317	2.26812	3.49462	4.96110
0.08000	1.44982	1.64174	2.07518	3.15859	4.48918
0.09000	1.36247	1.53259	1.91468	2.86943	4.06215
0.10000	1.28813	1.44008	1.77865	2.61834	3.67573
0.11000	1.22401	1.36047	1.66161	2.39861	3.32608
0.12000	1.16810	1.29110	1.55969	2.20507	3.00969
0.13000	1.11886	1.23003	1.47002	2.03360	2.72339
0.14000	1.07512	1.17577	1.39044	1.88092	2.46433
0.15000	1.03598	1.12720	1.31927	1.74435	2.22991
0.16000	1.00072	1.08342	1.25521	1.62168	2.01779
0.17000	0.96877	1.04374	1.19722	1.51109	1.82585
0.18000	0.93969	1.00759	1.14446	1.41105	1.65216
0.19000	0.91308	0.97450	1.09623	1.32026	1.49500
0.20000	0.88866	0.94409	1.05198	1.23764	1.35279
0.21000	0.86616	0.91605	1.01123	1.16224	1.22411
0.22000	0.84537	0.89010	0.97358	1.09327	1.10766
0.23000	0.82610	0.86603	0.93869	1.03003	1.00230
0.24000	0.80821	0.84365	0.90628	0.97191	0.90695
0.25000	0.79155	0.82278	0.87609	0.91840	0.82068
0.26000	0.77602	0.80329	0.84791	0.86903	0.74261
0.27000	0.76152	0.78505	0.82156	0.82340	0.67197
0.28000	0.74795	0.76795	0.79687	0.78115	0.60805
0.29000	0.73524	0.75190	0.77370	0.74198	0.55021
0.30000	0.72333	0.73682	0.75193	0.70560	0.49787
0.31000	0.71216	0.72264	0.73143	0.67176	0.45051
0.32000	0.70167	0.70928	0.71213	0.64025	0.40766
0.33000	0.69182	0.69669	0.69392	0.61086	0.36888
0.34000	0.68256	0.68482	0.67672	0.58342	0.33379
0.35000	0.67387	0.67362	0.66047	0.55778	0.30204
0.36000	0.66570	0.66305	0.64511	0.53378	0.27331
0.37000	0.65803	0.65308	0.63057	0.51131	0.24731
0.38000	0.65083	0.64367	0.61681	0.49024	0.22378
0.39000	0.64408	0.63479	0.60377	0.47046	0.20250
0.40000	0.63776	0.62641	0.59143	0.45190	0.18323
0.41000	0.63184	0.61851	0.57973	0.43445	0.16580
0.42000	0.62632	0.61107	0.56864	0.41804	0.15003
0.43000	0.62117	0.60407	0.55813	0.40259	0.13576
0.44000	0.61638	0.59749	0.54819	0.38805	0.12285
0.45000	0.61195	0.59131	0.53876	0.37436	0.11116
0.46000	0.60785	0.58552	0.52985	0.36145	0.10059
0.47000	0.60409	0.58011	0.52142	0.34928	0.09102
0.48000	0.60065	0.57507	0.51345	0.33781	0.08236
0.49000	0.59753	0.57039	0.50593	0.32700	0.07453
0.50000	0.59472	0.56606	0.49885	0.31680	0.06744

TABLE 2. (CONT.)

GAMMA = Y	10.000	7.500	5.000	2.500	0.000
0.51000	0.59222	0.56207	0.49219	0.30718	0.06102
0.52000	0.59003	0.55842	0.48593	0.29811	0.05522
0.53000	0.58813	0.55510	0.48007	0.28957	0.04997
0.54000	0.58654	0.55211	0.47460	0.28153	0.04522
0.55000	0.58525	0.54945	0.46951	0.27396	0.04092
0.56000	0.58427	0.54712	0.46479	0.26684	0.03702
0.57000	0.58359	0.54511	0.46045	0.26016	0.03350
0.58000	0.58322	0.54343	0.45647	0.25390	0.03032
0.59000	0.58316	0.54209	0.45286	0.24804	0.02744
0.60000	0.58342	0.54108	0.44961	0.24257	0.02483
0.61000	0.58402	0.54041	0.44672	0.23748	0.02247
0.62000	0.58494	0.54008	0.44421	0.23276	0.02033
0.63000	0.58621	0.54012	0.44207	0.22840	0.01840
0.64000	0.58785	0.54052	0.44030	0.22440	0.01665
0.65000	0.58985	0.54130	0.43892	0.22075	0.01507
0.66000	0.59224	0.54247	0.43794	0.21745	0.01364
0.67000	0.59504	0.54405	0.43736	0.21451	0.01234
0.68000	0.59827	0.54606	0.43720	0.21191	0.01117
0.69000	0.60195	0.54852	0.43749	0.20967	0.01011
0.70000	0.60611	0.55146	0.43823	0.20779	0.00916
0.71000	0.61078	0.55490	0.43945	0.20629	0.00829
0.72000	0.61600	0.55887	0.44118	0.20517	0.00751
0.73000	0.62181	0.56343	0.44345	0.20445	0.00680
0.74000	0.62824	0.56860	0.44630	0.20414	0.00616
0.75000	0.63537	0.57445	0.44978	0.20427	0.00558
0.76000	0.64325	0.58102	0.45392	0.20486	0.00506
0.77000	0.65195	0.58839	0.45879	0.20595	0.00458
0.78000	0.66155	0.59663	0.46446	0.20758	0.00416
0.79000	0.67216	0.60585	0.47102	0.20977	0.00377
0.80000	0.68389	0.61614	0.47855	0.21260	0.00342
0.81000	0.69687	0.62765	0.48716	0.21612	0.00311
0.82000	0.71127	0.64051	0.49701	0.22040	0.00283
0.83000	0.72728	0.65494	0.50824	0.22552	0.00257
0.84000	0.74515	0.67114	0.52107	0.23160	0.00235
0.85000	0.76517	0.68941	0.53574	0.23876	0.00214
0.86000	0.78770	0.71010	0.55255	0.24713	0.00196
0.87000	0.81320	0.73365	0.57190	0.25692	0.00179
0.88000	0.84227	0.76063	0.59428	0.26833	0.00165
0.89000	0.87565	0.79178	0.62034	0.28164	0.00152
0.90000	0.91434	0.82808	0.65094	0.29719	0.00140
0.91000	0.95966	0.87088	0.68724	0.31543	0.00130
0.92000	1.01341	0.92207	0.73091	0.33692	0.00122
0.93000	1.07831	0.98450	0.78434	0.36240	0.00114
0.94000	1.15922	1.06300	0.85132	0.39291	0.00108
0.95000	1.26688	1.16704	0.93819	0.42989	0.00103
0.96000	1.42811	1.31761	1.05680	0.47549	0.00098
0.97000	1.71001	1.56382	1.23140	0.53320	0.00095
0.98000	2.26367	2.01442	1.51341	0.60940	0.00093
0.99000	3.34945	2.86288	1.99990	0.71701	0.00092
1.00000	5.06151	4.19819	2.75823	0.87388	0.00091

TABLE 3. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 3.162

GAMMA = Y	3.162	2.372	1.581	0.791	0.000
0.00000	3.81696	3.57507	3.34636	3.20460	3.17206
0.02000	2.80297	2.78844	2.81461	2.90578	2.97840
0.04000	2.17631	2.28668	2.45079	2.66710	2.79666
0.06000	1.80743	1.96198	2.18331	2.46298	2.62610
0.08000	1.56959	1.73194	1.97231	2.28302	2.46605
0.10000	1.40161	1.55782	1.79915	2.12206	2.31587
0.12000	1.27504	1.42024	1.65355	1.97682	2.17494
0.14000	1.17549	1.30830	1.52901	1.84499	2.04272
0.16000	1.09484	1.21528	1.42115	1.72479	1.91866
0.18000	1.02810	1.13672	1.32681	1.61484	1.80229
0.20000	0.97196	1.06950	1.24365	1.51400	1.69312
0.22000	0.92413	1.01141	1.16987	1.42132	1.59072
0.24000	0.88296	0.96078	1.10407	1.33598	1.49469
0.26000	0.84725	0.91636	1.04513	1.25730	1.40463
0.28000	0.81608	0.87717	0.99214	1.18467	1.32019
0.30000	0.78875	0.84246	0.94438	1.11757	1.24104
0.32000	0.76472	0.81162	0.90123	1.05553	1.16685
0.34000	0.74355	0.78416	0.86218	0.99814	1.09732
0.36000	0.72489	0.75969	0.82680	0.94504	1.03219
0.38000	0.70848	0.73789	0.79475	0.89590	0.97118
0.40000	0.69409	0.71849	0.76570	0.85044	0.91406
0.42000	0.68155	0.70128	0.73942	0.80838	0.86060
0.44000	0.67071	0.68609	0.71569	0.76951	0.81057
0.46000	0.66146	0.67278	0.69431	0.73360	0.76379
0.48000	0.65372	0.66123	0.67514	0.70047	0.72007
0.50000	0.64742	0.65136	0.65805	0.66996	0.67922
0.52000	0.64252	0.64310	0.64293	0.64190	0.64109
0.54000	0.63900	0.63642	0.62971	0.61617	0.60553
0.56000	0.63687	0.63129	0.61833	0.59266	0.57239
0.58000	0.63613	0.62770	0.60874	0.57124	0.54153
0.60000	0.63682	0.62570	0.60092	0.55184	0.51285
0.62000	0.63902	0.62531	0.59487	0.53438	0.48621
0.64000	0.64281	0.62660	0.59061	0.51879	0.46152
0.66000	0.64831	0.62969	0.58818	0.50501	0.43868
0.68000	0.65568	0.63470	0.58764	0.49300	0.41759
0.70000	0.66513	0.64180	0.58910	0.48274	0.39817
0.72000	0.67692	0.65124	0.59267	0.47421	0.38035
0.74000	0.69139	0.66329	0.59852	0.46740	0.36404
0.76000	0.70900	0.67834	0.60688	0.46233	0.34920
0.78000	0.73033	0.69687	0.61801	0.45902	0.33574
0.80000	0.75618	0.71955	0.63227	0.45752	0.32364
0.82000	0.78761	0.74723	0.65013	0.45789	0.31282
0.84000	0.82614	0.78108	0.67218	0.46022	0.30326
0.86000	0.87396	0.82275	0.69923	0.46465	0.29491
0.88000	0.93439	0.87462	0.73236	0.47134	0.28774
0.90000	1.01287	0.94031	0.77313	0.48052	0.28172
0.92000	1.11903	1.02570	0.82386	0.49253	0.27683
0.94000	1.27202	1.14132	0.88825	0.50784	0.27304
0.96000	1.51320	1.30837	0.97313	0.52729	0.27035
0.98000	1.92877	1.57240	1.09338	0.55263	0.26874
1.00000	2.60860	1.99480	1.27715	0.58877	0.26820

TABLE 4. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 1.000

GAMMA = Y	1.000	0.750	0.500	0.250	0.000
0.00000	1.64582	1.50956	1.40346	1.33606	1.31298
0.02000	1.51594	1.42717	1.35569	1.30929	1.29324
0.04000	1.42507	1.36654	1.31767	1.28531	1.27402
0.06000	1.35344	1.31631	1.28431	1.26282	1.25531
0.08000	1.29352	1.27267	1.25407	1.24148	1.23711
0.10000	1.24193	1.23391	1.22623	1.22110	1.21939
0.12000	1.19675	1.19901	1.20040	1.20160	1.20217
0.14000	1.15671	1.16733	1.17629	1.18288	1.18542
0.16000	1.12093	1.13838	1.15370	1.16491	1.16915
0.18000	1.08878	1.11181	1.13248	1.14764	1.15335
0.20000	1.05976	1.08735	1.11250	1.13103	1.13801
0.22000	1.03347	1.06479	1.09369	1.11507	1.12312
0.24000	1.00962	1.04394	1.07594	1.09972	1.10868
0.26000	0.98795	1.02467	1.05920	1.08497	1.09469
0.28000	0.96825	1.00685	1.04341	1.07079	1.08113
0.30000	0.95036	0.99037	1.02852	1.05718	1.06801
0.32000	0.93412	0.97516	1.01449	1.04411	1.05531
0.34000	0.91943	0.96114	1.00128	1.03157	1.04304
0.36000	0.90616	0.94824	0.98886	1.01956	1.03118
0.38000	0.89426	0.93641	0.97720	1.00806	1.01974
0.40000	0.88363	0.92561	0.96628	0.99706	1.00870
0.42000	0.87422	0.91579	0.95608	0.98655	0.99807
0.44000	0.86599	0.90693	0.94658	0.97652	0.98783
0.46000	0.85888	0.89899	0.93776	0.96698	0.97799
0.48000	0.85288	0.89195	0.92961	0.95790	0.96855
0.50000	0.84795	0.88580	0.92212	0.94929	0.95949
0.52000	0.84409	0.88052	0.91528	0.94114	0.95081
0.54000	0.84128	0.87611	0.90909	0.93345	0.94251
0.56000	0.83953	0.87256	0.90353	0.92620	0.93459
0.58000	0.83883	0.86987	0.89861	0.91940	0.92705
0.60000	0.83922	0.86805	0.89432	0.91305	0.91988
0.62000	0.84071	0.86710	0.89067	0.90715	0.91307
0.64000	0.84333	0.86705	0.88766	0.90168	0.90663
0.66000	0.84713	0.86792	0.88529	0.89665	0.90055
0.68000	0.85216	0.86972	0.88358	0.89206	0.89483
0.70000	0.85849	0.87251	0.88254	0.88792	0.88947
0.72000	0.86620	0.87631	0.88218	0.88421	0.88447
0.74000	0.87539	0.88118	0.88252	0.88096	0.87982
0.76000	0.88618	0.88718	0.88358	0.87815	0.87552
0.78000	0.89873	0.89440	0.88540	0.87579	0.87157
0.80000	0.91322	0.90292	0.88801	0.87389	0.86797
0.82000	0.92987	0.91287	0.89145	0.87247	0.86472
0.84000	0.94898	0.92439	0.89578	0.87152	0.86181
0.86000	0.97093	0.93766	0.90108	0.87107	0.85925
0.88000	0.99620	0.95295	0.90744	0.87113	0.85703
0.90000	1.02547	0.97057	0.91497	0.87174	0.85515
0.92000	1.05972	0.99101	0.92387	0.87292	0.85362
0.94000	1.10044	1.01498	0.93441	0.87475	0.85243
0.96000	1.15029	1.04375	0.94706	0.87732	0.85157
0.98000	1.21518	1.08013	0.96286	0.88085	0.85106
1.00000	1.31104	1.13275	0.98547	0.88623	0.85089

TABLE 5. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 0.316

GAMMA = Y	0.316	0.237	0.158	0.079	0.000
0.00000	1.08068	1.06011	1.04519	1.03614	1.03311
0.02000	1.06867	1.05250	1.04071	1.03354	1.03113
0.04000	1.05961	1.04656	1.03699	1.03116	1.02919
0.06000	1.05184	1.04134	1.03361	1.02889	1.02730
0.08000	1.04490	1.03660	1.03047	1.02671	1.02544
0.10000	1.03859	1.03223	1.02751	1.02461	1.02362
0.12000	1.03281	1.02817	1.02471	1.02257	1.02185
0.14000	1.02747	1.02437	1.02205	1.02061	1.02012
0.16000	1.02251	1.02081	1.01952	1.01870	1.01843
0.18000	1.01790	1.01746	1.01710	1.01686	1.01677
0.20000	1.01361	1.01432	1.01479	1.01507	1.01516
0.22000	1.00962	1.01135	1.01259	1.01334	1.01359
0.24000	1.00589	1.00856	1.01049	1.01167	1.01206
0.26000	1.00243	1.00593	1.00849	1.01005	1.01057
0.28000	0.99921	1.00346	1.00658	1.00848	1.00913
0.30000	0.99622	1.00113	1.00475	1.00697	1.00772
0.32000	0.99345	0.99896	1.00302	1.00551	1.00635
0.34000	0.99090	0.99692	1.00137	1.00410	1.00502
0.36000	0.98856	0.99503	0.99981	1.00274	1.00373
0.38000	0.98642	0.99326	0.99833	1.00144	1.00249
0.40000	0.98448	0.99163	0.99693	1.00018	1.00128
0.42000	0.98274	0.99013	0.99561	0.99897	1.00011
0.44000	0.98119	0.98876	0.99437	0.99782	0.99898
0.46000	0.97982	0.98751	0.99321	0.99671	0.99790
0.48000	0.97865	0.98639	0.99213	0.99566	0.99685
0.50000	0.97766	0.98539	0.99112	0.99465	0.99584
0.52000	0.97685	0.98452	0.99020	0.99369	0.99487
0.54000	0.97623	0.98377	0.98935	0.99279	0.99395
0.56000	0.97580	0.98315	0.98858	0.99193	0.99306
0.58000	0.97555	0.98265	0.98790	0.99112	0.99221
0.60000	0.97550	0.98228	0.98728	0.99036	0.99140
0.62000	0.97563	0.98203	0.98676	0.98965	0.99063
0.64000	0.97596	0.98191	0.98630	0.98900	0.98990
0.66000	0.97649	0.98193	0.98593	0.98839	0.98921
0.68000	0.97722	0.98208	0.98565	0.98783	0.98856
0.70000	0.97816	0.98237	0.98544	0.98732	0.98795
0.72000	0.97932	0.98279	0.98533	0.98686	0.98738
0.74000	0.98070	0.98337	0.98529	0.98646	0.98685
0.76000	0.98232	0.98409	0.98535	0.98611	0.98636
0.78000	0.98419	0.98497	0.98550	0.98581	0.98590
0.80000	0.98632	0.98602	0.98575	0.98556	0.98549
0.82000	0.98872	0.98725	0.98610	0.98537	0.98512
0.84000	0.99143	0.98866	0.98655	0.98523	0.98478
0.86000	0.99448	0.99028	0.98711	0.98515	0.98449
0.88000	0.99789	0.99212	0.98780	0.98513	0.98423
0.90000	1.00172	0.99421	0.98862	0.98518	0.98401
0.92000	1.00604	0.99659	0.98959	0.98529	0.98384
0.94000	1.01097	0.99932	0.99074	0.98547	0.98370
0.96000	1.01669	1.00251	0.99210	0.98574	0.98360
0.98000	1.02363	1.00639	0.99380	0.98612	0.98354
1.00000	1.03339	1.01185	0.99621	0.98671	0.98352

TABLE 6. DIMENSIONLESS SHEAR FLOW Q FOR BETA = 0.000

GAMMA =	31.623	10.000	3.162	1.000	0.316
Y					
0.00000	6.63950	5.81699	3.16707	1.41909	1.04900
0.01000	4.20694	3.86110	2.55542	1.35704	1.04332
0.02000	2.66511	2.62210	2.15887	1.31255	1.03904
0.03000	1.92244	1.99087	1.90015	1.27618	1.03535
0.04000	1.58600	1.66963	1.71613	1.24482	1.03204
0.05000	1.41701	1.48585	1.57644	1.21707	1.02901
0.06000	1.31113	1.36296	1.46576	1.19213	1.02620
0.07000	1.23075	1.27040	1.37544	1.16947	1.02358
0.08000	1.16396	1.19596	1.30012	1.14873	1.02112
0.09000	1.10692	1.13411	1.23626	1.12962	1.01880
0.10000	1.05781	1.08177	1.18136	1.11194	1.01660
0.11000	1.01531	1.03688	1.13365	1.09551	1.01452
0.12000	0.97828	0.99797	1.09179	1.08020	1.01254
0.13000	0.94577	0.96390	1.05478	1.06589	1.01066
0.14000	0.91701	0.93384	1.02182	1.05250	1.00887
0.15000	0.89138	0.90710	0.99230	1.03993	1.00717
0.16000	0.86841	0.88317	0.96573	1.02812	1.00554
0.17000	0.84770	0.86163	0.94170	1.01702	1.00399
0.18000	0.82896	0.84216	0.91988	1.00656	1.00251
0.19000	0.81192	0.82448	0.90000	0.99670	1.00109
0.20000	0.79638	0.80837	0.88184	0.98741	0.99974
0.21000	0.78216	0.79364	0.86520	0.97865	0.99845
0.22000	0.76911	0.78015	0.84992	0.97038	0.99722
0.23000	0.75712	0.76775	0.83586	0.96257	0.99605
0.24000	0.74608	0.75635	0.82291	0.95521	0.99493
0.25000	0.73590	0.74583	0.81095	0.94827	0.99387
0.26000	0.72649	0.73613	0.79991	0.94172	0.99286
0.27000	0.71780	0.72717	0.78971	0.93555	0.99189
0.28000	0.70976	0.71888	0.78027	0.92974	0.99098
0.29000	0.70233	0.71123	0.77154	0.92428	0.99012
0.30000	0.69545	0.70414	0.76346	0.91915	0.98930
0.31000	0.68910	0.69760	0.75599	0.91433	0.98852
0.32000	0.68322	0.69156	0.74910	0.90982	0.98780
0.33000	0.67780	0.68598	0.74273	0.90561	0.98711
0.34000	0.67281	0.68084	0.73687	0.90169	0.98647
0.35000	0.66821	0.67612	0.73148	0.89804	0.98587
0.36000	0.66400	0.67179	0.72653	0.89466	0.98531
0.37000	0.66015	0.66783	0.72201	0.89155	0.98479
0.38000	0.65664	0.66423	0.71790	0.88868	0.98432
0.39000	0.65346	0.66096	0.71417	0.88607	0.98388
0.40000	0.65060	0.65802	0.71081	0.88370	0.98348
0.41000	0.64804	0.65540	0.70781	0.88158	0.98312
0.42000	0.64578	0.65307	0.70516	0.87968	0.98280
0.43000	0.64380	0.65105	0.70284	0.87802	0.98252
0.44000	0.64211	0.64930	0.70085	0.87659	0.98228
0.45000	0.64068	0.64784	0.69918	0.87538	0.98207
0.46000	0.63952	0.64665	0.69782	0.87439	0.98190
0.47000	0.63862	0.64572	0.69677	0.87362	0.98177
0.48000	0.63798	0.64507	0.69601	0.87308	0.98168
0.49000	0.63760	0.64467	0.69557	0.87275	0.98162
0.50000	0.63747	0.64454	0.69542	0.87264	0.98160

TABLE 6. (CONT.)

GAMMA = Y	31.623	10.000	3.162	1.000	0.316
0.51000	0.63760	0.64467	0.69557	0.87275	0.98162
0.52000	0.63798	0.64507	0.69601	0.87308	0.98168
0.53000	0.63862	0.64572	0.69677	0.87362	0.98177
0.54000	0.63952	0.64665	0.69782	0.87439	0.98190
0.55000	0.64068	0.64784	0.69918	0.87538	0.98207
0.56000	0.64211	0.64930	0.70085	0.87659	0.98228
0.57000	0.64380	0.65105	0.70284	0.87802	0.98252
0.58000	0.64578	0.65307	0.70516	0.87968	0.98280
0.59000	0.64804	0.65540	0.70781	0.88158	0.98312
0.60000	0.65060	0.65802	0.71081	0.88370	0.98348
0.61000	0.65346	0.66096	0.71417	0.88607	0.98388
0.62000	0.65664	0.66423	0.71790	0.88868	0.98432
0.63000	0.66015	0.66783	0.72201	0.89155	0.98479
0.64000	0.66400	0.67179	0.72653	0.89466	0.98531
0.65000	0.66821	0.67612	0.73148	0.89804	0.98587
0.66000	0.67281	0.68084	0.73687	0.90169	0.98647
0.67000	0.67780	0.68598	0.74273	0.90561	0.98711
0.68000	0.68322	0.69156	0.74910	0.90982	0.98780
0.69000	0.68910	0.69760	0.75599	0.91433	0.98852
0.70000	0.69545	0.70414	0.76346	0.91915	0.98930
0.71000	0.70233	0.71123	0.77154	0.92428	0.99012
0.72000	0.70976	0.71888	0.78027	0.92974	0.99098
0.73000	0.71780	0.72717	0.78971	0.93555	0.99189
0.74000	0.72649	0.73613	0.79991	0.94172	0.99286
0.75000	0.73590	0.74583	0.81095	0.94827	0.99387
0.76000	0.74608	0.75635	0.82291	0.95521	0.99493
0.77000	0.75712	0.76775	0.83586	0.96257	0.99605
0.78000	0.76911	0.78015	0.84992	0.97038	0.99722
0.79000	0.78216	0.79364	0.86520	0.97865	0.99845
0.80000	0.79638	0.80837	0.88184	0.98741	0.99974
0.81000	0.81192	0.82448	0.90000	0.99670	1.00109
0.82000	0.82896	0.84216	0.91988	1.00656	1.00251
0.83000	0.84770	0.86163	0.94170	1.01702	1.00399
0.84000	0.86841	0.88317	0.96573	1.02812	1.00554
0.85000	0.89138	0.90710	0.99230	1.03993	1.00717
0.86000	0.91701	0.93384	1.02182	1.05250	1.00887
0.87000	0.94577	0.96390	1.05478	1.06589	1.01066
0.88000	0.97828	0.99797	1.09179	1.08020	1.01254
0.89000	1.01531	1.03688	1.13365	1.09551	1.01452
0.90000	1.05781	1.08177	1.18136	1.11194	1.01660
0.91000	1.10692	1.13411	1.23626	1.12962	1.01880
0.92000	1.16396	1.19596	1.30012	1.14873	1.02112
0.93000	1.23075	1.27040	1.37544	1.16947	1.02358
0.94000	1.31113	1.36296	1.46576	1.19213	1.02620
0.95000	1.41701	1.48585	1.57644	1.21707	1.02901
0.96000	1.58600	1.66963	1.71613	1.24482	1.03204
0.97000	1.92244	1.99087	1.90015	1.27618	1.03535
0.98000	2.66511	2.62210	2.15887	1.31255	1.03904
0.99000	4.20694	3.86110	2.55542	1.35704	1.04332
1.00000	6.63950	5.81699	3.16707	1.41909	1.04900

Appendix B. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION FOR Q

Writing Equation (2-23a) in the form

$$Q(y) = K - \beta^2 y + \beta^2 I_1(y) + \gamma^2 I_2(y) \quad , \tag{B1}$$

where

$$I_1(y) = \int_0^y (y - u)\, Q(u)\, du \tag{B2}$$

$$I_2(y) = \int_0^1 Q(u) \ln |y - u|\, du \tag{B3}$$

we apply the trapezoidal rule for N equal subintervals of length $h = 1/N$. From Equation (B2) we have in general

$$I_1(y_n) = (h^2/2)\, [\, nQ_0 + 2 \sum_{p=1}^{n-1} (n - p)\, Q_p \,] \tag{B4}$$

where

$$y_n = u_n = nh \quad ,$$

and

$$Q_i = Q(y_i) \quad .$$

To facilitate treatment of the logarithmic singularity write Equation (B3) as the sum of four integrals

$$I_2(y_n) = I_{21} + I_{22} + I_{23} + I_{24} \quad ,$$

where

$$I_{21} = \int_0^{y_{n-1}} F du \quad , \qquad I_{22} = \int_{y_{n-1}}^{y_n} F du \quad ,$$

$$I_{23} = \int_{y_n}^{y_{n+1}} F du \quad , \qquad I_{24} = \int_{y_{n+1}}^{y_N} F du \quad ,$$

and

$$F(u, y_n) = Q(u) \ln |y_n - u| \quad .$$

Note that $y_N = 1$.

Since the singularity occurs when $u = y_n$, it seems appropriate to give the greatest weight to the function $Q(u)$ at $u = y_n$. Thus in applying the

Theorem of the Mean to this integral we extract the function Q(u) and let Q take on the value $Q(y_n) = Q_n$ in the expressions for I_{22} and I_{23}. The rest of the integrand is integrable in closed form, with the result that

$$I_{22} + I_{23} = 2hQ_n (\ln h - 1) \quad .$$

The expressions for I_{21} and I_{24} may be treated in the usual manner; in general, the quadrature formula for $I_2(y_n)$ becomes

$$I_2(y_n) = 2hQ_n(\ln h - 1) + (h/2) \{ Q_o \ln (hn) + 2 \sum_{p=1}^{n-2} Q_p \ln [h (n - p)] + Q_{n-1} \ln h + Q_{n+1} \ln h + \sum_{p=n+2}^{N-1} Q_p \ln [h (p - n)] + Q_N \ln (1 - hn) \} . \quad \text{(B5)}$$

Special versions of this expression must be written when $y_n = y_0, y_1, y_{N-1}$, and y_N, but these all follow the general pattern described. At this point we have the capacity to approximate Equation (B1) as a system of N + 1 linear equations in the N + 2 unknowns $Q_o, Q_1, Q_2, \ldots, Q_N$, K, by writing Equation (B1) successively at $y_0 = 0$, $y_1 = h$, $y_2 = 2h, \ldots,$ $y_N = Nh = 1$.

The values of Q_n must also satisfy the equilibrium condition (2-23b), which furnishes the necessary (N + 2) th equation

$$1 = \int_0^1 Q(y)\, dy \quad , \tag{B6}$$

From Equation (B6), applying the two-point quadrature formula we have

$$1 = (h/2)Q_0 + h \sum_{p=1}^{N-1} Q_p + (h/2)\, Q_N \quad . \tag{B7}$$

Combining Ecuation (B1) and (B7), we have a system of linear algebraic equations in Q_n and k, here written with all specialized forms given in detail:

$$(B_1 - 1)\, Q_0 + B_3 Q_1 + \sum_{p=2}^{N-1} V_{0p}\, Q_p + K = 0 \quad , \tag{B8a}$$

$$B_2 Q_0 + (2B_1 - 1)\, Q_1 + B_3 Q_2 + \sum_{p=3}^{N-1} V_{1p}\, Q_p$$

$$+ \; (V_{1N}/2)\, Q_N + K = \beta^2 h \quad , \tag{B8b}$$

$$(U_{20}/2)\, Q_0 + (2B_2 + B_3)\, Q_1 + (2B_1 - 1)\, Q_2$$

$$+ B_3 Q_3 + \sum_{p=4}^{N-1} V_{2p} Q_p + (V_{2N}/2)\, Q_N$$

$$+ K = \beta^2 h \cdot 2 \quad , \tag{B8c}$$

$$(U_{n0}/2)\, Q_0 + \sum_{p=1}^{n-2} U_{np} Q_p + (2B_2 + B_3)\, Q_{n-1} + (2B_1 - 1)\, Q_n$$

$$+ B_3 Q_{n+1} + \sum_{p=n+2}^{N-1} V_{np} Q_p + (V_{nN}/2)\, Q_N$$

$$+ K = \beta^2 hn \quad : \qquad (n = 3, \ldots N - 3) \quad , \tag{B8d}$$

$$(U_{N-2,\,0}/2)\, Q_0 + \sum_{p=1}^{N-4} U_{N-2,\,p} Q_p + (2B_2 + B_3)\, Q_{N-3}$$

$$+ (2B_1 - 1)\, Q_{N-2} + B_3 Q_{N-1} + (V_{N-2,\,N}/2)\, Q_N$$

$$+ K = \beta^2\, hL(N - 2) \quad , \tag{B8e}$$

$$(U_{N-1,\,0}/2)\,Q_0 + \sum_{p+1}^{N-3} U_{N-1,\,p}Q_p + (2B_2 + B_3)Q_{N-2}$$

$$+ (2B_1 - 1)\,Q_{N-1} + K = \beta^2 h(N-1) \quad , \qquad \text{(B8f)}$$

$$(U_{N0}/2)\,Q_0 + \sum_{p=1}^{N-2} U_{Np}Q_p + (2B_2 + B_3)\,Q_{N-1}$$

$$+ (B_1 - 1)\,Q_N + K = \beta^2 \cdot 1 \quad , \qquad \text{(B8g)}$$

$$(h/2)\,Q_0 + h\sum_{p=1}^{N-1} Q_p + (h/2)\,Q_N = 1 \quad , \qquad \text{(B8h)}$$

where

$$B_1 = \gamma^2 h\,(\ln h - 1) \quad ,$$

$$B_2 = \beta^2 h/2 \quad ,$$

$$B_3 = (\gamma^2 h \ln h)/2 \quad ,$$

$$U_{np} = h\{\beta^2 h(n - p) + \gamma^2 \ln [h(n - p)]\} \quad ,$$

$$V_{np} = \gamma^2 h \ln [h (p - n)] \quad .$$

The computing hardware which is used consists of a Control Data 3600 computer in conjunction with an x-y plotter. The software consists of a program in Fortran IV, and is presented on the following pages. Because of the characteristics of Fortran, it is necessary to index n and p from 1 instead of 0. The constant K is regarded as Q_{N+2} in the computation.

```
      PROGRAM JOINT
      DIMENSION Y(101),Q(101),BETA(10),GAMMA(10),LBL(15),NPC(4),
     A          PRM(4,3),AH(101,4),AV(101,4),DET(100),QK(100),EPSQR(10)
      COMMON/BLK1/A(102,102),C(102,1)
      COMMON/BLK2/CF(100,10)
      EQUIVALENCE (C,Q)
C     GENERATE A(I,J)
  100 READ 1001, N, MB, MG, NCG, LN, LSQDEG
 1001 FORMAT (6I10)
      READ 1002, (BETA(K), K=1,MB)
      READ 1002, (EPSQR(L), L=1,MG)
 1002 FORMAT (8F10.0)
      READ 2000, (LBL(M),M=1,13)
 2000 FORMAT (3A8)
      FN=N $ H=1.0/FN
      NCRV=MB*MG
      DO 121 K=1,MB
      DO 120 L=1,MG
      GAMMA(L)=EPSQR(L)*BETA(K)
      NNC=(L+(K-1)*MG)-1 $ NN1=NNC/NCG $ NNT=NCG*NN1
      IF (NNC-NNT) 2,1,2
    1 NC=0
    2 NC=NC+1
      N2=N+2
      DO 42 I=1, N2
      DO 41 J=1, N2
      A(I,J)=0.0
C     DIAG
      IF (I.EQ.J) GO TO 11
      GO TO 12
   11 B1=(GAMMA(L)**2)*H*(LOGF(H)-1.0)
      IF (I.GE.2.AND.I.LE.N) GO TO 27
      IF (I.EQ.1.OR.I.EQ.N+1) GO TO 28
      IF (I.EQ.N+2) GO TO 29
   12 IF (J.EQ.I-1.OR.J.EQ.I+1) GO TO 13
      GO TO 17
```

```
C     LOWER DIAG
   13 B2=((BETA(K)*H)**2)/2.0
      B3=((GAMMA(L)**2)*H*LOGF(H))/2.0
      IF (J.EQ.I-1) GO TO 14
      GO TO 15
   14 IF (I.GE.3.AND.I.LE.N+1) GO TO 30
      IF (I.EQ.2) GO TO 38
      IF (I.EQ.N+2) GO TO 33
C     UPPER DIAG
   15 IF (J.EQ.I+1) GO TO 16
      GO TO 17
   16 IF (I.GE.1.AND.I.LE.N-1) GO TO 31
      IF (I.EQ.N) GO TO 29
      IF (I.EQ.N+1) GO TO 39
   17 IF (J.LE.I-2) GO TO 21
C     UPPER TRI
      B4=(GAMMA(L)**2)*H*LOGF(H*(J-I))
      IF (J.LE.N) GO TO 34
      IF (J.EQ.N+1) GO TO 35
      IF (J.EQ.N+2) GO TO 39
C     LOWER TRI
   21 B5=H*((BETA(K)**2)*H*(I-J)+(GAMMA(L)**2)*LOGF(H*(I-J)))
      IF (J.GE.2.AND.I.LE.N+1) GO TO 36
      IF (J.EQ.1.AND.I.LE.N+1) GO TO 37
C     EQUIL
   25 IF (I.EQ.N+2) GO TO 26
      GO TO 40
   26 IF (J.EQ.1) GO TO 33
      IF (J.GE.2.AND.J.LE.N) GO TO 32
      GO TO 40
   27 A(I,J)=2.0*B1-1.0       $ GO TO 40
   28 A(I,J)=B1-1.0           $ GO TO 40
   29 A(I,J)=0.0              $ GO TO 40
   30 A(I,J)=2.0*B2+B3        $ GO TO 40
   31 A(I,J)=B3               $ GO TO 40
   32 A(I,J)=H                $ GO TO 40
```

```
   33 A(I,J)=H/2.0          $ GO TO 40
   34 A(I,J)=B4             $ GO TO 40
   35 A(I,J)=B4/2.0         $ GO TO 40
   36 A(I,J)=B5             $ GO TO 40
   37 A(I,J)=B5/2.0         $ GO TO 40
   38 A(I,J)=B2             $ GO TO 40
   39 A(I,J)=1.0
   40 CONTINUE
   41 CONTINUE
   42 CONTINUE
C     COMPUTE C(I)
      DO 43 I=1,N2
      C(I)=0.0
      IF (I.GE.1.AND.I.LE.N+1) C(I)=(BETA(K)**2)*H*(I-1)
      IF (I.EQ.N+2) C(I)=1
   43 CONTINUE
      CALL MATINV (N2,1,DETERM)
      NNC1=NNC+1
      QK(NNC1)=C(N2) $ DET(NNC1)=DETERM
      N1=N+1
      DO 104 I=1,N1
      Y(I)=(I-1)*H
      AH(I,NC)=Y(I)  $  AV(I,NC)=Q(I)
  104 CONTINUE
      PRINT 1003,NNC1
 1003 FORMAT (1H1,100X,I3)
      PRINT 1006, N, BETA(K), GAMMA(L)
      PUNCH 1006, N, BETA(K), GAMMA(L)
 1006 FORMAT ( 24X,8H    N = ,I10,/,24X,8H BETA = ,F10.3,/,24X,
     A         8HGAMMA = ,F10.3,//)
  105 PRINT 1008, (I,Y(I),Q(I),I=1,N1)
      PUNCH 1008, (I,Y(I),Q(I),I=1,N1)
 1008 FORMAT (30X,10HDISTANCE,Y,10X,12HSHEAR FLOW,Q,//,
     A        (26X,I3,X,2(E12.5,8X)))
      NPC(NC)=N1
      PRM(NC,2)=BETA(K) $ PRM(NC,3)=GAMMA(L) $ PRM(NC,1)=N
```

```
      IF (NC.NE.NCG.AND.NNC.NE.NCRV-1) GO TO 120
      NCG=NC
      CALL GRAPH1(AV,AH,NPC,PRM,LBL,101,NCG,NCRV,LN,LSQDEG)
  120 CONTINUE
  121 CONTINUE
      PRINT 1010, (NNC1,DET(NNC1),QK(NNC1),NNC1=1,NCRV)
 1010 FORMAT (1H1,29X,6HDETERM,14X,2HQK,//,(26X,I3,X,2(E12.5,8X)))
      DO 125 K=1,MB
      DO 125 L=1,MG
      NNLS=L+(K-1)*MG
  125 PUNCH 1012, NNLS,BETA(K),EPSQR(L),(CF(NNLS,I),I=1,10)
 1012 FORMAT (I10,/,2F12.5,/,(5(E16.8)))
      END
      SUBROUTINE MATINV (N,M,DETERM)
      DIMENSION IPIVOT(102),INDEX(102,2),PIVOT(102)
      COMMON/BLK1/A(102,102),B(102,1)
C       INTIALIZATION
   10 DETERM=1.0
   15 DO 20 J=1,N
   20 IPIVOT(J)=0
   30 DO 550 I=1,N
C     SEARCH FOR PIVOT ELEMENT
   40 AMAX=0.0
   45 DO 105 J=1,N
   50 IF (IPIVOT(J)-1) 60, 105, 60
   60 DO 100 K=1,N
   70 IF (IPIVOT(K)-1) 80, 100, 740
   80 IF (ABSF(AMAX)-ABSF(A(J,K))) 85, 100, 100
   85 IROW=J
   90 ICOLUM=K
   95 AMAX=A(J,K)
  100 CONTINUE
  105 CONTINUE
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
  130 IF (IROW-ICOLUM) 140, 260, 140
```

```
  140 DETERM=-DETERM
  150 DO 200 L=1,N
  160 SWAP=A(IROW,L)
  170 A(IROW,LD=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
  205 IF(M) 260, 260, 210
  210 DO 250 L=1, M
  220 SWAP=B(IROW,L)
  230 B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
  260 INDEX(I,1)=IROW
  270 INDEX(I,2)=ICOLUM
  310 PIVOT(I)=A(ICOLUM,ICOLUM)
  320 DETERM=DETERM*PIVOT(I)
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
  330 A(ICOLUM,ICOLUM)=1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  355 IF(M) 380, 380, 360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
C     REDUCE NON-PIVOT ROWS
  380 DO 550 L1=1,N
  390 IF(L1-ICOLUM) 400, 550, 400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM)=0.0
  430 DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  455 IF(M) 550, 550, 460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C     INTERCHANGE COLUMNS
  600 DO 710 I=1,N
  610 L=N+1-I
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
```

```
  630 JROW=INDEX(L,1)
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW)=A(K,JCOLUM)
  700 A(K,JCOLUM)=SWAP
  705 CONTINUE
  710 CONTINUE
  740 RETURN
  750 END
      SUBROUTINE GRAPH1(YV,XH,NPC,PRM,LBL,MNPC,NCG,NCRV,LN,LSQDEG)
      DIMENSION YV(MNPC,NCG),XH(MNPC,NCG),NPC(4),PRM(4,3),LBL(15)
      COMMON/GRA/LSD
      DATA (NCURVE=0)
      LSD=LSQDEG
C     INITIALIZATION
      CALL PLOT (0.,0.,0, 80., 80.)
      CALL PLOT (0.,-13.75,2)
      CALL PLOT (0.,0.,0)
      CALL PLOT (2.,3.,2)
      CALL PLOT (0.,0.,0)
      IUB=(NCRV/NCG)*20+20
      CALL PLOT (IUB,X,3)
C     GRID
      FN=NCG-1 $ YLBT=FN*0.25+11.0
      CALL CHAR (YLBT,0.0,LBL(1),8,0.0,.15,.1)
      CALL CHAR (YLBT,1.25,LBL(2),8,0.0,.15,.1)
      CALL CHAR (YLBT,2.50,LBL(3),8,0.0,.15,.1)
      CALL PLOT (0.,0.,2)
      FLN=10.0
      IF (LN.NE.0) FLN=0.1
      DO 2 I=2,10,2
      A=I-1
      CALL PLOT (0.0,A,2)
      CALL PLOT (FLN,A,1)
      CALL PLOT (FLN,A+1.,2)
```

```
      CALL PLOT (0.,A+1.,1)
    2 CONTINUE
      CALL PLOT (0.,0.,1)
      DO 3 I=2,10,2
      A=I-1
      CALL PLOT (A,0.0,2)
      CALL PLOT (A,FLN,1)
      CALL PLOT (A+1.,FLN,2)
      CALL PLOT (A+1.,0.,1)
    3 CONTINUE
      CALL PLOT (0.,0.,1)
      CALL PLOT (0.,0.,2)
C     SCALE X
      CALL CHAR (-1.0,2.00,LBL(4),8,0.0,.15,.1)
      CALL CHAR (-1.0,3.25,LBL(5),8,0.0,.15,.1)
      CALL CHAR (-1.0,4.50,LBL(6),8,0.0,.15,.1)
      CALL NUM (XH,XL,XS, NPC,NCG,MNPC)
      DTX=XL-XS $ IDTX=DTX $ IF (DTX.EQ.1.0) GO TO 52
      IF (IDTX) 51,52,51
   51 IDX=0 $ GO TO 55
   52 DO 54 I=1,2 $  TT=10.0**I
      IDTX1=DTX*TT $ IDX=I
      IF (IDTX1) 53,54,53
   53 GO TO 55
   54 CONTINUE
   55 CALL SCALE (XL,XS,IDX)
      C=(XL-XS)/10.0
      ENCODE (6,4,IS)XS
    4 FORMAT (F6.2)
      CALL CHAR (-0.25,-0.26,IS,6,0.0,1./8.,1./12.)
      B=XS
      DO 5 I=1,10
      FF=I $ F=FF-0.26 $ G=B+C
      ENCODE (6,4,JS)G
      CALL CHAR (-0.25,F,JS,6,0.0,1./8.,1./12.)
      B=G
```

```
    5 CONTINUE
C     SCALE Y
      CALL CHAR (2.00,-1.5,LBL(7),8,90.,.15,.1)
      CALL CHAR (3.25,-1.5,LBL(8),8,90.,.15,.1)
      CALL CHAR (4.50,-1.5,LBL(9),8,90.,.15,.1)
      CALL NUM (YV,YL,YS, NPC,NCG,MNPC)
      DTY=YL-YS $ IDTY=DTY
      IF (IDTY) 61,62,61
   61 IDY=0 $ GO TO 65
   62 DO 64 I=1,3 $  TT=10.0**I
      IDTY1=DTY*TT $ IDY=I
      IF (IDTY1) 63,64,63
   63 GO TO 65
   64 CONTINUE
   65 CALL SCALE (YL,YS,IDY)
      C=(YL-YS)/10.0
      ENCODE (8,7,KS)YS
    7 FORMAT (F8.3)
      CALL CHAR (0.0,-1.10,KS,8,0.0,1./8.,1./12.)
      B=YS
      DO 9 I=1,10
      F=I $ G=B+C
      ENCODE (8,7,LS)G
      CALL CHAR (F,-1.10,LS,8,0.0,1./8.,1./12.)
      B=G
    9 CONTINUE
C     PARAMETER LIST
      DO 20 I=1,3
      YLB=FN*0.25+10.25 $ XLB=3*(I-1) $ LL=I+9
      CALL CHAR ( YLB,XLB,LBL(LL),8,0.0,.15,.1)
      DO 20 NC=1,NCG
      FN1=NCG-NC $ YP=FN1*0.25+10.25 $ XP=XLB+1.25
      IF (I.NE.3) GO TO 17
      NCURVE=NCURVE+1
      ENCODE (3,100,NCVE)NCURVE
  100 FORMAT (I3)
```

```
      CALL CHAR (YP,9.870,NCVE,3,0.,1./8.,1./12.)
      IF (NCG.EQ.1) GO TO 17
      YSY=YP+0.1  $ XSY=9.0
      CALL SYMBOL (NC,YSY,XSY,80.,80.)
   17 NCM1=NC-1
      IF (NC.GT.1.AND.PRM(NC,I).EQ.PRM(NCM1,I)) GO TO 20
      ENCODE (8,15,JP1)PRM(NC,I)
   15 FORMAT (F8.3)
      CALL CHAR (  YP,  XP,JP1,8,0.,.15,.1)
   20 CONTINUE
      CALL PLOT (0.,0.,2,80.,80.)
      CALL CURVE1(YV,XH,NPC,MNPC,NCG,XL,XS,YL,YS)
      CALL PLOT (0.,0.,0,80.,80.)
      CALL CHAR (12.5,0.,LBL(13),8,0.,0.5,1./3.)
      CALL PLOT (20.,0.,2)
      CALL PLOT (20.,0.,-1)
      END
      SUBROUTINE NUM(X,Y,Z,N,M,MN)
      DIMENSION X(MN,M),N(4)
      Y=X(1,1)
      DO 3 J=1,M
      NJ=N(J)
      DO 3 I=1,NJ
      B=X(I,J)
      IF (Y-B)1,3,3
    1 Y=B
    3 CONTINUE
      Z=X(1,1)
      DO 6 J=1,M
      NJ=N(J)
      DO 6 I=1,NJ
      C=X(I,J)
      IF(Z-C)6,6,4
    4 Z=C
    6 CONTINUE
      END
```

```
      SUBROUTINE SCALE (CL,CS,N)                                        6010
      P=10.0**(N-1) $ I=P*CS $ A=I $ C2=A/P                             6020
      IF (C2.EQ.0.0) GO TO 2                                            6022
      IF (CS-C2) 1,8,1                                                  6023
    1 CS=(A-1.0)/P $ GO TO 8                                            6024
    2 IF (CS) 4,8,6                                                     6026
    4 CS=-10.0**(-(N-1))  $ GO TO 8                                     6028
    6 CS=0.0                                                            6030
    8 CONTINUE                                                          6035
      DC=CL-CS $ J=P*DC $ B=J $ C1=B/P                                  6038
      II=0                                                              6040
   10 P1=P*10.0**(-II) $ J1=P1*DC                                       6050
      IF (J1.LT.10) GO TO 15                                            6060
      II=II+1 $ GO TO 10                                                6070
   15 IF (II) 17,16,17                                                  6075
   16 IF (DC.EQ.C1.AND.J1.EQ.1) GO TO 20                                6080
   17 IF (J1.LT.1) CL=1.0/P1+CS                                         6083
      IF (J1.GE.1.AND.J1.LT.5) CL=5.0/P1+CS                             6085
      IF (J1.GE.5.AND.J1.LT.10) CL=10.0/P1+CS                           6090
   20 RETURN                                                            6095
      END                                                               6100
      SUBROUTINE CURVE1(YV,XH,NPC,MNPC,NCG,XL,XS,YL,YS)                 6110
      DIMENSION YV(MNPC,NCG),XH(MNPC,NCG),NPC(4)                        6120
      COMMON/CRV/Y(101),X(101),NCURVE                                   6130
      DATA (NCURVE=0)                                                   6134
      SY=100./((YL-YS)/8.)                                              6140
      SX=100./((XL-XS)/8.)                                              6150
      CALL PLOT (YS,XS,0,SY,SX)                                         6160
      DO 25 NC=1,NCG                                                    6170
      NCURVE=NCURVE+1                                                   6175
      K=NPC(NC)                                                         6180
      DO 3 J=1,K                                                        6190
      Y(J)=YV(J,NC)                                                     6200
    3 X(J)=XH(J,NC)                                                     6210
      CALL LSTSQ1(K,XL,XS)                                              6290
    9 DO 10 I=1,101                                                     6300
```

```
      IF (Y(I).GT.YL-0.005) Y(I)=YL
      IF (Y(I).LT.YS+0.005) Y(I)=YS
   10 CONTINUE
   15 CALL PLOT (Y(1),X(1),2,SY,SX)
      IF (Y(1).NE.YS.AND.Y(1).NE.YL) CALL SYMBOL (NC,Y(1),X(1),SY,SX)
      NP=K
      IF (K.LT.101) NP=101
      DO 20 I=2,NP $ J1=1
      I1=I+1 $ IM1=I-1
      YT1=(Y(I)-YS)*(YL-Y(I)) $ YT2=(Y(IM1)-YS)*(YL-Y(IM1))
      IF (YT1.EQ.0.0.AND.YT2.EQ.0.0) J1=2
      CALL PLOT (Y(I),X(I),J1,SY,SX)
      IF (YT1.EQ.0.0.AND.I.NE.NP) GO TO 17
      IF (YT1.NE.0.0.AND.I.EQ.NP) GO TO 19 $ GO TO 20
   17 IF (Y(IM1).EQ.YL.AND.Y(I1).LT.YL) GO TO 19
      IF (Y(IM1).LT.YL.AND.Y(I1).EQ.YL) GO TO 19
      IF (Y(IM1).GT.YS.AND.Y(I1).EQ.YS) GO TO 19
      IF (Y(IM1).EQ.YS.AND.Y(I1).GT.YS) GO TO 19 $ GO TO 20
   19 CALL SYMBOL (NC,Y(I),X(I),SY,SX)
   20 CONTINUE
   25 CONTINUE
      CALL PLOT (YS,XS,2,SY,SX)
      END
      SUBROUTINE LSTSQ1(K,XL,XS)
      DIMENSION W(101),R(2,101),C(10)
      COMMON/CRV/Y(101),X(101),NCURVE
      COMMON/GRA/N
      COMMON/BLK2/CF(100,10)
      PRINT 100
  100 FORMAT (13H1LSTSQ OUTPUT,//)
      DO 5 I=1,K
    5 W(I)=1.0
      CALL MCPALS (K,N,0.0,W,X,Y,R,C,IDEG)
      DO 6 I=1,10
    6 CF(NCURVE,I)=C(I)
      PRINT 105, NCURVE,(I,R(1,I),R(2,I),I=1,K)
```

```
105 FORMAT (4X,*CURVE NO.*,I3,/,42X,5HERROR,15X,10HFRAC ERROR,//,
   A        (38X,I3,X,2(E12.4,8X)))
    IDEG1=IDEG+1
    PRINT 110, IDEG,(C(I),I=1,IDEG1)
110 FORMAT (//,X,*IDEG=*,I2,/,X,*LSTSQ COEF*,/,(X,5(E16.8,X)))
    IF (K.GE.101) GO TO 12
    DO 10 J=1,101
    X(J)=(J-1)*0.01*(XL-XS)+XS $ Y(J)=0.0
    IF (X(J).EQ.0.0) GO TO 9
    DO 8 I=1,IDEG1
    Y(J)=Y(J)+C(I)*X(J)**(I-1)
  8 CONTINUE $ GO TO 10
  9 Y(J)=C(1)
 10 CONTINUE
 12 RETURN
    END
    SUBROUTINE SYMBOL (NC,YI,XI,SY,SX)
    R=0.04
    CALL PLOT (YI,XI,2,SY,SX)
    GO TO (1,2,3,4),NC
  1 CALL CIRCLE (R,YI,XI) $ GO TO 5
  2 CALL TRI (R,YI,XI)     $ GO TO 5
  3 CALL SQU (R,YI,XI)     $ GO TO 5
  4 CALL DIA (R,YI,XI)
  5 CALL PLOT (YI,XI,1,SY,SX)
    END
    SUBROUTINE CIRCLE (R,YI,XI)
    CALL PLOT(YI,XI,2,100.,100.)
    CALL PLOT(YI,XI+R,2)
    DO 12 I=10,360,10
    A=I*3.1415926536/180.
    X=R*COSF(A)+XI  $ Y=R*SINF(A)+YI
    CALL PLOT(Y,X,1)
 12 CONTINUE
    CALL PLOT(YI,XI,2)
    END
```

```
      SUBROUTINE TRI (R,YI,XI)
      CALL PLOT (YI,XI,2,100.,100.)
      CALL PLOT (-(2./3.)*0.866*R+YI,XI+R,2)
      CALL PLOT ((4./3.)*0.866*R+YI,XI,1)
      CALL PLOT (-(2./3.)*0.866*R+YI,XI-R,1)
      CALL PLOT (-(2./3.)*0.866*R+YI,XI+R,1)
      CALL PLOT (YI,XI,2)
      END
      SUBROUTINE SQU (R,YI,XI)
      CALL PLOT (YI,XI,2,100.,100.)
      CALL PLOT (YI-R,XI+R,2)
      CALL PLOT (YI+R,XI+R,1)
      CALL PLOT (YI+R,XI-R,1)
      CALL PLOT (YI-R,XI-R,1)
      CALL PLOT (YI-R,XI+R,1)
      CALL PLOT (YI,XI,2)
      END
      SUBROUTINE DIA (R,YI,XI)
      CALL PLOT (YI,XI,2,100.,100.)
      CALL PLOT (YI,XI+R,2)
      CALL PLOT (YI+R,XI,1)
      CALL PLOT (YI,XI-R,1)
      CALL PLOT (YI-R,XI,1)
      CALL PLOT (YI,XI+R,1)
      CALL PLOT (YI,XI,2)
      END
      SUBROUTINE MCPALS(M,N,EPS,W,X,Y,R,C,IDEG)
         DIMENSION W(M),X(M),Y(M),A(10,10),SUMXSQ(19),C(10),R(2,M),B(10)
         SUMXSQ(1)=B(1)=0
         NMX=N
         IF((M-N-1).LT.0) NMX=M-1
         NMX1=NMX+1
         DO 1 I=1,M
            R(2,I)=1.0
            B(1)=B(1)+Y(I)*W(I)
1           SUMXSQ(1)=SUMXSQ(1)+W(I)
```

```
      R(1,1)=B(1)
      NMN=1
      IF(EPS.EQ.0) NMN=NMX
      DO 10 NN=NMN,NMX
         N2=2*NN
         N1=NN+1
         N2 =N2-1
         IF(EPS.EQ.0) N21=1
         DO 2 J=N21,N2
            J1=J+1
            IF(J1.LE.NMX1) B(J1)=0
            SUMXSQ(J1)=0
            DO 2 I=1,M
               R(2,I)=R(2,I)*X(I)
               SUM=R(2,I)*W(I)
               IF(J1.LE.NMX1) R(1,J1)=B(J1)=B(J1)+SUM*Y(I)
2              SUMXSQ(J1)=SUMXSQ(J1)+SUM
         DO 3 I=1,N1
            J1=I-1
            DO 3 J=1,N1
3              A(I,J)=SUMXSQ(J1+J)
         CALL GAUSS(N1,A,B,C)
         DO 4 I=1,N1
4           B(I)=R(1,I)
         DO 8 I=1,M
            SUM=C(N1)
            DO 5 J=1,NN
5              SUM=X(I)*SUM+C(N1-J)
            SUM=Y(I)-SUM
            IF((ABSF(SUM).LT.EPS).OR.(NN.EQ.NMX)) GO TO 7
            DO 6 J=1,NMX1
6              R(1,J)=B(J)
            GO TO 10
7           R(1,I)=SUM
8           CONTINUE
         DO 9 I=1,M
```

```
9          R(2,I)=R(1,I)/Y(I)
         IDEG=NN
         RETURN
10     CONTINUE
       RETURN
     END
     SUBROUTINE GAUSS(M,A,B,C)
       DIMENSION A(10,10),B(M),C(M)
101    FORMAT (//53X,30H***SINGULAR MATRIX IN GAUSS***//)
       DO 6 K=1,M
         C(1)=0
         IMAX=K
         DO 1 I=K,M
           T=ABSF(A(I,K))
         IF(C(1).GE.T) GO TO 1
           C(1)=T
           IMAX=I
1          CONTINUE
         IF(C(1).NE.0) GO TO 2
           PRINT 101
           RETURN
2        IF(K.EQ.IMAX) GO TO 4
           J=IMAX
           T=B(K)
           B(K)=B(J)
           B(J)=T
           DO 3 L=1,M
             T=A(K,L)
             A(K,L)=A(J,L)
3            A(J,L)=T
4        I=K+1
         DO 5 J=I,M
           T=A(J,K)/A(K,K)
           B(J)=B(J)-B(K)*T
           DO 5 L=I,M
5            A(J,L)=A(J,L)-T*A(K,L)
```

```
6         CONTINUE
      J=M+1
      DO 8 K=1,M
         I=J-K
         T=0
         IMAX=I+1
         DO 7 L=IMAX,M
7           T=T+A(I,L)*C(L)
8        C(I)=(B(I)-T)/A(I,I)
      RETURN
    END
```