CERTAIN SUMMATION AND CUBATURE FORMULAS

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CERTAIN SUMMATION AND CUBATURE FORMULAS

by

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CERTAIN SUMMATION AND CUBATURE FORMULAS

1. INTRODUCTION

Certain approximation formulas in one variable have been presented [o] by which a sum is estimated by taking h times the sum of every h^{th} term. The value of these summation formulas of Lubbock and Woolhouse is that they contain remainder terms which give upper bounds for the error in calculation.

The major purpose of this paper is to obtain extensions into two variables of Lubbock's formulas of the first, second, and third type, and of Woolhouse's formula of the first type. Approximations of a double sum are obtained by taking hk times the sum of every h^{th} term in every k^{th} array of one of the variables. To these approximations are added certain corrective terms which involve finite differences or derivatives. Remainder terms are obtained, and examples illustrate the use of these double summation formulas.

Also a formula for approximating the value of a double integral, called a cubature formula, is obtained. It may be compared with Hardy's formula of mechanical quadrature in one variable. A remainder term is again available, together with an illustrative example.

We shall make extensive use of the notation of the calculus of finite differences. For one variable we define

$$\underset{h}{\Delta} f(x) = f(x+h) - f(x) \,, \tag{1}$$

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[o] J.F. Steffensen, Interpolation, Baltimore, The Williams and Wilkins Company, 1927, p. 138. Hereafter referred to as Steffensen.

(2) $$\underset{h}{\delta} f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right),$$

(3) $$\underset{h}{\square} f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}.$$

It follows that

(4) $$\underset{h}{\square}\,\underset{h}{\delta} f(x) = \frac{f(x+h) - f(x-h)}{2}.$$

The quantities defined in (1)-(4) are called the descend - ing difference, the central difference, the mean, and the mean central difference of f(x) respectively. Generally h = 1, in which case we write Δ, δ, $\square$, and $\square\delta$.

For two variables we define

(5) $$\begin{aligned}\Delta_x \Delta_y f(x;y) &= \Delta_x \left[f(x;y+1) - f(x;y)\right]\\ &= \Delta_y \left[f(x+1;y) - f(x;y)\right]\\ &= f(x+1;y+1) - f(x;y+1) - f(x+1;y) + f(x;y).\end{aligned}$$

The symbols, Δ_x and Δ_y, are commutative.

Again in one variable we define a displacement symbol E^h such that when it operates on f(x) we obtain f(x + h). That is, $E^h f(x) = f(x + h)$. Hence $\Delta f(x) = (E - 1)f(x)$, showing that the symbol Δ is a linear function of a displacement symbol. For descending dif- ferences of higher order, $\Delta^n f(x) = (E - 1)^n f(x)$.

For two variables we define

$$\Delta_x^m \Delta_y^n f(x;y) = (E_x - 1)^m (E_y - 1)^n f(x;y).$$

For m = n = 1, we obtain (5).

For the central differences we define

(6) $$\begin{aligned}\delta_x \delta_y f(x;y) &= \delta_y \left[f\left(x+\tfrac{1}{2};y\right) - f\left(x-\tfrac{1}{2};y\right)\right]\\ &= \delta_x \left[f\left(x;y+\tfrac{1}{2}\right) - f\left(x;y-\tfrac{1}{2}\right)\right]\end{aligned}$$

$$= f(x+\tfrac{1}{2}; y+\tfrac{1}{2}) - f(x-\tfrac{1}{2}; y+\tfrac{1}{2}) - f(x+\tfrac{1}{2}; y-\tfrac{1}{2}) + f(x-\tfrac{1}{2}; y-\tfrac{1}{2}),$$

and for the mean central differences

(7)
$$\Box\delta_x \Box\delta_y = \frac{1}{4}\Big[f(x+1; y+1) - f(x-1; y+1) - f(x+1; y-1) + f(x-1; y-1)\Big].$$

The treatment of the remainder terms in this paper depends on certain quantities called divided differences. For one variable we define

$$f(a_0, a_1) = \frac{f(a_0) - f(a_1)}{a_0 - a_1},$$

$$f(a_0, a_1, a_2) = \frac{f(a_0, a_1) - f(a_1, a_2)}{a_0 - a_2},$$

and finally

$$f(a_0, a_1 \cdots a_n) = \frac{f(a_0 \cdots a_{n-1}) - f(a_1 \cdots a_n)}{a_0 - a_n},$$

where this last expression is an n^{th} order divided difference of $f(x)$ with respect to the arguments $a_0, a_1, \ldots, a_n$. It may be shown that

$$f(a_0, a_1 \cdots a_n) = \sum_{i=0}^{n} \frac{f(a_i)}{(a_i - a_0)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)},$$

so that the value of a divided difference of $f(x)$ of $n + 1$ arguments depends upon the value of $f(x)$ at the $n + 1$ arguments.[o]

Similarly, for two variables,

$$f(a_0, a_1; b_0) = \frac{f(a_0; b_0) - f(a_1; b_0)}{a_0 - a_1},$$

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o Steffensen, p. 15.

$$f(a_0, a_1; b_0, b_1) = \frac{f(a_0, a_1; b_0) - f(a_0, a_1; b_1)}{b_0 - b_1}$$

$$= \frac{f(a_0; b_0) - f(a_1; b_0) - f(a_0; b_1) + f(a_1; b_1)}{(a_0 - a_1)(b_0 - b_1)} ,$$

or, in general,

$$f(a_0, a_1 \cdots a_n; b_0) = \frac{f(a_0 \cdots a_{n-1}; b_0) - f(a_1 \cdots a_n; b_0)}{a_0 - a_n} ,$$

and

$$f(a_0 \cdots a_n; b_0 \cdots b_n) = \frac{f(a_0 \cdots a_{n-1}; b_0 \cdots b_{n-1}) - f(a_0 \cdots a_{n-1}; b_1 \cdots b_n)}{(a_0 - a_n)(b_0 - b_n)}$$

$$+ \frac{-f(a_1 \cdots a_n; b_0 \cdots b_{n-1}) + f(a_1 \cdots a_n; b_1 \cdots b_n)}{(a_0 - a_n)(b_0 - b_n)} .$$

A formula of major importance° for the evaluation of a divided difference is

$$f(x, a_0 \cdots a_n; y, b_0 \cdots b_m) = \frac{1}{(n+1)!(m+1)!} f_{n+1, m+1}(\xi; \eta) ,$$

where $a_0 \le \xi \le a_n$ and $b_0 \le \eta \le b_m$ if x and y lie in the same closed intervals as ξ and η. The formula indicates that the order of the partial derivative for each variable is one less than the number of arguments for that same variable in the divided difference.

In the formulas that will be used, the following factorial notation will appear frequently.

(8) $x^{(\nu)} = x(x-1) \dots (x - \nu + 1)$,

° Steffensen, p. 205.

(9) $\quad x^{[2\nu]} = x^2(x^2 - 1)(x^2 - 4) \ldots \left[x^2 - (\nu - 1)^2\right]$,

(10) $\quad x^{[2\nu+1]} = x\left(x^2 - \frac{1}{4}\right)\left(x^2 - \frac{9}{4}\right) \cdots \left[x^2 - \frac{(2\nu-1)^2}{4}\right]$,

(11) $\quad x^{[2\nu]-1} = \dfrac{x^{[2\nu]}}{x}$,

(12) $\quad x^{[2\nu+1]-1} = \dfrac{x^{[2\nu+1]}}{x}$.

Function (8) is the descending factorial. Functions (9)-(12) are central factorials. It is seen that (9) and (12) are even functions, while (10) and (11) are odd functions. Each factorial is a polynomial of degree indicated by the exponents. For $\nu = 0$, (8), (9), and (12) are assigned the value of 1, and hence (10) is equal to x and (11) is equal to 1 / x.

The following Theorem of Mean Value for sums will be used repeatedly.

H_1) f(x) is continuous in the closed interval (a,b),

H_2) $\phi(x)$ does not change sign in (c,d),

H_3) $c \leq x_i \leq d$, $a \leq \xi_i \leq b$, $a \leq \bar{\xi} \leq b$,

C_1) $\displaystyle\sum_{i=1}^{n} \phi(x_i) f(\xi_i) = f(\bar{\xi}) \sum_{i=1}^{n} \phi(x_i)$.

Proof: Since f(x) is continuous in the closed interval (a,b) it assumes a maximum, M, and a minimum, m, in that interval. Then, if $\phi(x)$ is positive or zero (otherwise we take $-\phi(x)$),

$$m \sum_{i=1}^{n} \phi(x_i) \leq \sum \phi(x_i) f(\xi_i) \leq M \sum_{i=1}^{n} \phi(x_i) \quad ,$$

which would not be true except for H_2. Hence there exists an N where $m \leq N \leq M$ such that

$$N \sum_{i=1}^{n} \phi(x_i) = \sum_{i=1}^{n} \phi(x_i) f(\xi_i) .$$

But f(x), being continuous, equals N for some $x = \bar{\xi}$ in (a,b). C_1 is obtained by replacing N in the above equation by $f(\bar{\xi})$.

The Theorem of Mean Value may be extended to double sums. Its use will be illustrated in the following important example. Suppose we have the double sum

$$\sum_{s=0}^{h-1} \sum_{t=0}^{k-1} \frac{1}{p!\,q!} \left(\frac{s}{h}\right)^{(p)} \left(\frac{t}{k}\right)^{(q)} f_{p,q}(\xi_s ; \eta_t) .$$

The factorials, $\left(\frac{s}{h}\right)^{(p)}$, $\left(\frac{t}{k}\right)^{(q)}$, and hence their product, do not change sign within the limits of the summation for s and t. It is assumed throughout this paper that the function under consideration has continuous partial derivatives of sufficiently high order in x and y. Then the Theorem of Mean Value applies, and we have

$$\sum_{s=0}^{h-1} \sum_{t=0}^{k-1} \frac{1}{p!\,q!} \left(\frac{s}{h}\right)^{(p)} \left(\frac{t}{k}\right)^{(q)} f_{p,q}(\xi_s ; \eta_t) = \alpha_{pq} \, f_{p,q}(\xi ; \eta) ,$$

where

$$\alpha_{pq} = \sum_{s=0}^{h-1} \sum_{t=0}^{k-1} \frac{1}{p!\,q!} \left(\frac{s}{h}\right)^{(p)} \left(\frac{t}{k}\right)^{(q)} ,$$

and where min.$\{\xi_s\} \le \xi \le$ max.$\{\xi_s\}$ and min.$\{\eta_t\} \le \eta \le$ max.$\{\eta_t\}$. These minimum and maximum limits for ξ and η may be used because the end points of the closed intervals in H_1 of the Theorem may be chosen as the minimum and maximum values of ξ_s (and η_t).

two variables

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2. EXTENSION OF LUBBOCK'S FORMULA OF THE FIRST TYPE

The interpolation formula in descending differences for two variables may be written

$$(13)\qquad f\left(x+\tfrac{s}{h};\, y+\tfrac{t}{k}\right) = \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} \frac{1}{i!\,j!}\left(\tfrac{s}{h}\right)^{(i)}\left(\tfrac{t}{k}\right)^{(j)} \Delta_x^{i}\,\Delta_y^{j} f(x;y) + R,$$

where the remainder term may be expressed in terms of descending factorials and divided differences as follows:

$$(14)\qquad R = \left(\tfrac{s}{h}\right)^{(p)} f\left(x+\tfrac{s}{h}, x, x+1, \dots x+p-1;\, y+\tfrac{t}{k}\right)$$
$$+ \left(\tfrac{t}{k}\right)^{(q)} f\left(x+\tfrac{s}{h};\, y, y+1, \dots\dots y+q-1\right)$$
$$- \left(\tfrac{s}{h}\right)^{(p)}\left(\tfrac{t}{k}\right)^{(q)} f\left(x+\tfrac{s}{h}, x, x+1, \dots x+p-1;\, y+\tfrac{t}{k}, y, y+1, \dots y+q-1\right).$$

The remainder term may also be written [o]

$$(15)\qquad R = \frac{1}{p!}\left(\tfrac{s}{h}\right)^{(p)} f_{p,0}\left(\xi_s;\, y+\tfrac{t}{k}\right) + \frac{1}{q!}\left(\tfrac{t}{k}\right)^{(q)} f_{0,q}\left(x+\tfrac{s}{h};\, \eta_t\right)$$
$$- \frac{1}{p!\,q!}\left(\tfrac{s}{h}\right)^{(p)}\left(\tfrac{t}{k}\right)^{(q)} f_{p,q}\left(\xi_s';\, \eta_t'\right),$$

where $x \le \xi_s, \xi_s' \le x+p-1$, $y \le \eta_t, \eta_t' \le y+q-1$. $\xi_s, \xi_s', \eta_t, \eta_t'$ each depend on $x+s/h$ and $y+t/k$.

For $p = q = 3$, the interpolation formula (13) becomes

$$f\left(x+\tfrac{s}{h};\, y+\tfrac{t}{k}\right) = f(x;y) + \left(\tfrac{s}{h}\right)\Delta_x f(x;y) + \left(\tfrac{s}{h}\right)\left(\tfrac{s}{h}-1\right)\Delta_x^2 f(x;y)$$
$$+ \left(\tfrac{t}{k}\right)\Delta_y f(x;y) + \left(\tfrac{s}{h}\right)\left(\tfrac{t}{k}\right)\Delta_x\Delta_y f(x;y) + \left(\tfrac{s}{h}\right)\left(\tfrac{s}{h}-1\right)\left(\tfrac{t}{k}\right)\Delta_x^2\Delta_y f(x;y)$$
$$+ \left(\tfrac{t}{k}\right)\left(\tfrac{t}{k}-1\right)\Delta_y^2 f(x;y) + \left(\tfrac{s}{h}\right)\left(\tfrac{t}{k}\right)\left(\tfrac{t}{k}-1\right)\Delta_x\Delta_y^2 f(x;y)$$

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[o] Steffensen, p. 205.

$$+\left(\frac{s}{h}\right)\left(\frac{s}{h}-1\right)\left(\frac{t}{k}\right)\left(\frac{t}{k}-1\right)\Delta_x^2\,\Delta_y^2\,f(x;y)$$

$$+\frac{1}{3!}\left(\frac{s}{h}\right)\left(\frac{s}{h}-1\right)\left(\frac{s}{h}-2\right)f_{3,0}\left(\xi_s;\,y+\frac{t}{k}\right)$$

$$+\frac{1}{3!}\left(\frac{t}{k}\right)\left(\frac{t}{k}-1\right)\left(\frac{t}{k}-2\right)f_{0,3}\left(x+\frac{s}{h};\,\eta_t\right)$$

$$-\frac{1}{3!\,3!}\left(\frac{s}{h}\right)\left(\frac{s}{h}-1\right)\left(\frac{s}{h}-2\right)\left(\frac{t}{k}\right)\left(\frac{t}{k}-1\right)\left(\frac{t}{k}-2\right)f_{3,3}\left(\xi_s';\,\eta_t'\right).$$

We first sum (13) from $s = 0$ to $s = h - 1$ and from $t = 0$ to $t = k - 1$.

(16)
$$\sum_{s=0}^{h-1}\sum_{t=0}^{k-1} f\left(x+\frac{s}{h};\,y+\frac{t}{k}\right) = \sum_{s=0}^{h-1}\sum_{t=0}^{k-1}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\frac{1}{i!\,j!}\left(\frac{s}{h}\right)^{(i)}\left(\frac{t}{k}\right)^{(j)}\Delta_x^i\,\Delta_y^j\,f(x;y)$$
$$+\sum_{s=0}^{h-1}\sum_{t=0}^{k-1} R\,.$$

When s or t equals zero, the last term of (16) contains divided differences with repeated arguments which may be shown to be finite in value. Since the factorial coefficients vanish for s or t equal to zero, the last term of (16) is not affected by the equality of two of the arguments.

In the first right hand member of (16) it is permissible to interchange the first two summations with the last two since all the summations are finite. Then let

(17)
$$\alpha_{ij} = \sum_{s=0}^{h-1}\sum_{t=0}^{k-1}\frac{1}{i!\,j!}\left(\frac{s}{h}\right)^{(i)}\left(\frac{t}{k}\right)^{(j)}.$$

From (17), $\alpha_{00} = h\,k$. In (16) write the term containing α_{00} apart, obtaining

$$(18) \quad \sum_{s=0}^{h-1} \sum_{t=0}^{k-1} f\left(x+\frac{s}{h}; y+\frac{t}{k}\right) = hk\, f(x;y)$$

$$+ \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \alpha_{ij} \Delta_x^i \Delta_y^j f(x;y) + \sum_{s=0}^{h-1} \sum_{t=0}^{k-1} R,$$

where both i and j may not be zero simultaneously $(i = j \neq 0)$.

Now sum (18) from $x = 0$ to $x = m - 1$ and from $y = 0$ to $y = n - 1$. Since

$$\sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{s=0}^{h-1} \sum_{t=0}^{k-1} f\left(x+\frac{s}{h}; y+\frac{t}{k}\right) = \sum \sum f\left(\frac{\mu}{h}; \frac{\nu}{k}\right),$$

we have

$$(19) \quad \sum_{\mu=0}^{hm-1} \sum_{\nu=0}^{kn-1} f\left(\frac{\mu}{h}; \frac{\nu}{k}\right) = hk \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} f(\mu;\nu)$$

$$+ \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} \sum_{\substack{i=0 \\ (i=j\neq 0)}}^{p-1} \sum_{j=0}^{q-1} \alpha_{ij} \Delta_\mu^i \Delta_\nu^j f(\mu;\nu)$$

$$+ \sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{s=0}^{h-1} \sum_{t=0}^{k-1} R .$$

In the second term of the right hand member of (19) we interchange the first two summations with the last two and make use of the fact that

$$(20) \quad \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} \Delta_\mu^i \Delta_\nu^j f(\mu;\nu) = \Delta_\mu^{i-1} \Delta_\nu^{j-1} f(\mu;\nu) \Big|_0^m \Big|_0^n .$$

Then (19) becomes

$$(21)\qquad \sum_{\mu=0}^{hm-1}\sum_{\nu=0}^{kn-1} f\left(\tfrac{\mu}{h};\tfrac{\nu}{k}\right) = hk\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} f(\mu;\nu)$$

$$+\sum_{\substack{i=0\\(i=j\neq 0)}}^{p-1}\sum_{j=0}^{q-1} \alpha_{ij}\,\Delta_{\mu}^{i-1}\,\Delta_{\nu}^{j-1} f(\mu;\nu)\Big|_0^m\Big|_0^n$$

$$+\sum_{x=0}^{m-1}\sum_{y=0}^{n-1}\sum_{s=0}^{p-1}\sum_{t=0}^{q-1} R\,.$$

The remainder term in (21) needs special consideration. In (16) we find that the partial derivatives involved in the three terms of R are each assumed to be continuous functions. Furthermore the quantities $\left(\frac{s}{h}\right)^{(p)}$, $\left(\frac{t}{k}\right)^{(q)}$, and their product do not change sign for the values of s and t employed in the summations. It follows that the Theorem of Mean Value may be applied to each part of the remainder term. Using the abbreviated notation of (17) we have

$$(22)\qquad \sum_{s=0}^{h-1}\sum_{t=0}^{k-1} R = \alpha_{p0}\, f_{p,0}(\bar{\xi};y') + \alpha_{0q}\, f_{0,q}(x';\bar{\eta})$$

$$-\,\alpha_{pq}\, f_{p,q}(\bar{\xi}';\bar{\eta}'),$$

where $\min.\{\xi_s\}\le\bar{\xi}\le\max.\{\xi_s\}$, $\min.\{\eta_t\}\le\bar{\eta}\le\max.\{\eta_t\}$ and similarly for $\bar{\xi}'$ and $\bar{\eta}'$. Also $x\le x'\le x+(h-1)/h$, $y\le y'\le y+(k-1)/k$.

When the summations used in (19) are applied to (22) it is evident that there is a different $\bar{\xi}$, $\bar{\xi}'$, and x' for each x and a different $\bar{\eta}$, $\bar{\eta}'$, and y' for each y. The Theorem of Mean

Value may be applied once more since α_{p0} , α_{0q} , and α_{pq} are constant during summation over x and y. Thus

$$(23) \quad \sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{s=0}^{p-1} \sum_{t=0}^{q-1} R = mn\,\alpha_{p0}\, f_{p,0}(\xi_1;\eta_1)$$

$$+ mn\,\alpha_{0q}\, f_{0,q}(\xi_2;\eta_2) - mn\,\alpha_{pq}\, f_{p,q}(\xi_3;\eta_3) ,$$

where $0 \leq \xi_1, \xi_3 \leq m + p - 2$, $0 \leq \eta_2, \eta_3 \leq n + q - 2$, $0 \leq \xi_2 \leq m - 1/h$, $0 \leq \eta_1 \leq n - 1/k$.

In formulas (21) and (23) set $f(x;y) = F(hx;ky)$. Then

$$(24) \quad \sum_{u=0}^{hm-1} \sum_{v=0}^{kn-1} F(u;v) = hk \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} F(hu;kv)$$

$$+ \sum_{\substack{i=0 \\ (i=j\neq 0)}}^{p-1} \sum_{j=0}^{q-1} \alpha_{ij}\, \Delta_u^{i-1} \Delta_v^{j-1} F(hu;kv) \Big|_0^m \Big|_0^n$$

$$+ h^p mn\,\alpha_{p0}\, F_{p,0}(\xi_1;\eta_1) + k^q mn\,\alpha_{0q}\, F_{0,q}(\xi_2;\eta_2)$$

$$- h^p k^q mn\,\alpha_{pq}\, F_{p,q}(\xi_3;\eta_3) ,$$

where $0 \leq \xi_1, \xi_3 \leq h(m + p - 2)$, $0 \leq \eta_2, \eta_3 \leq k(n + q - 2)$, $0 \leq \xi_2 \leq hm - 1$, $0 \leq \eta_1 \leq kn - 1$.

Formula (24) is the first of the summation formulas that we have been seeking. It may be regarded as an extension into two variables of Lubbock's formula of the first type with a remainder term.[o]

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o Steffensen, p. 139.

By means of (24) a double sum may be approximated by taking hk times the sum of every h^{th} term of one independent variable corresponding to every k^{th} array of the other independent variable. This amounts to using the corner points only of h by k rectangles in the xy-plane. Corrections to this sum are determined by other quantities which involve finite descending differences and a remainder term which depends on partial derivatives of order p and q in x and y respectively.

In practical work the size of the remainder term should be estimated first for different values of p and q, m and n, and h and k in order that one may be aware from the start of the error in estimation. If the remainder term indicates an error in the sixth decimal place, it would be wise to carry all calculations to at least eight decimal places.

But first the coefficients, α_{ij}, in (24) must be determined. From their definition in (17)

$$\alpha_{ij} = \frac{1}{i!}\sum_{s=0}^{h-1}\left(\frac{s}{h}\right)^{(i)} \cdot \frac{1}{j!}\sum_{t=0}^{k-1}\left(\frac{t}{k}\right)^{(j)} = \Lambda_i \cdot \Lambda_j .$$

Thus α_{ij} may be readily determined from tables for Λ_i.°

For $h = k = 5$ we find

$$\alpha_{10} = \alpha_{01} = 10 \qquad \alpha_{12} = \alpha_{21} = -.8$$

$$\alpha_{20} = \alpha_{02} = -2 \qquad \alpha_{11} = 4$$

$$\alpha_{30} = \alpha_{03} = 1 \qquad \alpha_{22} = .16$$

$$\alpha_{33} = .04 .$$

Suppose we wish to calculate $\sum_{x=100}^{114}\sum_{y=100}^{114}\frac{1}{xy}$. Then

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° Steffensen, p. 140.

let $F(\mu;\nu) = \dfrac{1}{(\mu+100)(\nu+100)}$ and sum $\displaystyle\sum_{\mu=0}^{14}\sum_{\nu=0}^{14} F(\mu;\nu).$

Let h = k = 5, m = n = 3, p = q = 3. The remainder term is given by

$$\frac{5^3\cdot 3^2\,\alpha_{30}\,(-6)}{(\xi_1+100)^4(\eta_1+100)} + \frac{5^3\cdot 3^2\,\alpha_{03}\,(-6)}{(\xi_2+100)(\eta_2+100)^4} - \frac{5^6\cdot 3^2\,\alpha_{33}\,(36)}{(\xi_3+100)^4(\eta_3+100)^4}\,.$$

To obtain the upper limit for the error we choose $\xi_1 = \xi_2 = \xi_3 = \eta_1 = \eta_2 = \eta_3 = 0$ The negative sign of the error indicates that the calculated sum will overestimate the real sum by an amount which may be calculated to be not more than .00000135⁺.

The values of F(hμ ;kν) $10^9 = U_{\mu,\nu}$ are given in Table I. Values marked in red are used directly in the calculations; the others are needed to calculate some of the descending differences.

μ		$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
	0	100000 ×	95238 ×	90909 ×	86957 ×	83333
	1	95238 ×	90703 ×	86580 ×	82816 ×	79365
μ	2	90909 ×	86580 ×	82645 ×	79051 ×	75758
	3	86957 ×	82816 ×	79051 ×	75614 ×	72464
	4	83333	79365	75758	72464	69444

Table I

The first term in the right member of (24) becomes

(25) $$25\sum_{\mu=0}^{2}\sum_{\nu=0}^{2} F(5\mu;5\nu) = 25(.000818802) \quad .02047005.$$

Only the nine values in the upper left hand corner of Table I enter into this sum.

For i = 0 we have

$$(26)\qquad \sum_{j=1}^{2} \alpha_{0j}\, \Delta_\mu^{-1} \Delta_\nu^{j-1} F(h\mu;k\nu)\Big|_0^3\Big|_0^3$$

$$= 10 \quad \left(U_{0,3} \quad U_{1,3} \quad U_{2,3} - U_{0,0} - U_{1,0} - U_{2,0} \right)$$

$$-2\,\Delta_\nu \left(U_{0,3} \quad U_{1,3} \quad U_{2,3} - U_{0,0} - U_{1,0} - U_{2,0} \right).$$

The first term may be obtained from Table I while the second term may be obtained from Table II which has been constructed from Table I.

	μ	$\Delta_\nu U_{\mu,0}$	$\Delta_\nu U_{\mu,3}$
	0	- 4762	- 3624
	1	- 4535	- 3451
μ	2	- 4329	- 3293
	3	- 4141	- 3150
	4	- 3968	- 3020

Table II

Then (26) becomes

$$10(-.000037323) - 2(.000003258) \quad = \quad -.000379746.$$

For $i = 1$ we have

$$(27)\qquad \sum_{j=0}^{2} \alpha_{1j}\, \Delta_\nu^{j-1} F(h\mu;k\nu)\Big|_0^3\Big|_0^3 = 10\, \Delta_\nu^{-1} F(h\mu;k\nu)\Big|_0^3\Big|_0^3$$

$$+ 4\, F(h\mu;k\nu)\Big|_0^3\Big|_0^3 \quad - .8\, \Delta_\nu F(h\mu;k\nu)\Big|_0^3\Big|_0^3 .$$

The first term in (27) has the same value as the first term in (26) since $1 / xy$ is symmetric in x and y together. Relation (27) becomes

$$-.00037323 + 4(U_{3,3} - U_{0,3} - U_{3,0} \quad U_{0,0})$$

$$- .8\, \Delta_\nu (U_{3,3} - U_{0,3} - U_{3,0} \quad U_{0,0})$$

$$= -.00037323 + .0000068 + .0000001176 = -.0003663124 .$$

For $i = 2$, (24) gives

$$(28) \quad \sum_{j=0}^{2} \alpha_{2j} \Delta_\mu \Delta_\nu^{j-1} F(h\mu; k\nu)\Big|_0^3\Big|_0^3 = -2\, \Delta_\mu \Delta_\nu^{-1} F(h\mu; k\nu)\Big|_0^3\Big|_0^3$$

$$-.8\, \Delta_\mu F(h\mu; k\nu)\Big|_0^3\Big|_0^3 + .16\, \Delta_\mu \Delta_\nu F(h\mu; k\nu)\Big|_0^3\Big|_0^3 .$$

The first and second terms of (28) have the same values as the second term of (26) and the third term of (27) respectively. To evaluate the third term of (28) we form Table III from Table II.

		$\Delta_\mu \Delta_\nu U_{\mu,0}$	$\Delta_\mu \Delta_\nu U_{\mu,3}$
μ	0	$227^\times$	$173^\times$
	3	$173^\times$	$130^\times$

Table III

Then (28) becomes

$$- .000006516 + .0000001176 + .16(.000000011) = - .00000639664 .$$

Adding the results from (25), (26), (27), and (28) gives

$$\sum_{x=100}^{114} \sum_{y=100}^{114} \frac{1}{xy} = .019717592 .$$

In this particular problem the answer might have been obtained by considering

$$\sum_{x=100}^{114} \sum_{y=100}^{114} \frac{1}{xy} = \left(\sum_{x=100}^{114} \frac{1}{x} \right)^2 = (.140416149)^2$$

$$= .019716695^-,$$

which indicates that the actual error is almost 9 in the 7^{th} decimal place while the remainder term gave an upper bound of slightly more than 1 in the 6^{th} decimal place. The negative sign of the error as calculated from the remainder term foretold that the calculated sum overestimated the actual double sum.

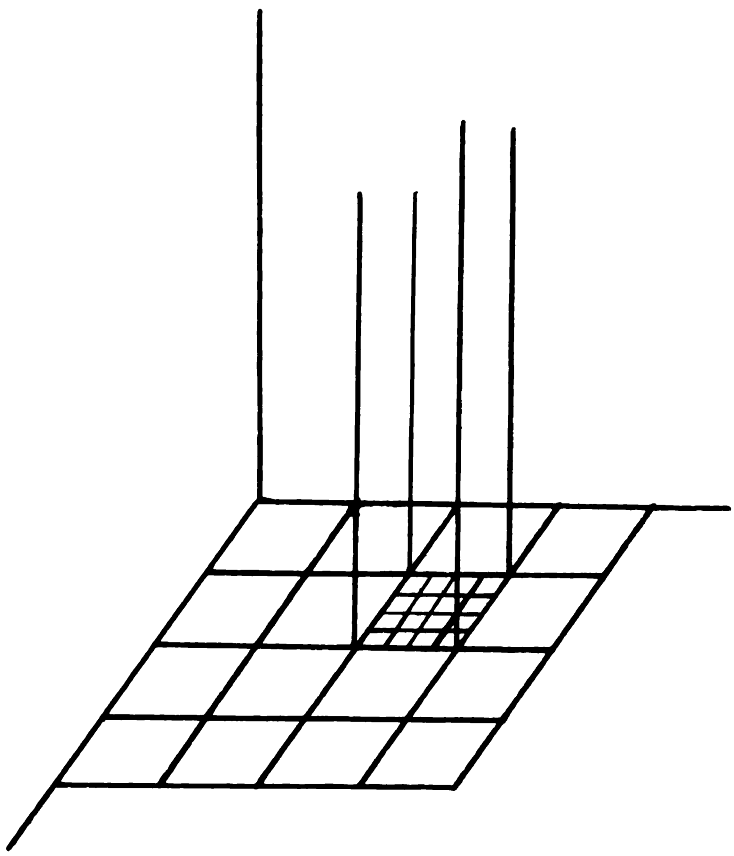

Figure 1

Figure 1 illustrates the technique of summing by formula (24). The functional values in the left hand member of (24) would be represented by ordinates (not drawn in the figure) erected at the points of intersection of the small network of rectangles. (In the figure, $h = k = 4$). The first term in the right member of (24) is calculated from the values of the ordinates erected at the corner points of the larger network of rectangles. The corrective terms are then based on descending differences which are determined directly from the values of these latter ordinates.

3. EXTENSION OF LUBBOCK'S FORMULA OF THE SECOND TYPE

An interpolation formula [o] in two variables involving the mean and central differences may be written

$$(29)\quad f\left(x+\tfrac{s}{h};\, y+\tfrac{t}{k}\right) = \sum_{i=0}^{p-1}\left[\frac{1}{(2i-1)!}\left(\frac{s}{h}\right)^{[2i]-1} \Box\, \delta_x^{2i-1} + \frac{1}{(2i)!}\left(\frac{s}{h}\right)^{[2i]} \delta_x^{2i}\right]$$

$$\cdot \sum_{j=0}^{q-1}\left[\frac{1}{(2j-1)!}\left(\frac{t}{k}\right)^{[2j]-1} \Box\, \delta_y^{2j-1} + \frac{1}{(2j)!}\left(\frac{t}{k}\right)^{[2j]} \delta_y^{2j}\right]$$

$$+ R,$$

where the terms in the symbolic product operate on $f(x;y)$. The meaning of formula (29) for $p = q = 2$ has been indicated in the above reference. For later convenience, however, we choose to write the remainder term as

$$(30)\quad R = \left(\frac{s}{h}\right)^{[2p]-1} f\left(x+\tfrac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\, y+\tfrac{t}{k}\right)$$

$$+ \left(\frac{t}{k}\right)^{[2q]-1} f\left(x+\tfrac{s}{h};\, y+\tfrac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)$$

$$- \left(\frac{s}{h}\right)^{[2p]-1}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x+\tfrac{s}{h}, x, \cdots x\pm(p-1);\, y+\tfrac{t}{k}, y, \cdots y\pm(q-1)\right),$$

where the first term contains a divided difference of $2p$ arguments in x and one in y; the second, one in x and $2q$ in y; and the third, $2p$ in x and $2q$ in y.

We define

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o Steffensen, pp. 145, 209.

$$(31) \qquad \chi_{2i,2j} = \frac{1}{(2i)!\,(2j)!} \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \; \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} \left(\frac{s}{h}\right)^{[2i]} \left(\frac{t}{k}\right)^{[2j]},$$

where the summations refer to s and t which take on all values at unit intervals from $-(h - 1)/2$ to $(h - 1)/2$ and from $-(k - 1)/2$ to $(k - 1)/2$ respectively.

Now sum (29) over these values of s and t.

$$(32) \qquad \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \; \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} f\left(x+\frac{s}{h};\, y+\frac{t}{k}\right) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \chi_{2i,2j}\, \delta_x^{2i}\, \delta_y^{2j} f(x;y) + \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \; \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} R,$$

where the other terms in the product have vanished since $\left(\frac{s}{h}\right)^{[2i]-1}$ and $\left(\frac{t}{k}\right)^{[2j]-1}$ are odd functions.

Next sum (32) from $x = 0$ to $x = m - 1$ and from $y = 0$ to $y = n - 1$, and write the term containing $\chi_{0,0}$ ($= hk$) apart. But since

$$\sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \; \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} f\left(x+\frac{s}{h};\, y+\frac{t}{k}\right) = \sum_{\mu=-\frac{(h-1)}{2}}^{hm-\frac{(h+1)}{2}} \; \sum_{\nu=-\frac{(k-1)}{2}}^{kn-\frac{(k+1)}{2}} f\left(\frac{\mu}{h};\, \frac{\nu}{k}\right)$$

we may write

$$(33) \qquad \sum_{\mu=-\frac{(h-1)}{2}}^{hm-\frac{(h+1)}{2}} \; \sum_{\nu=-\frac{(k-1)}{2}}^{kn-\frac{(k+1)}{2}} f\left(\frac{\mu}{h};\, \frac{\nu}{k}\right) = hk \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} f(\mu;\nu)$$

$$+ \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} \underset{(i=j\neq 0)}{\sum_{i=0}^{p-1} \sum_{j=0}^{q-1}} \chi_{2i,2j}\, \delta_\mu^{2i}\, \delta_\nu^{2j} f(\mu;\nu)$$

$$+\sum_{x=0}^{m-1}\sum_{y=0}^{n-1}\sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}}\sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} R\,.$$

In the second right hand member of (33) interchange the first two summations with the last two and make use of the fact

$$\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\delta_\mu^{2i}\,\delta_\nu^{2j} f(\mu;\nu) = \delta_\mu^{2i-1}\,\delta_\nu^{2j-1} f(\mu;\nu)\Big|_{-\frac{1}{2}}^{m-\frac{1}{2}}\Big|_{-\frac{1}{2}}^{n-\frac{1}{2}}\,.$$

Then (33) becomes

$$(34)\qquad \sum_{\mu=-\frac{(h-1)}{2}}^{hm-\frac{(h+1)}{2}}\ \sum_{\nu=-\frac{(k-1)}{2}}^{kn-\frac{(k+1)}{2}} f\left(\frac{\mu}{h};\frac{\nu}{k}\right) = hk\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} f(\mu;\nu)$$

$$+\sum_{\substack{i=0\\(i=j\neq 0)}}^{p-1}\sum_{j=0}^{q-1} K_{2i,2j}\,\delta_\mu^{2i-1}\,\delta_\nu^{2j-1} f(\mu;\nu)\Big|_{-\frac{1}{2}}^{m-\frac{1}{2}}\Big|_{-\frac{1}{2}}^{n-\frac{1}{2}}$$

$$+\sum_{x=0}^{m-1}\sum_{y=0}^{n-1}\sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}}\sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} R\,,$$

where the remainder term needs special consideration.

In (32) the remainder term appears as a double sum of three separate terms and remains unchanged when s is replaced by $-s$, or t is replaced by $-t$, or both. For the first term replace s by $-s$ and average it with the original. For the second term replace t by $-t$ and average it with the original. For the third term three other equivalent forms may be obtained by replacing s by $-s$, t by $-t$, and both s by $-s$ and t by $-t$; the arithmetic mean of these four equivalent forms, if we omit writing the double summation sign of (32), is

$$\frac{1}{4}\Big[\left(\frac{s}{h}\right)^{[2p]-1}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x+\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y+\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)$$

$$-\left(\frac{s}{h}\right)^{[2p]-1}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x-\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y+\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)$$

$$-\left(\frac{s}{h}\right)^{[2p]-1}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x+\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y-\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)$$

$$+\left(\frac{s}{h}\right)^{[2p]-1}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x-\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y-\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)\Big],$$

which becomes

$$\frac{1}{2}\left(\frac{s}{h}\right)^{[2p]}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x\pm\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y+\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right)$$

$$-\frac{1}{2}\left(\frac{s}{h}\right)^{[2p]}\left(\frac{t}{k}\right)^{[2q]-1} f\left(x\pm\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y-\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right),$$

or

$$\left(\frac{s}{h}\right)^{[2p]}\left(\frac{t}{k}\right)^{[2q]} f\left(x\pm\frac{s}{h}, x, x\pm 1, \cdots x\pm(p-1);\ y\pm\frac{t}{k}, y, y\pm 1, \cdots y\pm(q-1)\right).$$

Since a divided difference of $(2p + 1)$ arguments in x and $(2q + 1)$ arguments in y may be expressed in terms of partial derivatives of the $(2p)^{th}$ order in x and $(2q)^{th}$ order in y, this part of the remainder term from (32) may be written

$$\sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}}\ \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} \frac{1}{(2p)!\,(2q)!}\left(\frac{s}{h}\right)^{[2p]}\left(\frac{t}{k}\right)^{[2q]} f_{2p,2q}(\xi_s';\ \eta_t'),$$

where $x - p + 1 \le \xi_s' \le x + p - 1$ and $y - q + 1 \le \eta_t' \le y + q - 1$.

Using similar properties of divided differences on the averages for the first two terms of the remainder term in (32) we may write

$$(35)\quad \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} \left[\frac{1}{(2p)!}\left(\frac{s}{h}\right)^{[2p]} f_{2p,0}\left(\xi_s; y+\frac{t}{k}\right) + \frac{1}{(2q)!}\left(\frac{t}{k}\right)^{[2q]} f_{0,2q}\left(x+\frac{s}{h}; \eta_t\right) - \frac{1}{(2p)!\,(2q)!}\left(\frac{s}{h}\right)^{[2p]}\left(\frac{t}{k}\right)^{[2q]} f_{2p,2q}\left(\xi'_s; \eta'_t\right)\right],$$

where $x - p + 1 \le \xi_s, \xi'_s \le x + p - 1$, $y - q + 1 \le \eta_t, \eta'_t \le y + q - 1$.

The central factorials in (35) do not change sign in the intervals of summation for s and t; and since the partial derivatives are assumed continuous, the Theorem of Mean Value may be applied to (35), giving

$$(36)\quad mn\, X_{2p,0}\, f_{2p,0}(\bar{\xi}; y') + mn\, X_{0,2q}\, f_{0,2q}(x'; \bar{\eta}) - mn\, X_{2p,2q}\, f_{2p,2q}(\bar{\xi}'; \bar{\eta}'),$$

where $x - p + 1 \le \bar{\xi}, \bar{\xi}' \le x + p - 1$, $y - q + 1 \le \bar{\eta}, \bar{\eta}' \le y + q - 1$,

and $x - \frac{1}{2} + 1/h \le x' \le x + \frac{1}{2} - 1/h$, $y - \frac{1}{2} + 1/k \le y' \le y + \frac{1}{2} - 1/k$.

When the summations introduced in (33) are applied to (36), the Theorem of Mean Value may be used again, giving

$$(37)\quad mn\, X_{2p,0}\, f_{2p,0}(\xi_1; \eta_1) + mn\, X_{0,2q}\, f_{0,2q}(\xi_2; \eta_2) - mn\, X_{2p,2q}\, f_{2p,2q}(\xi_3; \eta_3),$$

where $1 - p \le \xi_1, \xi_3 \le m + p - 2$, $1 - q \le \eta_2, \eta_3 \le n + q - 2$,

and $1/2h - \frac{1}{2} \le \xi_2 \le m - \frac{1}{2} - 1/2h$, $1/2k - \frac{1}{2} \le \eta_1 \le n - \frac{1}{2} - 1/2k$.

Finally, in (34) and (37) replace $f(u;v)$ by $F\left(hu + \frac{h-1}{2}; kv + \frac{k-1}{2}\right)$. Then

$$(38)\quad \sum_{u=0}^{hm-1} \sum_{v=0}^{kn-1} F(u;v) = hk \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} F\left(hu+\frac{h-1}{2}; kv+\frac{k-1}{2}\right)$$

$$+ \sum_{\substack{i=0 \\ (i=j\neq 0)}}^{p-1} \sum_{j=0}^{q-1} X_{2i,2j}\, \delta_u^{2i-1} \delta_v^{2j-1} F\left(hu+\frac{h-1}{2}; kv+\frac{k-1}{2}\right)\Bigg|_{-\frac{1}{2}}^{m-\frac{1}{2}}\Bigg|_{-\frac{1}{2}}^{n-\frac{1}{2}}$$

$$+ h^{2p} mn \chi_{2p,0} F_{2p,0}(\xi_1; u_1) + k^{2q} mn \chi_{0,2q} F_{0,2q}(\xi_2; u_2)$$

$$- h^{2p} k^{2q} mn \chi_{2p,2q} F_{2p,2q}(\xi_3; u_3) ,$$

where $h(1 - p) + (h - 1)/2 \leq \xi_1, \xi_3 \leq h(m + p - 2) + (h - 1)/2,$

(39) $k(1 - q) + (k - 1)/2 \leq u_2, u_3 \leq k(n + q - 2) + (k - 1)/2,$

$0 \leq \xi_2 \leq hm - 1, \quad 0 \leq u_1 \leq kn - 1.$

Formula (38) is the second of the summation formulas that we have been seeking. It may be regarded as an extension of Lubbock's formula of the second type with a remainder term [o]. Formula (38) enables one to estimate a double sum by taking hk times the sum of a series of terms which would be situated near the midpoints of the sides of h by k rectangles. This estimation is to be corrected by terms involving central differences and by a remainder term involving partial derivatives. In comparison with (24), formula (38) has the advantage that only central differences of odd order are employed. In practice it is easier to apply if h and k are odd integers for then the value of F need be obtained at integral points only. For $h = 2c$, $k = 2d$, the values of $F(2c\mu + c - \frac{1}{2};\ 2d\nu + d - \frac{1}{2})$ are not usually found in the given table.

Before formula (38) may be used, the coefficients, $\chi_{2i,2j}$, must be determined. From their definition in (31), we find

$$\chi_{2i,2j} = \frac{1}{(2i)!} \sum_{-\frac{(h-1)}{2}}^{\frac{h-1}{2}} \left(\frac{s}{h}\right)^{[2i]} \cdot \frac{1}{(2i)!} \sum_{-\frac{(k-1)}{2}}^{\frac{k-1}{2}} \left(\frac{t}{k}\right)^{[2j]}$$

$$= P_{2i} \cdot P_{2j} ,$$

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[o]Steffensen, p. 143.

where tables [o] for P_{2i} have already been calculated. For $h = k = 5$, we find

$$\chi_{2,0} = \chi_{0,2} = 1. \qquad \chi_{2,2} = .04$$
$$\chi_{4,0} = \chi_{0,4} = -.072 \qquad \chi_{2,4} = \chi_{4,2} = -.00288$$
$$\chi_{6,0} = \chi_{0,6} = .00928 \qquad \chi_{4,4} = .00020736$$
$$\chi_{6,6} = .000003444736 .$$

In order to calculate $\sum_{x=100}^{114}\sum_{y=100}^{114}\frac{1}{xy}$, let

$$F(\mu;\nu) = \frac{1}{(\mu+100)(\nu+100)} \qquad \text{and sum} \quad \sum_{\mu=0}^{14}\sum_{\nu=0}^{14} F(\mu;\nu).$$

Let $h = k = 5$, $m = n = 3$, $p = q = 3$. The remainder term becomes

$$\frac{5^6\cdot 3^2\chi_{6,0}\,6!}{(\xi_1+100)^7(\eta_1+100)} + \frac{5^6\, 3^2\chi_{0,6}\,6!}{(\xi_2+100)(\eta_2+100)^7} - \frac{5^{12}\, 3^2\chi_{6,6}\,(6!)^2}{(\xi_3+100)^7\,(\eta_3+100)^7} .$$

To obtain the upper limit for the error, we choose

$$\xi_1 = \xi_3 = \eta_2 = \eta_3 = -8\,, \quad \xi_2 = \eta_1 = 0 .$$

Hence the error in calculation is positive in sign and is slightly less than 4 in the 10^{th} decimal place. That is, the sum calculated by formula (38) should underestimate the actual total slightly.

Table IV contains the values of $F(5\mu+2;5\nu+2)\times 10^9 = U_{\mu,\nu}$ The nine central values determine the first term in (38) which is

(40) $\qquad 25(.000788390) = .01970975$.

Since the original entries are accurate to nine places only, (40) may be in error in the 8^{th} decimal place, a result which is larger than the error indicated by the remainder term. This can only

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[o] Steffensen, p. 145.

be remedied by increasing the accuracy of Table IV. We shall, however, carry the present problem through in order to illustrate the technique of calculating the central differences.

μ	$\nu = -2$	$\nu = -1$	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
-2	118147	112057	106564	101585	97050	92902	89095
-1	112057	106281	101071	96348	92047	88113	84502
0	106564	101071	96117$^{\times}$	91625$^{\times}$	87535$^{\times}$	83794	80360
1	101585	96348	91625$^{\times}$	87344$^{\times}$	83445$^{\times}$	79879	76605
2	97050	92047	87535$^{\times}$	83445$^{\times}$	79719$^{\times}$	76313	73185
3	92902	88113	83794	79879	76313	73051	70057
4	89095	84502	80360	76605	73185	70057	67186

Table IV

In Table V the following pattern has been used for $\mu = -2$ to $\mu = 4$, except for the first column for which the values may be found in each row of Table IV.

$$\begin{array}{llll}
U_{\mu,-2} & & & \\
 & \delta U_{\mu,-1\frac{1}{2}} & & \\
U_{\mu,-1} & & \delta^2 U_{\mu,-1} & \\
 & \delta U^{\times}_{\mu,-\frac{1}{2}} & & \delta^3 U^{\times}_{\mu,-\frac{1}{2}} \\
U_{\mu,0} & & \delta^2 U_{\mu,0} & \\
 & \delta U_{\mu,\frac{1}{2}} & & \\
U_{\mu,1} & & & \\
 & \delta U_{\mu,1\frac{1}{2}} & & \\
U_{\mu,2} & & \delta^2 U_{\mu,2} & \\
 & \delta U^{\times}_{\mu,2\frac{1}{2}} & & \delta^3 U^{\times}_{\mu,2\frac{1}{2}} \\
U_{\mu,3} & & \delta^2 U_{\mu,3} & \\
 & \delta U_{\mu,3\frac{1}{2}} & & \\
U_{\mu,4} & & &
\end{array}$$

In Table VI the pattern for $\nu = -\frac{1}{2},\ 2\frac{1}{2}$ is

$$\begin{array}{ll} \delta U_{-1,\nu} & \\ & \delta\delta U^{\times}_{-\frac{1}{2},\nu} \\ \delta U_{0,\nu} & \end{array}$$

$$\begin{array}{ll} \delta U_{2,\nu} & \\ & \delta\delta U^{\times}_{2\frac{1}{2},\nu} \\ \delta U_{3,\nu} & \end{array} .$$

In Table VII the pattern for $\nu = -\frac{1}{2},\ 2\frac{1}{2}$ is

$$\begin{array}{llll} \delta^3 U_{-2,\nu} & & & \\ & \delta\delta^3 U_{-1\frac{1}{2},\nu} & & \\ \delta^3 U_{-1,\nu} & & \delta^2\delta^3 U_{-1,\nu} & \\ & \delta\delta^3 U_{-\frac{1}{2},\nu} & & \delta^3\delta^3 U_{-\frac{1}{2},\nu} \\ \delta^3 U_{0,\nu} & & \delta^2\delta^3 U_{0,\nu} & \\ & \delta\delta^3 U_{\frac{1}{2},\nu} & & \\ \delta^3 U_{1,\nu} & & & \\ & \delta\delta^3 U_{1\frac{1}{2},\nu} & \delta^2\delta^3 U_{2,\nu} & \\ \delta^3 U_{2,\nu} & & & \delta^3\delta^3 U_{2\frac{1}{2},\nu} \\ & \delta\delta^3 U_{2\frac{1}{2},\nu} & \delta^2\delta^3 U_{3,\nu} & \\ \delta^3 U_{3,\nu} & & & \\ & \delta\delta^3 U_{3\frac{1}{2},\nu} & & \\ \delta^3 U_{4,\nu} & & & \end{array} .$$

For $i = 0,\ j = 1$ in formula (38) we have

$$(41) \qquad \chi_{0,2}\, \delta_\mu^{-1}\, \delta_\nu F(5\mu+2;\, 5\nu+2)\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}$$

$$= \delta_\nu \left[U_{0,2\frac{1}{2}} + U_{1,2\frac{1}{2}} + U_{2,2\frac{1}{2}} - U_{0,-\frac{1}{2}} - U_{1,-\frac{1}{2}} - U_{2,-\frac{1}{2}} \right],$$

which, from Table V, is

$- 3741 - 3566 - 3406 -(-4954) -(-4723) - (4512) = .000003476$.

For $i = 1,\ j = 0$ we obtain the same value.

$\mu = -2$			$\mu = -1$			$\mu = 0$			$\mu = 1$		
-6090			-5776			-5493			-5237		
	597			566			539			514	
-5493		-83	-5210		-79	$-4954^{\times}$		$-77^{\times}$	$-4723^{\times}$		$-72^{\times}$
	514			487			462			442	
-4979			-4723			-4492			-4281		
-4535			-4301			-4090			-3899		
	387			367			349			333	
-4148		-46	-3934		-44	$-3741^{\times}$		$-42^{\times}$	$-3566^{\times}$		$-41^{\times}$
	341			323			307			292	
-3807			-3611			-3434			-3274		

$\mu = 2$			$\mu = 3$			$\mu = 4$		
-5003			-4789			-4593		
	491			470			451	
$-4512^{\times}$		$-69^{\times}$	-4319		-66	-4142		-64
	422			404			387	
-4090			-3915			-3755		
-3726			-3566			-3420		
	320			304			292	
$-3406^{\times}$		$-42^{\times}$	-3262		-36	-3128		-35
	278			268			257	
-3128			-2994			-2871		

Table V

$\nu = -\frac{1}{2}$		$\nu = 2\frac{1}{2}$	
-5210		-3934	
	$256^{\times}$		$193^{\times}$
-4954		-3741	
-4512		-3406	
	$193^{\times}$		$144^{\times}$
-4319		-3263	

Table VI

$\nu = -\frac{1}{2}$				$\nu = 2\frac{1}{2}$			
-83				-46			
	4				2		
-79		-2		-44		0	
	$2^{\times}$		$5^{\times}$		$2^{\times}$		$-1^{\times}$
-77		3		-42		-1	
	5				1		
-72				-41			
	3				-1		
-69		0		-42		7	
	$3^{\times}$		$-1^{\times}$		$6^{\times}$		$-12^{\times}$
-66		-1		-36		-5	
	2				1		
-64				-35			

Table VII

For $i = 0$, $j = 2$, we have

(42) $$\chi_{0,4}\ \delta_\mu^{-1}\ \delta_\nu^{3}\ F(5\mu+2;\ 5\nu+2)\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}\ \Big|_{-\frac{1}{2}}^{2\frac{1}{2}}$$

$$= (-.072)\ \delta_\nu^{3}\left[U_{0,2\frac{1}{2}} + U_{1,2\frac{1}{2}} + U_{2,2\frac{1}{2}} - U_{0,-\frac{1}{2}} - U_{1,-\frac{1}{2}} - U_{2,-\frac{1}{2}}\right],$$

which, from Table V, is equal to -.000000006696. Similarly for $i = 2$, $j = 0$.

For $i = j = 1$, formula (38) gives

(43) $$\chi_{2,2}\ \delta_\mu\ \delta_\nu\ F(5\mu+2;\ 5\nu+2)\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}\ \Big|_{-\frac{1}{2}}^{2\frac{1}{2}},$$

which, by use of Table VI, is equal to

(.04)(144 - 193 - 193 + 256) = .00000000056 .

For $i = 1$, $j = 2$ we have

(44) $$\chi_{2,4}\ \delta_\mu\ \delta_\nu^{3}\ F(5\mu+2;\ 5\nu+2)\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}\ \Big|_{-\frac{1}{2}}^{2\frac{1}{2}},$$

which, by use of Table VII, is equal to

(-.00288)(6 - 3 - 2 + 2) = -.00000000000864 .

Similarly for $i = 2$, $j = 1$.

For $i = j = 2$, formula (38) gives

(45) $$\chi_{4,4}\ \delta_\mu^{3}\ \delta_\nu^{3}\ F(5\mu+2;\ 5\nu+2)\Big|_{-\frac{1}{2}}^{2\frac{1}{2}}\ \Big|_{-\frac{1}{2}}^{2\frac{1}{2}}$$

$$= (.00020736)\ \delta_\mu^{3}\ \delta_\nu^{3}\left[U_{2\frac{1}{2},2\frac{1}{2}} - U_{2\frac{1}{2},-\frac{1}{2}} - U_{-\frac{1}{2},2\frac{1}{2}} + U_{-\frac{1}{2},-\frac{1}{2}}\right],$$

which, by use of Table VII, is equal to .0000000000010368 .

The sum to nine decimal places of terms (40) - (45) is .019716689, while the actual sum is .019716695⁻ so that the underesti-

mation is 6 in the 9^{th} decimal place. In view of the accuracy of the original set of values this error may not be considered extreme.

The applications of formulas (24) and (38) to $\sum_{x=100}^{114} \sum_{y=100}^{114} \frac{1}{xy}$ show that the latter gives, in this particular case, the greater accuracy when each is written out to the same number of terms. The coefficients in (38) decrease more rapidly than those of (24), but a comparison of the two remainder terms, for the same p and q, involves consideration of partial derivatives of different order. The upper limits for the two errors will depend on the particular function used and the ranges of summation.

4. EXTENSION OF LUBBOCK'S FORMULA OF THE THIRD TYPE

Another interpolation formula [o] in two variables involving the mean and central differences may be written

(46)
$$f\left(x+\tfrac{1}{2}+\tfrac{s}{h};\, y+\tfrac{1}{2}+\tfrac{t}{k}\right) = \sum_{i=0}^{p-1}\left[\frac{1}{(2i)!}\left(\frac{s}{h}\right)^{[2i+1]-1} \square\, \delta_x^{2i} + \frac{1}{(2i+1)!}\left(\frac{s}{h}\right)^{[2i+1]} \delta_x^{2i+1}\right]$$
$$\cdot \sum_{j=0}^{q-1}\left[\frac{1}{(2j)!}\left(\frac{t}{k}\right)^{[2j+1]-1} \square\, \delta_y^{2j} + \frac{1}{(2j+1)!}\left(\frac{t}{k}\right)^{[2j+1]} \delta_y^{2j+1}\right]$$
$$+ R,$$

where the terms in the symbolic product operate on $f(x+\frac{1}{2};\, y+\frac{1}{2})$. The meaning of this formula for $p = q = 2$ has been indicated in the above reference. The remainder term may be written

(47)
$$R = \frac{1}{(2p)!}\left(\frac{s}{h}\right)^{[2p+1]-1} f_{2p,0}\left(\xi_s;\, y+\tfrac{1}{2}+\tfrac{t}{k}\right) + \frac{1}{(2q)!}\left(\frac{t}{k}\right)^{[2q+1]-1} f_{0,2q}\left(x+\tfrac{1}{2}+\tfrac{s}{h};\, \eta_t\right)$$
$$- \frac{1}{(2p)!\,(2q)!}\left(\frac{s}{h}\right)^{[2p+1]-1}\left(\frac{t}{k}\right)^{[2q+1]-1} f_{2p,2q}\left(\xi_s';\, \eta_t'\right),$$

where $x - p + 1 \leq \xi_s \leq x + p$, $y - q + 1 \leq \eta_t \leq y + q$, and similarly for ξ_s' and η_t'.

We define

(48)
$$S_{2i,2j} = \frac{1}{(2i)!\,(2j)!} \sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} \left(\frac{s}{h}\right)^{[2i+1]-1}\left(\frac{t}{k}\right)^{[2j+1]-1},$$

where s and t take on values at unit intervals from $-(h - 2)/2$ to $(h - 2)/2$ and from $-(k - 2)/2$ to $(k - 2)/2$ respectively.

Now sum (46) over these values of s and t.

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[o] Steffensen, pp. 210, 221.

(49)
$$\sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} f\left(x+\tfrac{1}{2}+\tfrac{s}{h};\, y+\tfrac{1}{2}+\tfrac{t}{k}\right) =$$

$$+ \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} S_{2i,2j} \, \square \, \delta_x^{2i} \, \square \, \delta_y^{2j} f\left(x+\tfrac{1}{2};\, y+\tfrac{1}{2}\right)$$

$$+ \sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R,$$

where the particular method of summation has caused the terms in the product involving the odd functions $\left(\frac{s}{h}\right)^{[2i+1]}$ and $\left(\frac{t}{k}\right)^{[2j+1]}$ to vanish.

Add $\displaystyle\sum_{t=-\frac{k}{2}}^{\frac{k-2}{2}} f\left(x;\, y+\tfrac{1}{2}+\tfrac{t}{k}\right) + \sum_{s=-\frac{h}{2}}^{\frac{h-2}{2}} f\left(x+\tfrac{1}{2}+\tfrac{s}{h};\, y\right) - f(x;y)$

to both members of (49). Keep the term containing $S_{0,0} = (h-1)(k-1)$ apart and make use of the fact

(50)
$$\square_x \square_y f\left(x+\tfrac{1}{2};\, y+\tfrac{1}{2}\right) = f(x;y) + \frac{\Delta_x f(x;y)}{2} + \frac{\Delta_y f(x;y)}{2} + \frac{\Delta_x \Delta_y f(x;y)}{4}.$$

Then (49) may be written

(51)
$$\sum_{s=-\frac{h}{2}}^{\frac{h-2}{2}} \sum_{t=-\frac{k}{2}}^{\frac{k-2}{2}} f\left(x+\tfrac{1}{2}+\tfrac{s}{h};\, y+\tfrac{1}{2}+\tfrac{t}{k}\right) =$$

$$\sum_{t=-\frac{k}{2}}^{\frac{k-2}{2}} f\left(x;y+\tfrac{1}{2}+\tfrac{t}{k}\right) + \sum_{s=-\frac{h}{2}}^{\frac{h-2}{2}} f\left(x+\tfrac{1}{2}+\tfrac{s}{h};y\right) - f(x;y)$$

$$+(h-1)(k-1)\left[f(x;y) + \frac{\Delta_x f(x;y)}{2} + \frac{\Delta_y f(x;y)}{2} + \frac{\Delta_x \Delta_y f(x;y)}{4}\right]$$

$$+\sum_{\substack{i=0 \\ (i=j\neq 0)}}^{p-1} \sum_{j=0}^{q-1} S_{2i,2j} \,\square\, \delta_x^{2i} \,\square\, \delta_y^{2j} f\left(x+\tfrac{1}{2};y+\tfrac{1}{2}\right)$$

$$+\sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R \,.$$

Now sum (51) from x = 0 to x = m - 1 and from y = 0 to y = n - 1 and make a slight change in notation.

$$\text{(52)} \quad \sum_{u=0}^{hm-1} \sum_{v=0}^{kn-1} f\left(\frac{u}{h};\frac{v}{k}\right) = \sum_{u=0}^{m-1} \sum_{v=0}^{kn-1} f\left(u;\frac{v}{k}\right)$$

$$+\sum_{u=0}^{hm-1} \sum_{v=0}^{n-1} f\left(\frac{u}{h};v\right) - \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u;v)$$

$$+(h-1)(k-1)\sum_{u=0}^{m-1} \sum_{v=0}^{n-1}\left[f(u;v) + \frac{\Delta_u f(u;v)}{2} + \frac{\Delta_v f(u;v)}{2} + \frac{\Delta_u \Delta_v f(u;v)}{4}\right]$$

$$+\sum_{u=0}^{m-1} \sum_{v=0}^{n-1} \sum_{\substack{i=0 \\ (i=j=0)}}^{p-1} \sum_{j=0}^{q-1} S_{2i,2j} \,\square\, \delta_u^{2i} \,\square\, \delta_v^{2j} f\left(u+\tfrac{1}{2};v+\tfrac{1}{2}\right)$$

$$+\sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R \,.$$

From Lubbock's formula of the third type in one variable[o] we have

$$(53) \qquad \sum_{\nu=0}^{kn-1} f\left(\mu;\frac{\nu}{k}\right) = k\sum_{\nu=0}^{n-1} f(\mu;\nu) + \frac{k-1}{2} f(\mu;\nu)\Big|_0^n$$

$$+ \sum_{j=1}^{q-1} Q_{2j}\,\square\,\delta_\nu^{2j-1} f(\mu;\nu)\Big|_0^n$$

$$+ n\,Q_{2q}\,f_{0,2q}(\mu;\eta_t''),$$

and

$$(54) \qquad \sum_{\mu=0}^{hm-1} f\left(\frac{\mu}{h};\nu\right) = h\sum_{\mu=0}^{m-1} f(\mu;\nu) + \frac{h-1}{2} f(\mu;\nu)\Big|_0^m$$

$$+ \sum_{i=1}^{p-1} Q_{2i}\,\square\,\delta_\mu^{2i-1} f(\mu;\nu)\Big|_0^m$$

$$+ m\,Q_{2p}\,f_{2p,0}(\xi_s'';\nu),$$

where $1 - p \le \xi_s'' \le m + p - 1$ and $1 - q \le \eta_t'' \le n + q - 1$. Also

$$Q_{2i} = \frac{1}{(2i)!}\sum_{s=-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \left(\frac{s}{h}\right)^{[2i+1]-1}.$$

It follows from (48) that

$$(55) \qquad S_{2i,2j} = Q_{2i}\cdot Q_{2j}.$$

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o Steffensen, p. 146.

We also make use of the following:

$$(56)\quad \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\frac{\Delta_u f(u;v)}{2} = \frac{1}{2}\sum_{v=0}^{n-1} f(u;v)\Big|_0^m ,$$

$$(57)\quad \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\frac{\Delta_v f(u;v)}{2} = \frac{1}{2}\sum_{u=0}^{m-1} f(u;v)\Big|_0^n ,$$

$$(58)\quad \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\frac{\Delta_u \Delta_v f(u;v)}{4} = \frac{1}{4} f(u;v)\Big|_0^m\Big|_0^n ,$$

$$(59)\quad \sum_{u=0}^{m-1}\sum_{v=0}^{n-1}\Box\,\delta_u^{2i}\,\Box\,\delta_v^{2j} f\left(u+\tfrac{1}{2};v+\tfrac{1}{2}\right) = \Box\,\delta_u^{2i-1}\,\Box\,\delta_v^{2j-1} f(u;v)\Big|_0^m\Big|_0^n .$$

Substituting the results of (53), (54), (56), (57), (58), and (59) in (52) gives, after some simplification,

$$(60)\quad \sum_{u=0}^{hm-1}\sum_{v=0}^{kn-1} f\left(\frac{u}{h};\frac{v}{k}\right) = hk\sum_{u=0}^{m-1}\sum_{v=0}^{n-1} f(u;v) + \frac{h(k-1)}{2}\sum_{u=0}^{m-1} f(u;v)\Big|_0^n$$

$$+\frac{k(h-1)}{2}\sum_{v=0}^{n-1} f(u;v)\Big|_0^m + \frac{(h-1)(k-1)}{4} f(u;v)\Big|_0^m\Big|_0^n$$

$$+\sum_{u=0}^{m-1}\sum_{j=1}^{q-1} Q_{2j}\,\Box\,\delta_v^{2j-1} f(u;v)\Big|_0^n + \sum_{v=0}^{n-1}\sum_{i=1}^{p-1} Q_{2i}\,\Box\,\delta_u^{2i-1} f(u;v)\Big|_0^m$$

$$+\sum_{\substack{i=0\\(i=j\neq 0)}}^{p-1}\sum_{j=0}^{q-1} S_{2i,2j}\,\Box\,\delta_u^{2i-1}\,\Box\,\delta_v^{2j-1} f(u;v)\Big|_0^m\Big|_0^n$$

$$+ \sum_{\mu=0}^{m-1} n\, Q_{2q}\, f_{0,2q}(\mu; u_t'') + \sum_{\nu=0}^{n-1} m\, Q_{2p}\, f_{2p,0}(\xi_s''; \nu)$$

$$+ \sum_{x=0}^{m-1} \sum_{y=0}^{n-1} \sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R \; .$$

The remainder term in (60), which consists of the last three terms, may be simplified. Upon application of the Theorem of Mean Value the first two of these terms may be written

(61) $$m\, n\, Q_{2p}\, f_{2p,0}(\xi_1; u_1) + m\, n\, Q_{2q}\, f_{0,2q}(\xi_2; u_2),$$

where $0 \le \xi_1 \le m - 1$, $1 - q \le u_1 \le n + q - 1$, and $1 - p \le \xi_2 \le m + p - 1$, $0 \le u_2 \le n - 1$.

To evaluate the last term of the remainder term in (60) we first apply the Theorem of Mean Value to the remainder term of (51). Then

(62) $$\sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}} \sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R = S_{2p,0}\, f_{2p,0}(\bar\xi; y') + S_{0,2q}\, f_{0,2q}(x'; \bar u) - S_{2p,2q}\, f_{2p,2q}(\bar\xi'; \bar u'),$$

where $x - p + 1 \le \bar\xi, \bar\xi' \le x + p$, $y - q + 1 \le \bar u, \bar u' \le y + q$, and $x + 1/h \le x' \le x + 1 - 1/h$, $y + 1/k \le y' \le y + 1 - 1/k$.

When the summations over x and y are introduced, the Theorem of Mean Value may be applied once more, and the final form for this part of the remainder term becomes

$$(63)\quad \sum_{x=0}^{m-1}\sum_{y=0}^{n-1}\sum_{-\frac{(h-2)}{2}}^{\frac{h-2}{2}}\sum_{-\frac{(k-2)}{2}}^{\frac{k-2}{2}} R = m\,n\,S_{2p,0}\,f_{2p,0}(\xi_3;\eta_3)$$

$$+\,m\,n\,S_{0,2q}\,f_{0,2q}(\xi_4;\eta_4) - m\,n\,S_{2p,2q}\,f_{2p,2q}(\xi_5;\eta_5),$$

where $1 - p \le \xi_3, \xi_5 \le m + p - 1$, $1 - q \le \eta_4, \eta_5 \le n + q - 1$, and $1/h \le \xi_4 \le m - 1/h$, $1/k \le \eta_3 \le n - 1/k$.

In (60), (61), and (63) set $f(u;v) = F(hu;kv)$. Then

$$(64)\quad \sum_{u=0}^{hm-1}\sum_{v=0}^{kn-1} F(u;v) = h\,k\sum_{u=0}^{m-1}\sum_{v=0}^{n-1} F(hu;kv)$$

$$+\,\frac{h(k-1)}{2}\sum_{u=0}^{m-1} F(hu;kv)\Big|_0^n + \frac{k(h-1)}{2}\sum_{v=0}^{n-1} F(hu;kv)\Big|_0^m$$

$$+\,\frac{(h-1)(k-1)}{4}\,F(hu;kv)\Big|_0^m\Big|_0^n$$

$$+\sum_{u=0}^{m-1}\sum_{j=1}^{q-1} Q_{2j}\,\Box\,\delta_v^{2j-1}F(hu;kv)\Big|_0^n + \sum_{v=0}^{n-1}\sum_{i=1}^{p-1} Q_{2i}\,\Box\,\delta_u^{2i-1}F(hu;kv)\Big|_0^m$$

$$+\sum_{\substack{i=0\\(i=j\neq 0)}}^{p-1}\sum_{j=0}^{q-1} S_{2i,2j}\,\Box\,\delta_u^{2i-1}\,\Box\,\delta_v^{2j-1}F(hu;kv)\Big|_0^m\Big|_0^n$$

$$+\,h^{2p}\,m\,n\,Q_{2p}\,F_{2p,0}(\xi_1;\eta_1) + k^{2q}\,m\,n\,Q_{2q}\,F_{0,2q}(\xi_2;\eta_2)$$

$$+\,h^{2p}\,m\,n\,S_{2p,0}\,F_{2p,0}(\xi_3;\eta_3) + k^{2q}\,m\,n\,S_{0,2q}\,F_{0,2q}(\xi_4;\eta_4)$$

$$-\,h^{2p}k^{2q}\,m\,n\,S_{2p,2q}\,F_{2p,2q}(\xi_5;\eta_5),$$

where $0 \le \xi_1 \le h(m-1)$, $k(1-q) \le \eta_1, \eta_4, \eta_5 \le k(n+q-1)$, $h(1-p) \le \xi_2, \xi_3, \xi_5 \le h(m+p-1)$, $0 \le \eta_2 \le k(n-1)$, $1 \le \xi_4 \le hm-1$, $1 \le \eta_3 \le kn-1$.

Formula (64) is the third of the summation formulas that we have been seeking. It may be regarded as an extension of Lubbock's formula of the third type with a remainder term °. The summing is done at the corner points as in formula (24) which has the same first term as formula (64). For identical values of p and q, more terms are used in estimating by means of (64) than by means of (24) or (38). The corrective terms in the present formula involve the less known mean central differences. The remainder terms of (24) and (64) are not directly comparable because they contain partial derivatives of different orders, although it will be noticed that the remainder in (64) involves two extra terms. The remainder term in (38) is the best of the three.

We shall illustrate the application of formula (64) to

$\sum_{x=100}^{114} \sum_{y=100}^{114} \frac{1}{xy}$ when $h = k = 5$, $m = n = 3$, and $p = q = 2$. The following coefficients may be found from relation (55) and tables for Q_{2i}. °°

$$Q_2 = -.4 \qquad S_{2,0} = S_{0,2} = -1.6$$
$$Q_4 = .0736 \qquad S_{2,2} = .16$$
$$S_{4,4} = .00541696 \qquad S_{0,4} = S_{4,0} = .2944 .$$

The value of the remainder term is

$$\frac{5^4\, 3^2 (.0736)\, 24}{(\xi_1+100)^5(\eta_1+100)} + \frac{5^4\, 3^2 (.0736)\, 24}{(\xi_2+100)(\eta_2+100)^5}$$

° Steffensen, p. 146.

°° Steffensen, p.147.

$$\frac{5^4 3^2 (.2944)\,24}{(\xi_3+100)^5(\kappa_3+100)} + \frac{5^4 3^2 (.2944)\,24}{(\xi_4+100)(\kappa_4+100)^5}$$

$$- \frac{5^8 3^2 (.00541696)\,576}{(\xi_5+100)^5(\kappa_5+100)^5},$$

where $F(\mu;\nu) = \dfrac{1}{(\mu+100)(\nu+100)}$ and we are summing $\sum_{\mu=0}^{14}\sum_{\nu=0}^{14} F(\mu;\nu)$.

To obtain the upper limit for the error we choose $\xi_1 = \kappa_2 = 0$, $\xi_2 = \xi_3 = \xi_5 = \kappa_1 = \kappa_4 = \kappa_5 = -5$, and $\xi_4 \cdot \xi_3 = 1$. The error is then indicated as being less than 2 in the 7^{th} decimal place.

Values of $F(h\mu;k\nu) \times 10^9 = U_{\mu,\nu}$ are recorded in Table VIII.

μ	$\nu = -1$	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$	$\nu = 4$
-1	110803	105263	100251	95694	91533	87719
0	105263	100000	95238	90909	86957	83333
1	100251	95238	90703	86580	82816	79365
2	95694	90909	86580	82645	79051	75758
3	91533	86957	82816	79051	75614	72464
4	87719	83333	79365	75758	72464	69444

Table VIII

The first term in (64) has a value of

(65) $\quad 25(.000818802) = .02047005$.

The second term becomes

(66) $\quad 10(U_{0,3} - U_{0,0} + U_{1,3} - U_{1,0} + U_{2,3} - U_{2,0})$

$= -.00037323,$

and the third term has the same value.

The fourth term is

(67) $$4(U_{3,3} - U_{0,3} - U_{3,0} \quad U_{0,0}) = .0000068 \ .$$

The fifth term in (64) may be written

(68) $$Q_2 \,\square\, \delta_\nu F(0,5\nu)\Big|_0^3 + Q_2 \,\square\, \delta_\nu F(5;5\nu)\Big|_0^3 + Q_2 \,\square\, \delta_\nu F(10;5\nu)\Big|_0^3$$

$$= -.4 \,\square\, \delta_\nu \left[U_{0,3} + U_{1,3} + U_{2,3} - U_{0,0} - U_{1,0} - U_{2,0}\right],$$

which, from Table IX, is equal to $-.4(.000003503) = -.0000014012$.
The sixth term has the same value.

μ	$\square\delta_\nu U_{\mu,0}$	$\square\delta_\nu U_{\mu,3}$
-1	- 5276	-3988
0	- 5012	-3788
1	- 4774	-3608
2	- 4557	-3444
3	- 4358	-3294
4	- 4177	-3157

Table IX

μ	$\square\delta_\mu \square\delta_\nu U_{\mu,0}$	$\square\delta_\mu \square\delta_\nu U_{\mu,3}$
0	251	190
3	190	144

Table X

For $i = 0$, $j = 1$, the seventh term in (64) may be written

(69) $$S_{0,2} \,\square\, \delta_\mu^{-1} \,\square\, \delta_\nu F(h\mu;k\nu)\Big|_0^3\Big|_0^3$$

$$= -.8 \,\square\, \delta_\nu \left(U_{0,\nu} + 2U_{1,\nu} + 2U_{2,\nu} + U_{3,\nu}\right)_0^3 ,$$

which, from Table IX, is equal to

$$- .8 \left[(-.3788) + 2(-3608) + 2(-3444) + (-3294)\right.$$
$$\left. -(-5012) - 2(-4774) - 2(-4557) - (-4358)\right] = -.0000054768 \ .$$

Similarly for $i = 1$, $j = 0$.

For $i = j = 1$, formula (64) gives us

$$(70) \qquad .16 \ \Box\delta_u \ \Box\delta_v \left(U_{3,3} - U_{3,0} - U_{0,3} + U_{0,0} \right),$$

which, from Table I, is equal to .0000000024 .

Combining the results of (65)-(70) gives us .0197166364 which underestimates the actual sum by slightly less than 6 in the 8^th decimal place. This checks with the fact that the error as indicated by the remainder term is positive and has an upper limit of 2 in the 7^th decimal place. A larger error in estimation was made when formula (24) was used with larger values of p and q. Because of this, the upper bound for the error should be calculated from the remainder term before any of the tables of differences needed in the summation formulas are set up.

5. EXTENSION OF WOOLHOUSE'S FORMULA OF THE FIRST TYPE

In one variable Woolhouse's formula of the first type is based on the same principle as the Lubbock formulas except that the former employs derivatives instead of differences in its corrective terms.[o] Its derivation depends upon writing the Euler-MacLaurin summation formula in two different forms and then eliminating the integral between them. Since the Euler-MacLaurin summation formula in two independent variables with a remainder [oo] has been recently developed, it is now possible to obtain an extension of Woolhouse's formula.

For later convenience we define a polynomial in two independent variables x and y, $B_{m,n}(x,y)$, such that

(71) $$\Delta_x \Delta_y B_{m,n}(x,y) = m\,n\,x^{m-1}y^{n-1},$$

and

(72) $$D_x^i D_y^j B_{m,n}(x,y) = \frac{m!\,n!}{(m-i)!\,(n-j)!} B_{m-i,n-j}(x,y),$$

where the first subscript of B denotes the degree of the polynomial in x, and the second the degree in y. D_x^i represents the i^{th} derivative with respect to x.

Such polynomials are called Bernoulli product polynomials from the fact that

$$B_{m,n}(x,y) = B_m(x)\cdot B_n(y),$$

where $B_m(x)$ is the Bernoulli polynomial of m^{th} degree in one variable.[ooo]

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[o] Steffensen, p. 148.

[oo] W.D. Baten, A Remainder for the Euler-MacLaurin Summation Formula in Two Independent Variables, American Journal of Mathematics, Vol.LIV, April, 1932, p. 265. Hereafter referred to as Baten.

[ooo] Steffensen, p. 120.

The values of the product polynomial for $x = y = 0$ are the values of the Bernoulli product numbers which are designated by the symbol $B_{m,n}$. It follows that $B_{m,n} = B_m \cdot B_n$ where B_m is the m^{th} Bernoulli number in one variable.

We also define a function, $\bar{B}_{m,n}(x,y)$, of period 1 in x and y which equals $\dfrac{B_{m,n}(x,y)}{m!\,n!}$ for $0 \leq x < 1$ and $0 \leq y < 1$. For all x and y $\bar{B}_{m,n}(x+1, y+1) = \bar{B}_{m,n}(x,y)$. The properties of this function have been discussed elsewhere.[o] It has immediate use in the Euler-MacLaurin summation formula in two variables which may be written

$$(73) \quad \sum_{x=0}^{m-1}\sum_{y=0}^{n-1} g(x,y) = \sum_{i=0}^{p}\sum_{j=0}^{q} \frac{B_{i,j}}{i!\,j!}\, G_{i,j}(x,y)\Big|_0^m\Big|_0^n$$

$$- \sum_{i=0}^{p} \int_0^n \bar{B}_{i,q}(0,r)\, G_{i,q+1}(x,r)\,dr \Big|_0^m$$

$$- \sum_{j=0}^{q} \int_0^m \bar{B}_{p,j}(t,0)\, G_{p+1,j}(t,y)\,dt \Big|_0^n$$

$$- \int_0^m\int_0^n \bar{B}_{p,q}(t,r)\, G_{p+1,q+1}(t,r)\,dt\,dr,$$

where $G_{i,j}(x,y) = g_{i-1,j-1}(x,y)$. The function $g(x,y)$ is continuous in x and y and possesses continuous derivatives of all orders in x and y which are integrable. The last three terms of (73) constitute

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o Baten, p.268.

the remainder term.

In (73) let $g(x,y) = f(hx,ky)$. Then we have

$G_{i,j}(x,y) = h^{i-1}k^{j-1}F_{i,j}(hx,ky)$ where $F_{i,j}(hx,ky) = f_{i-1,j-1}(hx,ky)$.

After multiplying all members by hk we obtain

$$(74)\quad hk\sum_{x=0}^{m-1}\sum_{y=0}^{n-1} f(hx,ky) = \sum_{i=0}^{p}\sum_{j=0}^{q} h^i k^j \frac{B_{i,j}}{i!\,j!} F_{i,j}(x,y)\Big|_0^{hm}\Big|_0^{kn}$$

$$-\sum_{i=0}^{p}\int_0^{kn} h^i k^q \bar{B}_{i,q}\left(0,\tfrac{r}{k}\right) F_{i,q+1}(x,r)\,dr\Big|_0^{hm}$$

$$-\sum_{j=0}^{q}\int_0^{hm} h^p k^j \bar{B}_{p,j}\left(\tfrac{t}{h},0\right) F_{p+1,j}(t,y)\,dt\Big|_0^{kn}$$

$$-h^{p+1}k^{q+1}\int_0^m\int_0^n \bar{B}_{p,q}(t,r)\,F_{p+1,q+1}(ht,kr)\,dt\,dr.$$

Another variation of (73) may be obtained by replacing $g(x,y)$ by $f(x,y)$ and $G_{ij}(x,y)$ by $F_{ij}(x,y)$ and also by replacing m by hm and n by kn. Subtracting (74) from this latter variation, we obtain

$$(75)\quad \sum_{x=0}^{hm-1}\sum_{y=0}^{kn-1} f(x,y) = hk\sum_{x=0}^{m-1}\sum_{y=0}^{n-1} f(hx;ky)$$

$$-\sum_{i=0}^{p}\sum_{j=0}^{q}\left(h^i k^j - 1\right)\frac{B_{i,j}}{i!\,j!} F_{i,j}(x,y)\Big|_0^{hm}\Big|_0^{kn}$$

$$- \sum_{i=0}^{p} \int_0^{kn} (h^i k^q - 1)\, \bar{B}_{i,q}\left(0, \frac{\tau}{k}\right) F_{i,q+1}(x,\tau)\, d\tau \Big|_0^{hm}$$

$$- \sum_{j=0}^{q} \int_0^{hm} (h^p k^j - 1)\, \bar{B}_{p,j}\left(\frac{t}{k}, 0\right) F_{p+1,j}(t,y)\, dt \Big|_0^{kn}$$

$$+ h^{p+1} k^{q+1} \int_0^m \int_0^n \bar{B}_{p,q}(t,\tau)\, F_{p+1,q+1}(ht, k\tau)\, dt\, d\tau$$

$$- \int_0^{hm} \int_0^{kn} \bar{B}_{p,q}(t,\tau)\, F_{p+1,q+1}(t,\tau)\, dt\, d\tau .$$

Formula (75) may be compared to Woolhouse's formula of the first type in one variable. It will be noticed that the first term in the right member of (75) is identical with the corresponding term in formulas (24) and (64). The second term is the corrective term which involves integrals and derivatives of $f(x,y)$. Since the term in this double sum corresponding to $i = j = 0$ is 0, the evaluation of a double integral is avoided. The remainder term consists of the last four terms and offers the same difficulties as the remainder term of the Euler-MacLaurin summation formula in two variables. For actual use formula (75) is not as convenient as desired. It seems that further work might be done in modifying (75) so that the remainder term would be more adaptable to numerical evaluation. We shall, however, illustrate the use of the formula for a simple example where the remainder term is zero.

To sum $\sum_{x=0}^{9} \sum_{y=0}^{9} x^2 y^2$ let $h = k = 5$, $m = n = 2$, $p = q = 3$. Since $f(x,y) = x^2 y^2$, we have $F(x,y) = \frac{x^3 y^3}{9}$ and

$$F_{10} = \frac{x^2 y^3}{3} \qquad F_{01} = \frac{x^3 y^2}{3}$$

$$F_{20} = \frac{2 x y^3}{3} \qquad F_{02} = \frac{2 x^3 y}{3}$$

$$F_{21} = 2 x y^2 \qquad F_{12} = 2 x^2 y$$

$$F_{11} = x^2 y^2 \qquad F_{22} = 4 x y \ .$$

It may be verified that all terms of (75) containing third order partial derivatives vanish. We also need

$$B_{10} = B_{01} = -\tfrac{1}{2}, \ B_{20} = B_{02} = \tfrac{1}{6}, \ B_{12} = B_{21} = -\tfrac{1}{12}, \ B_{11} = \tfrac{1}{4} \ .$$

The first term in (75) is 65(625) = 15625. The second term becomes, from symmetry,

$$2(-4)\left(-\tfrac{1}{2}\right) \frac{x^2 y^3}{3} \Bigg|_0^{10} \Bigg|_0^{10} + 2(-24)\left(\tfrac{1}{12}\right) \frac{2 x y^3}{3} \Bigg|_0^{10} \Bigg|_0^{10}$$

$$+ 2(-124)\left(-\tfrac{1}{24}\right) 2 x^2 y \Bigg|_0^{10} \Bigg|_0^{10} - 24\left(\tfrac{1}{4}\right) x^2 y^2 \Bigg|_0^{10} \Bigg|_0^{10} - 624\left(\tfrac{1}{144}\right) 4 x y \Bigg|_0^{10} \Bigg|_0^{10} ,$$

which equals 65600. Hence the double sum is 81225 which checks.

6. EXTENSION OF HARDY'S FORMULA OF MECHANICAL QUADRATURE

In one variable Hardy's formula of mechanical quadrature may be obtained as a special case of the Cotes' formula which depends upon integration of the Lagrange interpolation identity between fixed limits.[o] A set of formulas for mechanical cubature may be obtained in a similar manner by integrating the Lagrange interpolation formula for two variables between fixed limits for x and y. We assume the Lagrange interpolation identity [oo]

$$f(x;y) = \sum_{\mu=-r}^{r} \sum_{\nu=-s}^{s} \frac{P_\mu(x)\,\bar{P}_\nu(y)}{P_\mu(\mu)\,\bar{P}_\nu(\nu)} f(\mu;\nu) + R, \tag{76}$$

where

$$P_\mu(x) = \frac{x^{[2r+2]-1}}{x-\mu}, \qquad \bar{P}_\nu(y) = \frac{y^{[2s+2]-1}}{y-\nu},$$

and

$$\begin{aligned} R = {} & x^{[2r+2]-1} f(x,0,\pm 1, \cdots \pm r; y) \\ & + y^{[2s+2]-1} f(x; y, 0, \pm 1, \cdots \pm s) \\ & - x^{[2r+2]-1}\, y^{[2s+2]-1} f(x,0,\pm 1, \cdots \pm r; y, 0, \pm 1, \cdots \pm s). \end{aligned}$$

We now integrate (76) from $x = -m$ to $x = m$ and from $y = -n$ to $y = n$ and make the following substitutions.

$$U_\mu = \int_{-m}^{m} \frac{P_\mu(x)}{P_\mu(\mu)}\,dx, \qquad \bar{U}_\nu = \int_{-n}^{n} \frac{\bar{P}_\nu(y)}{\bar{P}_\nu(\nu)}\,dy.$$

Then

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o Steffensen, pp. 154, 167-168.

oo Steffensen, p. 224.

$$(77)\qquad \int_{-n}^{n}\int_{-m}^{m} f(x;y)\,dx\,dy = \sum_{\mu=-r}^{r}\sum_{\nu=-s}^{s} U_\mu \bar{U}_\nu f(\mu;\nu) + R\ .$$

If we choose unity as the intervals of integration for x and y by letting $f(x;y) = F(\frac{x}{2m};\frac{y}{2n})$, then (77) may be written

$$(78)\qquad \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} F(x;y)\,dx\,dy = V_0\bar{V}_0 F_{00} + V_0\sum_{\nu=1}^{s}\bar{V}_\nu F_{\pm 0\nu}$$

$$+ \bar{V}_0\sum_{\mu=1}^{r} V_\mu F_{\pm\mu 0} + \sum_{\mu=1}^{r}\sum_{\nu=1}^{s} V_\mu\bar{V}_\nu F_{\pm\mu\nu} + R,$$

where

$$V_\mu = \frac{1}{2m}U_\mu\ ,\qquad \bar{V}_\nu = \frac{1}{2n}\bar{U}_\nu\ ,$$

$$F_{\pm\mu 0} = F\left(\frac{\mu}{2m},0\right) + F\left(\frac{-\mu}{2m},0\right),\quad F_{\pm 0\nu} = F\left(0,\frac{\nu}{2n}\right) + F\left(0,\frac{-\nu}{2n}\right),$$

$$F_{\pm\mu\nu} = F\left(\frac{\mu}{2m},\frac{\nu}{2n}\right) + F\left(\frac{\mu}{2m},\frac{-\nu}{2n}\right) + F\left(\frac{-\mu}{2m},\frac{\nu}{2n}\right) + F\left(\frac{-\mu}{2m},\frac{-\nu}{2n}\right).$$

The remainder term in (78) may be shown to have the form

$$R = O_{2m}^{[2r+2]} F_{2r+2,0}(\xi_1;\eta_1) + O_{2n}^{[2s+2]} F_{0,2s+2}(\xi_2;\eta_2)$$

$$- O_{2m}^{[2r+2]}\ O_{2n}^{[2s+2]}\ F_{2r+2,2s+2}(\xi_3;\eta_3),$$

where each ξ and η lies within the finite limits of integration, and

$$\overset{[2k]}{O}_{2m} = \frac{2}{(2k)!\,(2m)^{2k+1}} \int_0^m x^{[2k]}\,dx \ .$$

Values for the coefficients V_μ , $\bar{V}_\nu$, and $\overset{[2k]}{O}_{2m}$ may be obtained from tables.[o] If, in formula (78), we let $r = m = 3$, $s = n = 3$, we obtain

(79) $$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} F(x;y)\,dx\,dy = \frac{1}{705600}\Big[73984\ F_{00}$$

$$+ 7344\left(F_{\pm 10} + F_{\pm 01}\right) + 58752\left(F_{\pm 20} + F_{\pm 02}\right)$$

$$+ 11152\left(F_{\pm 30} + F_{\pm 03}\right) + 729\ F_{\pm 11}$$

$$+ 46656\ F_{\pm 22} + 1681\ F_{\pm 33} + 5832\left(F_{\pm 21} + F_{\pm 12}\right)$$

$$+ 1107\left(F_{\pm 31} + F_{\pm 13}\right) + 8856\left(F_{\pm 32} + F_{\pm 23}\right)\Big]$$

$$- .0^9 64\left[F_{8,0}(\xi_1;u_1) + F_{0,8}(\xi_2;u_2) + .0^9 64\ F_{8,8}(\xi_3;u_3)\right],$$

where $.0^9 64$ means $.00000000064$.

To the right hand member of (79) we add and subtract the

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[o] Steffensen, p. 158.

quantity $\frac{1476}{705600} \Delta_x^6 \Delta_y^6 F_{-3-3}$. The subtracted part we put with the remainder term and make the substitution

$$\Delta_x^6 \Delta_y^6 F_{-3-3} = \frac{F_{6,6}(\xi_4;\eta_4)}{6^6 \cdot 6^6}.$$

The added part is combined with the functional values after the following substitution has been made

(80)
$$\Delta_x^6 \Delta_y^6 F_{-3-3} = 400 F_{00} - 300(F_{\pm 10} + F_{\pm 01})$$
$$+ 120(F_{\pm 20} + F_{\pm 02}) - 20(F_{\pm 30} + F_{\pm 03})$$
$$+ 225 F_{\pm 11} + 36 F_{\pm 22} + F_{\pm 33}$$
$$- 90(F_{\pm 21} + F_{\pm 12}) + 15(F_{\pm 31} + F_{\pm 13})$$
$$- 6(F_{\pm 32} + F_{\pm 23})$$

to give

(81)
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x;y)\, dx\, dy = \frac{1}{705600} \Big[664384 F_{00}$$
$$- 435456(F_{\pm 10} + F_{\pm 01}) + 235872(F_{\pm 20} + F_{\pm 02})$$
$$- 18368(F_{\pm 30} + F_{\pm 03}) + 332829 F_{\pm 11}$$
$$+ 99792 F_{\pm 22} + 3157 F_{\pm 33}$$

$$-127008\left(F_{\pm 21}+F_{\pm 12}\right) + 23247\left(F_{\pm 31}+F_{\pm 13}\right)\Big]$$

$$-.0^{9}64\left[F_{8,0}(\xi_1;u_1)+F_{0,8}(\xi_2;u_2)+.0^{9}64\,F_{8,8}(\xi_3;u_3)\right]-.0^{12}961\,F_{6,6}(\xi_4;u_4)\,.$$

Formula (81) is to be compared with Hardy's formula in one variable. Cubature formula (79) makes use of 49 ordinates. Formula (81) has the advantage that 8 ordinates, $F_{\pm 32}+F_{\pm 23}$, have been dropped out, although the remainder term does contain one extra term. Table XI indicates which 41 of the 49 ordinates are used. The method of this section might be used to eliminate any set of 1, 4, or 8 ordinates used in (79), but the substitution (80) leaves the coefficients in the most convenient form.

We shall illustrate the use of formula (81) in evaluating the double integral

$$\int_2^3\int_2^3 \frac{x}{x^2+y^2}\,dx\,dy\,.$$

For the region of integration under consideration it may be shown [o] that the upper bound for the error is determined from the fact

$$\left|F_{2\mu,2\nu}(\xi;u)\right| \le \frac{(2\nu+2\mu)!}{2^{3\nu+3\mu+1}\sqrt{2}}\,.$$

Hence the upper bound for the error is

$$.0^{9}64\left[\frac{2\cdot 8!}{2^{13}\sqrt{2}}+\frac{16!}{2^{25}\sqrt{2}}\right]+.0^{12}961\left[\frac{12!}{2^{19}\sqrt{2}}\right] < .000000006.$$

[o] Steffensen, p. 229.

The functional values needed to carry through the cubature are presented in Table XI.

x	$y = 2$	$y = 2\frac{1}{6}$	$y = 2\frac{1}{3}$	$y = 2\frac{1}{2}$	$y = 2\frac{2}{3}$	$y = 2\frac{5}{6}$	$y = 3$
2	$\frac{1}{4}$		$\frac{18}{85}$	$\frac{8}{41}$	$\frac{9}{50}$		$\frac{2}{13}$
$2\frac{1}{6}$		$\frac{3}{13}$	$\frac{78}{365}$	$\frac{39}{197}$	$\frac{78}{425}$	$\frac{39}{229}$	
$2\frac{1}{3}$	$\frac{21}{85}$	$\frac{84}{365}$	$\frac{3}{14}$	$\frac{84}{421}$	$\frac{21}{113}$	$\frac{84}{485}$	$\frac{21}{130}$
$2\frac{1}{2}$	$\frac{10}{41}$	$\frac{45}{197}$	$\frac{90}{421}$	$\frac{1}{5}$	$\frac{90}{481}$	$\frac{45}{257}$	$\frac{10}{61}$
$2\frac{2}{3}$	$\frac{6}{25}$	$\frac{96}{425}$	$\frac{24}{113}$	$\frac{96}{481}$	$\frac{3}{16}$	$\frac{96}{545}$	$\frac{24}{145}$
$2\frac{5}{6}$		$\frac{51}{229}$	$\frac{102}{485}$	$\frac{51}{257}$	$\frac{102}{545}$	$\frac{3}{17}$	
3	$\frac{3}{13}$		$\frac{27}{130}$	$\frac{12}{61}$	$\frac{27}{145}$		$\frac{1}{6}$

Table XI.

The required value, apart from the remainder term, is

$$\frac{1}{705600}\Big[\,664384\left(\frac{1}{5}\right) - 435456\left(\frac{84}{421}+\frac{90}{421}+\frac{96}{481}+\frac{90}{481}\right)$$

$$+\,235872\left(\frac{39}{197}+\frac{45}{197}+\frac{51}{257}+\frac{45}{257}\right)$$

$$-\,18368\left(\frac{8}{41}+\frac{10}{41}+\frac{12}{61}+\frac{10}{61}\right)$$

$$+\,332829\left(\frac{21}{113}+\frac{3}{14}+\frac{24}{113}+\frac{3}{16}\right)$$

$$+ \; 99792\left(\frac{39}{229} + \frac{3}{13} + \frac{51}{229} + \frac{3}{17}\right)$$

$$+ \; 3157\left(\frac{2}{13} + \frac{1}{4} + \frac{3}{13} + \frac{1}{6}\right)$$

$$- \; 127008\left(\frac{84}{485} + \frac{78}{425} + \frac{78}{365} + \frac{84}{365} + \frac{96}{425} + \frac{102}{485} + \frac{102}{545} + \frac{96}{545}\right)$$

$$+ \; 23247\left(\frac{21}{130} + \frac{9}{50} + \frac{18}{85} + \frac{21}{85} + \frac{6}{25} + \frac{27}{130} + \frac{27}{145} + \frac{24}{145}\right)\Bigg]$$

$$= .200021345^{+} .$$

Since the value of the integral is .200021343, the actual error is only 2 in the 9th decimal place while the upper bound for the error is 6 in the 9th decimal place.

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