APPLICATION OF A FIRST ORDER AUTOREGRESSIVE MODEL TO SOME ECONOMIC RELATIONS

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APPLICATION OF A FIRST ORDER AUTOREGRESSIVE MODEL TO SOME ECONOMIC RELATIONS

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ABSTRACT

In this study, examples of the estimation of supply and demand relations for agricultural products were examined with a view to obtain some indications of the importance of inefficiencies in traditional least squares procedures when disturbances are autocorrelated. A statistical test of autocorrelation was applied to a number of previously-fitted relations. A modified least squares procedure was used to re-estimate parameters of those relations for which the test rejected the null hypothesis of serial independence of disturbances.

The test applied was the Durbin-Watson test of serial independence of disturbances. The modified least squares method suggested was based on a first order autoregressive model. In this model, disturbances are no longer regarded as statistically independent but are considered to be generated by a first order autoregressive process, ($u_t = \rho u_{t-1} + v_t$, where u and v denote elements of non-independent and independent disturbances respectively, and ρ is a constant which is often called the autocorrelation coefficient.)

In all, there were nineteen regression equations tested by the Durbin-Watson test. Only in five cases, the hypothesis of zero autocorrelation at the 10% significance level was accepted. In ten cases, the test was indeterminate; and in four cases, the null hypothesis was rejected.

Regarding the results of applying a first order autoregressive model to those cases where a significant autocorrelation of disturbances

was established, the sum of squared residuals from a regression line was substantially reduced, as expected, from that obtained from the usual least squares estimates; and different values of parameters were obtained.

When the autocorrelation coefficient ρ is set equal to +1 in a first order autoregressive model, it is called the first difference model. Since the first difference model has been frequently fitted by economist to eliminate some of unfavorable effects of positive autocorrelation, it was also applied in this study to those cases where a significant positive autocorrelation had been established to see what effects it would have on estimates of regression coefficients and the sum of squared residuals. In three cases, the sum of squared residuals was increased; and in one case, it was reduced. However, there seemed to be a tendency for the regression coefficients estimated from the first differences of variables to be a closer approximation of the true values of regression coefficients when there was an indication of positive autocorrelation of disturbances.

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CHAPTER I

INTRODUCTION

Scope of Study

Generally speaking, the ultimate goal of studies like the present one is to contribute to the improvement of statistical analysis of the economic relations determining movement of price and quantity. If the statistical analysis of such economic relations could be more accurate and reliable, the significance of its application to both public and private policy would be considerably enhanced.

In this thesis, the author reports the results of some empirical investigations into one of the many problems that may arise when some of the assumptions on which statistical analysis is based are unrealistic. These assumptions may be roughly classified into several categories. 1) An investigator may assume that he has completely and appropriately specified all the relevant variables in his model. 2) He may assume that the variables are measured relatively free from error. 3) He may assume that the variables can be described by a system of single equation (i.e., the system that contains only one dependent variable.) 4) He has to make an assumption with regard to a form of algebraic equation to be fitted to data. 5) He may regard that disturbances[1] are statistically independent, (i.e., no autocorrelation is involved.)

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[1] Disturbances are random variables expressing the difference between the observed values of the dependent variables and their expected values. On the other hand, residuals are differences between observed and calculated values of the dependent variables.

When any of the above assumptions cannot be maintained, it tends to cast doubt on the usefulness of statistical analysis. In the present study the author forcuses his attention on the problem of autocorrelated disturbances. In order to concentrate on the problem, he has ignored the difficulties arising from the inadequacy of other assumptions at present. However, it should be obvious that for the purpose of estimating structural parameters it is necessary to find a method of dealing simultaneously with all other difficulties.

Immediate Objectives

Since it has been known that much efficiency is lost by current methods of estimation and prediction if autocorrelation of disturbances exists, the author tries to see the extent to which autocorrelation is involved in some current formulations of economic relations. It is hoped that this attempt will call the attention of economists to the seriousness of the problem.

It has been suggested that to regain the lost efficiency some modification of the usual least squares method of estimation is required. Therefore, a modified method of the usual least squares regression is applied to several empirical studies, in which there is evidence to suspect that autocorrelation of disturbances exists, to estimate structural coefficients. The modified method employed in the present study is a first order autoregressive model[2]. It is the author's intention to find out

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[2] In this model disturbances are no longer regarded as statistically independent but are assumed to be generated by a first order autoregressive scheme, $u_t = \rho u_{t-1} + v_t$, where u and v stand for non-independent and independent disturbances respectively, and ρ denotes a constant called the autocorrelation coefficient. Sometimes it is called a first order Markoff model.

how much improvement in estimating structural coefficients of these models can be made even by a slight relaxation of the assumption with respect to disturbances as is done in applying a first order autoregressive model.

Some economists have often applied the first difference model[3] when they encountered positive autocorrelation of disturbances. They claim that the first difference model will eliminate some of unfavorable effects of autocorrelation. However, one statistician[4] recommends that one must be cautious in applying the first difference model. He reports that the application of the first difference model does not always result in an increased efficiency of estimation. These different opinions as to the advisability of the use of the first difference model are checked in the present study. In short, the usual least squares estimates of regression coefficients and sum of squared residuals are compared with those obtained from the first order autoregressive model and with those obtained from the first difference model.

Finally, in the procedure suggested in this study (i.e., the application of a first order autoregressive model in estimating structural coefficients) it is necessary to minimize the sum of squared residuals. Like any other problem of maximum and minimum, one must ascertain whether the minimum obtained is the absolute minimum or a relative minimum within

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[3] The first difference model is the same as a first order autoregressive model in which ρ is set equal to +1. See the footnote in the previous page.

[4] G.S.Watson gives an extensive discussion of the problem of autocorrelation in his Ph.D dissertation, Serial Correlation, Department of Applied Statistics, North Carolina State College, Raleigh, N.C.

an interval examined. In other words, one must determine if a multiple-minimum situation esists.

CHAPTER II

THE METHOD OF ANALYSIS[1]

Selection of Some Empirical Studies

In this study the author proposes to test empirically some of the new ideas developed by statisticians and economists in regard to the problem of autocorrelated disturbances. For this purpose some interesting empirical economic models were selected. In all of the models selected the usual least squares method of estimation and prediction was employed.

Some efforts were made in the course of selection of these models so that difficulties arising from the inadequacy of the statistical assumptions, other than the one concerned with the problem of autocorrelated disturbances, might be of secondary importance. The selection was also limited to studies for which original data were available. These studies include work done at the U.S.D.A., and the California, North Carolina and Michigan Agricultural Experiment Stations.

Significance Test of Serial Independence of Disturbances

In investigating the extent to which autocorrelation of disturbances is involved in a group of the selected empirical studies, it seems natural to apply a statistical test to them. Another objective of the present study is to analyze the effects of substituting a first order autoregressive model for the usual least squares model in some empirical studies. Before carrying out this substitution, it is desirable to decide

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1 This method of analysis was outlined to the author by Professor Hildreth.

on those studies where autocorrelation of disturbances at the significance level concerned can be established.

For these reasons, the application of a significance test of serial independence of disturbances is considered necessary . Although there are several tests of this kind available, the Durbin-Watson test[2] is considered most satisfactory for the present study. Other commonly used tests such as von Neuman's ratio test are not strictly appropriate because they are designed to be applied to observed variables.

When the Durbin-Watson test indicates significant autocorrelation of disturbances in a least squares regression model at the 10% significance level, that model is chosen for further investigation (i.e., to substitute a first order autoregressive model for the ordinary least squares model.) When the tests are indeterminate or do not indicate significant autocorrelation of disturbances, no further steps are taken at present.

The choice of the 10% significance level as a dividing line is rather arbitrary. Since the number of the studies tested by the Durbin-Watson test is limited, the author has to choose the 10% level as the significance level concerned so that there will be sufficient number of studies to which a first order autoregressive model may be applied.

<u>Least Squares Estimates of Regression Coefficients</u>

Before a first order autoregressive model applied in this study is introduced, it is desirable to discuss briefly the derivation of

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2 Durbin, J. and Watson, G.S. "Testing for Serial Correlation in Least Squares Regression I," Biometrika, Vol. 37, pp.409-428, "Testing for Serial Correlation Least Squares Regression II," Biometrika, Vol.38, pp.159-178.

regression coefficients estimates by the usual least squares method. This discussion probably will help a reader to see how the two models are related.

Observations are available on a dependent variable, y_t, $t=1,2,..,T$, and certain fixed variables z_{tk}, for $t=1,2,....,T$, and $k=1,2,.......,K$. y_t is assumed to depend upon z_{tk} in the following manner:

$$y_t = \pi_1 z_{t1} + \pi_2 z_{t2} + \cdots\cdots + \pi_k z_{tk} + u_t \quad , \qquad (1)$$

where u_t is an element of independent disturbances with mean zero. For the T observations, the system may be written in vector form:

$$\begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_T \end{pmatrix} = \begin{pmatrix} z_{11} & \cdots\cdots & z_{1K} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_{T1} & \cdots\cdots & z_{TK} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \cdot \\ \cdot \\ \cdot \\ \pi_K \end{pmatrix} + \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_T \end{pmatrix} \qquad (2)$$

The column vector $\begin{pmatrix} \pi_1 \\ \cdot \\ \cdot \\ \cdot \\ \pi_K \end{pmatrix}$ is a set of regression coefficients to be estimated in the above system.

If the $u_1, u_2, \ldots\ldots, u_T$ are independent and have a common distribution with finite variance, then best linear unbiased estimates of the regression coefficients can be obtained by minimizing the sum of squared residuals. If the $u_1, \ldots., u_T$ are normally distributed the resulting magnitudes are also maximum-likelihood estimates.

Matrix equation (2), above, may be written more compactly

$$Y = Z\pi + U, \qquad (3)$$

where the matrix Z is of order T x K and rank K.

Let the sum of squared residuals to be minimized be denoted by S.

$$S = (Y - Z\pi)'(Y - Z\pi)$$

$$= Y'Y - 2\pi'Z'Y + \pi'Z'Z\pi \quad . \tag{4}$$

When the first derivative of S is taken with respect to π, and is set equal to zero; a set of normal equations are obtained.

$$\frac{\partial S}{\partial \pi} = -2Z'Y + 2Z'Z\pi = 0$$

$$= Z'Z\pi - Z'Y = 0 \quad . \tag{5}$$

Hence the estimator $\hat{\pi}$ is as follows:

$$\hat{\pi} = (Z'Z)^{-1}(Z'Y) \tag{6}$$

This estimator is based on the assumption that disturbances are independent. In the model to be considered in the present study, the above assumption is somewhat relaxed. The disturbances are no longer regarded as serially independent, but are related through a first order autoregressive process. The new assumption will lead to a slightly different estimator of the regression coefficients.

<u>First Order Autoregressive Model</u>

Probably it is necessary to explain first why disturbances are assumed to be generated by a first order autoregressive process. This assumption is made mainly because we attach a priority to simplicity at our first attempt in attacking the problem of autocorrelation. Eventually the validity of the assumption must be checked, although it has not been done in this study.

Let the new regression model be written as

$$Y = Z\pi + W \tag{7}$$

$$W = \rho W^* + V \qquad |\rho| < 1 \quad . \tag{8}$$

Equation (7) is similar to (3) except W is generated according to the first order autoregressive process (8). W, W*, and V stand for column vectors $\begin{pmatrix} w_1 \\ \vdots \\ w_T \end{pmatrix}$, $\begin{pmatrix} w_o \\ \vdots \\ w_{T-1} \end{pmatrix}$, and $\begin{pmatrix} v_1 \\ \vdots \\ v_T \end{pmatrix}$ respectively;

W and W* denote autocorrelated disturbances and V denote independent random disturbances. ρ is a constant called the autocorrelation coefficient. The autoregressive scheme is assumed to be stationary, (i.e., the absolute value of the autocorrelation coefficient is less than one.) This appears to be reasonable in the light of real market situation, because when all the exogenous variables are stabilized the price of a commodity would usually be expected to be stable or at least show bounded variation.

Substituting (8) into (7), the dependent variable Y may be expressed as

$$Y = Z\pi + \rho W^* + V. \tag{9}$$

Since $W^* = Y^* - Z^*\pi$, (10)

where $Y^* = \begin{pmatrix} y_o \\ \cdot \\ \cdot \\ \cdot \\ y_{T-1} \end{pmatrix}$ and $Z^* = \begin{pmatrix} z_{o1} & \cdots & z_{oK} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_{T-1,1} & \cdots & z_{T-1,K} \end{pmatrix}$.

Now substituting (10) into (9), a new regression equation is obtained.

$$(Y - \rho Y^*) = (Z - \rho Z^*)\pi + V. \tag{11}$$ [3]

Equation (11) is essentially in the same form as Equation (3). If ρ were

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3 If one wishes to estimate Y from a given Z, Z* and Y*, an appropriate form would be to use the relation $Y = \rho Y^* + (Z - \rho Z^*)\pi$, where π are estimated from (11).

known, Equation (11) could be regarded as an ordinary least squares regression model whose dependent and independent variables are ($Y - \rho Y^*$) and ($Z - \rho Z^*$) respectively. If ρ were assumed to be 0 in Equation (11), i.e., to say disturbances are non-autocorrelated, Equation (11) would then be identical with the familiar regression model such as Equation (3). If ρ were assumed to be +1. Equation (11) would be the first difference model.

A difficulty in applying Equation (11) to the empirical studies selected for the present investigation is that the exact value of the autocorrelation coefficient ρ for each case is not known in advance, (although there is reasonable ground to make an assumption that it lies between -1 and +1.) Therefore a trial and error method was adopted to find the ρ which will minimize the sum of squared residuals under the assumption of first order autoregressive scheme.

First, several possible values of ρ , either between 0 and +1 or between 0 and -1 depending on whether disturbances are positively or negatively autocorrelated, are selected. When each assumed value of ρ is inserted in Equation (11), the new variables ($Y - \rho Y^*$) and ($Z - \rho Z^*$) can be constructed; then the least squares method can be applied to them to estimate regression coefficients π . The estimator of π is now a function of ρ .

To find the appropriate ρ , a sum of squared residuals, corresponding to each π that have been estimated for an assumed value of ρ , is calculated. These sums of squared residuals are then compared and the ρ , for which the sum of squared residuals is a minimum, is selected as a best approximation. In short, the sum of squared residuals which is a function of π , hence a function of ρ , is minimized with respect to ρ .

In general the estimator of π may be expressed as follows. When the sum of squared residuals of Equation (11) is minimized with respect to π in the same manner as in Steps (4) and (5) in pp.7-8, the estimator of π is obtained. Note it is expressed as a function of ρ.

$$\tilde{\pi}(\rho) = \left[(Z - \rho Z^*)'(Z - \rho Z^*)\right]^{-1}\left[(Z - \rho Z^*)'(Y - \rho Y^*)\right] \qquad (12)$$

Substituting this estimator back to Equation (11), an expression for residuals is derived.

$$\tilde{V}(\rho) = (Y - \rho Y^*) - (Z - \rho Z^*)\tilde{\pi}. \qquad (13)$$

The sum of squared residuals then is

$$\tilde{V}'\tilde{V}(\rho) = (Y'Y - 2\rho Y'Y^* + \rho^2 Y^{*\prime}Y^*) - \tilde{\pi}'(Z - \rho Z^*)'(Y - \rho Y^*), \qquad (14)$$

and the least squares estimates of ρ and π are the pair of values, say $\hat{\rho}$ and $\hat{\pi} = \tilde{\pi}(\hat{\rho})$, that minimize $\tilde{V}'\tilde{V}(\rho)$.[4] If the elements of V are normally distributed, then $\hat{\rho}$ and $\hat{\pi}$ are also maximum-likelihood estimation.

Summary

In summarizing this chapter, the main objectives of the present study are briefly restated. It is to examine the regression coefficients estimates and sum of squared residuals as functions of ρ , within the interval $|\rho| < 1$, for each of the selected group of regression models.

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[4] As it has been mentioned in the very last paragraph of the first chapter, one has to be careful to see whether a minimum value of the function obtained is the absolute minimum or a relative minimum. It is desirable from a standpoint of estimation of regression coefficients if it can be proven in general that there is always a unique value of ρ , $|\rho| < 1$, for which the value of the function becomes the absolute minimum.

The author is interested in comparing the regression coefficients estimates and the sums of squared residuals which are obtained from setting $\rho = 0$, $\hat{\rho}$, and +1. As stated before, when ρ is assumed to be 0, a first order autoregressive model is identical with the ordinary least squares regression model; when ρ is set equal to +1, it becomes the first difference model.

CHAPTER III

REVIEW OF SOME RELATED STUDIES

Introduction

This chapter is devoted to the review of a few papers relevant to the present study. The Durbin-Watson significance test of serial correlation, Cochrane and Orcutt's investigation of application of least squares regression to relationships containing autocorrelated disturbances, Marshall and Hirshleifer's report on effects of applying a first order Markoff process in estimating a supply curve of female labor during the last war, Gurland's comments on the assumption of a first order Markoff process, and Watson' theoretical study of serial correlation are discussed in that order.

The Durbin-Watson Significance Test of Serial Independence of Disturbances

As it has been mentioned in Chapter I, the chief purpose of this study is to see effects of assuming that disturbances are autocorrelated through a first order autoregressive scheme in some interesting practical least squares regression models in which there is evidence to suspect that disturbances are autocorrelated. To detect these least squares regression models, the Durbin-Watson test is used.

In applying the Durbin-Watson test to examine the possibility of autocorrelated disturbances, a test statistic is first calculated. The test statistic chosen for the Durbin Watson test is

$$d = \frac{\sum_{t=2}^{T} (\tilde{u}_t - \tilde{u}_{t-1})^2}{\sum_{t=1}^{T} (\tilde{u}_t)^2}$$

where $\tilde{u}_t$ denotes the residual from a regression line for the tth period. If the disturbances were positively autocorrelated, d would tend to be relatively small; if the disturbances were negatively autocorrelated d would tend to be relatively large.

In case of testing, say, positive autocorrelation, it would seem natural to determine a critical value of d, say $\bar{d}$. If the observed value of d is less than $\bar{d}$, it may be inferred that positive autocorrelation is established at the significance level concerned. Durbin and Watson have shown that exact critical values of this kind cannot be obtained. However, it is possible to calculate upper and lower bounds to the critical values.[1] These bounds depend on the sample size, the number of regression vectors, the significance level and the definition of the test statistic. The lower and upper bounds are called d_L and d_u respectively. If the observed value of d is less than d_L, it is concluded that the value is significant; while if the observed value of d is greater than d_u, it is concluded that the value is not significant at the significance level concerned. If d lies between d_L and d_u, the test is inconclusive.

Sometimes it is necessary to test negative autocorrelation. To make the test, d is calculated as before. If d is greater than $4 - d_L$, there is a significant evidence of negative autocorrelation; if d is less

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[1] These bounds are tabulated in "Testing for Serical Correlation in Least Squares Regression II," Biometrika, Vol. 38, pp.159-178.

than $4 - d_u$, there is not significant evidence; otherwise the test is inconclusive.

Since there is no a priori knowledge as to the sign of autocorrelation of disturbances in all the least squares regression models examined in this study, two-tailed tests are used instead of single-tailed tests just described. For the two tailed test, the test statistic d will be significant at the significance level concerned, if d is less than d_L or greater than $4 - d_L$; d is non-significant if it lies between d_L and $4 - d_u$, and it is inconclusive otherwise. The values of d_L and d_u at the 10%, 5% and 2% significance levels for a two-tailed test can be obtained from the 5%, 2.5%, and 1% significance level values of d_L and d_u respectively in Durbin and Watson's article in Biometrika.

Results of applying this two-tailed test to a number of linear regression models are presented in the next chapter.

Cochrane and Orcutt's Sampling Study of the Problem of Autocorrelated Disturbances in Applying Least Squares Regression[2]

Cochrane and Orcutt constructed several sets of artificial data in which the autocorrelation was built in according to some prescribed error generating process. From these data, they examined the properties of a regression model containing autocorrelated disturbances. When the least squares method was applied to these artificial data, the first effect they noticed was a "bias toward randomness". This means that the residuals estimated by the least squares regression do not seem to

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2 Cochrane, D. and Orcutt, G.H. "Application of Least Squares Regression to Relationships Containing Autocorrelated Error Terms," Journal of American Statistical Association, Vol. 44, 1949, pp. 32-61.

indicate significant autocorrelation when the disturbances actually are autocorrelated. Despite the small smaple size (each artificial time series consists of 20 observations), their results show very clearly that the least squares estimates of the regression coefficients are very inefficient if the disturbances are highly autocorrelated.

To make an efficient analysis when it is known that the disturbances are autocorrelated, they suggested that a first order or second order autoregressive process[3] be fitted to the disturbances. They further suggested that in many economic regression analyses the disturbances could be made approximately uncorrelated by taking the first difference of the time series. This is one of the points that the present study is trying to examine.

Estimation of a Supply Curve of Female Labor by Fitting a First Order Autoregressive Process to Disturbances

Marshall and Hirshleifer[4], in their study of female labor supply during the World War II, estimated a supply function in which disturbances were assumed to be generated by a first order autoregressive process. Their system is expressed as:

$$y_t = \alpha_o + \sum_{i=1}^{K} \alpha_i z_{it} + u_t , \qquad t = 1,2,\ldots\ldots,T$$

$$u_t = \rho u_{t-1} + v_t , \qquad 1 \geqslant |\rho|$$

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[3] $u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + v_t$ is called a second order autoregressive process.

[4] Marshall, A.W. and Hirshleifer, J. The Supply of Female Labor in World War II. The Rand Corporation, Santa Monica, California.

where v_t is an element of disturbances which are assumed to be normally and independently distributed (N.I.D.) with $(0, \sigma^2)$. They derived the maximum likelihood estimators of α's and ρ from the joint probability density function of $y_1, \ldots\ldots\ldots\ldots, y_T$,

$$F(y_1, \ldots\ldots, y_T) = \prod_{t=1}^{T} f(v_t)$$

where

$$v_t = y_t - \alpha_0 - \sum_{i=1}^{K} \alpha_i z_{it} - P(y_{t-1} - \alpha_0 - \sum_{i=1}^{K} \alpha_i z_{i,t-1})$$

From the assumption that v_t is N.I.D. $(0, \sigma^2)$

$$f(v_t) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-v_t^2}{2\sigma^2}}$$

Hence

$$F(y_1, \ldots\ldots\ldots, y_T) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-v_t^2}{2\sigma^2}}$$

The logarithm of the likelihood function is expressed as:

$$L(\alpha, \rho) = \frac{T}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^{T} v_t(\alpha, \rho)^2$$

If ρ is known in the likelihood function, the estimator of α_i, $i = 0, 1, 2, \ldots\ldots, K$, will be the same as the ones obtained from the least squares regression of new variables $(y_t - \rho y_{t-1})$ and $(z_{it} - \rho z_{i,t-1})$, $i = 1, 2, \ldots, K$. This is the well known fact that the maximum likelihood estimation is equivalent to the least squares estimation of parameters if disturbances are N.I.D. $(0, \sigma^2)$.

When ρ is unknown, the maximum likelihood estimates of α_i, and ρ are the values of α_i, and ρ obtained by solving the following system of simultaneous equations.

$$\frac{\partial L(\alpha, \rho)}{\partial \alpha_0} = \frac{1}{\sigma^2} \sum_{t=1}^{T} \{v_t(\alpha, \rho)\}\{1 - \rho\} = 0$$

$$\frac{\partial L(\alpha,\rho)}{\partial \alpha_i} = \frac{1}{\sigma^2}\sum_{t=1}^{T}\{v_t(\alpha,\rho)\}\{z_{i,t-1} - \rho z_{i,t-1}\} = 0, \quad i = 1,\ldots,K.$$

$$\frac{\partial L(\alpha,\rho)}{\partial \rho} = \frac{1}{\sigma^2}\sum_{t=1}^{T}\{v_t(\alpha,\rho)\}\{y_{t-1} - \alpha_0 - \sum_{i=1}^{K}\alpha_i z_{i,t-1}\}$$

There are k+2 equations and the same number of unknown in the system. To solve it, Marshall and Hirshleifer suggested the iterative procedure:

1. Assume two values for ρ between -1 and +1.
2. For each such value, the first k+1 equations become simple linear expressions in the other unknown parameters, the α's, and so may be solved in well-known ways.
3. Substitute each set of solutions derived at the previous step in the k+2th equation, and compute $\frac{\partial L}{\partial \rho}$.

In general, $\left[\frac{\partial L}{\partial \rho}\right]_{\rho=1}$ and $\left[\frac{\partial L}{\partial \rho}\right]_{\rho=-1}$ will be two different values. With these two points the parameters of a straight line, $A + B\rho = 0$, can be obtained. Solving it for ρ, an improved estimate for ρ is obtained. Using this improved ρ and one of the first two ρ's the same procedure can be repeated until $\frac{\partial L}{\partial \rho}$ becomes as close to 0 as desired. They indicated that in their procedure convergence is very rapid. However, they did not report any provision that they might have made to check if $\frac{\partial L}{\partial \rho}$ would become 0 for more than one value of ρ . It is possible that $\frac{\partial L}{\partial \rho}$ may become zero for more than one value of ρ , since $L(\alpha,\rho)$ is a function of ρ of high order.

Marshall and Hirshleifer reported that in their case the first difference model would achieve an excellent approximation to the results given by the first order autoregressive model.

Gurland's Study of Effects of Certain Types of Specification Bias Concerning the Disturbances[5]

In linear regression models in which the disturbances are autocorrelated, it is often assumed that these are given by an autoregressive process. Gurland investigated the loss of efficiency of estimators of the regression parameters due to incorrect specification of a generating process of the disturbances. He asserts that although the regression parameters are still unbiased regardless of whether the covariance matrix of disturbances is correctly specified, the corresponding loss of efficiency becomes very serious if the process is incorrectly assumed to be stationary. Gurland is skeptical about the assumption that the absolute value of ρ is less than one in a first order Markoff process as it has been assumed in Cochrane and Orcutt's empirical study.[6]

Watson's Findings on the Performances of Regression Analysis Containing Autocorrelated Disturbances

In this section a part of Watson's Ph.D. dissertation which is relavant to this study is reviewed.

Since it was difficult to obtain accurate point estimates of the autocorrelation coefficient of the first order autoregressive process directly, and also difficult to determine the form of the error process, Watson compared the performances of regression analyses based on different

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5 Gurland, J. "An Example of Autocorrelated Disturbances in Linear Regression." Econometrica. Vol.22, 1954, pp.218-227.

6 It does not seem unreasonable to assume $|\rho|\leq 1$ in the light of real market situations. When all the exogenous variables in the model are stabilized, it is likely that an explosive case of $|\rho|<1$ will not occur.

error covariance matrices. He discussed five different cases in which each represents a different combination of assumed error process and true error process.[7]

	Assumed Error Process	True Error Process
1.	A given 1st order autoregressive process	A different 1st order autoregressive process
2.	A given 1st order autoregressive process	A 2nd order autoregressive process
3.	A given 1st order moving average process	A different 1st order moving average process
4.	A given 1st order moving average process	A 2nd order moving average process
5.	A given 1st order autoregressive process	A 1st order moving average process

In case 1, both assumed and true error processes are of first order autoregressive, but the first serial correlation of the true error process ρ is not equal to that of the assumed process θ. However, the absolute values of ρ and θ are both less than one. Watson reported

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7 Generally speaking, there are two types of error generating processes; (1) autoregressive process and (2) moving average process.

A sequence $\{u_t\}$ is autoregressive of order K if it is generated by the non-homogeneous difference equation,

$$u_t + \rho_1 u_{t-1} + \dots\dots\dots + \rho_K u_{t-K} = \varepsilon_t ,$$

where ε_t is N.I.D. with $(0, \sigma^2)$. The autocorrelation of the disturbances is uniquely determined by the ρ 's; conversely the ρ 's are uniquely determined from the covariance matrix of the disturbances.

Another type of error generating process is the process of moving average of order K which is expressed as;

$$u_t = \varepsilon_t + \rho_1 \varepsilon_{t-1} + \dots\dots\dots + \rho_K \varepsilon_{t-K} ,$$

where ε_t is N.I.D.(o, σ^2) random variable. The difficulty in handling this type of error process is that although the autocorrelation is uniquely determined from the ρ 's, and the ρ 's are not uniquely determined from the covariance matrix of the disturbances.

two results with respect to estimate of variance. 1) if θ is less than ρ the downward bias in the variance estimate predominates, but if θ is greater than ρ, the upward bias is more important. 2) The downward bias is far more violent. His conclusion is that in guessing the ρ, it would be better to overestimate than to underestimate because it would err in the conservative side.

As to the efficiency of regression coefficients estimates, a high value of θ tends to reduce efficiency unless ρ also happens to be equal to one. When ρ is close to +1 or -1, then θ has to be very close to ρ if reasonable degree of efficiency is desired. This is the point that tends to cast doubt on the advisability of the first difference model.

In case 2, the true error process is a stationary second order autoregressive scheme with parameters λ and μ, and the assumed error process is the same as in case 1. The fractional bias of the variance estimates has not been examined, because no formulae were found. Watson stated that the second order autoregressive scheme did not seem to have a definite pattern in influencing the efficiency of regression coefficients estimates.

Cases 3 and 4 have results similar to those of cases 1 and 2 respectively. In general, Watson found that in the moving average scheme it is more important to determine the order of the process than in the case of the autoregressive scheme.

In case 5, Watson considered the simplest case of incorrectly assumed error form. In this case even when ρ and θ, first serial correlations of the true and assumed process respectively, are quite close together, bias in the estimate of variance is still great, because the two error processes have very different correlograms.[8] The efficiency of regression coefficients estimates is high when θ is a close approximation of ρ.

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[8] Correlogram: graph of the error scheme in which ρ_i is plotted against i.

CHAPTER IV

APPLICATION OF THE DURBIN-WATSON TEST AND A FIRST ORDER AUTOREGRESSIVE MODEL TO SOME EMPIRICAL REGRESSION MODELS

Introduction

This chapter is to repot results of applying the general procedure which was outlined in Chapter II to a number of previously-fitted regression models. First, the Durbin-Watson test results are reported together with a brief description of each of the regression models tested. Secondly, regression coefficients and sum of squared residuals estimated from a first order autoregressive model are presented in tables and diagrams.

First order autoregressive model was applied to those equations which the Durbin-Watson test indicated significant autocorrelation of disturbances at the 10% significance level. The 10% significance level was tentatively chosen as a deciding point because the number of examples analyzed is not sufficiently large enough to permit the author to set the significance test at the 5% level in deciding whether there is significant evidence of autocorrelation in regression equations.

Results of Applying the Durbin-Watson Test

1. The first one examined was Meinken's econometric model of the wheat industry.[1] In his study, he formulated and carried out a statistical

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[1] Meinken, K.W. The Demand and Price Structure for Wheat. Technical Bulletin No.1136, November, 1955, United States Department of Agriculture.

fitting of six equations that together represent the economic forces which determine domestic and world prices for wheat and domestic utilization during the marketing year for food, feed, storage, and export. Two of the six equations were chosen for the present investigation. One equation (denoted by 1a)[2] related the per capita domestic use of wheat and wheat product for food to the average wholesale price of NO.2 hard Red Winter wheat at Kansas City and to a few other relevant economic variables.[3] Another equation (1b) related average wholesale price of wheat at Liverpool to world production of wheat and the wholesale price level of 45 raw materials in England.[4] The latter is supposed to represent a price-quantity relationship of wheat in the world market, because prior to 1940 Liverpool was the leading wheat market of the world due to the fact that the United Kingdom was the largest single importer of wheat. In these two equations, analyses were based on annual data of the years between the two world wars.

The equation (1a) was fitted by the limited-information method because it includes more than one endogenous variable. Meinken also fitted the same equation by the least squares method. The regression coefficients estimated from the two methods are very similar. Meinken said that a

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2 For convenience, each equation examined by the Durbin-Watson test is identified by a combination of an Arabic numeral and an alphabetical letter in the subsequent discussion. The numeral and letter refer to the model and equation respectively, e.g., 1a means the equation (a) in the Meinken's wheat model.

3 Meinken. op.cit., p.40, Equation(3.1).

4 ibid., p.40, Equation (8).

larger bias in the least squares equation probably would have been indicated had his model been fitted by the full-information maximum likelihood method.

The Durbin-Watson test was applied to the least squares equation. The observed test statistic fell within the lower part of the inconclusive zone both at the 5% and 10% significance levels. Hence no further analysis was carried out at this time.

The equation (lb), was fitted directly by the least squares method since all the variables on the right side of the equation are considered to be predetermined. The test indicated significant positive autocorrelation at the 10% level, and was inconclusive at the 5% level. A first order autoregressive model was applied to this equation.

2. Next examined was French's "Price and Production Outlook for Apples"[5]. His model includes price-quantity relationships both at the local (Michigan) and national levels. In the present study, only the demand and supply functions at the national level were investigated.

On the demand side, French obtained three different equations from slight different sets of variables. He chose the first one[6] for actual use in the estimating model since it imposes less arbitrary restrictions on the coefficients. All three equations were estimated by

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[5] French, B.C. The Long-Term Price and Production Outlook for Apples in the United States and Michigan. Michigan State University Agricultural Experiment Station, Technical Bulletin 255, April, 1956.

[6] ibid., p.7.

a single equation model of least squares regression. The use of the single model seems to be justifed because disposable income, the volume of competing fruits, and total production may be considered as predetermined for all practical purposes. However, a question may be raised with regard to the quantity of apples actually marketed being treated as predetermined, since it may vary with a rise and fall in prices. French got around this difficulty by using net production per capita excluding export as an output variable, and also omitting those years in which a significant amount was not harvested.[7] The coefficients of the demand equation were estimated from annual data for the 20-year period 1930-1953, excluding the three war years. Since the first demand equation (2a) was used by French for prediction, it was tested by the Durbin-Watson test. The test indicated no significant autocorrelation at either the 5% or 10% level.

The regression coefficients of the supply function (2b) were also estimated by the single equation model.[8] It seems to be appropriate to use the single equation model in this case because a considerable length of time elapses from planting to harvesting, and therefore, the supply may be regarded as a function of lagged price. The test indicated that there was also no significant autocorrelation in this equation at the 5% and 10% levels. No further analyses were carried out in this apple study at present.

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[7] French reports that the ommission of those years did not result in any significant difference in the regression coefficients when compared with those obtained from all years included.

[8] French, op.cit., p.15.

3. Thr third case was Hoos' report on the market situation of tomatoes and tomatoes products in California.[9] He used a single equation model to explain variations of California f.o.b. price of canned tomatoes for the years 1926-27 through 1953-54 excluding 1941-47. Among several single equations formulated, Hoos selected the euqation (1) in p.46 (denoted 3a in this study) as the best statistical demand function because of its simplicity in interpretation and trade use. It was tested by the Durbin-Watson test and the result was inconclusive at both the 5% and 10% significance levels. Therefore, the equation (3a) was not further analyzed.

4. Another of Hoos' statistical demand studies examined was concerned with market situations of California lemons.[10] A single equation method was used to estimate a demand function. Years analyzed were from 1921-22 through 1948-49 (excluding the war years.) Hoos considered demand equations of summer lemons (4a) and winter lemons (4b) separately because they reflect different market characteristics. The test was inconclusive when applied to the summer lemons demand function, and it indicated no significant autocorrelation at either the 5% or 10% level when applied to the winter lemons demand function. Hence no first order autoregressive model was applied to either of the equations.

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9 Hoos, Signey, Tomatoes and Tomato Products: Economic Trend and f.o.b. Price Relationships. California Agricultural Experiment Station, Mimeographed Report No. 185, March, 1956.

10 Hoos, S. and Seltzer, R.E. Lemons and Lemon Products: Changing Economic Relationships, 1951-52. California Agricultural Experiment Station Bulletin 729.

5. Jerry Foytik studies characteristics of demand for California plums extensively.[11] Instead of treating California plums as consisting a homogenous product in the sense used by the theorist, he classified them into several quality-size categories, and examined the demand characteristics of plums with respect to each category. First, he estimated regression coefficients of demand functions for three varieties of plums, viz., early, midseason, and late, from annual data for those years from 1922 to 1947 excluding the war years, using the single equation least squares method. Secondly, the variations in weekly prices are related to weekly auction sales and a few other relevant variables. Finally, the influence of one size of plums on another size was studied.

Only the varietal aspect of Foytik's study was examined in the current study because it may be considered a typical statistical demand analysis. The application of the Durbin-Watson test to the first equation (5a)[12] indicated that one may accept the hypothesis that the significant autocorrelation of disturbances is absent either at the 5% or 10% level. As to the second equation (5b)[13], the test was inconclusive. In the third equation (5c)[14], the test indicated no significant autocorrelation at the 5% level and was inconclusive at the 10% level. A first order auto-

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11 Foytik, Jerry. Characteristics of Demand for California Plums. Hilgardia, Vol. 20, NO.20, April, 1951, California Agricultural Experiment Station.

12 ibid., p.428, Equation (1)

13 ibid., p.431, Equation (2)

14 ibid., p.435, Equation (3)

regressive model was applied to the equation (5b) and results were presented in next section. (Originally it was not planned to analyze any equation in which the test was inconclusive. But in this case, the test was erroneously considered to be significant at first and further analysis was done.)

It is interesting to note that Foytik himself has tested his three varietal demand equations for the autocorrelation of residuals. He applied the von Neumman's ratio test (the ratio of mean square successive difference to variance) to them. His own test indicated that absence of autocorrelation in the residuals may be accepted as a suitable hypothesis for all three varietal demand equations.

6. Linstrom and King made a study of the factors that influence prices received by growers of slicing cucumbers and green peppers in North Carolina.[15] The demand function of cucumbers, which they considered as representative among many they formulated, was estimated from the observations which had been transformed to the first difference of logarithms.[16] Since it has not been considered how to take account of autocorrelated disturbances explicitly in this type of equation, no investigation was carried out.

As to green peppers, the equation (6a)[17] describing the influence of New York wholesale price on the local (Clinton, N.C.) market prices

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15 Linstrom, I.A. and King, R.A. The Demand for North Carolina Slicing Cucumbers and Green Peppers.

16 ibid., p.11.

17 ibid., p.40.

was tested by the Durbin-Watson test. The coefficients of this equation were estimated from daily data for the 23-day period, June 8 - July 18, 1953. The test indicated significant positive autocorrelation of disturbances at the 5% and 10% levels. A first order autoregressive model was applied to the equation (6a).

7. Quackenbush and Shaffer estimated the coefficients of a demand function (7a) for ice cream that are purchased for home consumption by the single equation method of least squares regression.[18] Data was obtained from the M.S.U. consumer panel. They cover 30 4-week periods from March 18, 1951 to July 11, 1953. Testing this equation by the Durbin-Watson test indicated significant positive autocorrelation at both the 5% and 10% levels. Therefore, this equation was further analyzed.

8. Hoos set up a sigle equation model to analyze f.o.b. prices of the Pacific Coast canned fruits and their relations to the major factors affecting them.[19] Annual data were used. They covered the years 1924-25 through 1955-56 (excluding 1941-42 through 1946-47.) His three demand equations for canned cling peaches, canned pears, and canned apricots respectively were examined by the Durbin-Watson test. The test applied to the equation for canned cling peaches (8a) indicated no

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18 Quackenbush, G.G. and Shaffer, J.D. Factors Affecting Purchases of Ice Cream for Home Use. Michigan State University Agricultural Experiment Station. Technical Bulletin 249, April 1955.

19 Hoos, S. F.O.B. Price Relationships, 1955-56 Pacific Coast Canned Fruits. California Agricultural Experiment Station, Mimeographed Report, No. 189, July 1956.

significant autocorrelation of disturbances at the 5% and 10% levels.[20] The test applied to the equation for canned pears (8b) indicated the similar test result.[21] The test was inconclusive when applied to the equation for canned apricots, (8c).[22]

9. The last one examined was "Relation between Auction Prices and Supplies of California Frech Bartlett Pears" by Hoos and Shear.[23] They formulated five statistical demand equations (single equation type) in an attempt to explain market characteristics of California frech Bartlett pears, and chose the Equation (5)[24] as a representative demand equation (denoted 9a in this study). The test, when applied to this equation, was inconclusive both at the 5% level and 10% level.

In Appendix B, they published the results of the statistical analysis of the relations of pears to its competing fruits such as plums, peaches, and oranges. They estimated the coefficients of a dozen demand equations by the least squares regression analysis. Due to time limit, only three of them were tested by the Durbin-Watson test. The test applied to the first equation (9b) in Table 11, which expresses the price of pears as a function of pear unloads; the unloads of plums; an index of New York state factory wages; and "time", was inconclusive. So was the test applied

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20 Hoos, op.cit., p.26.

21 ibid., p.31.

22 ibid., p.36.

23 Hoos, D. and Shear, S.W. Relation Between Auction Prices and Supplies of California Frech Bartlett Pears. Hilgardia, Vol. 14, No.5, January, 1942, California Agricultural Experiment Station.

24 ibid., p.280.

to the second equation (9c) in the same Table, which expresses the prices of plums as a function of pear unloads; index of New York State factory wages; and "time". When the quantity of pears shipped is treated as dependent variable as in the first equation (9d) of Table 12, the test indicated that the disturbances are negatively autocorrelated at the 10% level and the test was indeterminate at the 5% level. In the Hoos' pear study, only the equation (9d) was examined by applying a first order autoregressive model.

In the following table a summary of the Durbin-Watson test is given.

Table 1
Application of the Durbin-Watson Test

Equation No.		No. of Independent Variables	No. of Observa- tions	Calculated Test Statistic	Limits of Relevant Boundary 5% d_L	5% d_u	10% d_L	10% d_u	Test Results at the 10% Level
1a		4	17	0.8479	0.68	1.77	0.78	1.90	I
1b	#	2	15	0.8922	0.83	1.40	0.95	1.54	R
2a		3	20	1.7789	0.89	1.55	1.00	1.68	N
2b		1	19	1.7446	1.06	1.28	1.18	1.49	N
3a		3	23	1.2822	0.97	1.54	1.08	1.66	I
4a		4	24	1.8586	0.91	1.66	1.01	1.78	N
4b		5	23	1.0540	0.80	1.80	0.90	1.92	I
5a		2	25	2.3840	1.10	1.43	1.21	1.55	N
5b	#	5	25	1.2830	0.86	1.77	0.95	1.89	I
5c		4	25	1.6507	0.94	1.65	1.04	1.77	I
6a	#	2	23	0.9146	1.06	1.42	1.17	1.54	R
7a	#	3	30	1.0389	1.12	1.54	1.21	1.65	R
8a		3	26	1.6580	1.04	1.54	1.14	1.65	N
8b		3	23	2.4885	0.97	1.54	1.08	1.66	I
8c		3	26	1.4400	1.04	1.54	1.14	1.65	I
9a		4	25	1.1625	0.94	1.65	1.04	1.77	I
9b		4	15	2.7744	0.59	1.84	0.69	1.97	I
9c		4	15	3.0567	0.59	1.84	0.69	1.97	I
9d	#	4	15	3.3476	0.59	1.84	0.69	1.97	R

Abbreviations:

d_L : Lower limit.

d_u : Upper limit.

N : Null hypothesis not rejected.

I : Indeterminate.

R : Null hypothesis rejected.

\# : A firsr order autoregressive model has been applied.

Regression Coefficients and Sums of Squared Residuals Estimated from First Estimated from A First Order Autoregressive Model

When the Durbin-Watson test was applied, a significant autocorrelation of disturbances was established at the 10% level in four cases. They are (1b) Meinken's demand equation for the world wheat market, (6a) Linstrom and King's demand equation relating local (Clinton, N.C.) green peppers prices to the New York wholesale prices, (7a) Quackenbush and Shaffer's demand equation for ice cream bought for home use in Lansing area, (9d) Hoos' demand equation relating the variations of unloads of pears to wholesale prices of pears and plums.

A first order autoregressive model, as explained in Chapter II, was applied to these four cases. Results of the computations were tabulated and diagrammed.

One of Foytik's demand equation for mid-season plums (5b) was also analyzed and reported in the same manner as the above four cases. Originally it was not planned to apply a first order autoregressive model to those cases where the Durbin-Watson test indicated inconclusive results. The Foytik's demand equation belongs to this category. However, an error was made when the Durbin-Watson test statictic for the Foytik's equation was first calculated, and it was considered that the significant autocorrelation of disturbances was established. Subsequently, a first order autoregressive model was fitted. It is felt that the results are interesting enough to be presented here.

In each of the five cases, one may notice some discrepancies between the original regression coefficients estimates reported in each author's study and the regression coefficients estimated from assuming $\rho = 0$ in a first order autoregressive model. (This is the same as the traditional least squares model.) This is probably due to a difference in the number of observation used in estimation. The first order autoregressive model as applied in the present study are based on the observations whose number is one less than those from which the original regression coefficients estimates were obtained.[25]

1b) Meinken's World Wheat Demand Equation[26]

Since the Durbin-Watson test indicated significant positive autocorrelation, values of ρ assumed in a first order autoregressive process were taken from the interval, 0 to +1. The first approximation to the value of $\hat{\rho}$ that would minimize the sum of squared residuals under the assumption was 0.7. By plotting the sums of squared residuals against ρ, it was realized that the minimum value of the sum of squared residuals would most likely be between $\rho = 0.6$ and 0.7. Repeating the same procedure for several different values of ρ between 0.6 and 0.7, it was found that the sum of squared residuals would be minimized when ρ was set equal to 0.67.

One can see from Table 2 and Fig. 1 that although the first difference model would increase the sum of squares of residual from that of the original regression model, the values of parameters seem to approach the true values.

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25 In a first order autoregressive model, lagged values of variables are introduced. Hence the number of observation is reduced by one. See the Appendix.

26 Meinken, op.cit., p.41, Equation (8), $P_w = 142 - 0.36 S_w + 1.1 I_w$.

The regression equation obtained from the first order autoregressive model is as as follows:

$$P_w = 162 - 0.046\, S_w + 1.35\, I_w$$

where

P_w : Average wholesale price of wheat at Liverpool, England, per bushel.

S_w : World production of wheat plus stocks about August 1, excluding Russia and China but including net exports from Russia, million bushels.

I_w : Index of wholesale prices of 45 raw materials in England (1910-14 = 100).

It is suitable to use the above equation for prediction of long term price. However, if one has observations on this year's independent variables and wants to predict next year's dependent variable, it would be better to use the equation of the following form. This second equation is an alternative way of expressing the first one.[27]

$$P_w = 54 - 0.67\, P_w^* - 0.046(S_w - 0.67\, S_w^*) + 1.35(I_w - 0.67\, I_w^*)$$

where

P_w^*, S_w^* and I_w^* stand for lagged values of the respective variables.

In the following diagram, the sum of squared residuals and the regression coefficients estimates are plotted as functions of ρ . Data used are presented in the table.

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27 See footnote 3 in p.9.

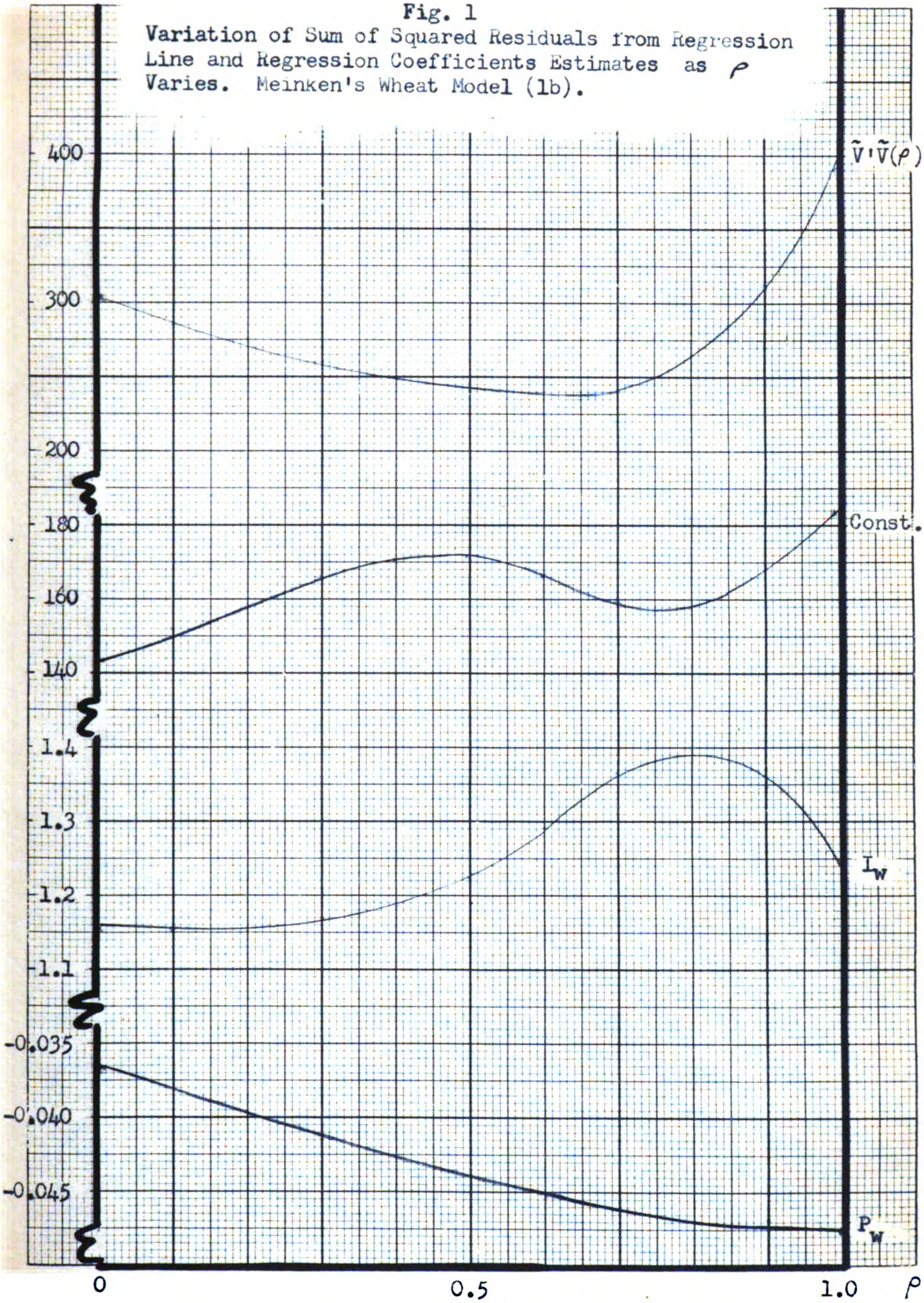
Fig. 1
Variation of Sum of Squared Residuals from Regression Line and Regression Coefficients Estimates as ρ Varies. Meinken's Wheat Model (1b).
400
300
200
180
160
140
1.4
1.3
1.2
1.1
-0.035
-0.040
-0.045
0
0.5
1.0
ρ
Ṽ'Ṽ(ρ)
Const.
I_W
P_W

Table 2

Estimates of Regression Coefficients and Sum of Squared Residuals as ρ Varies: Meinken's Wheat Demand Equation (1b).

Autocorrelation Coefficients	Regression Coefficients			Sum of Squared Residuals $V'V(\rho)$
	Constant	P_w	I_w	
0	144.92	-0.0371	1.155	301.437
0.1	152.81	-0.0387	1.150	287.958
0.2	160.26	-0.0403	1.150	276.633
0.3	166.49	-0.0419	1.159	266.720
0.4	170.39	-0.0433	1.181	257.433
0.5	170.69	-0.0444	1.224	248.016
0.6	166.59	-0.0453	1.291	239.185
0.65	163.29	-0.0456	1.330	236.281
0.66	162.61	-0.0456	1.338	236.054
0.67	161.93	-0.0457	1.345	235.974
0.68	161.28	-0.0457	1.353	236.070
0.69	160.65	-0.0458	1.360	236.348
0.7	160.07	-0.0458	1.367	236.185
0.8	158.55	-0.0463	1.399	257.801
0.9	170.71	-0.0470	1.345	313.158
1.0	183.82	-0.0478	1.245	390.126

6a) Linstrom and King's Demand equation for Green Peppers in North Carolina.[28]

Applying a first order autoregressive model in the similar manner as in the previous case, the first approximation to $\hat{\rho}$ was found to be 0.9. When a diagram was drawn as before, it was suspected that $\hat{\rho}$ would be somewhere between 0.8 and 0.9. After further computations, the sum of squared residuals was found to be the smallest when ρ was set equal to 0.88.

[28] Linstrom and King, op. cit., p.40, Equation (5.1), $X_1 = -0.3117 + 0.7144\ X_2 - 0.0000\ X_3$.

The sign of the coefficient of X_2 which was derived from the first order autoregressive model does not agree with Linstrom and King's assumption that the local and the New York wholesale market prices of green peppers move toward the same direction. This probably is due to the fact that the disturbances are generated by a more general scheme than a a first order autoregressive scheme, or some relevant variables were overlooked when Linstrom and King formulated their model.

The regression equation derived from the first order autoregressive model is

$$X_1 = 0.6773 - 0.0383\ X_2 + 0.00005\ X_3$$

or alternatively stated,

$$X_1 = 0.0813 - 0.88\ X_1^* - 0.0383\ (X_2 - 0.88\ X_2^*) + 0.00005(\ X_3 - 0.88\ X_3^*)$$

where

X_1 : The daily weighted average price for peppers at Clinton, N.C. in dollars per bushel.

X_2 : The simple Average daily price paid for North Carolina peppers of good quality on the New York wholesale market in dollars per bushel.

X_3 : Suppy on the Clinton Market in bushels.

Fig. 2
Variation of Sum of Squared Residuals and Regression Coefficients Estimates as ρ Varies. Linstrom and King's Demand Equation for Green Pepper. (6a)

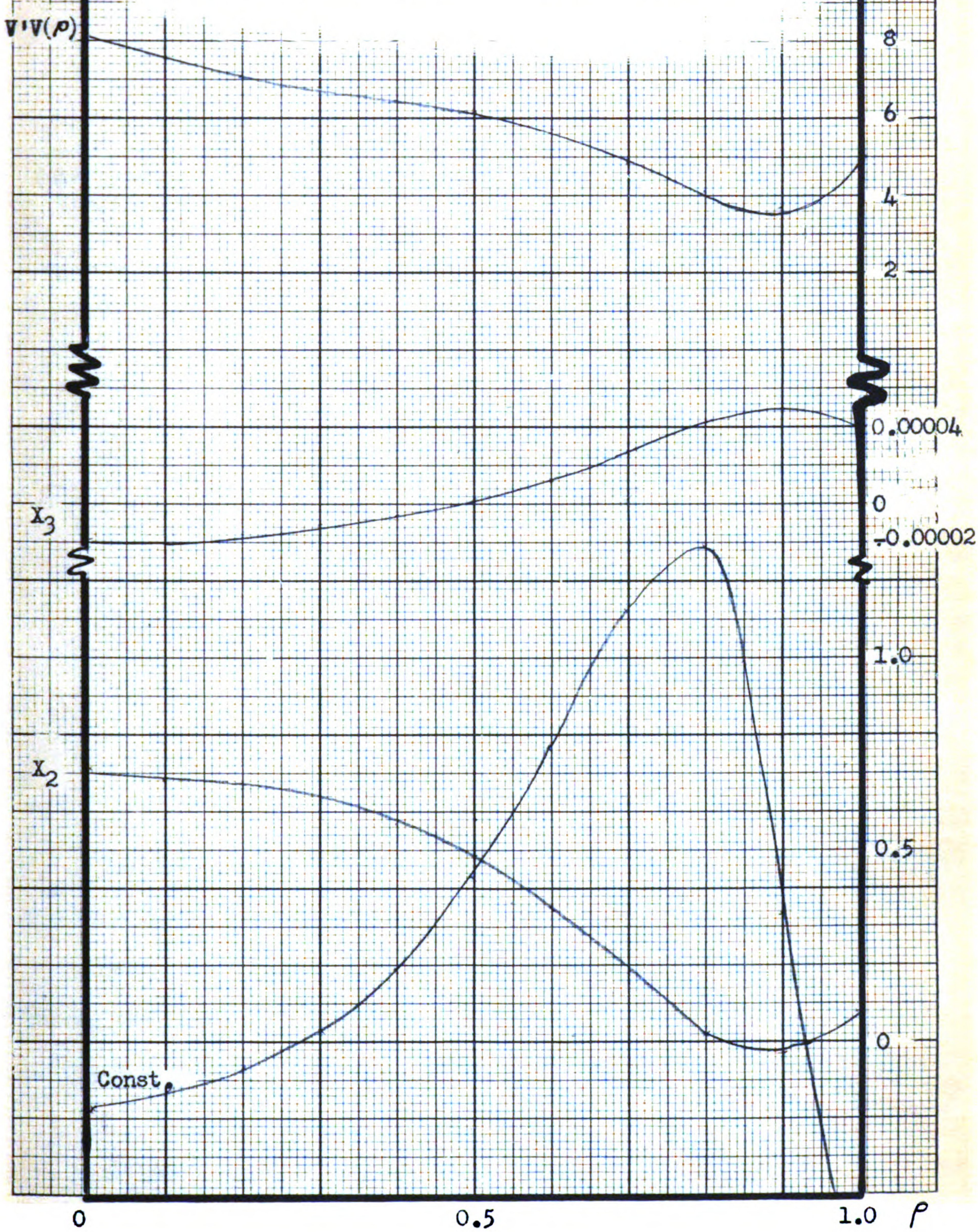

Table 3

Regression Coefficients and Sum of Squares

Residuals for Different Values of ρ.

Linstrom and King's Demand Study of Green

Peppers (6a).

Autocorrelation Coefficient	Regression Coefficients Constant	X_2	X_3	Sum of Squared Residuals
0	-0.1716	0.7043	-0.000019	8.098
0.1	-0.1352	0.6904	-0.000017	7.576
0.2	-0.0734	0.6679	-0.000015	7.174
0.3	0.0274	0.6324	-0.000011	6.861
0.4	0.1870	0.5760	-0.000006	6.580
0.5	0.4295	0.4877	0.000002	6.237
0.6	0.7656	0.3559	0.000013	5.699
0.7	1.1262	0.1846	0.000027	4.880
0.8	1.2294	0.0242	0.000042	4.009
0.84	1.0683	-0.0168	0.000046	3.787
0.85	0.9927	-0.0241	0.000046	3.752
0.86	0.9103	-0.0300	0.000047	3.727
0.87	0.8049	-0.0346	0.000047	3.711
0.88	0.6773	-0.0383	0.000048	3.705
0.89	0.5229	-0.0399	0.000048	3.711
0.9	0.3351	-0.0406	0.000048	3.728
0.95	-1.7185	-0.0437	0.000048	3.962
1	-1.7470	0.0932	0.000040	5.193

7a) Quackenbush and Shaffer's Demand Function for Ice Cream Purchased for Home Use in Lansing Area. [29]

Going through the same procedure as described in the previous two cases, the first approximation of $\hat{\rho}$ turned out to be 0.4. After some more computations, 0.41 was selected as a closest approximation to $\hat{\rho}$.

29 Quackenbush and Shaffer, op. cit., p. 10,
$X_1 = 0.1860 - 1.1134 X_2 + 0.0035 X_3 + 0.0037 X_4$

In this case, the first difference transformation of variables made the sum of squared residuals greater than the originally-fitted model. Yet the values of regression coefficients did not approach those of the true parameters. Probably it would not be appropriate to use the first difference model in this case. The appropriate regression equation obtained under the assumption of a first order autoregressive model is

$$X_1 = 0.1593 - 0.8929\ X_2 + 0.0032\ X_3 + 0.0036\ X_4$$

or
$$X_1 = 0.0940 - 0.43\ X_1^* - 0.8929\ (X_2 - 0.43\ X_2^*) + 0.0032\ (X_3 - 0.43\ X_3^*) + 0.0036\ (X_4 - 0.43\ X_4^*)$$

where

X_1 : Pints of ice cream per capita.

X_2 : Price per pint.

X_3 : Weekly family income.

X_4 : Mean Temperature, Lansing.

Fig. 3
Variation of Sum of Squared Residuals and Regression Coefficients Estimates as ρ Varies. Quackenbush and Shaffer's Demand Equation for Ice cream. (7a)

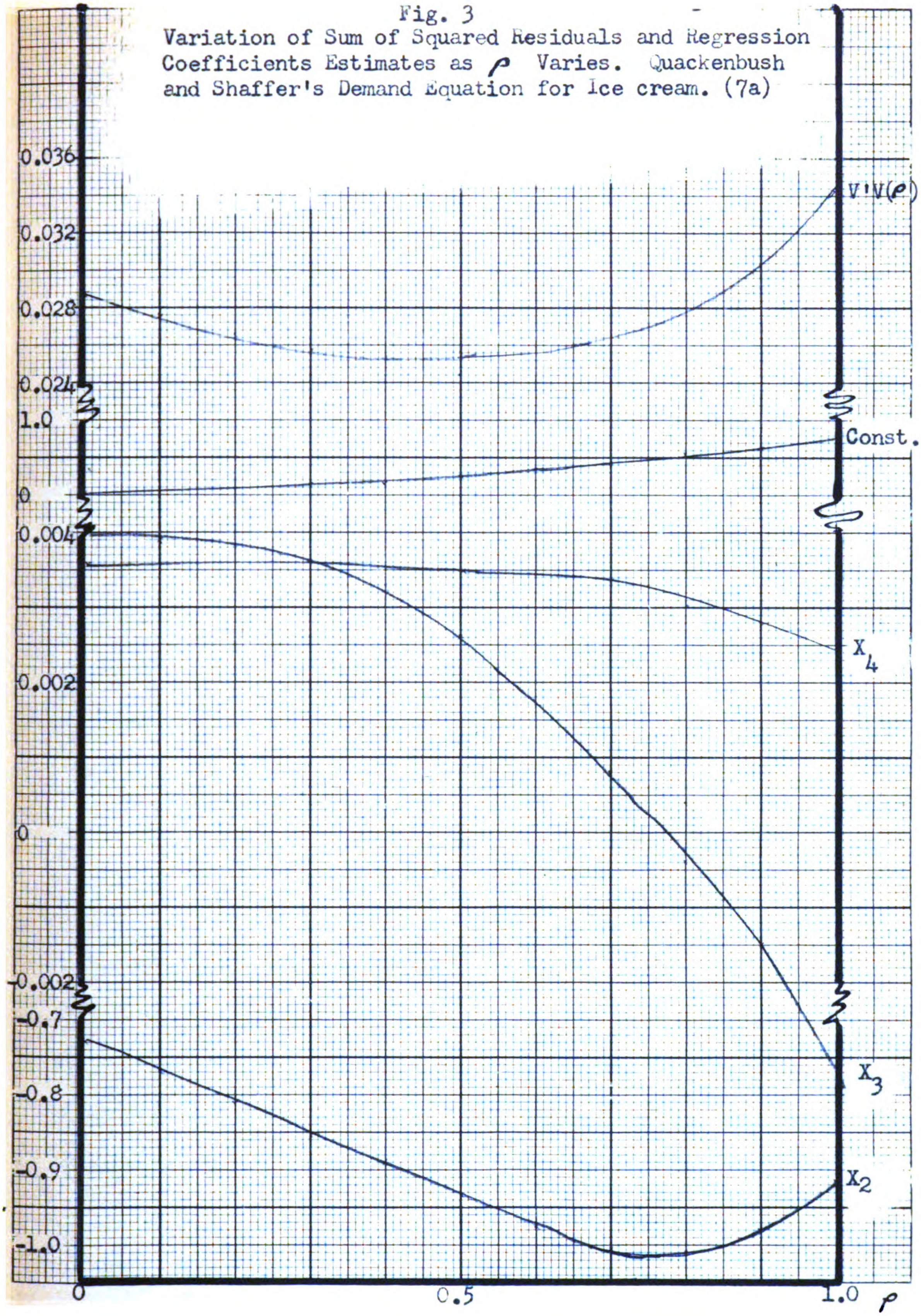

Table 4

Regression Coefficients and Sum of Squared Residuals for Different Values of ρ. Quackenbush and Shaffer's Demand Study of Ice Cream (7a).

Autocorrelation Coefficient	Regression Coefficient				Sum of squares Residuals
	Constant	X_2	X_3	X_4	
0	0.0441	-0.7378	0.0040	0.0036	0.029521
0.1	0.0588	-0.7726	0.0039	0.0036	0.027668
0.2	0.0803	-0.8102	0.0038	0.0036	0.026394
0.3	0.1114	-0.8502	0.0036	0.0036	0.025743
0.38	0.1461	-0.8835	0.0033	0.0036	0.025461
0.40	0.1565	-0.8920	0.0033	0.0036	0.025134
0.42	0.1725	-0.9004	0.0031	0.0035	0.025459
0.43	0.1734	-0.9047	0.0031	0.0035	0.025468
0.5	0.2202	-0.9341	0.0026	0.0035	0.025622
0.6	0.3062	-0.9725	0.0018	0.0034	0.026024
0.7	0.4120	-1.0001	0.0008	0.0033	0.026469
0.8	0.5235	-1.0064	-0.0004	0.0030	0.027198
0.9	0.6155	-0.9827	-0.0015	0.0028	0.029036
1	0.7574	-0.9292	-0.0032	0.0027	0.034178

9d) Hoos' Demand Equation for Pears.[30]

When the Durbin-Watson test was applied, significant negative autocorrelation was established. It was, therefore, assumed that the true value of $\hat{\rho}$ would lie between 0 and -1. The first and second approximations to $\hat{\rho}$ were -0.6 and -0.66 respectively. The first difference model was also fitted. There was a subtantial increase in the sum of squared residuals without noticeable change in regression parameters.

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[30] Hoos and Shear, op. cit., p.295, Equation (1),
$X_1 = 163.41667 - 19.17780\ Y_1 - 15.60794\ Y_2 + 8.51542\ W - 10.83552\ T.$

This case seems to suggest that one must be cautious in applying the first difference model, if there is a reason to suspect that disturbances are negatively autocorrelated. The regression equation derived from the first order autoregressive model in which ρ was set equal to -0.66 is as follows:

$$X_1 = 173.25 - 22.459\, Y_1 - 17.351\, Y_2 + 9.366\, W - 11.106\, T$$

or $$X_1 = 287.60 + 0.66X_1^* - 22.459(Y_1 + 0.66Y_1^*) - 17.351(Y_2 + 0.66Y_2^*)$$
$$+ 9.355\, (W + 0.66W^*) - 11.106(T + 0.66T^*)$$

where

X_1 : New York unloads of pears (in 100 tons.)

Y_1 : New York wholesale prices of pears (in dollars per 100 pounds.)

Y_2 : New York wholesale prices of plums (in dollars per 100 pounds.)

W : New York state factory wages, June-August (dollars per week.)

T : "Time" in years (origin, July-August, 1931).

Fig.4

Variation of Sum of Squared Residuals and Regression Coefficients Estimates as ρ Varies. Hoos' Demand Equation for Pears. (9d)

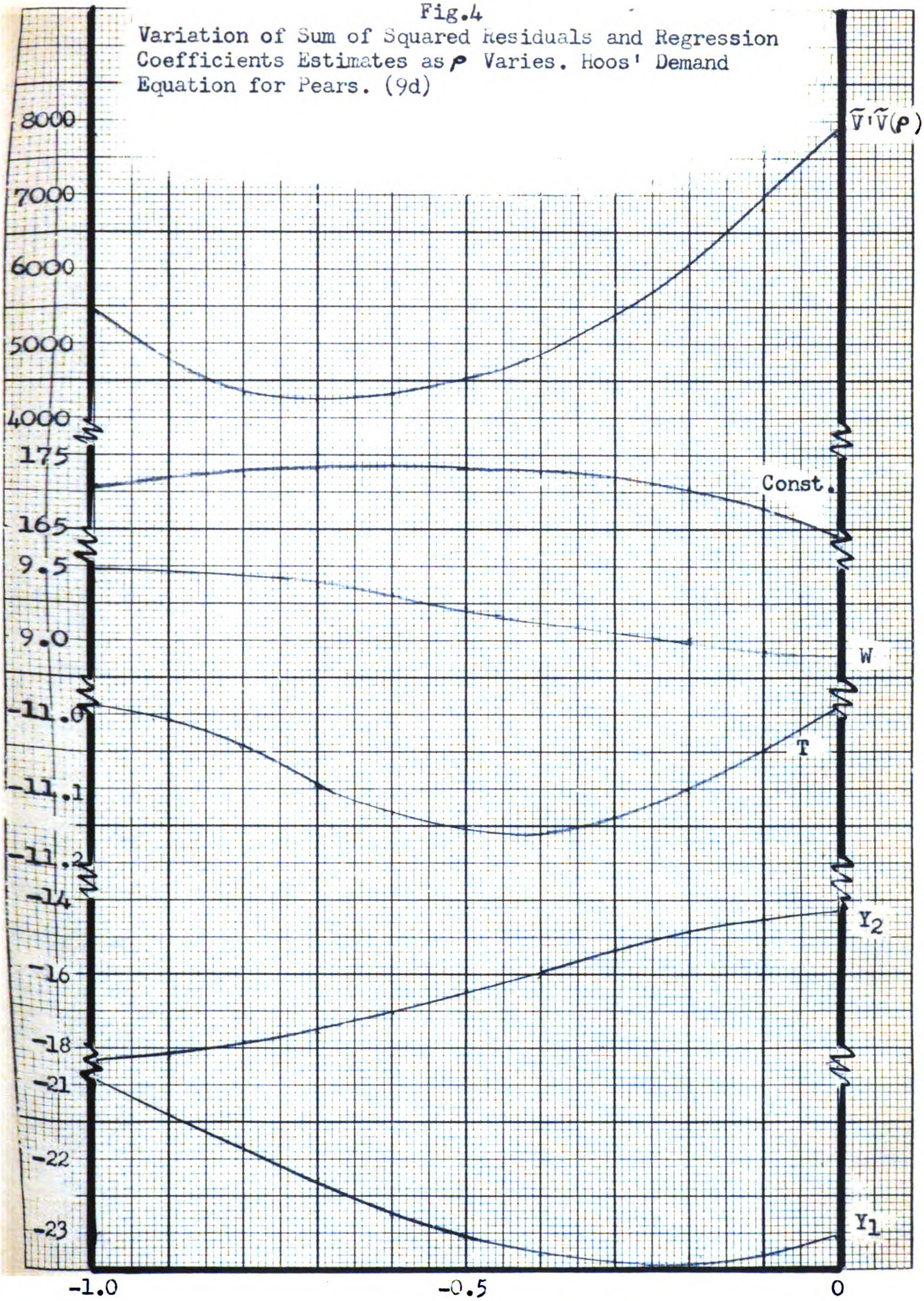

Table 5

Regression Coefficients and Sum of Squared Residuals for Different Values of ρ . Hoos' Demand Equation for Pears (9d)

Autocorrelation Coefficient	Regression Coefficients: Constant	Y_1	Y_2	W	T	Sum of Squared Residuals
0	164.70	-23.140	-14.129	8.958	-10.982	7840.549
-0.1	167.91	-23.318	-14.447	8.959	-11.053	6840.105
-0.2	170.13	-23.436	-14.825	9.000	-11.108	6008.607
-0.3	171.78	-23.454	-15.308	9.070	-11.144	5340.961
-0.4	172.81	-23.340	-15.862	9.156	-11.161	4841.266
-0.5	173.30	-23.087	-16.450	9.244	-11.155	4510.176
-0.6	173.34	-22.719	-17.028	9.321	-11.131	4349.346
-0.65	173.38	-22.504	-17.299	9.354	-11.113	4329.218
-0.66	173.25	-22.459	-17.351	9.360	-11.112	4311.043
-0.67	173.10	-22.415	-17.404	9.366	-11.106	4316.735
-0.7	173.07	-22.278	-17.552	9.382	-11.094	4359.631
-0.8	172.58	-21.810	-17.991	9.422	-11.048	4345.262
-0.9	172.00	-21.356	-18.331	9.444	-11.001	4904.161
-1.0	171.38	-20.943	-18.575	9.449	-10.954	5437.704
+1.0	111.60	-21.450	-13.978	10.470	-10.345	27040.092

5b) **Foytik's Demand Equation for Midseason California Plums.**[31]

Originally it was thought that there was a significant positive autocorrelation at the 10% level in the disturbances of this regression model, and a first order autoregressive model was fitted accordingly. Later when the Durbin-Watson test statistic was recomputed, an error was discovered, and it was decided that test result was inconclusive. However, since a part of computations which had been finished showed some interest-

31 Foytik, op.cit., p.431, Equation (2),

$P_2 = 1,7271 - 0.0270\ Q_2 + 0.0112\ I - 0.131\ T + 0.0125\ Q_1 - 0.0311\ S_2.$

ing results, the investigation was continued.

In applying a first order autoregressive model, different possible values of ρ ranging from -0.5 to $+1.0$ were inserted in the error process. The closest approximation to $\hat{\rho}$ was 0.48. Parameters estimated from the first difference model were quite close to those derived from the first order autoregressive model. The regression equation derived from the first order autoregressive model is as follows:

$$P_2 = 1.9649 - 0.0264\, Q_2 + 0.01126\, I - 0.0161\, T + 0.00792\, Q_1 - 0.03232\, S_2$$

or alternatively stated,

$$P_2 = 1.0217 - 0.48\, P_2^* - 0.02646\, (Q_2 - 0.48\, Q_2^*) + 0.01126\, (I - 0.48\, I^*)$$
$$- 0.0161\, (T - 0.48\, T^*) + 0.00792\, (Q_1 - 0.48\, Q_1^*) - 0.03232\, (S_2 - 0.48\, S_2^*)$$

where,

P_2 : New York-Chicago auction price for midseason varieties, dollars per crate.

Q_2 : New York-Chicago auction sales of midseason varieties, in 10,000 equivalent crates.

I : Index of U.S. non-agricultural income payments, May-October average, 1935-1939 = 100.

T : Time in years, with origin at 1921.

Q_1 : New York-Chicago auction sales of early varieties in 10,000 equivalent crates.

S_2 : Supply of early peaches (total production in Ga., S.C., N.C., and Ark.,) in million bushels.

Fig. 5

Variation of Sum of Squared Residuals and Regression Coefficients Estimates as ρ Varies.

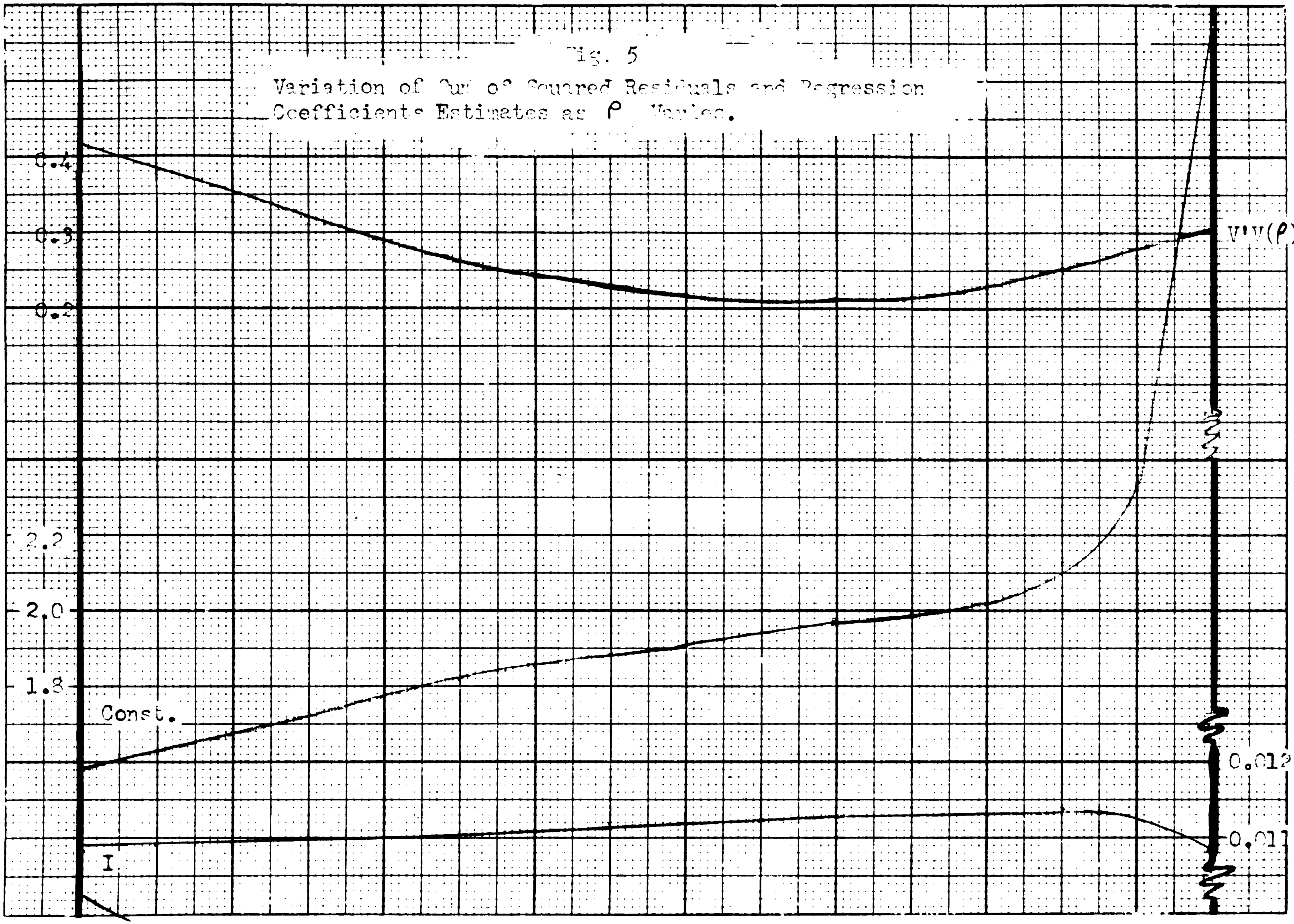

Table 6

Regression Coefficients and Sum of Squared Residuals for Different Values of ρ. Foytik's Demand Equation of Mid-Season Plums (5b).

Autocorrelation Coefficient	Regression Coefficients						Sum of Squares Residuals
	Constant	Q_2	I	T	Q_1	S_2	$V'V(\rho)$
-0.5	1.5915	-0.02579	0.01086	-0.01577	0.01462	-0.02657	0.41227
-0.4	1.6344	-0.02621	0.01090	-0.01558	0.01426	-0.02758	0.37650
-0.3	1.6807	-0.02658	0.01095	-0.01537	0.01376	-0.02862	0.34446
-0.2	1.7275	-0.02684	0.01101	-0.01513	0.01309	-0.02960	0.31563
-0.1	1.7726	-0.02699	0.01107	-0.01491	0.01230	-0.03043	0.28976
0	1.8146	-0.02702	0.01112	-0.01474	0.01142	-0.03109	0.26696
0.1	1.8526	-0.02696	0.01116	-0.01465	0.01053	-0.03157	0.24721
0.2	1.8864	-0.02684	0.01120	-0.01469	0.00968	-0.03190	0.23130
0.3	1.9165	-0.02670	0.01123	-0.01492	0.00892	-0.03212	0.21974
0.4	1.9438	-0.02656	0.01125	-0.01543	0.00831	-0.03225	0.21314
0.47	1.9620	-0.02646	0.01126	-0.01600	0.00795	-0.03230	0.21235
0.48	1.9649	-0.02646	0.01126	-0.01610	0.00792	-0.03232	0.21182
0.49	1.9675	-0.02646	0.01126	-0.01621	0.00787	-0.03232	0.21192
0.5	1.9702	-0.02645	0.01126	-0.01632	0.00783	-0.03233	0.21208
0.51	1.9729	-0.02644	0.01126	-0.01644	0.00779	-0.03234	0.21229
0.6	1.9993	-0.02635	0.01128	-0.01786	0.00750	-0.03238	0.21698
0.7	2.0386	-0.02629	0.01130	-0.02065	0.00729	-0.03241	0.22816
0.8	2.1108	-0.02626	0.01133	-0.02651	0.00717	-0.03241	0.24578
0.9	2.3293	-0.02623	0.01140	-0.04455	0.00712	-0.03240	0.26989
1.0	2.9728	-0.02636	0.01087	-0.09170	0.00754	-0.03254	0.30854

CHAPTER V

SUMMARY AND CONCLUSIONS

Since one of the objectives of this study is to see how important the autocorrelation of disturbances is in the selected group of previously-fitted regression models, each of the regression models was tested by the Durbin-Watson test of serial independence of disturbances. Results were presented in the previous chapter. Among nineteen regression equations tested, the hypothesis of zero autocorrelation in disturbances at the 10% significance level may not be rejected in only seven of them. In four cases, significant autocorrelation was established , and subsequently a first order autoregressive model was fitted to them. In the remaining nine cases, the test was indeterminate. Although no further test was applied to these nine cases at this time, it seems reasonable to expect that autocorrelation was a factor in some of the cases.

From these results, it is evident that some of the regression models tested are highly autocorrelated, either positively or negatively. These autocorrelated disturbances may account for a part of the loss of efficiency in estimation and prediction by the current method of least squares regression.

Another objective of this study is to compare the least squares estimates of regression coefficients and the sum of squared residuals with those estimated from the first order autoregressive model, and with those estimated from the first difference model. For this purpose, the relevant estimates of regression coefficients and sum of squared residuals for each of the five cases are summarized as follows:

Table 7

Regression Coefficients and Sum of Squared Residuals Estimated from the Ordinary Regression Model, the First Order Autoregressive Model, and the First Difference Model.

Equation No.		Regression Coefficients						Sum of Squared Residuals
(1b)		Constant	S_w	I_w				
	ρ = 0	144.92	-0.0371	1.155				301.437
	ρ = 0.67	161.93	-0.0457	1.345				235.974
	ρ = 1	183.82	-0.0478	1.245				390.126
(6b)		Constant	X_2	X_3				
	ρ = 0	-0.1716	0.7043	-0.000019				8.098
	ρ = 0.88	0.6773	-0.0383	0.000048				3.705
	ρ = 1	-1.747	0.0932	0.000040				5.193
(7a)		Constant	X_2	X_3	X_4			
	ρ = 0	0.0441	-0.7378	0.0040	0.0036			0.029521
	ρ = 0.41	0.1593	-0.8929	0.0032	0.0036			0.025134
	ρ = 1	0.7574	-0.9292	-0.0032	0.0027			0.034178
(9d)		Constant	Y_1	Y_2	W	T		
	ρ = 0	164.70	-23.140	-14.129	8.958	-10.982		7840.549
	ρ = -0.65	173.25	-22.459	-17.351	9.360	-11.112		4011.043
	ρ = -1.0	171.38	-20.943	-18.575	9.449	-10.954		5437.704
	ρ = 1.0	111.60	-21.450	-13.978	10.570	-10.345		27040.092
(5b)		Constant	Q_2	I	T	Q_1	S_2	
	ρ = 0	1.8146	-0.02702	0.01112	-0.01474	0.01142	-0.03109	0.26696
	ρ = 0.48	1.9646	-0.02646	0.01126	-0.01610	0.00792	-0.03232	0.21182
	ρ = 1	2.9728	-0.02636	0.01087	-0.09170	0.00754	-0.03254	0.30854

According to the above table, in three of four cases where a positive autocorrelation was indicated the first difference transformation tends to bring the estimated parameters closer to those derived from the first order autoregressive model, although the sums of squared residuals were increased compared with the original least squares models. In the case (9d) where the significant negative autocorrelation was established, it seems to suggest that if the sign of autoccorelation is mistaken the result will be very serious. In applying the first difference model, one must first ascertain the sign of the autocorrelation, and must remember that even in the case of a positive autocorrelation the first difference transformation does not always eliminate unfavorable effects of autocorrelation of disturbances.

Last of the main onjectives was to examine sum of squared residuals as a function of the assumed autocorrelation coefficients. This curve was plotted in the top of each of the five graphs in the previous chapter. Each curve seems to have only one minimum within the interval examined. If it can be proven in general that there is always a unique autocorrelation coefficient that minimizes the sum of squared residuals, the procedure suggested by Marshall and Hirshleifer (briefly discussed in Chapter III) may be used to estimate regression coefficients and an autocorrelation coefficient.

To find out how effective a first order autoregressive model is, one must test it against empirical data and use it for actual predictions in the similar manner as one does to any other statistical estimation of economic relations.

The present study is concerned only with effects of applying a first order autoregressive model to some single equation models in which there is a significant evidence of autocorrelated disturbances. For further investigation into the problem of autocorrelation, it may be interesting to extend this kind of empirical study to economic models in which there are simultaneous relationships between the variables. Probably it is also worthwhile to look for a more efficient computing procedure because a considerable amount of time was spent in inverting a number of moment matrices by the Dolittle method in this study. Professor Hildreth suggested that efforts should be made to find distribution functions of the estimate of the autocorrelation coefficient and regression coefficients in the future. This will enable us to find confidence regions and perform tests of significance. He also suggested that the assumption of a first order autoregressive model should be checked against still more general models such as a second order autoregressive model or some other entirely different disturbances generating process.

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APPENDIX

COMPUTING PROCEDURES

SUPPLEMENTARY TO CHAPTER II

APPENDIX

The least squares computing procedures followed in estimating regression coefficients and a sum of squared residuals by assuming disturbances to be generated through a first order autoregressive model are described with an illustration.

In general, the first step in a regression analysis of this type is to compute the moments of variables being analyzed. The moment matrices are specified in Equation (12), Chapter II. It is expressed as follows:

$$\tilde{\pi}(\rho) = \left[(Z - \rho Z^*)'(Z - \rho Z^*)\right]^{-1} \left[(Z - \rho Z^*)'(Y - \rho Y^*)\right] . \qquad (A1)$$

Hence the moments are $(Z - \rho Z^*)'(Z - \rho Z^*)$ and $(Z - \rho Z^*)'(Y - \rho Y^*)$. If they are denoted by $M(\rho)$ and $N(\rho)$ respectively, (A1) may be re-written as,

$$\tilde{\pi}(\rho) = M(\rho)^{-1}N(\rho). \qquad (A2)$$

To refresh our memory, symbols used in (A1) are again identified here. Z and Y stand for observations on independent variables and a dependent variable. They are of the following matrices.

$$Z = \begin{pmatrix} z_{11} & \cdots & z_{1K} \\ \vdots & & \vdots \\ z_{T1} & \cdots & z_{TK} \end{pmatrix} \qquad Z^* = \begin{pmatrix} z_{o1} & \cdots & z_{oK} \\ \vdots & & \vdots \\ z_{T-1,1} & \cdots & z_{T-1,K} \end{pmatrix}$$

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} \qquad Y^* = \begin{pmatrix} y_1 \\ \vdots \\ y_{T-1} \end{pmatrix}$$

To faciliate the computation, $M(\rho)$ and $N(\rho)$ are expanded.

$$M(\rho) = Z'Z - 2\rho Z*'Z + \rho^2 Z*'Z* \qquad (A3)$$

$$N(\rho) = Z'Y + \rho(Z*Y + Z'Y*) + \rho^2 Z*'Y* \qquad (A4)$$

When Z'Z, Z*'Z etc., are computed a pair of quadratic of ρ can be obtained. From these quadratics, desired moment matrices may be derived by inserting any value ρ , $|\rho| < 1$, in them. This procedure is illustrated with the Meinken's demand equation for the world wheat market. In the Meinken's demand equation for the world wheat market (it is called 1b in this study), Z,Y,Z*,Y* are as follows. They are expressed in terms of deviations from sample means.

	Z		Y		Z*		Y*
	S_w	I	P_w		S^*_w	I*	P^*_w
1925	-667.14	36.84	60.64	1924	-694.79	46.10	69.04
26	-458.14	32.24	54.94	25	-564.79	32.40	52.94
27	-288.14	28.44	44.34	26	-355.79	27.80	46.84
28	92.86	22.24	19.54	27	-185.79	24.00	36.24
29	-54.14	9.44	21.44	28	195.21	17.80	11.40
30	285.86	-13.86	-29.76	29	48.21	5.00	13.44
31	329.86	-20.16	-34.06	30	388.21	-18.30	-37.86
32	272.86	-24.16	-35.16	31	432.21	-24.60	-42.16
33	362.86	-21.46	-41.06	32	375.21	-28.60	-43.26
34	93.86	-19.16	-30.46	33	465.21	-25.90	-49.16
35	-77.14	-15.96	-22.46	34	196.21	-23.60	-38.56
36	-304.14	-1.26	14.34	35	25.21	-20.40	-30.56
37	-225.14	-1.56	14.04	36	-201.79	-5.70	6.24
38	635.86	-11.56	-36.36	37	-122.79	-6.00	5.94

	Z'Z		Z'Y	Z'Z*		Z'Y*
Z { S_w	1708457.7	-78163.0	-153751.6	1181104.4	-70979.4	-106729.8
Z { I	-78163.0	6191.4	10055.4	-93543.9	6719.9	10526.2
Y { P_w			17628.1			16051.4

	Z*'Z		Z*'Y	Z*'Z*		Z*'Y
Z* { S^*_w	1181104.4	-93543.9	-150266.3	1792902.4	-104737.6	-180510.0
Z* { I*	-70979.4	6719.9	10337.1	-104737.6	8336.1	13029.9
Y* { P^*_w			16051.4			21337.1

Compute sums of squares and cross products of all the variables and fill them in proper places in the brackets below the table. For instance, the 2 x 2 matrix in the upper left corner is Z'Z or $\begin{pmatrix} \sum_t s_{tw}^2 & \sum_t s_{tw} i_t \\ \sum_t i_t s_{tw} & \sum_t i_t^2 \end{pmatrix}$. It is immediately noticed that this matrix is symmetrical. One has to compute only those elements on the main diagonal and those above the main diagonal. Those elements below the main diagonal may be computed to check the accuracy of computations. The 2 x 1 vector is Z*'Y or $\begin{pmatrix} \sum_t s_{tw} p_{tw} \\ \sum_t i_t \, p_{tw} \end{pmatrix}$. The single element is $\sum_t p_{tw}^2$. Inserting the values of the sums of squares and cross products into the two quadratics (A3) and (A4), the following expressions are derived.[1]

$$M(\rho) = \begin{bmatrix} 1708457.714 & -78163.026 \\ & 6191.412 \end{bmatrix} - \rho \begin{bmatrix} 2362208 & -164523.310 \\ & 13439.800 \end{bmatrix} + \rho^2 \begin{bmatrix} 1792902.357 & -104737.600 \\ & 8336.080 \end{bmatrix}$$

$$N(\rho) = \begin{bmatrix} 153751.614 \\ 10055.394 \end{bmatrix} - \rho \begin{bmatrix} -256956.156 \\ 20853.282 \end{bmatrix} + \rho^2 \begin{bmatrix} -180510.007 \\ 13029.970 \end{bmatrix}$$

For each value of ρ inserted, say ρ_1, a pair of moment matrices $M(\rho_1)$ and $N(\rho_1)$ can be obtained from the above quadratics.

Next step is to invert $M(\rho)$ and multiply this inverse with $N(\rho)$. In carrying out this computation, it is convenient to normalize $M(\rho)$ and $N(\rho)$. The normalization used here is to convert $M(\rho)$ and $N(\rho)$ into correlation matrices. A normalizing factor for m_{ii} (a typical element

---

1 Since the matrices in $M(\rho)$ are symmetrical, the terms below the main diagonal are not filled.

on the main diagonal of $M(\rho)$) is simply the reciprocal of m_{ii} itself, so that all elements on the main diagonal become 1 when normalized. To find a normalizing facor of m_{ij} (a typical element above or below the main diagonal), take the square roots of m_{ii} and m_{jj} and form a product of their reciprocals. For n_i (a typical element in $N(\rho)$), a normalizing factor used is $1/\sqrt{m_{ii}}\sqrt{Y(\rho)}$, where $Y(\rho)$ is defined as,

$$Y(\rho) = Y'Y - 2\rho Y'Y^* + \rho^2 Y^{*\prime}Y \qquad (A5)$$ [2]

This process is illustrated with the case where ρ is set equal to 0.1 in the Meinken's wheat model.

Since

$$M(0.1) = \begin{bmatrix} 1490165.852 & -62758.073 \\ & 4930.793 \end{bmatrix} \qquad N(0.1) = \begin{bmatrix} -129861.098 \\ 8100.370 \end{bmatrix}$$

and $Y(0.1) = 14631.184$,

the normalizing factors for $M(0.1)$ and $N(0.1)$ are as follows:

$$\frac{1}{\sqrt{1490165.852}\sqrt{1490165.852}} \text{ for } m_{11} \qquad \frac{1}{\sqrt{1490165.852}\sqrt{4930.793}} \text{ for } m_{12}$$

$$\frac{1}{\sqrt{4930.793}\sqrt{4930.793}} \text{ for } m_{22}$$

$$\frac{1}{\sqrt{1490165.852}\sqrt{14631.184}} \text{ for } n_1 \qquad \frac{1}{\sqrt{4930.793}\sqrt{14631.184}} \text{ for } n_2$$

The correlation matrices are obtained by multiplying each element in $M(0.1)$ and $N(0.1)$ by its respective normalizing factor. They are denoted by $\tilde{M}(0.1)$ and $\tilde{N}(0.1)$.

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[2] In Meinken's case, $Y(\rho) = 17628.094 - \rho\, 32102.816 + \rho^2\, 21337.112$

$$\tilde{M}(0.1) = \begin{bmatrix} 1 & -0.732139625 \\ & 1 \end{bmatrix} \qquad N(0.1) = \begin{bmatrix} -0.879472038 \\ 0.953688977 \end{bmatrix}$$

Now invert $\tilde{M}(0.1)$ and perform the matrix multiplication $\tilde{M}(0.1)^{-1}\,\tilde{N}(0.1)$ as follows:

$$\begin{bmatrix} 2.155304473 & 1.577983808 \\ & 2.155304473 \end{bmatrix} \begin{bmatrix} -0.879472038 \\ 0.953688977 \end{bmatrix} = \begin{bmatrix} -0.390624254 \\ 0.667697482 \end{bmatrix}$$

Elements in the column vector on the right side of the equality sign are normalized regression coefficients estimates. To transform them back to the original variables, each regression coefficient estimate π_i, i=1,2, is multiplied by $\sqrt{Y(0.1)}\,/\sqrt{m_{ii}}$.

$$-0.390624254 \times 0.099088311 = -0.038706298$$

$$0.667697482 \times 1.722587729 = 1.150167490$$

Therefore, estimates of the regression coefficients of the original variables s_w and I are approximately -0.039 and 1.2 when the autocorrelation coefficient ρ is set equal to 0.1.

The third step is to calculate the sum of squared residuals as described in Equation (14) in Chapter II. The sum of squared residuals $\tilde{V}'\tilde{V}(\rho)$ is expressed as,

$$\tilde{V}'\tilde{V}(\rho) = Y'Y - 2\rho Y'Y^{*} + \rho^2 Y^{*}{}'Y^{*} - \pi'N(\rho) \qquad \text{(A6)}$$

or

$$= Y(\rho) - \pi'N(\rho) \;.$$

In the Meinken's model where ρ is assumed to be 0.1, $\tilde{V}'\tilde{V}(0.1)$ is

$$= 14631.184 - \begin{bmatrix} -0.038706298 \\ 1.150167490 \end{bmatrix} \begin{bmatrix} -129861.098 \\ 8100.370 \end{bmatrix}$$

$$= 287.958$$

The final step is to calculate the mean or constant (π_0). Here the sums of dependent variables and independent variables are espressed in terms of original observations, and T denotes the number of observations.

$$\pi_0 = \frac{1}{T+1}\left[\sum_t y_t - (\pi_1 \sum_t z_{t1} + \ldots\ldots\ldots\ldots\ldots + \pi_K \sum_t z_{tK})\right]$$

In the Meinken's case,

$$\pi_0 = \frac{1}{15}\left[1531 - (-0.038706298 \times 65284 + 1.150167490 \times 1668.3)\right]$$
$$= 152.8164$$

In summary, the regression equation obtained under the assumption, $\rho = 0.1$, in the Meinken's wheat model and the corresponding sum of squares of residual are as follows:

$$P_w = 153 - 0.039\, S_w + 1.2\, I + U$$

$$U = 0.1\, U^* + V$$

and its sum of squared residuals is 287.958.

Date Due

~~MAR 30 1961~~			