MEAN CURVATURE FLOW IN HIGHER CODIMENSION

By

Andrew Allen Cooper

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

MATHEMATICS

2011

ABSTRACT

MEAN CURVATURE FLOW IN HIGHER CODIMENSION

By

Andrew Allen Cooper

In this work, we consider the mean curvature flow of compact submanifolds of Riemannian manifolds. If the flow becomes singular in finite time, we show how to produce a smooth singularity model, the "smooth blow-up" of the singularity. This construction relies upon a compactness theorem for families of submanifolds with bounded second fundamental form which we establish.

Using the smooth blow-up, we establish that if the contraction of the mean curvature with the second fundamental form is bounded, the flow may be continued. We also show that in case the singularity is of type I, the mean curvature must blow up and that in the type II case, the mean curvature must blow up, if at all, at a strictly slower rate than the full second fundamental form.

We also use the smooth blow-up to investigate Lagrangian mean curvature flow in Calabi-Yau manifolds. In particular we show that the singularities of the Lagrangian mean curvature flow are modelled either by zero Maslov class or monotone Lagrangian flows in Euclidean space.

ACKNOWLEDGMENT

Many thanks to my adviser, Jon G. Wolfson, for his constant mentorship and assistance throughout my years as a graduate student at Michigan State. I also wish to acknowledge the role of Drs. Ron Fintushel and Jeanne Wald in encouraging me to pursue research mathematics. The fruitful discussions I have had with Drs. Ben Schmidt, Xiaodong Wang, and Thomas Parker have also been invaluable in the research in this thesis.

I also wish to acknowledge the generous support of the National Science Foundation under grants DMS 0353717 and DMS 0604759, and of the MSU Department of Mathematics.

TABLE OF CONTENTS

Chapter 1

Introduction

1.1 The Mean Curvature Flow

Given an immersion of a manifold into a Riemannian manifold, $F : M^m \to (N^{m+n}, h)$, we may consider its second fundamental form $\mathrm{II} = (h_{ij\alpha})$, which is the projection of the Hessian D^2F to the normal bundle of $F(M)$:

$$h_{ij\alpha} = D^2_{ij}F_\alpha \tag{1.1}$$

Here and in the sequel we use Latin indices for coordinates on M, and equivalently for tangential directions of $F(M)$, and Greek indices for directions normal to $F(M)$. The immersion induces a metric $g = F^*h$. We will use g_{ij} for this metric and g^{ij} for its inverse.

The *mean curvature* of the immersion is normal vector given by tracing the second fundamental form in the tangential direction:

$$H_\alpha = g^{ij}h_{ij\alpha} \tag{1.2}$$

A *mean curvature flow* (MCF) from the initial immersion F_0 is a one-parameter family of immersions $F(t)$ which satisfies:

$$\begin{aligned} \frac{\partial}{\partial t}F(t) &= H(F(t)) \\ F(0) &= F_0 \end{aligned} \tag{1.3}$$

One can see easily that equation (1.3) is parabolic (up to choice of tangential reparametrization), with principal symbol given by the ordinary Laplacian. The corresponding elliptic problem, $H \equiv 0$, is the classical minimal-surface problem. Indeed, area-minimisation is the motivation for studying MCF. Computing the first variation at a vector field V of the area functional $\int_{F(M)} d\mathcal{H}^m$, we have:

$$\delta_V \int_{F(M)\cap D} d\mathcal{H}^m = -\int_{F(M)\cap D} H \cdot V d\mathcal{H}^m \tag{1.4}$$

In particular, equation (1.3) is the downward gradient of the area functional.

1.2 Overview of the Literature

The mean curvature flow has been most successfully studied in the case of hypersurfaces, beginning with Huisken's theorem that convex hypersurfaces shrink to round points [32] and Grayson's theorem that embedded plane curves shrink to round points [19], and culminating in Huisken-Sinestrari's topological classification of 2-convex hypersurfaces [31].

In the hypersurface setting, there is only one normal direction, so the second fundamental form is a symmetric real-valued two-tensor, and the mean curvature is a real-valued function. This greatly simplifies the analysis of the flow. In particular, Hamilton's maximum principle for tensors [24] has been used to show that various positivity conditions on the second fundamental form are preserved by the flow [30] [29].

In the Lagrangian setting, Neves and Groh-Schwarz-Smoczyk-Zehmisch have investigated singularity formation [41] [42] [20]. Much of the Lagrangian mean curvature flow literature involves finding conditions which guarantee infinite-time existence of the flow, with convergence at infinite time to a known minimal submanifold. In particular, the flow of Lagrangian graphs has been studied extensively by Smoczyk and Wang, among others [52] [56] [55] [57].

In general codimension, Andrews-Baker have shown that submanifolds sufficiently close to the round sphere collapse to round points [3].

In this thesis, we will establish the groundwork for a theory of finite-time singularities of the mean curvature flow in general codimension. As an application, we will establish some results about the singularities of compact Lagrangian mean curvature flows. In future work, we hope to use our smooth singularity models to prove a surgery theorem like those of Huisken-Sinestrari and Hamilton-Perelman.

1.3 Summary of Known and New Results

1.3.1 Singularity Models for the Mean Curvature Flow

To understand the structure of the singularities of the mean curvature flow, we need a theory of singularity models.

The standard technique in the study of singularities of the mean curvature has been the *tangent flow* construction, which shares many similarities with the tangent cone construction in the theory of minimal submanifolds. In particular, tangent flows are not expected to be smooth objects. The tangent flow is smooth when the singularity is mild [27]. The success of Huisken-Sinestrari in remaining in the smooth category for their surgery theorem [31] relies upon the fact that their assumption on the initial data ("2-convex") forces the singularity to be mild, so that they may use the tangent flow to obtain a smooth singularity model.

In the study of Ricci flow, no extrinsic technique such as the tangent flow is available, and the theory of Riemannian manifolds with singularities is somewhat less tractable than the theory of submanifolds with singularities [9]. Thus to model singularities of the Ricci flow, Hamilton, aided by Perelman's celebrated non-collapsing theorem [44], constructs smooth limit objects [25]. The success of the Hamilton-Perelman program for Ricci flow relies upon deducing the properties of Ricci flows which arise as singularity models [12]. In particular, in dimension 3 such limit flows are highly restricted.

We adapt Hamilton's ideas to build smooth singularity models for the mean curvature flow. In particular, we establish the following theorem:

Theorem 1.3.1. *Let $F : M^m \times [0, T) \to (N^{m+n}, h)$ be a compact mean curvature flow in a Riemannian manifold with bounded geometry, with singularity at time $T < \infty$. Then there exists a mean curvature flow $F_\infty : M_\infty \times (-\infty, C) \to \mathbb{R}^{m+n}$ which models the singularity of the flow F, where $C = \sup_{M\times[0,t]} |\,\mathrm{II}(p, t)|^2(T - t)$. F_∞ has the following properties:*

- *If $C < \infty$ then F_∞ has second fundamental form bounded by 1 up to time 0.*
- *If $C = \infty$ then F_∞ has a second fundamental form bounded by 1 for all time.*
- *F_∞ has at least one point p_∞ with $|\mathrm{II}(p_\infty, 0)| = 1$.*

We call F_∞ a smooth blow-up *of the flow $F : M \times [0, T) \to (N, h)$.*

By "models" here, we mean that, after rescaling so that the second fundamental form has maximum norm 1, F_∞ is a C^∞ approximation to the developing singularity, in the sense described in Chapter 2.

We will also show that, if the ambient manifold is $\mathbb{R}^{m+n}$ and $C < \infty$, a smooth blow-up is essentially equivalent to a tangent flow.

The work in establishing the above theorem is to prove a compactness theorem for smooth mean curvature flows. To this end we establish a Cheeger-Gromov-type theorem for immersions, in particular showing that a pointwise bound on the second fundamental form is all that is needed to ensure convergence in the smooth category.

Chen-He have also considered the project of a compactness theorem for mean curvature flows and constructing smooth singularity models for their singularities [10]. In particular they provide sufficient conditions on (N, h) to be able to generalize some aspects of the tangent flow construction.

1.3.2 Singular Time of the Mean Curvature Flow

In general one expects, because MCF is the downward gradient of the area functional, that if the initial submanifold is close to minimal, the flow is expected to converge to a minimal submanifold in infinite time.

If the ambient manifold N is the euclidean space $\mathbb{R}^{m+n}$, all minimal submanifolds have infinite area. Thus starting from closed initial data, it is impossible to converge to a minimal submanifold. In fact all MCFs in euclidean space starting from closed initial data will become singular at some time $T < \infty$.

We have the following characterization of such singular times:

Theorem 1.3.2 (Huisken [32]). *Let $F_t : M \to (N, h)$ be a compact mean curvature flow on the maximal interval $[0, T)$, $T < \infty$. Then*

$$\lim_{t \to T} \sup_M |\mathrm{II}(\cdot, t)|^2 = \infty \tag{1.5}$$

where $\mathrm{II}(t)$ is the second fundamental form of $F_t(M)$.

Theorem 1.3.2 is similar in content and proof to those of the following theorems from other parabolic geometric flows:

Theorem 1.3.3 (Hamilton [23]). *Let $(M, g(t))$ be a compact Ricci flow on the maximal interval $[0, T)$, $T < \infty$. Then*

$$\lim_{t \to T} \sup_M |\mathrm{Rm}(\cdot, t)|^2 = \infty \tag{1.6}$$

a where $\mathrm{Rm}(t)$ is the Riemann curvature tensor of $g(t)$.

Theorem 1.3.4 (Streets-Tian [53]). *Let $(M, J, g(t))$ be a compact pluriclosed flow on the maximal interval $[0, T)$, $T < \infty$. Then*

$$\lim_{t \to T} \max\{\sup_M |\Omega(\cdot, t)|^2, \sup_M |T(\cdot, t)|^2, \sup_M |\nabla T(\cdot, t)|^2\} = \infty \tag{1.7}$$

where $\Omega(t)$ and $T(t)$ are the curvature and torsion, respectively, of the Chern connection of the pair $(g(t), J)$.

Remark 1. Streets-Tian have recently refined this result to show that in fact either the torsion T or a potential function they call ϕ must blow up at a singular time [54].

It is natural to ask whether finite-time singularities can be characterized more weakly. For the Ricci flow, finite-time singularities are characterized by the blow-up of the Ricci tensor:

Theorem 1.3.5 (Sesum [49]). *Let $(M, g(t))$ be a compact mean Ricci flow on the maximal interval $[0, T)$, $T < \infty$. Then*

$$\lim_{t \to T} \sup_{M} |\mathrm{Ric}|^2 = \infty \tag{1.8}$$

In the case of the mean curvature flow, we will establish that the tensor

$$A_{ij} = H^{\alpha} h_{ij\alpha} \tag{1.9}$$

which is a trace of the square of the second fundamental form, must blow up at the singular time. The tensor A is important because it is the evolution of the induced metric g:

$$\frac{\partial}{\partial t} g_{ij} = -2A_{ij} \tag{1.10}$$

That is, we will establish in Chapter 3 the following theorem:

Theorem 1.3.6. *Let $F_t : M^m \to (N, h)$ be a compact mean curvature flow on the maximal interval $[0, T)$. Then*

$$\lim_{t \to T} \sup_{M} |A|^2 = \infty \tag{1.11}$$

In case the singularity is mild, we can get even better characterizations of finite singular times:

Theorem 1.3.7 (Enders-Müller-Topping [16], Le-Sesum [38]). *Let $(M, g(t))$ be a compact Ricci flow on the maximal interval $[0, T)$, $T < \infty$. Suppose the singularity is of type I. Then*

$$\lim_{t \to T} \sup_M |R(t)|^2 = \infty \tag{1.12}$$

where $R(t)$ is the scalar curvature of $g(t)$.

Note that $R = \operatorname{tr} \operatorname{Ric}$. The analogous quantity in mean curvature flow is $\operatorname{tr} A = |H|^2$. We will show in Chapter 3 that Theorem 1.3.7 also analogizes to the mean curvature flow:

Theorem 1.3.8. *Let $F_t : M^m \to (N, h)$ be a compact mean curvature flow on the maximal interval $[0, T)$. Suppose the singularity is of type I. Then*

$$\lim_{t \to T} \sup_M |H|^2 = \infty \tag{1.13}$$

Remark 2. Theorems 1.3.6 and 1.3.8, as well as Corollary 2.1.11, appear in [14]. Theorem 1.3.8 was independently established by Le-Sesum in [39].

1.3.3 Lagrangian Mean Curvature Flow

If the initial submanifold is a Lagrangian $L : \Sigma^m \to \mathbb{C}^m$, or more generally in a Calabi-Yau manifold, then the mean curvature flow preserves the Lagrangian condition. Thus we may consider how the Lagrangian geometry evolves under the flow. We are motivated to study Lagrangian mean curvature flow as a method of obtaining and understanding obstructions to minimal Lagrangians (called *special Lagrangian* submanifolds) which have a long history, see e.g. [26] [47] [48] [58].

Two natural classes of Lagrangians are *zero Maslov class* and *monotone* Lagrangians. For embedded Lagrangians, these conditions relate to holomorphic discs whose boundary lies in the Lagrangian submanifold. The relationship between mean curvature flow and holomorphic discs has been explored by Groh-Schwarz-Smoczyk-Zehmisch [20].

Chen-Li established some basic properties of the tangent flows of Lagrangian mean curvature flows [11]. Using this, Neves and Groh-Schwarz-Smoczyk-Zehmisch established the following theorems:

Theorem 1.3.9 (Neves [42], Groh-Schwarz-Smoczyk-Zehmisch [20]). *Suppose $L : \Sigma \times [0,T) \to \mathbb{C}^m$ is a compact Lagrangian mean curvature flow which is initially monotone, that is, $[\lambda_0] = C_0[h_0]$ where λ_0 and h_0 are the Liouville and Maslov forms of $L(0)$.*

If the singular time $T < \frac{1}{2}C_0$, then any tangent flow to L is a collection of minimal Lagrangian cones.

Theorem 1.3.10 (Neves [41]). *Suppose $L : \Sigma \times [0,T) \to \mathbb{C}^m$ is a zero Maslov class Lagrangian mean curvature flow, $T < \infty$. Then any tangent flow to L is a collection of minimal Lagrangian cones.*

Using the smooth blow-up, we are able to show that monotone and zero Maslov flows are the only possibilities for singularities of the Lagrangian mean curvature flow:

Theorem 1.3.11. *Suppose $L : \Sigma^m \times [0,T) \to (X^{2m}, \omega, J)$ is a compact Lagrangian mean curvature flow in a Calabi-Yau manifold. If the singularity is of type I, then any smooth blow-up is a monotone Lagrangian mean curvature flow. If the singularity is of type II, then any smooth blow-up is a zero Maslov class Lagrangian mean curvature flow.*

Chapter 2

Singularity Models for the Mean Curvature Flow

To understand singularities of the flow, we employ a rescaling technique. Given a mean curvature flow $F : M \times [a, b] \to \mathbb{R}^{m+n}$, a time $t_0 \in [a, b)$, and a point $x_0 \in \mathbb{R}^{m+n}$, and a positive real number α, we can parabolically rescale F about (x_0, t_0) by α:

$$\tilde{F}(p, s) = \alpha \left[F\left(p, t_0 + \frac{s}{\alpha^2} \right) - x_0 \right] \tag{2.1}$$

The map $\tilde{F}$ is a mean curvature flow from M into $\mathbb{R}^{m+n}$, defined on the interval $\Big[-\alpha^2(t_0 - a), \alpha^2(b - t_0)\Big)$.

We will consider sequences (α_j, x_j, t_j), such that $\alpha_j \to \infty$, and try to take a limit of the corresponding rescales $\tilde{F}_j$. Hence we must establish compactness properties for immersions. In particular, prove the following theorem:

Theorem 2.0.12. *Let M_k^m be smooth closed m-manifolds and (N_k, h_k) smooth Riemannian $m+n$-manifolds such that $\left|\nabla^\ell \operatorname{Rm}(N_k, h_k)\right| \le C_\ell$, $0 \le \ell \le \ell_0$, and $\operatorname{inj}(N_k, h_k) \ge \eta > 0$. Suppose $F_k : (M_k, p_k) \to (N_k, h_k, x_k)$ are a sequence of pointed immersions of M_k into (N_k, h_k) such that the second fundamental forms and their covariant derivatives are bounded pointwise, i.e. $\left|\nabla^\ell \operatorname{II}_k\right| \le C_\ell$, $0 \le \ell \le \ell_0$. Then there exist a C^{ℓ_0+1} m-manifold (M_∞, p_∞) and a complete Reimannian manifold $(N_\infty, h_\infty, x_\infty)$ such that:*

1. *M_∞ admits an exhausting sequence $W_1 \subset W_2 \subset \cdots$ of relatively compact open sets and embeddings $\phi_k : (W_k, p_\infty) \hookrightarrow (M_k, p_k)$, such that for any $R > 0$, the $F_k^* h_k$-metric ball $B(p_k, R)$ is contained in $\phi_k(W_k)$ for all $k \ge k_0(R)$*

2. *N_∞ admits an exhausting sequence $V_1 \subset V_2 \subset \cdots$ of relatively compact open sets and embeddings $\psi_k : (V_k, x_\infty) \hookrightarrow (N_k, x_k)$, such that for any $R > 0$, the h_k-metric ball $B(x_k, R)$ is contained in $\psi_k(V_k)$ for all $k \ge k_0(R)$*

3. *$\psi_k^* h_k \to h_\infty$ on compact sets in the $C^{\ell_0+1,\gamma}$ topology for any $0 \le \gamma < 1$*

4. *$\phi_k(W_k) \subset \psi_k(V_k)$.*

5. *$\psi_k^{-1} \circ F_k \circ \phi_k \to F_\infty$ on compact sets in the $C^{\ell_0+1,\gamma}$ topology for any $0 \le \gamma < 1$*

6. *$(M_\infty, F_\infty^* h_\infty)$ is a complete Riemannian manifold*

Here the $C^{\ell_0+1,\gamma}$ topology is that given by isometrically embedding N_∞ into some Euclidean space $\mathbb{R}^K$ and equipping M_∞ with a background metric.

We will then use this theorem to establish a compactness theorem for mean curvature flows and take limits of rescales (2.1).

2.1 Compactness and the Second Fundamental Form

A celebrated theorem of Cheeger and Gromov states that families of Riemannian manifolds with uniform C^ℓ bounds on the curvature tensor and a uniform lower bound on the injectivity radius are precompact in a certain sense:

Theorem 2.1.1 (Cheeger-Gromov [21]). *Let (N_k, h_k, x_k) be a sequence of complete pointed Riemannian manifolds such that $\left|\nabla^\ell \operatorname{Rm}(N_k, h_k)\right| \leq C$ for each $1 \leq \ell \leq \ell_0$ and $\operatorname{inj}(h_k) \geq \eta > 0$. Then there is complete C^{ℓ_0+1} Riemannian manifold $(N_\infty, h_\infty, x_\infty)$ such that*

1. *N_∞ admits a sequence of relatively compact open sets $V_1 \subset V_2 \subset \cdots \subset N_\infty$ which exhausts N_∞ and embeddings $\psi_k : (V_k, x_\infty) \hookrightarrow (N_k, x_k)$, such that for each $R > 0$ the h_k-metric ball $B(x_k, R)$ is contained in $\psi_k(V_k)$ for all $k \geq k_0(R)$*

2. *$\psi_k^* h_k \to h_\infty$ in the $C^{\ell_0+1,\gamma}$ topology on compact subsets of N_∞, for any $0 \leq \gamma < 1$*

This theorem has been used extensively in the theory of singularities of the Ricci flow [25]. Our goal in this section is to establish an analogous compactness theorem for Riemannian immersions.

Given an immersion $F : M \to (N, h)$ of a compact m-manifold M, we may equip M with a background Riemannian metric and isometrically embed (N, h) into some Euclidean space $\mathbb{R}^K$. This allows us to consider the space $C^\ell(M, N)$ of C^ℓ maps from M to N. The curvature of the image submanifold $F(M)$ is invariant under reparametrization of M; thus bounds on the curvature of $F(M)$ do not allow us to appeal directly to the Arzela-Ascoli theorem for compactness of families of immersions $F : M \to N$. In fact by composing with a diffeomorphism of M, we may make any derivative of F arbitrarily large without changing the extrinsic curvature. The content of the Theorem 2.0.12 is that this diffeomorphisminvariance can be corrected for in a way that allows us to use Arezela-Ascoli, albeit at the cost of possible topological change.

We refer to convergence as in the conclusion of Theorem 2.0.12 as *convergence in* $C^{\ell_0+1,\gamma}$ *in the geometric sense.* We note that in case $m = 0$, $M_k = \{p_k\}$, our theorem recovers the Cheeger-Gromov theorem.

2.1.1 Langer Charts

The idea of the proof of the Theorem 2.0.12 is due essentially to Langer [35]. We will go over the construction in detail for the case when the ambient manifold is Euclidean, and then indicate how the construction can be extended to an arbitrary Riemannian manifold with bounded geometry.

2.1.1.1 Euclidean Case

We begin by considering the case of the graph of a map $f : \mathbb{R}^m \to \mathbb{R}^n$, as in [46]. We need to compare the standard square-norm of certain objects, e.g. $\left|D^2 f\right|^2 = \sum_{\substack{1\leq\alpha\leq n \\ 1\leq i,j\leq m}} \left(\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j}\right)^2$, with the norms of the tensors II and ∇ II in the metric g induced by the immersion. To keep the norms straight, in this section we use $|\cdot|$ for the standard square-norm and $|\cdot|_g$ for the norm in g:

$$\begin{aligned} |\mathrm{II}|_g^2 =& h_{ij\alpha} h_{kl\beta} g^{\alpha\beta} g^{ik} g^{jl} \\ |\nabla\,\mathrm{II}|_g^2 =& \nabla_i h_{jk\alpha} \nabla_p h_{qr\beta} g^{ip} g^{jq} g^{kr} g^{\alpha\beta} \end{aligned} \tag{2.2}$$

Lemma 2.1.2. *Let* $f : D_r^m \to \mathbb{R}^n$ *be a* C^2 *function on the disc of radius* r*. Then*

$$\left|D^2 f\right|^2 \leq (1 + |Df|^2)^3\, |\mathrm{II}|_g^2$$

where II *is the second fundamental form of the graph of* f*.*

Proof. The graph of f has immersion map $F(x_1, \ldots, x_m) = (x_1, \ldots, x_m, f_1, \ldots, f_n)$. We use the following tangent and normal frames, where $1 \leq i \leq m$ and $1 \leq \alpha \leq n$:

$$\begin{aligned} e_i =& (0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f_1}{\partial x_i}, \ldots, \frac{\partial f_n}{\partial x_i}) = (0, \ldots, 0, 1, 0, \ldots, 0, D_i f) \\ \nu_\alpha =& (-\frac{\partial f_\alpha}{\partial x_1}, \ldots, -\frac{\partial f_\alpha}{\partial x_m}, 0, \ldots, 0, 1, 0, \ldots, 0) = (-Df_\alpha, 0, \ldots, 0, 1, 0, \ldots, 0) \end{aligned} \tag{2.3}$$

These choices induce the metric on the tangent bundle of the graph, which we denote by g with Latin indices:

$$g_{ij} = e_i \cdot e_j = \delta_{ij} + D_i f \cdot D_j f \tag{2.4}$$

We also get a metric on the normal bundle, which we denote by g with Greek indices:

$$g_{\alpha\beta} = \nu_\alpha \cdot \nu_\beta = \delta_{\alpha\beta} + Df_\alpha \cdot Df_\beta \tag{2.5}$$

We will use g^{ij} to denote the inverse matrix to g_{ij} and $g^{\alpha\beta}$ to denote the inverse to $g_{\alpha\beta}$. We compute the second fundamental form. Note that $D^2 F = (0, D^2 f)$. So we have

$$\begin{aligned} \mathrm{II}(e_i, e_j) =& \mathrm{proj}^{\perp}(D^2 F(e_i, e_j)) \\ =& (D^2_{ij} F \cdot \nu_\beta) g^{\alpha\beta} \nu_\alpha \\ =& \frac{\partial^2 f_\beta}{\partial x_i \partial x_j} g^{\alpha\beta} \nu_\beta \end{aligned} \tag{2.6}$$

In components, $h_{ij\alpha} = \dfrac{\partial^2 f_\alpha}{\partial x_i \partial x_j}$.

Then the norm-squared of the second fundamental form is

$$|\mathrm{II}|_g^2 = \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \frac{\partial^2 f_\beta}{\partial x_k \partial x_l} g^{\alpha\beta} g^{ik} g^{jl} \quad . \tag{2.7}$$

We can think of $|\mathrm{II}|_g^2$ as the norm-squared of $D^2 f$ in the metric g as opposed to the standard metric. We will compare $g^{\alpha\beta}$ and g^{ij} to the standard metric by giving estimates for the eigenvalues of $g^{\alpha\beta}$ and g^{ij}. To do this we estimate the eigenvalues of g_{ij} and $g_{\alpha\beta}$.

Since $g_{\alpha\beta} = \delta_{\alpha\beta} + Df_\alpha \cdot Df_\beta$, we have that each eigenvalue λ of $g_{\alpha\beta}$ has

$$1 \leq \lambda \leq 1 + |Df|^2 \tag{2.8}$$

and similiarly the eigenvalues μ of g_{ij} are bounded by

$$1 \leq \mu \leq 1 + |Df|^2. \tag{2.9}$$

Thus the eigenvalues of the inverse matrices $g^{\alpha\beta}$ and g^{ij} are bounded away from zero and infinity:

$$\begin{aligned} 1 \geq \lambda^{-1} &\geq \frac{1}{1+|Df|^2} \\ 1 \geq \mu^{-1} &\geq \frac{1}{1+|Df|^2} \end{aligned} \tag{2.10}$$

So we can estimate

$$
\begin{aligned}
|\mathrm{II}|_g^2 =& \frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \frac{\partial^2 f_\beta}{\partial x_k \partial x_l} g^{\alpha\beta} g^{ik} g^{jl} \\
\geq & \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i,j \leq m}} \left(\frac{\partial^2 f_\alpha}{\partial x_i \partial x_j} \right)^2 \frac{1}{(1+|Df|^2)(1+|Df|^2)^2} \\
=& \left| D^2 f \right|^2 \frac{1}{(1+|Df|^2)^3}
\end{aligned}
\tag{2.11}
$$

which establishes our lemma.

□

We may similarly bound the higher derivatives of f in terms of Df and the covariant derivatives of II:

Lemma 2.1.3. *For any $\ell \geq 2$, we can bound $\left| D^\ell f \right|$ in terms of $|Df|$, $\left| D^2 f \right|$, . . ., $|D^{\ell-1} f|$, $\left| \nabla^{\ell-2}\, \mathrm{II} \right|_g$, and absolute constants depending on m, n, and ℓ. In particular, the $\ell = 3$ case is*

$$
\left| D^3 f \right| \leq (1+|Df|^2)^2 \, |\nabla\, \mathrm{II}|_g + \left(2\sqrt{2m + 4\sqrt{mn} + n} \right) \left| D^2 f \right|^2 |Df|
$$

Proof. We will do the $\ell = 3$ computation explicitly; the others are similar but more tedious.

As in the proof of Lemma 2.1.2, we start by estimating $|\nabla\, \mathrm{II}|_g$ below. To do this, we need to compute the Christoffel symbols for the tangent and normal bundles. First we compute the tangential Christoffel symbols. We compute $\nabla_{e_i} e_j$, the projection to the tangent space of $D_{e_i} e_j$:

$$
\begin{aligned}
\nabla_{e_i} e_j &= \mathrm{proj}^T (D_{e_i} e_j) \\
&= \mathrm{proj}^T \left(\frac{\partial}{\partial x_i}(0, \ldots, 0, 1, 0, \ldots, 0, D_j f)\right) \\
&= \mathrm{proj}^T ((0, D^2_{ij} f)) \\
&= g^{kl}((0, D^2_{ij} f) \cdot e_l) e_k \\
&= g^{kl}(D^2_{ij} f \cdot D_l f) e_k
\end{aligned}
\tag{2.12}
$$

so $\Gamma^k_{ij} = g^{kl}(D^2_{ij} f \cdot D_l f)$. Similarly to compute $\Gamma^\beta_{i\alpha}$:

$$
\begin{aligned}
\nabla_{e_i} \nu_\alpha &= \mathrm{proj}^\perp (D_{e_i} \nu_\alpha) \\
&= \mathrm{proj}^\perp \left(\frac{\partial}{\partial x_i}(-Df_\alpha, 0, \ldots, 0, 1, 0, \ldots, 0)\right) \\
&= \mathrm{proj}^\perp \left(-\frac{\partial^2 f_\alpha}{\partial x_i \partial x_1}, \ldots, -\frac{\partial^2 f_\alpha}{\partial x_i \partial x_m}, 0\right) \\
&= g^{\beta\gamma}\left(\left(-\frac{\partial^2 f_\alpha}{\partial x_i \partial x_1}, \ldots, -\frac{\partial^2 f_\alpha}{\partial x_i \partial x_m}, 0\right) \cdot \nu_\gamma\right) \nu_\beta \\
&= g^{\beta\gamma}\left(\sum_r \frac{\partial^2 f_\alpha}{\partial x_i \partial x_r} \frac{\partial f_\gamma}{\partial x_r}\right) \\
&= g^{\beta\gamma}(D^2_{i\cdot} f_\alpha \cdot D f_\gamma) \nu_\beta
\end{aligned}
\tag{2.13}
$$

so $\Gamma^\beta_{i\alpha} = g^{\beta\gamma}(D^2_{i\cdot} f_\alpha \cdot D f_\gamma)$.

Then $|\nabla \mathrm{II}|_g^2$ is given by

$$\begin{aligned}|\nabla \mathrm{II}|_g^2 =&(\frac{\partial h_{jk\alpha}}{\partial x_i} + h_{lk\alpha}\Gamma_{ij}^l + h_{jl\alpha}\Gamma_{ik}^l + h_{jk\beta}\Gamma_{i\alpha}^\beta) \\ &\cdot (\frac{\partial h_{qr\gamma}}{\partial x_p} + h_{lr\gamma}\Gamma_{pq}^l + h_{ql\gamma}\Gamma_{pr}^l + h_{qr\delta}\Gamma_{p\gamma}^\delta) g^{ip} g^{jq} g^{kr} g^{\alpha\gamma} \\ \geq& \frac{1}{(1+|Df|^2)^4}(\frac{\partial}{\partial x_i} h_{jk\alpha} + h_{lk\alpha}\Gamma_{ij}^l + h_{jl\alpha}\Gamma_{ik}^l + h_{jk\beta}\Gamma_{i\alpha}^\beta)^2\end{aligned} \tag{2.14}$$

By (2.14) and Cauchy-Schwarz, we have

$$(1+|Df|^2)^4 \, |\nabla \mathrm{II}|_g^2 + 2\left|D^3 f\right| |B| \geq \left|D^3 f\right|^2 \tag{2.15}$$

where $B_{ijk\alpha} = h_{lk\alpha}\Gamma_{ij}^l + h_{jl\alpha}\Gamma_{ik}^l + h_{jk\beta}\Gamma_{i\alpha}^\beta$. It will suffice to bound $|B|$ above. Our estimates (2.10) for the eigenvalues of g^{ij} and $g^{\alpha\beta}$ imply $\left|g^{ij}\right|^2 \leq m$ and $\left|g^{\alpha\beta}\right|^2 \leq n$.

$$\begin{aligned}|B|^2 =& \sum_{\substack{1\leq\alpha\leq n \\ 1\leq i,j,k\leq m}} (h_{lk\alpha}\Gamma_{ij}^l + h_{jl\alpha}\Gamma_{ik}^l + h_{jk\beta}\Gamma_{i\alpha}^\beta)^2 \\ =& \sum_{\substack{1\leq\alpha\leq n \\ 1\leq i,j,k\leq m}} (\frac{\partial f_\alpha}{\partial x_l \partial x_k} g^{ls}(D_{ij}^2 f \cdot D_s f) + \frac{\partial f_\alpha}{\partial x_j \partial x_l} g^{ls}(D_{ik}^2 f \cdot D_s f) \\ &+ \frac{\partial f_\beta}{\partial x_j \partial x_k} g^{\beta\gamma}(D_{i\cdot}^2 f_\alpha \cdot Df_\gamma))^2 \\ \leq& \, 2\left|D^2 f\right|^2 \left|g^{ij}\right|^2 \left|D^2 f\right|^2 |Df|^2 \\ &+ 4\left|D^2 f\right|^2 \left|g^{ij}\right| \left|g^{\alpha\beta}\right|^2 |Df|^2 + \left|D^2 f\right|^2 \left|g^{\alpha\beta}\right|^2 \left|D^2 f\right|^2 |Df|^2 \\ \leq& \left|D^2 f\right|^4 |Df|^2 (2m + 4\sqrt{mn} + n)\end{aligned} \tag{2.16}$$

Thus we have

$$\left|D^3 f\right|^2 \leq (1+|Df|^2)^4 \, |\nabla \, \mathrm{II}|_g^2 + \left(2\sqrt{2m+4\sqrt{mn}+n}\right) \left|D^3 f\right| \left|D^2 f\right|^2 |Df| \qquad (2.17)$$

The claimed estimate for $\left|D^3 f\right|$ follows from this and the quadratic formula. □

Next we want to realize any immmersion $F : M \to \mathbb{R}^{m+n}$ as a collection of graphs over discs.

We introduce the following notation and notions, following [35]. Given $q \in M$, denote by A_q any Euclidean isometry which takes $F(q)$ to the origin and $T_{F(q)}F(M)$ to the plane $\{(x_1, \ldots, x_m, 0)\}$. Let π be the projection of $\mathbb{R}^{n+m}$ to the plane $\{(x_1, \ldots, x_m, 0)\}$. Define the *Langer chart at* q $U_{r,q} \subset M$ to be the component of $(\pi \circ A_q \circ F)^{-1}(D_r)$ which contains q.

We call $F : M \to \mathbb{R}^{n+m}$ a (r, α)-immersion if for each $q \in M$ there is some $f_q : D_r^m \to \mathbb{R}^n$ with $Df_q(0) = 0$ and $|Df_q| \leq \alpha$ so that $A_q \circ F(U_{r,q}) = \mathrm{graph}(f_q)$.

Lemma 2.1.4. *Let $\alpha > 0$. Then for any C^2-immersed submanifold $F : M^m \to \mathbb{R}^{n+m}$ and any r satisfying*

$$r \leq \frac{\alpha}{\left(1+\alpha^2\right)^{3/2}} \frac{1}{\sup_M |\mathrm{II}|_g}$$

F is a (r, α)-immersion.

Proof. Let $q \in M$ be arbitrary. Every submanifold is locally a graph over its tangent plane; thus $A_q(F(U_{r,q}))$ can be written as a graph over D_r for small enough r. So we set $S_q = \sup\{r | F(U_{r,q}) = \text{graph}(f_{r,q})\}$. For any large K, if $F(U_{r,q}) = \text{graph}(f_{r,q})$ and $|Df_{r,q}| \leq \frac{K}{2}$, we can extend $f_{r,q}$ to have a larger domain and still $|Df| \leq K$. Thus we have

$$\lim_{r \to S_q} \inf_f \sup_{D_r} |Df| = \infty \tag{2.18}$$

where the infimum is taken over all f with $Df(0) = 0$ of which $A_q F(U_{r,q})$ is a graph. Thus for our given α there exists some $r_q, f_q : D_{r_q} \to \mathbb{R}^n$ with $\sup_{D_{r_q}} |Df_q| = \alpha$. Now we use the fundamental theorem of calculus and Lemma 2.1.2 to get

$$\alpha = \sup_{D_{r_q}} |Df_q| \leq r \sup_{D_{r_q}} \left|D^2 f_q\right| \leq r_q \left(1 + \alpha^2\right)^{3/2} \sup_{D_{r_q}} \left|\text{II}_{f_q}\right|_g \tag{2.19}$$

which implies that

$$r_q \geq \frac{\alpha}{\left(1 + \alpha^2\right)^{3/2}} \frac{1}{\sup_{D_{r_q}} \left|\text{II}_{f_q}\right|_g} \geq \frac{\alpha}{\left(1 + \alpha^2\right)^{3/2}} \frac{1}{\sup_M |\text{II}|_g}. \tag{2.20}$$

So for r less than the right-hand side of (2.20), there is some $f : D_r \to \mathbb{R}^{m+n}$ of which $A_q F(U_{r,q})$ is the graph, with $Df(0) = 0$ and $|Df| \leq \alpha$. □

We will also make use of the following lemma, which relates the Langer atlas to the metric structure of $(M, F^* dx^2)$.

Lemma 2.1.5. *Let $\alpha \leq \sqrt{3}$. In a (r, α)-immersion $F : M^m \to \mathbb{R}^{m+n}$, for any $0 < \rho \leq \frac{r}{2}$, $\ell \in \mathbb{N}$, $q_0 \in M$, we have that any metric ball $B(q_0, \ell\frac{\rho}{2}) \subset M$ can be covered using at most K^ℓ Langer charts $U_{p,\frac{\rho}{4}}$ of radius $\frac{\rho}{4}$, such that $p \in B(q_0, \ell\rho)$, where $K = K(m, \alpha)$ is a constant depending only on the dimension m and the constant α. Moreover, we can assume that if*

$\ell_1 \leq \ell_2$, the covering of $B(q_0, \ell_2 \frac{\rho}{2})$ in K^{ℓ_2} Langer charts of radius $\frac{\rho}{4}$ contains the K^{ℓ_1} Langer charts used to cover $B(q_0, \ell_1 \frac{\rho}{2})$.

Proof. We proceed by induction. If $\ell = 2$, we have $B(q_0, \rho) \subset\subset U_{q_0,\rho} \subset U_{q_0,2\rho}$. Since $\rho \leq \frac{r}{2}$, the projection of $F(U_{q_0,2\rho})$ to $T_{F(q_0)}F(M)$ is the m-ball of radius 2ρ. Let K be the number of m-balls of radius $\frac{\rho}{4\sqrt{1+\alpha^2}}$ which can cover $D^m_{2\rho}$. Each ball of radius $\frac{\rho}{4\sqrt{1+\alpha^2}}$ is contained in the projection of some $U_{p,\frac{\rho}{4}}$, $p \in U_{q_0,2\rho}$. Thus we have that $B(q_0, \rho)$ can be covered by K Langer charts of radius $\frac{\rho}{4}$. It is clear that the centers of these Langer charts can be taken to lie in $B(q_0, \rho)$.

Now suppose $B(q_0, \ell\frac{\rho}{2}) \subset \bigcup_{i=1}^{K^\ell} U_{p_i,\frac{\rho}{4}}$. Then $B(q_0, (\ell+1)\frac{\rho}{2})$ is contained in a $\frac{\rho}{2}$ neighborhood of $\bigcup_{i=1}^{K^\ell} U_{p_i,\frac{\rho}{4}}$. On the other hand, the $\frac{\rho}{2}$ neighborhood of each $U_{p_i,\frac{\rho}{4}}$ is contained in $B(p_i, \sqrt{1+\alpha^2}\frac{\rho}{4} + \frac{\rho}{2})$. Since $\alpha \leq \sqrt{3}$, we have

$$B(q_0, (\ell+1)\frac{\rho}{2}) \subset \bigcup_{i=1}^{K^\ell} B(p_i, \rho) \tag{2.21}$$

Each term in the union on the right-hand side can, by the definition of K, be covered by K Langer charts of radius $\frac{\rho}{4}$, centered within distance ρ of one of the p_i. This completes the inductive step. □

2.1.1.2 General Case

The general case of Lemma 2.1.2 is:

Lemma 2.1.6. *Let (N^{m+n}, h) a Riemannian manifold, x a point in N, $y : \mathbb{R}^{m+n} \to (U, x)$ a coordinate chart of N. If $f : D_r^m \to \mathbb{R}^n$ is a C^2 function, then there exists C depending on $|Dy|$ and $|D(y^{-1})|$ so that*

$$\left|D^2 f\right|^2 \leq C(1 + |Df|^2)^3 \, |\mathrm{II}|^2$$

where II *is the second fundamental form of* $y(\operatorname{graph} f)$.

Proof. Let $\overline{\mathrm{II}}$ be the second fundamental form of graph f considered as a submanifold of $(\mathbb{R}^{m+n}, y^* h)$. Then $|\mathrm{II}| = \left|\overline{\mathrm{II}}\right|$, so we will compute $\left|\overline{\mathrm{II}}\right|^2$ as in the proof of Lemma 2.1.2. We will abuse notation and write h for $y^* h$. We will write $\overline{g}_{ij}$ for the metric induced on graph f by h, and $\overline{g}_{\alpha\beta}$ for the metric on the normal bundle of graph f with respect to h. We let θ be the least eigenvalue of h and Θ the greatest eigenvalue of h.

The tangent bundle of graph f is spanned, as before, by

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0, D_i f)$$

So the induced metric is

$$\begin{aligned} \overline{g}_{ij} =& h(e_i, e_j) \\ =& h_{ij} + h(D_i f, D_j f) + h_{pj} \frac{\partial f_p}{\partial x_i} + h_{iq} \frac{\partial f_q}{\partial x_j} \end{aligned} \tag{2.22}$$

and the eigenvalues $\overline{\mu} = \overline{g}(X,X)$ of $\overline{g}_{ij}$ are therefore bounded by

$$\begin{aligned} \theta \leq h(X,X) \leq \overline{\mu} \ &\leq h(X,X)\left(1+|Df|_h+|Df|_h^2\right) \\ &\leq \Theta\left(1+\Theta^{\frac{1}{2}}|Df|+\Theta|Df|^2\right) \end{aligned} \tag{2.23}$$

where $|\cdot|_h$ denotes the norm induced by h.

The normal bundle N_h is characterised by $N_h = \{X|h(X,e_i)\} = 0$. Equivalently $N_h = h^{-1}N_{dx^2}$, where N_{dx^2} is the normal bundle of the graph with respect to the standard metric dx^2 and we consider h as a bundle map over the identity $h : T\mathbb{R}^{m+n} \to T\mathbb{R}^{m+n}$. We may thus take a normal frame $\overline{\nu}_\alpha = h^{-1}(\nu_\alpha)$. Then to compute the eigenvalues of the normal metric $\overline{g}_{\alpha\beta}$, we consider $X \in \mathbb{R}^n$ with $|X|^2 = 1$:

$$\begin{aligned} \overline{g}(X,X) = \overline{g}_{\alpha\beta}X^\alpha X^\beta =& h(\overline{\nu}_\alpha, \overline{\nu}_\beta)X^\alpha X^\beta \\ =& h(h^{-1}(\nu_\alpha), h^{-1}(\nu_\beta))X^\alpha X^\beta \\ =& \nu_\alpha \cdot h^{-1}(\nu_\beta)X^\alpha X^\beta \\ =& (-D(X\cdot f), X)\cdot y^{-1}(-D(X\cdot f), X) \\ =& h^{-1}((-D(X\cdot f), X), (-D(X\cdot f), X)) \end{aligned} \tag{2.24}$$

Thus we have

$$\frac{1}{\Theta} \leq \overline{g}(X,X) \leq \frac{1}{\theta}(1+|Df|^2) \tag{2.25}$$

Now just as in the proof of Lemma 2.1.2, we use (2.23) and (2.24) to bound the Hessian of f in terms of $|\overline{\mathrm{II}}|$, $|Df|$, and θ, Θ. The eigenvalues of h^{-1} are clearly controlled by $|Dy|$ and $|Dy^{-1}|$. $\square$

Similarly one can extend Lemma 2.1.3 to a general ambient manifold:

Lemma 2.1.7. *Let* (N, h)*, x, and y be as above. If* $f : D_r^m \to \mathbb{R}^n$ *is a* C^ℓ *function, then we can bound* $\left|D^\ell f\right|$ *in terms of* $|Df|, \ldots, \left|D^{\ell-1} f\right|$, $\left|\nabla^{\ell-2}\, \mathrm{II}\right|$, $|Dy|, \ldots, \left|D^{\ell-1} y\right|$, *and* $\left|D(y^{-1})\right|, \ldots, \left|D^{\ell-1}(y^{-1})\right|$*, where* II *and* ∇ *are the second fundamental form and covariant derivative on* $y(\mathrm{graph}\, f)$*.*

The proof of Cheeger-Gromov's theorem involves the following proposition, which is analogous to our Lemma 2.1.4. An exposition can be found in chapter 10, section 3 of [45].

Proposition 2.1.8. *Suppose* (N^{m+n}, h) *is a Riemannian manifold with* $\mathrm{inj}(N, h) \geq \eta > 0$ *and* $\left|\nabla^\ell \, \mathrm{Rm}(N, h)\right| \leq C$ *for* $1 \leq \ell \leq \ell_0$*. Then there exist* r_0 *and* Q *depending on* C, η, ℓ_0, m, n *such that for any* $0 < r \leq r_0$*, each* $x \in N$ *admits a chart* $y_x : (U_x, x) \to (\mathbb{R}^{m+n}, 0)$ *such that* $y_x(U_x)$ *contains* $D_r^{m+n} \subset \mathbb{R}^{m+n}$ *and such that* $|Dy_x|, \ldots, \left|D^{\ell_0+2} y_x\right|$ *and* $\left|D(y_x^{-1})\right|, \ldots, \left|D^{\ell_0+2}(y_x^{-1})\right|$*, and the derivatives of the transition maps are all bounded by* Q

Moreover we may take a subatlas with the property that the centers of the charts are some definite $0 < \delta \leq \frac{r_0}{4}$ *apart.*

We refer to such an atlas as the Cheeger-Gromov atlas.

We are now ready to prove the version of Lemma 2.1.4 for a general ambient manifold. Toward this end, given $q \in M$, $r > 0$, let the Langer chart $U_{r,q}$ be the component of $F^{-1}(y_{F(q)}(\pi^{-1}(D_r^m)))$ which contains q, where $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is projection to the first m coordinates.

Lemma 2.1.9. *Let* $F : M^m \to (N^{m+n}, h)$ *be an immersion such that* $|\mathrm{II}| \leq C_1$, $\mathrm{inj}(N, h) \geq \eta > 0$*, and* $|\mathrm{Rm}(N, h)| \leq C_2$*. For any* $\alpha > 0$*, there is* $r_1 > 0$ *depending on* $C_1, C_2, m, n, \alpha, \eta$

such that for any $0 < r \leq r_1$, $q \in M$, $y_{F(q)}^{-1}(F(U_{r,q})) = \operatorname{graph} f$ *for some* $f : D_r^m \to \mathbb{R}^n$ *with* $|Df| \leq \alpha$.

Proof. Let $\alpha > 0, q \in M$ be arbitrary. For small r, $y_{F(q)}^{-1}(F(U_{r,q}))$ is a graph over D_r^m. So let S_q be the supremum of such r. If $S_q < r_0$, then the argument in the proof of Lemma 2.1.4 gives a lower bound on S_q depending only on α, C_1, C_2, and in particular independent of q.

If $S_q = r_0$, we can write $y_{F(q)}^{-1}(F(U_{r_0,q}))$ as a graph of some $f : D_{r_0}^m \to \mathbb{R}^n$. If $|Df| \leq \alpha$, we are done. If $\sup_{D_{r_0}^m} |Df| > \alpha$, there is some smaller disc D_r^m with $\sup_{D_r^m} |Df| = \alpha$; then the argument in the proof of Lemma 2.1.4 gives a lower bound on S_q depending only on α, C_1, C_2. $\square$

2.1.2 Proof of Theorem 2.0.12

2.1.2.1 Euclidean Case

We now prove the following theorem, which is the special case of Theorem 2.0.12 when $(N_k, h_k) \equiv (\mathbb{R}^{m+n}, dx^2)$.

Theorem 2.1.10. *Let* M_k^m *be a sequence of smooth m-manifolds. Suppose* $F_k : (M_k, p_k) \to (\mathbb{R}^{m+n}, 0)$ *is an immersion of* M_k *into* $\mathbb{R}^{m+n}$ *such that the second fundamental forms* II_k *and their covariant derivatives* $\nabla^\ell \, \mathrm{II}_k$ *are bounded pointwise,* $1 \leq \ell \leq \ell_0$. *Then there exists a smooth m-manifold* (M_∞, p_∞) *which admits a sequence of relatively compact open subsets* $W_1 \subset W_2 \subset \cdots$ *which exhausts* M_∞ *and embeddings* $\phi_k : (W_k, p_\infty) \hookrightarrow (M_k, p_k)$ *such that:*

1. $F_k \circ \phi_k$ *subconverge in the* $C^{\ell_0+1,\gamma}$ *topology for any* $0 \leq \gamma < 1$ *on compact subsets of* M_∞ *to some* $F_\infty : (M_\infty, p_\infty) \to (\mathbb{R}^{m+n}, 0)$.

2. *For each $R > 0$, the metric ball $B_k(p_k, R) \subset (M_k, F_k^* dx^2)$ is contained in $\phi_k(W_k)$ for all $k \geq k_0(R)$.*

3. *$(M_\infty, F_\infty^* dx^2)$ is a complete Riemannian manifold.*

In case M_∞ is compact we may take one of the W_k to be M_∞ itself.

Before proving this theorem, we note that given Lemmas 2.1.4 and 2.1.5, the proof is essentially finished already. This is because we have shown we can choose a parametrization, at least on Langer charts of a definite positive size, in which each immersion is the graph a function which has small first derivative and bounded higher derivatives; thus Arzela-Ascoli guarantees convergence on each Langer chart. By passing to a subsequence we can add Langer charts so that the convergence agrees on overlaps. The following merely formalizes this argument.

Proof. Let $\alpha < \frac{1}{10}$, r given by the lemmas, and $\delta = \frac{r}{10}$.

For each $\ell \in \mathbb{N}$, define

$$W_{\ell,k} = \bigcup_{i=1}^{K^\ell} U_{q_k^i, \frac{3\delta}{4}} \tag{2.26}$$

where the q_k^i are those points, given by Lemma 2.1.5, for which $B(p_k, \ell\frac{\delta}{2}) \subset \bigcup U_{q_k^i, \frac{\delta}{4}}$. Let $U_k^i = U_{q_k^i, \delta}$ and $\tilde{U}_k^i = U_{q_k^i, \frac{3\delta}{4}}$.

Fix ℓ. For each $1 \leq i \leq K^\ell$, $|F_k(q_k^i)| \leq d_k(q_k^i, p_k) \leq \ell\delta$. Lemma 2.1.4 gives Euclidean isometries A_k^i which take $F_k(q_k^i)$ to the origin and $T_{F_k(q_k^i)} F_k(M_k)$ to $\mathbb{R}^m \times \{0\}$. Since these Euclidean isometries are bounded, a subsequence of them must converge, for each i, to some A_∞^i, which is a Euclidean isometry; moreover since there are finitely many i for a fixed ℓ, this convergence may be taken to be uniform in i. In particular the A_k^i are a Cauchy sequence.

Lemma 2.1.4 also produces $f_k^i : D_\delta^m \to \mathbb{R}^n$ so that graph $f_k^i = A_k^i \circ F_k(U_k^i)$, so that $\left|Df_k^i\right| \leq \alpha$ and $\left|D^2 f_k^i\right| \leq \left(1+\alpha^2\right)^{\frac{3}{2}} C$. Since the $\{A_k^i\}$ are a Cauchy sequence, for k, k' large enough depending on ϵ, $A_k^i \circ (A_k^i)^{-1}$ is ϵ-close to the identity on $\mathbb{R}^{m+n}$. Thus we may take k, k' large enough that $A_{k'}^i \circ (A_k^i)^{-1}(\text{graph } f_k^i|_{D_{\frac{3\delta}{4}}})$ is a graph over $D_{\frac{3\delta}{4}}$. For any two indices $i, j \leq K^\ell$ with $U_k^i \cap U_k^j$ nonempty, we have that

$$A_k^j \circ (A_k^i)^{-1}(\text{graph } f_k^i|_{\pi \circ A_k^i(F_k(U_k^i \cap U_k^j))}) = \text{graph } f_k^j|_{\pi \circ A_k^j(F_k(U_k^i \cap U_k^j))}. \tag{2.27}$$

We can therefore take k, k' large enough so that

$$A_{k'}^i \circ (A_k^j)^{-1}(\text{graph } f_k^j|_{\pi \circ A_k^j(F_k(\tilde{U}_k^i \cap \tilde{U}_k^j))}) \tag{2.28}$$

is a graph over $D_{\frac{3\delta}{4}} \cap (\pi \circ A_{k'}^j(F_{k'}(\tilde{U}_{k'}^i \cap \tilde{U}_{k'}^j)))$.

The previous paragraph allows us to choose $k_0(\ell)$ so that for any $k, k' \geq k_0$, $F_k(W_{\ell,k})$ is a graph over $F_{k'}(W_{\ell,k'})$. In particular, $W_{\ell,k}$ and $W_{\ell,k'}$ are diffeomorphic and we can write W_ℓ unambiguously. We write $\phi_{\ell,k}$ for the identification of W_ℓ with $W_{\ell,k} \subset M_k$. By construction, $W_\ell \subset W_{\ell+1}$. In fact, since the M_k are without boundary, for each W_ℓ we have $\overline{W}_\ell \subset W_{\ell'}$ for some $\ell' > \ell$. We therefore can pass to a subsequence so that $\overline{W}_\ell \subset W_{\ell+1}$.

By construction it is clear that $\phi_{\ell+1,k}|_{W_\ell} = \phi_{\ell,k}$, so we can pass to a diagonal subsequence $\phi_k = \phi_{k,k} : W_k \hookrightarrow M_k$. Setting $M_\infty = \bigcup_{k=1}^\infty W_k$, we have the claimed M_∞ exhausted by the sequence $\{W_k\}$.

We now prove the convergence of $F_k \circ \phi_k$ on compact sets in $C^{\ell_0+1,\gamma}(M_\infty, \mathbb{R}^{m+n})$ for any $0 \leq \gamma < 1$. Given a compact set $C \subset M_\infty$, C is contained in some W_K. $F_k(\phi_k(W_k)) = F_k(W_{k,k})$ is a graph over $F_{k_0}(W_{k_0,k_0})$ for $k \geq k_0(\ell)$; moreover by construction the function of which it is a graph has first derivative bounded by 2α and higher derivatives up to order ℓ_0+2 bounded by Lemma 2.1.3. Thus by Arzela-Ascoli, the $F_k \circ \phi_k$ converge in $C^{\ell_0+1,\gamma}(W_\ell, \mathbb{R}^{m+n})$ for any $0 \le \gamma < 1$. The limit maps $F_\infty : W_\ell \to \mathbb{R}^{m+n}$, by construction, agree. So we have the claimed $F_\infty : M_\infty \to \mathbb{R}^{m+n}$. This completes the proof of Theorem 2.1.10. □

Remark 3. The Cheeger-Gromov charts given by Proposition 2.1.8 are exactly analogous to the Langer charts $U_{r,q}$. The injectivity bound is used by Cheeger-Gromov to ensure these charts can be taken to be of a definite size; here we are able to exploit, via Lemma 2.1.2, the bound on II to achieve this purpose.

In fact we have

Corollary 2.1.11 (to the proof of Theorem 2.1.10)**.** *The injectivity radius of the induced metric F^*h of an immersed submanifold $F : M \to (N,h)$ is bounded below:*

$$\mathrm{inj}(M, F^*h) \ge \frac{C}{\sup_M |\mathrm{II}|}$$

where C depends on the injectivity radius of (N,h). In case $(N,h) = (\mathbb{R}^{m+n}, dx^2)$, $C = \frac{1}{2\sqrt{2}}$.

Proof. We prove the Euclidean case; the general case is similar. Taking $\alpha = 1$ and r given by Lemma 2.1.4, we have for any $q \in M$ that $B(q,r) \subset U_{r,q}$ is a graph over the tangent plane at $F(q)$. Thus $\mathrm{inj}(q) \ge r$. □

2.1.2.2 General Case

The proof of Theorem 2.0.12 proceeds along the same lines as in Theorem 2.1.10, using the lemmas in section 2.1.1.2 in place of those in section 2.1.1.1. M_∞ is constructed as the union of limits of Langer charts and N_∞ is constructed as the union of limits of Cheeger-Gromov charts.

Remark 4. We could prove convergence as above given good-enough integral bounds ($L^p, p > m$) on the second fundamental form, as in [35]. In the first inequality of (2.19), we would need to use the Sobolev inequality instead of the fundamental theorem.

2.2 Topological Finiteness Theorems

Before considering applications of Theorem 2.0.12 to the mean curvature flow, which is our main purpose for it, we discuss in this section some topological finiteness theorems which may be of independent interest.

We begin by relating $C^{\ell,\alpha}$ geometric convergence in the sense of Theorem 2.0.12 to convergence in the function space $C^{\ell,\alpha}(M,N)$.

Proposition 2.2.1. *Let $\{M_k\}$ be a family of smooth m manifolds and $\{N_k\}$ a family of smooth $m+n$ manifolds. If the Riemannian immersions $F_k : (M_k, p_k) \to (N_k, h_k, x_k)$ converge in $C^{\ell,\alpha}$ in the geometric sense to $F_\infty : (M_\infty, p_\infty) \to (N_\infty, h_\infty, x_\infty)$, with M_∞ and N_∞ compact, then $\psi_k^{-1} \circ F_k \circ \phi_k$ converge to F_∞ in $C^{\ell,\alpha}(M_\infty, N_\infty)$.*

Proof. Since M_∞ and N_∞ are compact, $M_\infty = W_k$ and $N_\infty = V_k$ in the tail of the sequence; thus it makes sense to consider $\psi_k^{-1} \circ F_k \circ \phi_k$ in $C^{\ell,\alpha}(M_\infty, N_\infty)$. Then Theorem 2.0.12 gives the result. □

The implicit function theorem gives the following, which says that the set of immersions which are regularly homotopic to a given immersion is open in $C^{1,\alpha}$.

Proposition 2.2.2. *Let M^m, N^{m+n} be smooth compact manifolds, $F \in C^{1,\alpha}(M,N)$ an immersion. Then there is $\epsilon(F) > 0$ such that $||G - F||_{C^{1,\alpha}} \leq \epsilon$ implies that G is an immersion, which is regular-homotopic to F through $C^{1,\alpha}$ immersions.*

In particular, the intersection of $C^{1,\gamma}(M,N)$ with each regular-homotopy class is open in $C^{1,\gamma}(M,N)$.

We now apply Theorem 2.0.12 and Propositions 2.2.1 and 2.2.2 to obtain a topological finiteness theorem, somewhat analogous to the results in Cheeger's thesis [8]. We make the following definitions to allow us to state the finiteness theorem.

Definition 2.2.1. Two immersions $F, G : M \to N$ are *conjugate regular homotopic* if there exist diffeomorphisms $\phi : M \to M$ and $\psi : N \to N$ so that $\psi^{-1} \circ F \circ \phi$ is regular isotopic to G. Two embeddings $F, G : M \hookrightarrow N$ are *conjugate ambient isotopic* if there exist diffeomorphisms ϕ, ψ so that $\psi^{-1} \circ F \circ \phi$ is ambient isotopic to G.

Theorem 2.2.3. *Consider the class $\mathcal{F}$ of immersions $F : M \to (N, h)$ which satisfy, for some C_1, C_2, C_3, C_4, η,*

- $\mathrm{vol}(N,h) \leq C_1$, $\mathrm{inj}(N,h) \geq \eta$, $|\mathrm{Rm}(N,h)| \leq C_2$
- $\mathrm{vol}(M, F^*h) \leq C_3$, $|\mathrm{II}(F(M),h)| \leq C_4$

Then there are finitely many diffeomorphism types of M, finitely many diffeomorphism types of N, and finitely many conjugate regular homotopy classes of F represented in $\mathcal{F}$.

Proof. For any immersion satisfying the hypothesized bounds, (M, F^*h) is a Riemannian manifold with bounded curvature, volume, and by Corollary 2.1.11, injectivity radius. It follows from a standard Riemannian argument that each such (M, F^*h) has bounded diameter. Cheeger's theorem states that there are finitely many diffeomorphism types of such M.

So we restrict our attention conjugate regular homotopy classes of immersions from some M_0 into some N_0. Then the proof of Theorem 2.0.12 allows us to reparametrize F as a (r, α) immersion; in particular, the reparametrized F is bounded in C^0 by the diameter bound, bounded in C^1 since it is a (r, α)-immersion, and bounded in C^2 by the assumed bound on the second fundamental form.

That is, up to reparametrization the class $\mathcal{F}$ is bounded in $C^2(M_0, N_0)$. Hence it is compact in $C^{1,\gamma}(M_0, N_0)$ for any $0 \leq \gamma < 1$. On the other hand, each regular homotopy class is open in $C^{1,\gamma}(M_0, N_0)$. The theorem follows. □

By fixing a target manifold, we get a finiteness theorem for regular homotopy classes up to parametrization of the domain:

Theorem 2.2.4. *For any compact Riemannian manifold (N, h), let $\mathcal{F}_{(N,h)}$ be the class of immersions $F : M \to (N, h)$ which satisfy $\mathrm{vol}(F(M)) \leq C_1$, $|\mathrm{II}(F(M))| \leq C_2$. There are finitely many regular homotopy classes, up to parametrization of the domain, represented in $\mathcal{F}_{(N,h)}$.*

To state Theorem 2.2.3 in a manner more topologically useful, we fix the diffeomorphism type of M and state the contrapositive to obtain:

Theorem 2.2.5. *Let $\mathcal{C} = \{c_i\}$ be a collection of regular homotopy classes of maps $F : M \to (N, h)$, up to diffeomorphism of M. If $\mathcal{C}$ is infinite, then there is no choice of immersed representatives $F_i \in c_i$ which satisfies $\mathrm{vol}(F_i(M)) \leq C_1$, $\left|\mathrm{II}(F_i(M))\right| \leq C_2$.*

Similarly, we may prove finiteness theorems for ambient isotopy classes of embeddings $F : M \hookrightarrow (N, h)$. Since embeddedness is fragile, we require uniformity in the following sense:

Definition 2.2.2. The embedding constant of an immersion $F : M \to (N, h)$ is

$$\kappa(F) = \sup_{p,q \in M} \frac{d_g(p, q)}{d_h(F(p), F(q))} \tag{2.29}$$

where d_g is the distance function on M induced by $g = F^*h$ and d_h is the distance function on N induced by h.

F is an embedding if and only if $\kappa(F)$ is finite. F is totally geodesic if and only if $\kappa(F) = 1$.

Proposition 2.2.6. *Let M^m be a smooth compact manifold, $F \in C^{1,\alpha}(M, N)$ an embedding. Then there is $\epsilon(F) > 0$ such that $||G - F||_{C^{1,\alpha}} \leq \epsilon$ implies that G is an embedding which is ambient-isotopic to F.*

In particular, the intersection of $C^{1,\alpha}(M, N)$ with each ambient isotopy class is open in $C^{1,\alpha}(M, N)$.

Proof. The proof is the same as that of Proposition 2.2.2, since an immersion which is locally ambient isotopic to an embedding must be an embedding which is ambient isotopic. □

Theorem 2.2.7. *Consider the class $\mathcal{F}_{emb}$ of embeddings $F : M \to (N, h)$ which satisfy, for some $C_1, C_2, C_3, C_4, C_5, \eta$*

- $\mathrm{vol}(N, h) \leq C_1$, $\mathrm{inj}(N, h) \geq \eta$, $|\mathrm{Rm}(N, h)| \leq C_2$
- $\mathrm{vol}(M, F^*h) \leq C_3$, $|\mathrm{II}(F(M), h)| \leq C_4$, $\kappa(F) < C_5$

Then there are finitely many diffeomorphism types of M, finitely many diffeomorphism types of N, and finitely many conjugate ambient isotopy classes of F represented in $\mathcal{F}_{emb}$.

Proof. The only difference between the proof of this theorem and Theorem 2.2.3 is we must assume the embeddings are uniform so that the class $\mathcal{F}_{\mathrm{emb}}$ will be closed. □

Similarly, there are ambient-isotopy versions of Theorems 2.2.4 and 2.2.5.

To conclude this section, we give examples of infinite collections of homotopy classes which have immersive representatives. First consider $M = T^2$, $N = T^5$. By Whitney's theorem, every map $F : M \to N$ is homotopic to an immersion. Moreover, since T^2 and T^5 are Eilenberg-Maclane spaces, we have $[M, N] = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^5)$.

Similarly, we may consider two hyperbolic manifolds $M^m = \mathbb{H}^m/\Gamma$, $N^n = \mathbb{H}^{m+n}/\Lambda$, where Γ is a lattice in $SO(m, 1)$ and Λ is a lattice in $SO(m + n, 1)$. If $n \geq m$, Whitney's theorem says that every map from M to N is homotopic to an immersion. The homotopy classes of maps from M to N are given by $\mathrm{Hom}(\Gamma, \Lambda)$. Γ and Λ can be chosen so that $\mathrm{Hom}(\Gamma, \Lambda)$ is infinite.

Or consider the case of a simply-connected four-manifold X with non-torsion $H_2(X)$. By the theorem of Hurewicz, $|\pi_2(X)| = \infty$; Theorem 2.2.5 says that in order to realize each one of these classes, the immersion must be allowed to have either arbitrarily large volume or arbitrarily large curvature.

We also note that in the case $M = S^1$ and N is closed, every homotopy class admits a geodesic representative, so our finiteness theorems imply that for each $L > 0$ there are at most finitely many distinct homotopy classes whose (shortest) geodesic representatives have length less than L.

2.3 Singularity Models

We now use Theorem 2.0.12 to construct singularity models for compact mean curvature flows $F : M \times [0,T) \to (N,h)$.

First we state a compactness theorem for mean curvature flows, which follows directly from Theorem 2.0.12:

Theorem 2.3.1. *Suppose that $F_j : M_j \times [\alpha,\omega] \to (N_j, h_j)$ are compact mean curvature flows such that $\left|\mathrm{II}_j(t)\right| \le C$ for all j and all $t \in [\alpha,\omega]$ and $\left|\nabla^\ell \mathrm{II}_j(0)\right| \le C_\ell$ for each ℓ, and such that (N_j, h_j) have uniformly bounded geometry. Then there is a mean curvature flow $F_\infty : M_\infty \times [\alpha,\omega] \to (N_\infty, h_\infty)$ such that for each $t \in [\alpha,\omega]$, $F_j(t)$ subconverges in C^ℓ in the geometric sense to $F_\infty(t)$, for any ℓ; moreover this convergence is uniform in t.*

Proof. By the smoothness estimate for the mean curvature flow, the uniform bound on the second fundamental form gives uniform bounds on all its derivatives as well. Thus at each $t \in [\alpha,\omega]$, we may apply Theorem 2.0.12 to get $M_\infty(t)$, $F_\infty(t)$, and $(N_\infty(t), h_\infty(t))$. The time-derivatives $\frac{\partial^\ell}{\partial t^\ell} F_j$ are, by the flow equation, uniformly bounded; thus $F_\infty(t) : M_\infty(t) \to (N_\infty(t), h_\infty(t))$ are a smooth one-parameter family. Moreover, the construction of $M_\infty(t)$ and $(N_\infty(t), h_\infty(t))$ and the maps ϕ_j and ψ_j relies only on the curvature and injectivity bounds, which are uniform, so we may take M_∞, (N_∞, h_∞), and ϕ_j, ψ_j independent of time.

It is clear that $F_\infty : M_\infty \times [\alpha,\omega] \to (N_\infty, h_\infty)$ is a mean curvature flow. $\square$

2.3.1 The Smooth Blow-up

Now suppose that $F : M \times [0,T) \to (N,h)$ is a compact mean curvature flow, (p_j, t_j) are a sequence of points and times (the *central sequence*), and $\alpha_j \nearrow \infty$ are a sequence of positive numbers such that $\limsup_j \frac{\sup_{M\times[0,t_j]}|\mathrm{II}|}{\alpha_j} < \infty$. Then the rescales

$$\tilde{F}_j(s) : F(t_j + \frac{s}{Q_j^2}) \to (N, \alpha_j^2 h, F(p_j, t_j)) \tag{2.30}$$

form a sequence as in Theorem 2.3.1 for any $[\alpha, \omega] \subset (-\alpha_j^2 t_j, 0]$. Note that if the geometry of (N,h) is bounded, then the Cheeger-Gromov limit of $(N, \alpha_j^2 h, F(p_j, t_j))$ is $(\mathbb{R}^{m+n}, \mathrm{dx}^2)$.

To construct models for the singularities of the mean curvature flow, we must correctly pick the central sequence (p_j, t_j) and the scale factors α_j. The choices we make are inspired by those used by Hamilton for the Ricci flow [12] [25].

The construction depends on how severe the singularity is.

Proposition 2.3.2. *For any compact mean curvature flow $F : M \times [0,T) \to (N,h)$ with singular time $T < \infty$ we have*

$$\max_M |\mathrm{II}(\cdot,t)| \geq \frac{C}{\sqrt{T-t}} \tag{2.31}$$

where the constant C depends on the initial submanifold M_0.

The blow-up rate (2.31) is that of a shrinking sphere or cylinder; it represents the mildest sort of singularity that the MCF can encounter. We define

Definition 2.3.1. The mean curvature flow $F : M \times [0,T) \to (N,h)$ achieves a type I singularity at T if

$$\sup_{M\times[0,T)} |\mathrm{II}|^2 (T-t) < \infty$$

Otherwise we say the singularity is of type II.

First consider the case of a type II singularity. For any sequence $\tilde{t}_j \nearrow T$, let $p_j \in M$ be such that

$$(\tilde{t}_j - t_j) \left|\mathrm{II}(p_j, t_j)\right|^2 = \max_{M\times[0,\tilde{t}_j]} (\tilde{t}_j - t) \left|\mathrm{II}(p, t)\right|^2 \tag{2.32}$$

Set $Q_j = \left|\mathrm{II}(p_j, t_j)\right|$. By the type II assumption, $(\tilde{t}_j - t_j)Q_j^2 \to \infty$, so for any t there is j large enough that $t \in (-Q_j^2 t_j, (\tilde{t}_j - t_j)Q_j^2)$. For such j, we compute

$$\begin{aligned} \left|\mathrm{II}_j(p, t)\right|^2 = Q_j^{-2} \left|\mathrm{II}(p, t_j + \frac{t}{Q_j^2})\right|^2 &\leq Q_j^2 \frac{(\tilde{t}_j - t_j) \left|\mathrm{II}(p_j, t_j)\right|^2}{\tilde{t}_j - (t_j + \frac{t}{Q_j^2})} \\ &= \frac{(\tilde{t}_j - t_j)Q_j^2}{(\tilde{t}_j - t_j)Q_j^2 - t} \end{aligned} \tag{2.33}$$

The right-hand side of this inequality approaches 1 as $j \to \infty$, hence is bounded by a continuous function of t. Therefore we may apply Theorem 2.3.1 to the F_j to extract a limit mean curvature flow F_∞.

If the singularity is of type I, we pick $t_j = \tilde{t}_j$ and p_j so that $Q_j = \left|\mathrm{II}(p_j, t_j)\right| = \max_{M\times[0,t_j]} |\mathrm{II}|$. Then $Q_j \to \infty$.

Since M is compact, in either case we have that, after passing to a subsequence, $p_j \to \overline{p}$. We choose the rescales $\tilde{F}_j$ about the central sequence $(\overline{p}, t_j)$.

$$\tilde{F}_j(s) = F(t_j + \frac{s}{Q_j^2}) \to (N, Q_j^2 h, F(\overline{p}, t_j)) \tag{2.34}$$

In the type I case, each $\tilde{F}_j$ has second fundamental form bounded by 1 on the interval $[-Q_j^2 t_j, 0]$. In the type II case, $\tilde{F}_j$ has second fundamental form bounded by 1 on the interval $[-Q_j^2 t_j, Q_j^2(\tilde{t}_j - t_j)]$

Theorem 2.3.3. *The geometric limit of the rescaled sequence (2.34) is mean curvature flow $F_\infty : M_\infty \times (-\infty, C) \to \mathbb{R}^{m+n}$. Here $C = 0$ if the singularity is of type I and $C = \infty$ if the singularity is of type II.*

Moreover, we have $|\mathrm{II}_\infty(p_\infty, 0)| = 1$.

Proof. Note that since (N, h) has bounded geometry, the Cheeger-Gromov limit of $(N, Q_j^2 h)$ is $(\mathbb{R}^{m+n}, \mathrm{dx}^2)$.

The only thing left to prove is that $|\mathrm{II}_\infty(p_\infty, 0)| = 1$. For a fixed k, notice that the rescaled metric $g_k(0) = F_{t_k}^*(Q_k^2 h)$ is a metric on M. Let B_k denote the metric ball in the metric $g_k(0)$. Since $p_j \to p$, we have that for any $R > 0$, $p_j \in B_k(\overline{p}, R)$ for all $j \geq j_0(k, R)$. By geometric convergence the metrics $\{g_j(0)\}$ have the Cauchy property that $B_k(\overline{p}, R) \subset B_j(\overline{p}, 2R)$.

On the other hand, p_j is a point where $\left|\mathrm{II}_j(p_j, 0)\right| = 1$. Thus in the tail of the sequence there is a point of curvature 1 within $2R$ of $\overline{p}$. This condition clearly persists to the limit, so there is a point of curvature 1 within $2R$ of p_∞. But R was arbitrary, so letting $R \to 0$ we see that $|\mathrm{II}_\infty(p_\infty, 0)| = 1$. □

We refer to the MCF $F_\infty : M_\infty \times (-\infty, C) \to \mathbb{R}^{m+n}$ as a *smooth blow-up* of the original flow $F : M \times [0, T) \to (N, h)$.

Though we have stated the construction of the smooth blow-up for compact mean curvature flows, note that the construction will also work provided the singularity is of *compact type*:

Definition 2.3.2. We say that a mean curvature flow $F : M \times [0,T) \to (N,h)$ has a compact-type singularity at $T < \infty$ if:

- $\lim_{t \to T} \sup_M |\mathrm{II}(t)| = \infty$
- For any $t_j \nearrow T$, there exist p_j with $\left|\mathrm{II}(p_j, t_j)\right| = \sup_{M \times [0,t_j)} |\mathrm{II}|$ and $p_j \to p$

Remark 5. In general smooth blow-ups are nonunique, since Theorem 2.3.1 only gives subsequential convergence.

Remark 6. The diffeomorphisms ϕ_k in the construction of the smooth blow-up amount to choosing the "correct" parametrization of regions of the domain submanifold M which are becoming singular. Huisken-Sinestrari, in order to carry out their surgery theorem, explicitly construct such a parametrization of the singular region by means of a nearby shrinking cylinder [31]. The import of Theorem 2.3.1 is that such a parametrization can always be found.

2.3.2 Comparison to the Tangent Flow

The smooth blow-up is inspired Hamilton's idea for singularity models for the Ricci flow [12]. In previous literature on the mean curvature flow, singularities have been understood using a rescaling procedured called the *tangent flow*, which we now describe.

To produce a tangent flow, we work in the category of *Brakke flows*, i.e. one-parameter families of integral currents which are locally maximally area-decreasing [5] [33]. A mean curvature flow is, a fortiori, a Brakke flow. We have the following theorem due to Brakke, which follows from the compactness theorem for integral currents of Federer-Fleming.

Theorem 2.3.4 (Brakke, [5])**.** *Let $T_k(t)$ be a sequence of Brakke flows on $[\alpha, \omega]$. Then $T_k(t)$ subconverge as integral currents to a Brakke flow $T_\infty(t)$ on $[\alpha, \omega]$.*

Given a compact mean curvature flow $F_t : M \times [0, T) \to \mathbb{R}^{m+n}$, there is some point $x_0 \in \mathbb{R}^{m+n}$ such that $\lim_{t \to T} F(p, t) = x_0$ for some $p \in M$ with $\lim_{t \to T} |\mathrm{II}(p, t)| = \infty$. We say that the singularity of the flow occurs at x_0. If $t_j \nearrow T$ and $Q_j = \sup_{M \times [0, t_j]} |\mathrm{II}|$, we define

$$\overline{F}_j(p, s) = Q_j^2 \left[F\left(p, T + \frac{s}{Q_j^2} \right) - x_0 \right] \tag{2.35}$$

and call a subsequential Brakke flow limit of $\overline{F}_j$ a *tangent flow* with center (x_0, T) of the original flow F_t.

The primary advantage of using the tangent flow construction is that all Brakke flows that arise as tangent flows satisfy an elliptic equation called the *self-shrinker* equation.

Definition 2.3.3. Given a mean curvature flow $M(t)$ and any $(x_0, t_0) \in \mathbb{R}^{m+n} \times \mathbb{R}$, we define Huisken's monotonic quantity

$$\Theta_{M, x_0, t_0}(t) = \int_{M(t)} (4\pi(t_0 - t))^{-\frac{m}{2}} e^{\frac{-|x - x_0|^2}{4(t_0 - t)}} \, d\mathcal{H}^m \tag{2.36}$$

Theorem 2.3.5 (Huisken [27])**.** *Huisken's monotonic quantity is monotone along a smooth mean curvature flow. In particular it satisfies:*

$$\frac{d}{dt} \Theta_{M, x_0, t_0}(t) = - \int_{M(t)} \left| H + \frac{1}{2(t_0 - t)} (x - x_0)^\perp \right|^2 (4\pi(t_0 - t))^{-\frac{m}{2}} e^{\frac{-|x - x_0|^2}{4(t_0 - t)}} \, d\mathcal{H}^m$$

here $(x - x_0)^\perp$ is the projection of the vector $x - x_0$ to the normal bundle of M.

Flows for which Θ is constant are called *self-shrinking*. In fact the mean curvature flow with intial data satisfying the elliptic equation

$$H = \alpha x^{\perp} \tag{2.37}$$

for some $\alpha < 0$ are necessarily self-shrinking; we call a submanifold, or more generally an integral current, satisfying (2.37) a *self-shrinker*. We have the following theorem:

Theorem 2.3.6 (Huisken [27]). *Any tangent flow to a mean curvature flow is a self-shrinking flow.*

The self-shrinking condition imposes fairly strong restrictions, as in the following theorem:

Theorem 2.3.7 (Huisken [28]). *A smooth mean-convex self-shrinking hypersurface must be one of the following:*

- *a round sphere*
- *a round cylinder*
- $\Gamma \times \mathbb{R}^{m-1}$, *where* Γ *is one of the Abresch-Langer curves [1]*

Huisken [27] showed that in the type I case, the tangent flow construction in fact yields a smooth limit. We now show that this construction is the same as the smooth blow-up.

Proposition 2.3.8. *Suppose that* $F_t : M \times [0, T) \to \mathbb{R}^{m+n}$ *is a compact mean curvature flow with type I singularity at* T. *Then the smooth blow-up of* F_t *is a self-shrinking flow.*

Proof. The proof is the same as the proof of Theorem 2.3.6, with the necessary changes enabled by the type I assumption.

Given the central sequence $\{(\overline{p}, t_j)\}$, set $x_j = F(\overline{p}, t_j)$. Then there is a subsequential limit $x_0 = \lim_j x_j$. We compute:

$$
\begin{aligned}
|x_j - x_0| &= \left| \int_{t_j}^{T} H(\overline{p}, s) ds \right| \\
&\leq \int_{t_j}^{T} |H(\overline{p}, s)| ds \\
&\leq \int_{t_j}^{T} C(T-s)^{-\frac{1}{2}} ds \\
&= C(T-t_j)^{\frac{1}{2}} \leq \frac{C'}{Q_j}
\end{aligned}
\tag{2.38}
$$

Thus $\{Q_j(x_0 - x_j)\}$ is a bounded sequence, so that again passing to a subsequence, we have some $\overline{x} = \lim_j Q_j(x_0 - x_j)$.

Set $\alpha_j = Q_j^2(T - t_j)$. Then each M_j exists on $(-Q_j^2 t_j, \alpha_j)$. By the type I assumption, we can pass to a subsequence so that the limit $\lim_j \alpha_j = C$ exists. We consider Huisken's monotonic quantity centered at $(\overline{x}, C)$:

$$
\Theta_{M_\infty, \overline{x}, C}(s) = \int_{M_\infty(s)} (4\pi(C-s))^{-\frac{m}{2}} e^{-\frac{|x-\overline{x}|^2}{4(C-s)}} d\mathcal{H}^m \tag{2.39}
$$

Given any $-Q_j^2 t_j < a < b < \alpha_j$ and a compact set $K \subset \mathbb{R}^{m+n}$, we have by the scaling properties of Huisken's quantity:

$$\int_a^b \int_{M_j(s)\cap K} \left| H + \frac{(x - Q_j x_0)^\perp}{2(\alpha_j - s)} \right|^2 (4\pi(\alpha_j - s))^{-\frac{m}{2}} e^{-\frac{|x-Q_j x_0|^2}{4(\alpha_j - s)}} d\mathcal{H}^m ds$$
$$= \int_{t_j+\frac{a}{Q_j^2}}^{t_j+\frac{b}{Q_j^2}} \int_{M(t)\cap(Q_j^{-1}K+x_j)} \left| H + \frac{(x - x_0)^\perp}{2(T - t)} \right|^2 (4\pi(T - t))^{-\frac{m}{2}} e^{-\frac{|x-x_0|^2}{4(T-t)}} d\mathcal{H}^m dt \tag{2.40}$$

We can estimate the right-hand side of (2.40) by integrating over all of $M(t)$ and applying Theorem 2.3.5:

$$\int_{t_j+\frac{a}{Q_j^2}}^{t_j+\frac{b}{Q_j^2}} \int_{M(t)\cap(Q_j^{-1}K+x_j)} \left| H + \frac{(x - x_0)^\perp}{2(T - t)} \right|^2 (4\pi(T - t))^{-\frac{m}{2}} e^{-\frac{|x-x_0|^2}{4(T-t)}} d\mathcal{H}^m dt$$
$$\leq \int_{t_j+\frac{a}{Q_j^2}}^{t_j+\frac{b}{Q_j^2}} \int_{M(t)} \left| H + \frac{(x - x_0)^\perp}{2(T - t)} \right|^2 (4\pi(T - t))^{-\frac{m}{2}} e^{-\frac{|x-x_0|^2}{4(T-t)}} d\mathcal{H}^m dt \tag{2.41}$$
$$= \Theta_{M,x_0,T}(t_j + \frac{a}{Q_j^2}) - \Theta_{M,x_0,T}(t_j + \frac{b}{Q_j^2})$$

Since $t_j + \frac{a}{Q_j^2}$ and $t_j + \frac{b}{Q_j^2}$ both approach T as $j \to \infty$, we have by Theorem 2.3.5 that the right-hand side of (2.41) goes to 0 as $j \to \infty$.

On the other hand, the left-hand side of (2.40) approaches

$$\int_a^b \int_{M_\infty(s)\cap K} \left| H + \frac{(x - \overline{x})^\perp}{2(C - s)} \right|^2 (4\pi(C - s))^{-\frac{m}{2}} e^{-\frac{|x-\overline{x}|^2}{4(C-s)}} d\mathcal{H}^m ds \tag{2.42}$$

Since K, a, and b were arbitrary we have that for almost every s and almost every $x \in M_\infty(s)$ that $\left|H + \frac{(x-\overline{x})^\perp}{2(C-s)}\right|^2 = 0$. Thus M_∞ is a self-shrinking flow with center $(\overline{x}, C)$. $\square$

We therefore have the following characterization of singularity types in case $(N, h) = (\mathbb{R}^{m+n}, \mathrm{dx}^2)$.

Corollary 2.3.9. *The singularity of a compact mean curvature flow $F_t : M \to \mathbb{R}^{m+n}$ is of type I if and only if it admits a smooth blow-up which becomes extinct in finite time.*

If the singularity of the flow $F : M \times [0, T) \to \mathbb{R}^{m+n}$ is of type II, its tangent flow is expected to have singularities. In particular, as shown by examples of Neves [42], it is possible that a tangent flow can consist of a union of (non smooth) minimal Lagrangian cones. In this case, it is unclear in what sense the tangent flow approximates the developing singularity of the flow F. However, such an approximation is essential for a surgery theorem as in Hamilton's program for the Ricci flow. Note, by contrast, that the smooth blow-up approximates the developing singularity in the C^∞ sense. This is precisely the sort of approximation that Huisken-Sinestrari use to prove their surgery theorem for 2-convex hypersurfaces.

Colding-Minicozzi have a program to understand mean curvature flows of surfaces immersed in $\mathbb{R}^3$ using the tangent flow [13]. Instead of surgering near singular times, they flow to the singular time and jump to a "nearby" Brakke flow constructed using the tangent flow. Their aim is to establish that, for generic initial data, this nearby Brakke flow is in fact a smooth mean curvature flow, and moreover that Huisken's quantity does not increase in the jump. This project is ongoing.

Remark 7. Baker's thesis [4] contains many of the same ideas as this chapter, including a construction essentially equivalent to the smooth blow-up. We first became aware of these results in May 2011.

A compactness theorem for immersions, quite similar to our Theorem 2.1.10, is attributed by Baker in his thesis [4] to Breuning. Breuning's result requires an additional hypothesis, called the *local volume bound*, which allows a more direct application of Langer's approach in [35]. Our Lemma 2.1.5 allows us to avoid this consideration.

Chapter 3

Characterizing the Singular Time

In this chapter we prove Theorems 1.3.6 and 1.3.8. The proof is based on Sesum's proof of Theorem 1.3.5.

We will use the singularity models constructed in the last chapter and the following lemmas concerning one-parameter families of Riemannian metrics:

Lemma 3.0.10 (Glickenstein [18])**.** *Suppose a one-parameter family of complete Riemannian manifolds $(M, g(t))$ is uniformly continuous in t, that is, for any $\epsilon > 0$ there is $\delta > 0$ so that for any t_0, $(1-\epsilon)g(t_0) \leq g(t) \leq (1+\epsilon)g(t_0)$ for $t \in [t_0, t_0+\delta]$. Then for any $p \in M$, $r > 0$, the metric balls centered at p satisfy:*

$$B_{g(t_0)}\left(p, \frac{r}{\sqrt{1+\epsilon}}\right) \subseteq B_{g(t)}(p, r) \subseteq B_{g(t_0)}\left(p, \frac{r}{\sqrt{1-\epsilon}}\right)$$

Proof. Let $p, q \in M$. Let $\gamma : [0, S] \to M$ be a minimising geodesic from p to q for the metric $g(t_0)$. Then the distance $d_{g(t_0)}(p, q)$ in the metric $g(t_0)$ satisfies

$$
\begin{aligned}
d_{g(t_0)}(p,q) &= \int_0^S |\dot{\gamma}|_{g(t_0)}(s)ds \\
&\geq \frac{1}{\sqrt{1+\epsilon}} \int_0^S |\dot{\gamma}|_{g(t)}(s)ds \\
&\geq \frac{1}{\sqrt{1+\epsilon}} d_{g(t)}(p,q)
\end{aligned}
\tag{3.1}
$$

This immediately implies

$$
B_{g(t_0)}\left(p, \frac{r}{\sqrt{1+\epsilon}}\right) \subset B_{g(t)}(p, r). \tag{3.2}
$$

The other inclusion is analogous. □

Lemma 3.0.11 (Hamilton [23]). *Let $(M, g(t))$ be a one-parameter family of compact Riemannian manifolds defined for $t \in [0, T)$. Suppose that*

$$
\int_0^T \max_{M_t} \left| \frac{\partial g}{\partial t} \right|_{g(t)} dt < \infty
$$

Then the metrics $g(t)$ are uniformly equivalent and converge pointwise as $t \to T$ to a continuous positive-definite metric $g(T)$.

We will be considering volumes of metric balls, so we state the following evolution equation:

Lemma 3.0.12. *The evolution of the volume form dvol_t of the induced metric $g(t)$ of a mean curvature flow is*

$$
\frac{\partial}{\partial t} \mathrm{dvol}_t = -|H|^2 \, \mathrm{dvol}_t
$$

3.1 Proof of Theorem 1.3.6

Proof of Theorem 1.3.6. Suppose, for the purpose of contradiction, that $F_t : M^m \to (N, h)$ is a mean curvature flow of compact submanifolds on the maximal interval $[0, T)$, and $|A| \leq C$ all along the flow.

First note that since $|A|$ is bounded, and $|A|$ has scaling degree -2, any smooth blow-up $F_\infty : M_\infty \to \mathbb{R}^{m+n}$ for the flow will necessarily have $A \equiv 0$, hence $|H|^2 = \operatorname{tr} A = 0$. That is, the smooth blow-up is a minimal submanifold, and in particular is stationary in time. Thus we work at time $s = 0$. Denote the pullback metric $F_\infty^* \,\mathrm{dx}^2$ by g_∞

We will consider the volume growth of metric balls in (M_∞, g_∞) to obtain a contradiction. Let us use the following conventions for balls and volumes. $B_\infty(\rho)$ will denote the metric ball in g_∞ centered at p_∞; $B_j(\rho)$ will denote the metric ball in $F_j(0)^*(\mathrm{dx}^2)$ centered at p_j; $B_{t_j}(\rho)$ will denote the metric ball in $F_{t_j}^*(\mathrm{dx}^2)$ centered at p_j. vol_∞ will denote the volume form of g_∞; vol_j will denote the volume form of $F_j(0)^*(\mathrm{dx}^2)$; vol_{t_j} will denote the volume form of $F_{t_j}^*(\mathrm{dx}^2)$. Note that

$$\begin{aligned} B_j(\rho) &= B_{t_j}\left(\frac{\rho}{Q_j}\right) \\ \mathrm{vol}_j &= Q_j^m \, \mathrm{vol}_{t_j} \end{aligned} \tag{3.3}$$

We have, for any $r > 0$

$$\begin{aligned} \frac{\mathrm{vol}_\infty \left(B_\infty(r)\right)}{r^m} &= \lim_j \frac{\mathrm{vol}_j \left(B_j(r)\right)}{r^m} \\ &= \lim_j \frac{\mathrm{vol}_{t_j} \left(B_{t_j}\left(\frac{r}{Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} \end{aligned} \tag{3.4}$$

The evolution of g is

$$\partial_t g_{ij} = -2A_{ij} \tag{3.5}$$

so we have $|\partial_t g| \leq C$, and in particular g is uniformly continuous in time in the sense of Lemma 3.0.10.

Thus we may apply Lemma 3.0.10 to estimate the metric balls at any time t_j by the metric ball at time t_{j_0}, so long as $t_j - t_{j_0} \leq \delta$. Since $t_j \to T$, we can pick a j_0 so that this condition holds for all $j \geq j_0$. So we can estimate (3.4) by:

$$\lim_j \frac{\mathrm{vol}_{t_j} \left(B_{t_j}\left(\frac{r}{Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} \leq \lim_j \frac{\mathrm{vol}_{t_j} \left(B_{t_{j_0}}\left(\frac{r}{\sqrt{1-\epsilon}Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} \tag{3.6}$$

The evolution of the volume form shows that the flow is pointwise volume-reducing. So $\mathrm{vol}_{t_j} \leq \mathrm{vol}_{t_{j_0}}$ for $j \geq j_0$. Thus we can estimate (3.6) by

$$\lim_j \frac{\mathrm{vol}_{t_j}\left(B_{t_{j_0}}\left(\frac{r}{\sqrt{1-\epsilon}Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} \leq \lim_j \frac{\mathrm{vol}_{t_{j_0}}\left(B_{t_{j_0}}\left(\frac{r}{\sqrt{1-\epsilon}Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} = (1-\epsilon)^{-\frac{m}{2}} \lim_j \frac{\mathrm{vol}_{t_{j_0}}\left(B_{t_{j_0}}\left(\frac{r}{\sqrt{1-\epsilon}Q_j}\right)\right)}{\left(\frac{r}{\sqrt{1-\epsilon}Q_j}\right)^m}. \tag{3.7}$$

The only dependence of the right hand side on j is in the Q_j.

The limit on the right hand side of (3.7) is the local volume comparison at p_{j_0} for the Riemannian manifold $\left(M, F^*_{t_{j_0}}(\mathrm{dx}^2)\right)$. It is well-known that this limit is ω_m, the volume of the Euclidean unit m-ball. Therefore we have

$$\frac{\mathrm{vol}_\infty(B_\infty(r))}{r^m} \leq (1-\epsilon)^{-\frac{m}{2}} \omega_m \tag{3.8}$$

Since ϵ was arbitrary, we have shown $\mathrm{vol}_\infty(B_\infty(r)) \leq \omega_m r^m$.

To show the reverse inequality, we make a similar argument starting from (3.4), this time using the first inclusion of Lemma 3.0.10. We now seek to estimate vol_{t_j} *below* by $\mathrm{vol}_{t_{j_0}}$. Since we have assumed $|H| \leq C$, the evolution of vol implies that

$$\mathrm{vol}_{t_j} \geq e^{-C^2(t_j - t_{j_0})} \mathrm{vol}_{t_{j_0}} \tag{3.9}$$

and taking j_0 large enough we may ensure that $e^{-C^2(t_j - t_{j_0})} \geq 1 - \epsilon$. Then we can estimate (3.4) by

$$\lim_j \frac{\mathrm{vol}_{t_j}\left(B_{t_j}\left(\frac{r}{Q_j}\right)\right)}{\left(\frac{r}{Q_j}\right)^m} \geq \lim_j (1-\epsilon)(1+\epsilon)^{-\frac{m}{2}} \frac{\mathrm{vol}_{t_{j_0}}\left(B_{t_{j_0}}\left(\frac{r}{\sqrt{1+\epsilon}Q_j}\right)\right)}{\left(\frac{r}{\sqrt{1+\epsilon}Q_j}\right)^m} \tag{3.10}$$

Again we can take the limit in j to get

$$\frac{\mathrm{vol}_\infty(B_\infty(r))}{r^m} \geq (1-\epsilon)(1+\epsilon)^{-\frac{m}{2}}\omega_m \tag{3.11}$$

Since ϵ was arbitrary we have shown

$$\mathrm{vol}_\infty(B_\infty(r)) \geq \omega_m r^m \tag{3.12}$$

Now we are ready to obtain the contradiction. Since F_∞ is a minimal immersion with $|\mathrm{II}(p_\infty)|^2 = 1$, we have by Gauss's equation that the scalar curvature $R(p_\infty)$ is given by

$$R(p_\infty) = |H(p_\infty)|^2 - |\mathrm{II}(p_\infty)|^2 = -1 \tag{3.13}$$

On the other hand, the scalar curvature at p_∞ is related to the volume growth of balls centered at p_∞ [17]:

$$\mathrm{vol}_\infty(B_\infty(r)) = \omega_m r^m \left(1 - \frac{R_\infty(p_\infty)}{6(m+2)} r^2 + O(r^3)\right) \tag{3.14}$$

Equation (3.14), together with (3.12), implies that $R_\infty(p_\infty) = 0$, which is our desired contradiction. $\square$

3.2 Proof of Theorem 1.3.8

We now prove Theorem 1.3.8. In fact we can prove that H must blow up under a slightly more-general condition than type I, namely that

$$|\mathrm{II}(t)|^p\,(T-t) \leq C \text{ for some } p > 1 \tag{3.15}$$

It is unknown, however, whether there are examples of mean curvature flows which satisfy estimate (3.15) but which are not of type I.

Theorem 3.2.1. *Let $F_t : M^m \to (N, h)$ be a compact mean curvature flow on the maximal interval $[0, T)$ satisfying (3.15). Then*

$$\lim_{t\to T} \sup_M |H|^2 = \infty$$

Proof. We want to emulate the proof of Theorem 1.3.6. Suppose $|H| \leq C$ all along the flow. Then as before, the smooth blow-up F_∞ is a minimal immersion.

Condition (3.15), together with the bound on $|H|$, implies that

$$\begin{aligned}\int_0^T \max\left|\frac{\partial}{\partial t} g\right| ds =& 2\int_0^T \max|A| ds \\ &\leq C\int_0^T \max|\mathrm{II}|\, ds \\ &\leq C'\int_0^T (T-s)^{-\frac{1}{p}} ds < \infty\end{aligned} \tag{3.16}$$

Thus we may apply Lemma 3.0.11 to see that the metrics $g(t)$ approach a limit C^0 metric $g(T)$. In particular, the balls of the metrics $g(t)$ are equivalent in the sense of Lemma 3.0.10 and we may proceed just as in the proof of Theorem 1.3.6. □

We also have the following corollary of Propositon 2.3.8 which relates to the *rate* at which $|H|$ blows up in the type I case. Le-Sesum have an independent proof of this result [37].

Corollary 3.2.2. *Let $F_t : M^m \to \mathbb{R}^{m+n}$ be a compact mean curvature flow on the maximal interval $[0, T)$ with a type I singularity at T. Then $|H|$ and $|\mathrm{II}|$ blow up at the same rate, that is,* $\limsup_{t \to T} \dfrac{\sup_M |H|}{\sup_M |\mathrm{II}|} > 0$.

Proof. Suppose for a contradiction that $|H|$ blows up more slowly than $|\mathrm{II}|$. Then the smooth blow-up F_∞ will be minimal. Since the singularity is type I, F_∞ will also be a self-shrinking flow. Thus

$$x^\perp = 2(C - s)H = 0 \tag{3.17}$$

everywhere on F_∞. In particular F_∞ is a cone. The only smooth minimal cones in a Euclidean space are planes, so in fact F_∞ is flat. On the other hand, F_∞ admits at least one non-flat point. This contradiction establishes the corollary. □

We end this chapter with the following question:

Question 1. Is it possible that a compact mean curvature flow with a finite time singularity has $|H|$ bounded along the flow?

Le-Sesum have shown that $||H||_{L^\alpha(M \times [0,T))}$ must blow up at a finite-time singularity, for any $\alpha \geq n + 2$ [36]. The corresponding question for Ricci flow, namely understanding when pointwise and integral bounds on the scalar curvature R are enough to guarantee the extension of the Ricci flow, has been investigated by X. Cao, Sesum-Tian, and Z. Zhang, among others [6] [50] [59].

Chapter 4

Lagrangian Mean Curvature Flow

Throughout this chapter, Σ^m will be a compact smooth manifold, $L_t : \Sigma \to (X^{2m}, \omega, J)$ will be a mean curvature flow in a Calabi-Yau manifold with maximal existence interval $[0, T)$, with L_0 Lagrangian. We will use $\langle \cdot, \cdot \rangle$ to denote the Riemannian metric induced by (ω, J) and $g = L^* \langle \cdot, \cdot \rangle$ will be the pullback of this metric.

4.1 Preliminaries

We begin by recalling some facts about the geometry of Lagrangian submanifolds $L : \Sigma^m \to \mathbb{C}^m$.

If $\{x_1, \ldots, x_m\}$ are coordinates on Σ, then $\{\partial_i = \dfrac{\partial L}{\partial x_i}\}$ span TL. Since L is Lagrangian, $\{\nu_i = J\partial_i\}$ span NL. We always compute in terms of such a frame, and use Latin indices throughout. In particular, we set

$$h_{ijk} = \langle D_{\partial_i}\partial_j, \nu_k\rangle = -\omega(D_{\partial_i}\partial_j, \partial_k) \tag{4.1}$$

$$H_k = g^{ij} h_{ijk} \tag{4.2}$$

Lemma 4.1.1. *The tensor* $\mathrm{II} = h_{ijk}$ *is totally symmetric.*

Proof. Symmetry in the first two indices holds in general by the definition of the second fundamental form. Since $d\omega = 0$ and ω is compatible with the metric, we have

$$\begin{aligned} 0 &= d\omega(\partial_i, \partial_j, \partial_k) \\ &= \omega(D_{\partial_k}\partial_i, \partial_j) + \omega(\partial_i, D_{\partial_k}\partial_j) \\ &= -h_{kij} + h_{kji} \end{aligned} \tag{4.3}$$

which establishes symmetry in the second and third indices. □

Definition 4.1.1. A *Liouville form* for the Kähler manifold (X, ω, J) is any one-form η on X with $d\eta = 2\omega$. We call $\lambda = L^*\eta$ the *Liouville form* of L. In case (X, ω, J) is the standard Kähler structure on $\mathbb{C}^m$, we may take $\eta = p_i dq^i - q_i dp^i$, where $\{p_i + \sqrt{-1}q_i\}$ are coordinates for $\mathbb{C}^m$.

Note that if $L : \Sigma \to X$ is a Lagrangian submanifold, then $\lambda = L^*\eta$ is closed, since $L^*\omega = 0$.

Definition 4.1.2. The Maslov form h is the one-form dual, with respect to ω, to the mean curvature vector $H = H^i\nu_i$, that is, $h = L^*(H \lrcorner \omega)$. That is, $h = H_i\,\mathrm{dx}^i$.

In case (X, ω, J) is the standard $\mathbb{C}^m$, then the choice of η above shows that λ is the one-form dual to the vector $L^*(-Jx^\perp)$, where $x^\perp$ is the projection of the position vector x to the normal bundle of L.

When (X, ω, J) is Kähler-Einstein, by Codazzi's equation and the contracted Bianchi identity, we have that $\nabla_i H_j = \nabla_j H_i$, hence h is closed.

The following lemma allows us to consider *Lagrangian* mean curvature flow.

Lemma 4.1.2. *Mean curvature flow preserves the Lagrangian condition in a Kähler-Einstein manifold, i.e. if the initial submanifold L_0 is Lagrangian, so is each time slice L_t.*

Proof. The Lagrangian condition is $L^*\omega = 0$. We compute

$$\begin{aligned} \frac{\partial}{\partial t} L^*\omega &= L^*(\mathcal{L}_H \omega) \\ &= L^* d(H \lrcorner \omega) + L^*(H \lrcorner d\omega) \\ &= dh + L^*(H \lrcorner d\omega) \end{aligned} \tag{4.4}$$

Both terms are zero, since $dh = 0$ and $d\omega = 0$. □

Lemma 4.1.3 ([15], [40])**.** *If $L : \Sigma \to (X, \omega, J)$ is a Lagrangian submanifold of a Calabi-Yau manifold, then there is a smooth function $\beta : \Sigma \to S^1$, called the* Lagrangian angle*, with $h = d\beta$.*

Remark 8. The Lagrangian angle can be defined by the relation

$$L^* \Re(\Omega) = e^{i\beta} \,\mathrm{dvol} \tag{4.5}$$

where Ω is the unit holomorphic $(m, 0)$ form of the Calabi-Yau manifold (X, ω, J) and dvol is the volume element of (Σ, g).

Definition 4.1.3. The class $[\lambda] \in H^1(\Sigma)$ is called the *period* or *Liouville class* of the immersion L. $[h] \in H^1(\Sigma)$ is the *Maslov class* of the immersion L.

If $[h] = 0$, or equivalently if β is a real-valued function, we say L has *zero Maslov class*.

If $[\lambda] = 0$, or equivalently if $\lambda = d\phi$ for some smooth real-valued ϕ, we say L is *exact*.

Remark 9. Suppose $H^1(X) = 0$. If η_1 and η_2 are two different Liouville forms on (X, ω, J) inducing λ_1 and λ_2 on Σ, then $d(\eta_1 - \eta_2) = \omega - \omega = 0$, so $\eta_1 - \eta_2 = df$ for some function f.

$$\int_\gamma \lambda_1 - \lambda_2 = \int_{L \circ \gamma} \eta_1 - \eta_2 = \int_{L \circ \gamma} df = 0 \tag{4.6}$$

Therefore $[\lambda] \in H^1(\Sigma)$ is independent of the choice of η. In particular, this is true when (X, ω, J) is the standard $\mathbb{C}^m$.

We will use the Maslov class and period to study the singularities of the flow. We begin by recalling the following computations of Smoczyk [51]:

Lemma 4.1.4 ([51]). *The Maslov form and Liouville form evolve according to*

$$\frac{\partial}{\partial t} h = dd^* h \tag{4.7}$$

$$\frac{\partial}{\partial t} \lambda = dd^* \lambda - 2h \tag{4.8}$$

where d^* is the negative adjoint to d. In particular,

$$\frac{\partial}{\partial t} [h] = 0 \tag{4.9}$$

$$\frac{\partial}{\partial t} [\lambda] = -2[h] \tag{4.10}$$

We will also use the scaling properties of λ and h:

Lemma 4.1.5. *Let $L : \Sigma \to (X, \omega, J)$ be a Lagrangian submanifold and $\tilde{L} : \Sigma \to (X, \alpha^2\omega, J)$ be its α-rescale. Then $\tilde{\lambda} = \alpha^2\lambda$ and $\tilde{h} = h$, where $\tilde{\lambda}$ and $\tilde{h}$ are the Liouville and Maslov forms of $\tilde{L}$.*

Proof. If η is a Liouville form for (X, ω, J), then $\alpha^2\eta$ is a Liouville form for $(X, \alpha^2\omega, J)$. So

$$\tilde{\lambda} = L^*(\alpha^2\eta) = \alpha^2 L^*\eta = \alpha^2\lambda \tag{4.11}$$

To see how h scales, note $\tilde{h}_{ijk} = -\alpha^2\omega(D_{\partial_i}\partial_j, \partial_k) = \alpha^2 h_{ijk}$. Then

$$\begin{aligned} \tilde{h} = \tilde{H}_i dx^i = \tilde{g}^{jk}\tilde{h}_{ijk}dx^i \\ = \alpha^{-2}g^{jk}\alpha^2 h_{ijk}dx^i = g^{jk}h_{ijk}dx^i = H_i dx^i = h \end{aligned} \tag{4.12}$$

□

Note that from Lemma 4.1.5 it follows that the Lagrangian angle β is scale-invariant and the primitive ϕ of the Liouville form has scaling degree 2, since the d operator is scale-invariant.

4.2 Lagrangian Singularities

In this section we use symplectic-topology invariants to investigate the structure of singularities of compact Lagrangian submanifolds of $\mathbb{C}^m$. In particular, we will prove the following theorem:

Theorem 4.2.1. *Let $L_\infty : \Sigma_\infty \to \mathbb{C}^m$ be the smooth blow-up of a Lagrangian mean curvature flow.*

If the singularity is of type I, then any smooth blow-up is monotone, i.e. $[\lambda_\infty]$ *is a positive multiple of* $[h_\infty]$.

If the singularity is of type II, then any smooth blow-up has $[h_\infty] = 0$ *and* $[\lambda_\infty] = 0$.

Proof. First we compute how the Liouville form of the smooth blow-up relates to the Liouville form of the original flow. Given a loop γ in Σ_∞, γ embeds in $\Sigma_j = \Sigma$ as $\phi_j\gamma$ for all j large enough. In fact, by geometric convergence the $\phi_j\gamma$ will be homotopic in the tail of the sequence. We call its homology class $[\gamma_0]$. h and λ will denote the Maslov and Liouville forms of the original flow, h_j and λ_j will denote the Maslov and Liouville forms of the j^{th} rescale, and h_∞ and λ_∞ will denote the Maslov and Liouville forms of Σ_∞. If $\eta \in H^1(\Sigma)$, we write $\eta.[\gamma]$ for the evaluation of η on $[\gamma]$.

We have by Lemma 4.1.4 that, if $[h].[\gamma] \neq 0$,

$$\frac{[\lambda].[\gamma]}{[h].[\gamma]}(t) = \frac{[\lambda].[\gamma]}{[h].[\gamma]}(0) - 2t \tag{4.13}$$

Therefore, we can compute $\dfrac{[\lambda_\infty].[\gamma]}{[h_\infty].[\gamma]}$ as follows, using Lemma 4.1.5:

$$\begin{aligned}\frac{[\lambda_\infty].[\gamma]}{[h_\infty].[\gamma]}(s) &= \lim_j \frac{[\lambda_j].[\phi_j\gamma]}{[h_j].[\phi_j\gamma]}(s) \\ &= \lim_j Q_j^2 \frac{[\lambda].[\gamma_0]}{[h].[\gamma_0]}\left(t_j + \frac{s}{Q_j^2}\right) \\ &= \lim_j Q_j^2 \left(\frac{[\lambda].[\gamma_0]}{[h].[\gamma_0]}(0) - 2t_j - 2\frac{s}{Q_j^2}\right) \\ &= \lim_j Q_j^2 \left(\frac{[\lambda].[\gamma_0]}{[h].[\gamma_0]}(0) - 2t_j\right) - 2s\end{aligned} \tag{4.14}$$

The assumption that γ is a loop which survives to Σ_∞ means that the limit on the right hand side of (4.14) must exist. This is only possible if the singularity is of type I, and $\frac{[\lambda].[\gamma_0]}{[h].[\gamma_0]}(0) = 2T$. Thus the loops which survive to Σ_∞ all have the same initial value of $\frac{[\lambda].[\gamma]}{[h].[\gamma]}$. Using $s = 0$ in (4.14), we have that $\frac{[\lambda_\infty].[\gamma]}{[h_\infty].[\gamma]}(0) = \lim_j 2Q_j^2(T - t_j) = 2C$, where C is the best type I constant. In particular $\frac{[\lambda_\infty].[\gamma]}{[h_\infty].[\gamma]}$ is independent of the choice of $[\gamma] \in H_1(\Sigma_\infty)$, hence L_∞ is a monotone flow.

In the type II case, the above necessarily requires $[h_\infty] = 0$. Since $[h]$ is scale-invariant and also constant along the flow, we have

$$\begin{aligned} 0 = [h_\infty].[\gamma](s) &= \lim_j [h_j].[\phi_j\gamma](s) \\ &= \lim_j [h].[\gamma_0]\left(t_j + \frac{s}{Q_j^2}\right) \\ &= \lim_j [h].[\gamma_0](0) = [h].[\gamma_0](0) \end{aligned} \tag{4.15}$$

so that we must have $[h].[\gamma_0] = 0$. In particular, $[\lambda].[\gamma_0]$ must be constant in time. On the other hand, we compute $[\lambda_\infty].[\gamma]$:

$$\begin{aligned} [\lambda_\infty].[\gamma](s) &= \lim_j [\lambda_j].[\phi_j\gamma](s) \\ &= \lim_j Q_j^2[\lambda].[\gamma_0]\left(t_j + \frac{s}{Q_j^2}\right) \\ &= \lim_j Q_j^2[\lambda].[\gamma_0](0) \end{aligned} \tag{4.16}$$

In order for the left-hand side to be finite, we must therefore have $[\lambda].[\gamma_0] = 0$. Then (4.16) gives that $[\lambda_\infty].[\gamma] = 0$. This completes the proof of the theorem. □

Remark 10. In case (X, ω, J) is the standard $\mathbb{C}^m$, note that the type I case of Theorem 4.2.1 is redundant. This is because self-shrinking flows are a fortiori monotone, and Proposition 2.3.8 guarantees smooth blow-ups of type I singularities are self-shrinking.

Corollary 4.2.2. *If Σ_∞ is a smooth blow-up of a type II singularity, there exist smooth functions $\phi, \beta : \Sigma_\infty \times (-\infty, \infty) \to \mathbb{R}$ such that:*

$$\nabla\beta = JH \tag{4.17}$$

$$\nabla\phi = -Jx^\perp \tag{4.18}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\beta = 0 \tag{4.19}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = -2\beta \tag{4.20}$$

We refer to β as the Lagrangian angle *and ϕ as the* normal potential *of Σ_∞.*

Proof. The first two claims are immediate from Theorem 4.2.1, since h is the dual one-form to JH and λ is the dual one-form to $-Jx^\perp$. The latter two follow in the same way from the evolution equations for h and λ:

$$\frac{\partial}{\partial t}d\beta = dd^*d\beta = d\Delta\beta \tag{4.21}$$

$$\frac{\partial}{\partial t}d\phi = dd^*d\phi - 2d\beta = d\Delta\phi - 2d\beta \tag{4.22}$$

d and $\frac{\partial}{\partial t}$ commute, so after adding a time-dependent constant to β and ϕ we have the claimed equations for β and ϕ (this argument is standard and appears in [42].) □

In fact β grows at worst linearly in space and time and ϕ grows at worst quadratically in space and time:

Lemma 4.2.3. *For any* $p_0, t_0 \in M_\infty \times (-\infty, \infty)$ *we have:*

- $|\beta(p,t) - \beta(p_0,t_0)| \leq C|t-t_0| + d_{t_0}(p,p_0)$
- $|\phi(p,t) - \phi(p_0,t)| \leq C|t-t_0|^2 + C|t-t_0| d_{t_0}(p,p_0) + \frac{1}{2} d_{t_0}(p,p_0)^2 + m|t-t_0| + d_{t_0}(p,p_0)|L(p_0,t)|$

Proof. The smooth blow-up has $|\nabla\beta| = |H| \leq 1$, so β is at worst linear in space. Also,

$$\begin{aligned} |\frac{\partial}{\partial t}\beta| = |\Delta\beta| &\leq \sqrt{m}|\nabla^2\beta| \\ &= \sqrt{m}|\nabla H| \leq C \end{aligned} \tag{4.23}$$

The claimed bound on β follows by integrating first in time and then in space.

Similarly, we have for any $q \in \Sigma_\infty$ that

$$\begin{aligned} |\nabla\phi(q)| = |L_\infty(q,t)^\perp| &\leq |L_\infty(q,t)| \\ &\leq |L_\infty(q,t) - L_\infty(p_0,t)| + |L_\infty(p_0,t)| \\ &\leq d_t(q) + |L_\infty(p_0,t)| \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} |\frac{\partial}{\partial t}\phi| = |\Delta\phi - 2\beta| &\leq \sqrt{m}|\nabla^2\phi| + 2|\beta| \\ &= \sqrt{m}|\nabla x^T| + 2|\beta| \\ &= m + 2|\beta| \end{aligned} \tag{4.25}$$

The claimed bound for ϕ follows by using the bound for β and integrating first in time and then in space. ∎

4.2.1 A Conjecture

The self-shrinker equation is a soliton equation for the mean curvature flow, that is, self-shrinking flows evolve by scaling. There are also examples of self-expanding flows and self-translating flows even in the Lagrangian category [2] [34] [43]. The minimal submanifold equation is also clearly a soliton equation for the flow. In the case of the Ricci flow, it is known that in dimension 3, all singularity models must satisfy the *gradient shrinking soliton* equation. In general we expect that rescaling procedures such as the smooth blow-up construction ought to result in solitons under the flow. Thus we confront the question of which solitons can arise as singularity models.

Notice that Theorem 4.2.1 gives the Lagrangian case of the following conjecture, *up to cohomology*:

Conjecture 1. Every singularity of a compact mean curvature flow is modeled either by a self-shrinker or a minimal submanifold, depending on the type of the singularity.

In fact we expect to be able to prove Conjecture 1 in the Lagrangian case, using the fact that β is a well-controlled eternal solution of the heat equation and a backward limit technique. More explicitly, since β satisfies the heat equation, we have

$$\frac{d}{dt}\int_{\Sigma}\beta^2(q,t)u_{p_0,t_0}(q,t)\,\mathrm{dvol}_t(q) = -2\int_{\Sigma}|\nabla\beta(q,t)|^2\,u_{p_0,t_0}(q,t)\,\mathrm{dvol}_t(q) \tag{4.26}$$

where u_{p_0,t_0} is the fundamental solution of the conjugate heat equation centered at (p_0,t_0) [7] [22]. The monotone quantity $\int_{\Sigma}\beta^2(q,t)u_{p_0,t_0}(q,t)\,\mathrm{dvol}_t(q)$ is invariant under parabolic rescaling. It is a standard approach, given such a quantity, to construct rescaling limits for which the quantity is constant. In our setting, this would mean $\nabla\beta = H \equiv 0$.

In non-Lagrangian settings, however, there is not a clear path to a proof of this conjecture.

In case the ambient manifold (N, h) is Euclidean, Proposition 2.3.8 is the type I case of Conjecture 1. The question of which manifolds admit a Huisken's monotonic quantity will be the subject of future research; it is expected that Proposition 2.3.8 can be extended to other ambient manifolds.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Uwe Abresch and Joel Langer. The normalized curve shortening flow and homothetic solutions. *Journal of Differential Geometry*, 23(2):175–196, 1986.

[2] Henri Anciaux. Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$. *Geometriae Dedicata*, 120(1):37–48, 2006.

[3] Ben Andrews and Charles Baker. Mean curvature flow of pinched submanifolds to spheres. *Journal of Differential Geometry*, 85:357–395, 2010.

[4] Charles Baker. *The mean curvature flow of submanifolds of high codimension*. PhD thesis, Australian National University, 2010.

[5] Kenneth Brakke. *The Motion of a Surface by its Mean Curvature*. Number 20 in Mathematical Notes. Princeton University Press, 1978.

[6] Xiaodong Cao. Curvature pinching estimate and singularities of the Ricci flow. arxiv:math.DG/1010.6064, 2010.

[7] Albert Chau, Luen-Fai Tam, and Chengjie Yu. Psuedolocality for the Ricci flow and applications. *Canadian Journal of Mathematics*, 63(1):55–85, 2011.

[8] Jeff Cheeger. *Comparison and Finiteness Theorems for Riemannian Manifolds*. PhD thesis, Princeton University, 1967.

[9] Jeff Cheeger and Mikhael Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. *Journal of Differential Geometry*, 23:309–346, 1985.

[10] Jingyi Chen and Weiyong He. A note on the singular time of mean curvature flow. *Mathematische Zeitschrift*, 266(4):921–931, 2010.

[11] Jingyi Chen and Jiayu Li. Singularity of mean curvature flow of Lagrangian submanifolds. *Inventiones Mathematicae*, 156:25–51, 2004.

[12] Bennett Chow, Peng Lu, and Lei Ni. *Hamilton's Ricci Flow*. Number 77 in Graduate Studies in Mathematics. AMS, 2006.

[13] Tobias Colding and William Minicozzi. Generic mean curvature flow I: generic singularities. arxiv:math.DG/0908.3788, 2009.

[14] Andrew A. Cooper. A characterization of the singular time of the mean curvature flow. *Proceedings of the AMS*, to appear.

[15] Pierre Dazord. Sur la géométrie des sous-fibrés et des feuilletages Lagrangiens. *Annales Scientifiques de l'École Normale Supérieure*, 14(4):465–480, 1981.

[16] Joerg Enders, Reto Müller, and Peter Topping. On type I singularities in Ricci flow. arxiv:math.DG/1005.1624, 2010.

[17] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian Geometry*. Springer-Verlag, 1987.

[18] David Glickenstein. Precompactness of solutions to the Ricci flow. *Geometry and Topology*, 7:487–510, 2003.

[19] Matthew A. Grayson. The heat equation shrinks embedded plane curves to round points. *Journal Differential Geometry*, 26:285–314, 1987.

[20] Konrad Groh, Matthias Schwarz, Knut Smoczyk, and Kai Zehmisch. Mean curvature flow of monotone Lagrangian submanifolds. *Mathematische Zeitschrift*, 257(2):295–327, 2007.

[21] Mikhael Gromov. *Structures Métriques pour les Varietés Riemanniennes*. Number 1 in Textes Mathematiques. Fernand-Nathan, 1981.

[22] Christine Guenther. The fundamental solution on manifolds with time-dependent metrics. *Journal of Geometric Analysis*, 12(3):425–436, 2002.

[23] Richard Hamilton. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17:255–306, 1982.

[24] Richard Hamilton. Four-manifolds with positive curvature operator. *Journal of Differential Geometry*, 24:153–179, 1986.

[25] Richard Hamilton. A compactness property for solutions of the Ricci flow. *American Journal of Mathematics*, 117:545–572, 1995.

[26] Reese Harvey and H. Blaine Lawson, Jr. Calibrated geometries. *Acta Mathematica*, 148(1):47–157, 1981.

[27] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *Journal of Differential Geometry*, 31:285–299, 1990.

[28] Gerhard Huisken. Local and global behaviour of hypersurfaces moving by mean curvature. *Proceedings of Symposia in Pure Mathematics*, 54(1):175–191, 1993.

[29] Gerhard Huisken and Carlo Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Mathematica*, 183:45–70, 1999.

[30] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calculus of Variations and Partial Differential Equations*, 8:1–14, 1999.

[31] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow with surgeries of two-convex hypersurfaces. *Inventiones Mathematicae*, 2008.

[32] Gerhard Husiken. Flow by mean curvature of convex hypersurfaces into spheres. *Journal of Differential Geometry*, 20:237–266, 1984.

[33] Tom Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Memoirs of the American Mathematical Society*, 108(520), 1994.

[34] Dominic Joyce, Yng-Ing Lee, and Mao-Pei Tsui. Self-similar solutions and translating solitons for Lagrangian mean curvature flow. *Journal of Differential Geometry*, 84(1):127–161, 2010.

[35] Joel Langer. A compactness theorem for surfaces with L_p-bounded second fundamental form. *Mathematische Annalen*, 270:223–234, 1985.

[36] Nam Le and Natasa Sesum. On the extension of the mean curvature flow. *Mathematische Zeitschrift*, 267(3-4):583–604, 2009.

[37] Nam Le and Natasa Sesum. Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers. arxiv:math/1011.5245v1, 2010.

[38] Nam Le and Natasa Sesum. Remarks on curvature behavior at the first singular time of the Ricci flow. arxiv:math.DG/1005.1220, 2010.

[39] Nam Le and Natasa Sesum. The mean curvature at the first singular time of the mean curvature flow. *Annales I. H. Poincare - AN*, to appear.

[40] Jean-Marie Morvan. Classe de maslov d'une immersion Lagrangienne et minimalité. *Comptes Rendus de Academie des Sciences Paris*, 292:633–636, 1981.

[41] Andre Neves. Singularities of Lagrangian mean curvature flow: zero-Maslov class case. *Inventiones Mathematicae*, 168:449–484, 2007.

[42] Andre Neves. Singularities of Lagrangian mean curvature flow: monotone case. *Mathematical Research Letters*, 17(1):109–126, 2010.

[43] Andre Neves and Gang Tian. Translating solutions to Lagrangian mean curvature flow. arxiv:math.DG/0711.4341, 2007.

[44] Grisha Perelman. The entropy formula for the Ricci flow and its geometric applications. arxiv:math/0211159, 2002.

[45] Peter Petersen. *Riemannian Geometry*. Number 171 in Graduate Texts in Mathematics. Springer-Verlag, second edition, 2006.

[46] Robert C. Reilly. On the Hessian of a function and the curvatures of its graph. *Michigan Math Journal*, 20:373–383, 1973.

[47] Rick Schoen and Jon Wolfson. Minimizing area among Lagrangian surfaces: The mapping problem. *Journal of Differential Geometry*, 58(1):1–86, 2001.

[48] Rick Schoen and Jon Wolfson. Mean curvature flow and Lagrangian embeddings. Unpublished, 2003.

[49] Natasa Sesum. Curvature tensor under the Ricci flow. *American Journal of Mathematics*, pages 1315–1324, 2005.

[50] Natasa Sesum and Gang Tian. Bounding scalar curvature and diameter along the Kähler Ricci flow. *Journal of the Institute for Mathematics of Jussieu*, 7:575–587, 2008.

[51] Knut Smoczyk. Angle theorems for the Lagrangian mean curvature flow. *Mathematische Zeitschrift*, 240:849–883, 2002.

[52] Knut Smoczyk and Mu Tao Wang. Mean curvature flows of Lagrangian submanifolds with convex potentials. *Journal of Differential Geometry*, 62(2):243–257, 2002.

[53] Jeffrey Streets and Gang Tian. A parabolic flow of pluriclosed metrics. *International Mathematics Research Notices*, 2010(16):3101–3133, 2010.

[54] Jeffrey Streets and Gang Tian. Regularity theory for pluriclosed flow. *Comptes Rendus de Academie des Sciences Paris*, 349(1):1–4, 2011.

[55] Mu Tao Wang. Deforming area preserving diffeomorphism of surfaces by mean curvature flow. *Mathematical Research Letters*, 8(5-6):651–662, 2001.

[56] Mu Tao Wang. Mean curvature flow of surfaces in Einstein four-manifolds. *Journal of Differential Geometry*, 57(2):301–338, 2001.

[57] Mu Tao Wang. Subsets of Grassmannians preserved by mean curvature flow. *Communications in Analysis and Geometry*, 13(5):981–998, 2005.

[58] Jon G. Wolfson. Minimal Lagrangian diffeomorphisms and the Monge-Ampere equation. *Journal of Differential Geometry*, 45:335–373, 1997.

[59] Zhou Zhang. Scalar curvature behavior for finite-time singularity of Kähler Ricci flow. *Michigan Math Journal*, 59:419–433, 2010.