ANALYTIC FUNCTIONS OVER A CERTAIN HYPERCOMPLEX ALGEBRA

Thesis for the Degree
of
Master of Science

Samuel Woodard Stewart
1937

Functions

Mathematics

To Professor J. E. Powell, for whose instruction, advice, and encouragement I am deeply grateful.

A N A L Y T I C   F U N C T I O N S
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H Y P E R C O M P L E X   A L G E B R A

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CONTENTS

Chapter III

ELEMENTARY FUNCTIONS

ANALYTIC FUNCTIONS

over a

CERTAIN HYPERCOMPLEX ALGEBRA

INTRODUCTION

This paper is a study of the algebra of a four-component hypercomplex variable which is commutative and associative, and of analytic functions of that variable. After a discussion of the algebra involved, the 'Cauchy-Riemann' equations are derived; Stokes' theorem is extended to four dimensions; Cauchy's integral theorem and integral formula are demonstrated; and properties of certain elementary functions are considered.

That the present variable was studied rather than the quaternion is due to the fact that analytic functions of a quaternion are limited to linear functions. Even the simple function $w = z^2$ is not analytic when z is a quaternion. Quaternion algebra has the advantage, however, of having no nilfactors.

The study made here follows to a certain extent a paper on general hypercomplex variables by P. W. Ketchum[1]. The present paper, however, gives explicit

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[1] Ketchum, P. W., "Analytic Functions of Hypercomplex Variables", Transactions of the American Mathematical Society, Volume 30, 1928, pages 641-667.

statements and proofs of results for the four-component variable, where Ketchum's paper is restricted, by its consideration of the n-component variable, to more general statements and indications of proofs.

In the introduction to his article, Ketchum lists several references to original papers by those who began the study of functions of hypercomplex variables. Four of these references, the ones most closely related to the present work, are given in the bibliography.

CHAPTER I

THE HYPERCOMPLEX ALGEBRA

1. <u>The variable</u>. The four-component variable used will be denoted by

$$z = x_1 + ix_2 + jx_3 + kx_4 .$$

The units are 1 (one), i, j, and k, with the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	k	-1	-i
k	k	-j	-i	1

It is to be especially noted that

$$i^2 = j^2 = -1 , \qquad 1^2 = k^2 = 1 .$$

The x's will be restricted to be real numbers.

It is easily shown that addition and multiplication are associative and commutative, and that multiplication is distributive. The factor law does not always hold true, for $(i + j)(i - j) = i^2 - j^2 = -1 + 1 = 0$, although $i + j \neq 0$ and $i - j \neq 0$. Such quantities as $i + j$ and $i - j$, for which a second factor can be found which makes the product zero though neither factor equals zero, are called <u>nilfactors</u>. We shall find that it is possible to specify all nilfactor values of the variable. In case a product is zero and it can be shown that no factor is a nilfactor, then the factor law may be applied.

2. Definitions. A determinant which occurs in the discussion of the algebra is

$$X = \begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix}.$$

The absolute value of z is defined to be

$$|z| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}.$$

In the discussion of functions, w will be used to represent the dependent variable. That is,

$$w = f(z) = u_1 + iu_2 + ju_3 + ku_4 ,$$

where $u_p = u_p(x_1, x_2, x_3, x_4)$. Similar to the determinant X, U will represent the determinant

$$U = \begin{vmatrix} u_1 & u_2 \\ u_3 & u_4 \end{vmatrix}.$$

The absolute value of a function will be denoted by

$$|w| = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^{1/2}.$$

The product of z by a real number, a scalar, a, is defined to be

$$az = ax_1 + iax_2 + jax_3 + kax_4.$$

The quotient of z by a real number different from zero is defined to be the product of z and 1/a,

$$\frac{z}{a} = \frac{x_1}{a} + i\frac{x_2}{a} + j\frac{x_3}{a} + k\frac{x_4}{a} .$$

3. Inequalities. Beginning with the well-known inequality of real variables,

$$a^2 + b^2 \geqq 2ab ,$$

it can be shown that:

$$(1) \qquad \big||z_1| - |z_2|\big| \leqq |z_1 + z_2| \leqq |z_1| + |z_2| ,$$

$$(2) \qquad X \leqq |X| \leqq |x_1x_4| + |x_2x_3| \leqq \frac{1}{2}|z|^2,$$

$$(3) \qquad |z| \leqq |x_1| + |x_2| + |x_3| + |x_4| \leqq 2|z| .$$

The second part of (3) is demonstrated by the same method used to prove the corresponding inequality of complex variables.

4. Product. The product of two numbers is,

$$\begin{aligned} w = u_1 + iu_2 + ju_3 + ku_4 &= z_1z_2 \\ &= (x_{11}x_{21} - x_{12}x_{22} - x_{13}x_{23} + x_{14}x_{24}) \\ (4) \qquad &+ i(x_{11}x_{22} + x_{12}x_{21} - x_{13}x_{24} - x_{14}x_{23}) \\ &+ j(x_{11}x_{23} - x_{12}x_{24} + x_{13}x_{21} - x_{14}x_{22}) \\ &+ k(x_{11}x_{24} + x_{12}x_{23} + x_{13}x_{22} + x_{14}x_{21}) . \end{aligned}$$

This product is symmetrical; that is, if x_{1p} and x_{2p} are interchanged the result remains the same. The absolute value of the product is

$$\begin{aligned} |w| = |z_1z_2| &= (|z_1|^2|z_2|^2 + 4X_1X_2)^{1/2} \\ (5) \qquad &\leqq \sqrt{2}\,|z_1||z_2| . \end{aligned}$$

The inequality follows from the inequality (2).

5. <u>Conjugate</u>. The conjugate, $\bar{z}$, of a number z is a number such that the product, $w = z\bar{z}$, has only real terms. That is,

$$u_2 = u_3 = u_4 = 0.$$

Given z, in order to find $\bar{z}$ let z_1 of (4) equal $\bar{z}$ and z_2 equal z. Then we have three equations to solve for the components $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$ of $\bar{z}$; they are,

$$\begin{aligned} u_2 &= x_2\bar{x}_1 + x_1\bar{x}_2 - x_4\bar{x}_3 - x_3\bar{x}_4 = 0 , \\ u_3 &= x_3\bar{x}_1 - x_4\bar{x}_2 + x_1\bar{x}_3 - x_2\bar{x}_4 = 0 , \\ u_4 &= x_4\bar{x}_1 + x_3\bar{x}_2 + x_2\bar{x}_3 + x_1\bar{x}_4 = 0 . \end{aligned} \tag{6}$$

These equations are sufficient to determine the ratios of the $\bar{x}$'s. The values of the $\bar{x}$'s are proportional to the three-rowed determinants, alternately plus and minus, obtained by dropping the first, second, etc., columns successively from the matrix of the coefficients.[2] On simplifying these determinants, we find that

$$\begin{aligned} c\bar{x}_1 &= x_1|z|^2 - 2x_4X , \\ c\bar{x}_2 &= -x_2|z|^2 - 2x_3X , \\ c\bar{x}_3 &= -x_3|z|^2 - 2x_2X , \\ c\bar{x}_4 &= x_4|z|^2 - 2x_1X , \end{aligned}$$

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[2] Bôcher, Maxime, <u>Introduction to Higher Algebra</u>, page 47, theorem IV.

where c is a real proportionality factor different from zero. Taking c equal to one, we shall mean by the conjugate of z,

$$(7)\qquad \begin{aligned} \bar{z} = (x_1|z|^2 - 2x_4X) - i(x_2|z|^2 + 2x_3X) \\ - j(x_3|z|^2 + 2x_2X) + k(x_4|z|^2 - 2x_1X) , \end{aligned}$$

which may also be written as

$$(7')\qquad \begin{aligned} \bar{z} = |z|^2(x_1 - ix_2 - jx_3 + kx_4) \\ - 2X(x_4 + ix_3 + jx_2 + kx_1) . \end{aligned}$$

The product of a number and its conjugate reduces to

$$(8)\qquad w = z\bar{z} = |z|^4 - 4X^2.$$

In case z is a nilfactor, the rank of the matrix of coefficients of equations (6) is two and the solution for the $\bar{x}$'s given above is not valid. The components of $\bar{z}$ then are properly the same as those of the conjugate nilfactor.[3]

6. Quotient. The quotient of two numbers, z_1 and z_2, is defined to be a number w such that

$$z_1 = wz_2.$$

Multiplying each member of this equation by the conjugate of z_2, we get

$$z_1\bar{z}_2 = wz_2\bar{z}_2.$$

Dividing each member by $(z_2\bar{z}_2)$, a real number, we

3 For a demonstration of the statements of this paragraph, see section 10.

find for w, using (8),

$$(9) \qquad w = \frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^4 - 4X_2^2} .$$

Let

$$C = (x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + x_{14}x_{24}) ,$$

$$D = (x_{11}x_{22} - x_{12}x_{21} + x_{13}x_{24} - x_{14}x_{23}) ,$$

$$E = (x_{11}x_{23} + x_{12}x_{24} - x_{13}x_{21} - x_{14}x_{22}) ,$$

$$F = (x_{11}x_{24} - x_{12}x_{23} - x_{13}x_{22} + x_{14}x_{21}) .$$

Then, after simplification,

$$(10) \qquad w = \frac{z_1}{z_2} = \frac{1}{(|z_2|^4 - 4X_2^2)}\Big[(C|z_2|^2 - 2FX_2) - i(D|z_2|^2 + 2EX_2) - j(E|z_2|^2 + 2DX_2) + k(F|z_2|^2 - 2CX_2)\Big] .$$

We also find that the absolute value of the quotient is

$$|w| = \left|\frac{z_1}{z_2}\right| = \left[\frac{|z_1|^2|z_2|^2 - 4X_1X_2}{|z_2|^4 - 4X_2^2}\right]^{1/2} .$$

7. <u>Determination</u> <u>of</u> <u>a</u> <u>nilfactor</u>. A number z_1 different from zero is called a nilfactor if there exists a second number z_2 different from zero such that their product is zero. The number z_2 will then be called the <u>conjugate</u> <u>nilfactor</u> to z_1.

If $w = z_1 z_2$, then for a known number z_1 to be a nilfactor we must be able to find a z_2 such that $w = 0$; that is,

$$u_1 = u_2 = u_3 = u_4 = 0 .$$

Taking the values of the u's from equation (4), we get the four equations:

$$(12) \qquad \begin{aligned} u_1 &= x_{11}x_{21} - x_{12}x_{22} - x_{13}x_{23} + x_{14}x_{24} = 0, \\ u_2 &= x_{12}x_{21} + x_{11}x_{22} - x_{14}x_{23} - x_{13}x_{24} = 0, \\ u_3 &= x_{13}x_{21} - x_{14}x_{22} + x_{11}x_{23} - x_{12}x_{24} = 0, \\ u_4 &= x_{14}x_{21} + x_{13}x_{22} + x_{12}x_{23} + x_{11}x_{24} = 0. \end{aligned}$$

In order for there to exist solutions not all zero for the components of z_2, the determinant of the coefficients (the known x_1's) must equal zero. <u>The necessary and sufficient condition</u>, then, <u>that</u> z_1 <u>be a nilfactor is that</u>

$$(13) \qquad D = \begin{vmatrix} x_{11} & -x_{12} & -x_{13} & x_{14} \\ x_{12} & x_{11} & -x_{14} & -x_{13} \\ x_{13} & -x_{14} & x_{11} & -x_{12} \\ x_{14} & x_{13} & x_{12} & x_{11} \end{vmatrix} = 0 \; .$$

On expanding the determinant, we get

$$(14) \qquad D = |z_1|^4 - 4X_1^2 \; .$$

From $D = 0$, we find

$$|z_1|^2 = \pm 2X_1 \; .$$

That is,

$$x_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2 = \pm 2x_{11}x_{14} \mp 2x_{12}x_{13} ,$$

$$x_{11}^2 \mp 2x_{11}x_{14} + x_{14}^2 + x_{12}^2 \pm 2x_{12}x_{13} + x_{13}^2 = 0,$$

or

$$(x_{11} \mp x_{14})^2 + (x_{12} \pm x_{13})^2 = 0 .$$

Therefore,

(15) $$x_{11} \mp x_{14} = 0 \quad \text{and} \quad x_{12} \pm x_{13} = 0 ,$$

$$x_{11}^2 = x_{14}^2 \qquad\qquad x_{12}^2 = x_{13}^2 .$$

Adding,

$$x_{11}^2 + x_{12}^2 = x_{13}^2 + x_{14}^2 = x_{11}^2 + x_{13}^2 = x_{12}^2 + x_{14}^2.$$

Let

$$m = x_{11}^2 + x_{12}^2 .$$

Then

(16) $$2m = |z_1|^2 , \qquad m = \frac{1}{2}|z_1|^2 \neq 0 ,$$

since z_1 cannot be zero.

Also from (15),

(17) $$x_{11}x_{13} + x_{12}x_{14} = \mp x_{11}x_{12} \pm x_{12}x_{11} = 0 .$$

Taking the signs in (15) one at a time, we have two possibilities:

Case I,

(18) $$x_{14} = x_{11} , \qquad x_{13} = -x_{12} ,$$

$$X_1 = x_{11}^2 + x_{12}^2 = m ,$$

$$|z_1|^2 = 2(x_{11}^2 + x_{12}^2) = 2m = 2X_1 ,$$

(19) $$z_1 = r + is - js + kr .$$

Case II,

(20) $$x_{14} = -x_{11}\ , \qquad x_{13} = x_{12}\ ,$$

$$X_1 = -x_{11}^2 - x_{12}^2 = -m\ ,$$

$$|z_1|^2 = 2(x_{11}^2 + x_{12}^2) = 2m = -2X_1\ ,$$

(21) $$z_1 = r + is + js - kr\ ,$$

where r and s are any real parameters not both zero.

In either case,

(22) $$m = |X_1|\ .$$

We are now able to state the theorem: <u>The necessary and sufficient condition that z_1 be a nilfactor is that its components satisfy either</u> (18) <u>or</u> (20). This is true because (18) and (20) follow directly from D = 0, and each gives D = 0.

8. <u>Solution for the conjugate nilfactor</u>. When D = 0, that is, when z_1 is a nilfactor, then D is of rank two. This can be shown by consideration of the second- and third-order minors of D. The two-by-two determinant in the upper left-hand corner of D equals m, which is never zero, (16). The three-by-three determinant in the upper left-hand corner, for example, reduces to

$$x_{11}|z_1|^2 - 2x_{14}X_1\ .$$

Considering the equations given under either case I or case II, we find that this determinant is zero. The other three third-order determinants having m as first minor reduce to similar values and are likewise equal to zero. Therefore, all three-rowed determinants of D are zero, and D is of rank two.[4]

As D is of rank two, we can solve equations (12) for two of the x_2's in terms of the other two. Solving for x_{23} and x_{24}, we get

$$x_{23} = \frac{1}{m}\Big[x_{21}(x_{11}x_{13} + x_{12}x_{14}) + x_{22}(x_{11}x_{14} - x_{12}x_{13})\Big].$$

In this expression, the first quantity in parentheses is zero by (17), the second is X_1. By (22), $m = |X_1|$. Therefore,

$$(23) \qquad x_{23} = \frac{X_1}{|X_1|}\, x_{22} \, .$$

Similarly,

$$(24) \qquad x_{24} = -\frac{X_1}{|X_1|}\, x_{21} \, .$$

Referring to the equations given under cases I and II, we have, under case I,

$$(25) \qquad x_{23} = x_{22}\,, \qquad \text{and} \qquad x_{24} = -x_{21}\,,$$

$$(26) \qquad z_2 = p + iq + jq - kp\,;$$

and under case II,

$$(27) \qquad x_{23} = -x_{22}\,, \qquad \text{and} \qquad x_{24} = x_{21}\,,$$

$$(28) \qquad z_2 = p + iq - jq + kp\,,$$

where p and q are real parameters, not both zero.

---

4 Bôcher, Theorem I, page 54.

9. <u>Summary of nilfactors</u>. Let a, b, c, d be any real numbers such that

$$a^2 + b^2 \neq 0 \text{ , and } c^2 + d^2 \neq 0 \text{ .}$$

Then, by equations (19) and (21),

$$z_1 = a + ib \mp jb \pm ka$$

is a nilfactor, and, by equations (26) and (28),

$$z_2 = c + id \pm jd \mp kc$$

is a conjugate nilfactor to z_1, regardless of the relative values of a, b, c, and d. This can be verified by direct multiplication of z_1 and z_2, the product being zero.

If z_1 is a nilfactor, a conjugate nilfactor is most simply found by changing the signs of x_{13} and x_{14}.

It follows from the associative law of multiplication that the product of a nilfactor by any hypercomplex number, not zero and not a conjugate nilfactor of the given number, is itself a nilfactor. It should be noted that if X is zero, z cannot be a nilfactor.

10. <u>Relation of conjugate and nilfactor</u>. If z is a nilfactor, then D is of rank two as shown above. The matrix of coefficients of equations (6) is the same, when x_p in (6) equals x_{1p} in D, as the last three rows of D, and is therefore of rank two.

This renders the solutions for the components of $\bar{z}$ given in (7) invalid. The last two of equations

(6) are of rank two and may be solved for $\bar{x}_3$ and $\bar{x}_4$ in terms of $\bar{x}_1$, $\bar{x}_2$, and the x's. The components of the conjugate then are found to be the same as those of the conjugate nilfactor. This is evident by comparison of the last two of equations (6) with the last two of (12). The name "conjugate nilfactor" was chosen for this reason.

The product of a nilfactor and its conjugate is necessarily zero, by the argument of the preceding paragraph. Therefore, it is necessary to bar division by nilfactors as well as division by zero. A nilfactor in the denominator of a fraction has the same effect as a zero.

CHAPTER II

DIFFERENTIATION AND INTEGRATION

11. Limits; continuity. The definition of a limit is the same as that used in complex variable. The necessary and sufficient condition that a sequence or a function approach a limit is that each component of the general term of the sequence, or each conjugate function of the function, approach the corresponding component of the limit. The usual theorems on the sum, product, and quotient of limits hold true, except that nilfactor, as well as zero, values of the limit of the divisor are barred.

The definitions of continuity and uniform continuity likewise remain the same. The necessary and sufficient condition for a function to be continuous is that the conjugate functions be continuous. Other theorems on continuity and on limits also carry over from complex variable.[5]

12. The derivative. As in functions of a complex variable, the derivative of a function is defined to be the following limit, if it exists:

$$\frac{dw}{dz} = f'(z) = \underset{\Delta z = 0}{L} \frac{f(z + \Delta z) - f(z)}{\Delta z} . \tag{29}$$

---

5 Ketchum, page 643, footnote, states that in Townsend, Functions of a Complex Variable, every theorem of Chapter II, pages 20-41, is true for hypercomplex variables.

A function which has a derivative at $z = a$ is said to be regular at a. If it is regular at every point of a region, it is said to be holomorphic in that region. It is called an analytic function if it is holomorphic in some region.

If a function has a derivative, that derivative must be independent of the method of approach of $(z + \Delta z)$ to z. By setting Δz equal, in turn, to Δx_1, $i\Delta x_2$, $j\Delta x_3$, and $k\Delta x_4$, and taking the limit as Δz approaches zero, we find the following values for the derivative, if the partial derivatives of the conjugate functions exist:

$$
\begin{aligned}
\frac{dw}{dz} &= \frac{\partial \mu_1}{\partial x_1} + i\frac{\partial \mu_2}{\partial x_1} + j\frac{\partial \mu_3}{\partial x_1} + k\frac{\partial \mu_4}{\partial x_1}, \\
&= \frac{\partial \mu_2}{\partial x_2} - i\frac{\partial \mu_1}{\partial x_2} + j\frac{\partial \mu_4}{\partial x_2} - k\frac{\partial \mu_3}{\partial x_2}, \\
&= \frac{\partial \mu_3}{\partial x_3} + i\frac{\partial \mu_4}{\partial x_3} - j\frac{\partial \mu_1}{\partial x_3} - k\frac{\partial \mu_2}{\partial x_3}, \\
&= \frac{\partial \mu_4}{\partial x_4} - i\frac{\partial \mu_3}{\partial x_4} - j\frac{\partial \mu_2}{\partial x_4} + k\frac{\partial \mu_1}{\partial x_4}.
\end{aligned}
\tag{30}
$$

On equating the corresponding components of these four values of the derivative, we find that the necessary conditions for a function to have a derivative are that the partial derivatives of the conjugate functions exist and satisfy the equations,

$$
\begin{aligned}
&\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = \frac{\partial u_4}{\partial x_4}\,, \\
&\frac{\partial u_1}{\partial x_2} = -\frac{\partial u_2}{\partial x_1} = \frac{\partial u_3}{\partial x_4} = -\frac{\partial u_4}{\partial x_3}\,, \\
&\frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_4} = -\frac{\partial u_3}{\partial x_1} = -\frac{\partial u_4}{\partial x_2}\,, \\
&\frac{\partial u_1}{\partial x_4} = -\frac{\partial u_2}{\partial x_3} = -\frac{\partial u_3}{\partial x_2} = \frac{\partial u_4}{\partial x_1}\,.
\end{aligned}
\tag{31}
$$

These are the equations corresponding to the Cauchy-Riemann equations in functions of a complex variable. We shall call them the Cauchy-Riemann equations here also.

By substitution from these equalities in the values of the derivative given above, we can write other expressions for the derivative, each involving the partial derivatives of only one of the u's. They are,

$$
\begin{aligned}
\frac{dw}{dz} &= \frac{\partial u_1}{\partial x_1} - i\frac{\partial u_1}{\partial x_2} - j\frac{\partial u_1}{\partial x_3} + k\frac{\partial u_1}{\partial x_4}\,, \\
&= \frac{\partial u_2}{\partial x_2} + i\frac{\partial u_2}{\partial x_1} - j\frac{\partial u_2}{\partial x_4} - k\frac{\partial u_2}{\partial x_3}\,, \\
&= \frac{\partial u_3}{\partial x_3} - i\frac{\partial u_3}{\partial x_4} + j\frac{\partial u_3}{\partial x_1} - k\frac{\partial u_3}{\partial x_2}\,, \\
&= \frac{\partial u_4}{\partial x_4} + i\frac{\partial u_4}{\partial x_3} + j\frac{\partial u_4}{\partial x_2} + k\frac{\partial u_4}{\partial x_1}\,.
\end{aligned}
\tag{32}
$$

<u>If the conjugate functions u_p have continuous first partial derivatives satisfying the Cauchy-Riemann equations in a region, then $w = f(z)$ has a derivative $f'(z)$ in that region.</u>

Since u_p has continuous partial derivatives, it is totally differentiable, and

$$\Delta u_p = \frac{\partial u_p}{\partial x_1}\Delta x_1 + \frac{\partial u_p}{\partial x_2}\Delta x_2 + \frac{\partial u_p}{\partial x_3}\Delta x_3 + \frac{\partial u_p}{\partial x_4}\Delta x_4 + \epsilon_{p1}\Delta x_1 + \epsilon_{p2}\Delta x_2 + \epsilon_{p3}\Delta x_3 + \epsilon_{p4}\Delta x_4 ,$$

where the ϵ's converge uniformly to zero as Δz approaches zero.[6] That is,

$$|\epsilon_{pi}| < \epsilon \quad \text{for} \quad |\Delta x_i| \leqq |\Delta z| < \delta .$$

Then

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u_1 + i\,\Delta u_2 + j\,\Delta u_3 + k\,\Delta u_4}{\Delta z}$$

$$= \frac{1}{\Delta z}\left[\sum_{p=1}^{4}\frac{\partial u_1}{\partial x_p}\Delta x_p + i\sum_1^4\frac{\partial u_2}{\partial x_p}\Delta x_p + j\sum_1^4\frac{\partial u_3}{\partial x_p}\Delta x_p + k\sum_1^4\frac{\partial u_4}{\partial x_p}\Delta x_p\right] + \bar{\epsilon} ,$$

for $|\Delta z| < \delta$. The absolute value of $\bar{\epsilon}$ is less than 16ϵ because

$$\left|\frac{\Delta x_p}{\Delta z}\right| \leqq 1 , \quad \text{and} \quad |\epsilon_{pi}| < \epsilon .$$

---

[6] Pierpont, J., <u>Theory of Functions of Real Variables</u>, Volume I, Article 426, pages 271, 2.

Substituting in the above difference quotient from equations (31), we can reduce it to

$$\frac{\Delta w}{\Delta z} = \left[\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} + j\frac{\partial u_3}{\partial x_1} + k\frac{\partial u_4}{\partial x_1} \right] \frac{\Delta z}{\Delta z} + \bar{\epsilon} .$$

Taking the limit as Δz approaches zero, we get

$$f'(z) = \frac{dw}{dz} = \frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} + j\frac{\partial u_3}{\partial x_1} + k\frac{\partial u_4}{\partial x_1} ,$$

since we know that the partial derivatives exist and that $|\bar{\epsilon}| < 16\epsilon$, while ϵ approaches zero with Δz. The value of $f'(z)$ found is exactly the first of equations (30); so the theorem is proved.

As in functions of complex variables, it can be shown that if a function has a derivative, that derivative is continuous. From this it follows easily that the first partial derivatives of the conjugate functions u_p are continuous. Accepting this as true, we can state the following theorem: <u>The necessary and sufficient condition that a function</u> $w = f(z)$ <u>be holomorphic in a given region is that the first partial derivatives of the conjugate functions exist, are continuous, and satisfy equations</u> (31).

The rules for differentiation of a sum, a product, and a quotient remain the same, as can be shown from the definition. The derivative of z is found from the definition to be 1. Starting with this, we can show by induction that the derivative of z^n is $nz^{(n-1)}$, and that the derivative of a rational function may be found as usual, except at nilfactor values of the denominator.

13. Stokes' theorem in S_4. By an argument similar to the proof of Green's theorem in the plane,[7] it can be shown that on a curved surface with curvilinear coordinates u and v, we have the Lemma: *If $P(u,v)$, $Q(u,v)$, $\frac{\partial P}{\partial v}$, and $\frac{\partial Q}{\partial u}$ are continuous in a region S' and C is an ordinary curve on the surface in S' bounding a region S of the surface, then*

$$(33) \qquad \iint_S \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du\, dv = \int_C (P\, du + Q\, dv) \; .$$

With the use of this lemma we get Stokes' Theorem: *If $P_i(x_1, x_2, x_3, x_4)$, $i = 1, 2, 3, 4$, are continuous functions with continuous first partial derivatives in a region R of S_4, C is an ordinary closed curve in R, S is a surface in R, bounded*

[7] Townsend, E. J., *Functions of a Complex Variable*, pages 54-56.

by C, with parametric equations $x_i = x_i(u,v)$, $i = 1, 2, 3, 4$, where the first partial derivatives $\frac{\partial x_i}{\partial u}$ and $\frac{\partial x_i}{\partial v}$ exist and are continuous and the second partial derivatives $\frac{\partial^2 x_i}{\partial u \partial v} = \frac{\partial^2 x_i}{\partial v \partial u}$ exist, then

$$\int_C (P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + P_4 dx_4)$$

$$(34) \qquad \begin{aligned} = \iint_S \Big[&\pm \left(\frac{\partial P_2}{\partial x_1} - \frac{\partial P_1}{\partial x_2} \right) dx_1 dx_2 \\ &\pm \left(\frac{\partial P_3}{\partial x_2} - \frac{\partial P_2}{\partial x_3} \right) dx_2 dx_3 \\ &\pm \left(\frac{\partial P_4}{\partial x_3} - \frac{\partial P_3}{\partial x_4} \right) dx_3 dx_4 \\ &\pm \left(\frac{\partial P_1}{\partial x_3} - \frac{\partial P_3}{\partial x_1} \right) dx_3 dx_1 \\ &\pm \left(\frac{\partial P_2}{\partial x_4} - \frac{\partial P_4}{\partial x_2} \right) dx_4 dx_2 \\ &\pm \left(\frac{\partial P_4}{\partial x_1} - \frac{\partial P_1}{\partial x_4} \right) dx_1 dx_4 \Big] . \end{aligned}$$

The $\pm$ signs used above do not indicate ambiguity, but a difficulty in determining the proper signs of the Jacobians which occur in the proof. The result as stated is sufficient for the purpose desired in the proof of Cauchy's first integral theorem given in section 15.

Let

$$x_{pu} = \frac{\partial x_p}{\partial u}, \quad x_{puv} = \frac{\partial^2 x_p}{\partial u \partial v}, \text{ etc.}$$

Then, by the lemma, (33),

$$\int_C P_1 dx_1 = \int_C P_1(x_{1u}du + x_{1v}dv)$$

$$= \int_C \left[(P_1 x_{1u})du + (P_1 x_{1v})dv\right]$$

$$= \iint_S \left[\frac{\partial}{\partial u}(P_1 x_{1v}) - \frac{\partial}{\partial v}(P_1 x_{1u})\right] du\, dv \;.$$

By writing out the partial derivatives indicated and performing the subtraction, we find that

$$\int_C P_1 dx_1 = - \iint_S \frac{\partial P_1}{\partial x_2} J\left(\frac{x_1, x_2}{u,\ v}\right) du\, dv$$

$$+ \iint_S \frac{\partial P_1}{\partial x_3} J\left(\frac{x_3, x_1}{u,\ v}\right) du\, dv$$

$$- \iint_S \frac{\partial P_1}{\partial x_4} J\left(\frac{x_1, x_4}{u,\ v}\right) du\, dv$$

$$= \mp \iint_{S_1} \frac{\partial P_1}{\partial x_2} dx_1 dx_2 \pm \iint_{S_4} \frac{\partial P_1}{\partial x_3} dx_3 dx_1$$

$$\mp \iint_{S_6} \frac{\partial P_1}{\partial x_4} dx_1 dx_4 \;,$$

where S_1, S_2, . . . S_6 are the projections of S on the x_1x_2, x_2x_3, x_3x_4, x_3x_1, x_4x_2, and x_1x_4 planes, respectively. Similarly,

$$\int_C P_2 dx_2 = \mp \iint_{S_2} \frac{\partial P_2}{\partial x_3} dx_2 dx_3$$

$$\pm \iint_{S_5} \frac{\partial P_2}{\partial x_4} dx_4 dx_2 \pm \iint_{S_1} \frac{\partial P_2}{\partial x_1} dx_1 dx_2 \;,$$

$$\int_C P_3 dx_3 = \mp \iint_{S_3} \frac{\partial P_3}{\partial x_4} dx_3 dx_4$$

$$\mp \iint_{S_4} \frac{\partial P_3}{\partial x_1} dx_3 dx_1 \pm \iint_{S_2} \frac{\partial P_3}{\partial x_2} dx_2 dx_3 ,$$

$$\int_C P_4 dx_4 = \pm \iint_{S_6} \frac{\partial P_4}{\partial x_1} dx_1 dx_4$$

$$\mp \iint_{S_5} \frac{\partial P_4}{\partial x_2} dx_4 dx_2 \pm \iint_{S_3} \frac{\partial P_4}{\partial x_3} dx_3 dx_4 .$$

Adding these four integrals, we get the equation of the theorem, with integration over S, since x_1 and x_2 take on the same values over S as over the projection S_1, etc.

14. Definition and properties of the integral. The integral is defined to be

$$(35) \qquad \int_C f(z)\, dz = \underset{ND=0}{L} \sum_{p=1}^{n} f(\xi_p)\, \Delta_p z ,$$

where

$$w = f(z) = u_1 + iu_2 + ju_3 + ku_4 ,$$

$$\Delta_p z = \Delta_p x_1 + i\, \Delta_p x_2 + j\, \Delta_p x_3 + k\, \Delta_p x_4$$

$$= z_p - z_{p-1} .$$

$$\xi_p = \xi_{p1} + i\, \xi_{p2} + j\, \xi_{p3} + k\, \xi_{p4}$$

is a value of z in the interval $\Delta_p z$ on C, the path of integration.

The product to be summed in (35), expanded, consists of sixteen terms, a typical one being

$$(- j\, u_4(\xi_p)\, \Delta_p x_2) .$$

If $f(z)$ is continuous, then the u's are continuous, and the limit of the sum of each of the sixteen terms exists. Therefore, the limit defining the integral exists. It can be evaluated in terms of real line integrals by the use of the following equation, each real integral being the limit of the sum of four of the sixteen terms mentioned above. This equation is

$$(36)\quad \begin{aligned}\int_C f(z)dz = &\int_C (u_1dx_1 - u_2dx_2 - u_3dx_3 + u_4dx_4)\\ &+ i\int_C (u_2dx_1 + u_1dx_2 - u_4dx_3 - u_3dx_4)\\ &+ j\int_C (u_3dx_1 - u_4dx_2 + u_1dx_3 - u_2dx_4)\\ &+ k\int_C (u_4dx_1 + u_3dx_2 + u_2dx_3 + u_1dx_4)\ .\end{aligned}$$

By argument from the definition of an integral, it can be shown that if $f(z)$ and $g(z)$ are integrable and c is a point on the path of integration, then:

$$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz\ ,$$

$$\int_a^b k\,f(z)\,dz = k\int_a^b f(z)\,dz\ ,$$

$$\int_a^b [f(z) + g(z)]\,dz = \int_a^b f(z)\,dz + \int_a^b g(z)\,dz\ ,$$

$$\int_a^b f(z)\,dz = \int_a^c f(z)\,dz + \int_c^b f(z)\,dz\ .$$

Also from the definition it is easily shown that

$$(37)\qquad \int_a^b |dz| = L\ ,$$

where L is the length of the path of integration.

By inequalities (1) and (5),

$$\left|\sum_{p=1}^{n} f(\xi_p)\,\Delta_p z\right| \leqq \Sigma\left|f(\xi_p)\,\Delta_p z\right|$$

$$\leqq \sqrt{2}\;\Sigma\left|f(\xi_p)\right|\cdot\left|\Delta_p z\right| .$$

Taking the limit as Δz approaches zero, we have, if $|f(z)|$ is integrable,

$$(38)\qquad \begin{aligned} \left|\int_a^b f(z)\,dz\right| &\leqq \sqrt{2}\int_a^b |f(z)|\cdot|dz| \\ &= \sqrt{2}\int_a^b M\,|dz| = \sqrt{2}\,M\int_a^b |dz| = \sqrt{2}\,ML , \end{aligned}$$

where M is the least upper bound of $|f(z)|$ on the path of integration.

15. <u>Cauchy's first integral theorem.</u> <u>If</u> $f(z)$ <u>is holomorphic in a region</u> R <u>and the partial derivatives</u> $\frac{\partial u_i}{\partial x_j}$ $(i, j = 1, 2, 3, 4)$ <u>exist and are continuous in</u> R, <u>and if</u> C <u>is a closed curve in</u> R, <u>then</u>

$$(39)\qquad \int_C f(z)\,dz = 0 .$$

Since $f(z)$ is holomorphic, then it is continuous and the partial derivatives satisfy equations (31). From the continuity of $f(z)$ we know that the integral exists. Applying Stokes' theorem four times, once to each of the integrals in the right member of equation (36), the u's being the P functions, we get an expression for $\int_C f(z)\,dz$ in terms of double integrals, over a surface S bounded by C, of such expressions as

$$\left(- \frac{\partial \mu_2}{\partial x_1} - \frac{\partial \mu_1}{\partial x_2} \right) .$$

There are twenty-four such integrals in all. On substitution from equations (31) in these expressions, each of them is seen to be zero. Therefore, the value of each double integral is zero and the conclusion of the theorem is proved.

The hypothesis that the partial derivatives $\frac{\partial \mu_i}{\partial x_j}$ are continuous is equivalent to requiring that $f'(z)$ be continuous. By a demonstration parallel to Goursat's proof of the theorem in complex variable, it is possible to complete the proof without requiring the continuity of $f'(z)$.[8]

A consequence of the theorem is, *Corollary* I: *The integral of* f(z) *has the same value along any path in* R *joining the same two points; that is, the integral is independent of the path of integration*.

Another consequence is, *Corollary* II: *In a region in which* f(z) *is holomorphic except at certain points, a path of integration can be deformed into any other path without affecting the value of the integral provided that in the process the path is*

---

[8] Townsend, pages 66-71.

cut by no singularities.

16. Cauchy's integral formula. The values of z which make $(z - b)$ a nilfactor are

$$z = b_1 + r + i(b_2 + s) + j(b_3 \mp s) + k(b_4 \pm r) ,$$

where the b_p are the components of b, and r and s are parameters, not both zero. The points representing these values of z lie in the planes A and B, and include all points of the planes but the point b which is their only point of intersection.

Plane A is

$$x_1 = x_1 , \qquad x_3 = - x_2 + (b_3 + b_2) ,$$
$$x_2 = x_2 , \qquad x_4 = x_1 + (b_4 - b_1) .$$

Plane B is

$$x_1 = x_1 , \qquad x_3 = x_2 + (b_3 - b_2) ,$$
$$x_2 = x_2 , \qquad x_4 = - x_1 + (b_4 + b_1) .$$

The planes E and F,

$$x_3 = b_3 , \qquad x_4 = b_4 ,$$

and

$$x_2 = b_2 , \qquad x_4 = b_4 ,$$

intersect planes A and B at the point b only. With the planes A, B, E, and F specified, we can state the following theorem, Cauchy's integral formula: If $f(z)$ is holomorphic in the simply connected region R, b is any point in R, and C is an ordinary curve in R encircling, but not intersecting, the planes A and B, then: (1) <u>if</u> C <u>can</u> <u>be</u> <u>deformed</u> <u>into</u> <u>a</u> <u>curve</u> C_1 <u>about</u> <u>the</u> <u>point</u> <u>b</u> <u>in</u> <u>the</u> <u>plane</u> E <u>without</u> <u>cutting</u> <u>planes</u> A <u>or</u> B,

$$f(b) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - b}\, dz \; ; \tag{40}$$

(2) <u>if</u> C <u>can</u> <u>be</u> <u>deformed</u> <u>into</u> <u>a</u> <u>curve</u> C_2 <u>about</u> <u>the</u> <u>point</u> b <u>in</u> <u>the</u> <u>plane</u> F <u>without</u> <u>cutting</u> <u>planes</u> A <u>or</u> B,

$$f(b) = \frac{1}{2\pi j} \int_C \frac{f(z)}{z - b}\, dz \; . \tag{41}$$

We see that

$$F(z) = \frac{f(z)}{z - b}$$

is holomorphic in R except on planes A and B. Therefore, it is holomorphic on planes E and F except at b.

Case (1), C deformable into C_1: In plane E, let K_1 be the circle

$$x_1 + ix_2 = b_1 + ib_2 + r\, e^{i\theta} ,$$

$$x_3 = b_3 , \qquad x_4 = b_4 ,$$

where r is a constant, θ a parameter. Then, by corollary II on page 26,

$$\int_C \frac{f(z)}{z - b}\, dz = \int_{K_1} \frac{f(z)}{z - b}\, dz$$

$$= \int_{K_1} \frac{f(b)}{z - b}\, dz + \int_{K_1} \frac{f(z) - f(b)}{z - b}\, dz \; .$$

Because f(z) is holomorphic in R, it is regular, and hence continuous, at b. Therefore,

$$|f(z) - f(b)| < \epsilon \quad \text{for} \quad |z - b| < \delta \ .$$

On K_1, $|z - b| = r$. Pick $r < \delta$. Then, by equation (38),

$$\left| \int_{K_1} \frac{f(z) - f(b)}{z - b}\, dz \right| \leqq \sqrt{2}\ ML$$

$$= \sqrt{2}\ \left(\frac{\epsilon}{r}\right)(2\pi r) = 2^{3/2}\pi\,\epsilon \ .$$

Therefore,

$$\left| \int_{K_1} \frac{f(z) - f(b)}{z - b}\, dz \right| = 0,$$

and

$$\int_{K_1} \frac{f(z) - f(b)}{z - b}\, dz = 0 \ .$$

Setting

$$z - b = r\, e^{i\theta} \ ,$$

for z on K_1, we find that

$$\int_{K_1} \frac{f(b)}{z - b}\, dz = f(b) \int_0^{2\pi} i\ d\theta = 2\pi i\ f(b).$$

Combining these two results, we have

$$\int_C \frac{f(z)}{z - b}\, dz = 0 + 2\pi i\ f(b),$$

$$f(b) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - b}\, dz \ ,$$

as was to be shown.

Case (2), C deformable into C_2: In plane F, let K_2 be the circle

$$x_1 + jx_3 = b_1 + jb_3 + r\, e^{j\theta} \ ,$$

$$x_2 = b_2 \ , \qquad x_4 = b_4 \ ,$$

where r is constant, θ a parameter. By the same argument as for the first case, replacing i by j, K_1 by K_2, we can show that

$$f(b) = \frac{1}{2\pi j} \int_C \frac{f(z)}{z - b} \, dz \, .$$

CHAPTER III

ELEMENTARY FUNCTIONS

17. Definitions. The exponential function, circular functions, and hyperbolic functions are defined by the series which represent them in complex variable. The addition formulas for these functions remain valid for the hypercomplex variable, as can be verified from the expansions.

From the series we find that

$$e^{jx} = \cos x + j \sin x ,$$

$$e^{kx} = \cosh x + k \sinh x ,$$

$$\sin jx = j \sinh x , \qquad \cos jx = \cosh x ,$$

$$\sin kx = k \sin x , \qquad \cos kx = \cos x .$$

These equations remain true when j is everywhere replaced by i.

18. The exponential function. Using the addition theorem and the equations of the preceding section, we find that

$$\begin{aligned} w = e^z &= e^{(x_1 + ix_2 + jx_3 + kx_4)} \\ &= e^{(x_1 + ix_2)} e^{(jx_3 + kx_4)} \\ &= e^{x_1}(\cos x_2 \cos x_3 \cosh x_4 \\ &\qquad + \sin x_2 \sin x_3 \sinh x_4) \end{aligned}$$

$$
\begin{aligned}
& + i\, e^{x_1}(\sin x_2 \cos x_3 \cosh x_4 \\
& \qquad - \cos x_2 \sin x_3 \sinh x_4) \\
(42) \qquad & + j\, e^{x_1}(\cos x_2 \sin x_3 \cosh x_4 \\
& \qquad - \sin x_2 \cos x_3 \sinh x_4) \\
& + k\, e^{x_1}(\sin x_2 \sin x_3 \cosh x_4 \\
& \qquad + \cos x_2 \cos x_3 \sinh x_4)\ .
\end{aligned}
$$

From this value, after simplification, we have

$$(43) \qquad |w|^2 = e^{2x_1} \cosh(2x_4)\ ,$$

and

$$(44) \qquad U = \frac{1}{2}\, e^{2x_1} \sinh(2x_4)\ .$$

Therefore,

$$
\begin{aligned}
& |w|^4 - 4U^2 \\
(45) \qquad & = e^{4x_1} \cosh^2(2x_4) - e^{4x_1} \sinh^2(2x_4) \\
& = e^{4x_1} \neq 0\ ,
\end{aligned}
$$

and we can assert that e^z is never a nilfactor. Using equation (30) we find, as would be expected, that the derivative of e^z is again e^z.

19. <u>The circular functions</u>. By the use of the addition formula for $\sin(A + B)$, we arrive at the following expression for sin z:

$$w = \sin z = \sin(x_1 + ix_2 + jx_3 + kx_4)$$

$$
\begin{aligned}
&= \sin(x_1 + ix_2)\cos(jx_3 + kx_4) \\
&\qquad + \cos(x_1 + ix_2)\sin(jx_3 + kx_4) \\
&= (\sin x_1 \cosh x_2 \cosh x_3 \cos x_4 \\
&\qquad - \cos x_1 \sinh x_2 \sinh x_3 \sin x_4) \\
&+ i(\sin x_1 \cosh x_2 \sinh x_3 \sin x_4 \\
&\qquad + \cos x_1 \sinh x_2 \cosh x_3 \cos x_4) \\
&+ j(\cos x_1 \cosh x_2 \sinh x_3 \cos x_4 \\
&\qquad + \sin x_1 \sinh x_2 \cosh x_3 \sin x_4) \\
&+ k(\cos x_1 \cosh x_2 \cosh x_3 \sin x_4 \\
&\qquad - \sin x_1 \sinh x_2 \sinh x_3 \cos x_4) \,.
\end{aligned}
\tag{46}
$$

The values of $|w|^2$ and U for $w = \sin z$ reduce to,

$$
\begin{aligned}
|w|^2 &= -\sin^2 x_1 \sin^2 x_4 - \cos^2 x_1 \cos^2 x_4 \\
&\qquad + \sinh^2 x_2 \sinh^2 x_3 + \cosh^2 x_2 \cosh^2 x_3 \\
&= -\frac{1}{2}\begin{vmatrix} \cos(2x_1) & \cosh(2x_2) \\ \cosh(2x_3) & \cos(2x_4) \end{vmatrix} ,
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
U &= \cos x_1 \sin x_1 \cos x_4 \sin x_4 \\
&\qquad - \cosh x_2 \sinh x_2 \cosh x_3 \sinh x_3 \\
&= \frac{1}{4}\begin{vmatrix} \sin(2x_1) & \sinh(2x_2) \\ \sinh(2x_3) & \sin(2x_4) \end{vmatrix} .
\end{aligned}
\tag{48}
$$

Computing the expression

$$|w|^4 - 4U^2$$

and setting it equal to zero, we find the condition that sin z be a nilfactor to be

$$\begin{aligned} &\sin^2 x_1 + \sinh^2 x_2 + \cosh^2 x_3 - \cos^2 x_4 \\ (49) \qquad &= \pm 2(\sin x_1 \cosh x_2 \cosh x_3 \sin x_4 \\ &\quad - \cos x_1 \sinh x_2 \sinh x_3 \cos x_4) \ . \end{aligned}$$

Starting with the second values in (47) and (48), the condition is found to be

$$\begin{aligned} &\cos(4x_1) + \cosh(4x_2) + \cosh(4x_3) + \cos(4x_4) \\ (49') \qquad &= 4\left[\cos(2x_1) \cosh(2x_2) \cosh(2x_3) \cos(2x_4) \right. \\ &\quad \left. - \sin(2x_1) \sinh(2x_2) \sinh(2x_3) \sin(2x_4)\right] \ . \end{aligned}$$

We observe that two sets of values of z that make sin z a nilfactor, a few of the values being zeros of the function, are

$$x_1 = x_4 = \frac{k\pi}{2} \ , \qquad x_2 = x_3 \ ,$$

where k is zero or an integer, and

$$x_1 = x_4 \ , \qquad x_2 = x_3 = 0 \ .$$

The expressions for cos z corresponding to those of (46) through (49) for sin z are similar and may be found in the same way.

Finding the derivatives of sin z and cos z by equation (30), we can show that

$$\frac{d}{dz}\sin z = \cos z \ ,$$

and

$$\frac{d}{dz}\cos z = -\sin z \ .$$

From these equations, the derivatives of the other circular functions may be found.

20. The logarithm. Let $w = \log z$ be defined by the equation

$$e^w = e^{(u_1 + iu_2 + ju_3 + ku_4)}$$

$$= z = x_1 + ix_2 + jx_3 + kx_4 \ .$$

Then, using equation (42), we find for the x's,

$$x_1 = e^{u_1}(\cos u_2 \cos u_3 \cosh u_4 + \sin u_2 \sin u_3 \sinh u_4) \ ,$$

$$x_2 = e^{u_1}(\sin u_2 \cos u_3 \cosh u_4 - \cos u_2 \sin u_3 \sinh u_4) \ ,$$

$$x_3 = e^{u_1}(\cos u_2 \sin u_3 \cosh u_4 - \sin u_2 \cos u_3 \sinh u_4) \ ,$$

$$x_4 = e^{u_1}(\sin u_2 \sin u_3 \cosh u_4 + \cos u_2 \cos u_3 \sinh u_4) \ .$$

From these equations we compute

$$(50) \qquad 2(x_1x_4 - x_2x_3) = 2X = e^{2u_1} \sinh(2u_4) \ ,$$

(51) $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = |z|^2 = e^{2u_1} \cosh(2u_4) ,$$

(52) $$x_1^2 - x_2^2 + x_3^2 - x_4^2 = e^{2u_1} \cos(2u_2) ,$$

(53) $$x_1^2 + x_2^2 - x_3^2 - x_4^2 = e^{2u_1} \cos(2u_3) .$$

Subtracting the square of (50) from that of (51), we have,

$$e^{4u_1} = |z|^4 - 4X^2 ,$$

and

(54) $$u_1 = \frac{1}{4}\log(|z|^4 - 4X^2) .$$

Solving (52) for u_2 and (53) for u_3, we find

$$u_2 = \frac{1}{2} \text{ arc cos}\frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{e^{2u_1}}$$

(55) $$= \frac{1}{2} \text{ arc cos}\left[\frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{(|z|^4 - 4X^2)^{1/2}}\right] ,$$

and

(56) $$u_3 = \frac{1}{2} \text{ arc cos}\left[\frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{(|z|^4 - 4X^2)^{1/2}}\right] .$$

Dividing (50) by (51) and solving for u_4, we discover its value to be

$$u_4 = \frac{1}{2} \text{ arc tanh}\left[\frac{2X}{|z|^2}\right]$$

(57) $$= \frac{1}{4} \log\left[\frac{|z|^2 + 2X}{|z|^2 - 2X}\right] .$$

The complete expression for log z is found by use of equations (54) through (57) to be,

$$
w = \log z
$$

$$
= \frac{1}{4} \log(|z|^4 - 4X^2)
$$

$$
+ i \frac{1}{2} \text{arc} \cos\left[\frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{(|z|^4 - 4X^2)^{1/2}}\right]
$$

(58)

$$
+ j \frac{1}{2} \text{arc} \cos\left[\frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{(|z|^4 - 4X^2)^{1/2}}\right]
$$

$$
+ k \frac{1}{4} \log\left[\frac{|z|^2 + 2X}{|z|^2 - 2X}\right] .
$$

It is obvious from this expression for the logarithm that every nilfactor value of z, as well as the point z = 0, is a singular point of the function. The quantity

(59) $$|z|^4 - 4X^2 ,$$

which is zero when z is zero or a nilfactor occurs either in the denominator or inside a real logarithm in every conjugate function. This is true because both factors of (59) appear in the value of u_4, and one of them is zero whenever (59) is.

21. Comparison with functions of a complex variable. It is easily seen, by a review of the functions considered in this chapter, that zero and nilfactor values of the argument together take the place in our study of the zero in the study of func-

tions of a complex variable. We have seen that corresponding to the non-existence, for a finite argument, of a zero of e^z in complex variable, we have that e^z is never zero or a nilfactor in our function theory. Corresponding to the singularity at zero of log z in complex variable, we find that log z is not defined for a zero or nilfactor argument. If we should write out the expression for tan z as the quotient of sin z by cos z, it would be evident that the function has singularities at nilfactor, as well as zero, values of cos z.

In general, x_2 and x_3, the second and third components of our four-component variable, play the part in this paper that the imaginary term does in complex variable, while x_4 goes with x_1 in taking over the role played by the real term. This is seen in the absolute value of e^z, where only x_1 and x_4 occur. It is seen again in the logarithm, where it is the second and third conjugate functions which give the function its multiple-valued character.

BIBLIOGRAPHY

Bôcher, Maxime, Introduction to Higher Algebra, New York, Macmillan, 1907.

Ketchum, P. W., "Analytic Functions of Hypercomplex Variables", Transactions of the American Mathematical Society, Volume 30, 1928, pages 641-667.

Pierpont, J., Theory of Functions of Real Variables, Volume I, Boston, Ginn, 1905.

Townsend, E. J., Functions of a Complex Variable, New York, Henry Holt, 1915.

Historical references given by Ketchum:

Autonne, Journal de Mathématiques, (6), Volume 3, 1907, page 55.

Berloty, Thesis, Paris, 1886.

Scheffers, Leipziger Berichte, Volume 45, 1893, page 828.

Weierstrass, Göttinger Nachrichten, 1884, page 395.

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