RESIDUES OF A POLYGENIC FUNCTION

Thesis for the Degree of M. A.

John A. Straw

1937

Functions, Polygenic
Serie

Mathematics

ACKNOWLEDGMENT

I wish to thank Dr. J. E. Powell for his suggestions and criticisms in the preparation of this thesis.

RESIDUES OF A POLYGENIC FUNCTION

A Thesis

Submitted to the Faculty

of

MICHIGAN STATE COLLEGE

of

AGRICULTURE AND APPLIED SCIENCE

In Partial Fulfillment of the

Requirements for the Degree

of

Master of Arts

by

John Arthur Straw

1937

CONTENTS

RESIDUES OF A POLYGENIC FUNCTION

1. INTRODUCTION

In the study of analytic functions of a complex variable the theory of residues plays a very important role. In the study of polygenic functions a corresponding theory should prove valuable. Professor V. C. Poor gave a definition for the residue of a polygenic function but did not carry his theory very far.*

It is the purpose of this paper to formulate a definition for the residue of a polygenic function. The definition used gives, except for a certain type of pole, all the results which Poor obtained. It enables one to consider a more general class of polygenic functions than Poor considered and to arrive at a few more conclusions than he stated. The definition holds for a fairly general class of polygenic functions. Many of the theorems (or corresponding theorems) on residues of analytic functions

---

* V. C. Poor, <u>Residues of Polygenic Functions</u>, Transactions of the American Mathematical Society, Vol. 32 (1930), pp. 216-222.

can be proved, in a modified form, for polygenic functions; for one particular type of polygenic function all the theorems on residues of analytic functions will hold. The definition has the fault, however, of not carrying over theorems in which an integral about a curve enclosing more than one pole is concerned.

First there is given a brief discussion of polygenic functions, followed by a discussion of the types of poles and zeros of such functions. This is followed by a summary of Poor's definition and results. We then state our definition and the results obtained by it. This definition is then applied to three particular types of polygenic functions. Other possibilities are mentioned and some suggestions offered.

2. POLYGENIC FUNCTIONS

A function of a complex variable is said to be "regular" or "analytic" at a point if it is defined, single-valued, and possesses a unique derivative at the point and in the neighborhood of the point. If a function which is analytic in any region or in the neighborhood of any point (or even on the arc of a curve) is extended analytically as far as possible, then the function and all its function elements obtained by analytic continuation is called a "monogenic" function. A "polygenic" function of a complex variable is one that is not monogenic; that is, a polygenic function is one which is not analytic in any region or in the neighborhood of any point, although it might have a derivative at several points.

Let w be a function of the complex variable $z = x + iy$. In this paper we consider

$$w = f(z) = u(x,y) + iv(x,y),$$

where u and v are continuous and have continuous first partial derivatives. If m represents the slope of the curve along which Δz approaches zero, the derivative of w with respect to z can be written*

$$\frac{dw}{dz} = \lim_{\Delta z = 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \frac{u_x \Delta x + u_y \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i(v_x \Delta x + v_y \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \frac{u_x + u_y \frac{\Delta y}{\Delta x} + \epsilon_1 + \epsilon_2 \frac{\Delta y}{\Delta x} + i\left(v_x + v_y \frac{\Delta y}{\Delta x} + \epsilon_3 + \epsilon_4 \frac{\Delta y}{\Delta x}\right)}{1 + i\frac{\Delta y}{\Delta x}}$$

$$= \frac{u_x + m u_y + i(v_x + m v_y)}{1 + im} = \frac{u_x + i v_x + m(u_y + i v_y)}{1 + im},$$

since

$$m = \frac{dy}{dx} = \lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \frac{\Delta y}{\Delta x} \quad \textit{and} \quad \lim_{\substack{\Delta x = 0 \\ \Delta y = 0}} \epsilon_i = 0 \qquad (i = 1, 2, 3, 4).$$

For this derivative to be unique it must be independent of m. It is easily shown that the necessary and sufficient condition for this derivative to be independent of m is that the functions u and v satisfy the Cauchy-Riemann equations.

---

* E. R. Hedrick, Non-Analytic Functions of a Complex Variable, Bulletin of the American Mathematical Society, XXXIX, No. 2, Feb. (1933), pp. 75-96.

The above expression for the derivative can be written in a more descriptive and possibly more useful form. Let θ be the angle between the x axis and the curve of approach. Then $m = \tan\theta$ and

$$\frac{dw}{dz} = \frac{u_x + iv_x + (u_y + iv_y)\tan\theta}{1 + i\tan\theta} = \frac{(u_x+iv_x)\cos\theta + (u_y+iv_y)\sin\theta}{\cos\theta + i\sin\theta}$$

$$= e^{-i\theta}\left[(u_x+iv_x)\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) + (u_y+iv_y)\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\right].$$

Therefore we have

$$(1) \qquad \frac{dw}{dz} = D[f(z)] + P[f(z)]\,e^{-2i\theta},$$

where

$$(2) \qquad \begin{aligned} D[f(z)] &= \tfrac{1}{2}\left[u_x + v_y + i(v_x - u_y)\right] \\ P[f(z)] &= \tfrac{1}{2}\left[u_x - v_y + i(v_x + u_y)\right] \end{aligned}.$$

It will be noticed that if the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

are satisfied, then $\frac{dw}{dz}$ is unique, and

$$\frac{dw}{dz} = D[f(z)].$$

If we let

$$z = x + iy, \qquad \bar{z} = x - iy$$

then

$$x = \frac{z+\bar{z}}{2}, \qquad y = \frac{z-\bar{z}}{2i},$$

and we have

$$w = f(z) = u(x,y) + i\,v(x,y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + i\,v\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = F(z,\bar{z}).$$

Now

$$\frac{\partial F(z,\bar{z})}{\partial z} = \frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$= (u_x + i\,v_x)\frac{\partial x}{\partial z} + (u_y + i\,v_y)\frac{\partial y}{\partial z}$$

$$= (u_x + i\,v_x)\left(\frac{1}{2}\right) + (u_y + i\,v_y)\left(\frac{1}{2i}\right)$$

$$= \frac{1}{2}\left[u_x + v_y + i(v_x - u_y)\right] = D\left[f(z)\right],$$

and

$$\frac{\partial F(z,\bar{z})}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$= (u_x + i\,v_x)\frac{\partial x}{\partial \bar{z}} + (u_y + i\,v_y)\frac{\partial y}{\partial \bar{z}}$$

$$= (u_x + iv_x)\left(\frac{1}{2}\right) + (u_y + iv_y)\left(\frac{-1}{2i}\right)$$

$$= \frac{1}{2}\left[u_x - v_y + i(u_y + v_x)\right] = P[f(z)] .$$

Therefore

(3) $$\frac{dw}{dz} = D[f(z)] + P[f(z)]e^{-2i\theta} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial \bar{z}} e^{-2i\theta}$$

$$= \frac{\partial F(z,\bar{z})}{\partial z} + \frac{\partial F(z,\bar{z})}{\partial \bar{z}} e^{-2i\theta} .$$

If the polygenic function

$$w = f(z) = u(x,y) + iv(x,y)$$

is such that u and v are continuous and have continuous first partial derivatives, then the function

$$w = F(z, \bar{z}),$$

obtained as above, will be continuous and have continuous first partial derivatives in some regions A and $\bar{A}$.*

---

* By the regions A and $\bar{A}$ we shall mean "conjugate regions"; that is, regions in the z plane and in the $\bar{z}$ plane such that if z is in A then $\bar{z}$ is in $\bar{A}$ and conversely.

Such a function of two complex variables is called an analytic function of the two variables.* We shall hereafter speak of such functions as being analytic in z and $\bar{z}$ in the regions A and $\bar{A}$ and shall consider only this type of polygenic function. We will also use the terminology "a function is analytic in z and $\bar{z}$ in the neighborhoods of a and $\bar{a}$", meaning that the regions A and $\bar{A}$ contain a and $\bar{a}$ respectively.

3. POLES AND ZEROS OF POLYGENIC FUNCTIONS

In analytic function theory we consider residues at poles of our function. In connection with residues of the logarithmic derivative we are interested in both poles and zeros.

To aid in the definition and discussion of poles and zeros of a polygenic function, we list six types of factors which are zero when $z = a$ and $\bar{z} = \bar{a}$. They are

---

* E. Goursat and E. R. Hedrick, Functions of a Complex Variable, Vol. 2, part 1, Ginn and Company, (1916) page 219.

(a) $(z-a)^{l}$,

(b) $(\bar{z}-\bar{a})^{k}$,

(c) $(z-a)^{l}(\bar{z}-\bar{a})^{k}$,

(d) $\left[\alpha(z-a)+\beta(\bar{z}-\bar{a})\right]^{p}$,

(e) $\left[(z-a)^{l}f_1(z,\bar{z})+(\bar{z}-\bar{a})^{k}f_2(z,\bar{z})\right]^{p}$,

(f) $\sum_{i=1}^{n}(z-a)^{l_i}(\bar{z}-\bar{a})^{k_i}f_i(z,\bar{z})$.

In all the above types the exponents are considered to be positive integers; in type (e) we shall assume that $f_1(a,\bar{a})\neq 0$ and $f_2(a,\bar{a})\neq 0$; in type (f) we assume that $f_i(a,\bar{a})\neq 0$ for all i's. We note that these six types are, for the most part, in the order of increasing generality. Types (a) and (b) are readily seen to be special cases of the other types. Type (c) is seen to be a special case of (f) while (d) is a special case of (e). Type (e), in turn, is a special case of type (f), since when we expand type (e) we get just a sum of terms of the type given in (f). Type (f) then is the most general type of factor of the polygenic function $F(z,\bar{z})$ which is zero when $z=a$ and $\bar{z}=\bar{a}$ since it includes all the others and also any combination of sums and products of the others. The most general factor which could occur in a numerator and which is zero when $z=a$ and $\bar{z}=\bar{a}$ is of type (c) if we agree that when we have a factor of type (d), (e), or (f) we will expand and separate into factors of type (c).

We now define a pole and zero for the function

$$(4) \qquad F(z,\bar z) = \frac{(z-a)^{\ell}(\bar z-\bar a)^{K} f(z,\bar z)}{\sum_{i=1}^{n}(z-a)^{\ell_i}(\bar z-\bar a)^{K_i} f_i(z,\bar z)},$$

where all the exponents are positive integers, $n>1$, $f(a,\bar a)\neq 0$, and $f_i(a,\bar a)\neq 0$ for all i's. This function has the most general types of factors, in both numerator and denominator, which are zero when $z=a$ and $\bar z=\bar a$. Let m be the least of all the numbers $\ell_i + K_i$. Then if

$$m > \ell + K,$$

we will say that $F(z,\bar z)$ has a pole of order $m-(\ell+K)$ at $z=a$ and $\bar z=\bar a$. If

$$m < \ell + K,$$

we will say that $F(z,\bar z)$ has a zero of order $(\ell+K)-m$ at $z=a$ and $\bar z=\bar a$.

A particular case of (4) is

$$F(z,\bar z) = \frac{(z-a)^{\ell}(\bar z-\bar a)^{K} f(z,\bar z)}{\left[(z-a)^{\ell_1} f_1(z,\bar z)+(\bar z-\bar a)^{\ell_2} f_2(z,\bar z)\right]^{p}}.$$

We shall consider this function as having more than one term in the denominator, the terms being obtained by expanding the denominator. Let m be the smaller of $\ell_1 p$ and $\ell_2 p$. Then for this function the definition above gives,

if $m > l+k$, a pole of order $m-(l+k)$ at $z = a$ and $\bar{z} = \bar{a}$;

and if $m < l+k$, a zero of order $(l+k)-m$ at $z = a$ and $\bar{z} = \bar{a}$.

For the still more special function,

$$F(z,\bar{z}) = \frac{(z-a)^{l}(\bar{z}-\bar{a})^{k} f(z,\bar{z})}{\left[\alpha(z-a)+\beta(\bar{z}-\bar{a})\right]^{p}},$$

we have,

if $p > l+k$, a pole of order $p-(l+k)$ at $z = a$ and $\bar{z} = \bar{a}$;

and if $p < l+k$, a zero of order $(l+k)-p$ at $z = a$ and $\bar{z} = \bar{a}$.

In section 5 we define the residue (at $z = a$) of the function

$$(5) \qquad F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{l}(\bar{z}-\bar{a})^{k}},$$

where $f(a,\bar{a}) \neq 0$ and where k and l are any integers, positive, negative, or zero. From our definition above we see that,

if $k > 0$ and $l > 0$, there is a pole of order $k+l$ at $z = a$ and $\bar{z} = \bar{a}$;

if $k > 0$ and $l < 0$, there is either a pole of order $k-l$ or a zero of order $l-k$ at $z = a$ and $\bar{z} = \bar{a}$. We note from the discussion above that the function (5) includes all types of zeros, since k and l can be negative, but includes only poles of type (c).

Type (d), which is a special case of type (e), is of special interest and presents a new type of difficulty. Such a pole arises from the factor $(\alpha z+\beta\bar{z}+\gamma)^p$. This factor is zero at $z=a$ and $\bar{z}=\bar{a}$ when and only when $\gamma=-\alpha a-\beta\bar{a}$, so that the factor reduces to $[\alpha(z-a)+\beta(\bar{z}-\bar{a})]^p$ If $|\alpha|\neq|\beta|$, there is an isolated pole at $z=a$. However, if $|\alpha|=|\beta|$ so that $\beta=\alpha e^{i\phi}$, then $\alpha(z-a)+\beta(\bar{z}-\bar{a})$ vanishes at every point on the line through the point $z=a$ whose slope is $\tan\frac{\phi+\pi}{2}$. To show this we set $z-a=re^{i\theta}$; then

$$\bar{z}-\bar{a}=x-iy-(a_1-ia_2)=x-a_1-i(y-a_2)$$

$$=\overline{x-a_1+i(y-a_2)}=\overline{x+iy-(a_1+ia_2)}=\overline{z-a}=re^{-i\theta},$$

where $\overline{z-a}$ denotes the conjugate of $z-a$.

Hence

$$\alpha(z-a)+\beta(\bar{z}-\bar{a})=\alpha re^{i\theta}+\alpha e^{i\phi}re^{-i\theta}$$

$$=\alpha r(e^{i\theta}+e^{i\phi}e^{-i\theta})=\frac{\alpha r}{e^{i\theta}}(e^{2i\theta}+e^{i\phi}).$$

This expression vanishes whenever

$$e^{2i\theta}=-e^{i\phi}=e^{i\pi}e^{i\phi}=e^{i(\pi+\phi)},$$

or when

$$2i\theta=i(\pi+\phi)\ ;\quad \theta=\frac{\pi+\phi}{2}.$$

Therefore, when $|\alpha| = |\beta|$, there is not just a pole at $z = a$ and $\bar{z} = \bar{a}$ but a line of poles, the line passing through $z = a$ of course.

4. POOR'S DEFINITION AND RESULTS*

It is natural, and in most cases probably desirable, to model definitions for polygenic functions after those for analytic functions. Poor does this to some extent in defining the residue of a polygenic function.

The residue at the point $z = a$ of the analytic function $f(z)$ is usually defined by the expression

$$R_a = \frac{1}{2\pi i} \int_C f(z)\, dz ,$$

where C is a closed curve lying in a region in which $f(z)$ is analytic except at a; C encloses the point $z = a$ but encloses no singular point of $f(z)$ other than $z = a$. This definition is not a satisfactory one for polygenic functions because the integral is dependent on the curve C.

---

* V. C. Poor, Residues of Polygenic Functions, Transactions of the American Mathematical Society, Vol. 32, (1930), pp. 216-222.

Let C be taken as a circle of radius r about $z=a$ (or an arbitrary curve within such a circle) such that the only singularity of the polygenic function $f(z)$ enclosed by C is the point $z=a$. Poor defines

$$\operatorname*{Res}_{z=a} f(z) = \lim_{r=0} \frac{1}{2\pi i}\int_C f(z)\,dz$$

Using this definition Poór states and proves the following two theorems:

__Theorem 4.1__: If $f(z)$ is a polygenic function in a region A and has no singularity at $z=a$ or in the neighborhood of $z=a$, then the residue of $f(z)$ at $z=a$ is zero.

__Theorem 4.2__: If the polygenic function $F(z)$ has no singularity at $z=a$ or in the neighborhood of $z=a$, the residue of the function $\frac{F(z)}{z-a}$ at $z=a$ is equal to $F(a)$.

For the function $\frac{1}{az+b\bar{z}}$ he shows that if

$$|b|>|a|, \qquad \operatorname*{Res}_{z=0} f(z) = \frac{1}{b}\,;$$

$$|a|>|b|, \qquad \operatorname*{Res}_{z=0} f(z) = \frac{1}{a}\,;$$

$$|b|=|a|, \qquad \operatorname*{Res}_{z=0} f(z) = \frac{a+b}{2ab}.$$

The results in the first two cases seem somewhat unsatisfactory since they involve only b or only a. Referring to page 12, we see that in the first two cases there is an isolated pole at $z=\bar{z}=0$, whereas in the third case there is a line of poles.

Poor also shows that if $f(t)$ is a polygenic function, then

$$f(z) = \lim_{r=0} \frac{1}{2\pi i} \int_C \frac{f(t)\,dt}{t-z}$$

and

$$\frac{\partial f(z)}{\partial z} = \lim_{r=0} \frac{1}{2\pi i} \int_C \frac{\partial}{\partial z}\left(\frac{f(t)}{t-z}\right) dt = \lim_{r=0} \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-z)^2}\, dt.$$

Similarly

$$f(\bar{z}) = \lim_{r=0} \frac{1}{2\pi i} \int_{\bar{C}} \frac{-f(\bar{t})}{\bar{t}-\bar{z}}\, d\bar{t}$$

and

$$\frac{\partial f(\bar{z})}{\partial \bar{z}} = \lim_{r=0} \frac{1}{2\pi i} \int_{\bar{C}} \frac{\partial}{\partial \bar{z}}\left(\frac{-f(\bar{t})}{\bar{t}-\bar{z}}\right) d\bar{t} = \lim_{r=0} \frac{1}{2\pi i} \int_{\bar{C}} \frac{-f(\bar{t})}{(\bar{t}-\bar{z})^2}\, d\bar{t}.$$

We arrive at similar results in terms of partial derivatives of the polygenic function.

5. DEFINITION OF A RESIDUE

Let us write our polygenic function as a function of the two complex variables z and $\bar{z}$. As mentioned above we will consider only functions of the type which are analytic in z and $\bar{z}$ except for poles. We shall limit our study to poles of types (a), (b), and (c) on page 9.

We shall consider a function $F(z,\bar{z})$ which can be written as the sum of a finite number of terms of the type

$$\frac{\phi_{\ell k}(z,\bar{z})}{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}},$$

where k and l are any integers, positive, negative, or zero, and where $\phi_{\ell k}(z,\bar{z})$ is analytic in z and $\bar{z}$ in the neighborhoods of a and $\bar{a}$ and $\phi_{\ell k}(a,\bar{a}) \neq 0$. We shall define the residue of the sum of a finite number of terms to be the sum of the residues of the terms. It will then be necessary to consider only the residue of a single term. We consider the general term by examining the function

$$(5) \qquad F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}},$$

where k and l are any integers and $f(z,\bar{z})$ satisfies the conditions given above for $\phi_{\ell k}(z,\bar{z})$. Let C and $\bar{C}$ be two closed curves lying entirely in A and $\bar{A}$ and enclosing a and $\bar{a}$ respectively. A and $\bar{A}$ are conjugate regions, containing a and $\bar{a}$, in which $F(z,\bar{z})$ is analytic in z and $\bar{z}$ except at a and $\bar{a}$. C and $\bar{C}$ enclose no other singular points of $F(z,\bar{z})$ except a and $\bar{a}$. Our definition of the residue at z = a of the polygenic function (5) is

$$(6) \qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) \equiv \frac{1}{2\pi i(k+\ell-1)!}\Big[0!\,{}_{k+\ell-1}C_0 \int_{\bar{C}} \frac{\partial^{k+\ell-1} f(a,\bar{z})}{\partial z^{k+\ell-1}} \cdot \frac{d\bar{z}}{\bar{z}-\bar{a}}$$

$$+ (1)!\,{}_{k+\ell-1}C_1 \int_{\bar{C}} \frac{\partial^{k+\ell-2} f(a,\bar{z})}{\partial z^{k+\ell-2}} \cdot \frac{d\bar{z}}{(\bar{z}-\bar{a})^2} + \cdots\cdots$$

$$+ (k-1)!\ {}_{k+\ell-1}C_{k-1} \int_{\bar{C}} \frac{\partial^{\ell} f(a,\bar{z})}{\partial z^{\ell}} \cdot \frac{d\bar{z}}{(\bar{z}-\bar{a})^{k}} + 0!\ {}_{k+\ell-1}C_{k+\ell-1} \int_{C} \frac{\partial^{k+\ell-1} f(z,\bar{a})}{\partial \bar{z}^{k+\ell-1}} \cdot \frac{dz}{z-a}$$

$$+ (1)!\ {}_{k+\ell-1}C_{k+\ell-2} \int_{C} \frac{\partial^{k+\ell-2} f(z,\bar{a})}{\partial \bar{z}^{k+\ell-2}} \cdot \frac{dz}{(z-a)^{2}} + \cdots + (\ell-1)!\ {}_{k+\ell-1}C_{\ell-1} \int_{C} \frac{\partial^{k} f(z,\bar{a})}{\partial \bar{z}^{k}} \cdot \frac{dz}{(z-a)^{\ell}} \Big]$$

$$= \frac{1}{2\pi i\,(k+\ell-1)!} \Big[\sum_{i=1}^{k} (i-1)!\ {}_{k+\ell-1}C_{i-1} \int_{\bar{C}} \frac{\partial^{k+\ell-i} f(a,\bar{z})}{\partial z^{k+\ell-i}} \cdot \frac{d\bar{z}}{(\bar{z}-\bar{a})^{i}}$$

$$+ \sum_{i=1}^{\ell} (i-1)!\ {}_{k+\ell-1}C_{i-1} \int_{C} \frac{\partial^{k+\ell-i} f(z,\bar{a})}{\partial \bar{z}^{k+\ell-i}} \cdot \frac{dz}{(z-a)^{i}} \Big].$$

where we shall agree that,

if $k \le 0$, the first summation will be zero;

if $\ell \le 0$, the second summation will be zero;

any term containing a partial derivative of negative order will be regarded as being zero. A partial derivative of zero order is just the function itself.

For the particular function

(5a) $$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{\ell}},$$

our definition gives

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i\,(\ell-1)!} \sum_{i=1}^{\ell} (i-1)!\ {}_{\ell-1}C_{i-1} \int_{C} \frac{\partial^{\ell-i} f(z,\bar{a})}{\partial \bar{z}^{\ell-i}} \cdot \frac{dz}{(z-a)^{i}}.$$

For the function

$$\text{(5b)} \qquad F(z,\bar z) = \frac{f(z,\bar z)}{(\bar z - \bar a)^K} \;,$$

we have

$$\operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{2\pi i\,(K-1)!} \sum_{i=1}^{K} (i-1)!\; {}_{K-1}C_{i-1} \int_C \frac{\partial^{K-i} f(a,\bar z)}{\partial z^{K-i}} \cdot \frac{d\bar z}{(\bar z - \bar a)^i} \,.$$

For the function

$$\text{(5c)} \qquad F(z,\bar z) = \frac{(\bar z - \bar a)^K f(z,\bar z)}{(z-a)^\ell} = \frac{f(z,\bar z)}{(z-a)^\ell (\bar z - \bar a)^{-K}} \;,$$

we have, if $\ell - K \geq 1$,

$$\operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{2\pi i\,(\ell-K-1)!} \sum_{i=1}^{\ell} (i-1)!\; {}_{\ell-K-1}C_{i-1} \int_C \frac{\partial^{\ell-K-i} f(z,\bar a)}{\partial \bar z^{\ell-K-i}} \cdot \frac{dz}{(z-a)^i} \;;$$

if $\ell - K < 1$, $\operatorname*{Res}_{z=a} F(z,\bar z) = 0$.

For the still more particular function

$$\text{(5d)} \qquad F(z,\bar z) = \frac{f(z,\bar z)}{z-a} \;,$$

we get

$$\operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{2\pi i} \int_C \frac{f(z,\bar a)}{z-a}\, dz = f(a,\bar a).$$

Similarly for

$$\text{(5e)} \qquad F(z,\bar z) = \frac{f(z,\bar z)}{\bar z - \bar a} \;,$$

we have

$$\operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{2\pi i} \int_{\bar C} \frac{f(a,\bar z)}{\bar z - \bar a}\, dz = f(a,\bar a).$$

For

(5f) $$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^2},$$

we have

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i}\left[\int_c \frac{\partial f(z,\bar{a})}{\partial \bar{z}}\cdot\frac{dz}{z-a} + \int_c \frac{f(z,\bar{a})}{(z-a)^2}\,dz\right]$$

$$= \frac{\partial f(a,\bar{a})}{\partial \bar{z}} + \frac{\partial f(a,\bar{a})}{\partial z}.$$

For

(5g) $$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(\bar{z}-\bar{a})^2},$$

we have

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i}\left[\int_c \frac{\partial f(a,\bar{z})}{\partial z}\cdot\frac{d\bar{z}}{\bar{z}-\bar{a}} + \int_c \frac{f(a,\bar{z})}{(\bar{z}-\bar{a})^2}\,d\bar{z}\right]$$

$$= \frac{\partial f(a,\bar{a})}{\partial z} + \frac{\partial f(a,\bar{a})}{\partial \bar{z}}.$$

For

(5h) $$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)(\bar{z}-\bar{a})},$$

we have

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i}\left[\int_c \frac{\partial f(a,\bar{z})}{\partial z}\cdot\frac{d\bar{z}}{\bar{z}-\bar{a}} + \int_c \frac{\partial f(z,\bar{a})}{\partial \bar{z}}\cdot\frac{dz}{z-a}\right]$$

$$= \frac{\partial f(a,\bar{a})}{\partial z} + \frac{\partial f(a,\bar{a})}{\partial \bar{z}}.$$

These examples suggest the following theorem.

THEOREM 5.1: *If, in the polygenic function (5), k and l are positive integers and $f(z,\bar z)$ is analytic in z and $\bar z$ in the neighborhoods of a and $\bar a$, then*

$$\text{(7)}\qquad \operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{(k+l-1)!}\sum_{i=1}^{k+l} {}_{k+l-1}C_{i-1}\,\frac{\partial^{k+l-1} f(a,\bar a)}{\partial z^{k+l-i}\,\partial \bar z^{i-1}}\,.$$

For, by (6),

$$\operatorname*{Res}_{z=a} F(z,\bar z) = \frac{1}{(k+l-1)!}\Big[\sum_{i=1}^{k}(i-1)!\,{}_{k+l-1}C_{i-1}\Big(\frac{1}{2\pi i}\int_{\bar c}\frac{\partial^{k+l-i} f(a,\bar z)}{\partial z^{k+l-i}}\cdot\frac{d\bar z}{(\bar z-\bar a)^i}\Big)$$

$$+\sum_{i=1}^{l}(i-1)!\,{}_{k+l-1}C_{i-1}\Big(\frac{1}{2\pi i}\int_{c}\frac{\partial^{k+l-i} f(z,\bar a)}{\partial \bar z^{k+l-i}}\cdot\frac{dz}{(z-a)^i}\Big)\Big]$$

$$=\frac{1}{(k+l-1)!}\Big[\sum_{i=1}^{k}(i-1)!\,{}_{k+l-1}C_{i-1}\Big(\frac{1}{(i-1)!}\cdot\frac{\partial^{k+l-1} f(a,\bar a)}{\partial z^{k+l-i}\,\partial \bar z^{i-1}}\Big)$$

$$+\sum_{i=1}^{l}(i-1)!\,{}_{k+l-1}C_{i-1}\Big(\frac{1}{(i-1)!}\,\frac{\partial^{k+l-1} f(a,\bar a)}{\partial \bar z^{k+l-i}\,\partial z^{i-1}}\Big)\Big]$$

$$=\frac{1}{(k+l-1)!}\sum_{i=1}^{k+l}{}_{k+l-1}C_{i-1}\,\frac{\partial^{k+l-1} f(a,\bar a)}{\partial z^{k+l-i}\,\partial \bar z^{i-1}}\,.$$

6. THEOREMS ON RESIDUES

Using our definition we now state and prove several theorems corresponding to the common theorems of analytic function theory. Our first theorem is

THEOREM 6.1: *If, in the polygenic function*

$$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}},$$

the function $f(z,\bar{z})$ *is analytic in z and* $\bar{z}$ *in the neighborhoods of a and* $\bar{a}$ *and if* $f(a,\bar{a}) \neq 0$, *then the residue of* $F(z,\bar{z})$ *at* $z=a$ *is zero for values of k and l such that* $k+\ell \leq 0$.

For, since $k+\ell \leq 0$, all partial derivatives which could occur in definition (6) are of negative orders. We have agreed that any term containing partial derivatives of negative orders will be zero. Therefore our conclusion follows.

THEOREM 6.2: *If, in the polygenic function*

$$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}},$$

the function $f(z,\bar{z})$ *is analytic in z and* $\bar{z}$ *in the neighborhoods of a and* $\bar{a}$ *and if* $f(a,\bar{a}) \neq 0$, *then the residue of* $F(z,\bar{z})$ *at* $z=a$ *is equal to the sum of the coefficients of*

$$\frac{(z-a)^{p}}{(\bar{z}-\bar{a})^{p+1}}, \quad (p=0,1,\cdots k-1),$$

<u>plus the sum of the coefficients of</u>

$$\frac{(\bar{z}-\bar{a})^{q}}{(z-a)^{q+1}}, \quad (q=0,1,\cdots\cdots l-1),$$

<u>in the expansion of</u> $F(z,\bar{z})$ <u>in powers of</u> z-a <u>and</u> $\bar{z}-\bar{a}$.

Since $f(z,\bar{z})$ is analytic in z and $\bar{z}$ in the neighborhoods of a and $\bar{a}$, we have

$$f(z,\bar{z}) = f(a,\bar{a}) + \frac{\partial f(a,\bar{a})}{\partial z}(z-a) + \frac{\partial f(a,\bar{a})}{\partial \bar{z}}(\bar{z}-\bar{a}) + \cdots\cdots$$

$$+ \frac{1}{(k+l-1)!}\Big[\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-1}}(z-a)^{k+l-1} + {}_{k+l-1}C_{k-1}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{l}\,\partial \bar{z}^{k-1}}(z-a)^{l}(\bar{z}-\bar{a})^{k-1}$$

$$+ {}_{k+l-1}C_{k}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{l-1}\,\partial \bar{z}^{k}}(z-a)^{l-1}(\bar{z}-\bar{a})^{k} + \cdots\, {}_{k+l-1}C_{k+l-1}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial \bar{z}^{k+l-1}}(\bar{z}-\bar{a})^{k+l-1}\Big] + \cdots.$$

And

$$F(z,\bar{z}) = \frac{f(z,\bar{z})}{(z-a)^{l}(\bar{z}-\bar{a})^{k}} = \frac{f(a,\bar{a})}{(z-a)^{l}(\bar{z}-\bar{a})^{k}}$$

$$+ \frac{\partial f(a,\bar{a})}{\partial z}\cdot\frac{1}{(z-a)^{l-1}(\bar{z}-\bar{a})^{k}} + \frac{\partial f(a,\bar{a})}{\partial \bar{z}}\cdot\frac{1}{(z-a)^{l}(\bar{z}-\bar{a})^{k-1}} + \cdots\cdots$$

$$+ \frac{1}{(k+l-1)!}\Big[{}_{k+l-1}C_{0}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-1}}\cdot\frac{(z-a)^{k-1}}{(\bar{z}-\bar{a})^{k}} + \cdots\, {}_{k+l-1}C_{k-1}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{l}\,\partial \bar{z}^{k-1}}\cdot\frac{1}{\bar{z}-\bar{a}}$$

$$+ {}_{k+l-1}C_{k}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{l-1}\,\partial \bar{z}^{k}}\cdot\frac{1}{z-a} + \cdots\, {}_{k+l-1}C_{k+l-1}\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial \bar{z}^{k+l-1}}\cdot\frac{(\bar{z}-\bar{a})^{l-1}}{(z-a)^{l}}\Big] + \cdots.$$

The coefficients named in the theorem are

$$(8) \qquad \frac{1}{(k+l-1)!}\sum_{i=1}^{k+l} {}_{k+l-1}C_{i-1}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-i}\,\partial \bar{z}^{i-1}} \; .$$

By (6)

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{(k+l-1)!}\Bigg[\sum_{i=1}^{k}(i-1)!\;{}_{k+l-1}C_{i-1}\left(\frac{1}{2\pi i}\int_{\bar{c}}\frac{\partial^{k+l-i} f(a,\bar{z})}{\partial z^{k+l-i}}\cdot\frac{d\bar{z}}{(\bar{z}-\bar{a})^{i}}\right)$$

$$+\sum_{i=1}^{l}(i-1)!\;{}_{k+l-1}C_{i-1}\left(\frac{1}{2\pi i}\int_{c}\frac{\partial^{k+l-i} f(z,\bar{a})}{\partial \bar{z}^{k+l-i}}\cdot\frac{dz}{(z-a)^{i}}\right)\Bigg]$$

$$= \frac{1}{(k+l-1)!}\Bigg[\sum_{i=1}^{k}(i-1)!\;{}_{k+l-1}C_{i-1}\left(\frac{1}{(i-1)!}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-i}\,\partial \bar{z}^{i-1}}\right)$$

$$+\sum_{i=1}^{l}(i-1)!\;{}_{k+l-1}C_{i-1}\left(\frac{1}{(i-1)!}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial \bar{z}^{k+l-i}\,\partial z^{i-1}}\right)\Bigg]$$

$$= \frac{1}{(k+l-1)!}\Bigg[\sum_{i=1}^{k}{}_{k+l-1}C_{i-1}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-i}\,\partial \bar{z}^{i-1}} + \sum_{j=k+1}^{k+l}{}_{k+l-1}C_{j-1}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial \bar{z}^{j-1}\,\partial z^{k+l-j}}\Bigg]$$

$$= \frac{1}{(k+l-1)!}\sum_{i=1}^{k+l}{}_{k+l-1}C_{i-1}\,\frac{\partial^{k+l-1} f(a,\bar{a})}{\partial z^{k+l-i}\,\partial \bar{z}^{i-1}} \, ,$$

and this is expression (8).

In the two special cases,

$$F(z,\bar{z}) = \frac{1}{(z-a)(\bar{z}-\bar{a})} \quad \text{and} \quad F(z,\bar{z}) = \frac{(\bar{z}-\bar{a})}{(z-a)},$$

our definition gives a residue of zero, and these results agree with the conclusion of the above theorem.

The notation used in our definition of a residue is such that an interpretation is needed for poles at the point at infinity. To avoid difficulties of notation we shall define the residue of $F(z,\bar{z})$ at $z=\infty$ to be equal to the residue of the transformed function $G(z',\bar{z}')$ at $z=0$, where we apply the reciprocal transformations

$$z = \frac{1}{z'}, \quad \bar{z} = \frac{1}{\bar{z}'},$$

to $F(z,\bar{z})$ to get $G(z',\bar{z}')$. Using this definition we have

<u>THEOREM 6.3: If, in the polygenic function</u>

$$F(z,\bar{z}) = z^{\ell}\bar{z}^{\kappa} f(z,\bar{z}),$$

<u>the function $f(z,\bar{z})$ is analytic in z and $\bar{z}$ in the neighborhood of the point at infinity and if</u>

$$\lim_{\substack{z=\infty \\ \bar{z}=\infty}} f(z,\bar{z}) \neq 0,$$

<u>then the residue of $F(z,\bar{z})$ at $z=\infty$ is equal to the negative of the sum of the coefficients of</u>

$$\frac{z^{p}}{\bar{z}^{p+1}}, \; (p=0,1,\cdots\ell),$$

plus the negative of the sum of the coefficients of

$$\frac{\bar{z}^{q}}{z^{q+1}}, \quad (q = 0, 1, \cdots K),$$

in the expansion of $F(z,\bar{z})$ in the neighborhood of $z = \infty$. Applying the transformations

$$z = \frac{1}{z'}, \quad \bar{z} = \frac{1}{\bar{z}'},$$

to $F(z,\bar{z})$ we have

$$G(z',\bar{z}') = F\left(\frac{1}{z'}, \frac{1}{\bar{z}'}\right) = \frac{f\left(\frac{1}{z'}, \frac{1}{\bar{z}'}\right)}{z'^{\ell}\bar{z}'^{K}} = \frac{g(z',\bar{z}')}{z'^{\ell}\bar{z}'^{K}},$$

where $g(z',\bar{z}')$ is analytic in z' and $\bar{z}'$ in the neighborhood of the origin and where $g(0,0) \neq 0$. We must also apply the above transformations to the differentials of z and $\bar{z}$. Thus

$$dz = -\frac{1}{z'^2}\,dz', \quad d\bar{z} = -\frac{1}{\bar{z}'^2}\,d\bar{z}',$$

in the integrals of (6). Keeping this in mind, theorem (6.2) says that the residue of $G(z',\bar{z}')$ at $z' = 0$ is equal to the negative of the sum of the coefficients of

$$\frac{\bar{z}'^{p+1}}{z'^{p}}, \quad (p = 0, 1, \cdots \ell),$$

plus the negative of the sum of the coefficients of

$$\frac{z'^{q+1}}{\bar{z}'^{q}}, \quad (q = 0, 1, \cdots K),$$

in the expansion of $G(z',\bar{z}')$ in the neighborhood of the origin. But these terms are respectively the terms

$$\frac{z^p}{\bar{z}^{p+1}}, \quad (p=0,1,\cdots \ell),$$

and

$$\frac{\bar{z}^q}{z^{q+1}}, \quad (q=0,1,\cdots k),$$

in the expansion of $F(z,\bar{z})$ in the neighborhood of the point at infinity.

The expression for the derivative of a polygenic function, equation (3), suggests a definition for the "logarithmic derivative" of a polygenic function. In order to make this derivative unique (independent of θ) we shall say that

$$(9) \qquad \text{Log. Der. } F(z,\bar{z}) = \frac{D[f(z)]+P[f(z)]}{f(z)} = \frac{\dfrac{\partial F(z,\bar{z})}{\partial z}+\dfrac{\partial F(z,\bar{z})}{\partial \bar{z}}}{f(z)}.$$

Then we have

<u>THEOREM</u> 6.4: <u>If, in the polygenic function</u>

$$F(z,\bar{z}) = (z-a)^{\ell}(\bar{z}-\bar{a})^{k} f(z,\bar{z})$$

<u>the function</u> $f(z,\bar{z})$ <u>is analytic in z and $\bar{z}$ in the neighborhood of a and $\bar{a}$ and if</u> $f(a,\bar{a}) \neq 0$, <u>then for k and l different from zero, the logarithmic derivative of</u> $F(z,\bar{z})$ <u>has a simple pole at $z=a$ and a simple pole at $\bar{z}=\bar{a}$; furthermore, for any integral values of k and l, the residue of the logarithmic derivative of</u> $F(z,\bar{z})$ <u>is equal to</u> $k+\ell$.

Since

$$\frac{\partial F(z,\bar{z})}{\partial z} = (\bar{z}-\bar{a})^{k}\left[(z-a)^{\ell}\frac{\partial f(z,\bar{z})}{\partial z} + \ell(z-a)^{\ell-1}f(z,\bar{z})\right],$$

then

$$\frac{\dfrac{\partial F(z,\bar{z})}{\partial z}}{F(z,\bar{z})} = \frac{\dfrac{\partial f(z,\bar{z})}{\partial z}}{f(z,\bar{z})} + \frac{\ell}{z-a}.$$

Since

$$\frac{\partial F(z,\bar{z})}{\partial \bar{z}} = (z-a)^{\ell}\left[(\bar{z}-\bar{a})^{k}\frac{\partial f(z,\bar{z})}{\partial \bar{z}} + k(\bar{z}-\bar{a})^{k-1}f(z,\bar{z})\right],$$

then

$$\frac{\dfrac{\partial F(z,\bar{z})}{\partial \bar{z}}}{F(z,\bar{z})} = \frac{\dfrac{\partial f(z,\bar{z})}{\partial \bar{z}}}{f(z,\bar{z})} + \frac{k}{\bar{z}-\bar{a}}.$$

By (9)

$$\text{Log. Der. } F(z,\bar{z}) = \frac{\dfrac{\partial f(z,\bar{z})}{\partial z}}{f(z,\bar{z})} + \frac{\ell}{z-a} + \frac{\dfrac{\partial f(z,\bar{z})}{\partial \bar{z}}}{f(z,\bar{z})} + \frac{k}{\bar{z}-\bar{a}}.$$

This proves our first conclusion. Also we have

$$\operatorname*{Res}_{z=a} \text{Log. Der. } F(z,\bar{z}) = \operatorname*{Res}_{z=a} \frac{\dfrac{\partial f(z,\bar{z})}{\partial z}}{f(z,\bar{z})} + \operatorname*{Res}_{z=a} \frac{\dfrac{\partial F(z,\bar{z})}{\partial \bar{z}}}{f(z,\bar{z})}$$

$$+ \operatorname*{Res}_{z=a} \frac{\ell}{z-a} + \operatorname*{Res}_{z=a} \frac{k}{\bar{z}-\bar{a}}.$$

The first two terms are zero because the functions

$$\frac{\dfrac{\partial f(z,\bar{z})}{\partial z}}{f(z,\bar{z})} \quad \text{and} \quad \frac{\dfrac{\partial f(z,\bar{z})}{\partial \bar{z}}}{f(z,\bar{z})}$$

are analytic in z and $\bar{z}$ in the neighborhoods of a and $\bar{a}$. For, by hypothesis, $f(z,\bar{z})$ satisfies this condition and so its first partial derivatives will also. Also, by hypothesis, $f(a,\bar{a}) \neq 0$. The functions,

$$\frac{\dfrac{\partial f(z,\bar{z})}{\partial z}}{f(z,\bar{z})} \quad \text{and} \quad \frac{\dfrac{\partial f(z,\bar{z})}{\partial \bar{z}}}{f(z,\bar{z})},$$

thus satisfy the hypotheses of theorem 6.1 where $K=\ell=0$ and so the residues of these terms are zero. Applying (6) to the term $\frac{\ell}{z-a}$ we get

$$\operatorname*{Res}_{z=a} \frac{\ell}{z-a} = \frac{1}{2\pi i}\int_C \frac{\ell}{z-a}\,dz = \ell \,.$$

For the term $\frac{K}{\bar{z}-\bar{a}}$ we get

$$\operatorname*{Res}_{z=a} \frac{K}{\bar{z}-\bar{a}} = \frac{1}{2\pi i}\int_C \frac{K}{\bar{z}-\bar{a}}\,d\bar{z} = K \,.$$

Therefore

$$\operatorname*{Res}_{z=a} \text{Log. Der.}\, F(z,\bar{z}) = K+\ell.$$

This theorem includes both of the following theorems on analytic functions:

<u>THEOREM</u> A: <u>If, in the analytic function</u>

$$F(z) = (z-a)^{\ell} f(z), \qquad (\ell > 0)$$

<u>the function $f(z)$ is analytic in the neighborhood of $z=a$ and if $f(a)\neq 0$, then the logarithmic derivative of $F(z)$ has a simple pole at $z=a$ with residue equal to 1.</u>

<u>THEOREM</u> B: <u>If, in the analytic function</u>

$$F(z) = \frac{f(z)}{(z-a)^{\ell}}, \qquad (\ell > 0),$$

<u>the function $f(z)$ is analytic in the neighborhood of $z = a$ and if $f(a) \neq 0$, then the logarithmic derivative of $F(z)$ has a simple pole at $z = a$ with residue equal to -1.</u>

The definition given here for the residue of a polygenic function has one serious objection. The theorems on residues in analytic function theory which involve the integration around a curve enclosing more than one pole do not seem to carry over in any suitable form. The reason for this is seen in the definition since it involves the substitution of "the" point at which there is a pole in the function $f(z, \bar{z})$ or its partial derivatives. Among the theorems from analytic function theory which do not carry over for this reason are:

<u>THEOREM</u> a: <u>If $f(z)$ is holomorphic in a region S except for a finite number of poles and if C is any curve lying entirely in S and enclosing all the poles of $f(z)$ then the integral, $\int_C f(z)dz$, is equal to $2\pi i$ times the sum of the residues of $f(z)$ in S.</u>

<u>THEOREM</u> b: <u>The sum of the residues of a rational function is zero.</u>

THEOREM c: The integral,

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz ,$$

taken in a positive sense around the boundary C of a closed region in which the "rational" function $f(z)$ is holomorphic except at a finite number of poles is equal to the number of zero points of $f(z)$ in this region diminished by the number of poles, each zero point and each pole being counted a number of times equal to its order.

7. THE FUNCTION $F(z,\bar{z}) = f_1(z) + f_2(\bar{z})$

In this and the next section we examine the three particular types of polygenic functions

(a) $$F(z,\bar{z}) = f_1(z) + f_2(\bar{z}) ,$$

(b) $$F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z) ,$$

(c) $$F(z,\bar{z}) = f_1(\bar{z}) + z f_2(\bar{z}) .$$

Kasner, in his geometric representation of the rectilinear second derivative of a polygenic function, mentions seven special types of functions, the three listed above, and four others which are special cases of these three.*

---

* Edward Kasner, The Second Derivative of a Polygenic Function, Transactions of the American Mathematical Society, Vol. 30, (1928), page 809.

It is seen that for the three types of functions listed above we have

(a) $F_{z\bar{z}}(z,\bar{z})=0$, (b) $F_{\bar{z}\bar{z}}(z,\bar{z})=0$, (c) $F_{zz}(z,\bar{z})=0$.

For type (a) we shall write

$$F(z,\bar{z}) = f_1(z) + f_2(\bar{z}) = \frac{\varphi_1(z)}{(z-a)^l} + \frac{\varphi_2(\bar{z})}{(\bar{z}-\bar{a})^k} ,$$

where k and l are any integers, and where $\varphi_1(z)$ and $\varphi_2(\bar{z})$ are analytic in z and $\bar{z}$ respectively in the neighborhoods of a and $\bar{a}$, and where $\varphi_1(a)\neq 0$, $\varphi_2(\bar{a})\neq 0$. Applying the definition to each term separately we get

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i\,(l-1)!}\left[(l-1)!\;{}_{l-1}C_{l-1}\int_C \frac{\varphi_1(z)}{(z-a)^l}\,dz\right]$$

$$+\frac{1}{2\pi i\,(k-1)!}\left[(k-1)!\;{}_{k-1}C_{k-1}\int_{\bar{C}} \frac{\varphi_2(\bar{z})}{(\bar{z}-\bar{a})^k}\,d\bar{z}\right]$$

$$=\frac{1}{2\pi i}\int_C \frac{\varphi_1(z)}{(z-a)^l}\,dz + \frac{1}{2\pi i}\int_{\bar{C}} \frac{\varphi_2(\bar{z})}{(\bar{z}-\bar{a})^k}\,d\bar{z} = \operatorname*{Res}_{z=a} f_1(z) + \operatorname*{Res}_{\bar{z}=\bar{a}} f_2(\bar{z}) ,$$

where, in the last line, $f_1(z)$ and $f_2(\bar{z})$ are considered as functions of the single variables z and $\bar{z}$ respectively. Thus for this function our definition gives a reasonable result.

Since the residue of $f_1(z)+f_2(\bar{z})$ turns out to be the sum of the residues of $f_1(z)$ and $f_2(\bar{z})$ considered as functions of the single variables z and $\bar{z}$ respectively, all of the theorems concerning residues of analytic functions will hold for this function. The proofs of the following theorems consist of first the statement,

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{z=a} f_1(z) + \operatorname*{Res}_{\bar{z}=\bar{a}} f_2(\bar{z}) \ ,$$

and then a statement or two from analytic function theory (usually a statement of the corresponding theorem for analytic functions).

<u>THEOREM</u> 7.1: <u>If, in the polygenic function</u>

$$F(z,\bar{z}) = f_1(z) + f_2(\bar{z}) = \frac{\varphi_1(z)}{(z-a)^{\ell}} + \frac{\varphi_2(\bar{z})}{(\bar{z}-\bar{a})^{k}} \ ,$$

<u>the functions</u> $\varphi_1(z)$ <u>and</u> $\varphi_2(\bar{z})$ <u>are analytic in z and $\bar{z}$ respectively in the neighborhoods of a and $\bar{a}$ and if</u> $\varphi_1(a) \neq 0$, $\varphi_2(\bar{a}) \neq 0$, <u>then if</u>

$$\ell \le 0,\ k>0, \qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{\bar{z}=\bar{a}} f_2(\bar{z}) \ ,$$

$$k \le 0,\ \ell>0, \qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{z=a} f_1(z) \ ,$$

$$k \le 0,\ \ell \le 0, \qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) = 0 \ ,$$

$$k=\ell=1, \qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) = \varphi_1(a) + \varphi_2(\bar{a}) \ .$$

THEOREM 7.2: If, in the polygenic function

$$F(z,\bar z)=f_1(z)+f_2(\bar z)=\frac{\varphi_1(z)}{(z-a)^\ell}+\frac{\varphi_2(\bar z)}{(\bar z-\bar a)^K},$$

the functions $\varphi_1(z)$ and $\varphi_2(\bar z)$ are analytic in z and $\bar z$ respectively in the neighborhoods of a and $\bar a$ and if $\varphi_1(a)\neq 0$, $\varphi_2(\bar a)\neq 0$, then the residue of $F(z,\bar z)$ at $z=a$ is equal to the coefficient of $(z-a)^{-1}$ in the expansion of $f_1(z)$ about $z=a$ plus the coefficient of $(\bar z-\bar a)^{-1}$ in the expansion of $f_2(\bar z)$ about $\bar z=\bar a$.

THEOREM 7.3: If the polygenic function

$$F(z,\bar z)=f_1(z)+f_2(\bar z)$$

is analytic in z and $\bar z$ in regions A and $\bar A$ except for a finite number of poles and if C and $\bar C$ are curves, lying in A and $\bar A$, enclosing no singularities of $f_1(z)$ or $f_2(\bar z)$ except the poles of $F(z,\bar z)$ in A and $\bar A$, then

$$\int_C f_1(z)\,dz+\int_{\bar C} f_2(\bar z)\,d\bar z$$

is equal to $2\pi i$ times the sum of the residues of $F(z,\bar z)$ in A and $\bar A$.

THEOREM 7.4: If, in the polygenic function

$$F(z,\bar z)=f_1(z)+f_2(\bar z)=z^\ell\varphi_1(z)+\bar z^K\varphi_2(\bar z),\qquad (K>0,\ \ell>0),$$

the functions $\varphi_1(z)$ and $\varphi_2(\bar z)$ are analytic in z and $\bar z$ respectively in the neighborhood of the point at infinity and if

$$\lim_{z=\infty}\varphi_1(z)\neq 0,\qquad \lim_{\bar z=\infty}\varphi_2(\bar z)\neq 0,$$

then the residue of $F(z,\bar{z})$ at $z=\infty$ is equal to the negative of the coefficient of z^{-1} in the expansion of $f_1(z)$ in the neighborhood of the point at infinity plus the negative of the coefficient of $\bar{z}^{-1}$ in the expansion of $f_2(\bar{z})$ in the neighborhood of the point at infinity.

THEOREM 7.5: If, in the polygenic function,

$$F(z,\bar{z}) = f_1(z) + f_2(\bar{z}),$$

the functions $f_1(z)$ and $f_2(\bar{z})$ are rational functions of z and $\bar{z}$ respectively, then the sum of the residues of $F(z,\bar{z})$ is zero.

We shall say that the logarithmic derivative of $F(z,\bar{z}) = f_1(z) + f_2(\bar{z})$ is equal to the sum of the logarithmic derivatives of $f_1(z)$ and $f_2(\bar{z})$. Using this definition we have the following two theorems:

THEOREM 7.6: If, in the polygenic function

$$F(z,\bar{z}) = f_1(z) + f_2(\bar{z}) = (z-a)^{\ell}\varphi_1(z) + (\bar{z}-\bar{a})^{K}\varphi_2(\bar{z}),$$

the functions $\varphi_1(z)$ and $\varphi_2(\bar{z})$ are analytic in z and $\bar{z}$ respectively in the neighborhoods of a and $\bar{a}$ and if $\varphi_1(a) \neq 0$, $\varphi_2(\bar{a}) \neq 0$, then if $K \neq 0$, $\ell \neq 0$, the logarithmic derivative of $F(z,\bar{z})$ has a simple pole at $z=a$ and a simple pole at $\bar{z}=\bar{a}$; furthermore for k and l any integers, the residue of the logarithmic derivative of $F(z,\bar{z})$ at $z=a$ is equal to $K+\ell$.

THEOREM 7.7: The sum of the integrals,

$$\frac{1}{2\pi i}\int_C \text{Log.Der.}\, f_1(z)\,dz + \frac{1}{2\pi i}\int_{\bar{C}} \text{Log.Der.}\, f_2(\bar{z})\,d\bar{z},$$

taken in the positive sense around the boundaries C and $\bar{C}$ of closed regions A and $\bar{A}$ in which the "rational functions" $f_1(z)$ and $f_2(\bar{z})$ are holomorphic in z and $\bar{z}$ respectively except at a finite number of poles is equal to the number of zero points of $f_1(z)$ in A plus the number of zero points of $f_2(\bar{z})$ in $\bar{A}$ diminished by the number of poles of $f_1(z)$ in A plus the number of poles of $f_2(\bar{z})$ in $\bar{A}$, each zero and each pole being counted a number of times equal to its order.

8. THE FUNCTIONS $F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z)$ AND $F(z,\bar{z}) = f_1(\bar{z}) + z f_2(\bar{z})$

For the particular function, type (b), page 30, we shall write

$$F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z) = \frac{\varphi_1(z)}{(z-a)^{\ell_1}} + \bar{z}\frac{\varphi_2(z)}{(z-a)^{\ell_2}},$$

where $a \neq 0$, where $\varphi_1(z)$ and $\varphi_2(z)$ are analytic in the neighborhood of $z = a$, and where $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$. Equation (6) gives as residue,

$$(10)\qquad \operatorname*{Res}_{z=a} F(z,\bar{z}) = \frac{1}{2\pi i\,(\ell_1-1)!}\left[(\ell_1-1)!\;{}_{\ell_1-1}C_{\ell_1-1}\int_C \frac{\varphi_1(z)}{(z-a)^{\ell_1}}\,dz\right]$$

$$+\frac{1}{2\pi i(\ell_2-1)!}\left[(\ell_2-2)!\,{}_{\ell_2-1}C_{\ell_2-2}\int_C \frac{\varphi_2(z)}{(z-a)^{\ell_2-1}}dz+(\ell_2-1)!\,{}_{\ell_2-1}C_{\ell_2-1}\int_C \frac{\bar{a}\,\varphi_2(z)}{(z-a)^{\ell_2}}dz\right]$$

$$=\frac{1}{2\pi i}\int_C \frac{\varphi_1(z)}{(z-a)^{\ell_1}}dz+\frac{1}{2\pi i}\int_C \frac{\varphi_2(z)}{(z-a)^{\ell_2-1}}dz+\frac{\bar{a}}{2\pi i}\int_C \frac{\varphi_2(z)}{(z-a)^{\ell_2}}dz$$

$$=\operatorname*{Res}_{z=a} f_1(z)+\operatorname*{Res}_{z=a}\frac{\varphi_2(z)}{(z-a)^{\ell_2-1}}+\bar{a}\operatorname*{Res}_{z=a} f_2(z),$$

where these residues are evaluated as for analytic functions of one variable. This result is also obtained if $f_1(z)+\bar{z}f_2(z)$ is combined into one term, using either the assumption $\ell_1 \geq \ell_2$ or $\ell_1 < \ell_2$. The expression (10) is as one would expect since

$$\bar{z}f_2(z)=\frac{\bar{z}\varphi_2(z)}{(z-a)^{\ell_2}}=\frac{(\bar{z}-\bar{a})\varphi_2(z)}{(z-a)^{\ell_2}}+\frac{\bar{a}\,\varphi_2(z)}{(z-a)^{\ell_2}}\ .$$

We note that in the special case where $a=0$, equation (10) gives as the residue

(10a) $$\operatorname*{Res}_{z=0} F(z,\bar{z})=\operatorname*{Res}_{z=0} f_1(z)+\operatorname*{Res}_{z=0}\frac{\varphi_2(z)}{z^{\ell_2-1}},$$

since if $a=0$, then $\bar{a}=0$.

The function $f_1(\bar{z})+z f_2(\bar{z})$ is similar to the function above. Since this function is obtainable from the above function by merely interchanging z and $\bar{z}$, the conclusions for this function can be obtained by the interchange of z and $\bar{z}$ in the results for the function $f_1(z)+\bar{z}f_2(z)$.

By interchanging z and $\bar{z}$ in (10) we get the residue of

$$F(z,\bar{z}) = f_1(\bar{z}) + z f_2(\bar{z})$$

to be

$$\underset{z=a}{\operatorname{Res}} = \underset{\bar{z}=\bar{a}}{\operatorname{Res}} f_1(\bar{z}) + \underset{\bar{z}=\bar{a}}{\operatorname{Res}} \frac{\varphi_2(\bar{z})}{(\bar{z}-\bar{a})^{k_2-1}} + a \underset{\bar{z}=\bar{a}}{\operatorname{Res}} f_2(\bar{z}). \tag{11}$$

The following theorems, which we state for the function $F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z)$ can be easily proved by using (10) and statements on residues of analytic functions.

<u>THEOREM</u> 8.1: <u>If, in the polygenic function</u>,

$$F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z) = \frac{\varphi_1(z)}{(z-a)^{l_1}} + \frac{\bar{z}\,\varphi_2(z)}{(z-a)^{l_2}}, \qquad (a \neq 0),$$

<u>the functions</u> $\varphi_1(z)$ <u>and</u> $\varphi_2(z)$ <u>are analytic in the neighborhood of</u> z = a <u>and if</u> $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$, <u>then if</u>

$$l_1 \leq 0,\ l_2 > 0, \quad \underset{z=a}{\operatorname{Res}} F(z,\bar{z}) = \underset{z=a}{\operatorname{Res}} \frac{\varphi_2(z)}{(z-a)^{l_2-1}} + \underset{z=a}{\operatorname{Res}} f_2(z);$$

$$l_1 > 0,\ l_2 \leq 0, \quad \underset{z=a}{\operatorname{Res}} F(z,\bar{z}) = \underset{z=a}{\operatorname{Res}} f_1(z);$$

$$l_1 \leq 0,\ l_2 \leq 0, \quad \underset{z=a}{\operatorname{Res}} F(z,\bar{z}) = 0;$$

$$l_1 = l_2 = 1, \quad \underset{z=a}{\operatorname{Res}} F(z,\bar{z}) = \varphi_1(a) + \bar{a}\,\varphi_2(a);$$

$$l_1 = 1,\ l_2 = 2, \quad \underset{z=a}{\operatorname{Res}} F(z,\bar{z}) = \varphi_1(a) + \varphi_2(a) + \bar{a}\,\frac{\partial \varphi_2(a)}{\partial z}.$$

THEOREM 8.2: If, in the polygenic function,

$$F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z) = \frac{\varphi_1(z)}{(z-a)^{\ell_1}} + \frac{\bar{z}\,\varphi_2(z)}{(z-a)^{\ell_2}}, \quad (a \neq 0),$$

the functions $\varphi_1(z)$ and $\varphi_2(z)$ are analytic in the neighborhood of $z = a$ and if $\varphi_1(a) \neq 0$, $\varphi_2(a) \neq 0$, then the residue of $F(z,\bar{z})$ at $z = a$ is equal to the coefficient of $(z-a)^{-1}$ in the expansion of $f_1(z)$ about $z = a$ plus the coefficient of $(z-a)^{-2}$ in the expansion of $f_2(z)$ about $z = a$ plus $\bar{a}$ times the coefficient of $(z-a)^{-1}$ in the expansion of $f_2(z)$ about $z = a$.

THEOREM 8.3: If, in the polygenic function,

$$F(z,\bar{z}) = f_1(z) + \bar{z} f_2(z) = z^{\ell_1}\varphi_1(z) + \bar{z}\, z^{\ell_2}\varphi_2(z), \quad (\ell_1 > 0, \ell_2 > 0),$$

the functions $\varphi_1(z)$ and $\varphi_2(z)$ are analytic in the neighborhood of the point at infinity and if

$$\lim_{z=\infty} \varphi_1(z) \neq 0, \quad \lim_{z=\infty} \varphi_2(z) \neq 0,$$

then the residue of $F(z,\bar{z})$ at $z = \infty$ is equal to the negative of the coefficient of z^{-1} in the expansion of $f_1(z)$ in the neighborhood of infinity plus the negative of the coefficient of z^{-2} in the expansion of $f_2(z)$ in the neighborhood of infinity.

In finding the residue of this function at $z = \infty$, we do as we did for the general function (see theorem 6.4). That is, we apply the transformations

$$z = \frac{1}{z'}, \quad \bar{z} = \frac{1}{\bar{z}'},$$

to $F(z,\bar{z})$, and define the residue of $F(z,\bar{z})$ at $z=\infty$ to be the residue of the transformed function $G(z',\bar{z}')$ at $z'=0$. Then in finding the residue of $G(z',\bar{z}')$ at $z'=0$, we note that $a'=0$ and so we use equation (10a).

THEOREM 3.4: If, in the polygenic function,

$$F(z,\bar{z}) = f_1(z) + \bar{z}f_2(z) = (z-a)^{\ell_1}\varphi_1(z) + \bar{z}(z-a)^{\ell_2}\varphi_2(z), \qquad (a \neq 0),$$

the functions $\varphi_1(z)$ and $\varphi_2(z)$ are analytic in the neighborhood of $z=a$ and if $\varphi_1(a)\neq 0$, $\varphi_2(a)\neq 0$, then if $\ell_1\neq 0$, $\ell_2\neq 0$, the logarithmic derivative of $F(z,\bar{z})$ has a simple pole at $z=a$; furthermore for ℓ_1 and ℓ_2 any integers, the residue of the logarithmic derivative is equal to $\ell_1+\ell_2$.

Here, as before, we say

$$\text{Log. Der. } F(z,\bar{z}) = \text{Log. Der. } f_1(z) + \text{Log. Der. } \bar{z}f_2(z).$$

Due to the fact that the residue of $F(z,\bar{z}) = f_1(z) + \bar{z}f_2(z)$ involves the term,

$$\bar{a} \operatorname*{Res}_{z=a} f_2(z),$$

which contains "the" pole $z=a$, the theorems on residues of analytic functions which depend on an integral along a closed curve enclosing more than just one pole do not hold. Thus theorems (a), (b), and (c) of section 6 do not carry over as they are stated.

It would seem possible, however, to prove theorems somewhat similar to the theorems of analytic functions. In particular, it probably is possible to show that if $f_1(z)$ and $f_2(z)$ are rational functions, then the sum of the residues is zero.

9. POSSIBLE GENERALIZATIONS

Our definition for the residue of a polygenic function, although limited to a few types of poles, need not be so restricted in the types of zeros included. In fact, we can include all types of zeros. For example, we can find the residue of the function,

$$F(z,\bar{z}) = \frac{[\alpha(z-a)+\beta(\bar{z}-\bar{a})]^p f(z,\bar{z})}{(z-a)^\ell(\bar{z}-\bar{a})^k},$$

where $[\alpha(z-a)+\beta(\bar{z}-\bar{a})]^p$ is zero for $z = a$ and $\bar{z} = \bar{a}$, by expanding the factor $[\alpha(z-a)+\beta(\bar{z}-\bar{a})]^p$ and separating our function into a sum of terms of the type given in equation (5) on page 16. Possibly a still more general definition can be found which will include all types of poles. Such a definition might be stated for the function

$$(4) \qquad F(z,\bar{z}) = \frac{(z-a)^\ell(\bar{z}-\bar{a})^k f(z,\bar{z})}{\sum_{i=1}^{m} (z-a)^{\ell_i}(\bar{z}-\bar{a})^{k_i} f_i(z,\bar{z})},$$

which we have seen includes all types of zeros and poles (see page 10). For this function we shall merely suggest the following definition. In the terms of the denominator of (4) let m be the largest of $l_i + k_i$. If there is just one such term, say $m = l_j + k_j$, we shall say that

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{z=a} \frac{(z-a)^{l}(\bar{z}-\bar{a})^{k} f(z,\bar{z})}{(z-a)^{l_j}(\bar{z}-\bar{a})^{k_j} f_j(z,\bar{z})} .$$

If there is more than one such term we will compare the absolute values of the functions $f_i(a,\bar{a})$ which occur in these terms of highest total exponent. If, of these terms, the function $f_n(a,\bar{a})$ has a larger absolute value than any of the others, we shall say that

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{z=a} \frac{(z-a)^{l}(\bar{z}-\bar{a})^{k} f(z,\bar{z})}{(z-a)^{l_n}(\bar{z}-\bar{a})^{k_n} f_n(z,\bar{z})} .$$

If the terms

$$f_{n_1}(a,\bar{a}), \cdots\cdots\cdots f_{n_m}(a,\bar{a}),$$

have the same maximum absolute value we shall define the residue to be

$$\operatorname*{Res}_{z=a} F(z,\bar{z}) = \operatorname*{Res}_{z=a} \frac{(z-a)^{l}(\bar{z}-\bar{a})^{k} f(z,\bar{z})}{(z-a)^{l_{n_1}}(\bar{z}-\bar{a})^{k_{n_1}} f_{n_1}(z,\bar{z})}$$

$$+ \cdots\cdots\cdots \operatorname*{Res}_{z=a} \frac{(z-a)^{l}(\bar{z}-\bar{a})^{k} f(z,\bar{z})}{(z-a)^{l_{n_m}}(\bar{z}-\bar{a})^{k_{n_m}} f_{n_m}(z,\bar{z})} .$$

This definition of the residue of (4) gives, for the function,

$$F(z,\bar{z})=\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{[\alpha(z-a)+\beta(\bar{z}-\bar{a})]^{p}},$$

if $|\alpha|>|\beta|$,

$$\operatorname*{Res}_{z=a}F(z,\bar{z})=\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{\alpha^{p}(z-a)^{p}}\quad;$$

if $|\alpha|<|\beta|$,

$$\operatorname*{Res}_{z=a}F(z,\bar{z})=\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{\beta^{p}(\bar{z}-\bar{a})^{p}}\quad;$$

if $|\alpha|=|\beta|$,

$$\operatorname*{Res}_{z=a}F(z,\bar{z})=\operatorname*{Res}_{z=a}\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{\alpha^{p}(z-a)^{p}}$$

$$+\operatorname*{Res}_{z=a}\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{\alpha^{p-1}\beta(z-a)^{p-1}(\bar{z}-\bar{a})}+\cdots+\operatorname*{Res}_{z=a}\frac{(z-a)^{\ell}(\bar{z}-\bar{a})^{k}f(z,\bar{z})}{\beta^{p}(\bar{z}-\bar{a})^{p}}.$$

10. BIBLIOGRAPHY

1. Hedrick, E. R., Non-Analytic Functions of a Complex Variable, Bulletin of the American Mathematical Society, XXXIX, No. 2, Feb. 1933, pp. 75-96.

2. Goursat, E. and Hedrick, E. R., Functions of a Complex Variable, Vol. 2, part 1, Ginn and Co., (1916), page 219.

3. Poor, V. C., Residues of Polygenic Functions, Transactions of the American Mathematical Society, Vol. 32, (1930), pp. 216-222.

4. Kasner, Edward, The Second Derivative of a Polygenic Function, Transactions of the American Mathematical Society, Vol. 30, (1928), page 809.

5. Poincare, Sur les Residus des Integrales Doubles, Acta Mathematica, Vol IX.