A STUDY OF THE VIBRATING REED AS A DEVICE FOR THE DETERMINATION OF VISCOELASTIC PROPERTIES

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This is to certify that the

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ABSTRACT

A Study of the Vibrating Reed as A Device for the Determination of Viscoelastic Properties

by William Wai Tung Seto

When cantilever specimens of different lengths are subjected to various values of frequency of vibration by an impressed forced vibration at the clamped end, the ratio of the amplitudes of oscillation of the free to the clamped end at the steady state condition can be found. If the phase lag of the free end behind the clamped end is also measured, the viscoelastic complex modulus can be calculated from the mathematical relation derived.

In addition to the internal damping of the material of the cantilever specimens, there is inherently air-damping when the cantilever specimens are oscillating. The effect and significance of this air-damping on the vibrating reed test of a material are also considered.

A brief outline of the simple linear viscoelastic theory is also included.

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Chapter I Introduction

Most engineering materials when subjected to loads do not behave as perfectly elastic solids. The assumptions of homogeneity, isotropy, and time-independent elasticity are in direct contrast to the phenomenon of fatigue, time and temperature-sensitive cohesive strength and creep behavior of real materials. Therefore, the analysis of the mechanical behavior of such materials requires information obtained through various studies concerned with the degree to which real materials differ from the ideal Hookean solid. In particular, for example, many of the physical "constants" related to stress analysis of a given material are sensitive to loading rates and thereby lead to a consideration of the so called "dynamic constants". A method which has been used widely in the determination of such constants is the so-called vibrating reed test. It is with a study of this test that the following investigation concerns itself.

Chapter II Object

The primary purpose of this investigation is to study analytically and experimentally the use of the vibrating reed as a means of determining viscoelastic properties. The experimental portion of the work was carried out using methyl methacrylate (commercially known as plexiglas or perspex) as the test material. Various influences such as the effect of air-damping, excitation frequency, and reed geometry were considered and included as part of the investigation.

A review of the literature dealing with the determination of the dynamic physical constants reveals a number of experimental and instrumentation techniques appropriate to the present work.

Among the many different driving mechanisms used as the driver to impart a sinusoidal oscillation to one end of the reed, a phonograph recording head was used by an early investigator, M. Horia of M. I. T. (1)* For vibrations with low and moderate damping, the "electrostatic method" for driving is preferrable, and has been employed in the determination of "inelastic losses" in some high polymers as a function of frequency and temperature. It makes use of a metal foil fixed on the free end of the reed. The metal foil is attracted by an alternating potential in the air-gap of a strong magnet.

Some investigators attain the required sinusoidal oscillation of the reed by subjecting it to an impressed force, supplied by an audio oscillator through

*Numbers in brackets refer to bibliography at the end of this report.

a mechanical linkage between a permanent magnet speaker and the centerline of the reed. Sustained oscillations of the reed can also be achieved by suitable adaptation of reciprocating apparatus designed for other purposes; such was the case taken by the present work.

As far as the measurement of the amplitude of vibration of the free end of the reed is concerned, some investigators used a stroboscope, synchronized with the frequency of oscillation of the reed, for easier viewing of the maximum points. A. W. Nolle (2) used a small telescope with crosshairs on it for this purpose; while L. E. Nielsen (3) employed a differential transformer to convert amplitude of mechanical oscillations to electrical potentials.

Another possible means of measuring the amplitude of vibration is by measuring the capacitance across two plates with a fixed piece of dielectric material attached to the reed and passing between the two fixed plates with the vibration of the reed thus causing a change of capacitance.

In the "electrostatic method" mentioned

above, the vibration of the end of the reed is observed by an optical method. The end of the reed interrupts a light beam. When the reed vibrates, this light beam is modulated, and in falling on a photo-cell, creates an A. C. signal proportional to the amplitude of vibration. This signal is then amplified by conventional means and fed into suitable recording apparatus.

Although it has only indirect connection with this work, it is interesting to note that based on viscoelastic behavior, a method has been developed for mathematical treatment of a sinusoidal rate of loading of fibers (4). Here the vertical displacements of a weight attached to the end of single filament of fiber is recorded by a movie camera, with a ground-glass as the screen and a synchronous motor as the clock. Initial displacement is given by means of an electromagnet, and temperature as well as humidity are controlled throughout the whole experiment.

On some occasions, the evaluation of the dynamic constants of a material can also be made from the measurements of the velocity of transmission and the attenuation of sound in the material. However, this has been chiefly applied to " low loss " materials, such as metals and some plastics. It consists of a signal generator, crystal driver, pickup, amplifier, and scope (5). This experiment is elaborate, extensive, and expensive. In addition, the material used for this investigation does not come under this " low loss " catalogue, and hence it was not used.

Chapter IV Experimental Work

The vibrating reed apparatus consists of two main components: namely a driving clamp to impart a sinusoidal displacement to one end of the reed, and a recording apparatus to measure the displacement of the free end.

In this work, the "calibrating beam", designed and built by the Michigan State Highway Department, was used as the driver to impart sinusoidal oscillations of various frequencies to the clamped end of the reed.

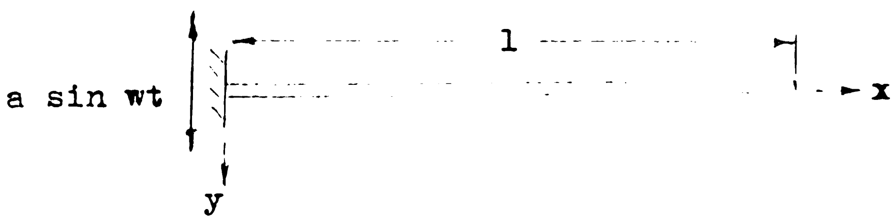
With regard to the measurement of the amplitude of vibration of the free end of the reed, much effort was spent in careful consideration and preliminary attempts to build and use apparatus such as the rotating drum, the photo-cell, capacitors, movie camera, and telescope with cross-hairs, all of which have been listed as methods of suitable merit in the previous section of this report. It was found, however, that with some care and operator experience, a "visual observation" method provided data of nearly equivalent to the indicated accuracy with a great saving of effort. In this, a straight edge mounted on a tripod-stand was brought close to the end of the vibrating reed for determining the value of the corresponding displacement. This is possible because the amplitude of vibration of the free end will reach a constant steady state magnitude after the transient motion of vibration has died out.

The ratio of the amplitudes of oscillation of the free to the clamped end at three different frequencies, namely 270 c.p.m., 435 c.p.m., and 645 c.p.m., were measured for different lengths of reed, the material of which were assumed to be statistically homogeneous and to have constant lateral dimensions.

Frequency of Vibration (c.p.m.)	Lengths of reed					
Width of reed (1 inch)	18	17	16	15	14	13
270	5.6	9.7	4.8	2.9	1.8	1.3
425	5.9	6.1	4.9	3.2	2.9	1.5
645	5.4	4.8	4.3	4.0	3.1	2.3
Width of reed ($\frac{3}{4}$ inch)						
	18	17	16	15	14	13
270	4.6	9.6	5.4	3.5	2.3	1.7
425	3.7	4.8	3.2	2.2	1.8	1.4
645	2.2	3.1	2.7	1.8	1.4	1.1

Chapter V Mathematical Analysis

The problem involved here is to obtain the response of a viscoelastic cantilever beam undergoing sinusoidal oscillations at the clamped end. Because of the time-dependent boundary conditions, it is convenient to employ the Mindlin-Goodman procedure (6) to solve the associated elastic beam problem and then by means of the elastic-viscoelastic correspondence principle to convert the results to obtain the solution for the viscoelastic beam. Thus for the beam shown in the following diagram,



Boundary conditions

$$y(0,t) = a \sin wt \quad \ldots\ldots\ldots \quad (1)$$

$$\left(\frac{\partial y}{\partial x}\right)_{x=0} = 0 \quad \ldots\ldots\ldots \quad (2)$$

$$\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=1} = 0 \quad \ldots\ldots\ldots \quad (3)$$

$$\left(\frac{\partial^3 y}{\partial x^3}\right)_{x=1} = 0 \quad \ldots\ldots\ldots \quad (4)$$

Initial conditions

$$y(0) = 0 \quad \ldots\ldots\ldots \quad (a)$$

$$\dot{y}(0) = 0 \quad \ldots\ldots\ldots \quad (b)$$

The differential equation for lateral vibration of beam is

$$\frac{\partial^2 y}{\partial t^2} + n^2 \frac{\partial^4 y}{\partial x^4} = 0 \qquad \text{and } n^2 = \frac{E\,I\,g}{A\,\gamma}$$

where E = Young's modulus of elasticity,

I = moment of inertia of cross-section,

A = cross-sectional area,

γ = weight of material per unit volume

Assume the solution

$$y(x,t) = e(x,t) + f(t)\,g(x) = e(x,t) + a \sin wt\, g(x)$$

Therefore, the beam equation becomes

$$n^2\left(\frac{\partial^4 e}{\partial x^4}\right) + \frac{\partial^2 e}{\partial t^2} = -\,n^2 a \sin wt \frac{\partial^4 g}{\partial x^4} + g a w^2 \sin wt$$

The boundary conditions become

$$e(0,t) = a \sin wt\,(1 - g(0)) \quad \dots\dots\dots \quad (1a)$$

$$\frac{\partial e(0,t)}{\partial x} = -\,a \sin wt\; g(0) \quad \dots\dots\dots\dots\dots \quad (2a)$$

$$\frac{\partial^2 e(1,t)}{\partial x^2} = -\,a \sin wt\; g\,(1) \quad \dots\dots\dots\dots\dots \quad (3a)$$

$$\frac{\partial^3 e(1,t)}{\partial x^3} = -\,a \sin wt\; g\,(1) \quad \dots\dots\dots\dots \quad (4a)$$

If g is

$$g = E \sin Bx + F \cos Bx + G \sinh Bx + H \cosh Bx$$

$$\text{where } B = \frac{w}{n}$$

It can be shown that the boundary conditions on e become zero if g is such that

$$H = 1 - F$$

$$E = -G$$

$$-E \sin Bl - F \cos Bl + G \sinh Bl + H \cosh Bl = 0$$

$$-E \cos Bl + F \sin Bl + G \cosh Bl + H \sinh Bl = 0$$

after solving the above equations,

$$g(l) = G(\sinh Bl - \sin Bl) + H(\cosh Bl - \cos Bl) + \cos Bl$$

When the transient motion has died out, i.e. $e(x,t) \to 0$, the steady state response of the beam will be

$$y(x,t) = a \sin wt \, g(x)$$

Therefore,

$$\frac{y(l,t)}{y(0,t)} = g(l) = \frac{\cos Bl + \cosh Bl}{1 + \cos Bl \cosh Bl}$$

which is the ratio of the amplitudes of vibration of the free to the clamped end, and

$$B = \left(\frac{m w^2}{E I} \right)^{\frac{1}{4}}$$

w = frequency of oscillation,

m = mass per unit length,

E = Young's modulus of elasticity.

If R denotes the ratio of the amplitudes of oscillation of the free to the clamped end, and θ the phase lag of the free end behind the clamped end, then

$$R e^{-i\theta} = \frac{\cos Bl + \cosh Bl}{1 + \cos Bl \cosh Bl}$$

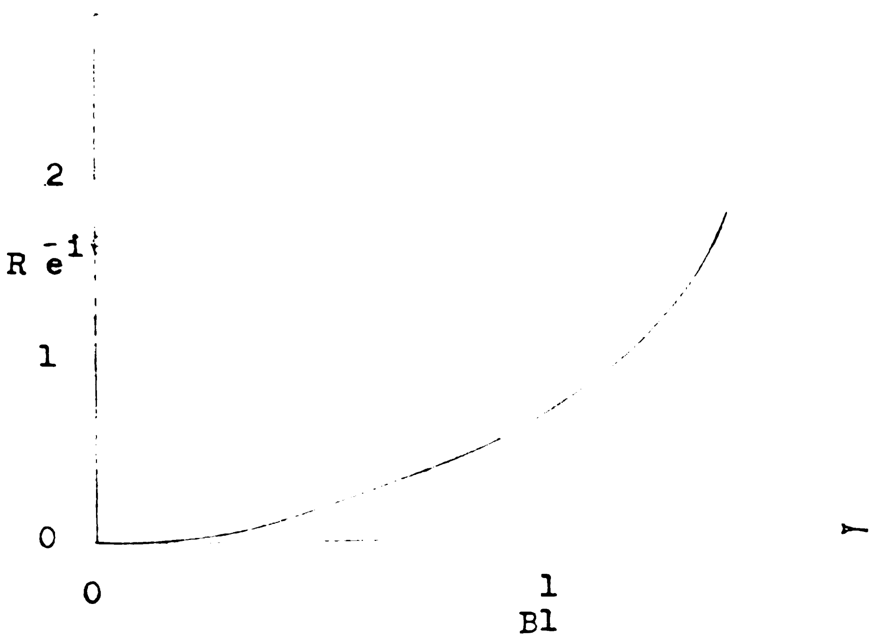
The solution for a viscoelastic material is given by replacing the elastic modulus E by the corresponding viscoelastic complex modulus. This is best obtained by the graphical method as shown here.

Sample calculation:

when $Bl = 1.00$, $\cos Bl = 0.5403$

$\cosh Bl = 1.5431$

$$R\,e^{-i\theta} = \frac{0.5403 + 1.5431}{1 + 0.5403 \times 1.5431} = \frac{2.0834}{1.8337} = 1.1361$$



Chapter VI Air - Damping

For a cantilever loaded by its own weight, the deflection of the free end is

$$d = \frac{W\ l^4}{8\ EI}$$

where W = weight of beam per unit length,

E = Young's modulus of elasticity,

I = moment of inertia of cross-section,

l = length of beam.

It can be shown that the fundamental natural frequency of free vibration of the cantilever beam loaded by its own weight is

$$w = \sqrt{\frac{8\ E\ I\ g}{W\ l^4}}$$

Let w_1 , w_2 be the natural frequencies of two cantilever beams made of the same material, having the same thickness, but different both in width and length.

But for rectangular cross-sections,

$$I_1 = \frac{b_1\ h^3}{12} \qquad I_2 = \frac{b_2\ h^3}{12}$$

For the same material,

$$E_1 = E_2 = E$$

$$W_1 = A_1\gamma = b_1h\gamma$$

$$W_2 = A_2\gamma = b_2h\gamma$$

Therefore,

$$w_1 = \sqrt{\frac{8\ E\ h^3b_1g}{\gamma\ hb_1l_1^4}} = \sqrt{\frac{8\ E\ h^2g}{\gamma\ l_1^4}}$$

$$w_2 = \sqrt{\frac{8\ E\ h^3b_2g}{\gamma\ hb_2l_2^4}} = \sqrt{\frac{8\ E\ h^2g}{\gamma\ l_2^4}}$$

Now if $l_1 = l_2$, then

$w_1 = w_2$, i. e. the two cantilever beams will vibrate freely at the same natural frequency. In other words, cantilever beams of the same length, thickness, and made of the same material will vibrate freely at the same frequency even though their widths are different. This is true when there is no damping force other than their own internal damping. However, if the amount of air-resistance is negligible compared to the amount of internal damping present in the system, the system, as a whole, will not be affected much.

The following table gives the values of natural frequency of free vibration of cantilever beams of various lengths and widths; they are, however, made of the material, methyl methacrylate, and are of the same thickness. These experimental results were obtained through the use of SR - 4 strain gages, Universal Strain Analyzer, and pen - and - ink oscillograph.

Length of Cantilever	Width of Cantilever 1	Width of Cantilever $\frac{3}{4}$
17 in.	5.11 c.p.s.	4.67 c.p.s.
16 in.	5.71 c.p.s.	5.42 c.p.s.
15 in.	6.31 c.p.s.	5.93 c.p.s.
14 in.	7.34 c.p.s.	6.67 c.p.s.
13 in.	8.34 c.p.s.	7.75 c.p.s.

These results indicate that air-damping cannot be ignored in this vibrating reed test of this particular material.

Another way for the determination of the influence of air-damping on the vibrating reed is the "logarithmic decrement" which is the logarithm of the two consecutive amplitudes of displacement in a decaying curve of vibration.

As has pointed out in the previous part of this section, $L_1 = L_2$ is a condition for the reeds to vibrate freely at the same frequency. In other words, when $L_1 = L_2$ the values of the logarithmic decrement determined from two decaying curve of free vibration should be the same.

Let D be the logarithmic decrement,

$$\text{therefore } D = \ln(A_1/A_2) = \ln(A_n/A_{n+1})$$

$$\text{or } D = \frac{\ln(A_1/A_{n+1})}{n}$$

In this manner, the following table was obtained.

Length of reed	Width of reed 3/4 inches	Width of reed 1 inch.
17 in.	D = 0.28	D = 0.31
16 in.	D = 0.30	D = 0.29
15 in.	D = 0.31	D = 0.28
14 in.	D = 0.30	D = 0.29
13 in.	D = 0.31	D = 0.32

Sample calculation:

When the reed length is 13 inches,

$A_1/A_5 = 12.4/3.7 = 3.35$ ..... (1)

$A_1/A_5 = 10.4/3.2 = 3.17$ ..... (2)

$A_1/A_5 = 14.6/4.5 = 3.24$ ..... (3)

$A_1/A_5 = 10.4/3.1 = 3.26$ ..... (4)

The average $A_1/A_5 = 3.25$

$4\ D = \ln 3.25,$

$D = 0.32$

The results show that reeds of different widths have different values of logarithmic decrement, i.e. the reeds are vibrating at different frequencies. This agrees with those obtained in the last section.

When air-damping is not negligible, the system becomes one with elastic constant k, internal damping C_I, and air-damping C_A. By solving the differential equation,

$$m\ddot{x} + (C_I + C_A)\dot{x} + kx = 0$$

it can be shown that the frequency of the system is

$$f = \sqrt{\frac{k}{m} - \left(\frac{C_I + C_A}{2m}\right)^2}$$

$$D = \left(\frac{C_I + C_A}{2m}\right) f$$

Therefore,

$$\frac{D_m}{D_n} = \frac{\left(\frac{C_{Im} + C_{Am}}{2m_m}\right) f_m}{\left(\frac{C_{In} + C_{An}}{2m_n}\right) f_n}$$

If there is no air-damping, $C_A = 0$, then

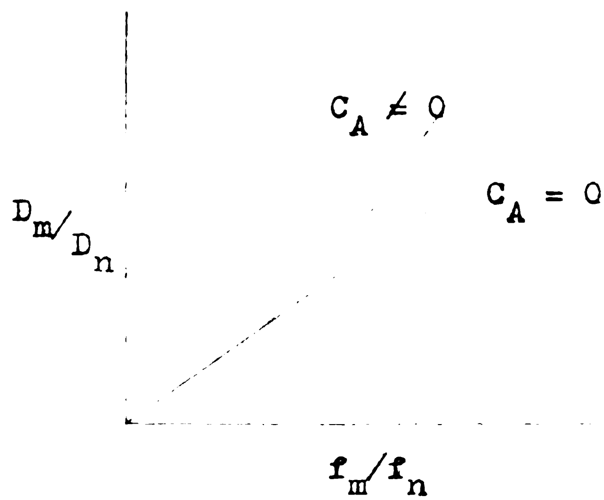
$$\frac{D_m}{D_n} = \frac{(C_{Im}/2m_m) f_m}{(C_{In}/2m_n) f_n}$$

For a constant length reed, this ratio should be a straight line for various widths.

When air-damping is not negligible, $C_A \neq 0$

$$\frac{D_m}{D_n} = \frac{\left[C_{Im} + C_A\, y(W^x)\right] / 2m_m}{\left[C_{In} + C_A\, y(W^x)\right] / 2m_n} \left(\frac{f_m}{f_n}\right)$$

The graph of $\left(\frac{D_m}{D_n}\right)$ versus $\left(\frac{f_m}{f_n}\right)$ is no longer a straight line. Therefore, the amount of air-damping present can be interpreted from this deviation.



If the vibrating reed test were carried out in a vacuum space, the effect of air-damping can be completely ignored. Or if the exact amount of internal damping of the test material is known from other source, the amount of air-damping on the reed can also be determined. An experiment, which includes little of the effect of air-damping, is to displace air by helium in a confined space where the vibrating reed test is being carried out.

Chapter VII Discussions

In a previous section of this report, it has been shown that the viscoelastic complex modulus of methyl methacrylate can be determined when the ratio of the amplitudes of oscillation of the free to the clamped end at steady state and the phase lag of the free end behind the clamped end are both known from measurements made by the vibrating reed test. In the experimental portion of this work, the ratios of the amplitudes of oscillation of the free to the clamped end at steady state for various lengths of cantilevers at three different values of frequencies were measured. With the values of the phase lag of the free end behind the clamped end, the viscoelastic complex modulus can thus be determined directly from the graphical solution presented.

From the experimental results of the free vibration of the cantilever beams, it is observed that cantilever beams of the same length, thickness, and made of the same material but different width exhibit different values of natural frequency of vibration which is in direct contrast to the results given by the mathematical analysis neglecting air-damping. This implies that air-damping does play a significant part in the overall damping of this particular system, and would therefore be of importance in practical applications.

Assuming that air-damping is proportional to the area of beam exposed, the cantilever of width 1 inch would encounter more air-damping than that by the cantilever of $\frac{3}{4}$ inch width; and consequently would vibrate at a slower pace. This is in agreement with the experimental results obtained.

Chapter VIII Appendix

Linear Viscoelastic Theory

Basically, this theory assumes that both elastic and viscous elements are involved in the resistance of a real body to deformation under load. Thus the response of a given material to various load conditions has associated with it the response of a mechanical model comprised of a suitable arrangement of elastic springs and Newtonian dashpots. A number of recent texts contain excellent accounts of the theory as it has developed to date (8,9). For the purpose of this study it is sufficient to point out that the simplest models available are those of a spring and dashpot in series (called Maxwell), and a spring and dashpot in parallel (called Kelvin). A diagrammatic sketch of each of these models is shown in Figure 2.

The stress-strain relations corresponding to these are

$$\dot{\epsilon} = \dot{\sigma}/E + \sigma/n \qquad \text{(Maxwell model)}$$

$$\sigma = \epsilon E + n\dot{\epsilon} \qquad \text{(Kelvin model)}$$

where σ = unit stress,

ϵ = unit strain,

E = Young's modulus of elasticity,

n = coefficient of viscosity,

$\dot{\tau}$ = stress rate, or rate of change of stress,

$\dot{\epsilon}$ = strain rate, or rate of change of strain,

When either of these models is subjected to a sinusoidally oscillating force of radian frequency ω, it is convenient to define a "complex modulus" having an in-phase component and a component 90 degree out-of-phase. The in-phase component corresponds to the elastic characteristic of the material, and the out-of-phase component is related to the dissipation characteristic of the material.

The presence of a free dashpot in the Maxwell model indicates that it would best represent a material having "true flow" characteristic which produce residual deformations upon loading. Since plexiglas was observed not to be of this nature, the Kelvin model representation was chosen. For the Kelvin model the complex modulus is given as

$$Y(i\omega) = E + in\omega$$

It may be observed here that this complex modulus has a damping component proportional to frequency ω and n.

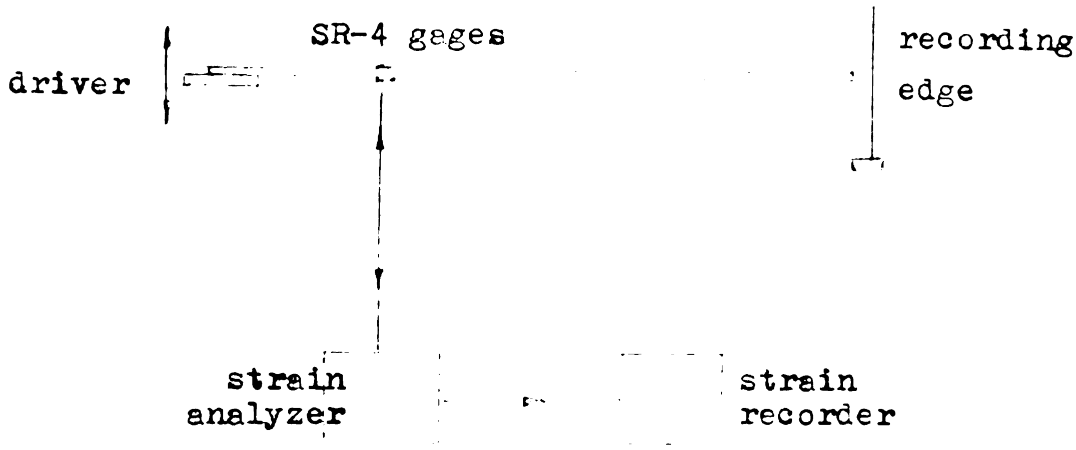


Figure 1 a. SCHEMATIC DIAGRAM OF EXPERIMENTAL WORK

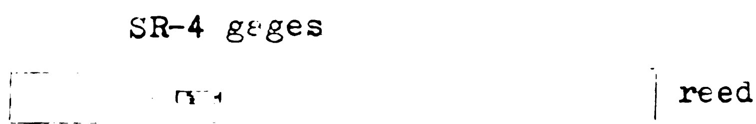


Figure 1 b. POSITION OF SR-4 STRAIN GAGES ON REED

Figure 2. SPRING DASHPOT MODELS OF LINEAR VISCOELASTIC MATERIALS:

(a) MAXWELL MODEL,

(b) KELVIN MODEL.

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