CHARACTERIZATIONS OF INNER PRODUCT SPACES

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

JOHN ARTHUR OMAN

1969

THESIS

This is to certify that the

thesis entitled

CHARACTERIZATIONS OF INNER
PRODUCT SPACES

presented by

John Arthur Oman

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

L M Kelly
Major professor

Date June 27, 1969

O-169

ESIS

THESIS

ABSTRACT

CHARACTERIZATIONS OF INNER PRODUCT SPACES

By

John Arthur Oman

One of the central problems in the study of metric spaces is that of deciding when a space is isometric to a well known space. The characterization of inner product spaces among normed linear spaces is an important special case of this problem. Perhaps the best known of these characterizations is the Jordan and von Neumann theorem which states that a normed linear space is an inner product space if and only if $\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + \| y \|^2)$ for all vectors x and y in the space. In elementary terms this is the assertion that a normed linear space is an inner product space if and only if the sum of the squares of the edges of each parallelogram is equal to the sum of the squares of its diagonals.

Subsequent to the publication of the Jordan and von Neumann theorem an extensive literature has appeared in which a variety of well known properties of Euclidean spaces have been shown to characterize inner product spaces. This thesis gives a historical survey and summary of this literature and continues the program.

Typical theorems proved in the thesis, the first an extension of the Jordan and von Neumann result, are the following.

THESIS

THEOREM: Let X be a normed linear space. Then X is an inner product space iff there exist a cone K with nonempty interior and $0 < \mu < 1$ such that for $x, y \in K$ there exists $0 < \lambda < 1$ so that the identity

$$\mu(1-\mu) \| \lambda x+(1-\lambda)y \|^2 + \lambda(1-\lambda) \| \mu x-(1-\mu)y \|^2 =$$
$$\lambda\mu(\lambda+\mu-2\mu\lambda) \| x \|^2 + (1-\lambda)(1-\mu)(\lambda+\mu-2\mu\lambda) \| y \|^2$$

is satisfied.

THEOREM: Let X be a real normed linear space and K be a closed, bounded, convex subset with nonempty interior. Then the following are equivalent:

1. X is an inner product space and K is a sphere
2. K has the property that for each pair of hyperplanes H_1 and H_2 supporting K at x and y respectively and each $r \in H_1 \cap H_2$ with r, x, and y linearly dependent then $\| x-r \| = \| y-r \|$.

In addition a fairly detailed study of generalizations of the inner product and orthogonality is carried out. New characterizations of those complex normed linear spaces admitting symmetric projectional orthogonality are obtained. Many of the results in the thesis are closely related to those of M. M. Day and R. C. James.

THESIS

CHARACTERIZATIONS OF INNER PRODUCT SPACES

BY

John Arthur Oman

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1969

TO

JONE

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor L. M. Kelly for his helpful suggestions and guidance during the research. He would also like to thank D. A. Senechalle for the use of his unpublished material, J. Quinn for his helpful discussions, and the quidance committee for their suggestions in the final preparation of the manuscript.

THESIS

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

TABLE OF CONTENTS

		PAGE
	LIST OF FIGURES	v
	LIST OF SYMBOLS	vi
CHAPTER I.	INTRODUCTION	1
	1. The Problem	1
	2. Basic Definitions	1
	3. History	4
CHAPTER II.	GENERALIZATIONS OF THE INNER PRODUCT	18
CHAPTER III.	SYMMETRIC ORTHOGONALITY	31
CHAPTER IV.	LOCALIZATION OF IDENTITIES	49
	1. Preliminary Results	49
	2. Topological	51
	3. Restriction	73
CHAPTER V.	CHARACTERIZATIONS OF INNER PRODUCT SPACES	77
	1. Characterizations Using Convex Sets	77
	2. A Conjecture by Hopf	85
	3. On a Theorem by Lorch	91
	4. On Ficken's Theorem	93
	5. Comments on Principle 5.1	95
	BIBLIOGRAPHY	104
	APPENDIX A	109
	APPENDIX B	112

FIGURE

3.1 . .
3.2 . .
3.3 . .
3.4 . .
4.1 . .
4.2 . .
4.3 . .
4.4 . .
4.5 . . .
4.6 . . .
4.7 . . . .
4.8 . . . .
5.1 . . . .
5.2 . . .
5.3 . . .

LIST OF FIGURES

FIGURE	PAGE
3.1	35
3.2	36
3.3	45
3.4	48
4.1	58
4.2	62
4.3	64
4.4	66
4.5	68
4.6	70
4.7	72
4.8	73
5.1	80
5.2	81
5.3	87

THESIS

LIST OF SYMBOLS

R	. . . .	real numbers
C	. . . .	complex numbers
F	. . . .	either R or C
$\overline{\alpha}$	. . . .	complex conjugate
$\mathrm{Re}\,\alpha$	. . . . . .	real part of α
$\mathrm{Im}\,\alpha$	. . . . . .	imaginary part of α
$x, y, z \ldots$	. . . .	usually vectors
$\left.\begin{matrix} a, b, c, \ldots \\ \alpha, \beta, \gamma, \ldots \end{matrix}\right\}$	. . . .	usually scalars
$S(x, \rho)$	. . . . .	ball with center x and radius ρ
$\mathrm{Lin}\{H\}$	. . . . .	linear span of H
X^*	. . . . . .	dual space of X
i.p.s.	. . . . . .	inner product space
$(\mid)$	. . . . .	inner product
$\alpha[x, y]$	. . . . .	Clarkson angle measure - page 9
$\langle x, y \rangle$	. . . . .	Sundaresan angle measure - page 9
$\vdots$ ro $\vdots$	. . . . . .	real projectional orthogonality - page 10
$\vdots$ ko $\vdots$	. . . . . .	complex projectional orthogonality
$\perp$	. . . .	either $\vdots$ ro $\vdots$ or $\vdots$ ko $\vdots$
$N(\; ; \;)$	. . . . . .	Gateaux derivative of the norm - page 13
$[\; , \;]$	. . . . . .	semi-inner-product - page 18
$M(\; , \;)$	. . . . . .	see page 22
$L(\; , \;)$	. . . . . .	see page 24

THESIS

X x Y

A(λ,μ,

C(x_1,.

INT(K,

dim X

D(X)

E-set

l_1 pla

$X \times Y$	Cartesian product of sets
$A(\lambda,\mu,r)$	see page 52
$C(x_1,\ldots,x_n)$ . .	convex cone over $x_1,\ldots,x_n$
$\mathrm{INT}(K)$	interior of K
$\dim X$	dimension of X
$D(X)$	see page 84
E-set	see page 84
l_1 plane	see page 26

THESIS

1. T

F

devote

which

system

but th

as non

spaces

a stan

system

this t

the gi

acteri

spaces

of the

2. B

T

so tha

to be

metric

pletene

Blument

INTRODUCTION

1. The Problem

From the time of Euclid much time and effort have been devoted to the study and creation of axiomatic systems which describe Euclidean geometry. Not only have these systems added to our understanding of Euclidean geometries, but they have also led us to consider such important concepts as non-Euclidean geometries, metric spaces, and topological spaces. When one considers these more general axiom systems a standard problem is that of augmenting the given axiom system to obtain one describing Euclidean geometry. In this thesis we are concerned with this augmentation when the given spaces are normed linear spaces, i.e. the characterization of inner product spaces among normed linear spaces. A more detailed description of the contributions of the thesis is found at the end of the chapter.

2. Basic Definitions

The following definitions and notations are given so that we may define more precisely some of the problems to be discussed in this paper. For the definitions of metric spaces, metric convexity, external convexity, completeness, and related concepts the reader is referred to Blumenthal [8]. Zaanen [77] , or any other standard text

on normed linear spaces may be used for the definitions and elementary properties of a normed linear space over the field F where F is either the real numbers R or the complex numbers C. When no confusion should arise the notation $\| \ \|$ is used for the norm in several spaces simultaneously.

The notation $S(x,\rho)$ in a metric space M refers to the ball with center x and radius ρ (i.e. the set of all z in M whose distance from x is at most ρ) and the term sphere with center x and radius ρ refers to the set of all z in M whose distance from x is exactly ρ. In particular, the unit sphere of a normed linear space is the sphere with center 0 and radius 1. The notation X^* is used to denote the norm dual of the normed linear space X and for $H \subset X$ the linear span of H is denoted $\mathrm{Lin}\{H\}$. A normed linear space is strictly convex if $\|x+y\| = \|x\| + \|y\|$ implies $x = \lambda y$ for some $\lambda \in R$.

Since there is some disagreement on the next terms, we state the following definitions.

<u>DEFINITION 1.1:</u> A normed linear space X over F is called an <u>inner</u> <u>product</u> <u>space</u> or an i.p.s. if there exists $(\ | \ ) : X \times X \to F$ satisfying for all $x,y,z \in X$ and $\alpha \in F$:

I-1. $(x+y|z) = (x|z) + (y|z)$

I-2. $(\alpha x|z) = \alpha(x|z)$

I-3. $(x|z) = \overline{(z|x)}$ (if $F = C$ then $\overline{\alpha}$ denotes

complex conjugation).

I-4. $(x \mid x) = \| x \|^2$

The terms pre-Hilbert spaces or generalized Euclidean spaces are also used for inner product spaces.

DEFINITION 1.2: A complete normed linear space is called a Banach space and a complete inner product space is called a Hilbert space.

DEFINITION 1.3: Two normed linear spaces X, Y are called isomorphic if there exists a map $\phi : X \longrightarrow Y$ such that :

1. ϕ is linear.
2. ϕ is one-to-one and onto.
3. There exist $m, M \in R$ such that for all $x \in X$ $m \| x \| \leq \| \phi(x) \| \leq M \| x \|$.

If $m = M = 1$ in definition 1.3 then ϕ is an isometry and we usually make no distinction between X and Y. Such a map is also called a congruence.

As mentioned the primary problem to be considered is:

PROBLEM 1.4: To find necessary and sufficient conditions for a normed linear space to be an inner product space.

This problem has many generalizations, some of which are discussed later. The following are three of these generalizations and references to them.

THESIS

PROBLEM 1.5: To find necessary and sufficient conditions for a normed linear space to be isomorphic to an inner product space. ([10], [39], [40], [48], [58], [73], [74])

PROBLEM 1.6: To find necessary and sufficient conditions for a metric space to be an inner product space. ([4], [6], [7], [19], [27], [42], [67], [76])

PROBLEM 1.7: To find necessary and sufficient conditions for a normed linear space to be (isometric to) a space of real functions whose p^{th} powers are Lesbeque integrable. ([28], [70])

3. History

The history of this problem is almost as old as the definition of an inner product space. What is probably one of the most important of all results, besides being one of the earliest, is that due to P. Jordan and J. von Neumann [37] ,

THEOREM 1.8: A normed linear space X is an i.p.s. iff
(J) $\|x\|^2 + \|y\|^2 = 1/2 (\|x+y\|^2 + \|x-y\|^2)$ $x,y \in X$.

An immediate corollary to theorem 1.8 is:

COROLLARY 1.9: A normed linear space X is an i.p.s. iff every two-dimensional subspace of X is an i.p.s.

The original proof of theorem 1.8 verifies that $(x \mid y) = 1/4 \, (\| x+y \|^2 - \| x-y \|^2)$ is an inner product. (J) is usually called the "parallelogram law" due to its interpretation in the plane as an identity between the sides of a parallelogram and its diagonals. A second geometric interpretation of (J) is to consider it as the functional relation between the length of the median ($\frac{x+y}{2}$) and the lengths of the sides of the triangle (vertices $0,x,y$).

Thus one way to generalize 1.8 is to restrict the triangles to be isosceles. Even more generally Day [18] has shown theorem 1.10.

THEOREM 1.10: A normed linear space X is an i.p.s. iff $x,y \in X$ and $\| x \| = \| y \| = 1$ imply $\| x+y \|^2 + \| x-y \|^2 \; r \; 4$ where r may be any of the relations $=$, $\geq$, $\leq$.

A second way to generalize 1.8 which was used by Senechalle [64] is to assume there is a relationship (not necessarily Euclidean) between the length of a median of a triangle and the lengths of the sides of that triangle.

THEOREM 1.11: A normed linear space X is an inner product space iff there exists $f : [0,2] \longrightarrow [0,2]$ such that $f(\| x+y \|) = \| x-y \|$ whenever $x,y \in X$ and $\| x \| = \| y \| = 1$.

The proofs of theorems 1.10 and 1.11 are quite different from that of theorem 1.8 in that they are very gometric in character. The next few paragraphs help link Euclidean

THESIS

geometry and the characterization problem.

Any n-dimensional real normed linear space has a representation as a Minkowski space. A Minkowski space is obtained by considering Euclidean n-space and determining a norm by

$$\| x \| = \inf\{\lambda \geq 0 \mid x/\lambda \in S\}$$

where S is a convex body which is symmetric about 0. The boundary of S is the unit sphere of the space. Day in [18] (also Kubota [46]) characterizes those bodies S which determine inner product spaces.

<u>THEOREM 1.12:</u> If X is a real two (three) dimensional Minkowski space then X is an inner product space iff the unit sphere of X is an ellipse (ellipsoid).

Since most characterizations of inner product spaces reduce to the two or three dimensional problem they are often very closely related to characterizations of ellipses or ellipsoids. Thus it is often possible to reformulate a characterization of inner product spaces as a characterization of ellipsoids or ellipses and conversely.

The theorem corresponding to 1.8 in metric spaces has been proven by Blumenthal [6].

<u>THEOREM 1.13:</u> Let M be a complete, convex, externally convex metric space. If $p,q,r \in M$ and $pr = qr = 1/2\, pq$ imply $2ps^2 + 2qs^2 = 4sr^2 + pq^2$ for any $s \in M$ then M is a Hilbert space. (pq denotes the distance from p to q.)

Theorems 1.14 - 1.18 may also be considered generalizations of 1.8. They postulate the existence of some norm equality or inequality to characterize inner product spaces.

<u>THEOREM 1.14:</u> (Day [19]) Let X be a normed linear space. Then X is an inner product space iff for each pair $x,y \in X$ with $\| x \| = \| y \| = 1$ there exist λ and μ such that $0 < \lambda, \mu < 1$ and $(\lambda+\mu-2\mu\lambda)(\lambda\mu+(1-\mu)(1-\lambda))\ r$ $\mu(1-\mu)\| \lambda x + (1-\lambda)y \|^2 + \lambda(1-\lambda) \| \mu x - (1-\mu)y\|^2$ where r is either $\geq$, $=$, or $\leq$.

Freese [27] and Kay [42] have partially generalized theorem 1.14 to metric spaces.

<u>THEOREM 1.15:</u> (Carlsson [13]) Let X be a normed linear space and $a_\nu \neq 0$, b_ν, c_ν, $\nu = 1, \ldots, m$ be real numbers such that (b_ν, c_ν) and (b_μ, c_μ) are linearly independent for $\nu \neq \mu$ and $\Sigma\, a_\nu b_\nu^2 = \Sigma\, a_\nu c_\nu^2 = \Sigma\, a_\nu b_\nu c_\nu = 0$. Then X is an inner product space iff $\Sigma\, a_\nu \| b_\nu x + c_\nu y \|^2 = 0 \quad x,y \in X$.

<u>THEOREM 1.16:</u> (Schoenberg [62]) A normed linear space X is an inner product space iff the Ptolemaic inequality holds. (i.e. $\| x-y\| \| z-w \| + \| x-w \| \| y-z \| \geq \| x-z \| \| y-w \|$ for $x,y,z,w \in X$.)

<u>THEOREM 1.17:</u> (Ficken [26]) A normed linear space X is an inner product space iff $\| x \| = \| y \| = 1$ implies

$\| ax+by \| = \| bx+ay \|$ for all $a,b \in R$.

THEOREM 1.18: (Lorch [50]) A normed linear space X is an inner product space iff there exists $\gamma \in R$, $\gamma \neq 0,1$ such that $\| x \| = \| y \|$ implies $\| x+\gamma y \| = \| \gamma x+y \|$.

Theorem 1.19 was given by Kakutani [38] for real normed linear spaces and extended to complex normed linear spaces by Bohenblust [9]. It is an often used theorem and is closely related to the concept of othogonality (to be discussed later) and to extension problems with linear functionals. The original proof of Kakutani is based on a characterization of ellipsoids due to Blaschke [3].

THEOREM 1.19: Let X be a normed linear space of dimension at least three. Then X is an inner product space iff for each two-dimensional subspace Y of X there exists a projection of norm 1 from X to Y.

A useful concept in plane geometry is that of an angle and its measure. Thus it is not surprising that several people have tried to extend this concept to normed linear spaces. While the angle concept carries over to any real linear space without difficulty there seems to be no unique natural measure to associate with an angle. Here are two measures which have received some study.

DEFINITION 1.20: Given $x, y \in X$, a normed linear space, the Clarkson angle measure between them is

$$a[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

DEFINITION 1.21: Let x be a nonzero vector in a normed linear space X and H be a linear subset of X. The Sundaresan angle measure between x and H is given by

$$\langle x,H\rangle = \sin^{-1} \frac{1}{\|x\|} \inf_{y \in H} \|x-y\|.$$

For a more complete discussion see Clarkson [14], Schaeffer [61], and Sundaresan [71], [72]. Both Schaeffer and Sundaresan give characterizations of inner product spaces based on these angles.

More important to us, however, will be definitions of orthogonality. Rather than trying to measure all angles we content ourselves with defining when two vectors are orthogonal. The first definition, due to Carlsson [13], contains definitions 1.23 and 1.24 as special cases.

DEFINITION 1.22: Let X be a real normed linear space and a_γ, b_γ, c_γ, $\gamma = 1, \ldots, m$ be a fixed collection of real numbers satisfying $\Sigma a_\gamma b_\gamma^2 = \Sigma a_\gamma c_\gamma^2 = 0$ and $\Sigma a_\gamma b_\gamma c_\gamma = 1$. Two vectors $x, y \in X$ are said to be orthogonal iff $\Sigma a_\gamma \| b_\gamma x + c_\gamma y \|^2 = 0$.

DEFINITION 1.23: The special case, $m=3$ and $1 = -a_1 = a_2 = a_3 = b_1 = b_2 = -c_1 = c_3$ and $b_3 = c_2 = 0$

THESIS

(i.e. $\| x \|^2 + \| y \|^2 = \| x-y \|^2$) is known as Pythagorean orthogonality.

DEFINITION 1.24: The special case $m = 2$, $2a_1 = -2a_2 = b_1 = b_2 = c_1 = -c_2 = 1$ (i.e. $\| x+y \| = \| x-y \|$) is called isosceles orthogonality.

A definition of orthogonality originally given by Birkhoff [2] and studied extensively by James [33] is related to definition 1.21. This type of orthogonality is studied in greater detail.

DEFINITION 1.25: Let X be a real (complex) normed linear space and $x,y \in X$. Then x is real (complex) projectional orthogonal to y, denoted $x : ro : y$ ($x : ko : y$) iff $\| x \| \leq \| x-ay \|$ for $a \in R$ ($a \in C$). (When it is clear whether X is real or complex the notation $x \perp y$ is used.)

It should be noted that in an inner product space all of the above definitions coincide with the usual definition, i.e. x orthogonal $y \iff (x|y) = 0$. Definitions 1.26-1.29 are properties of orthogonality in an inner product space which may be postulated for any of the above orthogonalities.

DEFINITION 1.26: Orthogonality is said to be left (right) additive if x orthogonal to y and z orthogonal to y (x orthogonal to z) imply $x+z$ orthogonal to y (x orthogonal to $y+z$).

DEFINITION 1.27: Orthogonality is said to be left (right) unique if for $x, y \in X$ there exists a unique $a \in F$ such that $x+ay$ is orthogonal to y (y is orthogonal to $x+ay$).

DEFINITION 1.28: Orthogonality is left (right) homogeneous if x orthogonal to y implies λx orthogonal to y (x orthogonal to λy) for $\lambda \in F$.

DEFINITION 1.29: Orthogonality is symmetric if x orthogonal to y implies y orthogonal to x.

Results based on definitions 1.22-1.24 and 1.26-1.29 may be found in [13], [16], [18], [35], [53], [54], [68]. Most of these are special cases of theorem 1.30 or theorem 1.31.

THEOREM 1.30: (Carlsson [13]) A real normed linear space X is an inner product space iff x orthogonal to y (i.e. $\Sigma\, a_v \,\| b_v x + c_v y \|^2 = 0$) implies

$$\lim_{n\to\infty} 1/n\, \Sigma\, a_v \,\| n b_v x + c_v y \|^2 = 0.$$

THEOREM 1.31: If any one type of orthogonality implies another then X is an inner product space.

Theorems 1.32 and 1.33 combine many of the results on projectional orthogonality which were proven by James [34] in the real case and von den Steinen [68] in the complex case.

THESIS

THEOREM 1.32:

1. Projectional orthogonality is homogeneous.
2. Given $x \in X$ there exists a closed hyperplane H such that $x \perp H$.
3. Given $x,y \in X$ there exist $a,b \in F$ such that $x+ay \perp y$ and $x \perp bx+y$.
4. $x \perp y$ iff there exists a continuous linear functional f such that $f(x) = \|f\| \|x\|$ and $f(y) = 0$.
5. If $x \perp y$ then x and y are linearly independent.

THEOREM 1.33: If X is a normed linear space of at least dimension 3 then the following are equivalent:

1. The norm of X is induced by an inner product.
2. For every closed subspace $S \neq 0$ of X there exists a projection P of norm 1 whose image is S.
3. For every closed hyperplane H of X there exists a projection P of norm 1 whose image is H.
4. For every closed hyperplane H of X there exists an element $x \in X$ with $x \neq 0$ and $H \perp x$.
5. The relation $\perp$ is left unique and for every $x \in X$ there exists a closed hyperplane H with $H \perp x$.
6. The relation $\perp$ is left additive.
7. The relation $\perp$ is symmetric.

Of particular interest in theorem 1.33 is condition 7. While conditions 2 - 6 are rather easily seen to be insufficient to characterize two-dimensional inner product spaces, the fact that 7 also does not characterize two-dimensional

inner product spaces is not as immediately clear. The spaces in which orthogonality is symmetric do form a rather nice class which has been characterized by Day [18]. These spaces are examined in more detail in chapter 3.

Definition 1.34 was given by James [34] and is a powerful tool in the study of orthogonality in normed linear spaces.

<u>DEFINITION 1.34:</u> Let X be a real normed linear space and $x,y \in X$. $N_{\pm}(x;y) = \pm \lim_{n\to\pm\infty} \|nx+y\| - \|nx\|$. If $N_{+}(x;y) = N_{-}(x;y)$ for all y then the norm of X is said to be <u>Gateaux differentiable</u> at x and $N(x;y) = N_{+}(x;y) = N_{-}(x;y)$.

<u>THEOREM 1.35:</u> Let X be a real normed linear space, $x,y,z \in X$ and $t \in R$.

1. $N_{+}(x;y+z) \leq N_{+}(x;y) + N_{+}(x;z)$.
2. $N_{+}(x;ty) \leq tN_{+}(x;y) \quad t \geq 0$
3. $N_{+}(x;x) = \|x\|$
4. $|N_{+}(x;y)| \leq \|y\|$
5. $N_{+}(x;y) = -N_{-}(x;-y)$
6. $x \perp ax+y$ iff $N_{-}(x;y) \leq -a\|x\| \leq N_{+}(x;y)$

<u>THEOREM 1.36:</u> If X is a real normed linear space and the norm of X is Gateaux differentiable (i.e. Gateaux differentiable at each point $x \in X$) then $N(x;\ )$ is a bounded linear functional on X.

THESIS

Now we are able to give still another generalization of theorem 1.8.

THEOREM 1.37: (Hopf [32]) A real normed linear space X is an i.p.s. iff there exists $f : R \times R \longrightarrow R$ such that $x \perp y$ implies $f(\| x \|, \| y \|) = \| x+y \|$.

The remainder of this section is not as important to the paper but is given for completeness. Included is a collection of interesting results based on approaches different from those previously mentioned.

Theorem 1.38 (Klee [44]) and theorem 1.39 (Comfort and Gordon [15]) are similar in that they depend rather explicitly on the "geometrical" properties of unit spheres.

THEOREM 1.38: For a normed linear space X the following assertions are equivalent:

1. X is an inner product space or is two-dimensional.
2. Whenever $\epsilon > 0$ and K is a convex subset of S the unit sphere of X, then S contains a translate of K whose distance from the origin is $< \epsilon$.

THEOREM 1.39: Let X be a real normed linear space of dimension at least three. The following are equivalent:

1. X is an inner product space.
2. For each three points $x_1, x_2, x_3 \in X$ and ρ_1, ρ_2, ρ_3 positive numbers with $\bigcap S(x_i, \rho_i) \neq \phi$ it follows that

$\cap S(x_1, \rho_1) \cap F \neq \phi$ where $F = \{x_1 + x \mid x = a(x_2 - x_1) + b(x_3 - x_1),\ \ a, b \in R\}$.

The geometrical significance of condition 2 is that for three spheres with nonempty intersection D the plane of centers intersects D.

The Hahn-Banach theorem is probably one of the best known theorems in functional analysis. Theorem 1.40 (Kakutani [38]) shows that a strengthened Hahn-Banach theorem characterizes inner product spaces.

THEOREM 1.40: Let X be a normed linear space. Then X is an inner product space iff for each closed linear subspace Y there exists a linear map $F: Y^* \dashrightarrow X^*$ (the dual spaces of Y and X) such that for $f \in Y^*$ then $F(f)$ is a norm preserving extension of f.

In a similar vein several people have studied the possibilities of extending contractions (Schonbeck [63]), isometries (Edelstein and Thompson [24]), or bilinear forms (Hayden [31]).

THEOREM 1.41: (Edelstein and Thompson) Let X and Y be real normed linear spaces, X be strictly convex, and $\dim X \geq 2$. Then X and Y are inner product spaces iff for $D \subset X$ then each isometry $f : D \dashrightarrow Y$ can be extended to an isometry $F : X \dashrightarrow Y$.

THESIS

be

All

Dv

pr

TH

is

the

used

with

to t

"gen

norm

prod

to t

side

symme

a non

New c

real

orthog

sional

orthogo

Many characterizations of inner product spaces may be regarded as expressing the "homogeneity" of the space. All generalizations of theorem 1.8 are of this type. Dvoretzky [22], Gromov [30], and Senechalle [65] have expressed homogeneity in terms of the "sameness" of subspaces.

THEOREM 1.42: (Senechalle) A real normed linear space X is an i.p.s. iff all two-dimensional subspaces are isometric.

In this section an effort has been made to relate the history of the problem and state background results used in the remaining chapters. We conclude the chapter with a few remarks outlining the contributions of this thesis to the general program.

Lumer [51] defined semi-inner-products which are "generalized inner products" that can be defined on arbitrary normed linear spaces. Another type of "generalized inner product" based on projectional orthogonality and its relation to the characterization of inner product spaces is considered in chapter 2.

In chapter 3 we discuss the two-dimensional spaces with symmetric projectional orthogonality (see theorem 1.33) and a non constructive characterization of such spaces is obtained. New constructions (see Day [18] and Busemann [12]) for all real two-dimensional normed linear spaces with symmetric real orthogonality and a class of examples of complex two-dimensional normed linear spaces with symmetric complex projectional orthogonality are given. These seem to be the first explicit

examples of non inner product spaces with symmetric complex projectional orthogonality.

In chapter 4 we consider the general problem of characterizing inner product spaces by assuming that various of the known characterizing identities (theorems 1.8, 1.10, and 1.14) hold only on restricted subsets of the vectors of the space. One of the interesting new results of chapter 4 is that the Jordan and von Neumann identity is still characterizing if it is postulated only on the vectors of a single cone. This answers and extends a conjecture of my colleague J. Quinn. This and the other localizations given in the chapter do not appear to have received previous study.

The metatheorem enunciated by Lorch and others to the effect that any Euclidean metric property adjoined to the axioms of a real normed linear space is enough to force the space to be Euclidean (inner product) is considered in the final chapter. While all characterizations are essentially of this type, in the last section of chapter 5 we are especially interested in this metatheorem. Among other results in the chapter we offer new proofs for the theorem of Ficken, answer a conjecture of Hopf (theorem 5.7) and settle a question raised by L. M. Kelly (theorem 5.2). With respect to methodology as well as the result itself this last theorem is one of the most interesting in the thesis.

GENERALIZATIONS OF THE INNER PRODUCT

In this chapter we determine a weaker set of axioms than I-1,..., I-4 for an inner product space and examine functions in arbitrary normed linear spaces which have some of the properties of inner products.

Lumer [51] defined the concept of a semi-inner-product in an arbitrary normed linear space. First Lumer and then Giles [29] used the semi-inner-product to reproduce the functional analysis of inner product spaces.

DEFINITION 2.1: Let X be a real normed linear space. Then $[\ ,\ ] : X \times X \to R$ is a semi-inner-product (s.i.p.) if for all $x, y, z \in X$ and $\lambda \in R$ it satisfies

S-1. $[x+y, z] = [x, z] + [y, z]$.

S-2. $[\lambda x, y] = \lambda [x, y]$

S-3. $[x, x] = \| x \|^2$

S-4. $| [x, y] |^2 \leq [x, x][y, y]$.

The following construction due to Giles puts a semi-inner-product on any normed linear space. If $x \in X$ and $\| x \| = 1$ by the Hahn-Banach theorem there exists $f_x \in X^*$ such that $\| f_x \| = f_x(x) = 1$. For each x on the unit sphere choose such an f_x and define $[y, x] = f_x(y)$. Now extend homogeneously to all vectors x. Note that this s.i.p. has the additional property that $[x, \lambda y] = \lambda [x, y]$.

Given a real normed linear space we are interested in functions $\phi : X \times X \longrightarrow R$ which satisfy some or all of the following properties.

G-1. $\phi(x+z,y) = \phi(x,y) + \phi(z,y) \qquad x,y,z \in X$

G-2. $\phi(x+y,y) = \phi(x,y) + \phi(y,y) \qquad x,y \in X$

G-3. $\phi(x,y+z) = \phi(x,y) + \phi(x,z) \qquad x,y,z \in X$

G-4. $\phi(x,x+y) = \phi(x,x) + \phi(x,y) \qquad x,y \in X$

G-5. $\phi(\lambda x,y) = \lambda\, \phi(x,y) \qquad \lambda \in R, \quad x,y \in X$

G-6. $\phi(x,\lambda y) = \lambda\, \phi(x,y) \qquad \lambda \in R, \quad x,y \in X$

G-7. $\phi(x,ty) = \phi(tx,y) = t\, \phi(x,y) \qquad t \geq 0, \quad x,y \in X$

G-8. $\phi(-x,y) = \phi(x,-y) = -\phi(x,y) \qquad x,y \in X$

G-9. $\phi(x,y) = \phi(y,x) \qquad x,y \in X$

G-10. $\phi(x,y) = 0 \Rightarrow \phi(y,x) = 0 \qquad x,y \in X$

G-11. $|\, \phi(x,y)\, | \leq \|x\|\,\|y\| \qquad x,y \in X$

G-12. There exists $k > 0$ such that

$$|\, \phi(x,y)\, | \leq k\, \|x\|\,\|y\| \qquad x,y \in X$$

G-13. $\phi(x,x) = \|x\|^2 \qquad x \in X$

If ϕ satisfies G-1, G-5, G-9, G-13 then it is an inner product and if ϕ satisfies G-1, G-5, G-11, G-13 then ϕ is a semi-inner-product. Theorem 2.2 gives a weakened set of axioms for an inner product space.

<u>THEOREM 2.2:</u> Suppose X is a normed linear space and there exists $\phi : X \times X \longrightarrow R$ satisfying G-2, G-4, G-8, and G-13. Then X is an inner product space.

<u>Proof:</u>

Using G-2

$$\phi(y,x+y) = \phi(-x,x+y) + \phi(x+y,x+y)$$

and applying G-4 and G-8

$$\phi(y,x) + \phi(y,y) = -\phi(x,x)-\phi(x,y) + \phi(x+y,x+y).$$

Likewise

$$\phi(x,x)-\phi(x,y) = \phi(x-y,x-y) + \phi(y,x)-\phi(y,y)$$

Adding, we obtain

$$\phi(x+y,x+y) + \phi(x-y,x-y) = 2(\phi(x,x) + \phi(y,y)).$$

Which by G-13 implies

$$\| x+y \|^2 + \| x-y \|^2 = 2(\| x \|^2 + \| y \|^2).$$

Hence by theorem 1.8 X is an i.p.s.

q.e.d.

We remark that ϕ need not be the inner product on the space since it need not be symmetric. The inner product is given by $(x \mid y) = 1/2\ (\phi(x,y) + \phi(y,x))$, however. Consider, for example, a complex inner product space with inner product $(\mid)$. Let $\phi(x,y) = \mathrm{Re}(x \mid y) + \mathrm{Im}\ (x \mid y)$. Then ϕ satisfies the hypothesis of theorem 2.2 but is not a real inner product on the space.

<u>COROLLARY 2.3:</u> Let X be a real normed linear space and $\phi : X \times X \longrightarrow R$ satisfy G-2, G-8, G-9, and G-13. Then ϕ is an inner product on X.

Next we look at the relationship of G-1, ..., G-13 and that of orthogonality (unless otherwise specified orthogonality will mean projectional orthogonality in the remainder of the paper).

THESIS

<u>THEOREM 2.4:</u> Let X be a real normed linear space and $\phi : X \times X \to R$ satisfy G-2, G-5, G-11, G-13. Then for $x,y \in X$ and $y \neq 0$, $y \perp x - \frac{\phi(x,y)}{\| y \|^2} y$.

<u>Proof:</u>

For $b \neq 0$ then

$$\begin{aligned}\phi(ax+by,y) &= b\,\phi(\tfrac{a}{b}x + y,y)\\ &= b\,\phi(\tfrac{a}{b}x,\ y) + b\,\phi(y,y)\\ &= a\,\phi(x,y) + b\,\phi(y,y).\end{aligned}$$

Thus $\phi(\ ,y)$ is a linear functional of norm $\| y \|$ on Lin $\{x,y\}$ (the linear span of x and y). By the Hahn-Banach theorem it has a norm preserving extension to X. But $\phi(x - \frac{\phi(x,y)}{\| y \|^2} y,y) = 0$ so by theorem 1.32

$y \perp x - \frac{\phi(x,y)}{\| y \|^2} y$.

q.e.d.

<u>COROLLARY 2.5:</u> If X is a real normed linear space and ϕ is a semi-inner-product then $y \perp x - \frac{\phi(x,y)}{\| y \|^2} y$. If X is Gateaux differentiable then $y \perp x - \alpha_0 y$ iff $\alpha_0 = \frac{\phi(x,y)}{\| y \|^2}$ and $\phi(x,y) = \| x \| N(x;y)$ for all $x,y \in X$.

Corollary 2.5 is an extension of theorem 2 of Giles.

A natural question to ask at this point concerns the possibility of defining a generalized inner product, ϕ, such that $x - \alpha_0 y \perp y$ iff $\alpha_0 = \frac{\phi(x,y)}{\| y \|^2}$. Since orthogonality

is not in general symmetric such a ϕ will in general not satisfy G-2, G-5, G-11, and G-13. However, we can make ϕ satisfy G-2, G-5, G-12, and G-13. The remainder of this chapter deals with the definition of such a ϕ and examines some of its properties. With this end in mind we begin with the following definitions.

DEFINITION 2.6: Let X be a normed linear space and $x, y \in X$.

$$M(x,y) = \left\{ \alpha \in F \mid \| x-\alpha y \| \leq \| x-\beta y \| \quad \beta \in F \right\}$$

Theorems 2.7 and 2.8 give the existence and basic properties of $M(x,y)$.

THEOREM 2.7: $M(x,y)$ exists for all $x, y \in X$ and for $y \neq 0$, $M(x,y)$ is compact and convex.

Proof:

For any $x, y \in X$ the function $f : F \longrightarrow R$ given by $f(\alpha) = \| x-\alpha y \|$ is a continuous convex function. If $y = 0$ then $M(x,y) = F$ and hence exists. If $y \neq 0$ then for $| \beta | > 2 \dfrac{\| x \|}{\| y \|}$ we have

$$\| x-\beta y \| \geq | \, \| x \| - | \beta | \, \| y \| \, | > \| x \|.$$

By the continuity of f it attains its minimum on $| \beta | \leq 2 \dfrac{\| x \|}{\| y \|}$ and hence on all of F. Moreover, this shows the compactness of $M(x,y)$. The convexity follows from the triangle inequality.

q.e.d.

THESIS

<u>THEOREM 2.8:</u> 1. $M(tx,y) = t\,M(x,y)$ $\qquad t \neq 0$

2. $M(x,y) = t\,M(x,ty)$ $\qquad t \neq 0$

3. $M(ax+by,y) = a\,M(x,y) + b$ $\qquad a,b \in F$

<u>Proof:</u>

1. Let $\alpha \in M(x,y)$. Then

$$\| x - \alpha y \| \leq \| x - \beta y \| \qquad \beta \in F,$$
$$\| tx - (\alpha t) y \| \leq \| tx - (\beta t) y \| \qquad \beta \in F,$$
$$\| tx - (\alpha t) y \| \leq \| tx - \beta y \| \qquad \beta \in F.$$

Thus $\alpha t \in M(tx,y)$ or $t\,M(x,y) \subset M(tx,y)$. But $M(tx,y) = t\,t^{-1}\,M(tx,y) \subset t\,M(t^{-1}tx,y) = t\,M(x,y)$ and hence $t\,M(x,y) = M(tx,y)$.

2. Let $\alpha \in M(x,y)$. Then

$$\| x - \alpha y \| \leq \| x - \beta y \| \qquad \beta \in F,$$
$$\| x - (\alpha t^{-1})(ty) \| \leq \| x - (\beta t^{-1})(ty) \| \qquad \beta \in F,$$
$$\| x - (\alpha t^{-1})(ty) \| \leq \| x - \beta (ty) \| \qquad \beta \in F.$$

Thus $t^{-1}M(x,y) \subset M(x,ty)$. But $M(x,ty) = t^{-1}tM(x,ty) \subset t^{-1}M(x,tt^{-1}y) = t^{-1}M(x,y)$ and hence $t^{-1}M(x,y) = M(x,ty)$.

3. Note that $M(y,y) = 1$ for $y \neq 0$. Hence if $a = 0$, $M(ax+by,y) = aM(x,y)+b$. If $a \neq 0$ and $\alpha \in M(ax+by,y)$ then

$$\| ax+by-\alpha y \| \leq \| ax+by-\beta y \| \qquad \beta \in F,$$
$$\| x - \frac{(\alpha-b)}{a} y \| \leq \| x - \frac{(\beta-b)}{a} y \| \qquad \beta \in F,$$
$$\| x - \frac{(\alpha-b)}{a} y \| \leq \| x - \beta y \| \qquad \beta \in F.$$

Hence $M(ax+by,y) \subset a\,M(x,y) + b$.

If $\alpha \in a\,M(x,y) + b$ then

$$\| x - \frac{(\alpha-b)}{a} y \| \leq \| x - \beta y \| \qquad \beta \in F,$$
$$\| x - \frac{(\alpha-b)}{a} y \| \leq \| x - \frac{(\beta-b)}{a} y \| \qquad \beta \in F,$$
$$\| ax+by-\alpha y \| \leq \| ax+by-\beta y \| \qquad \beta \in F.$$

THESIS

Hence $M(ax+by,y) = a\,M(x,y) + b$.

q.e.d.

If X is strictly convex then $M(x,y)$ reduces to a point for every $x,y \in X$. In general, however, it may be larger. In any case we define some candidates for the desired finction.

<u>DEFINITION 2.9:</u> Let X be a normed linear space and $x,y \in X$. $L(x,y) = a\|y\|^2$ where $a \in M(x,y)$ and $|a| = \inf\{|\alpha| \mid \alpha \in M(x,y)\}$.

If X is real then define

$$L_{\pm}(x,y) = \begin{cases} a\|y\|^2 & y \neq 0 \\ 0 & y = 0 \end{cases} \quad \text{where } a = \mp\inf\{\mp\alpha \mid \alpha \in M(x,y)\}.$$

If X is a complex normed linear space then there exist $L_C(x,y)$ and $L_R(x,y)$ where X is considered first as a complex space and secondly as a real space. $L_C(x,y)$ is a complex function and $L_R(x,y)$ is a real function. See appendix A.

With the above theorems and remarks in mind we can now state the basic properties of $L(\ ,\ )$ and $L_{\pm}(\ ,\ )$.

<u>THEOREM 2.10:</u>

1. $L(x,x) = \|x\|^2$
2. $x \perp y$ iff $L(x,y) = 0$
3. $L(tx,y) = L(x,ty) = t\,L(x,y)$
4. $L_{\pm}(tx,y) = L_{\pm}(x,ty) = \begin{cases} t\,L_{\pm}(x,y) & t \geq 0 \\ t\,L_{\mp}(x,y) & t < 0 \end{cases}$
5. $L_{\pm}(x+y,y) = L_{\pm}(x,y) + L_{\pm}(y,y)$
6. $x : ro : y$ iff $L_{-}(x,y) \leq 0 \leq L_{+}(x,y)$

7. If X is strictly convex then $L(\, , \,)$ satisfies G-2, G-5, G-6, G-12, G-13.

8. If X is an i.p.s. then $L(x,y) = L_{\pm}(x,y) = (x \mid y)$.

Suppose X is strictly convex and $\phi : X \times X \longrightarrow R$ satisfies G-2, G-5, G-13 and $\phi(x,y) = 0 \Rightarrow x : ro : y$ then $\phi(x,y) = L(x,y)$. To see this simply note $\phi(x - \frac{\phi(x,y)}{\| y \|^2} y, y) = 0$ which implies $L(x - \frac{\phi(x,y)}{\| y \|^2} y, y) = 0$ so $L(x,y) = \phi(x,y)$ by part 7 of theorem 2.10. Hence $L(\, , \,)$ has the desired property that $L(x,y) = 0$ iff $x \perp y$ and if X is strictly convex it is the only possible function.

An immediate corollary of theorem 2.10, theorem 2.2. and corollary 2.3 is the following.

<u>COROLLARY 2.11:</u> If X is strictly convex the following are equivalent:

1. X is an i.p.s.
2. $L_R(y,x) = L_R(x,y)$.
3. $L(\, , \,)$ satisfies G-4.

It should be noted that it would be sufficient to assume $L_R(y,x) = L_R(x,y)$ for $\| x \| = \| y \| = 1$ since the homogeneity would imply $L_R(y,x) = L_R(x,y)$ for all $x,y \in X$. Theorem 1.33 allows us to state another corollary to 2.10.

COROLLARY 2.12: If X is a normed linear space of dimension at least three then the following are equivalent:

1. X is an i.p.s.
2. $L(\ ,\ )$ satisfies G-1.
3. $L(\ ,\ )$ satisfies G-9.
4. $L(\ ,\ )$ satisfies G-10.

When X is strictly convex, corollary 2.12 is a special case of the theorem by Rudin and Smith [60] .

We continue to look at the properties of $L(x,y)$. From theorem 2.8 it follows that $|\ L_{\pm}(x,y)\ | \leq 2\ \|x\|\ \|y\|$. In general this is the best possible bound since it is attained in the l_1 plane (the l_1 plane is the Minkowski plane with the norm $\|(x,y)\| = |x| + |y|$). By rounding the sides on the l_1 unit sphere, a strictly convex normed linear space can be obtained where the bound is arbitrarily close to 2. The following does provide a stronger theorem than 2.8, however.

THEOREM 2.13: $|\ L_{\pm}(x,y)\ |\ |\ L_{\pm}(y,x)\ | \leq \|x\|^2\ \|y\|^2$

Proof:

Let $a \in M(x,y)$ and $b \in M(x,y)$. If $M(x,y) = M(y,x) = 0$ the result is trivial. If not we may assume $a,b \neq 0$. Then

$$\|x-ay\| = |a|\ \|y-a^{-1}x\| \geq |a|\ \|y-bx\|$$
$$= |ab|\ \|x-b^{-1}y\| \geq |ab|\ \|x-ay\|.$$

Hence $1 \geq |ab|$ and the theorem is proven (if $\|x-ay\| = 0$ then $x = ay$ and $L_{\pm}(x,y) = L_{\pm}(y,x) = \|x\|\ \|y\|$).

q.e.d.

We now use $L_{\pm}(x,y)$ to characterize symmetric orthogonality.

THEOREM 2.14: If X is a real normed linear space then $| L_{\pm}(x,y) | \leq \| x \| \| y \|$ for all $x,y \in X$ iff X has symmetric orthogonality.

Proof:

Suppose $| L_{\pm}(x,y) | \leq \| x \| \| y \|$ for all $x,y \in X$. Let $x,y \in X$ such that $x \perp y$ and $\| x \| = \| y \| = 1$. Then $1 \in M(ax+y,y)$ for all a since $0 \in M(x,y)$. Hence $\| y \| = 1 \leq \max | L_{\pm}(ax+y,y) | \leq \| ax+y \| \| y \| = \| ax+y \|$.

Suppose X has symmetric orthogonality. Let $\alpha \in M(x,y)$. If $\alpha = 0$ then $|\alpha| \leq \| x \| \| y \|$. If $\alpha \neq 0$ then

$$\| x-\alpha y \| \leq \| x-\beta y \| \quad \text{for all} \quad \beta \in F$$

and by assumption

$$\| y \| \leq \| y-\beta(x-\alpha y) \| \quad \text{for all} \quad \beta \in F.$$

Let $\beta = -\alpha^{-1}$. Then

$$\| y \| \leq \| y+\alpha^{-1}(x-\alpha y) \| = \frac{\| x \|}{| \alpha |}$$

$$|\alpha| \| y \|^2 \leq \| x \| \| y \| .$$

q.e.d.

From the definitions of $N_{\pm}(x;y)$ and $L_{\pm}(x,y)$ and by theorem 1.35 we may conclude the following.

THEOREM 2.15: Let X be a real normed linear space. Then $L_+(x,y) = b \| y \|^2$ and $L_-(x,y) = a \| y \|^2$ iff $N_+(x-by;y) \geq 0 \geq N_-(x-ay;y)$ and $N_+(x-\alpha y;y) = N_-(x-\alpha y;y) = 0$ for $\alpha \in (a,b)$.

<u>THEOREM 2.16:</u> If X is a real normed linear space then X has symmetric orthogonality iff $L_+(x,y) = \|y\| N_+(y;x)$ and $L_-(x,y) = \|y\| N_-(y;x)$.

While we are characterizing spaces consider the following.

<u>THEOREM 2.17:</u> Let X be a real normed linear space. Then X is strictly convex iff $L(x+y,y) = L(x,y) + L(y,y)$ for all $x,y \in X$.

<u>Proof:</u>

If X is strictly convex then $L(x+y,y) = L(x,y) + L(y,y)$ by theorem 2.10.

Suppose $L(x+y,y) = L(x,y) + L(y,y)$ for all $x,y \in X$. Let $x,y \in X$ and $y \neq 0$. Then $x - \frac{L_-(x,y)}{\|y\|^2}\, y \perp y$ and $x - \frac{L_+(x,y)}{\|y\|^2}\, y \perp y$.

Hence $0 = L\left(x - \frac{L_-(x,y)}{\|y\|^2}\, y, y\right) = L(x,y) - L\left(\frac{L_-(x,y)}{\|y\|^2}\, y, y\right) = L(x,y) - L_-(x,y)$.

$0 = L\left(x - \frac{L_+(x,y)}{\|y\|^2}\, y, y\right) = L(x,y) - L_+(x,y)$.

q.e.d.

If X is a complex normed linear space it is natural to wonder if there is a relation between $L_R(x,y)$ and $L_C(x,y)$. Theorem 2.18 answers this question as well as characterizing inner product spaces. See appendix A.

THESIS

THEOREM 2.18: If X is a strictly convex complex normed linear space the following are equivalent:

1. X is an i.p.s.
2. $L_R(x,y) = \text{Re } L_c(x,y)$ for all $x,y \in X$.
3. $L_R(x,y) = \text{Re } L_c(y,x)$ for all $x,y \in X$.

Proof:

Clearly 1 implies 2 and 1 implies 3 are trivial, for given a complex inner product $(x|y)$ then $\text{Re } (x|y)$ is a real inner product.

Also, if the complex dimension of X is one, then X is an i.p.s. in any case.

Now assume the complex dimension of X is two, so that the real dimension of X is four. Since $L_c(x,y)$ is complex additive which implies $L_R(x,y)$ is real additive in the first argument (second argument) since $L_R(x,y) = \text{Re } L_c(x,y)$ $(L_R(x,y) = \text{Re } L_c(y,x))$. By corollary 2.11 X is a real i.p.s. and hence a complex i.p.s.

If the complex dim $X > 2$ then every two-dimensional subspace is an i.p.s. so X is an i.p.s.

q.e.d.

The final result in this section is a new proof of theorem 1.17.

THEOREM: Let X be a normed linear space. Then X is an i.p.s. iff $\| x \| = \| y \|$ implies $\| x+\alpha y \| = \| \alpha x+y \|$ for all $\alpha \in R$.

THESIS

Proof:

The necessity of the second condition is obvious. In proving the sufficiency we may assume X is a real space since if it is complex then it is a complex i.p.s. iff it is a real i.p.s. Also we will not repeat the section of Ficken's argument which proves X is strictly convex. Using corollary 2.11 and the remark following it will suffice to show $L(x,y) = L(y,x)$ whenever $\| x \| = \| y \| = 1$.

Suppose $\| x \| = \| y \| = 1$ and $L(x,y) = a$. Then

$\| x-ay \| \leq \| x-\alpha y \|$ for all $\alpha \in R$.

But $\| x-ay \| = \| y-ax \|$ and $\| x-\alpha y \| = \| y-\alpha x \|$. Hence $\| x-ay \| \leq \| x-\alpha y \|$ for all $\alpha \in R$ or $L(x,y) = L(y,x)$.

q.e.d.

In some instances the function $L(\, , \,)$, like the semi-inner-product or the Gateaux derivative, can be used in an arbitrary normed linear space in much the same way as the inner product is used in inner product spaces. There are several results in later chapters which are based on $L(\, , \,)$.

As we have mentioned (theorem 1.33) there exist two-dimensional normed linear spaces in which orthogonality is symmetric and which are not inner product spaces. (Recall orthogonality means projectional orthogonality unless otherwise specified.) Because the class of spaces with symmetric orthogonality is so closely related to the class of inner product spaces and since it provides counterexamples to various conjectures we examine the class more carefully. Also many of the results in this chapter are used in the later chapters. Day [18] has given a construction from which all real two-dimensional normed linear spaces with symmetric orthogonality can be obtained. We give two similar constructions for all real two-dimensional normed linear spaces with symmetric orthogonality and in addition examine the possibility of constructing complex two-dimensional normed linear spaces with symmetric orthogonality. The chapter also includes several characterizations of spaces with symmetric orthogonality.

We begin by examining the geometric significance of orthogonality. In theorem 1.32 we found that $x \perp y$ iff there exists a continuous linear functional f such that $f(x) = \| f \| \| x \|$ and $f(y) = 0$. If $\| x \| = 1$ this would usually be stated geometrically that y was in a supporting hyperplane to the unit sphere at x. Thus if X is a real two-dimensional normed linear space and $x, y \in X$ satisfy

1. $\| x \| = \| y \| = 1$

2. $x \perp y$ and $y \perp x$

then there is a support line to the unit sphere at x parallel to y and a support line at y parallel to x. In terms of Minkowski spaces the problem of determining two-dimensional spaces with symmetric orthogonality reduces to the problem of determining centrally symmetric convex curves (unit spheres) for which diameters and support lines have this special relationship. Such curves are called Radon curves [57].

Using this geometric significance of orthogonality Day has shown the following.

<u>THEOREM 3.1:</u> Given any real two-dimensional space X there exist vectors $x, y \in X$ such that $x \perp y$ and $y \perp x$.

We now give a construction which produces all real two-dimensional normed linear spaces with symmetric orthogonality. By theorem 3.1 given any real two-dimensional normed linear space we can find $x, y \in X$ such that $\| x \| = \| y \| = 1$, $x \perp y$, and $y \perp x$. Following the notation of Day we call $\{ax+by \mid a,b \geq 0\}$ the first quadrant and similarly name the other three quadrants with the corresponding restrictions on a and b. Day has shown that X can be renormed in such a way that the two norms agree in the first quadrant and under this new norm X has symmetric orthogonality. He does this by constructing the second

quadrant of the new unit sphere. Moreover, if X had symmetric orthogonality originally the two norms agree. Thus we have constructed all two-dimensional real spaces with symmetric orthogonality.

The first construction which we give is that by Day except in more analytic terms.

CONSTRUCTION I:

Following Day we let X be any real two-dimensional normed linear space and $x, y \in X$ satisfy $\| x \| = \| y \| = 1$, $x \perp y$, and $y \perp x$. It is easy to show that for $0 \leq a < 1$ there exists a unique $f(a) \geq 0$ such that $\| ax+f(a)y \| = 1$. Moreover, f is continuous, convex, and satisfies $0 < f(a) \leq 1$, $f(0) = 1$. It is well known from the theory of such functions that at each point f has left and right derivatives (denoted $D_-^f(a)$ and $D_+^f(a)$ respectively) and that f is differentiable except for a countable set. Also f has a Schwartz derivative $f''(a) = \lim_{h \to 0} \dfrac{f(a+h) + f(a-h) - 2f(a)}{h^2}$ almost everywhere.

Let $z(a,m) = (am-f(a))^{-1}$ for $D_-^f(a) \geq m \geq D_+^f(a)$ and $0 \leq a < 1$. We claim that if X is renormed so that the unit sphere in the second quadrant has the form $\{ z(a,m)x + mz(a,m)y \mid 0 \leq a < 1,\ D_-^f(a) \geq m \geq D_+^f(a) \}$ then X has symmetric orthogonality with respect to the new norm. By Day's construction and our method of construction it suffices to calculate the slope of a support line to this curve at $z(a,m)x + mz(a,m)y$.

First we determine when $a \to z(a,m)$ is one-to-one. If $f'(a)$ does not exist then $a \to z(a,m)$ is one to many. Now suppose $a < b$ and $z(a,m_1) = z(b,m_2)$. Then $0 \geq m_1 \geq \frac{f(b) - f(a)}{b-a} \geq m_2$ from which it follows $z(a,m_1) = z(b,m_2)$ iff $m_1 = m_2$. Thus we can divide our calculations into several cases determined from these.

1. Suppose f', f'' both exist at "b" and $f''(b) \neq 0$. Then $m = f'(b)$ and the slope M of the support line at $(z(b,f'(b))x+(f'(b)z(b,f'(b))y$ is given by

$$M = \lim_{a \to b} \frac{mz(a,m) - f'(b)z(b,f'(b))}{z(a,m) - z(b,f'(b))}$$

$$= \lim_{a \to b} \frac{mf'(b) - f'(b)\frac{f(a) - f(b)}{a-b} + f(b)\frac{f'(b) - m}{b-a}}{-\frac{f(a) - f(b)}{a-b} + m + b\frac{f'(b) - m}{b-a}}$$

$$= \frac{(f'(b))^2 - (f'(b))^2 + f''(b)f(b)}{f'(b) - f'(b) + f''(b)b} = \frac{f(b)}{b}$$

which is what we desired.

2. Suppose f' does not exist at b and $D_-^f(b) > m > D_+^f(b)$. Then the slope M of the support line at $(z(b,m))x + (mz(b,m))y$ is again given by

$$M = \lim_{m_1 \to m} \frac{mz(b,m) - m_1z(b,m_1)}{z(b,m) - z(b,m_1)} = \lim_{m_1 \to m} \frac{(m-m_1)f(b)}{(m-m_1)b} = \frac{f(b)}{b}.$$

3. The remaining cases follow by continuity and are the points where there are non-unique support lines in the second quadrant (i.e. the points where $a \to z(a,m)$ is many to one). Thus flat spots on the unit sphere in the first quadrant correspond to corners on it in the second quadrant and corners in the first quadrant correspond to

THESIS

flat spots in the second quadrant. This completes the first construction.

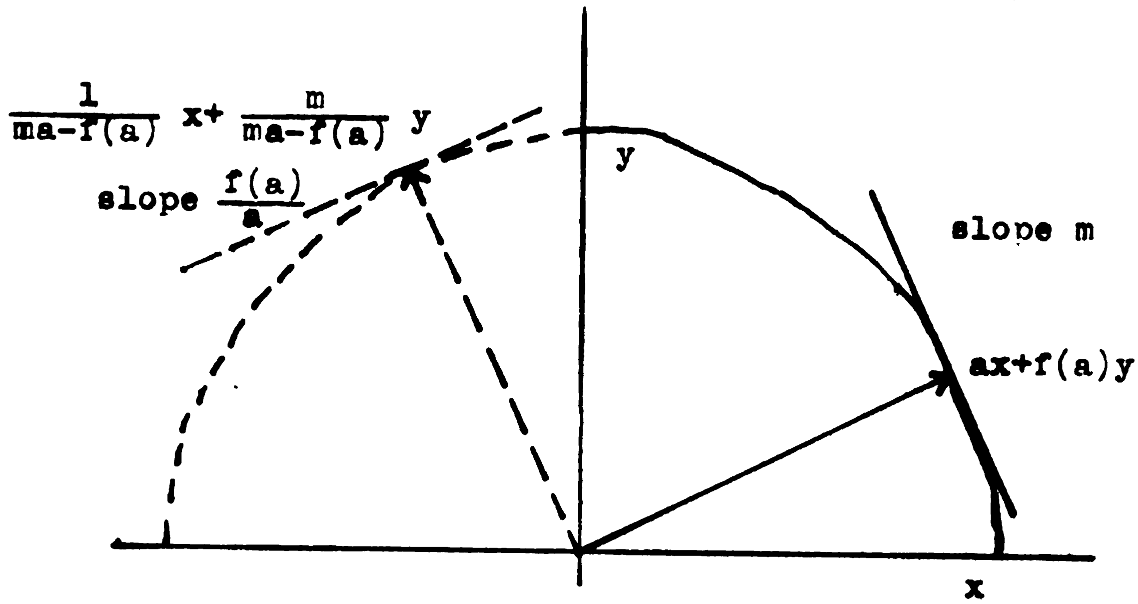

FIGURE 3.1

CONSTRUCTION II:

Construction I suggests this second construction which can be at least partially generalized to complex spaces. Again let X be any real two-dimensional normed linear space and $x,y \in X$ such that $\| x \| = \| y \| = 1$, $x \perp y$ and $y \perp x$. Then $h : R \longrightarrow R$ defined by $h(a) = \| x+ay \|$ is a continuous convex function which attains its minimum at $a = 0$. Again h restricted to the positive real numbers will determine the first quadrant of the unit sphere. This time we determine the second quadrant of the unit sphere of a new norm by assuming h is defined for only the positive reals and showing how to define it for the negative reals.

THESIS

Again h has left and right derivatives everywhere and is differentiable almost everywhere. Let $Z(a,m) = a - \frac{h(a)}{m}$ for $a \geq 0$ and $D_-^h(a) \leq m \leq D_+^h(a)$. Since h satisfies

1. $h(a) \geq 1, |a|$

2. $|h(b) - h(a)| \leq |b-a|$

we have $Z(a,m) \leq 0$ and $\frac{1}{m} \geq 1$. Let $\alpha = \lim_{a \to \infty} Z(a,m)$. Now we can extend h to the negative reals.

$$h(Z(a,m)) = \frac{1}{m} \quad \text{for } a \geq 0 \quad D_-^h(a) \leq m \leq D_+^h(a)$$

$$h(b) = 1 \qquad \text{if } \alpha \leq b \leq 0$$

To check the validity of this construction we shall reduce it to the first construction. If f is defined as in construction I then

$$f(1/h\ (a)) = a/h\ (a) \quad \text{for } a \geq 0$$

$$D_+^f(a) = Z(a, D_+^h\ (a)).$$

Thus $h(Z(a,m)) = \frac{1}{m}$ is equivalent to

$$\left\| \left(\frac{Z(a,m)}{h(a)} - \frac{a}{h(a)}\right)^{-1} x + (Z(a,m)) \left(\frac{Z(a,m)}{h(a)} - \frac{a}{h(a)}\right)^{-1} y \right\| = 1$$

which is construction I. Thus construction II is verified.

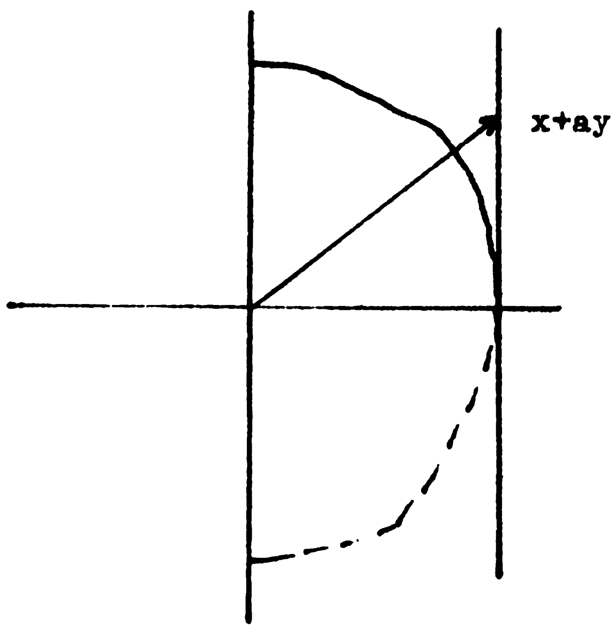

FIGURE 3.2

THESIS

Before we can attempt such a construction for complex spaces, however, we need some more results. Accordingly we obtain some characterizations of real and complex spaces with symmetric orthogonality. Lemmas 3.2 and 3.3 will give us the tools necessary to prove theorem 3.4 which is our main theorem. In the real case theorems 3.4 and 3.5 follow from Day's construction or construction I, but our proofs are independent of these constructions and equally valid for complex spaces.

<u>LEMMA 3.2:</u> Let X be a complex normed linear space and $x,y \in X$. Then $x : ro : y$ iff there exists a real number b such that $x : ko : bix+y$.

<u>Proof:</u>

Suppose there exists a real number b such that $x :$ $ko : bix+y$. Then there exists a complex continuous linear functional f such that $\| f \| = 1$, $f(x) = \| x \|$ and $f(bix+y) = 0$, (theorem 1.32) Then $\operatorname{Re} f$ is a real continuous linear functional such that $\| \operatorname{Re} f \| = 1$, $\operatorname{Re} f(x) = \| x \|$ and $\operatorname{Re} f(y) = 0$. Hence $x : ro : y$. (See appendix A)

Suppose $x : ro : y$. Then there exists a continuous real linear functional g such that $\| g \| = 1$, $g(x) = \| x \|$, and $g(y) = 0$. Define $f(z) = g(z) - ig(iz)$. Then f is a continuous complex linear functional such that $\| f \| = 1$ and $f(x) = \| x \|$. Let $b = -\dfrac{g(iy)}{\| x \|}$. Then $f(bix+y) = 0$ so $x : ko : bix+y$.

q.e.d.

THESIS

<u>LEMMA 3.3:</u> Let X and Y be normed linear spaces and $\phi : X \to Y$ ($\phi \neq 0$) be a linear map such that for $x_1, x_2 \in X$ and $x_1 \perp x_2$ then $\phi(x_1) \perp \phi(x_2)$. Then ϕ is continuous and $\phi/\| \phi \|$ is an isometry from X to $\phi(X)$.

<u>Proof:</u>

Suppose X and Y are complex spaces and $\perp$ denotes complex orthogonality. Then $x : ro : y \iff x : ko : bix+y \implies \phi(x) : ko : bi\phi(x)+\phi(y) \iff \phi(x) : ro : \phi(y)$ by lemma 3.2 for $x, y \in X$. Hence we may assume X and Y are real and $\perp$ denotes real orthogonality.

Next we assume $\dim X = 2$. By theorem 1.32 $\dim \phi(x) = 2$. Thus there exist $m, M > 0$ and $x_3, x_4 \in X$ such that $\| x_3 \| = \| x_4 \| = 1$, $m = \| \phi(x_3) \|$, $M = \| \phi(x_4) \|$ and $m \| x \| \leq \| \phi(x) \| \leq M \| x \|$ for all $x \in X$. Now choose $x_5, x_6 \in X$ such that $\| x_5 \| = \| x_6 \| = 1$, $x_3 \perp x_5$, and $x_4 \perp x_6$. Then

$$\frac{\| \phi(x_3) \|}{\| \phi(x_5) \|} \leq 1 \leq \frac{\| \phi(x_4) \|}{\| \phi(x_6) \|} .$$

By continuity there exist $x, z \in X$ such that $\| x \| = \| z \| = 1$, $x \perp z$ and $\| \phi(x) \| = \| \phi(z) \|$. Let $k = \| \phi(x) \|$.

Since $x \perp z$ for $-1 < a < +1$ there must exist a unique non-negative number $f(a)$ such that $\| ax+f(a)z \| = 1$. Likewise, since $\phi(x) \perp \phi(z)$ for $-1 < a < 1$ there exists a unique non-negative number $g(a)$ such that $\| a\phi(x) + g(a)\phi(z) \| = k$. Also there exists a dense set D of $-1 < a < +1$ such that both $f'(a)$ and $g'(a)$ exist.

For $a \in D$ the following must hold:

$ax + f(a)z \perp x + cz$ iff $f'(a) = c$

and $a\phi(x) + g(a)\phi(z) \perp \phi(x) + d\phi(z)$ iff $g'(a) = d$.

But $ax+f(a)z \perp x + cz$ implies $a\phi(x)+f(a)\phi(z) \perp \phi(x)+c\phi(z)$.

If $h(a) = k/\| a\phi(x) + f(a)\phi(z) \|$ for $a \in D$ then $g'(ah(a)) = f'(a)$ and $g(ah(a)) = h(a)f(a)$ whenever $-1< a\, h(a) < +1$ (for a near 0, $|a\, h(a)| < 1$ so these are nontrivial conditions). Since $1/h$ is convex, h' will exist almost everywhere so we can differentiate the second equation and

$$g'(a\, h(a))\, (h(a) + a\, h'(a)) = f'(a)h(a) + f(a)h'(a).$$

Combining this with the first equation

$$a\, f'(a)\, h'(a) = f(a)\, h'(a).$$

If h' does not vanish identically then

$$a\, f'(a) = f(a)$$

which has the solution $f(a) = ca$ for some constant c. But $f(0) = 1$ so clearly this is a contradiction. Hence $h'(a)$ must vanish identically or $h(a)$ is a constant. Thus $\| a\, \phi(x) + f(a)\, \phi(z) \| = k$ for $-1 < a < 1$ since $h(0) = 1$. This is sufficient to show $\phi/\| \phi \|$ is an isometry.

If $\dim X > 2$ let $x \in X$ be fixed and let $z \in X$. Then the above applies to $\mathrm{Lin}\{x,z\}$ so there exists a constant k_z such that $\| \phi(w) \| = k_z\| w \|$ for $w \in \mathrm{Lin}\{x,z\}$. However, $\| \phi(x) \| / \| x \|$ is fixed so k_z is constant over the whole space.

q.e.d.

<u>THEOREM 3.4:</u> Let X be a two-dimensional normed linear space. Then X has symmetric orthogonality iff there exists a linear isometry $\phi : X \longrightarrow X^*$ such that $[\phi(x)]\,(x) = 0$ for all $x \in X$.

THESIS

Proof:

First suppose X has symmetric orthogonality. Let $x, y \in X$, $\| x \| = \| y \| = 1$ and $x \perp y$. Define $\phi(ax+by)$ by $[\phi(ax+by)]\,(cx+dy) = ad-bc$. Then ϕ is a linear map such that $[\phi(ax+by)]\,(ax+by) = 0$. Suppose $ax+by \perp cx+dy$. Then there exists a linear functional f such that $f(cx+dy) = 0$ and $f(ax+by) = \| f \| \, \| ax+by \|$. But up to scalar multiples there is only one linear function such that $f(cx+dy) = 0$ so $| [\phi(cx+dy)]\,(ax+by) | = \| \phi(cx+dy) \| \, \| ax+by \|$. But $cx+dy \perp ax+by$ by assumption. So $| [\phi(ax+by)]\,(cx+dy) | = \| \phi(ax+by) \| \, \| cx+dy \|$. Now we identify X with X^{**} so that the above relations now imply $\phi(ax+by) \perp \phi(cx+dy)$ and $\phi(cx+dy) \perp \phi(ax+by)$. Then by lemma 3.2 there exists a constant k such that

$$\| \phi(ax+by) \| = k \, \| ax+by \| .$$

But $| [\phi(x)]\,(ax+by) | = |b| \leq \| ax+by \|$ so $\| \phi(x) \| \leq 1$. However $| [\phi(x)]\,(y) | = 1 = \| y \|$ so $\| \phi(x) \| = 1$ and $k = 1$.

Now suppose there exists an isometry $\phi : X \longrightarrow X^{*}$ such that $[\phi(x)]\,(x) = 0$ for all $x \in X$. Suppose $x \perp y$. Because X is two-dimensional it follows

$| [\phi(y)]\,(x) | = \| \phi(y) \| \, \| x \| = \| y \| \, \| x \|$. But

$$[\phi(x+y)]\,(x+y) = 0$$

so $[\phi(x)]\,(y) + [\phi(y)]\,(x) = 0$.

Hence $| [\phi(x)]\,(y) | = | [\phi(y)]\,(x) | = \| y \| \, \| x \|$ so $y \perp x$.

q.e.d.

THESIS

The fact that X is two-dimensional in theorem 3.4 is very important. Suppose X is an inner product space and of odd dimension greater than or equal to three. Then there can exist no continuous linear map $\phi : X \longrightarrow X^*$ such that $[\phi(x)](x) = 0$ since even dimensional spheres admit no continuous tangent fields. Thus the existence of ϕ is not, in general, necessary. Next consider a two-dimensional complex normed linear space with symmetric complex orthogonality. Then the existence of ϕ is given by theorem 3.4 but if X is not an inner product space, real orthogonality is not symmetric. Hence the existence of ϕ is not sufficient in general.

From theorem 3.4 we can prove the following characterizations:

<u>THEOREM 3.5:</u> If X is a normed linear space then the following are equivalent:

1. X has symmetric orthogonality.

2. If $x, y \in X$, $\| x \| = \| y \| = 1$ and $x \perp y$ then $ax + by \perp cx + dy$ iff $|ad - bc| = \| ax + by \| \| cx + dy \|$.

3. If $x, y \in X$, $\| x \| = \| y \| = 1$, $x+ay \perp y$ and $y+bx \perp x$ then $\| x + ay \| = \| y + bx \|$.

<u>Proof:</u>

$(1 \Rightarrow 2)$

If $\dim X > 2$ then X is an inner product space and this is an easy calculation. If $\dim X = 2$ then in proof of theorem 3.4 we proved

$ax+by \perp cx+dy$ iff $|ad-bc| = \| \phi(ax+by) \| \| cx+dy \|$ iff $|ad-bc| = \| ax+by \| \| cx+dy \|$ since ϕ is an isometry.

(2=>3)

Let x,y satisfy the hypothesis of 3. Then

$$x = \| x+ay \| \frac{x+ay}{\| x+ay \|} - ay$$

$$y+bx = b \| x+ay \| \frac{x+ay}{\| x+ay \|} + (1-ab)y.$$

Since $\frac{x+ay}{\| x+ay \|}$ and y satisfy the hypothesis of 2 we have

$\| y+bx \| = \| x \| \| y+bx \| =$

$| \| x+ay \| (1-ab) + ab \| x+ay \| | = \| x+ay \|.$

(3=>1)

Suppose $\| x \| = \| y \| = 1$, $x \perp y$ and $y+bx \perp x$. Then $\| x \| = \| y+bx \| = 1 = \| y \|$. Hence $y \perp x$.

q.e.d.

As usual we like to interpret our theorem geometrically and part 3 of theorem 3.5 may be expressed as the equality of the altitudes to the equal sides of an isosceles triangle.

The next theorem diverges from the goals of this section but we include it now because we have the tools to handle it and it includes concepts from both chaoters 2 and 3.

<u>THEOREM 3.6:</u> Let X be a real normed linear space. Then $L_+(x,y) = 0$ implies $L_+(y,x) = 0$ iff X has symmetric orthogonality and X is strictly convex.

<u>Proof:</u>

If X has symmetric orthogonality and is strictly

THESIS

convex then if $x \perp y$ we have $L(x,y) = L(y,x) = 0$ and $L_+(x,y) = L(x,y)$ and $L_+(y,x) = L(y,x)$.

Suppose $L_+(x,y) = 0$ implies $L_+(y,x) = 0$. Suppose $x \perp z$. Then

$$L_+(x - L_+(x,z)z, z) = 0 \Rightarrow L_+(z,\ x - L_+(x,z)z) = 0$$

$$\begin{aligned} L_-(x - L_-(x,z)z, z) = 0 &\Rightarrow L_+(x - L_-(x,z)z, -z) = 0 \\ &\Rightarrow L_+(-z, x - L_-(x,z)z) = 0 \\ &\Rightarrow L_-(z, x - L_-(x,z)z) = 0 \end{aligned}$$

But this implies $z \perp x$ by theorem 1.35. Thus X has symmetric orthogonality. If $\dim X > 2$ then X is an inner product space and hence is strictly convex. If $\dim X = 2$, then if X is not strictly convex we note we can find $x,y \in X$ such that $\| x \| = \| y \| = 1$, $x \perp y$, $L_+(y,x) > 0$ but $L_+(x,y) = L_-(x,y) = L(x,y) = 0$ (this is obvious from construction I). Hence $L_+(x,y) = 0$ but $L_+(y,x) \neq 0$ which is a contradiction.

q.e.d.

Now we are ready to look at the two-dimensional complex normed linear spaces with symmetric orthogonality. We generalize construction II to complex spaces. This procedure is helpful in characterising these spaces and constructing specific examples but does not construct all two-dimensional complex normed linear spaces with symmetric orthogonality as construction II.

CONSTRUCTION III:

Let X be a two-dimensional complex normed linear

space with symmetric orthogonality. Let $x, y \in X$, $\| x \| = \| y \| = 1$, and $x \perp y$. Define $f : R \times R \longrightarrow R$ by $f(a,b) = \| x + (a+bi)y \|$. Then by theorem 3.5 $x+(a+bi)y \perp x+(c+di)y$ iff $|(a-c) + (b-d)i| = f(a,b)\, f(c.d)$.

Now consider (a,b) fixed. Then the function $h(p,q) = ((a-p)^2 + (b-q)^2)^{1/2} / f(p,q)$ has its absolute maximum at (c,d) in which case $h(c,d) = f(a,b)$. Suppose f is differentiable at (c,d). Let $D = ((a-c)^2 + (b-d)^2)^{1/2}$. Then $0 = h_1(c,d) = (a-c)(Df(c,d))^{-1} - Df_1(c,d)(f(c,d))^{-2}$ and $0 = h_2(c,d) = (b-d)(Df(c,d))^{-1} - Df_2(c,d)(f(c,d))^{-2}$.

Simplifying and solving

$$f^2(a,b) = (a-c)(f(c,d)f_1(c,d))^{-1} = (b-d)(f(c,d)f_2(c,d))^{-1}$$

Equating the last terms, resubstituting and simplifying we obtain

$$(3.7) \qquad f^2(a,b) = (f_1^2\,(c,d) + f_2^2\,(c,d))^{-1}$$

Now solving for a and b

$$(3.8) \qquad \begin{aligned} a &= c + (f(c,d)f_1(c,d))(f_1^2\,(c,d) + f_2^2\,(c,d))^{-1} \\ b &= d + (f(c,d)f_2\,(c,d))(f_1^2\,(c,d) + f_2^2\,(c,d))^{-1} \end{aligned}$$

Naturally these equations are symmetric in (a,b) and (c,d). They are also the two-dimensional analogues to the equation of definition in construction II. Equations 3.7 and 3.8 hold whenever f has partials at (c,d) which will be almost everywhere in the plane so we can pick up the rest of the plane by continuity.

We might be tempted to try to construct all complex spaces with symmetric orthogonality by assuming f is defined on the upper half of the plane and extending the

equations 3.7 and 3.8. It is possible, however, that 3.7 and 3.8 either do not extend f to all the plane or extend f in two different manners for some points in general. The reason for this is that (a,b) and (c,d) need not lie on a line through the origin.

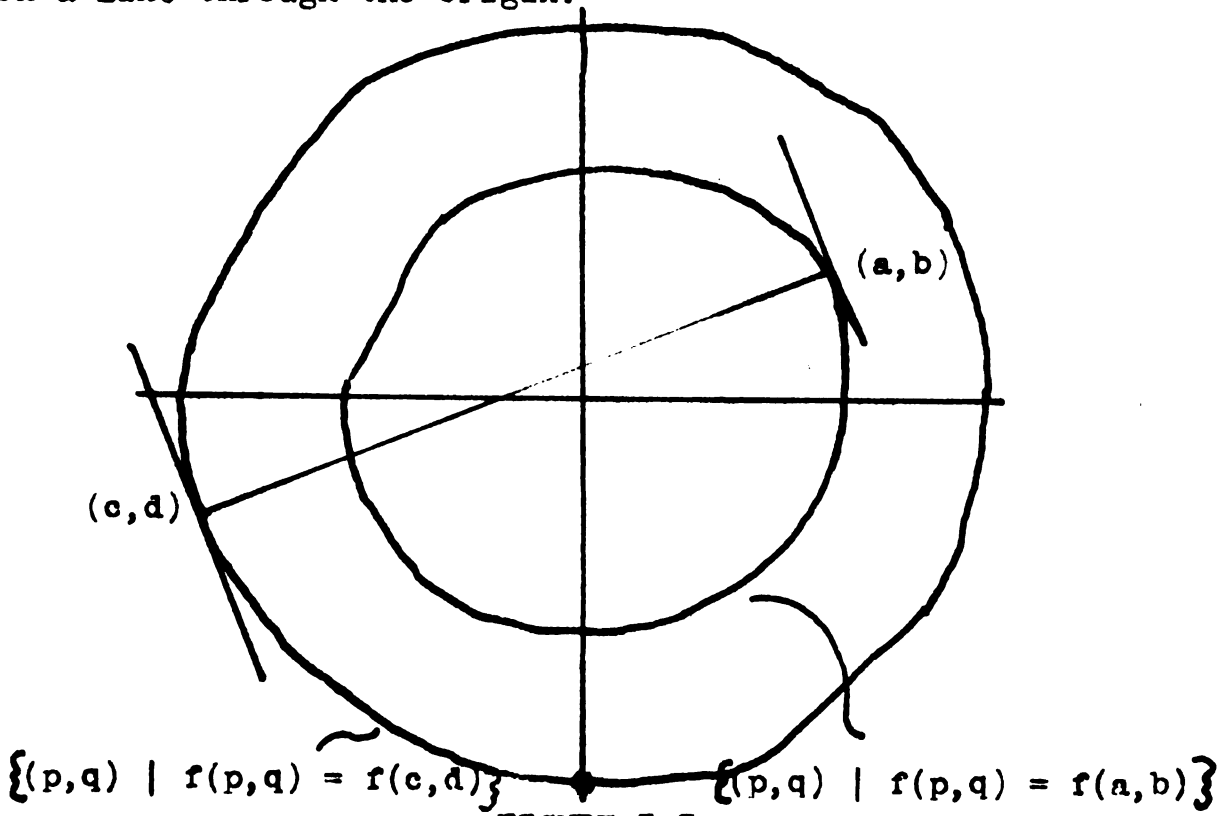

FIGURE 3.3

CONSTRUCTION IV:

We now use theorem 3.4 to generalize a construction by Thorp [15] to obtain spaces congruent to their duals. By doing so we are able to obtain examples of two-dimensional complex spaces with symmetric orthogonality and hence have examples of the functions described in construction III.

Let $\| \ \|$ be a norm on R^2 such that $(R^2, \| \ \|)$ has symmetric orthogonality, $\|(0,1)\| = \|(1,0)\| = 1$, $(0,1) \perp (1,0)$, and $\|(1,a)\| = \|(1,-a)\|$. (Such spaces

do exist. For example consider the Minkowski plane whose unit sphere is given by

$\{(r,s) \mid r^p + s^p = 1 \text{ if } rs \geq 0 \text{ and } r^{p/p-1} + s^{p/p-1} = 1 \text{ if } rs \leq 0\}$ where p is any integer greater than 2. Let $x = (2^{-1/p}, 2^{-1/p})$ and $y = (2^{1/p-1}, 2^{1/p-1})$ and use x, y as the coordinate axes.)

Let X, Y be any two normed linear spaces (where both are either real or complex) and define the norm on $X \times Y$ by $\|(x,y)\| = \|(\|x\|, \|y\|)\|$ for $x \in X$ and $y \in Y$. The only nontrivial condition to check, to see that this defines a norm, is the triangle inequality.

$$\begin{aligned}
\|(x_1+x_2, y_1+y_2)\| &= \|(\|x_1+x_2\|, \|y_1+y_2\|)\| \\
&\leq \|(\|x_1\| + \|x_2\|, \|y_1+y_2\|)\| \\
&\leq \|(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)\| \\
&\leq \|(\|x_1\|, \|y_1\|)\| + \|(\|x_2\|, \|y_2\|)\| \\
&= \|(x_1, y_1)\| + \|(x_2, y_2)\|.
\end{aligned}$$

The above inequalities follow from the triangle inequalities in X and Y and the fact that $(1,0) \perp (0,1)$ and $(0,1) \perp (1,0)$. We also notice X, Y are embedded isometrically as closed subspaces in $X \times Y$.

Now suppose X is any space such that X and X^{**} are congruent by a congruence $\phi : X \to X^{**}$. Let $X = X$ and $Y = X^*$. Let $Z = X \times X^*$ with the above norm. If $h \in X^*$ and $x \in X$ then the linear functional $(h, \phi(x))$ defined by $[(h,\phi(x))](y,g) = h(y) + [\phi(x)](g)$ belongs to Z^*. Moreover, all elements of Z^* are of this form. To determine the norm $|[(-h, \phi(x))](y,g)| = |-h(y) + [\phi(x)](g)|$ $y \in X$, $g \in X^*$

$$\leq |h(y)| + |[\phi(x)](g)|$$
$$\leq \|h\| \|y\| + \|x\| \|g\|$$
$$\leq \|(\|x\|, \|h\|)\| \ \|(\|y\|, -\|g\|)\|$$
$$= \|(\|x\|, \|h\|)\| \ \|(\|y\|, \|g\|)\|$$
$$= \|(x,-h)\| \ \|(y,g)\| .$$

Thus $\|(-h, \phi(x))\| \leq \|(x,h)\|$. Now choose $\|y\|$, $\|g\|$ so that $(\|y\|, -\|g\|) \perp (\|x\|, \|h\|)$ and $y_1 \in X$, $g_1 \in X^*$ so that $\|y_1\| = \|y\|$, $\|g_1\| = \|g\|$, $-h(y_1) \to \|h\| \|y\|$, and $[\phi(x)](g_1) \to \|g\| \|x\|$. Thus $\|(-h, \phi(x))\| = \|(x,h)\|$ since $|[(-h,\phi(x))](y,g)| \to \|(x,-h)\| \|(y,g)\|$. Hence the mapping $\psi : Z \to Z^*$ given by $\psi(x,h) = (-h,\phi(x))$ is a congruence. Moreover, if X is reflexive and ϕ is the canonical isomorphism then the map ψ also maps a vector onto an annihilator of itself.

Next suppose X and Y are any linear spaces such that there exist congruences $\phi_1 : X \to X^*$ and $\phi_2 : Y \to Y^*$. Let $Z = X \times Y$ then again $Z^* = X^* \times Y^*$. Exactly as above the map $\psi : Z \to Z^*$ given by $\psi(x,y) = (\phi_1(x),\phi_2(y))$ is a congruence. Also if ϕ_1,ϕ_2 map each vector onto an annihilator of itself ψ will also.

By using combinations of the above results we may obtain spaces of any dimension congruent to their duals. We have not characterized such spaces since we have not even constructed l_1 for instance.

CONSTRUCTION V:

As a particular example let $X = C$ (i.e. the one dimensional complex space). Then $X \times X^*$ with the above norm is a two-dimensional complex space with symmetric orthogonality by theorem 3.4. This example is a particularly simple case of construction III since $f(\alpha,\beta) = f(|\alpha|,|\beta|)$ and all real two-dimensional subspaces generated by $(0,\beta),(\alpha,0)$ are isometric to $(R^2, \| \quad \|)$.

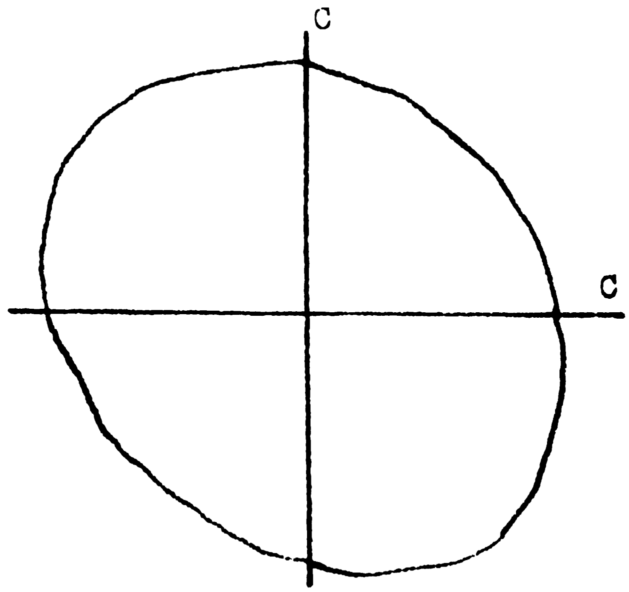

FIGURE 3.4

THESIS

1. Preliminary Results

Corollary 1.9 is an extremely important result because it allows us to reduce many characterizations to two-dimensional problems. For some of the theorems in this chapter we need stronger forms of 1.9 in order to make this reduction, however.

LEMMA 4.1: Let X and Y be normed linear spaces and suppose the norm on $Z = X \times Y$ is given by $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2$. Then Z is an inner product space iff both X and Y are inner product spaces.

Proof:

If Z is an inner product space then X and Y are embedded as subspaces of Z and hence are inner product spaces.

If X and Y are inner product spaces let $\phi_X(\ ,\ )$ and $\phi_Y(\ ,\ )$ be the inner products on X and Y, respectively. Define

$$((x_1,y_1)|(x_2,y_2)) = \phi_X(x_1,x_2) + \phi_Y(y_1,y_2).$$

It is easy to check this is an inner product on Z.

q.e.d.

LEMMA 4.2: Let X be a normed linear space. Then X is an inner product space iff there exist a hyperplane H and a vector x not in H such that H is an inner product

THESIS

space and every two-dimensional subspace of X which contains x is an inner product space.

Proof:

If X is an inner product space then H can be any hypersubspace H and any vector x not in H will do.

Suppose H and x exist. Let H' be any closed hypersubspace such that $x \perp H'$ (theorem 1.32). Then $X = \text{Lin}\{x\} \times H'$ where the norm is given by $\| (rx,h') \|^2 = \| rx \|^2 + \| h' \|^2$. By lemma 4.1 it will suffice to prove H' is an inner product space. If $h', g' \in H'$ there exist $r,s \in F$ such that $rx+h'$ and $sx+g'$ belong to H. Since the parallelogram law holds in H

$$\| (r+s)x + (h'+g') \|^2 + \| (r-s)x + (h'-g') \|^2 =$$
$$2 \| rx+h' \|^2 + 2 \| sx+g' \|^2$$

or

$$| r+s |^2 \| x \|^2 + \| h'+g' \|^2 + | r-s |^2 \| x \|^2 + \| h'-g' \|^2 =$$
$$2|r|^2 \| x \|^2 + 2\| h' \|^2 + 2|s|^2 \| x \|^2 + 2 \| g' \|^2.$$

Thus

$$\| h'+g' \|^2 + \| h'-g' \|^2 = 2 \| h' \|^2 + 2 \| g' \|^2 ,$$

and the parallelogram law also holds in H'. By theorem 1.8 H' is an inner product space.

q.e.d.

Lemma 4.2 is especially useful in three dimensions since hypersubspaces are two-dimensional. It may also be used as an inductive step for proving other generalizations of 1.9.

LEMMA 4.3: Let X be a normed linear space. Then X is an inner product space iff there exists a basis $\{y, x_\alpha\}$ for X such that every two-dimensional subspace of X which contains an x_α is an inner product space.

Proof:

If X is an inner product space any basis will do. Suppose such a basis exists. If the dimension of X is n then we will proceed by induction on n. If n is 1 or 2 the theorem is trivial. Suppose the theorem is true for $n = 1, 2, \ldots, m$. Let $n = m + 1$ and $y, x_1, \ldots, x_m$ be such a basis. Then $H = \text{Lin}\{y, x_1, \ldots, x_{m-1}\}$ is a hyper-subspace of X. By the induction hypothesis H is an inner product space. By lemma 4.2 X is an inner product space.

If the dimension of X is not finite then let H be a two-dimensional subspace of X. Since $\{y, x_\alpha\}$ is a basis there exists a positive integer N such that $H \subset \text{Lin}\{y, x_{\alpha_1}, \ldots, x_{\alpha_N}\}$ which by the above is an inner product space. Thus by 1.9 X is an inner product space.

q.e.d.

2. Topological

Now we look at a class of problems originally suggested by Dr. Charles MacCluer and Mr. Joseph Quinn. The main idea is to generalize the theorems from chapter one by assuming the hypothesis holds only locally. Locally here usually has a more topological meaning (i.e. the vectors

HESIS

are in some sense close) than a geometric meaning (i.e. the vectors form some specialized configuration). This appears to be a somewhat different approach than most which have appeared in the literature.

<u>DEFINITION 4.4</u>: Let X be a normed linear space and $(x,y) \in X \times X$. Let $0<\lambda,\mu<1$ and r be one of the relations $\geq$, $=$, or $\leq$. The pair (x,y) belongs to the class $A(\lambda,\mu,r)$ if and only if

$$\mu(1-\mu)\| \lambda x+(1-\lambda)y \|^2+\lambda(1-\lambda)\| \mu x-(1-\mu)y \|^2 \ r$$
$$\lambda\mu(\mu+\lambda-2\mu\lambda)\| x \|^2+(1-\lambda)(1-\mu)(\mu+\lambda-2\mu\lambda)\| y \|^2.$$

With this definition we may state the three conjectures with which most of this chapter is concerned.

<u>CONJECTURE 4.5</u>: Let X be a normed linear space. Then X is an inner product space iff there is a relation r and a set $K \subset X$ with the property that for $x,y \in K$ there exist $0<\lambda,\mu<1$ such that $(x,y) \in A(\lambda,\mu,r)$.

<u>CONJECTURE 4.6</u>: Let X be a normed linear space. Then X is an inner product space iff there is a relation r and $\epsilon > 0$ with the property that for all $x,y \in X$ satisfying $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$ then there exist $0<\lambda,\mu<1$ such that $(x,y) \in A(\lambda,\mu,r)$.

<u>CONJECTURE 4.7</u>: Let X be a normed linear space. Then X is an inner product space iff there is a relation r and

$\epsilon > 0$ with the property that for all $x,y \in X$ satisfying $\alpha[x,y] < \epsilon$ (i.e. $\| x/\|x\| - y/\|y\| \| < \epsilon$) then there exist $0<\lambda,\mu<1$ such that $(x,y) \in A(\lambda,\mu,r)$.

Actually all three of the conjectures as stated are false and what we are really interested in are the additional restrictions necessary in order to obtain true theorems. We prove some results, however, before giving counterexamples to the conjectures.

It should also be evident that some restrictions on K must be made in conjecture 4.5 if there is to be any hope of a result. Not only must K at least span X but it must in some sense contain at least one vector in every direction. One easy condition to place on K is that it have non-empty interior, but this is not the only one possible.

Before we begin the theorems we need another definition.

DEFINITION 4.8: A subset C of a linear space X is a cone iff for $x \in C$ and $a \geq 0$ then $ax \in C$. If C is convex then it is a convex cone.

If $x_1,\ldots,x_n \in X$ then $C(x_1,\ldots,x_n) = \{\Sigma a_i x_i \mid a_i \geq 0\}$ is a convex cone and if dimension of X is n and the x_i are independent then $C(x_1,\ldots,x_n)$ has nonempty interior.

The next theorem is a generalization of 1.14 and a special case of 4.5.

<u>THEOREM 4.9</u>: Let X be a normed linear space. Then X is an inner product space iff there exist a relation r and a subset $K \subset X$ with the properties that for each $x \in X$ there exists $\alpha \in R$ such that $\alpha x \in K$ and for $x,y \in K$ there exist $0<\lambda,\mu<1$ such that $(x,y) \in A(\lambda,\mu,r)$.

<u>Proof</u>:

The necessity of the last condition is obvious. This same statement holds for all of the theorems in this chapter and most of those in the next. Thus we will usually show only the sufficiency of the last condition.

Thus suppose K exists. By corollary 1.9 we may assume X is real and two-dimensional. First we prove the result when r is $\leq$. Let E be the minimal (in area) ellipse with center 0 and containing the unit sphere S. Then $E \cap S$ contains at least a pair of independent vectors x and y (and hence contains $-x$ and $-y$ also) and $E \cap S$ is compact since E and S both are. See appendix B for a discussion of E. Thus if $E \cap S \neq S$ there exist $r,s \in E \cap S$ such that $E \cap S \cap C(r,s) = \{r,s\}$. Also there exist $\alpha,\beta \in R$ with $\alpha r, \beta s \in K$. Finally if $|\ \ |$ is the norm determined by E then $|\ \ |$ is induced by an inner product and $\|x\| \geq |x|$ for all $x \in X$.

By hypothesis

$$\mu\lambda(\mu+\lambda-2\mu\lambda)\alpha^2+(1-\lambda)(1-\mu)(\mu+\lambda-2\mu\lambda)\beta^2 \geq$$
$$\mu(1-\mu)\|\lambda\alpha r+(1-\lambda)\beta s\|^2+\lambda(1-\lambda)\|\mu\alpha r-(1-\mu)\beta s\|^2 \geq$$
$$\mu(1-\mu)|\lambda\alpha r+(1-\lambda)\beta s|^2+\lambda(1-\lambda)|\mu\alpha r-(1-\mu)\beta s|^2 =$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)\alpha^2+(1-\lambda)(1-\mu)(\mu+\lambda-2\mu\lambda)\beta^2$$

Thus equality holds throughout so $\pm \dfrac{\lambda\alpha r+(1-\lambda)\beta s}{\|\lambda\alpha r+(1-\lambda)\beta s\|}$ and $\pm \dfrac{\mu\alpha r-(1-\mu)\beta s}{\|\mu\alpha r-(1-\mu)\beta s\|}$ belong to $E \cap S$. But at least one of the vectors belongs to $C(r,s)$ and is distinct from r and s contrary to the hypothesis. Hence $E \cap S = E$ and X is an inner product space.

If r is $\geq$ choose E to be the maximal inscribed ellipse in S and a similar proof holds. If r is $=$ this is a special case of either of the first two results.

q.e.d.

The classical example of a set K satisfying the condition imposed by this theorem is the boundary of any closed bounded set containing 0 as an interior point. Note that if f is any continuous linear functional then the boundary intersected with the half space $\{x \mid f(x) \geq 0\}$ also satisfies the hypothesis.

We omit the proof of the next lemma.

<u>LEMMA 4.10:</u> Let X be a normed linear space. If a subset $K \subset X$ has the property that for a relation r and for each $x,y \in K$ there exist $0 < \lambda, \mu < 1$ such that $(x,y) \in A(\lambda,\mu,r)$ then αK will also have the property for any scalar α.

The next lemma is intersting because we intuitively would like to attack conjectures 4.5–4.7 by this approach. It is less powerful, however, than later techniques we will use.

LEMMA 4.11: Let X be a normed linear space, K be a convex subset of X, and $w \in K$. If K has the property that there is a value $0 < \lambda < 1$ such that for $x,y \in K$ then $(x,y) \in A(\lambda, 1/2, =)$ then $K - w$ (the translation of K by w) has the same property.

Proof:

Let $x,y \in K$. Then

$$\| \lambda(x-w) + (1-\lambda)(y-w) \|^2 =$$
$$\frac{1}{\lambda(1-\lambda)} (\lambda\| \lambda x+(1-\lambda)y \|^2 + (1-\lambda)\| w \|^2 - \| \lambda(\lambda x+(1-\lambda)y)+(1-\lambda)w\|^2) =$$
$$\frac{1}{(1-\lambda)} \| \lambda x+(1-\lambda)y\|^2 + 1/\lambda\| w \|^2 - \frac{1}{(1-\lambda)\lambda} (\lambda\| \lambda x+(1-\lambda)w \|^2 +$$
$$(1-\lambda)\| \lambda y+(1-\lambda)w \|^2 - \lambda(1-\lambda)\| \lambda x-\lambda y \|^2) =$$
$$\frac{1}{1-\lambda} \|\lambda x+(1-\lambda)y \|^2 + 1/\lambda\| w \|^2 - \frac{1}{1-\lambda} (\lambda\| x \|^2 +(1-\lambda)\| w \|^2 -$$
$$\lambda(1-\lambda)\| x-w \|^2)-1/\lambda(\lambda\| y \|^2+(1-\lambda)\| w\|^2-\lambda(1-\lambda)\| y-w\|)+\lambda^2\| x-y\|^2 =$$
$$\lambda^2 \| x-y \|^2 + \frac{1}{1-\lambda} \| \lambda x+(1-\lambda)y \|^2 - \frac{\lambda}{1-\lambda} \| x \|^2 - \| y \|^2 + \lambda \| x-w \|^2 +$$
$$(1-\lambda) \| y-w \|^2.$$

Thus

$$1/4\| \lambda(x-w)+(1-\lambda)(y-w) \|^2+\lambda(1-\lambda)\| 1/2(x-w)-1/2(y-w) \|^2 =$$
$$1/4 \frac{1}{1-\lambda} (\lambda(1-\lambda)\| x-y\|^2+\| \lambda x+(1-\lambda)y\|^2-\lambda\| x\|^2-(1-\lambda)\| y\|^2) +$$
$$1/4(\lambda\| x-w\|^2+(1-\lambda)\| y-w\|^2)=$$
$$1/4(\lambda\| x-w \|^2 + (1-\lambda)\| y-w \|^2).$$

q.e.d.

THEOREM 4.12: Let X be a normed linear space. Then X is an inner product space iff there exist a set K with nonempty interior and a value $0 < \lambda < 1$ such that for $x,y \in K$ then $(x,y) \in A(\lambda, 1/2, =)$.

Proof:

Suppose $S(z,\epsilon) \subset K$. Then $S(z,\epsilon)$ satisfies the hypothesis of lemma 4.11 so $S(z,\epsilon) - z$ has the same property. But $S(z,\epsilon) - z$ satisfies the hypothesis of theorem 4.9 so X is an inner product space.

q.e.d.

The next theorem is a direct generalization of the last one but involves completely different techniques. Thus the theorems have been stated separately.

THEOREM 4.13: Let X be a normed space. Then X is an inner product space iff there exists a subset K of X with nonempty interior with the property that for $x,y \in K$ there exists $0 < \lambda < 1$ such that $(x,y) \in A(\lambda, 1/2, =)$.

Proof:

Suppose K exists. If $z \in \mathrm{INT}(K)$ then by lemma 4.10 we may assume $\| z \| = 1$ and it suffices to prove the theorem when K is a ball with center z.

We prove the theorem using several cases depending on the dimension of X. First suppose the real dimension of X is 2. Let $y \in S(z,\epsilon)$ such that $\| y \| = 1$ and $\| y-z \| < 2$. Then there exists a unique ellipse E with center 0 and which passes through y, z, $y-z/\| y-z \|$. (See appendix B) Let S be the unit sphere of X, $D = S \cap E$, and $|\ \ |$ be the norm determined by E, Then D and $D \cap C(y,z)$ are both closed and nonempty. Suppose $C(y,z) \cap D \neq C(y,z) \cap S$. Then there must exist $r,s \in C(y,z) \cap D$ such

that $C(r,s) \cap D = \{r,s\}$. Since X is two-dimensional there exist $0 < \delta, \sigma \leq 1$ and $0 \leq \mu, \gamma \leq 1$ such that $\delta r = \mu y + (1-\mu)z$ and $\sigma s = \gamma y + (1-\gamma)z$.

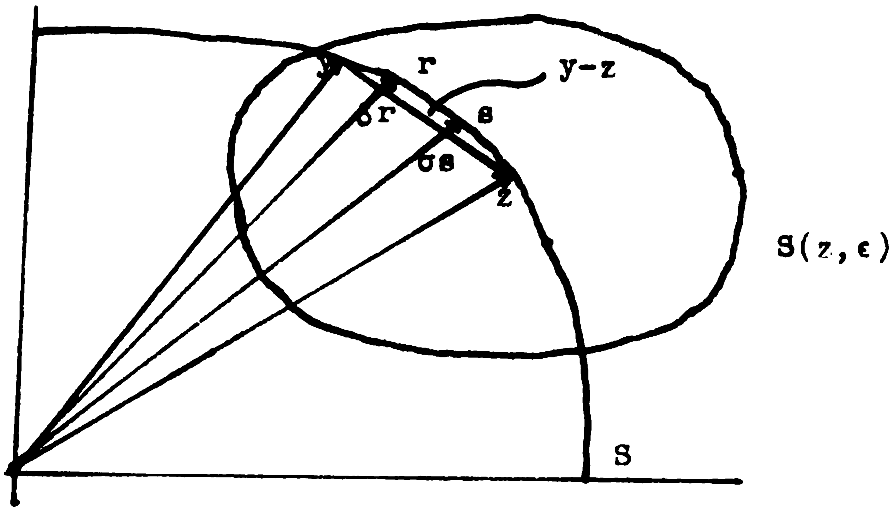

FIGURE 4.1

By convexity $\delta r, \sigma s \in S(z,\epsilon)$ so there exists $0 < \lambda < 1$ such that

$$\begin{aligned} \| \lambda\delta r+(1-\lambda)\sigma s \|^2 &= \lambda\| \delta r \|^2 + (1-\lambda)\| \sigma s \|^2 - \lambda(1-\lambda)(\mu-\gamma)^2\| y-z\|^2 \\ &= \lambda|\delta r|^2 + (1-\lambda)|\sigma s|^2 - \lambda(1-\lambda)(\mu-\gamma)^2|y-z|^2 \\ &= |\lambda\delta r + (1-\lambda)\sigma s|^2. \end{aligned}$$

Thus $\dfrac{\lambda\delta r + (1-\lambda)\sigma s}{\| \lambda\delta r + (1-\lambda)\sigma s \|} \in C(r,s) \cap D$ contrary to the choice of r and s so $C(y,z) \cap D = C(y,z) \cap S$.

Let $w = 1/2(z+y)$ and $x \in X$. There exists $\delta > 0$ such that $w+\delta x \in C(y,z) \cap S(z,\epsilon)$. Then by hypothesis there exists $0 < \lambda < 1$ such that

$$\begin{aligned} \lambda(1-\lambda)\| \delta x \|^2 &= \lambda\| w \|^2 + (1-\lambda)\| w+\delta x \|^2 - \| \lambda w+(1-\lambda)(w+\delta x)\|^2 \\ &= \lambda|w|^2 + (1-\lambda)|w+\delta x|^2 - |\lambda w+(1-\lambda)(w+\delta x)|^2 \\ &= \lambda(1-\lambda)|\delta x|^2. \end{aligned}$$

Hence $D = S = E$ and X is an inner product space.

If the real dimension of X is greater than two then since K has nonempty interior there exists a basis $\{x_\alpha\}$ of X such that $\{x_\alpha\} \subset \mathrm{Int}(K)$. But by the above argument each two-dimensional subspace containing x_α will be an inner product space. By lemma 4.3 X is an inner product space.

q.e.d.

In the next theorem we allow μ to take values other than 1/2 as it did in the previous theorem. However, we must make an additional hypothesis on K. Example 4.17 shows that this hypothesis is necessary.

<u>THEOREM 4.14:</u> Let X be a normed linear space. Then X is an inner product space iff there exist a convex cone K with nonempty interior and a fixed $0 < \mu < 1$ such that for all $x, y \in K$ there exists $0 < \lambda < 1$ with $(x,y) \in A(\lambda,\mu,=)$.

<u>Proof:</u>

Assume K exists and again assume X is two-dimensional. Choose $y, z \in K$ so that $\| y \| = \| z \| = 1$, $\| y-z \| < 1$, and $C(y,z) \subset K$. Let E be the unique ellipse with center 0 through $y, z, y-z/\| y-z \|$, and $|\ \ |$ be the norm determined by E. Let S be the unit sphere of X and $D = S \cap E$. If $D \cap C(y,z) \neq S \cap C(y,z)$ then there exist $r, s \in D \cap C(y,z)$ such that $D \cap C(r,s) = \{r, s\}$. There exist $\alpha, \beta \geq 0$ such that $\mu\alpha r - \beta(1-\mu)s = y - z$(here is where we use the fact K is a cone). Also there exists $0 < \lambda < 1$ such that

$$\mu(1-\mu)\|\lambda ar + (1-\lambda)\beta s\|^2 = \mu\lambda(\mu+\lambda-2\mu\lambda)\|ar\|^2$$
$$+(1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\|\beta s\|^2-\lambda(1-\lambda)\|\mu ar-(1-\mu)\beta s\|^2$$
$$= \mu\lambda(\mu+\lambda-2\mu\lambda)|ar|^2$$
$$+(1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)|\beta s|^2-\lambda(1-\lambda)|\mu ar-(1-\mu)\beta s|^2$$
$$= \mu(1-\mu)|\lambda ar+(1-\lambda)\beta s|^2.$$

Thus $\lambda ar+(1-\lambda)\beta s/\|\lambda ar+(1-\lambda)\beta s\| \in D \cap C(r,s)$ contrary to the choice of r and s so that $D \cap C(y,z) = S \cap C(y,z)$.

Let $x = ay+bz$. If $ab \geq 0$ then $\|ay+bz\| = |ay+bz|$ by the above. If $ab \leq 0$ then without loss of generality let $a \geq 0 \geq b$. Then there exists $0 < \lambda < 1$ such that

$$\lambda(1-\lambda)\|\mu(\frac{ay}{\mu}) - (1-\mu)(\frac{-bz}{1-\mu})\|^2 = \mu\lambda(\mu+\lambda-2\mu\lambda)\|y\|^2$$
$$+(1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\|z\|^2-\mu(1-\mu)\|\lambda(\frac{ay}{\mu})+(1-\lambda)(\frac{-bz}{1-\mu})\|^2$$
$$= \mu\lambda(\mu+\lambda-2\mu\lambda)|y|^2$$
$$+(1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)|z|^2-\mu(1-\mu)|\lambda(\frac{ay}{\mu})+(1-\lambda)(\frac{-bz}{1-\mu})|^2$$
$$= \lambda(1-\lambda)|ax+bz|^2 .$$

Thus $D = S = E$ and X is an inner product space.

If the dimension of X is greater than two then the result again follows from lemma 4.3 and the above argument.

q.e.d.

Lemma 4.15 enables us to draw conclusions on conjecture 4.7 from theorem 4.14.

<u>LEMMA 4.15:</u> Let X be a normed linear space and $x,y \in X$. Then $\alpha[z,w] \leq \alpha[x,y]$ for $z,w \in C(x,y)$.

<u>Proof:</u>

First note that we may assume $\|x\| = \|y\| = \|z\| = \|w\| = 1$. Let $z=ax+by$ and $w = cx+dy$ where $a,b,c,d \geq 0$

and $\| x-y \| = \epsilon = \alpha[x,y]$. Assume the notation was chosen so that $bc \geq ad$. First we show $\alpha[y,w] \leq \alpha[y,x]$. By the triangle inequality

$$\frac{c+d}{\epsilon} = \left\| \frac{c+d}{\epsilon} x \right\| = \left\| d\left(\frac{1}{\epsilon} x - \frac{1}{\epsilon} y\right) + \frac{1}{\epsilon}(cx+dy) \right\|$$
$$\leq d + \frac{1}{\epsilon}$$

and $1 = \| cx+dy \| \leq c+d$.

Hence $1 \leq c+d \leq d\epsilon+1$

Now consider two cases.

1. If $d \leq 1$

$$\alpha[y,w] = \| y-(cx+dy) \| = \| (1-d)(x-y)+(c+d-1)x \|$$
$$\leq (1-d)\epsilon+(c+d-1) \leq (1-d)\epsilon+d\epsilon = \epsilon.$$

2. If $d \geq 1$

$$\alpha[y,w] = \| y-(cx+dy) \| = \left\| \frac{c}{d} x + \left(1-\frac{1}{d}\right)(cx+dy) \right\|$$
$$\leq \frac{c}{d} + 1 - \frac{1}{d} = \frac{c+d-1}{d} \leq \frac{d\epsilon}{d} = \epsilon.$$

The assumption $bc \geq ad$ implies $z \in C(y,w)$ so the above implies $\alpha[z,w] \leq \alpha[y,w] \leq \alpha[y,x]$.

q.e.d.

<u>COROLLARY 4.16:</u> Let X be a normed linear space. Then X is an inner product space iff there exist $\epsilon > 0$ and $0 < \mu < 1$ such that for $x,y \in X$ with $\alpha[x,y] < \epsilon$ there exists $0 < \lambda < 1$ such that $(x,y) \in A(\lambda,\mu,=)$.

Now we give some examples which answer some of the questions connected with conjectures 4.5 - 4.7. They also illustrate some of the difficulties in extending theorems 4.12 - 4.15.

THESIS

EXAMPLE 4.17: Before actually giving the example we make a preliminary calculation. Suppose X is a normed space, $x, y \in X$, $0 < \lambda, \mu < 1$, $\| \lambda x+(1-\lambda)y \| = \lambda\| x \| +(1-\lambda)\| y \|$, and $\| \mu x-(1-\mu)y \| = | \mu\| x \| - (1-\mu)\| y \| |$. It is easily verified that $(x,y) \in A(\lambda,\mu,=)$. Geometrically our assumption means that $x/\| x \|$, $y/\| y \|$, $\lambda x+(1-\lambda)y/\| \lambda x+(1-\lambda)y \|$, and $\mu x-(1-\mu)y/\| \mu x-(1-\mu)y \|$ all lie on a flat spot of the unit sphere.

With this in mind consider the two dimensional normed linear space whose unit sphere S is determined by

$$(x,y) \in S \quad \text{iff} \quad \begin{cases} 4x^2 + 4/3\, y^2 = 1 & \text{and} \quad |y| \geq 3\,|x| \\ |x| + |y| = 1 & \text{and} \quad 3|x| \geq |y| \geq 1/3\,|x| \\ 4y^2 + 4/3\, x^2 = 1 & \text{and} \quad 3|y| \leq |x| \end{cases}$$

(i.e. this is the l_1 sphere whose corners have been rounded by ellipses with centers 0 and tangent to the sides of the l_1 sphere).

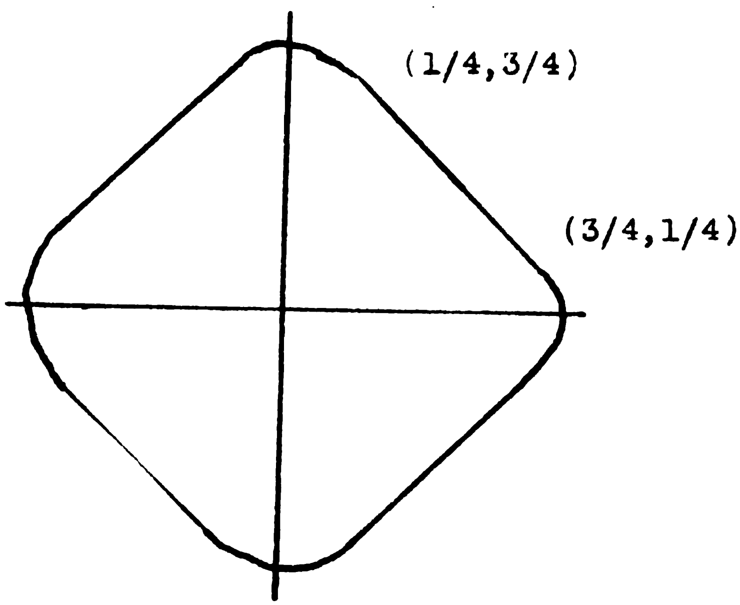

FIGURE 4.2

If $\alpha[x,y] < 1/4$ the $x/\|x\|$ and $y/\|y\|$ can have three relative positions. (1) both lie on a flat spot of the sphere and at least one is not at the end of the flat spot (2) both lie on an elliptical section of the sphere and at least one does not lie at the end of the elliptical section or (3) one lies on a flat spot and one lies on an adjacent elliptical section.

If either (1) or (2) happens then for any $0 < \lambda < 1$ and for $0 < \mu < 1$ chosen so that $\mu x-(1-\mu)y/\|\mu x-(1-\mu)y\|$ lies on the same flat spot or elliptical section as $x/\|x\|$ and $y/\|y\|$ we have $(x,y) \in A(\lambda,\mu,=)$ by the above remark. If (3) happens assume for definiteness that $x/\|x\|$ lies on a flat spot and $y/\|y\|$ lies on an adjacent elliptical section. If $0 < \lambda,\mu < 1$ are chosen so that $\lambda x+(1-\lambda)y/\|\lambda x+(1-\lambda)y\|$ and $\mu x-(1-\mu)y/\|\mu x-(1-\mu)y\|$ lie on the same flat spot (elliptical section) as $x/\|x\|$ $(y/\|y\|)$ then $(x,y) \in A(\lambda,\mu,\leq)$ ($(x,y) \in A(\lambda,\mu,\geq)$). By continuity it follows that we may choose λ and μ so that $(x,y) \in A(\lambda,\mu,=)$. This naturally disproves conjectures 4.6 and 4.7 as stated.

If we take $K = C((1/3,2/3),(2/3,1/3))$ we see that for $x,y \in K$ there exist $0 < \lambda,\mu < 1$ such that $(x,y) \in A(\lambda,\mu,=)$ so in theorem 4.14 "for a fixed μ" can not be replaced by "there exists a μ (dependent on x and y)". Also for $x,y \in K$ then $(x,y) \in A(\lambda,\mu,\geq)$ for all $0 < \lambda,\mu < 1$. Thus "=" in theorems 4.12-4.14 can not be replaced by "$\geq$".

Furthermore if we take $K = S((1/2,1/2),.01)$ and $\mu = .0001$ then for all $0 < \lambda < 1$ and $x,y \in K$ it follows that $(x,y) \in A(\lambda,\mu,=)$. Hence the term "cone" in theorem 4.11 can not be changed to "set with nonempty interior".

EXAMPLE 4.18: This example has many of the properties of example 4.13 and clearly illustrates how cones behave.

Let X be the Minkowski plane whose unit sphere is given by $\{(x,y) \mid x^2+y^2=1 \text{ if } xy \geq 0 \text{ and } |x|^3 + |y|^3 = 1 \text{ if } xy \leq 0\}$. Let $K = C((1,0),(0,1))$. Then for $x,y \in X$ and all $0 < \lambda, \mu < 1$ we have $(x,y) \in A(\lambda,\mu,\leq)$. Thus neither can "=" be replaced by "≤" in theorems 4.12–4.14.

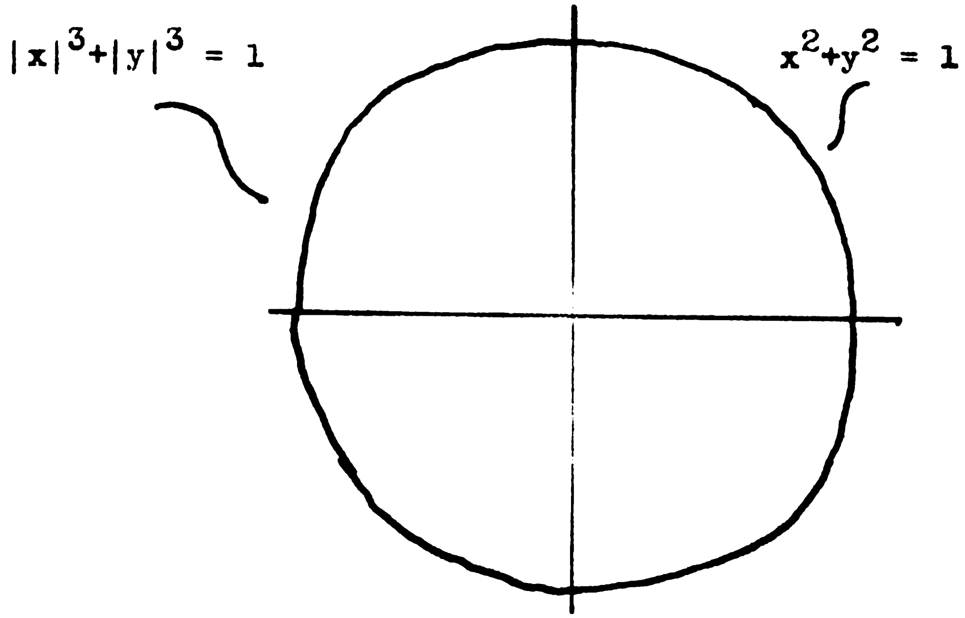

FIGURE 4.3

These results and examples settle conjecture 4.5 when K is a cone or set with nonempty interior. Still open is the following interesting case of conjecture 4.5.

What additional hypotheses are necessary for conjecture 4.5 to be true when K is the boundary of a convex set with nonempty interior?

While example 4.17 tells us that conjecture 4.6 is false it gives no idea of what additional hypotheses are necessary for the theorem to be valid. This conjecture has been much more difficult to attack than the other two. Corollary 4.21 will be one small step in this direction, however.

The next theorem is very much in the spirit of the previous ones but is of a slightly different nature. It is a localization of theorem 1.18 by Lorch.

<u>THEOREM 4.19:</u> Let X be a normed linear space. Then X is an inner product space iff there exist $0 < \lambda < 1/2$ and $\epsilon > 0$ such that for $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$ then $\| \lambda x+(1-\lambda)y \| = \| (1-\lambda)x+\lambda y \|$.

<u>Proof:</u>

Again it will suffice to prove the theorem when X has real dimension two.

Let $f(n)$ denote the sequence defined recursively by $f(0) = \lambda$ and $f(n) = (2\lambda-1)f(n-1) + (1-\lambda)$ for $n \geq 1$. Then

$$\begin{aligned}|1/2 - f(n)| &= |1/2 - ((2\lambda-1)f(n-1) + (1-\lambda))| \\ &= |1/2 - f(n-1)|\ |2\lambda-1| \\ &< |1/2 - f(n-1)\ |\end{aligned}$$

since $0 < |2\lambda-1| < 1$. Thus $f(n) \dashrightarrow 1/2$ and $0 < f(n) < 1$.

We wish to show by induction

$\| f(n)x + (1-f(n))y \| = \| (1-f(n))x + f(n)y \|$

for $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$. By the hypothesis of the theorem the induction hypothesis is true for $n = 0$. Suppose it is true for $n = 0,1,\ldots,m$. Then

$\| f(m+1)x + (1-f(m+1))y \| =$

$\| \lambda(f(m)x + (1-f(m))y) + (1-\lambda)((1-f(m))x + f(m)y) \| =$

$\| (1-\lambda)(f(m)x + (1-f(m))y) + \lambda((1-f(m))x + f(m)y) \| =$

$\| (1-f(m+1))x + f(m+1)y \|$

The middle equality follows from the induction hypothesis, the hypothesis of the theorem, and lemma 4.15.

Since $f(n) \to 1/2$ this also implies $x+y \perp x-y$.

Suppose X were not strictly convex. Then we could find $x,y \in X$ and $a,b \in R$ such that $\| x \| = \| y \| = 1$, $\| x-y \| < \epsilon$, $0 < a < b \neq 1-a$, $\| ax+by \| = 1$, and $\| \lambda_1 x + (1-\lambda_1)(ax+by) \| = 1$ for $0 \leq \lambda_1 \leq 1$. (See figure 4.4) Then $x+y \perp x-y$ but the unit sphere has a unique supporting hyperplane at $\dfrac{x+y}{\| x+y \|}$ and it is not in the direction of $x-y$. Hence X is strictly convex.

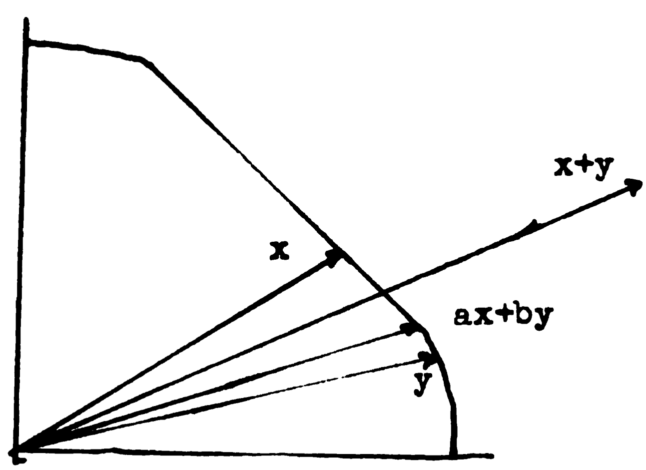

FIGURE 4.4

Now suppose $\| x \| = \| y \| = 1$, $\| x-y \| < \epsilon$, and $\| \lambda_1 x + (1-\lambda_1)y \| = \| \lambda_2 x + (1-\lambda_2)y \|$ where $\lambda_1 \neq \lambda_2$. Then $(\lambda_1+\lambda_2)x + (2 - (\lambda_1+\lambda_2))y \perp x-y$ and $x+y \perp x-y$. Since X is strictly convex $\lambda_1 + \lambda_2 = 2 - \lambda_1 - \lambda_2$ or $\lambda_2 = 1 - \lambda_1$. Thus $\| \lambda_1 x + (1-\lambda_1)y \| = \| (1-\lambda_1)x + \lambda_1 y \|$ for $0 \leq \lambda_1 \leq 1$ which implies $\| ax+by \| = \| bx+ay \|$ for $a,b \geq 0$.

Choose $0 < \epsilon' \leq \min(\epsilon,1)$ so that $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon'$ imply $\| x+y \| > 1$. Let $0 < a < 1$, $\| x \| = \| y \| = 1$, and $\| x-y \| < \epsilon'$. Then

$$\| ax+y \| = \| a \, \| x+y \| \frac{x+y}{\| x+y \|} + (1-a)y \|$$

$$= \| (1-a) \frac{x+y}{\| x+y \|} + a \| x+y \| y \|$$

or

$$\| ax+y \| \, \| x+y \| = \| (1-a)x + (1-a+a\| x+y \|^2)y \|.$$

Likewise

$$\| x+y \| = \| \, \| x+ay \| \frac{x+ay}{\| x+ay \|} + (1-a)y \|$$

$$= \| (1-a) \frac{x+ay}{\| x+ay \|} + \| x+ay \| y \|$$

or

$$\| ax+y \| \, \| x+y \| = \| x+ay \| \, \| x+y \|$$

$$= \| (1-a)x + (a-a^2+ \| x+ay \|^2)y \|.$$

Note $a - a^2 + \| x+ay \|^2 \geq 0$ and $1 - a + a \| x+y \| \geq 0$.

The function $g(b) = \| x+by \|$ has the property that $g(0) = 1$, $g(1) > 1$, and $g(-1) < 1$. Also it attains its minimum at a unique point b_0 and the above values show that $b_0 \leq 0$. Thus $\frac{1-a+a\| x+y \|^2}{1-a} = \frac{a-a^2 + \| x+ay \|^2}{1-a}$ since both are positive and have equal function values. Thus for $0 \leq a \leq 1$ we have $a\| x+y \|^2 + (1-a)^2 = \| x+ay \|^2$.

THESIS

Let $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon'$. There exists an unique ellipse E through the points x, y and $\dfrac{x+y}{\| x+y \|}$ with center 0. Let $| \quad |$ be the norm determined by E. Then

$$\begin{aligned} |ax+y|^2 &= a|x+y|^2 + (1-a)^2 \\ &= a\| x+y \|^2 + (1-a)^2 \\ &= \| ax+y \|^2 \end{aligned}$$

for $0 < a < 1$. Thus $E \cap S \cap C(x,y) = E \cap C(x,y)$. Now choose x', y' so that $x', \dfrac{x'+y'}{\| x'+y' \|} \in C(x,y)$, $\| x' \| = \| y' \| = 1$, and $\| x'-y' \| < \epsilon'$. Let E' be the unique ellipse through x', y', and $\dfrac{x'+y'}{\| x'+y' \|}$ with center 0.

Then $E' \cap S \cap C(x',y') = E' \cap C(x',y')$ and in particular $E' \cap S \cap C\left(x', \dfrac{x'+y'}{\| x'+y' \|}\right) = E' \cap C\left(x', \dfrac{x'+y'}{\| x'+y' \|}\right)$. Since E and E' are unique and agree on a section of curve we must have $E = E'$. This process may be continued and in a finite number of steps we have shown $E \cap S = E$. Thus X is an inner product space.

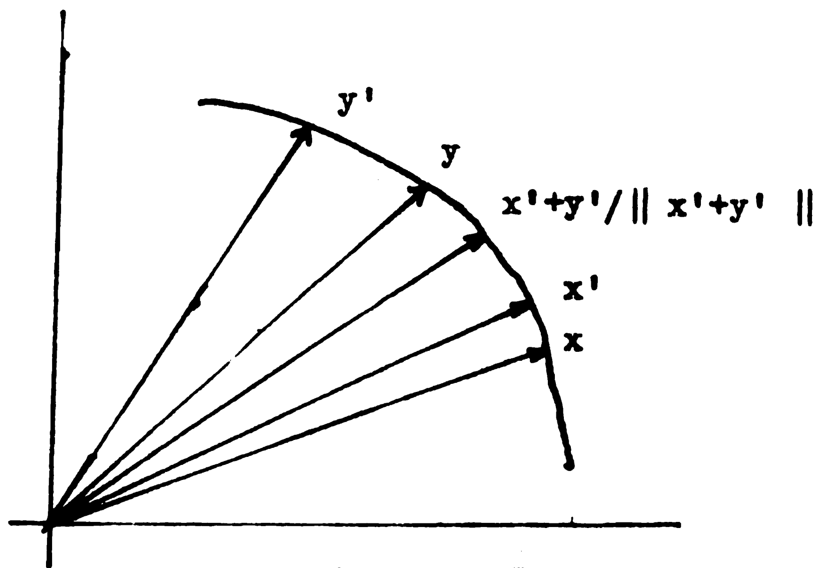

FIGURE 4.5

q.e.d.

<u>COROLLARY 4.20:</u> Let X be a normed linear space. Then X is an inner product space iff there exist $\epsilon > 0$, $0 < \lambda < 1/2$, and a function $f : [0,2] \to [0,2]$ such that for $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$ then $\| \lambda x+(1-\lambda)y \| = f(\| x+y \|)$.

<u>COROLLARY 4.21:</u> Let X be a normed linear space. Then X is an inner product space iff there exist $\epsilon > 0$ and $0 < \lambda < 1/2$ such that for $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$ then $(x,y) \in A(\lambda, 1/2, =)$.

Corollary 4.20 suggests a conjecture similar to conjecture 4.6 and which is even more localized than 4.6.

<u>CONJECTURE 4.22:</u> Let X be a normed linear space. Then X is an inner product space iff there exists $\epsilon > 0$ such that for $\| x \| = \| y \| = 1$ and $\| x-y \| < \epsilon$ then there exist $0 < \lambda, \mu < 1$, $\lambda \neq \mu$ so that

$$\mu(1-\mu) \| \lambda x+(1-\lambda)y \|^2 - \lambda(1-\lambda)\| \mu x + (1-\mu)y \|^2 \; r \; (\mu+\lambda-1)(\lambda-\mu)$$

where r is one of the relations $\geq$, $=$, $\leq$.

We will not pursue this conjecture but continue to work on conjecture 4.7.

<u>THEOREM 4.23:</u> A normed linear space X is an inner product space iff there are $0 < \lambda, \mu < 1$, $\epsilon > 0$, and a relation r such that for $\alpha[x,y] < \epsilon$ then $(x,y) \in A(\lambda,\mu,r)$.

THESIS

Proof:

Since the necessity of the last condition is obvious we will show only the sufficiency. Moreover, we may assume X has real dimension two. We will prove the case when r is $\leq$ since the others can be proven in the usual manner. Let S be the unit sphere of X, E be the maximal inscribed ellipse, $|\ \ |$ be the norm determined by E, and $D = S \cap E$. If $D \neq S$ there will exist $z, w \in D$ such that $C(z,w) \cap D = \{z,w\}$. Then there exist $a, b > 0$ such that the vectors $x = az+bw$ and $y = (\mu/1-\mu)az - (\lambda/1-\lambda)bw$ satisfy $\| x \| = 1$, $\| x-z \| < \epsilon/2$, and $\| y/\| y \| - z \| < \epsilon/2$. Then $\alpha[x,y] < \epsilon$ so that

$$\mu(1-\mu)\| \lambda x+(1-\lambda)y \|^2 + \lambda(1-\lambda)\| \mu x - (1-\mu)y \|^2 \leq$$
$$\lambda\mu(\lambda+\mu-2\mu\lambda)\| x \|^2 + (1-\lambda)(1-\mu)(\lambda+\mu-2\mu\lambda)\| y \|^2 \leq$$
$$\lambda\mu(\lambda+\mu-2\mu\lambda)|x|^2 + (1-\lambda)(1-\mu)(\lambda+\mu-2\mu\lambda)|y|^2 =$$
$$\mu(1-\mu)|\lambda x+(1-\lambda)y|^2 + \lambda(1-\lambda)|\mu x - (1-\mu)y|^2 =$$
$$\mu(1-\mu)\| \lambda x + (1-\lambda)y \|^2 + \lambda(1-\lambda)\| \mu x - (1-\mu)y \|^2.$$

Thus equality holds throughout so $x \in D$. But $x \in C(z,w)$ contrary to the choice of z and w. Hence $D = S$ and X is an inner product space.

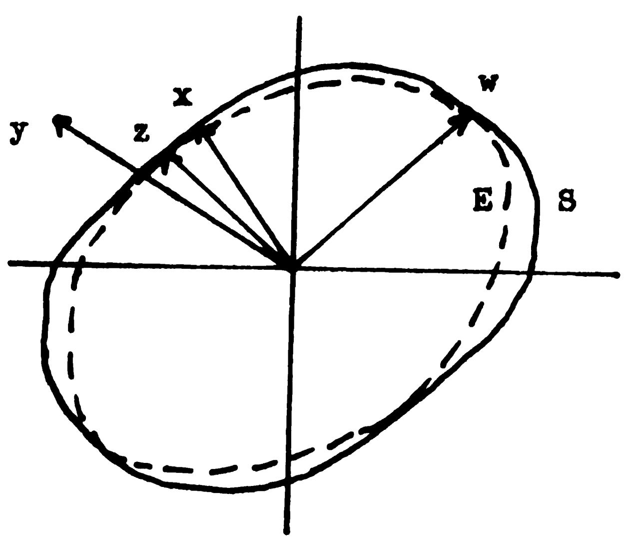

FIGURE 4.6

q.e.d.

<u>THEOREM 4.24:</u> Let X be a normed linear space. Then X is an inner product space iff there exist $0 < \mu < 1$, a relation r, and $\epsilon > 0$ such that for $x, y \in X$ satisfying $\alpha[x,y] < \epsilon$ then there exists $0 < \lambda < 1$ such that $(x,y) \in A(\lambda,\mu,r)$.

<u>Proof:</u>

If X is an inner product space then for all $0 < \lambda < 1$ and $x, y \in X$ we have $(x,y) \in A(\lambda,\mu,=)$.

Assume such a μ, ϵ, and r exist. We may assume X is real and two-dimensional. Let r be $\geq$.

What we wish to show is that if $\alpha[x,y] < \epsilon$ then $(x,y) \in A(\lambda,\mu,\geq)$ for all $0 < \lambda < 1$. By theorem 4.23 this would imply that X is an inner product space.

Let $\alpha[x,y] < \epsilon$. Then either there is an ellipse E with center 0 through $x/\|x\|$, $y/\|y\|$, and $\dfrac{\mu x-(1-\mu)y}{\|\mu x-(1-\mu)y\|}$ or $\|\mu x-(1-\mu)y\| = \mu\|x\| + (1-\mu)\|y\|$ or $\|\mu x-(1-\mu)y\| = |\mu\|x\| - (1-\mu)\|y\||$. If either of the last two cases happens it follows immediately that for all $0 < \lambda < 1$ then $(x,y) \in A(\lambda,\mu,\geq)$. Thus suppose E exists and let $|\ \ |$ be the norm determined by E.

If the assertion is not true then there will exist $r, s \in C(x,y)$ such that $\|r\| = |r| = \|s\| = |s| = 1$ and $\|ar+bs\| < |ar+bs|$ for $ab > 0$. Let $r = mx+ny$ and $s = px+qy$ where $m,n,p,q \geq 0$. Let $A = \dfrac{\mu}{1-\mu}\,\dfrac{(1-\mu)m+\mu n}{(1-\mu)p+\mu q}$.

Because $\alpha[r,s] < \epsilon$ there exists $0 < \lambda < 1$ such that

$\mu\lambda(\mu+\lambda-2\mu\lambda)|r|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)|As|^2 =$

$\mu\lambda(\mu+\lambda-2\mu\lambda)\|r\|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\|As\|^2 \leq$

$$\mu(1-\mu)\|\lambda r+(1-\lambda)As\|^2 + \lambda(1-\lambda)\|\mu r-(1-\mu)As\|^2 <$$
$$\mu(1-\mu)|\lambda r+(1-\lambda)As|^2 + \lambda(1-\lambda)|\mu r-(1-\mu)As|^2 =$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)|r|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)|As|^2$$

which is a contradiction.

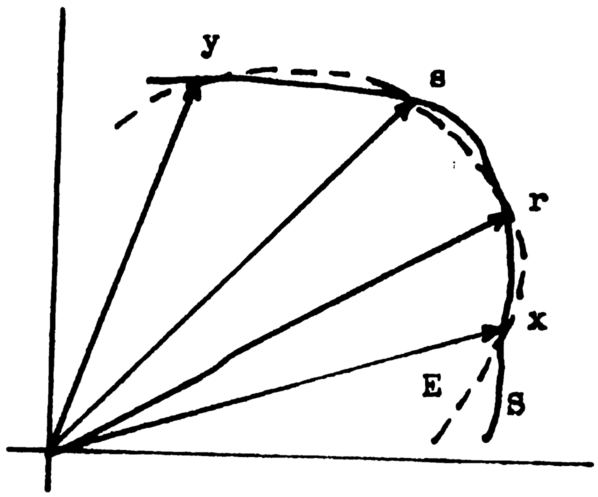

FIGURE 4.7

If r is $\leq$ then essentially the same proof as the above goes through with all inequalities reversed.

q.e.d.

This completes the results related to conjectures 4.5-4.7. The following table summarizes the results of this section. It lists the validity of conjectures 4.5-4.7 when various additional hypotheses on λ, μ, and r are given.

THESIS

λ	μ	r	Conjecture 4 5 Bodies	Cones	Conjecture 4.6	Conjecture 4.7
I	1/2	=	T	T	T ($\lambda \neq 1/2$)	T
D	1/2	=	T	T	?	T
I	I	=	F	T	?	T
D	I	=	F	T	?	T
D	D	=	F	F	F	F
I	I	$\geq$ or $\leq$	F	F	?	T
D	I	$\geq$ or $\leq$	F	F	?	T

Notation:

T -- true

F -- false

I -- λ (μ) independent of x and y

D -- λ (μ) dependent on x and y

FIGURE 4.8

3. Restriction

There is a second type of localization which is occasionally considered in normed linear spaces. This type of localization has been studied by Day and problems of this type were examined by Kolumban [45]. The actual conjecture we study is a variation of conjecture 4.5.

<u>CONJECTURE 4.25</u>: A normed linear space X is an inner product space iff there exist a nonempty set K and a relation r such that for $x \in K$ and $y \in X$ there are $0 < \lambda, \mu < 1$ such that $(x,y) \in A(\lambda,\mu,r)$.

Our previous work suggests that additional hypotheses on K, λ, μ, and r might be necessary.

Let X be an inner product space and Y be a normed linear space of dimension at least two. If $Z = X \times Y$ and the norm is given by $\| (x,y) \|^2 = \| x \|^2 + \| y \|^2$ then for all $(x_1,0)$, $(x_2,y) \in Z$ and all $0 < \lambda,\mu < 1$ we have $((x_1,0),(x_2,y)) \in A(\lambda,\mu,=)$. But if Y is not an inner product space then neither is Z. Thus a necessary assumption on K is that $\mathrm{Lin}\{K\}$ contains a hypersubspace.

Theorem 4.26 shows that this condition on K is also sufficient (at least with additional hypotheses on λ, μ, and r).

<u>THEOREM 4.26:</u> Let X be a normed linear space. Then X is an inner product space iff there exist a set K and $0 < \lambda,\mu < 1$ such that $\mathrm{Lin}\{K\}$ contains a hypersubspace of X and for $x \in K$ and $y \in X$ then $(x,y) \in A(\lambda,\mu,=)$.

<u>Proof:</u>

If X is an inner product space let $K = X$ and $0 < \lambda,\mu < 1$ be any values.

Suppose K exists. First note that it will suffice to prove the theorem when K is a basis for a hypersubspace. Secondly we observe that $\bigcup_{\alpha \in F} \alpha K$ has the same properties as K.

Now assume X is real two dimensional and by the above remarks let $K = \{\alpha x \mid \alpha \in R\}$. Let S be the unit sphere of X, E_1 be the minimal circumscribing ellipse of S with center 0, E_2 be the maximal inscribed ellipse

THESIS

with center 0, and $|\ |$ be the norm determined by E_1.

Now let $u, v \in E_1 \cap S$ be independent vectors and assume $u, v \neq x/\|x\|$. If $x = au+bv$ and $y = {}^{\mu a}/_{1-\mu}\, u - {}^{\lambda b}/_{1-\lambda}\, v$ then

$$\mu\lambda(\mu+\lambda-2\mu\lambda)\|x\|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\|y\|^2 =$$
$$\mu(1-\mu)\|\lambda x+(1-\lambda)y\|^2 + \lambda(1-\lambda)\|\mu x-(1-\mu)y\|^2 =$$
$$\mu(1-\mu)\ |\lambda x+(1-\lambda)y|^2 + \lambda(1-\lambda)\ |\mu x-(1-\mu)y|^2 =$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)|x|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)|y|^2 \leq$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)\|x\|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\ \|y\|^2.$$

Hence equality holds throughout so $\frac{x}{\|x\|} \in E_1 \cap S$. A similar argument shows that $\frac{x}{\|x\|} \in E_2 \cap S$.

Next suppose $u \in E_1 \cap S$, $w \in E_2 \cap S$, $u \neq w$, and $u, w \neq \frac{x}{\|x\|}$. Let $w = cx+du$ and then

$$\mu\lambda(\mu+\lambda-2\mu\lambda)\left\|\frac{c}{\lambda}x\right\|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\left\|\frac{d}{1-\lambda}u\right\|^2 =$$
$$\mu(1-\mu)\|w\|^2 + \lambda(1-\lambda)\left\|\frac{\mu c}{\lambda}x - \frac{(1-\mu)d}{1-\lambda}u\right\|^2 \geq$$
$$\mu(1-\mu)|w|^2 + \lambda(1-\lambda)\left|\frac{\mu c}{\lambda}x - \frac{(1-\mu)d}{1-\lambda}u\right|^2 =$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)\left|\frac{c}{\lambda}x\right|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\left|\frac{d}{1-\lambda}u\right|^2 =$$
$$\mu\lambda(\mu+\lambda-2\mu\lambda)\left\|\frac{c}{\lambda}x\right\|^2 + (1-\mu)(1-\lambda)(\mu+\lambda-2\mu\lambda)\left\|\frac{d}{1-\lambda}u\right\|^2.$$

Thus equality holds throughout so $w \in E_1 \cap S$.

Hence E_1 and E_2 are ellipses through $\frac{x}{\|x\|}$ and w with centers 0. But it is easy to see from the definitions of E_1 and E_2 that they have the same tangents at $\frac{x}{\|x\|}$ as S does. But there is an unique ellipse satisfying these conditions so $E_1 = E_2 = S$ and X is an inner product space.

Next assume X is complex two-dimensional and $K = \{\alpha x \mid \alpha \in C\}$. Let y be a vector independent of x and consider X as a four dimensional real space. Then

THESIS

Lin $\{ix, y, iy\}$ is a real three dimensional space, Lin $\{y, iy\}$ is an inner product space and every two-dimensional subspace through ix is an inner product space by the above result. Hence by lemma 4.2 Lin $\{ix, y, iy\}$ is an inner product space. By applying the lemma once more it follows that X is a real inner product space and hence a complex inner product space.

Finally if the dimension of X is greater than two then the above two cases and lemma 4.3 imply X is an inner product space.

q.e.d.

Finally we have an example to show that "=" in 4.25 cannot be replaced by "$\geq$".

<u>EXAMPLE 4.27:</u> Let X be the l_1 plane and $K = \{(0,1)\}$. It is easy to check that for all $(a,b) \in l_1$ then $((0,1),(a,b)) \in A(1/2,1/2,\geq)$.

Conjecture 4.25 like conjecture 4.6 has not yet been completely solved.

THESIS

CHARACTERIZATIONS OF INNER PRODUCT SPACES

In this chapter we continue to obtain characterizations of inner product spaces. Several of the theorems are direct generalizations of known results or new proofs of known results, but many are new theorems. Whenever possible geometric concepts are brought into the discussions.

1. Characterizations Using Convex Subsets

Theorem 5.2 is very representative of the theorems which are characterizations of inner product spaces. The proof is rather geometric and there is a geometric interpretation of the theorem. This theorem, like many of the characterizations, is a special case of the following principle.

<u>PRINCIPLE 5.1:</u> Let A be a theorem true in Euclidean geometry. A normed linear space X is an inner product space iff A is true in X.

Every high school student of plane geometry is familiar with the property of a circle that from each point p outside the circle there are two lines of tangency to the circle and the distances from p to the points of tangency are equal. It is not too difficult to show that this property characterizes the circle among the plane, closed, bounded

THESIS

convex sets with nonempty interior. We show in theorem 5.2 that a similar property characterizes the spheres in inner product spaces and characterizes the inner product spaces among normed linear spaces.

THEOREM 5.2: Let X be a real normed linear space and K be a closed, bounded, convex subset with nonempty interior. Then the following are equivalent:

1. X is an inner product space and K is a ball in X.

2. K has the property that for each pair of hyperplanes H_1 and H_2 supporting K at x and y respectively and each $r \in H_1 \cap H_2$ with r, x and y linearly dependent then $\| x-r \| = \| y-r \|$.

3. K has the property that for $f, g \in X^*$ and $x, y \in K$ satisfying $f(x) = \sup\limits_{z \in K} f(z)$ and $g(y) = \sup\limits_{z \in K} g(y)$ then

$$|g(y) - g(x)| \; \| f(y)x - f(x)y \| = |f(x) - f(y)| \; \| g(x)y - g(y)x \|.$$

Proof:

First we note the equivalence of 2 and 3. This equivalence is basically the equivalence of continuous linear functionals and closed hyperplanes which in this case is $H_1 = f^{-1}f(x)$ and $H_2 = g^{-1}g(y)$. If f and g are not linearly independent or $x = y$ then both 2 and 3 are trivial. If f and g are linearly independent and $x \neq y$ then the point r is given by $r = ax+by$ where

$$a = g(y)(f(y) - f(x))/(f(y)g(x) - g(y)f(x))$$

$$b = f(x)(g(x) - g(y))/(f(y)g(x) - g(y)f(x)).$$

THESIS

Now

$\| x-r \| = \| y-r \|$ iff

$$|g(y)-g(x)| \; \| f(y)x-f(x)y \| = |f(x)-f(y)| \; \| g(x)y-g(y)x \|.$$

That 1 implies 3 follows from the representation theorems for linear functionals in inner product spaces. Let $K = S(w,\rho)$, $f(x) = \sup_{z \in K} f(z)$, and $g(y) = \sup_{z \in K} g(z)$. There exist $u, v \in X$ and $\sigma, \delta > 0$ such that $\| u \| = \| v \| = 1$, $x = w+\rho u$, $y = w+\rho v$, $f(z) = (z|\sigma u)$, and $g(z) = (z|\delta v)$. By writing everything in terms of $(u|w)$, $(u|v)$, and $(v|w)$ it is a straightforward calculation to show

$$|g(y)-g(x)| \; \| f(y)x-f(x)y \| = |f(y)-f(x)| \; \| g(y)x-g(x)y \|.$$

To finish the proof it suffices to prove that 2 implies 1. We begin by proving the theorem when the dimension of X is two and using this to prove the theorem for higher dimensions. We consider X as a Minkowski plane and often use the language of plane geometry rather than that of normed linear spaces in describing the proof.

Let I_1 and I_2 be parallel support lines to K and $x_i \in I_i \cap K$ for $i = 1,2$. Furthermore let I_3, I_4 be support lines of K parallel to the line h determined by x_1 and x_2. Finally, let $x_i \in I_i \cap K$ for $i = 3,4$ and $a = I_1 \cap I_3$, $b = I_1 \cap I_4$, $c = I_2 \cap I_4$, $d = I_2 \cap I_3$ and k be the line determined by x_3 and x_4. The existence of such lines follows from standard theorems on convex bodies.

By hypothesis $\| a-x_1 \| = \| a-x_3 \|$, $\| b-x_1 \| = \| b-x_4 \|$, $\| c-x_2 \| = \| c-x_4 \|$, and $\| d-x_2 \| = \| d-x_3 \|$. Thus

THESIS

$$
\begin{aligned}
2\,\| a-b \| &= \| a-b \| + \| d-c \| \\
&= \| a-x_1 \| + \| x_1-b \| + \| c-x_2 \| + \| x_2 - d \| \\
&= \| a-x_3 \| + \| b-x_4 \| + \| c-x_4 \| + \| d-x_3 \| \\
&= \| a-d \| + \| c-b \| \\
&= 2\,\| a-d \|.
\end{aligned}
$$

Hence $\| c-d \| = \| a-d \| = \| a-b \| = \| b-c \|$. Also h, I_3, and I_4 are parallel so $\| a-x_1 \| = \| d-x_2 \|$ and $\| c-x_2 \| = \| b-x_1 \|$. These imply

$\| a-x_3 \| = \| a-x_1 \| = \| d-x_2 \| = \| d-x_3 \|$ and

$\| b-x_4 \| = \| b-x_1 \| = \| c-x_2 \| = \| c-x_4 \|$.

Finally $\| a-x_1 \| = \| a-x_3 \| = 1/2\| a-d \| = 1/2\| b-c \| = \| b-x_4 \| = \| b-x_1 \|$ and $\| d-x_2 \| = \| d-x_3 \| = 1/2\| a-d \| = 1/2\| c-b \| = \| c-x_2 \| = \| c-x_4 \|$. In conclusion x_i are the midpoints of the sides of the square with vertices a,b,c,d.

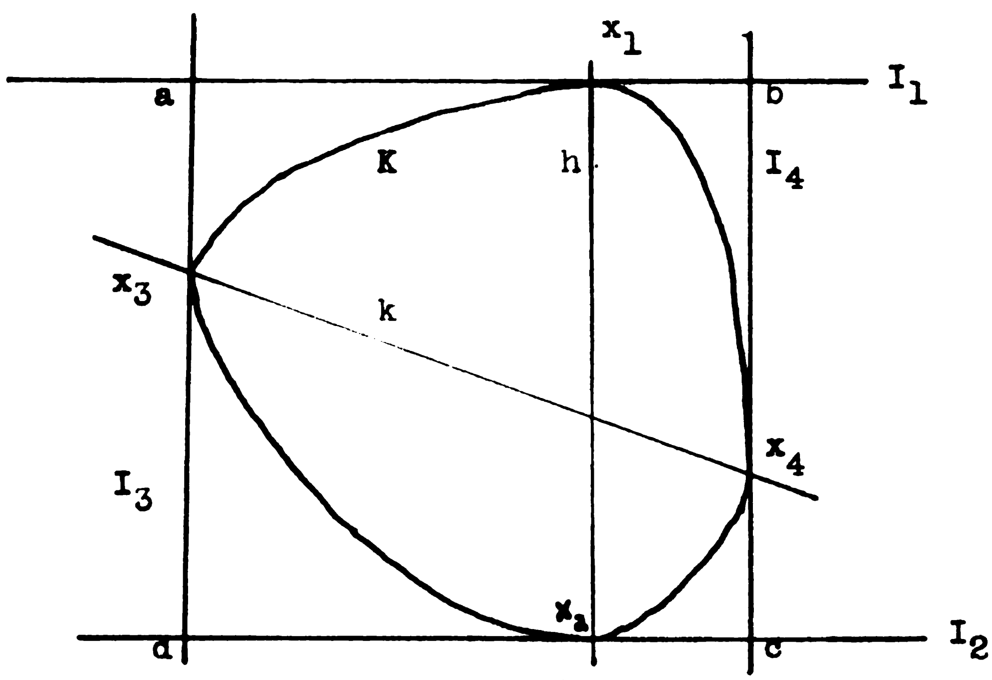

FIGURE 5.1

If $w = 1/2(x_1+x_2)$ and $\rho = 1/2\| x_1-x_2 \|$ then $x_3+x_4 = 2w$ and $\| x_i-w \| = \rho$ for $i=1,\ldots,4$. The norm $|\quad|$ given by

$|\alpha(x_1-x_2) + \beta(x_3-x_4)| = 2\rho(\alpha^2 + \beta^2)^{1/2}$ is determined by an inner product and $|x_1-x_2| = \| x_1-x_2 \| = |x_3-x_4| = \| x_3-x_4 \|$.

Let $z \neq x_1$ be a boundary point of K and j be a support line at z. Let $r = I_3 \cap j$, $s = I_2 \cap j$, $t = I_4 \cap j$ and assume the notation chosen so that z is between r and s. If $m = \frac{|r-z|}{\| r-z \|}$ then

$$|x_3-r| = \| x_3-r \| = \| r-z \| = \frac{|r-z|}{m}$$

$$|x_2-s| = \| x_2-s \| = \| s-z \| = \frac{|s-z|}{m}$$

and $$|x_4-t| = \| x_4-t \| = \| t-z \| = \frac{|t-z|}{m}$$

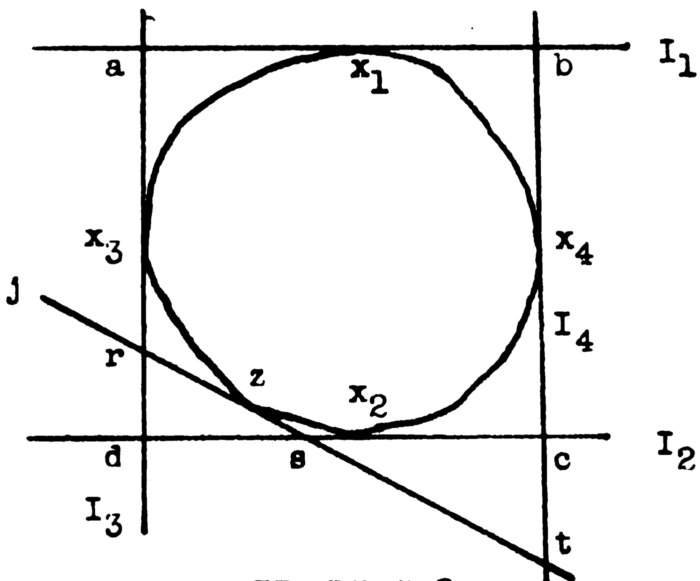

FIGURE 5.2

If $n = |r-x_3|$ and $q = |s-x_2|$ then by the above relations and similar triangles:

$$|r-s| = (n+q)m$$

$$|s-t| = (\rho+q)(n+q)m/(\rho-q)$$

$$|c-t| = (\rho-n)(\rho+q)/(\rho-q)$$

Thus

$$\frac{1}{m} = \frac{|t-x_4|}{|t-z|} = \frac{\rho + |c-t|}{|s-t| + |s-z|} = \frac{\rho + \frac{(\rho-n)(\rho+q)}{(\rho-q)}}{\frac{m(\rho+q)(n+q)}{(\rho-q)} + qm}$$

THESIS

or $1 = \dfrac{\rho(\rho-q)+(\rho-n)(\rho+q)}{q(\rho-q)+(\rho+q)(n+q)}$

which simplifies to $n = \rho\,\dfrac{\rho-q}{\rho+q}$.

But $|r-s|^2 = |r-d|^2 + |d-s|^2$

or $m^2(n+q)^2 = (\rho-n)^2 + (\rho-q)^2$

or $m^2 = \dfrac{(\rho-\rho\frac{\rho-q}{\rho+q})^2 + (\rho-q)^2}{(\rho\frac{\rho-q}{\rho+q}+q)^2} = 1.$

Similarly

$$|z|^2 = \left(\rho - (\rho-q)\,\frac{\rho\,\frac{(\rho-q)}{(\rho+q)}}{\rho\,\frac{\rho-q}{\rho+q}+q}\right)^2 + \left(\rho - \frac{\rho q - \rho q\,\frac{\rho-q}{\rho+q}}{\rho\,\frac{\rho-q}{\rho+q}+q}\right)^2$$
$$= \rho^2.$$

Thus $K = S(w,\rho)$ and X is an inner product space.

Now suppose the dimension of X is greater than two. Choose a basis $\{x_\alpha\}$ for X such that $\{x_\alpha\} \subset \mathrm{INT}(K)$. For any two-dimensional subspace L which contains an x_α then $K \cap L$ satisfies the hypotheses of the theorem and condition 2 so L is an inner product space by the above argument. Lemma 4.3 implies X is an inner product space.

It remains to show K is a sphere. First suppose the dimension of X is finite. If L is a two-dimensional subspace of X such that $L \cap \mathrm{INT}(K) \neq \emptyset$ then $L \cap K$ is a ball in L. Since K is compact there must exist a two-dimensional subspace L' such that the radius of this ball is maximal. Let w be the center of this ball and ρ be its radius. Choose $x,y \in L' \cap K$ so that $\|x-y\| = 2\rho$ and x and y are linearly dependent. Either x or y is not 0 so assume $x \neq 0$. If z is on the boundary of K and independent of x then $L'' \cap K$ satisfies the hypotheses

of the theorem.and condition 2 where $L'' = \text{Lin}\{z,x\}$. Hence $L'' \cap K$ is a ball (in L'') with radius $R \leq \rho$. But $2R \geq \| x-y \| = 2\rho$ so $R = \rho$ and the center of this ball is $w = 1/2(x+y)$. Thus $\| w-z \| = \rho$ and $K = S(w,\rho)$.

Finally suppose X is infinite dimensional. Let H_1 and H_2 be parallel supporting hyperplanes to K and $x_1 \in H_1 \cap K$. Let $y \in \text{INT}(K)$ and z be any point on the boundary of K. Then by the above argument $\text{Lin}\{x_1,x_2,y,z\} \cap K$ is a ball in $\text{Lin}\{x_1,x_2,y,z\}$. But $\text{Lin}\{x_1,x_2,y,z\} \cap H_1$ are parallel supporting hyperplanes to this ball so it has center $1/2(x_1+x_2)$ and radius $1/2\| x_1-x_2 \|$. Thus $\| 1/2(x_1+x_2)-z \| = 1/2\| x_1-x_2 \|$ and $K = S(1/2(x_1+x_2),1/2\| x_1-x_2\|)$.

q.e.d.

Condition 2 in 5.2 is somewhat weaker than assuming that from a point r outside of K then all support lines to K through r have the same length. The assumption that $x,y,$ and r are linearly dependent is particularly significant geometrically when $0 \in \text{INT}(K)$ but there is no need to assume this. Theorem 5.2 was stated for real spaces since complex spaces are usually considered as real spaces when supporting hyperplanes are discussed.

The natural way to generalize this theorem would be to use some subset of the class of all supporting hyperplanes of K. One method of doing this would be to add the condition in 2 that r must also lie on a given surface containing K. Theorem 5.5 shows that this condition no longer characterizes inner product spaces, however.

THESIS

DEFINITION 5.3: Let X be a real normed linear space and K be a closed convex subset with nonempty interior. A set K' is an E-set of K if for each $r \in K'$, $f,g \in X^*$, and $x,y \in K$ which satisfy

1. $f(x) = \sup_{z \in K} f(z)$
2. $g(y) = \sup_{z \in K} g(z)$
3. $f(r) = f(x)$, $g(r) = g(y)$, $r \in \mathrm{Lin}\{x,y\}$

then $\| r-x \| = \| r-y \|$.

DEFINITION 5.4: Let X be a real normed linear space. The set $D(X)$ denotes the vectors $r \in X$ such that there exist $f,g \in X^*$ and $x,y \in X$ satisfying

1. $\| x \| = \| y \| = 1$, $x \perp y$, $r \in \mathrm{Lin}\{x,y\}$
2. $|f(x)| = \| f \|$, $|g(y)| = \| g \|$
3. $f(r) = f(x)$, $g(r) = g(x)$.

THEOREM 5.5: If X is a real normed linear space then $D(X)$ is an E-set of the unit sphere iff X has symmetric orthogonality and is strictly convex.

Proof:

Suppose $D(X)$ is an E-set of the unit sphere. Let $\| x \| = \| y \| = 1$ and $x \perp y$. Choose $f,g \in X^*$ so that $f(x) = g(y) = \| f \| = \| g \| = 1$ and $f(y) = 0$. Let $r = x+(1-g(x))y$. Then by assumption $\| r-x \| = \| r-y \|$ so $|1-g(x)| = \| x-g(x)y \|$.
Now apply the hypothesis to $-x,y,-f,g$, and $r_1 = -x+(1+g(x))y$.

Then $\| r_1+x \| = \| r_1-y \|$ so $|1+g(x)| = \| x-g(x)y \|$. Hence $g(x) = 0$ and $y \perp x$. The fact that X must be strictly convex is obvious.

Suppose X has symmetric orthogonality and is strictly convex. Let $\| x \| = \| y \| = 1$ and $x \perp y$. Then there exist unique $f,g \in X^*$ such that $f(x) = g(y) = \| f \| = \| g \| = 1$. Moreover, $f(y) = g(x) = 0$ so $r = x+y$ satisfies $f(r) = f(x)$, $g(r) = g(y)$, and $\| r-x \| = \| r-y \|$. That all r are of this form follows from the symmetric orthogonality.

q.e.d.

With theorems 5.2 and 5.5 in mind we make the following conjecture which is a very difficult problem since it is unsolved even for the special case as a characterization of the circle in the Euclidean plane.

CONJECTURE 5.6: Let X be a real normed linear space and K be a closed, bounded, convex subset of X with nonempty interior. Then X is strictly convex with symmetric orthogonality and K is a sphere in X iff there exists a closed subset K' of X which contains K in its interior and whose boundary is an E-set of K.

2. A Conjecture by Hopf

The next theorem verifies a conjecture of Hopf [32] in normed spaces. It is also stronger than a result by

THESIS

Ohira which is a special case of theorem 1.31.

<u>THEOREM 5.7:</u> A real normed linear space X is an inner product space if and only if $\| x \| = \| y \| = 1$ and $x \perp y$ imply $\| x+y \| = \| x-y \|$.

<u>Proof:</u>

If X is an inner product space, $\| x \| = \| y \| = 1$, and $x \perp y$ then $\| x+y \| = \| x-y \| = 2$.

Suppose X has the above property. First we show that X has symmetric orthogonality. Let $\| x \| = \| y \| = \| ax+by \| = 1$, $x \perp y$, and $y \perp ax+by$. By appropriate choice of notation we may assume $a,b \geq 0$ so by orthogonality $1 \geq a \geq b \geq 0$. By assumption $\| x+y \| = \| x-y \|$ and $\| ax+(b+1)y \| = \| ax+(b-1)y \|$. Thus the convex function $g(r) = \| x+ry \|$ has the following properties: $g(0) = 1 \leq g(r)$ for $r \in R$, $g(1) = g(-1)$, and $g(\frac{b+1}{a}) = g(\frac{b-1}{a})$. But $\frac{b+1}{a} \geq 1$ and $0 \geq \frac{b-1}{a} \geq -1$. Hence $g(\frac{b+1}{a}) \geq g(1) \geq 1$ and $g(-1) \geq g(\frac{b-1}{a}) \geq 1$ which implies either $a = 1$ and and $b = 0$ or $\| x+ry \| = 1$ for $\frac{b+1}{a} \geq r \geq -1$. If $a = 1$ and $b = 0$ then $y \perp x$. If $\| x+ry \| = 1$ for $\frac{b+1}{a} \geq r \geq -1$ then $a^{-1} = \| x+\frac{b}{a} y \| = 1$ so it remains to determine b. Since $y \perp x+by$ and $1 = \| y \| = \| y+(x+by) \| = \| y - (x+by) \|$ we have $1 = \| y+r(x+by) \|$ for $-1 \leq r \leq +1$, $y+r(x+by) \perp x + by$ for $-1 \leq r \leq +1$, and $x+ry \perp y$ for $b+1 \geq r \geq -1$. In particular $x+(b+1)y \perp x + by$, $x+(b-1)y \perp y$, $x+(b-1)y \perp x + by$, and $x + (b+1)y \perp y$.

THESIS

But if $b \neq 0$ then the unit sphere at $x+(b-1)y$ has a unique tangent line (namely one in the direction of y) contrary to the above calculations. Hence $b = 0$ and $y \perp x$. Also note that these equations imply orthogonality is right unique so that X is strictly convex.

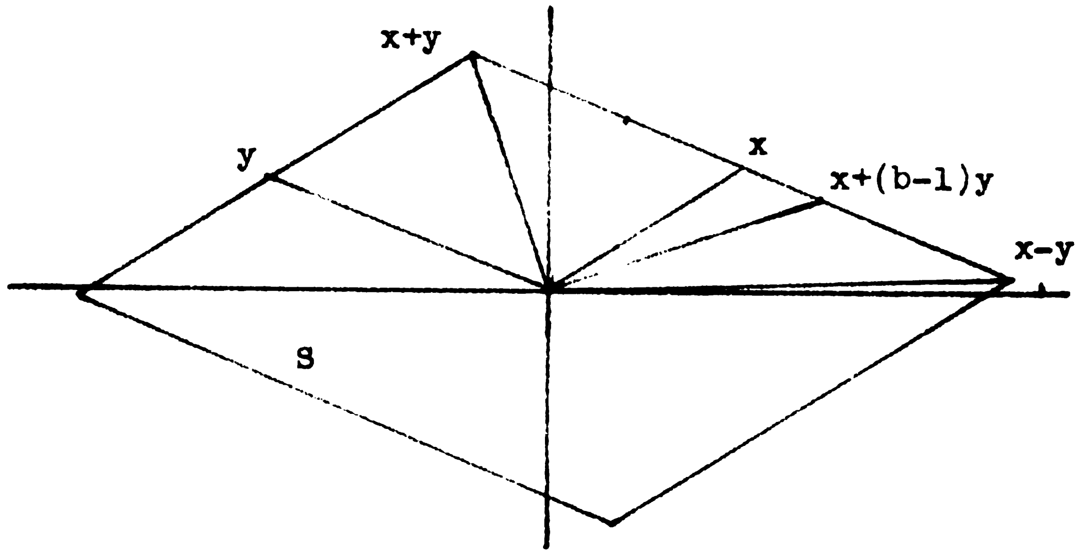

FIGURE 5.3

If the dimension of X is three or greater this implies X is an inner product space. Thus we now assume the dimension of X is two. We use construction I of chapter 3 (i.e. the points of X are those of R^2 and the norm is the one determined by the unit sphere S graphically represented by $\{(x,f(x)) \mid 0 \leq x \leq 1\}$ in the first quadrant and by $\{(^1/ xf'(x)-f(x),\ ^{f'(x)}/xf'(x)-f(x))\}$ in the second quadrant) to describe X. Also note that the norm of X is Gateaux differentiable so $f'(x)$ exists on $[0,1)$.

Choose $0<a,b,c,d,<1$ so that : $-a = {}^1/af'(a)-f(a)$, $f(b) = {}^{f'(b)}/bf'(b)-f(b)$, $f(c) = c$, and $f'(d) = -1$. What we show is that $a = b = c = d = {}^1/\sqrt{2}$ and that $(1,1) \perp (-1,1)$.

THESIS

Since we could start with any pair of orthogonal unit vectors as coordinate axes this implies that if $\| x \| = \| y \| = 1$ and $x \perp y$ then $\| x+y \| = \| x-y \| = \sqrt{2}$ and $x+y \perp x-y$.

Since $1/c = \| (1,1) \| = \| (-1,1) \| = d + f(d)$ this gives us one relationship between c and d. Also by convexity of the unit sphere the tangent line at $(d,f(d))$ must intersect $y = x$ at a point with x-coordinate at least c. The point of intersection is $(d+f(d)/2,\ d+f(d)/2)$ so $d+f(d)/2 \geq 1/d+f(d)$ or $d+f(d) \geq \sqrt{2}$. Hence $\| (1,1) \| \geq \sqrt{2}$ and by the above remark this also applies to all r,s with $\| r \| = \| s \| = 1$ and $r \perp s$. Thus

$\| (a+1/af'(a)-f(a),\ f(a) + f'(a)/af'(a)-f(a)) \| =$

$\| (0, 1/a) \| = 1/a \geq \sqrt{2}$ and

$\| (b - 1/bf'(b)-f(b),\ f(b) - f'(b)/bf'(b)-f(b)) \| =$

$\| (1/f(b) , 0) \| = 1/f(b) \geq \sqrt{2}.$

We now consider cases to see which relative orders of a,b,c,d are permissible.

Suppose $c < d$.

Case 1. If $a \leq c < d$ then

$c = -1/df'(d)-f(d) < -1/cf'(c)-f(c) \leq -1/af'(a)-f(a) = a$

which is a contradiction.

Case 2. If $c < d \leq a$ then

$a = -1/af'(a)-f(a) \leq -1/df'(d)-f(d) = c$

which is again a contradiction.

Case 3. If $c < a < d$ then

$f(a)/a < 1$ and $f'(a) > -1$. Hence

$$1 = af(a)-a^2f'(a) = a^2({}^{f(a)}/_{a} - f'(a)) < 2a^2 \leq 1$$

which is also a contradiction.

Now suppose $d < c$.

<u>Case 4.</u> If $b \leq d < c$ then

$c = f(c) < f(d) \leq f(b)$ and

$$\frac{f'(c)}{cf'(c)-f(c)} > \frac{f'(d)}{df'(d)-f(d)} = c \geq \frac{f'(b)}{bf'(b)-f(b)} = f(b)$$

which is a contradiction.

<u>Case 5.</u> If $d < c \leq b$ then $f(b) \leq f(c) = c < f(d)$ and

$$f(b) = \frac{f'(b)}{bf'(b)-f(b)} \geq \frac{f'(c)}{cf'(c)-f(c)} > \frac{f'(d)}{df'(d)-f(d)} = c$$

which is again a contradiction.

<u>Case 6.</u> If $d < b < c$ then $f'(b) < -1$ and $f(b) > b$.

Then $-1 > f'(b) = \dfrac{f^2(b)}{bf(b)-1}$

$$1-bf(b) < f^2(b)$$

$$0 < f^2(b) + bf(b) - 1 \leq 2f^2(b) - 1$$

$${}^1/\sqrt{2} < f(b)$$

which is also a contradiction.

Hence $c = d$ which implies $c = {}^1/\sqrt{2}$ which in turn implies $a = b = c$.

We may now complete the proof. From the above results we see that f on $[{}^1/\sqrt{2}\, , 1]$ determines the entire unit sphere. Moreover, $f'(x)$ exists except for $x = 1$ and $f''(x)$ exists almost everywhere but is never equal to zero. In particular for $x \in [{}^1/\sqrt{2}, 1)$ then

${}^1/\sqrt{2}\ (x + {}^1/_{xf'(x)-f(x)},\ f(x) + {}^{f'(x)}/_{xf'(x)-f(x)})$ and

${}^1/\sqrt{2}\ (x - {}^1/_{xf'(x)-f(x)},\ f(x) - {}^{f'(x)}/_{xf'(x)-f(x)})$ are

orthogonal unit vectors. For those points x where $f''(x)$ exists we have

$$\frac{f(x) - f'(x)/_{xf'(x)-f(x)}}{x - 1/_{xf'(x)-f(x)}} = \frac{\frac{d}{dx}\left(f(x) + f'(x)/_{xf'(x)-f(x)}\right)}{\frac{d}{dx}\left(x + 1/_{xf'(x)-f(x)}\right)}$$

which simplifies to $f''(x) = (xf'(x)-f(x))^3$. This differential equation must also satisfy the boundary conditions $f(1) = 0$, $f'(1) = -\infty$, $f(1/\sqrt{2}) = 1/\sqrt{2}$, and $f'(1/\sqrt{2}) = -1$. The equation is not very difficult to solve and by the boundary conditions has a unique solution. But we already know one solution, namely $f(x) = \sqrt{1 - x^2}$ and hence it is the solution. Thus X is an inner product space.

q.e.d.

There are two immediate corollaries.

COROLLARY 5.8: Let X be a real normed linear space. Then X is an inner product space iff there exists a constant c such that $\| x \| = \| y \| = 1$ and $x \perp y$ then $\| x+y \| = c$.

COROLLARY 5.9: Let X be a real normed linear space. Then X is an inner product space iff projectional orthogonality implies isosceles orthogonality.

It is interesting to note that von den Steinen [68] proved a complex version of theorem 5.7.

THESIS

3. On a Theorem of Lorch

The following theorem was given by Lorch [49], but does not appear to be very well known.

<u>THEOREM 5.10:</u> If X is a Banach space then X is an inner product space iff there exists $T: X \dashrightarrow X^*$ such that T is linear and bounded, T^{-1} exists and is bounded, and $[T(x)](x) = \|T(x)\| \|x\|$.

Similar theorems were also proven by Day [18] and Kakutani [38]. Such a property may be considered as a self adjointness property on X. Theorem 5.11 shows that many of the hypotheses of 5.10 can be removed or weakened. Also we give a proof which is independent of 5.10 and which is shorter than the original proof.

<u>THEOREM 5.11:</u> Let X be a normed linear space. There exists a linear operator $T: X \dashrightarrow X^*$ such that $[T(x)](x) = \|T(x)\| \|x\|$ for all x and $T \neq 0$ iff X is an inner product space.

<u>Proof:</u>

If X is an inner product space then the existence of T follows from the classical representation theorems for linear functionals.

Suppose T exists. First we wish to show T is one-to-one. If T is not one-to-one there exists $x \in X$ such

that $x \neq 0$, and $T(x) = 0$. Since $T \neq 0$ there exists $y \in X$, $y \neq 0$, and $T(y) \neq 0$. Now consider two cases. If $[T(y)](x) = 0$ let $r = 3\|y\| / \|x\|$. Because $T(rx+y) = T(y)$ we have

$$\|T(y)\| \, \|rx+y\| = [T(y)](rx+y) = [T(y)](y) = \|T(y)\| \, \|y\|$$

Since $\|T(y)\| \neq 0$

$$\|y\| = \|rx+y\| \geq \big| \, \|rx\| - \|y\| \, \big| = 2\|y\|$$

which is a contradiction.

If $[T(y)](x) \neq 0$ let $r = -\|T(y)\| \, \|y\| / [T(y)](x)$. Again $T(rx+y) = T(y)$ so $\|T(y)\| \, \|rx+y\| = [T(y)](rx+y) = 0$. Since $\|T(y)\| \neq 0$ we have $\|rx+y\| = 0$ which is a contradiction since x and y must have been linearly independent. Thus T must be one-to-one.

Now suppose the dimension of X is two. Then T must also be onto and bounded. First we wish to show that X is strictly convex and the norm is Gateaux differentiable. To show both of these it suffices to show $|[T(x)](y)| < \|T(x)\| \, \|y\|$ if x and y are linearly independent. Suppose to the contrary that there exist $x, y \in X$ such that $\|x\| = \|y\| = 1$, $\|x-y\| > 0$, and $[T(x)](y) = \|T(x)\|$. Then $x+y \perp x-y$ and there is an unique line which supports the unit sphere at $1/2(x+y)$. Thus $[T(x+y)](x-y) = 0 = [T(x)](x) + [T(y)](x) - [T(x)](y) - [T(y)](y)$ and hence $[T(y)](x) = [T(y)](y)$. But this implies $\|T(x-y)\| \, \|x-y\| = [T(x-y)](x-y) = 0$ which is a contradiction.

THESIS

Thus X is strictly convex and Gateaux differentiable and by the results of James [34] X^* has the same properties.

Since X is reflexive these imply in particular $r \perp s$ iff $[T(r)](s) = 0$ iff $T(s) \perp T(r)$. By theorem 3.1 we can find $x, y \in X$ such that $x \perp y$ and $y \perp x$. Then

$$\begin{aligned} ax + by \perp cx + dy \quad & \text{iff } [T(ax+by)](cx+dy) = 0 \\ & \text{iff } ac[T(x)](x) + bd[T(y)](y) = 0 \\ & \text{iff } [T(cx+dy)](ax+by) = 0 \\ & \text{iff } cx + dy \perp ax + by. \end{aligned}$$

Thus $x \perp y$ implies $T(x) \perp T(y)$ and by lemma 3.3 there exists k such that $\| T(x) \| = k \| x \|$ for $x \in X$. Hence

$$\begin{aligned} \| x+y \|^2 + \| x-y \|^2 &= k^{-1}([T(x+y)](x+y) + [T(x-y)](x-y)) \\ &= k^{-1}(2[T(x)](x) + 2[T(y)](y)) \\ &= 2(\| x \|^2 + \| y \|^2). \end{aligned}$$

Thus X is an inner product space.

Now suppose the dimension of X is greater than two. Let H be any two-dimensional subspace of X and $H' = T(H)$. By the hypothesis of the theorem $\| T(z)|_H \| = \| T(z) \|$ for $z \in H$ so that H' is congruent to H^*. By the two-dimensional case H is an inner product space and hence X is an inner product space.

q.e.d.

4. On Ficken's Theorem

In chapter 2 we gave an alternate proof of Ficken's theorem and from theorem 4.15 Ficken's theorem would follow as a corollary. We give two more proofs which are based on

classical characterizations of ellipsoids that can be found in Busemann [12].

<u>THEOREM 5.12:</u> A closed convex surface C is an ellipsoid iff the locus of the midpoints of any family of parallel chords in contained in a hyperplane.

<u>THEOREM 5.13:</u> A closed convex surface C is an ellipsoid iff there is a fixed point z inside of C such that for each pair $p,q \in C$ there exists an affine transformation of the space onto itself which maps C onto itself, takes p to q, and fixes z.

The proofs are mostly a matter of interpreting Ficken's theorem in terms of the above two theorems.

<u>THEOREM:</u> Let X be a normed linear space. Then X is an inner product space iff $\| ax+by \| = \| bx+ay \|$ for all $a,b \in R$ and $x,y \in X$ such that $\| x \| = \| y \| = 1$.

<u>Proof I:</u>

As usual we only prove the sufficiency of the condition and obviously we may assume that the dimension of X is two.

Let $x,y, ax+by, cx+dy \in S$ the unit sphere of X and $(ax+by) - (cx+dy) = \lambda(x-y)$ for some real number λ (i.e. the chord joining $ax+by$ to $cx+dy$ is parallel to the chord joining x to y). Since the locus of midpoints of chords parallel to $x-y$ goes through the origin it will

THESIS

suffice to show $(ax+by) + (cx+dy) = \mu(x+y)$ for some real number μ. We may assume that the segment joining $ax + by$ to $cx + dy$ does not lie on the unit sphere, since this case follows by continuity, so that $cx + dy$ is uniquely determined by $ax + by$. But $bx + ay$ has all of the properties attributed to $cx + dy$, whence $(cx+dy) + (ax+by) = (a+b)(x+y)$. Thus by theorem 5.12 S is an ellipse and hence X is an inner product space.

Proof II:

If $x, y \in X$ and $\|x\| = \|y\| = 1$ then by hypothesis the mapping $\phi : ax+by \longrightarrow bx+ay$ is an isometry on X and hence maps the unit sphere S onto itself. Thus ϕ is an affine transformation of the points of X onto themselves which maps S onto itself, x to y, and fixes 0. By theorem 5.13 S is an ellipsoid and thus X is an inner product space.

q.e.d.

For still another proof of the theorem see Day [20].

5. Comments on Principle 5.1

In this section we give some more specific examples based on principle 5.1. We begin by stating some theorems from plane geometry and reformulating them in terms of linear spaces. In order to do this we make the following definitions.

DEFINITION 5.14: Let X be a real normed linear space and $x,y \in X$. Then we may consider the triangle with vertices $0,x,y$ and let the base be the side determined by x and y. An altitude to the base is any vector $ax+by$ such that $ax + by \perp x - y$.

DEFINITION 5.15: Let X be a real normed linear space and $x,y \in X$. A vector $z = ax + by$ is said to lie on the bisector of the angle between x and y if the Sundaresan angles $<z,x>$ and $<z,y>$ are equal (i.e. $\min_{\alpha} \| z - \alpha x \| = \min_{\alpha} \| z - \alpha y \|$ so that the bisector may be considered as the set of points equidistant from the sides of the angle).

These definitions are used primarily as motivation and insight into the theorems discussed. With this in mind we state the following Euclidean theorems and their linear space analogues.

(5.16) For $1 \leq i \leq 9$ i-N is a real normed linear space analogue of the Euclidean theorem i-E.

1-E. The altitudes to the equal sides of an isosceles triangle are equal.

1-N. If $\| x \| = \| y \|$ and $x-by \perp y$ and $y-ax \perp x$ then $\| x-by \| = \| y-ax \|$.

2-E. The altitudes of an isosceles triangle are copunctal.

2-N. If $\| x \| = \| y \|$, $y-ax \perp x$, and $x-by \perp y$ then $a(1-b)x + b(1-a)y \perp x-y$.

3-E. The median to the base of an isosceles triangle is also an altitude.

3-N. If $\| x \| = \| y \|$ then $x + y \perp x - y$.

4-E. The median to the base of an isosceles triangle lies on the angle bisector of the vertex angle.

4-N. If $\| x \| = \| y \|$, $y-ax \perp x$, and $x-by \perp y$ then $\| x-by \| = \| y-ax \|$.

5-E. If a,b,c are the vertices of an isosceles triangle with $ab = ac$ and e,f are the feet of the altitudes to the equal sides then the median to the side ef of the triangle a,e,f bisects the angle at a.

5-N. If $\| x \| = \| y \|$, $x-ay \perp y$, and $y-bx \perp x$ then $|b| \; \| x-ay \| = |a| \; \| y-bx \|$.

6-E. If p is a point outside a circle C and if x,y are points on C such that the lines px and py are tangent to the circle then p lies on the bisector of the angle xpy.

6-N. If $\| x \| = \| y \|$, $x-ay \perp y$, $y-bx \perp x$, $x \perp y-cx$, and $y \perp x-dy$ then $| 1-d | \; \| y-bx \| = | 1-c | \; \| x-ay \|$.

7-E. If p is the intersection of the altitudes to the equal sides of an isosceles triangle then p lies on the bisector of the vertex angle.

7-N. If $\| x \| = \| y \|$, $x-ay \perp y$, and $y-bx \perp x$ then $a = b$.

8-E. If p is the intersection of the altitudes to the equal sides of an isosceles triangle then p lies on the median to the base.

THESIS

8-N. If $\| x \| = \| y \|$, $x-ay \perp y$, and $y-bx \perp x$ then $a = b$.

9-E. The bisectors of the angles of a triangle are copunctal.

9-N. If $x,y \in X$ there exists $e,f \in R$ such that

$$\min_{\alpha} \| ex + fy - \alpha x \| = \min_{\alpha} \| ex + fy - \alpha y \| = \min_{\alpha} \| ex+fy-\alpha(x-y) \|$$

There are naturally many more possibilities, but these seem to be rather representative in both the statement of results and the methods of proof. Note that 1-N and 4-N are the same and they were used in 3.5. Also note that 3-N is another way of saying isosceles orthogonality implies projective orthogonality. Corollary 2.11 is based on 7-N or 8-N. Finally 9-N is true in any real normed linear space.

Before giving more theorems based on 5.16 we prove the following lemma.

LEMMA 5.17: Let X be a real normed linear space which is strictly convex and has symmetric orthogonality. If $x,y \in X$, $a,b \in R$, $x - by \perp y$, and $y - ax \perp x$ then $ab \geq 0$.

Proof:

It will suffice to prove the result when the dimension of X is two since the higher dimensional cases are obvious.

Coordinatize so that $x = (\| x \|, 0)$, $z = (0,1)$, and $x \perp z$. Let $y = (c,d)$. Since X is strictly convex $c = a\| x \|$. From construction I in chapter 3 it follows that

THESIS

$(\|x\| - bc)\ (bd)\ (c)\ (d) \geq 0$

$(1 - ab)\ (ab) \geq 0$

but $|ab| \leq 1$ so $ab \geq 0$.

q.e.d.

The first theorem is based on 2-N.

THEOREM 5.18: Let X be a real normed linear space. Then X is an inner product space iff $x, y \in X$ and $a, b \in R$ satisfying $\|x\| = \|y\| = 1$, $y - ax \perp x$, and $x - by \perp y$ then $a(1-b)x + b(1-a)y \perp x - y$.

Proof:

To establish the necessity of the last condition we prove the stronger result that if $x, y \in X$, $y - ax \perp x$, and $x - by \perp y$ then $a(1-b)x + b(1-a)y \perp x - y$. It is easy to verify $a = {}^{(y|x)}/_{\|x\|^2}$, $b = {}^{(y|x)}/_{\|y\|^2}$, and that $(a(1-b)x + b(1-a)y \mid x - y) = 0$.

To establish the sufficiency of the condition suppose $\|x\| = \|y\| = 1$, $y - ax \perp x$, and $x - by \perp y$. Then the pair $x, -y \in X$ has the properties that $\|x\| = \|-y\| = 1$, $(-y) - (-a)x \perp x$, and $x - (-b)(-y) \perp -y$ so that $a(1+b)x - b(1+a)y \perp x + y$.

Now suppose $\|x\| = \|y\| = 1$, $y - ax \perp x$, $x \perp y$. If $a = 0$ then $y \perp x$. If $a \neq 0$ then by hypothesis $x \perp x + y$ and $x \perp x - y$. Thus $\text{Lin}\{x, y\}$ is an l_1 plane and hence $y \perp x$. In any case X has symmetric orthogonality.

For the remainder of the proof we may assume the dimenson of X is two. Also in the above argument the only

time orthogonality was not left unique was in the l_1 plane. But the l_1 plane does not have symmetric orthogonality so orthogonality in X is actually left unique. Hence X is strictly convex and the norm is Gateaux differentiable.

Suppose X is expressed as in Construction I in chapter 3. Then the unit sphere S in the first quadrant is given by $\{(x,f(x))\}$ and $f'(x)$ exists on $[0,1)$. Let $D = \{x \mid 0 \le x \le 1$ and $f'(x) = {}^{-x}/_{f(x)}$ or $x = 1\}$.

There exist $x_1, x_2 \in [0,1]$ such that $x_1^2 + f^2(x_1) \le x^2 + f^2(x) \le x_2^2 + f^2(x_2)$ for $x \in [0,1]$. If S is not a circle then at least one of the values x_1, x_2 is distinct from 0 and 1. By elementary considerations $x_1, x_2 \in D$. If $D \ne [0,1]$ there exist $x_3, x_4 \in [0,1]$ at least one of which is neither 0 or 1 and such that $[x_3, x_4] \cap D = \{x_3, x_4\}$. Let x be $(x_3, f(x_3))$ and y be $(x_4, f(x_4))$ and $|\ |$ be the norm determined by the unit circle. Then $x - by \perp y$ and $y - ax \perp x$ where $a = {}^{(y|x)}/_{|x|^2}$ and $b = {}^{(y|x)}/_{|y|^2}$ since $x_3, x_4 \in D$. Because x and y lie in the first quadrant and at least one does not lie on the coordinate axes it follows that $0 < a,b < 1$ since X is strictly convex. By hypothesis $a(1-b)x + b(1-a)y \perp x - y$. From the relationships on a and b we have $(x_5, f(x_5)) = {}^{a(1-b)x + b(1-a)y}/_{\| a(1-b)x+b(1-a)y\|}$ satisfies $x_3 < x_5 < x_4$ and $x_5 \in D$. But this is a contradiction to the choice of x_3 and x_4. Thus $D = [0,1]$ and $f'(x) = {}^{-x}/_{f(x)}$ for $x \in [0,1]$. Hence $f(x) = (1-x^2)^{1/2}$ and X is an inner product space.

q.e.d.

THESIS

Theorem 5.19 is based on 6-N.

<u>THEOREM 5.19:</u> If X is a real normed linear space then the following are equivalent:

1. X is an inner product space

2. If $x, y \in X$ and $a, b, c, d \in R$ satisfy $\| x \| = \| y \| = 1$, $x - ay \perp y$, $y - bx \perp x$, $x \perp y - cx$, and $y \perp y - dx$ then $| 1 - d | \, \| y - bx \| = | 1 - c | \, \| x - ay \|$.

<u>Proof:</u>

(1==>2) This is obvious since $a = b = c = d = (x|y)$.

(2==>1) Suppose $x \perp y$, $\| x \| = \| y \| = 1$, $y \perp x - dy$, and $y - bx \perp x$. By hypothesis $| 1 - d | \, \| y - bx \| = 1$. Now apply the hypothesis to $-x$ and y. Then $| 1 + d | \, \| y - bx \| = 1$. Hence $| 1 - d | = | 1 + d |$, whence $d = 0$. Thus X has symmetric orthogonality and is strictly convex.

Hence $a = d$, $b = c$ and by theorem 3.5 $\| y-bx \| = \| x-ay \|$. By theorem 2.14 $|a|, |b| \leq 1$. Thus condition 2 reduces $| 1 - a | = | 1 - b |$ whence $a = b$. Hence $L(x,y) = L(x,y)$ whenever $\| x \| = \| y \| = 1$ and by corollary 2.11 X is an inner product space.

q.e.d.

5-N provides the following theorem.

<u>THEOREM 5.20:</u> If X is a real normed linear space then the following are equivalent:

1. X is an inner product space.

THESIS

2. If $\| x \| = \| y \| = 1$, $x - ay \perp y$, and $y - bx \perp x$ then $|b| \; \| x - ay \| = |a| \; \| y - bx \|$

Proof:

(1==>2) This is again obvious since $a = b = (x|y)$.

(2==>1) Condition 2 clearly implies $a = 0$ iff $b = 0$ which implies X has symmetric orthogonality and is strictly convex. Since $\| x - ay \| = \| y - bx \|$ for such a space we have $|a| = |b|$. By lemma 5.17 this implies $a = b$ so by corollary 2.11 X is an inner product space.

q.e.d.

Since 3-N has already been used (theorem 1.31) we prove a slightly stronger theorem.

THEOREM 5.21: If X is a real normed linear space then the following are equivalent:

1. X is an inner product space.

2. If $x, y \in X$ satisfy $\| x \| = \| y \| = 1$, $x \perp y$, then $x + y \perp x - y$.

Proof:

(1==>2) This is obvious.

(2==>1) First note that $x \perp y$ and $\| x \| = \| y \| = 1$ imply $x \perp -y$ and $\| x \| = \| -y \| = 1$ so that $x + y \perp x - y$ and $x - y \perp x + y$.

It suffices to prove the theorem when the dimension of X is two. Choose $x, y \in X$ such that $x \perp y$, $y \perp x$, and $\| x \| = \| y \| = 1$. Assume the unit sphere S is given

THESIS

in polar coordinates by $(f(\theta),\theta)$ for $0 \leq \theta < 2\pi$, where $(1,0) = x$ and $(1,\pi/2) = y$. As θ varies from θ to $\pi/2$ one of the unit vectors orthogonal to it varies from $\pi/2$ to π and the sum of these two vectors varies in angle from $\pi/4$ to $3\pi/4$. But the sum and difference vectors are mutually orthogonal so this implies X has symmetric orthogonality.

Suppose $x \perp y$ and $\|x\| = \|y\| = 1$. Then

$$\frac{x+y}{\|x+y\|} + \frac{x-y}{\|x-y\|} \perp \frac{x+y}{\|x+y\|} - \frac{x-y}{\|x-y\|}$$

or

$$(1/\|x+y\| + 1/\|x-y\|)\, x + (1/\|x+y\| - 1/\|x-y\|)\, y \perp$$
$$(1/\|x+y\| - 1/\|x-y\|)\, x + (1/\|x+y\| + 1/\|x-y\|) y.$$

If $\|x+y\| \neq \|x-y\|$ this implies $ax + by \perp bx + ay$ where $x \perp y$, $\|x\| = \|y\|$, $a = 1/\|x+y\| + 1/\|x-y\|$, $b = 1/\|x+y\| - 1/\|x-y\|$, and $a,b \neq 0$. But obviously from construction I this cannot happen since orthogonal vectors cannot lie in the same quadrant. Thus $\|x+y\| = \|x-y\|$. By theorem 5.7 X is an inner product space.

q.e.d.

BIBLIOGRAPHY

THESIS

BIBLIOGRAPHY

1. L. Adauto da Justa Medeiros, "Existence of a scalar product in normed vector spaces", Gaz. Mat. (Lisbon) 22 (1961), no. 84/85, 1-6.

2. G. Birkhoff, "Orthogonality in linear metric spaces", Duke Math. J. 1 (1935), 169-172.

3. W. Blaschke, Kreis und Kugel, Leipzig (1916).

4. L. M. Blumenthal and E. Z. Andalafte, "Metric characterizations of Banach and Euclidean spaces", Fund. Math. 55 (1964), 23-55.

5. L. M. Blumenthal, "Note on normed linear spaces", La Revista de la real Acad. de Ciencias Exactias, 72 (1968), 307-310.

6. ——, "Extension of a theorem of Jordan and von Neumann", Pacific J. Math. 5 (1955), 161-167.

7. ——, "Generalized Euclidean space in terms of a quasi inner product", Amer. J. Math. 62 (1950), 686-698.

8. ——, Theory and Applications of Distance Geometry, Clarendon, Oxford (1953).

9. F. Bohenblust, "A characterization of complex Hilbert spaces", Portugaliae Mathematic 3 (1962), 103-109.

10. R. Bonic and F. Reis, "A characterization of Hilbert space", An Acad. Brasis. Ci. 38 (1966), 239-241.

11. T. Bonnesen and W. Fenchel, Theorie der konvexen Kopper, Springer Verlag (1934).

12. H. Busemann, The Geometry of Geodesics, Academic Press, New York (1955).

13. S. O. Carlsson, "Orthogonality in normed linear spaces", Ark. Mat. 4 (1962), 297-318.

14. Clarkson, "Uniformly convex spaces", Trans. Amer. Math. Soc. 40 (1936), 396-414.

THESIS

15. W.W. Comfort and H. Gordon, "Inner product spaces and the tri-spherical intersection property", Proc. Amer. Math. Soc. 12 (1961), 327-329.

16. H. F. Davis, "On isosceles orthogonality", Math. Mag. 32 (1958/59), 129-131.

17. M. M. Day, "Polygons circumscribed about closed convex curves", Trans. Amer. Math. Soc. 62 (1947), 315-319.

18. ———, "Some characterizations of inner product spaces", Trans. Amer. Math. Soc. 62 (1947), 320-337.

19. ———, "On criteria of Kasahara and Blumenthal for inner product spaces", Proc. Amer. Math. Soc. 10 (1959), 92-100.

20. ———, Normed Linear Spaces, Springer Verlag (1958).

21. D. G. Defigueredo and L. A. Karlovitz, "On the radial projection in normed spaces", Bull. Amer. Math. Soc. 73 (1967), 364-368.

22. A. Dvoretzky, "A characterization of Banach spaces isometric to inner product spaces", Proc. Coll. on Conv. (Copenhagen, 1965) pp. 61-66, Kobenhavns Univ. Mat. Inst. Copenhagen, 1967.

23. ———, "Some results on convex bodies and Banach spaces", Proc. Internat. Sympos. Linear Spaces, (Jerusalem, 1960), pp. 123-160. Jerusalem Academic Press, Jerusalem; Oxford: 1961.

24. M. Edelstein and A. C. Thompson, "Contractions, isometries, and some properties of inner product spaces", Nederl. Akad. Wetensch. Proc. Ser. A 70 = Indag. Math. 29 (1967), 326-331.

25. D. A. Ellis, "A modification of the parallelogram law characterization of Hilbert space", Math. Z. 59 (1953), 94-96.

26. F. A. Ficken, "Note on the existence of scalar products in normed linear spaces", Ann. of Math. (2) 45 (1944), 362-366.

27. R. W. Freese, "Criteria for inner product spaces", Proc. Amer. Math. Soc. 19 (1968), 953-958.

28. R. E. Fullerton, "Geometrical characterizations of certain function spaces", Proc. Internat. Sympos. Linear Spaces, (Jerusalem, 1960), pp. 227-236. Jerusalem Academic Press, Jerusalem; Oxford: 1961.

THESIS

29. J. R. Giles, "Classes of semi-inner-product spaces", Trans. Amer. Math. Soc. 129 (1967), 436-446.

30. M. L. Gromov, "On a geometric hypothesis of Banach", Izv. Akad. Nauk. SSSR, 31 (1967), 1105-1114.

31. T. L. Hayden, "The extension of bilinear functionals", Pacific J. Math. 22 (1967), 99-108.

32. E. Hopf, "Zur Kennzeichnung der Euklidischen Norm", Math. Z. 72 (1959/60), 76-81.

33. R. C. James, "Orthogonality in normed linear spaces", Duke Math. J. 12 (1945), 291-302.

34. -----, "Orthogonality and linear functionals in normed linear spaces", Trans. Amer. Math. Soc. 61 (1947), 265-292.

35. -----, "Inner products in normed linear spaces", Bull. Amer. Math. Soc. 53 (1947), 559-566.

36. J. T. Joichi, "More characterizations of inner product spaces", Proc. Amer. Math. Soc. 19 (1968), 1185-1186.

37. P. Jordan and J. von Neumann, "On inner products in linear metric spaces", Ann. of Math. (2) 36 (1935), 719-732.

38. S. Kakutani, "Some characterizations of Euclidean space", Jap. J. Math. 16 (1934), 93-97.

39. S. Kakutani and G. Mackey, "Two characterizations of real Hilbert space", Ann. of Math. (2) 45 (1944), 50-58.

40. -----, "Ring and lattice characterizations of complex Hilbert space", Bull. Amer. Math. Soc. 52 (1946), 727-733.

41. S. Kasahara, "A characterization of Hilbert space", Proc. Japan. Acad. 30 (1954), 846-848.

42. D. C. Kay, "A remark on R. W. Freese's characterization of inner product spaces", (unpublished).

43. W. A. Kirk and M. F. Smiley, "Another characterization of inner product spaces", Amer. Math. Monthly 71 (1964), 890-891.

44. V. Klee, "Circumspheres and inner products", Math. Scand. 8 (1960), 363-370.

THESIS

45. I. Kolumban, "A theorem regarding the best approximation in abstract space", Mathematica (cluj) 2 (25) (1960), 277-280.

46. T. Kubota, "Ein neuer Aufbau der euklidischen Geometrie in der affinen Ebene", Monatsh. Math. Phys. 48 (1939), 96-102.

47. V. V. Kuzimina, "A metric space with middle elements", Leningrad Inz. Ekonom. Inst. Trudy. Vyp. 63 (1966), 18-30.

48. J. Lindenstrauss, "On the modulus of smoothness and divergent series in Banach spaces", Michigan Math. J. 10 (1963), 241-252.

49. E. R. Lorch, "The Cauchy-Schwarz inequality and self-adjoint spaces", Ann. of Math. (2) 46 (1945), 468-473.

50. ——, "On certain implications which characterize Hilbert spaces", Ann. of Math. (2) 49 (1948), 522-532.

51. G. Lumer, "Semi-inner-product spaces", Trans. Amer. Math. Soc. 100 (1961), 29-43.

52. K. Ohira, "On a certain complete, separable and metric space", Mem. Fac. Sci. Kyusyu Univ. A. 6 (1951), 9-15.

53. ——, "On some characterizations of abstract Euclidean spaces by properties of orthogonality", Kumamoto J. Sci. Ser. A.1. (1952), 23-26.

54. Hazel Perfect, "Pythagorean orthogonality in a normed linear space", Proc. Edinburg Math. Soc. (2) 9 (1958), 168-169.

55. R. S. Phillips, "A characterization of Euclidean spaces", Bull. Amer. Math. Soc. 46 (1940), 930-933.

56. Niko Prijatelj, "Algebra characterization of Hilbert spaces", Publ. Dept. Math. (Ljubljana) 1 (1964), 43-57.

57. J. Radon, "Uber eine besondere Art ebener konvexer Kurven", Ber. Sachs. Akad. Wiss. Leipzig 68 (1916), 131-134.

58. M. M. Rao, "Characterizing Hilbert spaces by smoothness", Nederl. Akad. Wetensch. Proc. Ser. A 70 = Indag. Math. 29 (1967), 132-135.

59. H. Rubin and M. H. Stone, "Postulates for generalizations of Hilbert space", Proc. Amer. Math. Soc. 4 (1953), 611-616.

60. W. Rudin and K. T. Smith, "Linearity of best approximations: A characterization of ellipsoids", Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math. 23 (1961), 97-103.

61. J. J. Schaeffer, "Another characterization of Hilbert spaces", Studia Math. 25 (1965), 271-276.

62. J. Schoenberg, "A remark on M. M. Day's characterization of inner product spaces", Proc. Amer. Math. Soc. 3 (1952), 961-964.

63. S. O. Schonbeck, "Extensions of nonlinear contractions", Bull. Amer. Math. Soc. 72 (1966), 99-101.

64. D. A. Senechalle, "A characterization of inner product spaces", Proc. Amer. Math. Soc. 19 (1968), 1306-1312.

65. -----, "Isometry between two-dimensional subspaces implies an inner product", (unpublished).

66. A. Sobczyk, "Minkowski planes", Math. Ann. 173 (1967), 181-190.

67. W. L. Stamey, "On generalized euclidean and non-euclidean spaces", Pacific J. Math. 7 (1957), 1505-1511.

68. J. von den Steinen, "Zur Kennzeichnung der Euklidischen Norm vermoege einiger Eigenschaften der komplexen Projektionalorthogonalitaet", Math. Ann. 154 (1964), 166-170.

69. W. J. Stiles, "On inversions in normed linear spaces", Proc. Amer. Math. Soc. 20 (1969), 505-508.

70. F. E. Sullivan, "Norm characterizations of real L^p spaces", Bull. Amer. Math. Soc. 74 (1968), 153-154.

71. K. Sundaresan, "Orthogonality in normed linear spaces", J. Indian Math. Soc. 22 (1958), 99-107.

72. -----, "Characterizations of inner product spaces", Math. Student 29 (1961), 41-45.

73. -----, "Smooth Banach spaces", Bull, Amer. Math. Soc. 72 (1966), 520-521.

74. -----, "Smooth Banach spaces", Math. Ann. 173 (1967), 191-199.

75. E. O. Thorp, "Some Banach spaces congruent to their conjugates", J. London Math. Soc. 39 (1964), 703-705.

76. W. A. Wilson, "A relation between metric and Euclidean spaces", Amer. J. Math. 54 (1932), 505-517.

77. Zaanen, Linear Analysis, Interscience, New York (1953).

THESIS

APPENDICES

THESIS

APPENDIX A

Most of the geometric aspects (convexity, supporting hyperplanes, etc.) of normed linear spaces are most easily discussed in real normed linear spaces. However, much of the analytic theory (eigenvalues, spectral theory, etc.) finds a natural setting in complex normed linear space. The usual technique for studying the geometric aspects of a complex space is to "change" it into a real space. We wish to discuss how this change is made and prove some theorems based on it.

If X is a complex normed linear space we can associate a real normed linear space $\tilde{X}$ with it in the following manner. The vectors in $\tilde{X}$ are those of X, addition in $\tilde{X}$ is the same as that in X, scalar multiplication in $\tilde{X}$ is the same as multiplication by a real number in X, and finally the norm in $\tilde{X}$ is the same as the norm in X. It is a straightforward calculation that $\tilde{X}$ is a real normed linear space. Next we wish to determine the relation between the dual spaces of X and $\tilde{X}$.

<u>THEOREM A.1:</u> If $f \in X^*$ then $\operatorname{Re} f$ defined by $[\operatorname{Re} f](x) = \operatorname{Re}[f(x)]$ belongs to $\tilde{X}^*$ and $\|f\| = \|\operatorname{Re} f\|$. Furthermore, if $g \in \tilde{X}^*$ then the function f defined by $f(x) = g(x) - ig(ix)$ belongs to X^* and $\|f\| = \|g\|$.

<u>Proof:</u>

Let $f \in X^*$. Then $|\operatorname{Re} f(x)| \leq |f(x)| \leq \|f\|\,\|x\|$ so $\operatorname{Re} f \in \tilde{X}^*$ and $\|\operatorname{Re} f\| \leq \|f\|$. Let $x_1 \in X$ such that

THESIS

$\| x_1 \| = 1$ and $|f(x_1)| \to \| f \|$. Then if $\alpha_1 = \dfrac{\overline{f(x_1)}}{|f(x_1)|}$ we have $\| \alpha_1 x_1 \| = 1$ and $|\text{Re } f(\alpha_1 x_1)| \to \| f \|$. Hence $\| \text{Re } f \| = \| f \|$.

Let $g \in \tilde{X}^*$ and f be given by $f(x) = g(x) - ig(ix)$. For $\alpha \in C$ then

$$\begin{aligned} \alpha f(x) &= \alpha\,(g(x) - ig(ix)) \\ &= (\text{Re } \alpha)g(x) + (\text{Im } \alpha)g(ix) \\ &\quad + i((\text{Im } \alpha)g(x) - (\text{Re } \alpha)g(ix)) \\ &= g(\alpha x) - ig(\alpha i x) \\ &= f(\alpha x). \end{aligned}$$

Also

$$\begin{aligned} |\ f(x)\ | &= |\ g(x) - ig(ix)\ | \\ &\le |\ g(x)\ | + |\ g(ix)\ | \\ &\le 2\ \| g \|\ \| x \| \end{aligned}$$

so $\| f \| \le 2 \| g \|$ and $f \in X^*$. But $\text{Re } f \in \tilde{X}^*$ and $\text{Re } f = g$. Hence $\| f \| = \| \text{Re } f \| = \| g \|$ by the above.

q.e.d.

We can also determine the relation of X and $\tilde{X}$ in the characterization problem.

THEOREM A.2: X is a complex inner product space iff $\tilde{X}$ is a real inner product space.

Proof:

Suppose $(\ |\ )$ is an inner product on X. Then by straightforward calculation it can be shown that $\phi(x,y) = \text{Re } (x|y)$ is an inner product on $\tilde{X}$.

Now suppose $\phi(\ ,\ )$ is an inner product on $\tilde{X}$. We show $(x|y) = \phi(x,y) - i\phi(ix,y)$ is an inner product on X.

THESIS

Obviously $(\ |\ )$ satisfies I-1 and I-2 is satisfied by a calculation similar to the one in A-1.

For $\alpha \in C$ then

$$\begin{aligned}\|\alpha x\|^2 + 2\phi(\alpha x,\alpha y) + \|\alpha y\|^2 &= \phi(\alpha x+\alpha y,\alpha x+\alpha y)\\ &= \|\alpha x+\alpha y\|^2\\ &= |\alpha|^2\|x+y\|^2\\ &= |\alpha|^2\phi(x+y,x+y)\\ &= |\alpha|^2\{\|x\|^2 + 2\phi(x,y) + \|y\|^2\}\end{aligned}$$

or $\phi(\alpha x,\alpha y) = |\alpha|^2\phi(x,y)$.

Hence

$$\begin{aligned}(x|y) &= \phi(x,y) - i\phi(ix,y)\\ &= \phi(y,x) - i\phi(y,ix)\\ &= \phi(y,x) - i\phi(iy,-x)\\ &= \phi(y,x) + i\phi(iy,x)\\ &= \overline{(y|x)}\end{aligned}$$

Finally

$$\begin{aligned}2\|x\|^2 + 2\phi(ix,x) &= \phi(x+ix,x+ix)\\ &= \|x+ix\|^2\\ &= 2\|x\|^2\end{aligned}$$

or $\phi(ix,x) = 0$.

Hence $(x|x) = \phi(x,x) = \|x\|^2$ and $(\ |\ )$ is an inner product on X.

q.e.d.

Since X and $\mathfrak{X}$ are so closely connected, in the text we have taken the liberty of using such phrases as "consider X as a real normed linear space" or "X is a complex inner product space iff X is a real inner product space". Naturally what we mean is "consider $\mathfrak{X}$ instead of X" or "X is a complex inner product space iff $\mathfrak{X}$ is a real inner product space".

THESIS

APPENDIX B

Suppose X is a normed linear space with norm $\| \ \|$. Quite often we wish to define a second norm $| \ |$ on the vectors of X. Usually it is convenient to say more simply that we have defined a second norm on X. Moreover, we say $| \ |$ is induced by an inner product if there exists an inner product defined on the vectors of X such that $(x|x) = | x |^2$.

There is a technique which is very valuable for characterizing inner product spaces. Let X be a normed linear space with norm $\| \ \|$ and $\{x_1\} \subset X$. Suppose $| \ |$ is a second norm on X which is induced by an inner product and $| x_1 | = \| x_1 \|$. We then show by some method that $| x | = \| x \|$ for all $x \in X$. Thus we wish to discuss several methods of determining such a norm $| \ |$.

THEOREM B.1: Let X be a real two-dimensional normed linear space. Suppose $x,y \in X$ and $a,b \in R$ such that $\| ax \| + \| by \| > \| ax+by \| > | \, \| ax \| - \| by \| \, |$. Then there exists a norm $| \ |$ on X which is induced by an inner product such that $| x | = \| x \|$, $| y | = \| y \|$, and $| ax+by | = \| ax+by \|$.

Proof:

Define

$$(rx+sy|px+qy) = rp \| x \|^2 + (rq+ps)A + sq \| y \|^2$$

where $A = \dfrac{\| ax+by \|^2 - \| ax \|^2 - \| by \|^2}{2ab}$.

THESIS

Let $| rx+sy |^2 = (rx+sy|rx+sy)$. The only nontrivial condition to check is that $| rx+sy |$ is well defined. But
$$(rx+sy|rx+sy) = r^2 \| x \|^2 + 2rsA + s^2 \| y \|^2$$
$$> (r\| x \| - s \| y \|)^2$$
$$\geq 0$$
since $| A | < \| x \| \| y \|$.

q.e.d.

Suppose we consider X as a Minkowski space. Then B.1 is equivalent to the statement that there exists an ellipse E with center O through $\frac{x}{\| x \|}$, $\frac{y}{\| y \|}$, $\frac{ax+by}{\| ax+by \|}$ and $| \ \ |$ is the norm determined by E.

Theorem B.2 is due originally to Loewner and proofs of the two cases may be found in Day [18] and Schoenberg [62].

THEOREM B.2: Let S be a convex curve in the Euclidean plane which is symmetric about O. There exists an unique ellipse E with center O such that S is contained inside of E (E is contained inside of S) and E has the minimal area (maximal area) of all ellipses with this property. Moreover $E \cap S$ contains at least one pair of independent points.

The ellipse E is usually called the minimal circumscribed ellipse or the maximal inscribed ellipse, respectively.

Suppose X is a real two-dimensional normed linear space. If we consider X as a Minkowski space then the

THESIS

unit sphere satisfies the hypotheses on S in B.2 so that by theorem 1.12 we have the following.

<u>THEOREM B.3:</u> Let X be a real two-dimensional normed linear space. Then there exists a norm $|\ |$ on X which is induced by an inner product such that $|x| \leq \|x\|$ $(|x| \geq \|x\|)$ for all $x \in X$ and such that there exists a pair of linearly independent vectors $y, z \in X$ with $\|y\| = |y|$ and $\|z\| = |z|$.

In practice it is often convenient to refer to both the ellipse E and the norm $|\ |$. In doing so we are naturally considering X as a Minkowski space even if we do not explicitly mention this fact.

THESIS

THESIS

THESIS