STABILITY AND CONTROL OF NONLINEAR SINGULARLY PERTURBED SYSTEMS, WITH APPLICATION TO HIGH-GAIN FEEDBACK

By

Ali Saberi

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ABSTRACT

In part 1 of this thesis, for a general class of nonlinear singularly perturbed systems, a two-time-scale analysis and design procedure for stability, initial-value problem, stabilization and regulation is presented. It is shown that if the slow and fast dynamics satisfy an "Interaction Condition", then a decomposition base on the time scale structure of the system can be successfully performed for the purpose of analysis (stability, initial-value problem) or design (stabilization and regulation).

In part 2 of this thesis a problem of designing a robust decentralized control law, using local state or output feedback, for a class of large scale interconnected systems is studied. In the case that local states are available the proposed control law is static and is based on direct usage of high-gain local state feedback. On the other hand if the measurements of the local states are not available and the local observations are linear in the local states, a decentralized control law is proposed which is dynamics and employs high-gain local observer-based controllers.

To my parents and Valeh

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CHAPTER I

INTRODUCTION

Singularly perturbed systems often occur naturally due to the presence of small "parasitic" parameters, typically small time constants, masses, etc.,multiplying time derivatives or in more disguised forms, due to presence of high-gain feedback and weak coupling. In the early 1970s the chief purpose of singular perturbation, or more generally, the two-time-scale approach to analysis and design has been the alleviation of the high dimensionality and ill conditioning resulting from the interaction of slow and fast dynamic modes. However, in view of its rapid development and its role in control theory in late 70's, the role of singular perturbation has gone far beyond its early purpose, i.e. order reduction. For example singular perturbation methods prove extremely useful for the analysis of high-gain feedback systems, root locus analysis of multi-input multi-output linear systems, and synthesizing robust controller. In short,singular perturbation in automatic control had come of age. For an excellent introduction to the field, the reader should consult the 1976 survey paper [1].

The main theme of this thesis is the study the singular perturbation and its application to decentralized control. This study basically is divided in two parts. In the first part, which consists of Chapters 2, 3 and 4, several problem areas of singularly perturbed nonlinear system analysis and feedback control design are studied. The second part, which consists of Chapters 5 and 6, deals with designing decentralized controllers for interconnected systems using high-gain feedback.

Part 1

In this part the two-time-scale system decomposition provide a uniform framework within which Lyapunov stability, initial value problem, and stabilization and regulation of singularly perturbed nonlinear systems are studied. Each chapter in part 1 deals with one of these problem areas, and contains a survey of literature and comparison of our accomplishment in the discussed problem area with previous works.

A brief abstract of the chapters in part 1 is given in the following.

Chapter 2, "Lyapunov Stability"

Consider a nonlinear singularly perturbed system;

$$\dot{x} = f(x,y) \tag{1-1a}$$

$$\varepsilon\dot{y} = g(x,y,\varepsilon) \tag{1-1b}$$

where the origin (x=0, y=0) is the unique equilibirium point in the region of our interest. Assuming that $y = h(x)$ is a unique root of $g(x,y,0) = 0$, a reduced order system is obtained by setting $\varepsilon = 0$ in (1-1) to get:

$$\dot{x} = f(x,h(x)) \triangleq f_r(x). \tag{1-2}$$

A boundary layer system is defined as

$$\frac{dy}{d\tau} = g(x,y(\tau),0), \quad \tau = t/\varepsilon, \tag{1-3}$$

where x is treated as a fixed parameter. In chapter 2 we establish the asymptotic and exponential stability properties of the singularly perturbed system (1-1) for small ε from those of the reduced system (1-2) and the boundary layer system (1-3). The methodology in this study employs Lyapunov stability techniques. Estimates of domain of attractions, of upper bound on perturbation parameter, and of degree of exponential stability are also obtained. The stability result is illustrated by studying the stability of a synchronous generator connected to an infinite bus.

Chapter 3, "Closeness of the Trajectories of the Singularly Perturbed System to the Trajectories of its Slow and Fast Subsystems"

It is an old engineering practice to approximate the model of a physical system described by (1-1), which corresponds to a high-frequency model, by a low-frequency model, described by (1-2). In this chapter we study the justification of such an engineering simplification. Let $z_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ donote the solution of (1-1) defined on infinite interval $I = [t_0, \infty)$, with initial condition $z_\varepsilon(t_0) = (x_0, y_0)$. Moreover let $z_s(t) = (x_s(t), y_s(t))$ be the solution of (1-2) defined on I, with initial condition $z_s(t_0) = (x_0, h(x_0))$. A set of sufficient conditions is given under which $z_\varepsilon(t)$ uniformly converge to $z_s(t)$ on any closed subset of (t_0, ∞). Furthermore a boundary layer correction is provided to take care of the initial finite jump, which results from discrepancy of initial conditions $z_\varepsilon(t_0)$ and $z_s(t_0)$, in order to extend the convergence results to the interval I.

Chapter 4, "Stabilization and Regulation-Composite Control"

Chow and Kokotovic in their pioneer work [2] employed the decomposition of two-time-scale systems into separate slow and fast subsystems to promote a separation, with attendant simplification, in the design of state feedback controllers. The stabilization problem discussed in this chapter utilize this philosophy and the stability result of chapter 2 to perform a two-stage design of a stabilizing controller, so called composite-control, for a general nonlinear singularly perturbed system;

$$\dot{x} = f(x,y,u) \tag{1-3a}$$

$$\varepsilon\dot{y} = g(x,y,u,\varepsilon) \tag{1-3b}$$

This design procedure consists of following steps.

Step 1: Design a stabilizing controller $u_s = m(x)$, so called slow controller, for the slow subsystem;

$$\dot{x} = f(x,\ h(x,u_s),\ u_s), \tag{1-4}$$

where $h(x,u)$ is the unique root of $g(x,y,u,0) = 0$ in the region of our interest.

Step 2: Design a stablilizing controller $u_f = \Gamma(x,y)$, so called fast controller, for the fast subsystem;

$$\frac{dy}{d\tau} = g(x,y(\tau),\ u_s + u_f,\ 0) \tag{1-5}$$

Step 3: Form a composite control $u_c = u_s + u_f$.

In this chapter, under a set of conditions, it is established that the composite control is stabilizing.

For the regulator problem the same two-time-scale analysis and design procedure is employed. Consider a regulator problem (1-3) with the cost function

$$J = \int_0^\infty L(x,\ y,\ u)dt, \tag{1-6}$$

and initial condition $x(t_0) = x_0$, $y(t_0) = y_0$.

In step (1) of our design procedure we choose $u_s = M(x)$ as an optimal or near-optimal solution of the slow regulator (1-4) with the slow cost function

$$J_s = \int_0^\infty L(x,\ h(x,u_s),\ u_s) \tag{1-7}$$

and the initial condition $x(t_0) = x_0$. Next we proceed to steps 2 and 3 as before. In this chapter we establish a "near performance" property of the composite control obtained through above procedure. It is shown that this composite control is stabilizing and produces a finite cost, J_{uc}, which tends to the cost of the slow regulator as ε tends to zero. Furthermore, under different stability requirements on the fast subsystem, explicit upper and lower bounds on J_{uc} are obtained. Finally, for each result in this chapter an upper bound on the perturbation parameter ε is provided under which the result is valid.

Part II

Synthesizing a robust decentralized control, in the sense that the closed loop system can tolerate modeling errors and varing interconnections, involves high-gain feedback. The underlying philosophy of the design of a robust decentralized control scheme is that each isolated subsystem should maintain a large margin of stability to maintain the stability of the overall system at the presence of the interconnections, and this large margin of stability requires high-gain feedback. In [3] we illustrate the role of high-gain feedback in the problem area of decentralized control. In chapters 4 and 5 we proposed a design procedures using local high-gain state or output feedback for stabilizing a class of interconnected systems. A brief summary of these two chapters is given in the following:

Chapter 5, "Decentralized Control, Using Local High-Gain State Feedback"

Consider a nonlinear interconnected system

$$\dot{x}_i = f_i(x_i) + B_i(u_i + g_i(x_1 \ldots x_N)), \quad i = 1, \ldots N \qquad (1\text{-}8)$$

we prepare (1-8) for high-gain feedback by performing a local transformation $(y_i^T, z_i^T) = T_i x_i$, where T_i satisfies

$$T_i = \begin{pmatrix} 0 \\ G_i \end{pmatrix} \tag{1-9}$$

with G_i being a nonsingular matrix. Then (1-8) can be rewritten as

$$\dot{y}_i = \phi_i(y_i, z_i) \tag{1-10a}$$

$$\dot{z}_i = \pi_i(y_i, z_i) + G_i(u_i + g_i(x_1 \ldots x_N)). \tag{1-10b}$$

Now a reduced order isolated subsystem is defined as:

$$\dot{y}_i = \phi_i(y_i, v_i) \tag{1-11}$$

Where v_i is treated as an input vector. We assume that each isolated subsystem can solve a stabilization problem for the reduced-order system (1-11). Based on the local stabilization problem (1-11) a decentralized control law is designed, and it is shown that under some, basically smoothness, requirements the designed decentralized controller is stabilizing. The method is illustrated by linear and nonlinear examples and is compared with the existing results.

Chapter 6, "Decentralized Control, Using Local High-Gain Dynamic Output Feedback"

Consider an interconnected system

$$\dot{x}_i = A_i x_i + B_i(u_i + g_i(x_1 \ldots x_N)) + M_i(y_1 \ldots y_N) \tag{1-12a}$$

$$y_i = C_i x_i \tag{1-12b}$$

We assume that each isolated subsystem

$$\dot{x}_i = A_i x_i + B_i u_i \tag{1-13a}$$

$$y_i = C_i x_i \tag{1-13b}$$

is invertible and has all its transmission zeros in the left-half plane. Furthermore the mappings g_i and M_i are assumed to be Lipschitzian. In this chapter a local high-gain observer-based controller is designed, and it is shown that, under the above assumptions, the proposed decentralized controller is stabilizing.

CHAPTER II

LYAPUNOV STABILITY

1. Introduction

Stability properties of singularly perturbed systems have been investigated by several authors over the past two decades (see [1] for a survey). In [4-7] Lyapunov methods have been employed. The main idea is to consider two lower-order systems, the so-called reduced and boundary-layer systems. Assuming that each of the two systems is asymptotically stable and has a Lyapunov function, conditions are derived to guarantee that, for sufficiently small perturbation parameter, asymptotic stability of the singularly perturbed system can be established by means of a Lyapunov function which is composed as a weighted sum of the Lyapunov functions of the reduced and boundary-layer systems. The methods available in the literature have different conditions due to different smoothness assumptions, different classes of Lyapunov functions and different ways of obtaining a negative upperbound on the derivative of the composite Lyapunov function. Previous work, that is relevant to ours, is due to Grujic [5] and Chow [6]. Grujic employed composite Lyapunov methods (c.f. [8-10]) with linear-type Lyapunov functions to derive asymptotic stability conditions. He was concerned with establishing the existence of a composite Lyapunov function but did not use that Lyapunov function to investigate the stability properties of the system, like estimating the domain of attraction. Chow established the existence of a composite Lyapunov function and used it to obtain estimates of the domain of attraction. His method, however, is limited to a special case where the boundary-layer system is linear.

The new element in this work is the use of quadratic-type Lyapunov functions, which has been motivated by their successful use in studying the stability of interconnected systems [10]. For an asymptotically stable system $\dot{x} = f(x)$, the function $V(x)$ is said to be a quadratic-type Lyapunov function if $(\nabla_x V(x))^T f(x) \leq -\alpha\psi^2(x)$ and $\|\nabla_x V\| \leq \psi(x)$ where $\psi(x)$ is a positive definite function of x and α is a positive constant. The class of asymptotically stable systems that have quadratic-type Lyapunov functions is large; several interesting examples are given in [10]. Khalil [11] employed quadratic-type Lyapunov functions to study the special case that was studied by Chow [6], namely, linear boundary-layer systems. The results of [11] were promising and superior to the results of [6]. This work extends the results of [11] and improves over them. The extension is in studying a general case in which the boundary-layer system is nonlinear. The improvement comes through exploring the freedom in forming composite Lyapunov functions and using those Lyapunov functions to obtain an upper bound on the perturbation parameter, to estimate the domain of attraction or to estimate the degree of exponential stability.

In section 2.1, an asymptotic stability criterion for nonlinear autonomous systems is derived. An interesting feature of this criterion is that, under mild conditions, any weighted sum of quadratic-type Lyapunov functions for the reduced and boundary-layer systems is a quadratic-type Lyapunov function for the singularly perturbed system when the perturbation parameter is sufficiently small. It is shown that the choice of the weights of the composite Lyapunov function involves a trade-off between obtaining a large estimate of the domain of attraction and a large upper bound on the perturbation parameter. That trade-off is discussed and illustrated by an example. In section 2.2, the conditions are sharpened to give exponential stability. The composite Lyapunov function is used to obtain an estimate of the degree of exponential stability, which is shown, by means of an example, to be a very tight estimate. In section 2.3, the stability criterion is extended to non-autonomous systems. Application of the stability criterion to linear systems is carried out in section 3 where upper bounds on the perturbation parameter are obtained and compared with previous bounds due to Zien [12] and Javid [13]. As an illustration of the application of our method to physical systems in which the perturbation parameter may be fixed with given value, we apply the method in section 4 to estimate the domain of attraction of the stable equilibrium point of a synchronous generator connected to an infinite bus. The generator is represented by a three-dimensional model in which a field-flux decay is taken into consideration. The results are compared with previous ones due to Siddiqee [14].

2. Stability Criteria for Nonlinear Systems

2.1 Autonomous Systems: Asymptotic Stability

Consider the nonlinear singularly perturbed system[1]

$$\begin{aligned} \dot{x} &= f(x,y), & x \in B_x \subset R^n \\ \varepsilon\dot{y} &= g(x,y,\varepsilon), & y \in B_y \subset R^m, \ \varepsilon > 0. \end{aligned} \tag{2-1}$$

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[1] The symbol B_x indicates a closed sphere centered at $x = 0$; B_y is defined in the same way.

We assume that, in B_x and B_y, the origin ($x = 0$, $y = 0$) is the unique equilibrium point and (2-1) has a unique solution. A reduced system is defined by setting $\varepsilon = 0$ in (2-1) to obtain

$$\dot{x} = f(x,y) \tag{2-2a}$$

$$0 = g(x,y,0) \tag{2-2b}$$

Assuming that in B_x and B_y, (2-2b) has a unique root $y = h(x)$, the reduced system is rewritten as

$$\dot{x} = f(x,h(x)) \triangleq f_r(x) \tag{2-3}$$

A boundary-layer system is defined as

$$\frac{dy}{d\tau} = g(x,y(\tau),0) \tag{2-4}$$

where $\tau = t/\varepsilon$ is a stretching time scale. In (2-4), the vector $x \in R^n$ is treated as a fixed unknown parameter that takes values in B_x.

Our objective is to establish the stability properties of the singularly perturbed system (2-1), for small ε, from those of the reduced system (2-3) and the boundary-layer system (2-4). We are interested in cases when both (2-3) and (2-4) have quadratic-type Lyapunov functions. Our result shows, under mild assumptions, that, for sufficiently small ε, any weighted sum of the Lyapunov functions of the reduced and boundary-layer systems is a quadratic-type Lyapunov function for the singularly perturbed system (2-1).

We start by stating our assumptions.

(I) The reduced system (2-3) has a Lyapunov function $V : R^n \to R_+$ such that for all $x \in B_x$

$$(\nabla_x V(x))^T f_r(x) \leq -\alpha_1 \psi^2(x), \qquad \alpha_1 > 0,$$

where $\psi(x)$ is a scalar-valued function of x that vanishes at $x = 0$ and is different from zero for all other $x \in B_x$.

(II) The boundary-layer system (2-4) has a Lyapunov function $W(x,y) : R^n \times R^m \to R_+$ such that for all $x \epsilon B_x$ and $y \epsilon B_y$

$$(\nabla_y W(x,y))^T g(x,y,0) \leq -\alpha_2 \phi^2(y - h(x)), \qquad \alpha_2 > 0,$$

where $\phi(y - h(x))$ is a scalar-valued function of $(y - h(x)) \epsilon R^m$ that vanishes at $y = h(x)$ and is different from zero for all other $x \epsilon B_x$ and $y \epsilon B_y$.

(III) The following three inequalities hold $\forall x \epsilon B_x$ and $y \epsilon B_y$

(a) $(\nabla_x W(x,y))^T f(x,y) \leq c_1 \phi^2(y - h(x)) + c_2 \psi(x) \phi(y - h(x))$,

(b) $(\nabla_x V(x))^T [f(x,y) - f(x,h(x))] \leq \beta_1 \psi(x) \phi(y - h(x))$,

(c) $(\nabla_y W(x,y))^T [g(x,y,\varepsilon) - g(x,y,0)] \leq \varepsilon K_1 \phi^2(y - h(x))$ $+ \varepsilon K_2 \psi(x) \phi(y - h(x))$.

The constants c_1, c_2, β_1, K_1 and K_2 will be taken as nonnegative numbers [see Remark 2 for the case when some of these constants are negative]. Condition (I) guarantees that $x = 0$ is an asymptotically stable equilibrium point of the reduced system (2-3). Condition (III) plays the same role for the boundary-layer system (2-4) which has an equilibrium point $y = h(x)$; that is why in Condition (II) the comparison function $\phi(\cdot)$ depends on $y - h(x)$. It is important to notice that Condition (II) should hold uniformly in x, i.e., the positive constant α_2 should be independent of x. Inequalities (III-a) and (III-b) determine the permissible interaction between the slow and fast variables. They are basically smoothness requirements on f and g. One way to grasp their meaning is to consider a special case when the partial derivatives of V and W are bounded by ψ and ϕ, respectively, and $f_r(x)$ is bounded by ψ. In this special case, inequalities (III-a) and (III-b) follow from the Lipschitzian-like condition

$$|f(x,y_1) - f(x,y_2)| \leq L\phi(y_1 - y_2) \tag{2-5}$$

which simply says that the rate of growth of f in y cannot be faster than the rate of growth of $\phi(\cdot)$. Condition (III-a) will drop out if W is not a function of x. However, choosing W independent of x may be impossible or, if possible, would make Condition (II) very restrictive. Finally, inequality (III-c) determines the permissible dependence of g on ε. It will drop out if g is independent of ε. We believe that Conditions (I)-(III) are mild. One way to demonstrate that is to show that there is a wide class of systems for which these conditions hold. Lemma 1 does that.

Lemma 1: Consider a singularly perturbed system in which g is independent of ε. Let f,g and h be continuously differentiable in $\tilde{B}_x \times \tilde{B}_y$. Suppose that the reduced system is exponentially stable such that $\|x(t)\| \leq c_3 \|x(0)\| \exp(-c_4 t)$ whenever $x(0) \in \tilde{B}_x$. Let $B_x \subseteq \tilde{B}_x$ be a set such that every trajectory starting in B_x remains inside $\tilde{B}_x$. For every $x \in \tilde{B}_x$, suppose that the equilibrium point $\bar{y} = h(x)$ of the boundary-layer system is inside $\tilde{B}_y$ and is exponentially stable uniformly in x such that $\|y(\tau) - h(x)\| \leq c_5 \|y(0) - h(x)\| \exp(-c_6\tau)$ whenever $y(0) \in \tilde{B}_y$. Let $B_y \subseteq \tilde{B}_y$ be a set such that every trajectory starting in B_y remains inside $\tilde{B}_y$. Then, conditions (I)-(III) are satisfied in $B_x \times B_y$.

Proof: See Appendix.

We are now ready to state our stability criterion.

Theorem 1: Suppose that Conditions (I)-(III) hold; let d be a positive number such that $0 < d < 1$, and let $\varepsilon^*(d)$ be the positive number given by

$$\varepsilon^*(d) = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + [\beta_1(1 - d) + \beta_2 d]^2/4d(1 - d)},$$

where $\beta_2 = K_2 + C_2$, $\gamma = K_1 + C_1$,

then, for all $\varepsilon < \varepsilon^*(d)$, the origin $(x = 0,\ y = 0)$ is an asymptotically stable equilibrium point of (2-1) and

$$\nu(x,y) = (1 - d)V(x) + dW(x,y) \tag{2-6}$$

is a Lyapunov function for (2-1).

Proof: Let $\dot{\nu}(x,y)$ denote the derivative of ν along the trajectory of (2-1). We have

$$\begin{aligned}
\dot{\nu}(x,y) &= (1 - d)(\nabla_x V(x))^T f(x,y) + \frac{d}{\varepsilon}(\nabla_y W(x,y)^T g(x,y,\varepsilon) + d(\nabla_x W(x,y))^T f(x,y) \\
&= (1 - d)(\nabla_x V(x))^T f_r(x) + (1 - d)(\nabla_x V(x))^T [f(x,y) - f(x,h(x))] \\
&\quad + \frac{d}{\varepsilon}(\nabla_y W(x,y))^T g(x,y,0) + \frac{d}{\varepsilon}(\nabla_y W(x,y))^T [g(x,y,\varepsilon) - g(x,y,0)] \\
&\quad + d(\nabla_x W(x,y))^T f(x,y) \\
&\le -(1 - d)\alpha_1\psi^2(x) + (1 - d)\beta_1\psi(x)\phi(y - h(x)) - \frac{d}{\varepsilon}\alpha_2\phi^2(y - h(x)) \\
&\quad + dK_1\phi^2(y - h(x)) + dK_2\psi(x)\phi(y - h(x)) + dc_1\phi^2(y - h(x)) \\
&\quad + dc_2\psi(x)\phi(y - h(x)) \\
&= -\alpha_1(1 - d)\psi^2(x) + (1 - d)\beta_1\psi(x)\phi(y - h(x)) - d\left[\frac{\alpha_2}{\varepsilon} - \gamma\right]\phi^2(y - h(x)) \\
&\quad + d\beta_2\psi(x)\phi(y - h(x)) \\
&= -\begin{pmatrix}\psi(x) \\ \phi(y-h(x))\end{pmatrix}^T \mathbf{T} \begin{pmatrix}\psi(x) \\ \phi(y-h(x))\end{pmatrix}
\end{aligned}$$

where

$$\mathbf{T} = \begin{pmatrix} (1-d)\alpha_1 & -(1-d)\beta_1/2 - d\beta_2/2 \\ -(1-d)\beta_1/2 - d\beta_2/2 & d(\frac{\alpha_2}{\varepsilon} - \gamma) \end{pmatrix}$$

For asymptotic stability of (2-1), it is sufficient to require that $\mathbf{T}$ be a positive-definite matrix. For any choice of $d(0 < d < 1)$, $\mathbf{T}$ will be positive-definite when ε is sufficiently small. In particular, $\mathbf{T}$ is positive-definite for all $\varepsilon < \varepsilon^*(d)$.

Remark 1: If $V(x)$ and $W(x,y)$ are radially unbounded and $B_x \times B_y = R^n x R^m$, the origin ($x = 0$, $y = 0$) will be asymptotically stable in the large.

An interesting point in Theorem 1 is the arbitrariness of d. By allowing d to take any value on the interval (0,1), the composite Lyapunov function $\nu(x,y)$, as given by (2-6), covers all the possible linear combinations of V and W. According to Theorem 1, any one of these linear combinations is a Lyapunov function of the singularly perturbed system (2-1) as $\varepsilon \to 0$. What is more interesting is that for any given d, Theorem 1 provides us with the upper bound $\varepsilon^*(d)$. Fig. (1) shows a sketch of $\varepsilon^*(d)$ versus d. From that sketch we see that $\varepsilon^*(d)$ has maximum value ε^* at $d = d^*$. Straightforward calculations show that

$$\varepsilon^* = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + \beta_1 \beta_2} \tag{2-7}$$

and

$$d^* = \frac{\beta_1}{\beta_1 + \beta_2} \tag{2-8}$$

Corollary 1: Suppose that Conditions (I)-(III) hold and $\varepsilon < \varepsilon^*$, then the origin ($x = 0$, $y = 0$) is asymptotically stable.

In applying our stability criterion to study the stability of (2-1) when ε is known, we start by comparing ε with ε^*. If $\varepsilon > \varepsilon^*$, our criterion is not satisfied which means either that the origin is not asymptotically stable or that it is asymptotically stable but our criterion fails to detect that. On the other hand, if $\varepsilon < \varepsilon^*$ we conclude that the origin is asymptotically stable. The next step is to explore the freedom we have in choosing d. For any given ε, there is an interval (d_1, d_2) (see Fig. (1)) such that any $d \in (d_1, d_2)$ will be acceptable. As ε gets smaller, the interval (d_1, d_2) spreads out, tending eventually to (0,1) as $\varepsilon \to 0$. We can say that our stability criterion improves asymptotically as $\varepsilon \to 0$ in the sense that as $\varepsilon \to 0$ the criterion is always satisfied and there is greater freedom in forming the composite Lyapunov function.

The freedom we have in choosing d can be employed to obtain the largest possible estimate of the domain of attraction. The idea is illustrated by the synchronous machine example of section 4. In general, there is no systematic procedure for choosing d in order to obtain the best domain of attraction; such a choice is problem dependent. There is, however, a special case in which the choice of d is obvious. Suppose that the domain of attraction L_R, of the reduced systems and L_B, of the boundary-layer system, are given by

$$L_R = \{x \in B_x \mid V(x) \leq v_0\}$$

and

$$L_B = \{x \in B_x,\ y \in B_y \mid W(x,y) \leq w_0\},$$

then, an estimate of the domain of attraction of the singularly perturbed system (2-1) is given by

$$L = \{x \in B_x,\ y \in B_y \mid (1 - d)V(x) + dW(x,y) \leq \min((1 - d)v_0,\ dw_0)\} \tag{2-9}$$

It is apparent that the choice

$$d = \frac{v_o}{v_o + w_o} \tag{2-10}$$

will result in the largest set L^* which is given by

$$L^* = \{x \in B_x,\ y \in B_y \mid \frac{V(x)}{v_o} + \frac{W(x,y)}{w_o} \leq 1\} \tag{2-11}$$

The trade-off involved in choosing d is illustrated by the following example.

Example 1: Consider the second-order system

$$\dot{x} = y^2 - 2x^3, \qquad B_x = \{x \in R \mid |x| \leq 1\}, \tag{2-12a}$$

$$\varepsilon\dot{y} = x^3 - \tan y, \qquad B_y = \{y \in R \mid |y| \leq a\}, \text{ where a is} \tag{2-12b}$$

a constant less than, but arbitarily close to, $\pi/2$.

The reduced system is obtained by setting $\varepsilon = 0$ in (2-12b) to get

$$y = h(x) = \tan^{-1} x^3$$

which, when substituted in (2-12a), results in

$$\dot{x} = -2x^3 + (\tan^{-1} x^3)^2 \triangleq f_r(x) \tag{2-13}$$

In searching for a Lyapunov function V(x) for (2-13) we are guided by two requirements. First, V(x) should satisfy Condition (I). Second, $\frac{\partial V}{\partial x}$ should be bounded by $\psi(x)$. Although the second requirement does not appear explicitly in Conditions (I)-(III), it is helpful because, when it is satisfied, inequality (III-b) reduces to the Lipschitzian-like condition (2-5). The Lyapunov function $V(x) = \frac{1}{2} x^2$ satisfies Condition (I) with $\psi(x) = x^2$; however, $\frac{\partial V}{\partial x}$ grows faster than $\psi(x)$. So, it is not the Lyapunov function we are looking for. A Lyapunov function that satisfies both requirements is $V(x) = \frac{1}{4} x^4$. It satisfies Condition (I) with $\psi(x) = |x|^3$ and $\alpha_1 = 1$; at the same time $|\frac{\partial V}{\partial x}| = \psi(x)$. The set $L_R = \{x \in B_x | V(x) \le \frac{1}{4}\}$ is included in the domain of attraction of the reduced system. The boundary-layer system is defined by

$$\frac{dy}{d\tau} = x^3 - \tan y \qquad (2\text{-}14)$$

where x is treated as an unknown fixed parameter in B_x. Again in choosing the Lyapunov function W(x,y) we are guided by two requirements. First, W(x,y) should satisfy Condition (II). Second, $\frac{\partial W}{\partial x}$ should not grow faster than ϕ. A Lyapunov function that satisfies both requirements is $W(x,y) = \frac{1}{2}(y - \tan^{-1}x^3)^2$. It satisfies Condition (III) with $\phi(y - h(x)) = |y - \tan^{-1}x^3|$ and $\alpha_2 = 1$. The set $L_B = \{x \in B_x, y \in B_y | W(x,y) \le \frac{1}{2}(a - \frac{\pi}{4})^2\}$ is included in the domain of attraction of the boundary-layer system for every fixed x. Finally, it can be verified that the inequalities of Condition (III) hold with $c_1 = 2.64$, $c_2 = 6$, $\beta_1 = 2.356$, $K_1 = K_2 = 0$. So, $\varepsilon^*(d)$ is given by

$$\varepsilon^*(d) = \frac{1}{2.64 + [2.356(1 - d) + 6d]^2/4d(1 - d)} .$$

The largest upper bound $\varepsilon^* = 0.0596$ is obtained with the choice $d^* = \beta_1/(\beta_1 + \beta_2) = 0.282$ leading to a Lyapunov function

$$\nu(x,y) = 0.718\ V(x) + 0.282\ W(x,y)$$

A sketch of the corresponding estimate of the domain of attraction, as given by (2-9), is shown in Fig. (2). On the other hand, the largest set L^* contained in L_R and L_B is obtained with the choice $d = v_0/(v_0+w_0)$, d = 0.448 leading to a Lyapunov function

$$\nu(x,y) = 0.552\ V(x) + 0.448\ W(x,y).$$

The corresponding upper bound on ε is $\varepsilon^*(0.448) = 0.0534$. A sketch of L^* is shown in Fig. (2); of course, L^* contains the set obtained with $d^* = 0.282$. The price paid for enlarging the estimate of the domain of attraction is a reduction in the upper bound from 0.0596 to 0.0534.

As we pointed out in the introduction, there has been previous work on developing stability criteria for singularly perturbed systems using Lyapunov's method. Closely related to our work is the work of Chow [6] and that of Grujic [5]. We pause here to compare our work with the previous ones. Chow studied the stability of a special class of singularly perturbed systems in which f and g are linear in y. The stability criterion derived by Chow is quadratic in nature in the sense that the derivative of the Lyapunov function is bounded by a quadratic form. Motivated by the work of Chow, on one hand, and the work of Araki [10], on the other hand, we thought that a criterion similar to Chow's could be derived if we start with quadratic-type Lyapunov functions and perform composite stability analysis in a way similar to [9,10]. That has been done successfully [11] for the class of systems in which f and g are linear in y. This paper extends the results of [11] to systems in which f and g are nonlinear in y. The examples worked out in [11] show that our criterion produces estimates of the domain of attraction and of the upper bound on the perturbation parameter which are less conservative compared to Chow's criterion. So, our stability criterion is an extension and generalization of Chow's criterion and includes it as a special case. Grujic's criterion, however, is different from ours. There are two reasons for the difference. First, Grujic uses linear-type Lyapunov functions while we use quadratic-type Lyapunov functions. Second, different conditions are imposed on the interaction between the slow and fast variables. Generally speaking, we require stronger smoothness properties on f and g than those required by Grujic. As a result, our criterion is satisfied asymptotically as $\varepsilon \to 0$. This is not the case in Grujic's criterion. Beside requiring ε to be less than a certain upper bound ε^*, Grujic's criterion has a second requirement on the constants that appear in the inequalities that correspond to our Conditions (I)-(III). If that second requirement is not satisifed, the criterion is not satisfied even if $\varepsilon \to 0$. Since the basic philosophy in studying a singularly perturbed system [c.f. 1] is to establish its properties from those of the reduced and boundary-layer systems for sufficiently small ε, we think that our criterion better fits the purpose. To help the reader get a good picture of the difference between the two criteria, we solve an example that has been solved by Grujic [5]. We also make the same choice of the Lyapunov functions.

Example 2: Consider the absolute stability problem for the singularly perturbed system

$$\dot{x} = A_{11}x + A_{12}y + q_1\phi_1(\sigma_1), \tag{2-15}$$

$$\varepsilon\dot{y} = A_{21}x + A_{22}y + q_2\phi_2(\sigma_2), \tag{2-16}$$

where

$$\sigma_1 = c_{11}^T x + c_{12}^T y,$$

$$\sigma_2 = c_{21}^T x + c_{22}^T y.$$

Here $x \in R^n$, $y \in R^m$, $\sigma_i \in R$ and A_{ij}, c_{ij} and q_i are of appropriate dimensions. The nonlinearities $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are one-to-one, continuous, $\phi_i(0) = 0$ and satisfy the sector conditions

$$\frac{\phi_i(\sigma_i)}{\sigma_i} \in [0,k_i], \; i = 1,2, \qquad \forall \sigma_i \in (-\infty,\infty),$$

where $k_i \in (0,\infty)$. It is assumed that A_{11} is a Hurwitz matrix, the pair (A_{11},q_1) is controllable, the matrices $\hat{D}_{22}(0)$ and $\hat{D}_{22}(k_2)$ are negative definite, where

$$\hat{D}_{22}(k) = [A_{22} + k(c_{22}^T y) q_2 c_{22}^T] + [A_{22} + k(c_{22}^T y) q_2 c_{22}^T]^T$$

and that

$$c_{ii}^T A_{ii}^{-1} q_i \geq 0, \qquad i = 1,2.$$

The following numerical values are considered

$$A_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},\ q_1 = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix},\ c_{11} = \begin{pmatrix} -0.01 \\ 0.0 \end{pmatrix},\ A_{12} = I,$$

$$C_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ k_1 = 2,\ A_{21} = 10^{-3} I,\ C_{21} = \begin{pmatrix} 10^{-3} \\ 0 \end{pmatrix},\ K_2 = 1$$

$$A_{22} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix},\ q_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ c_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Grujic [5] studied the asymptotic stability of the origin for the given numerical values. For the reduced system

$$\dot{x} = A_{11} x + q_1 \phi_1 (c_{11}^T x)$$

he used a Lyapunov function $[V(x)]^{1/2}$ where

$$V(x) = x^T H_1 x + \theta \int_0^{c_{11}^T x} \phi_1(\sigma) d\sigma, \tag{2-17}$$

with $H_1 = \frac{1}{20} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $\theta = 1$.

For the boundary-layer system

$$\frac{dy}{d\tau} = A_{22} y + q_2 \phi_2 (c_{22}^T y)$$

he used a Lyapunov function $[W(y)]^{1/2}$ where

$$W(y) = y^T y \tag{2-18}$$

He verified that his stability criterion is satisfied for $\varepsilon < 0.52$. Let us consider applying our stability criterion to the same problem. In order to have a meaningful comparison with Grujic's solution, we use his choice of Lyapunov functions. The only difference is that we use $V(x)$ and $W(y)$ as Lyapunov functions rather than $V^{1/2}(x)$ and $W^{1/2}(y)$ since we work with quadratic-type Lyapunov functions. It can be verified that Conditions (I) and (II) of our criterion are satisfied with $\alpha_1=0.1$, $\psi(x) = \|x\|$, $\alpha_2 = 3.838$ and $\phi(y) = \|y\|$ (notice that in this problem $h(x) \equiv 0$). To verify Condition (III) we need to impose a stronger smoothness condition on the nonlinearities $\phi_i(\cdot)$. We require that $\phi_1(\cdot)$ and $\phi_2(\cdot)$ satisfy a Lipschitz condition, namely,

$$|\phi_i(\sigma_i) - \phi_i(\bar{\sigma}_i)| \leq L_i|\sigma_i - \bar{\sigma}_i|. \tag{2-19}$$

This condition is not needed in Grujic's work but it is essential in ours. With (2-19), it can be verified that Condition (III) is satisfied with $c_1 = c_2 = 0$, $\beta_1 = 0.3416 + 0.02\ L_1$, $K_1 = 0$ and $K_2 = 2\text{x}10^{-3}(1 + \sqrt{2}\ L_2)$. By Corollary 1, the origin is asymptotically stable for all $\varepsilon < \varepsilon^*$, where

$$\varepsilon^* = \frac{561.72}{(1 + 0.059\ L_1)(1 + \sqrt{2}\ L_2)} \tag{2-20}$$

From (2-20), we see that for values of L_1 and L_2 of order ten, or even a hundred, the upper bound obtained by our criterion is better than the 0.52 obtained by Grujic's criterion. Although we require a stronger smoothness condition on the nonlinearities, for a wide class of nonlinearities that satisfy our condition we get a less conservative result. This, however, is not the important point. The important difference is illustrated below. Let us resolve this problem for the same numerical values except for A_{21} and C_{21} which will be taken as $A_{21} = 0.1\ I$ and

$C_{21} = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}$ instead of $A_{21} = 10^{-3}$ I and $C_{21} = \begin{pmatrix} 10^{-3} \\ 0 \end{pmatrix}$, i.e. we allow for a stronger interaction between the slow and fast variables. Since changing the values of A_{21} and C_{21} does not affect the reduced and boundary-layer systems, the same choice of the Lyapunov functions will be used. It can be verified that Grujic's criterion is not satisfied because, in his notation, the condition $\zeta_1 + \xi_1 < 1$ does not hold. On the other hand, using our criterion we conclude that the origin is asymptotically stable for all $\varepsilon < \varepsilon^*$ where ε^* is given by

$$\varepsilon^* = \frac{5.61}{(1 + 0.059\, L_1)(1 + \sqrt{2}\, L_2)}.$$

Once Conditions (I)-(III) hold, our criterion is satisfied for sufficiently small ε irrespective of the numerical values of the constants appearing in the inequalities.

We conclude our discussion of the asymptotic stability of autonomous systems by pointing out some ideas that may lead to less conservative results.

Remark 2: In stating Conditions (I) and (II), the comparison functions $\psi(\cdot)$ and $\phi(\cdot)$ are not required to be positive-definite functions. It is merely required that they vanish only at the equilibrium points. As a result of that, the inequalities of Condition (III) may be satisfied with some of the constants on the right hand side being negative numbers. Such a situation will be the exception not the rule because in general one has to use norm inequalities in order to verify Condition (III). However, if it happens, it can be used to get less conservative results. It can be verified that Theorem 1 holds with the exception that we should say: for all ε satisfying

$$\frac{1}{\varepsilon} > \frac{1}{\varepsilon^*(d)} \tag{2-21}$$

instead of saying $\forall\varepsilon < \varepsilon^*(d)$ because $\varepsilon^*(d)$ could be negative, in which case the inequality (2-21) will be always satisfied since ε is positive.

<u>Example 3</u>: Consider the second-order system

$$\dot{x} = x - x^3 + y,$$

$$\varepsilon\dot{y} = -x - y.$$

With the choice $V(x) = \frac{1}{4}x^4$ and $W(x,y) = (1/2)(x+y)^2$, it can be verified that Conditions (I) to (III) hold with $\psi(x) = x^3$, $\phi(y - h(x)) = x + y$, $\alpha_1 = 1$, $\alpha_2 = 1$, $c_1 = 1$, $c_2 = -1$, $\beta_1 = 1$, $K_1 = K_2 = 0$. Thus, $\varepsilon^*(d)$ is given

$$\varepsilon^*(d) = \frac{1}{1 + \frac{1}{4d(1 - d)}[(1 - d) - d]^2}$$

The choice $d = \frac{1}{2}$ maximizes $\varepsilon^*(d)$ and yields $\varepsilon^* = 1$. So, the origin is aymptotically stable for all $\varepsilon < 1$. If we allow only for nonnegative numbers, we can take $\psi(x) = |x|^3$ and $\phi(y - h(x)) = |x + y|$. We will get the same numbers except $c_2 = 1$. Then,

$$\varepsilon^*(d) = \frac{1}{1 + \frac{1}{4d(1 - d)}[(1 - d) + d]^2}$$

The choice $d = \frac{1}{2}$ maximizes $\varepsilon^*(d)$ and yields $\varepsilon^* = 0.5$, which is more conservative.

The situation when we work with negative numbers is similar to the work of Michel and Miller [9] on studying stability of interconnected systems allowing for negative bounds on the interconnections.

<u>Remark 3</u>: The requirement $\psi(x) = 0$ in B_x iff $x = 0$ can be replaced by requiring that $\psi(0) = 0$ and the set $S = \{x | \psi(x) = 0\}$ does not contain a nontrivial trajectory of the reduced system.

Remark 4: Instead of using one comparison function for each of the reduced and boundary-layer systems, more than one comparison function may be used. This would be similar to the interconnected system stability criterion of [15]. In the synchronous machine example of section 4 we consider a case when two comparison functions are used with the boundary-layer system.

2.2 Autonomous Systems: Exponential Stability

Exponential stability result for (2-1) can be obtained by requiring stronger conditions. Suppose that Conditions (I)-(III) hold with comparision functions $\psi(\cdot)$ and $\phi(\cdot)$ which belong to class $\mathcal{K}$ functions [16], i.e., they are continuous and strictly increasing in B_x and B_y. In addition, suppose that V(x) and W(x,y) satisfy the inequalities

(IV) $e_1\psi^2(x) \leq V(x) \leq e_2\psi^2(x), \qquad \forall x \in B_x,$

(V) $e_3\phi^2(y - h(x)) \leq W(x,y) \leq e_4\phi^2(y - h(x)), \qquad \forall x \in B_x,\ \forall y \in B_y.$

Then, the conclusion of Theorem 1 holds with exponential stability replacing asymptotic stability. This follows from the fact that the composite Lyapunov function (2-6) and its derivative along the trajectories of (2-1) will have the same rate of growth. One case of particular interest is the case when $\psi(x) = \| x \|$ and $\phi(y - h(x)) = \| y - h(x) \|$. For this case, an estimate of the degree of exponential stability is obtained in Theorem 2.

Theorem 2: Suppose that Conditions (I)-(V) hold with $\psi(x) = \| x \|$ and $\phi(y - h(x)) = \| y - h(x) \|$, then the origin (x = 0, y = 0) is an exponentially stable equilibrium point of (2-1). Let α be any positive number such that $0 < \alpha < \alpha_1/2e_2$, then α is an estimate of the degree of exponential stability for all $\varepsilon < \varepsilon_1^*(\alpha)$ where $\varepsilon_1^*(\alpha)$ is given by

$$\varepsilon_1^*(\alpha) = \frac{(\alpha_1 - 2\alpha e_2)\alpha_2}{(\alpha_1 - 2\alpha e_2)(\gamma + 2\alpha e_4) + \beta_1\beta_2} \tag{2-22}$$

Proof: Consider a composite Lyapunov function $\nu(x,y)$ as in (2-6). Similar to the proof of Theorem (1), it can be shown that

$$\dot{\nu} \leq - \begin{pmatrix} \|x\| \\ \|y-h(x)\| \end{pmatrix}^T \tilde{T} \begin{pmatrix} \|x\| \\ \|y-h(x)\| \end{pmatrix} -2\alpha\, \nu$$

where

$$\tilde{T} = \begin{pmatrix} (1-d)(\alpha_1 - 2\alpha e_2) & -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 \\ -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 & d(\frac{\alpha_2}{\varepsilon} - \gamma - 2\alpha e_4) \end{pmatrix}$$

System (2-1) will be exponentially stable with degree α if $\tilde{T}$ is positive definite, which is the case for all $\varepsilon < \varepsilon_1^*(d,\alpha)$ where

$$\varepsilon_1^*(d,\alpha) = \frac{(\alpha_1 - 2\alpha e_2)\alpha_2}{(\alpha_1 - 2\alpha e_2)(\gamma + 2\alpha e_4) + [\beta_1(1-d) + \beta_2 d]^2/4d(1-d)}.$$

Recalling the discussion of section 2.1 on the choice of d, we see that $\varepsilon_1^*(d,\alpha)$ is maximum when $d = d^* = \beta_1/(\beta_1 + \beta_2)$. This choice of d yields $\varepsilon_1^*(\alpha)$ as given by (2-22).

In Theorem 2, the number $\alpha_1/2e_2$ is the estimate of the degree of exponential stability of the reduced system using the Lyapunov function V(x). The restriction $\alpha < \alpha_1/2e_2$ means that the estimate of the degree of exponential stability of the overall singularly perturbed system has to be less than the estimate for the reduced system, which is natural. It is interesting that any $\alpha \epsilon (0, \alpha_1/2e_2)$ is an estimate of the degree of exponential stability for sufficiently small ε. The upper bound $\varepsilon_1^*(\alpha)$ belongs to $(0,\varepsilon^*)$ with $\varepsilon_1^*(\alpha) \to \varepsilon^*$ as $\alpha \to 0$ while $\varepsilon_1^*(\alpha) \to 0$ as

$\alpha \to \alpha_1/2e_2$. When ε is known, the expression (2-22) can be used to obtain the estimate α as illustrated by the following example.

Example 4: Consider the stiff network shown in Fig. (3)

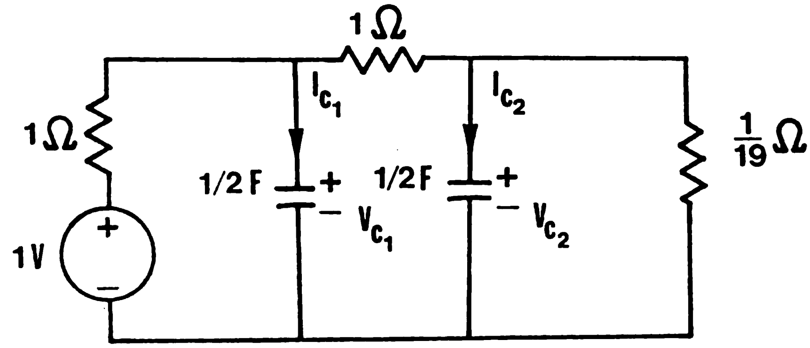


Fig. (3)

The maximum time constant of this network can be estimated using Lyapunov functions as the reciprocal of the estimate of the degree of exponential stability. This was done in [17] using a general Lyapunov function that does not take the stiffness into consideration. The estimated maximum time constant was 0.5 seconds which is within a factor of 2 of the actual maximum time constant (the actual time constants are $\tau_1 = 0.256$ and $\tau_2 = 0.025$). This estimate is good taking into consideration that it is based on a general algorithm. However, if stiffness is taken into consideration and Lyapunov functions are formed from singular perturbation point of view it is natural to expect better estimates. Theorem 2 will be employed to demonstrate that this is actually the case. Let $\bar{v}_{c1}$ and $\bar{v}_{c2}$ be the equilibrium values of v_{c1} and v_{c2}, respectively. Define

$x = v_{c1} - \bar{v}_{c1}$, $y = v_{c2} - \bar{v}_{c2}$ to get state equations in the form

$$\dot{x} = -4x + 2y$$

$$\varepsilon\dot{y} = 0.2x - 4y$$

where $\varepsilon = 0.1$. It can be verified that with the choice $V(x) = x^2$ and

$W(x,y) = (y - 0.05x)^2$, Conditions (I)-(V) are satisfied with $\psi(x) = |x|$, $\phi(y - h(x)) = |y - 0.05x|$, $\alpha_1 = 7.8$, $\alpha_2 = 8$, $c_1 = 0.2$, $c_2 = 0.39$, $\beta_1 = 4$, $K_1 = K_2 = 0$, $e_1 = e_2 = e_3 = e_4 = 1$. From (2-22) we get

$$\varepsilon_1^*(\alpha) = \frac{8(7.8 - 2\alpha)}{(7.8 - 2\alpha)(0.2 + 2\alpha) + 1.56}$$

Since ε is known to be 0.1, we choose α such that $\varepsilon_1^*(\alpha)$ is greater than 0.1. For example $\alpha = 3.8759$ yields $\varepsilon_1^* = 0.1979$ which is acceptable, so $\alpha = 3.8759$ is an estimate of the degree of exponential stability. The corresponding estimate of the maximum time constant is 0.258 seconds which is very close to the actual maximum time constant 0.256.

2.3 Nonautonomous Systems

The results of sections (2.1) and (2.2) for autonomous systems can be extended to nonautonomous systems with some additional assumptions. In order to identify the needed additional assumptions, we give the nonautonomous version of Theorem 1.

Consider the nonautonomous system

$$\dot{x} = \tilde{f}(t,x,z), \qquad (2\text{-}23a)$$

$$\varepsilon\dot{z} = \tilde{g}(t,x,z,\varepsilon), \qquad (2\text{-}23b)$$

where $t \in R_+$ and $(x = 0,\ z = 0)$ is the unique equilibrium point in $B_x \times B_z$. In this case it is more convenient to start by using a transformation

$$y = z - h(t,x) \qquad (2\text{-}24)$$

where $h(t,x)$ is the unique root satisfying

$$0 = \tilde{g}(t,x,h(t,x),0). \qquad (2\text{-}25)$$

It is assumed that $h(t,z)$ is continuously differentiable and that there is a nondecreasing real-valued function $p : R_+ \to R_+$ such that

$$\|h(t,x)\| \leq p(\|x\|) \quad \forall t \in R_+,\ \forall x \in B_x \qquad (2\text{-}26)$$

Hahn [18] showed that when (2-26) is satisfied h preserves uniform asyptotic stability. The transformed system is given by

$$\dot{x} = f(t,x,y) \tag{2-27a}$$

$$\varepsilon\dot{y} = g(t,x,y,\varepsilon) \tag{2-27b}$$

where

$$f(t,x,y) \triangleq \tilde{f}(t,x,y + h(t,x)),$$

and

$$g(t,x,y,\varepsilon) \triangleq \tilde{g}(t,x,y + h(t,x),\varepsilon) - \varepsilon \frac{\partial}{\partial t} h(t,x)$$

$$- \varepsilon(\nabla_x h(t,x))^T f(t,x,y + h(t,x))$$

It is sufficient to study uniform asymptotic stability of the equilibrium point of (2-27), namely, $(x = 0,\ y = 0)$. The reduced system is obtained by setting $\varepsilon = 0$ in (2-27b) to get $y = 0$ which, when substituted in (2-27a), results in

$$\dot{x} = f(t,x,0) \triangleq f_r(t,x). \tag{2-28}$$

The boundary-layer system is defined as

$$\frac{dy}{d\tau} = g(t,x,y(\tau),0) \tag{2-29}$$

where $t \in R_+$ and $x \in B_x$ are treated as fixed unknown parameters. We assume that there are positive-definite, decrescent[2] Lyapunov functions $V(t,x)$ and $W(t,x,y$, and class $\mathcal{K}$ functions $\psi(x)$ and $\phi(y)$ such that the following equalities hold $\forall t \in R_+$, $\forall x \in B_x$, $\forall y \in B_y$:

---

[2] $V(t,x)$ is said to be positive-definite if there is a strictly increasing continous real-valued function $v'(\|x\|$ such that $v'(0) = 0$ and $0 < v'(\|x\|) \leq V(t,x)\ \forall x \in B_x - \{0\}$, and is said to be decrescent if there is a strictly increasing continous real-valued function $v''(\|x\|)$ such that $v''(0) = 0$, $v''(x) > 0$ and $v(t,x) < v''(\|x\|)\ \forall x \in B_x - \{0\}$.

(I') $\frac{\partial}{\partial t} V(t,x) + (\nabla_x V(t,x))^T f_r(t,x) \leq -\alpha_1 \psi^2(x) \qquad \alpha_1 > 0,$

(II') $(\nabla_y W(t,x,y))^T g(t,x,y,0) \leq -\alpha_2 \phi^2(y), \qquad \alpha_2 > 0,$

(III'-a) $\frac{\partial}{\partial t} W(t,x,y) + (\nabla_x W(t,x,y))^T f(t,x,y) \leq c_1 \phi^2(y) + c_2 \psi(x) \phi(y),$

(III'-b) $(\nabla_x V(t,x))^T [f(t,x,y) - f(t,x,0)] \leq \beta_1 \psi(x) \phi(y),$

and

(III'-c) $(\nabla_y W(t,x,y))^T [g(t,x,y,\varepsilon) - g(t,x,y,0)] \leq \varepsilon K_1 \psi^2(y) + \varepsilon K_2 \psi(x) \phi(y).$

<u>Theorem 3</u>: Suppose that Conditions (I')-(III') hold; let d be any positive number such that $0 < d < 1$, and let $\varepsilon^*(d)$ be

$$\varepsilon^*(d) = \frac{\alpha_1 \alpha_2}{\alpha_1 \gamma + [\beta_1(1 - d) + \beta_2 d]^2 / 4d(1 - d)}$$

where $\beta_2 = K_2 + C_2$ and $\gamma = K_1 + C_1$,

then the origin ($x = 0$, $y = 0$) is a uniformly asymptotically stable equilibrium point of (2-27) and

$$\nu(t,x,y) = (1 - d)V(t,x) + dW(t,x,y)$$

is a Lyapunov function for (2-27).

3. Stability Criteria for Linear Systems

Consider the linear singularly perturbed system

$$\dot{x}(t) = A_{11}(t)x(t) + A_{12}(t)y(t), \qquad (2\text{-}30a)$$

$$\varepsilon\dot{y}(t) = A_{12}(t)x(t) + A_{22}(t)y(t), \qquad (2\text{-}30b)$$

where $x \in R^n$, $y \in R^m$ and A_{22} is nonsingular for all $t > 0$. In particular,

$$|\det(A_{22}(t)| \geq c_3 > 0, \qquad t > 0 \qquad (2\text{-}31)$$

It is well known [c.f., 19] that if the reduced system

$$\dot{x}(t) = A_o(t)x(t), \qquad (2\text{-}32)$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21},$$

is uniformly asymptotically stable and the boundary-layer system

$$\frac{dy}{d\tau} = A_{22}(t)y \tag{2-33}$$

is asymptotically stable uniformly in t, i.e.,

$$\text{Re } \lambda(A_{22}(t)) \leq -c_4 < 0 \quad \forall t > 0, \tag{2-34}$$

then the singularly perturbed system (2-30) is uniformly asymptotically stable for sufficiently small ε. A problem of practical significance is the determination of an upper bound ε^* such that the uniform asymptotic stability of (2-30) is guaranteed for all $\varepsilon < \varepsilon^*$. Zien obtained ε^* for time-invariant systems [12] and Javid obtained it for time-varying systems [13]. In both cases, ε^* was obtained in terms of the transition matrices $\phi_0(t,s)$ and $\phi_2(\tau,\sigma)$ of the reduced and boundary-layer systems, respectively. In [12] and [13], ε^* was computed by writing explicit expressions for the solution of (2-30). The same ε^* of [12, 13] can be derived using composite Lyapunov stability analysis with linear-type Lyapunov functions of the form

$$V(t,x) = \int_t^{\infty} \| \phi_0(\tau,t)x \| \, d\tau$$

for the reduced system, and a similar Lyapunov function for the boundary-layer system. In this section we apply the quadratic-type Lyapunov stability criterion of section 2 to compute ε^* and compare our ε^* with the previous ones.

3.1 Time-Invariant Systems

Consider (2-30) with constant matrices. Let Re $\lambda(A_0) < 0$, Re $\lambda(A_{22}) < 0$ and let $P_0 > 0$ and $P_2 > 0$ be the solutions of the Lyapunov equations

$$P_oA_o + A_o^TP_o = -2I_n, \tag{2-35}$$

$$P_2A_{22} + A_{22}^TP_2 = -2I_m. \tag{2-36}$$

It can be verified that

$$V(x) = \frac{1}{2}\, x^TP_ox \tag{2-37}$$

and

$$W(x,y) = \frac{1}{2}\,(y + A_{22}^{-1}A_{21}x)^TP_2(y + A_{22}^{-1}A_{21}x) \tag{2-38}$$

satisfy the conditions of Theorem 1 with $\psi(x) = \| x \|$ and $\phi(y-h(x)) = \| y + A_{22}^{-1}A_{21}x \|$. It follows that ε^* is given by

$$\varepsilon^* = \frac{1}{\gamma + \beta_1\beta_2} \tag{2-39}$$

where

$$\gamma = \| P_2A_{22}^{-1}A_{21}A_{12} \|, \tag{2-40}$$

$$\beta_1 = \| P_oA_{12} \|, \tag{2-41}$$

and

$$\beta_2 = \| P_2A_{22}^{-1}A_{21}A_o \|. \tag{2-42}$$

The upper bound obtained by Zien [12] is given by

$$\varepsilon^* = \frac{1}{M_2M_3 + M_4} \tag{2-43}$$

where

$$M_2 = \int_0^\infty \|\exp(A_ot)A_{12}\|\, dt, \tag{2-44}$$

$$M_3 = \int_0^\infty \|\exp(A_{22}t)A_{22}^{-1}A_{21}A_o\|\, dt, \tag{2-45}$$

and

$$M_4 = \int_0^\infty \|\exp(A_{22}t)A_{22}^{-1}A_{21}A_{12}\| \, dt \tag{2-46}$$

We observe that computing ε^* of (2-39) is much easier than computing ε^* of (2-43) because it requires merely solving algebraic Lyapunov equations; there is no need for finding transition matrices or performing integrations. To compare the two upper bounds, recall that P_0 and P_2 can be expressed as

$$P_0 = 2\int_0^\infty (\exp(A_0t))^T(\exp(A_0t))dt, \tag{2-47}$$

$$P_2 = 2\int_0^\infty (\exp(A_{22}t))^T(\exp(A_0t))dt. \tag{2-48}$$

Substituting (2-47) and (2-48) in (2-40)-(2-42) shows a great similarity between β_1, β_2 and γ on one hand and M_2, M_3 and M_4 on the other hand. There are two differences. First, the integrals in (2-40)-(2-42) depend quadratically on the transition matrices while those in (2-44)-(2-46) depend linearly on them. Second, in (2-40)-(2-42) integration is performed first followed by norm computations, while in (2-44)-(2-46) norms are computed first followed by integration. While we cannot make a general statement about the comparison between the two upper bounds, we expect, in view of the above differences, that ε^* of (2-39) will be less conservative in most cases than that of (2-43), especially in high-dimensional problems. This is the case in the example given below which was solved by Zien [12].

Example 5: Let

$$A_{11} = \begin{pmatrix} -0.2 & +0.2 \\ 0 & -0.5 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 & 0 \\ -k & 0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad k\epsilon[0,10]$$

ε^* of (2-39) is $\varepsilon^* = 0.2935$ while that of (2-43) is $\varepsilon^* = 0.0826$.

3.2 Time-Varying Systems

Consider the time-varying system (2-30) and suppose that the reduced system (2-32) is uniformly asymptotically stable and the boundary-layer system (2-33) is asymptotically stable uniformly in t. Let $P_o(t) > 0$ be the solution of the Lyapunov differential equation

$$-\dot{P}_o(t) = P_o(t)A_o(t) + A_o^T(t)P_o(t) + 2I, \qquad (2\text{-}49)$$

and $P_2(t) > 0$ be the solution of the algebraic Lyapunov equation

$$P_2(t)A_2(t) + A_2^T(t)P_2(t) = -2I_m. \qquad (2\text{-}50)$$

Condition (2-34) quarantees that $\inf_{t\geq 0} \lambda_{min}(P_2(t)) \geq c_5 > 0$. Following the treatment of nonautonomous systems that has been presented in section 2.3, we start by using a transformation

$$\tilde{y}(t) = y(t) + A_{22}^{-1}(t)A_{21}(t)x(t) \triangleq y(t) + L(t)x(t). \qquad (2\text{-}51)$$

The transformed system is

$$\dot{x}(t) = A_o(t)x(t) + A_{12}(t)\tilde{y}(t), \qquad (2\text{-}52a)$$

$$\varepsilon\dot{\tilde{y}}(t) = \varepsilon[\dot{L}(t) + L(t)A_o(t)]x(t) + [A_{22}(t) + \varepsilon L(t)A_{12}(t)]\tilde{y}(t). \qquad (2\text{-}52b)$$

It is assumed that $L(t)$ is uniformly bounded for all $t > 0$. This guarantees that (2-26) is satisfied. Therefore, it is sufficient to study the stability of (2-52). Moreover, it is assumed that A_o, A_{12}, A_{22}, $\dot{A}_{22}$ and $\dot{L}$ are uniformly bounded for all $t > 0$. This implies that $P_o(t)$, $P_2(t)$ and $\dot{P}_2(t)$ are uniformly bounded for all $t > 0$. Now, it can be verified that

$$V(t,x) = \frac{1}{2} x^T P_0(t) x \tag{2-53}$$

and

$$W(t,y) = \frac{1}{2} \tilde{y}^T P_2(t) \tilde{y} \tag{2-54}$$

satisfy the following conditions of Theorem 3 with $\psi(x) = \| x \|$ and $\phi(\tilde{y}) = \| \tilde{y} \|$. It follows that ε^* is given by

$$\varepsilon^* = \frac{1}{\gamma + \beta_1 \beta_2} \tag{2-55}$$

where

$$\gamma = \sup_t [\frac{1}{2} \| \dot{P}_2(t) \| + \| P_2(t) L(t) A_{12}(t) \|] , \tag{2-56}$$

$$\beta_1 = \sup_t [\| P_0(t) A_{12}(t) \|], \tag{2-57}$$

and

$$\beta_2 = \sup_t [\| P_2(t)(\dot{L}(t) + L(t) A_0(t)) \|]. \tag{2-58}$$

Based on arguments similar to those of the time-invariant case, it is expected that the upper bound ε^* given by (2-55) will, in most cases, be less conservative than the one given by Javid [13].

4. A Synchronous Machine Example

The stability criterion of section 2 will be employed to study the stability of a synchronous generator connected to an infinite bus. There has been quite a bit of literature on the use of Lyapunov's method to analyze transient stability of power systems (for a survey see [20] and for recent developments see [21]). In the standard approach, a generator is represented by a second-order model, usually referred to as the classical model, where the state variables are δ and $\omega = \dot{\delta}$; δ being the angle between the generated voltage and some reference. The need for higher models has been recognized [20] and several studies that employ higher order models have been conducted. Most of these studies are restricted to one machine connected to an infinite bus. The simplest higher order model that can be considered is a third-order model, known as the one-axis model, in which a field-flux decay E_q' is taken into consideration. Siddiqee [14] (see also [22]) studied the stability of one generator connected to an infinite bus using the one-axis model and came up with a Lyapunov function that can be used to estimate the domain of attraction. More recently, Sasaki [23] criticized Siddiqee's work on the basis that it did not take into consideration that the state variables have different speeds, and proposed an alternative approach. Between the work of Siddiqee and the criticism of Sasaki we found an interesting problem where the use of singular perturbations may be useful.

We consider a synchronous generator connected to an infinite bus through a transmission line (Fig. (4)).

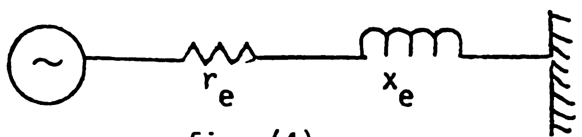


fig. (4)

The one-axis model (see [14], [20] or [23]) is given by

$$\tau_{do}'\dot{E}_q' = -\frac{(x_d + x_e)}{(x_d' + x_e)} E_q' + \frac{(x_d - x_d')}{(x_d' + x_e)} V \cos \delta + E_{FD}, \qquad (2\text{-}59a)$$

$$\dot{\delta} = \omega, \qquad (2\text{-}59b)$$

$$M\dot{\omega} = -D\omega + P_m - \frac{E_q'V}{(x_d' + x_e)} \sin \delta, \qquad (2\text{-}59c)$$

where

τ'_{do} = direct-axis open circuit transient time constant;

x_d = direct-axis synchronous reactance of the generator;

x'_d = direct-axis transient reactance of the generator;

E'_d = instantaneous voltage proportional to field-flux linkage;

δ = angle between voltage of infinite bus and E'_q;

D = damping coefficients of the generator;

E_{FD} = field voltage (assumed constant);

P_m = mechanical input to the generator (assumed constant);

ω = slip velocity of the generator;

x_e = reactance of the transmission line;

V = voltage of infinite bus.

In writing equation (2-59) we have employed some standard assumptions, namely that the transmission line resistance is negligible, a second order harmonic term on the right hand side of (2-59c), which results from saliency, is negligible and that $x'_d \simeq x'_q$ (q-axis synchronous reactance). Had the field-flux decay been neglected, E'_q would have been constant and equations (2-59b) and (2-59c) would give the classical model. Siddiqee [14] found a Lyapunov function, which will be given later in this section, and used it to estimate the domain of attraction. The knowlege of the domain of attraction can then be used to estimate the critical clearance time, which we briefly define by means of a simple example. Consider Fig. (5) in which a single machine is connected to an infinite bus through two parallel transmission lines A and B. Suppose that a short circuit fault takes place on line B. Equation (2-59) is written for the post-fault system with line B disconnected and an estimate of the domain of attraction is obtained. Trajectories of the faulted system are obtained by integrating equations describing the faulted system starting at the pre-fault conditions. The critical clearance time is taken as the time when the trajectories of the faulted system hit the boundary of the estimated domain of attraction. The idea is that if the fault is cleared any time before the critical clearance time, the state of the system will be inside the domain of attraction, hence the system trajectory will approach the stable equilibrium point.

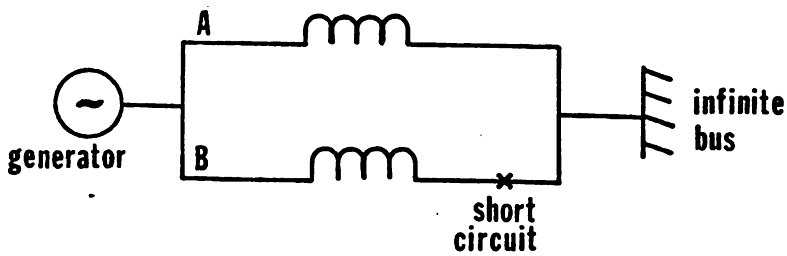


fig. (5)

Sasaki's criticism of Siddiqee's work was based on the observation that the time constant τ_{do}' is large relative to the critical clearance time (typically, τ_{do}' is greater than five seconds while the critical clearance time is less than one second). Therefore, E_q' will move slowly and changes in E_q' during the fault will be small (typically, less than 0.3 per unit). He suggested that treating E_q' as a state variable in estimating the domain of attraction might not be appropriate since extending the domain of attraction in the direction of the E_q'-axis, which would be at the expense of its extension in the direction of the δ-axis, would not be beneficial in computing the critical clearance time. He has proposed an alternative method in which equations (2-59b) and (2-59c) are used as a model for the machine but with E_q' treated as a varying parameter. He, then, gave a Lyapunov function that should work for all the values of E_q' and used it to estimate the domain of attraction. Sasaki's method, however, is not theoretically justified because he overlooked that changes in E_q' would result in changes in the equilibrium point of (2-59b) and (2-59c) and did not take those changes into consideration. It is interesting, however, that Sasaki's viewpoint of treating the slow variable as a varying parameter is identical to the way the boundary-layer system is defined in our singular perturbation method.

We will apply our stability criterion of section 2 to study the stability of the equilibrium of (2-59). The first task is to bring (2-59) into the singularly perturbed form. Let $\bar{E}_q'$, $\bar{\delta}$ and $\bar{\omega}$ denote the stable equilibrium point of (2-59). The state variables are taken as $x = E_q' - \bar{E}_q'$, $y_1 = \delta - \bar{\delta}$ and $y_2 = \omega - \bar{\omega}$, so that x will be the slow variable. In order to have the singularly perturbed form (1), we define ε as $\varepsilon = 1/\tau_{do}'$ and change the time scale from t to t/τ_{do}' to obtain

$$\dot{x} = -ax + b[\cos(y_1 + \bar{\delta}) - \cos \bar{\delta}] \triangleq f(x,y) \tag{2-60a}$$

$$\varepsilon\dot{y} = y_2 \triangleq g_1(x,y) \tag{2-60b}$$

$$\varepsilon\dot{y} = -\lambda y_2 - c[(1 + x)\sin(y_1 + \bar{\delta}) - \sin \bar{\delta}] \triangleq g_2(x,y) \tag{2-60c}$$

where

$$a = \frac{x_e + x_d}{x_e + x_d'},\ b = \frac{x_d + x_d'}{x_e + x_d'},\ c = \frac{1}{M(x_e + x_d')},\ \lambda = \frac{D}{M}.$$

The reduced system is obtained by setting $\varepsilon = 0$ in (2-60) to get

$$y = h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} = \begin{pmatrix} \sin^{-1}\left(\frac{\sin \bar{\delta}}{1+x}\right) - \bar{\delta} \\ 0 \end{pmatrix} \tag{2-61}$$

which when substituted in (2-60a) yields

$$\dot{x} = -ax + b[\cos(h_1(x) + \bar{\delta}) - \cos\bar{\delta}] \overset{\Delta}{=} -ax + bN(x) \tag{2-62}$$

The boundary-layer system is defined by

$$\frac{dy_1}{d\tau} = y_2, \tag{2-63a}$$

$$\frac{dy_2}{d\tau} = -\lambda y_2 + c[(1 + x)\sin(y_1 + \bar{\delta}) - \sin\bar{\delta}] \tag{2-63b}$$

In order for the reduced and boundary-layer systems to be well defined, as well as for other reasons that will be apparent later, we restrict x and y_1 to a region defined by the following inequalities

$$x \geq -\theta \tag{2-64a}$$

$$-(\pi + h_1(x) + 2\bar{\delta}) \leq y_1 \leq \pi - (h_1(x) + 2\bar{\delta}) \tag{2-64b}$$

$$[y_1 - h_1(x)][\sin(y_1 + \bar{\delta}) - \sin(h_1(x) + \bar{\delta})] \geq \eta|y_1 - h_1(x)|^2 \tag{2-64c}$$

where the positive numbers η and θ will be specified later; θ should satisfy

$$\left|\frac{\sin\bar{\delta}}{1 - \theta}\right| < 1$$

To study the stability of the reduced system (2-62) we choose

$$v(x) = \int_0^x [a\sigma - bN(\sigma)]d\sigma \tag{2-65}$$

as a Lyapunov function. From the definition of $N(x)$ it follows that $N(0) = 0$, $xN(x) > 0$ and

$$\left|\frac{\partial N}{\partial x}\right| = \left| \frac{1}{\sqrt{(1 + x)^2 - (\sin\bar{\delta})^2}} \left(\frac{\sin\bar{\delta}}{1 + x}\right)^2 \right| \leq K_N$$

where

$$K_N = \frac{1}{\sqrt{(1-\theta)^2 - (\sin\bar{\delta})^2}} \left(\frac{\sin\bar{\delta}}{1-\theta}\right)^2 .$$

We assume that the positive number θ can be chosen such that $K_N < (\frac{a}{b})$. If follows that $V(x)$ is positive-definite. The derivative of $V(x)$ along the trajectory of (2-62) is given by

$$V = \frac{\partial V}{\partial x}(-ax + bN(x)) = -(ax - bN(x))^2 .$$

Therefore, Condition (I) is satisfied with $\psi(x) = |ax - bN(x)|$ and $\alpha_1 = 1$. Next, we consider the boundary-layer system (2-63). This system has equilibrium at $\bar{y}_1 = h_1(x)$ and $\bar{y}_2 = 0$. We choose a Lure-type Lyapunov function $W(x,y)$ given by

$$W(x,y) = \frac{\lambda}{2}(y_1 - h_1(x)^2 + y_2(y_1 - h_1(x)) + \frac{n}{2\lambda} y_2^2 + \frac{nc}{\lambda}(1 + x)\int_{h_1(x)}^{y_1} M_x(\sigma)d\sigma \qquad (2\text{-}66)$$

where $n > 1$ and

$$M_x(\sigma) = \sin(\sigma + \bar{\delta}) - \sin(h_1(x) + \bar{\delta}).$$

Inequalities (2-64a) and (2-64b) guarantee that $W(x,y)$ is positive-definite. The derivative of $W(x,y)$ along the trajectories of (2-63) is given by

$$\frac{dW}{d\tau} = (\nabla_y W(x,y))^T g(x,y) = -(n-1)y_2^2 - c(1+x)(y_1 - h_1(x))M_x(y_1).$$

Using inequality (2-64c) we get

$$\frac{dW}{d\tau} \leq -(n-1)y_2^2 - c(1-\theta)n|y_1 - h_1(x)|^2$$

A comparison function for the boundary-layer system is taken as

$$\phi(y - h(x)) = \begin{pmatrix} |y_1 - h_1(x)| \\ \xi |y_2| \end{pmatrix}$$

where ξ is an arbitrary positive number that plays an important role in improving the upper bound. With the above choice of the comparison function and with the choice $\eta = 1 + c(1 - \theta)\eta\xi^2$, Condition (II) is satisfied with $\alpha_2 = c(1 - \theta)\eta$. It can be shown that Conditions (III-a) and (III-b) hold and we have

$$c_1 = \frac{1}{\xi}\left(\frac{1}{2} bK_1 + bK_1\lambda\xi\right), \quad c_2 = \frac{1}{\xi}\sqrt{K_1^2 + \xi^2\left(\lambda K_1 + \frac{nc}{\lambda}\right)^2}, \quad \beta_1 = b$$

where K_1 is an upper bound on $\frac{\partial h_1}{\partial x}$, which is given by

$$K_1 = \frac{\sin\bar{\delta}}{(1 - \theta)\sqrt{(1 - \theta)^2 - (\sin\bar{\delta})^2}}$$

Thus, all the conditions of Theorem 1 hold and we conclude that the equilibrium point of (2-59) is asymptotically stable for all $\varepsilon < \varepsilon^*(d)$ for any $d \in (0,1)$ where

$$\varepsilon^*(d) = \frac{c(1 - \theta)\eta\xi}{\frac{1}{2} bK_1 + bK_1\lambda\xi + \frac{1}{4d(1 - d)\xi}\left[b(1 - d)\xi + d\sqrt{K_1^2 + \xi^2\left(\lambda K_1 + \frac{nc}{\lambda}\right)^2}\right]^2}. \tag{2-67}$$

Moreover, $\nu(x,y)$, given by

$$\nu(x,y) = (1 - d)\int_0 [a\sigma - bN(\sigma)]d \;+ d[\tfrac{\lambda}{2}(y_1 - h_1(x))^2 + y_2(y_1 - h_1(x)) + \frac{n}{2\lambda} y_2^2 + \frac{nc}{\lambda}(1 + x)\int_{h_1(x)}^{y_1} M_x(\sigma)d\sigma], \quad (2\text{-}68)$$

is a Lyapunov function.

In (2-67) and (2-68), there are four unspecified positive parameters, namely, d, θ, η and ξ. The first three should be chosen to satisfy certain restrictions that were imposed throughout the derivation, while the fourth parameter ξ is arbitrary. In making these choices we are guided by the requirement that $\varepsilon^*(d)$ (given by (2-67)) should be greater than ε, and by our desire to obtain a large estimate of the domain of attraction. Let us illustrate that by a numerical example. Consider the numerical data used by Siddiqee [14]: $M = 147 \times 10^{-4}$, $x_d' = 0.3$, $x_e = 0.4$, $x_d = 1.15$, $\tau_{do}' = 6.6$, $P_m = 0.815$ and $E_{FD} = 1.22$, where M is in per unit power second squared per radian, τ_{do}' is in seconds and all other parameters are per unit (pu). In addition, we take $\lambda = \frac{D}{M} = 4\ (\text{Sec})^{-1}$. Choosing $d = 0.01$, $\theta = 0.42$, $\eta = 0.35$ and $\xi = 0.1$ yields $\varepsilon^*(0.01) = 0.1844$ which is acceptable since $1/\tau_{do}' = 0.1515$. Hence $\nu(x,y)$ is given by

$$\nu(x,y) = 0.99\, V(x) + 0.01\, W(x,y)$$

An estimate of the domain of attraction can be obtained by means of closed Lyapunov surfaces as in (2-9). However, a better estimate can be obtained by means of open Lyapunov surfaces [24]. Let C and D be two points on the line $x = -\theta$ such that for any point on the line CD the derivative $\dot{x}$ is positive and let $C_{max} = \min\{\nu(x_C,y_C),\ \nu(x_D y_D)\}$.

It can be shown that the set

$$\mathscr{L} = \{x,y \mid \nu(x,y) \leq C_{max} \text{ and } x \geq -\theta\}$$

is included in the domain of attraction. A sketch of the set $\mathscr{L}$ is shown in Fig. (6); shown also is a sketch of the domain of attraction obtained by Siddiqee's Lyapunov function. From Fig. (6), it is apparent that we succeeded in getting a better domain of attraction in the direction of the y_1-axis at the expense of reducing its size in the direction of the x-axis. The choice of the parameters d, θ, η and ξ, that we used here, was obtained after several trials. Further improvement is possible.

Let us use this example to explore another angle of using a singular perturbation decompositions in Lyapunov stability analysis. In the stability criterion of Theorem 1, singular perturbations have been employed for two different purposes. First, they were used to define separate reduced and boundary-layer systems whose Lyapunov functions were used to form a candidate Lyapunov function for the full system. This process simplifies the search for Lyapunov functions since it is done at the level of the lower-order subsystems. Second, the singularly perturbed form of the system together with inequalities (I)-(III) were used to show that the derivative of the candidate Lyapunov function is bounded by a negative upper bound. However, even if inequalities (I)-(III) do not hold, computing the derivative of the candidate Lyapunov function, obtained via singular perturbation decomposition, along the trajectories of the full system might still show that it is a valid Lyapunov function. Such computations will, in general, be difficult to carry out but in some special cases that might be possible. In the case of the synchronous machine example in-hand a candidate Lyapunov function for the boundary-layer system could be the energy-type Lyapunov function given by

$$\tilde{W}(x,y) = \frac{1}{2} y_2^2 + c(1 + x) \int_{h_1(x)}^{y_1} M_x(\sigma) d\sigma. \tag{2-69}$$

This function, though, does not satisfy the conditions of Theorem 1 since its derivative along the solution of boundary-layer systems is only negative semi-definite so that Condition II is not satisfied. However, we may still consider

$$\tilde{\nu}(x,y) = (1 - d)V(x) + d\tilde{W}(x,y) \tag{2-70}$$

as a candidate Lyapunov function. It is interesting to note that computing the derivative of $\tilde{\nu}$ along the trajectories of the full system (2-60) shows that there is a unique choice of d, namely $d = \frac{b}{b+c}$, which makes ν negative semidefinite. Moreover, it can be checked that the trivial solution is the only solution of $\dot{\tilde{\nu}} = 0$. Thus $\tilde{\nu}$ is a Lyapunov function for all $\varepsilon > 0$. What is more interesting is to observe that $\tilde{\nu}$ with the choice $d = \frac{(b+c)}{b} M$ is the Lyapunov function used by Siddiqee [14]. In comparing the two Lyapunov functions ν and $\tilde{\nu}$ we note that in $\tilde{\nu}$ there is no freedom in forming the composite Lyapunov function while in $\nu(x,y)$ the parameter d is free and can be used to generate a family of composite Lyapunov functions; that is why we were able to pick d that gave us a better shape for the domain of attraction. The price paid for that freedom in choosing d is the restriction of the range of ε to $(0, \varepsilon^*(d))$. Getting an acceptable $\varepsilon^*(d)$ is hinged on having $\lambda = D/M$ not too small. With the numerical values we used here we need $\lambda \geq 1$.

To illustrate the effect of λ, Fig. (7) shows estimates of the domain of attraction for $\lambda = 1,2$ and Table (1) gives corresponding values of d, θ, η, ξ and $\varepsilon^*(d)$.

λ	d	θ	η	ξ	$\varepsilon^*(d)$
1	0.0072	0.38	0.544	0.05	0.15186
2	0.007	0.4	0.45	0.05	0.169267

Table (1)

As λ gets smaller the improvement of the estimate of the domain of attraction over Siddiqee's result becomes less significant, because the choices of the parameters d, θ, η and ξ are primarily directed towards improving $\varepsilon^*(d)$. The requirement $\lambda \geq 1$ is realistic. Although for uncontrolled machines, when D accounts only for field damping, a typical value of λ would be as low as 0.2 [25], for a feedback controlled machine, in which D accounts for damping introduced by power system stabilizers, typical values of λ would be as high as 10 [26].

Appendix: Proof of Lemma 1

The proof of Lemma 1 is similar to the proof of the well-known Converse Theorem [27] except that extra work is needed to verify condition (III). Let $\mathscr{X}(t,x)$ be the trajectory of the reduced system (3) starting at initial point x at time t = 0, and let $s(\tau,y;x)$ be the trajectory of the boundary-layer system (4) starting at initial point y at time t = 0. By assumption, we have

$$\|\mathscr{X}(t,x)\| \leq c_3 \| x \| e^{-c_4 t}, \qquad \forall x \in B_x \tag{A-1}$$

and

$$\|s(\tau,y;x) - h(x)\| \leq c_5 \| y - h(x) \| e^{-c_6 \tau}, \quad \forall (x,y) \in B_x x B_y. \tag{A-2}$$

Consider conceptual Lyapunov functions V(x) and W(x,y) defined by

$$V(x) = \int_0^T \|\mathscr{X}(t,x)\|^2 dt \tag{A-3}$$

and

$$W(x,y) = \int_0^T \|s(\tau,y;x) - h(x)\|^2 d\tau. \tag{A-4}$$

It follows from the Converse Theorem [27] that

$$(\nabla_x V(x))^T f_r(x) \leq -\alpha_1 \| x \|^2, \qquad \forall x \in B_x, \tag{A-5}$$

$$\|\nabla_x V(x)\| \leq e_1 \|x\|, \qquad \forall x \in B_x, \tag{A-6}$$

$$(\nabla_y W(x,y))^T g(x,y) \leq -\alpha_2 \| y - h(x) \|^2, \quad \forall (x,y) \in B_x x B_y, \tag{A-7}$$

and

$$\|\nabla_y W(x,y)\| \leq e_2 \|y - h(x)\|, \qquad \forall (x,y) \in B_x x B_y. \tag{A-8}$$

Therefore, Conditions (I) and (II) are satisfied. To verify inequality (IIIa) consider

$$(\nabla_x W(x,y))^T f(x,y) = 2\int_0^T [s(\tau,y;x) - h(x)]^T[\nabla_x(s(\tau,y;x) - h(x))]f(x,y)d\tau.$$

Using that f, g and h are continuously differentialbe on the closed bounded set $B_x x B_y$, which implies that $\nabla_x(s - h)$ is uniformly bounded, we get that

$$(\nabla_x W(x,y))^T f(x,y) \leq (\mathcal{K}_1 \| y - h(x)\| + \mathcal{K}_2 \|x\|)\int_0^T \|s(\tau,y;x) - h(x)\| \, d\tau \qquad \text{(A-9)}$$

where f(x,y) was taken as

$$f(x,y) = f(x,y) - f(x,h(x)) + f(x,h(x)). \qquad \text{(A-10)}$$

Using (A-2) in (A-9) yields

$$(\nabla_x W(x,y))^T f(x,y) = \frac{c_5}{c_6}(1 - e^{-c_6 T})[\mathcal{K}_1 \| y - h(x)\|^2 + \mathcal{K}_2 \|x\| \, \|y - h(x)\|]$$

Hence, inequality (III-a) holds. Finally, (III-b) follows form (A-6) and the continuous differentiability of f w.r.t. y.

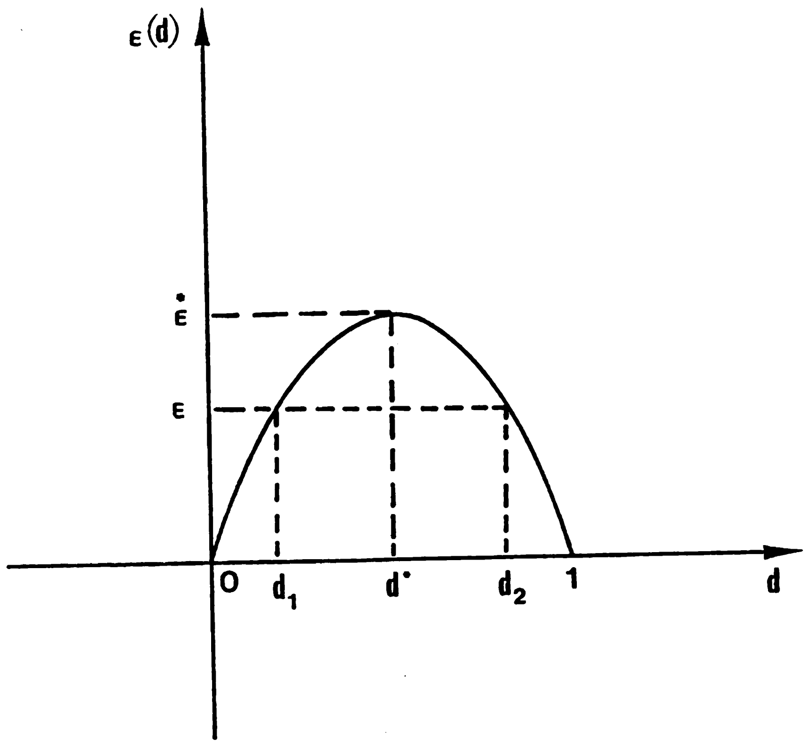


Fig. (1)

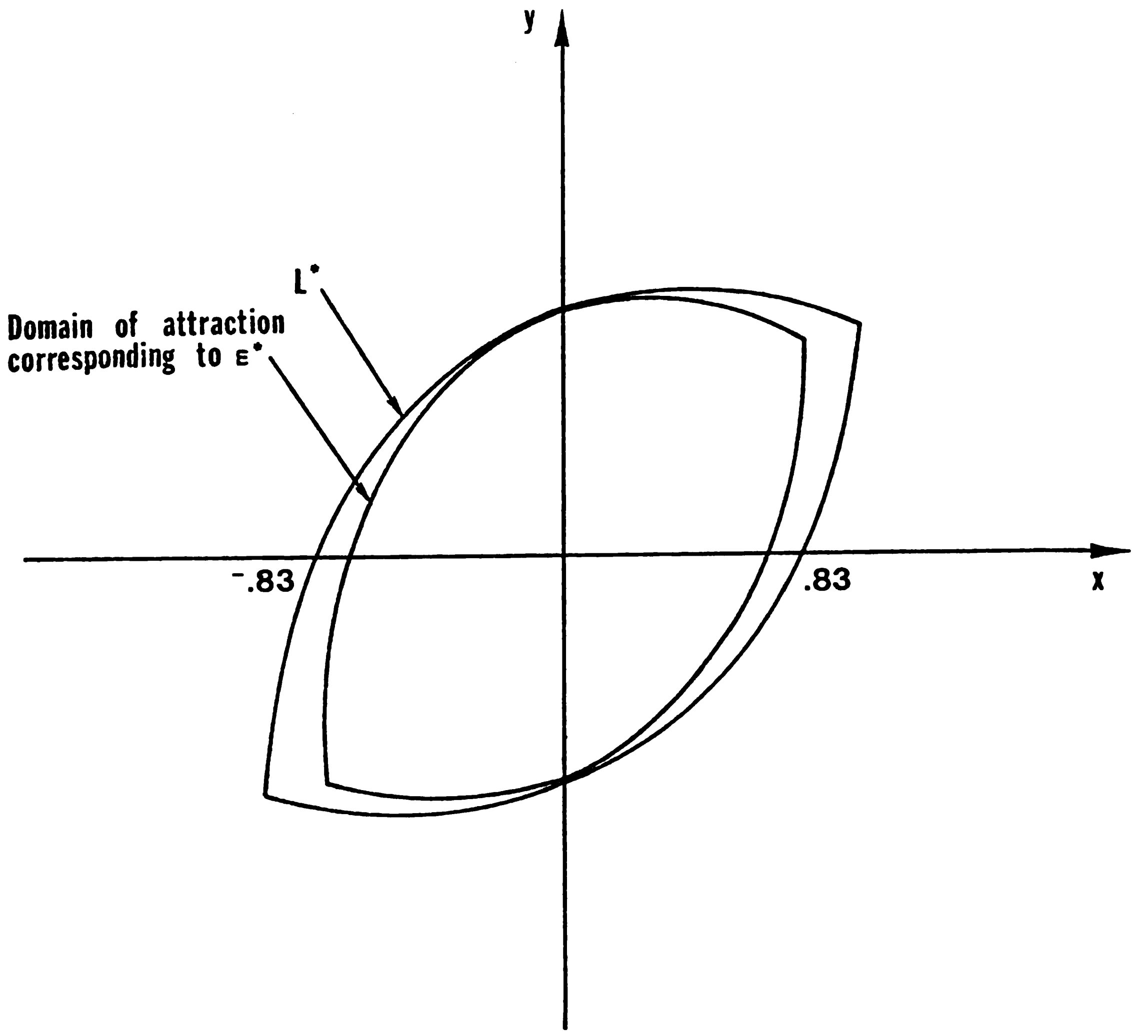
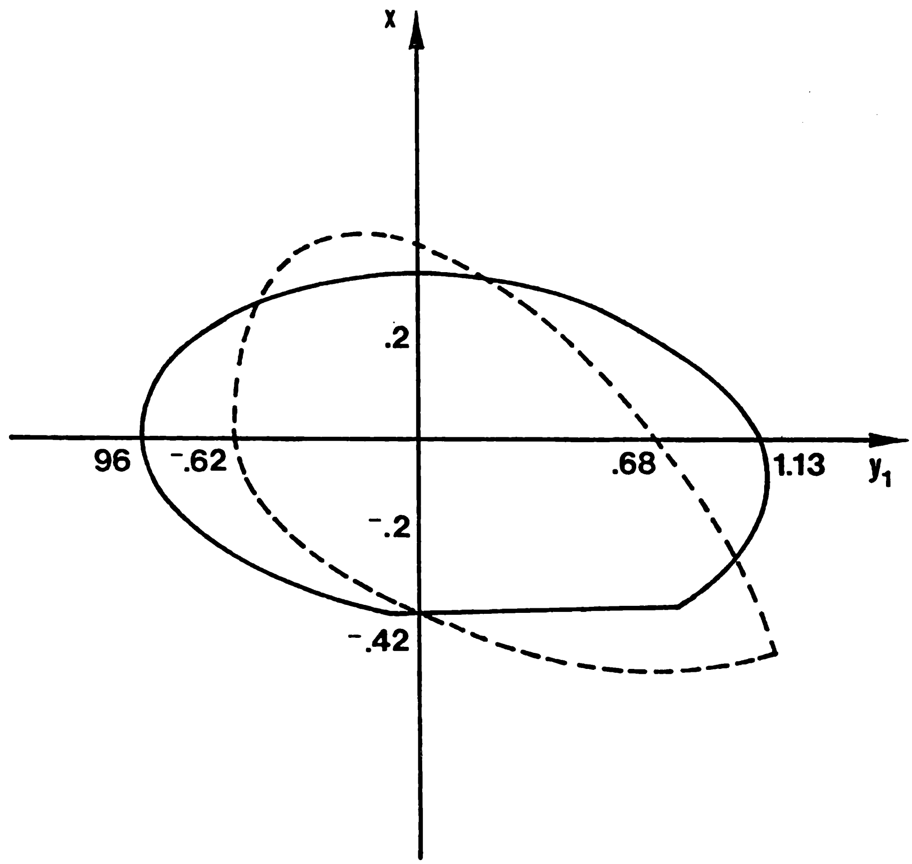


Fig. (2)



———— Our domain of attraction

- - - - - - - Siddiquee's domain of attraction

Fig. (6)

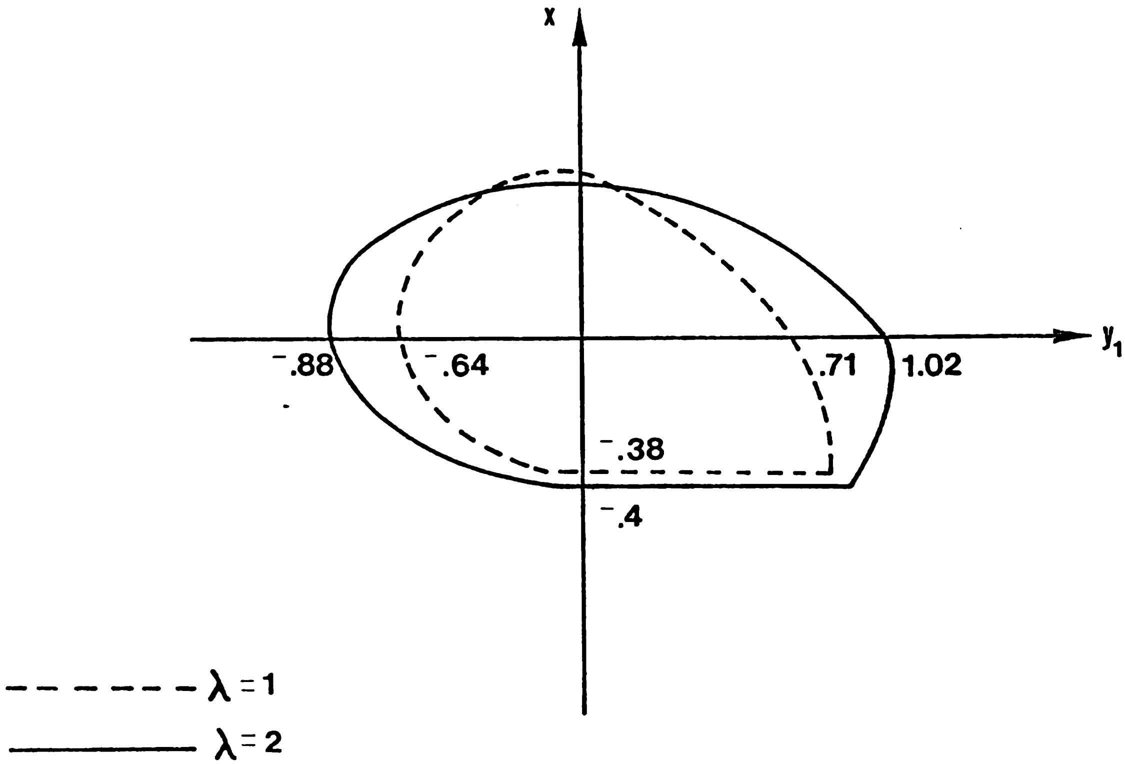


Fig. (7)

CHAPTER III

CLOSENESS OF THE TRAJECTORIES OF THE SINGULARLY PERTURBED SYSTEM TO THE TRAJECTORIES OF ITS SLOW AND FAST SUBSYSTEMS

1. Introduction

Simplifying a mathematical model of a physical system has been an old engineering practice. For instance in circuit theory parasitic reactances are often freely omitted from circuit models to form an approximate model. The reason for such a simplication is that parasitics not only increase the complexity of the system but often lead to equations with widely separated time constants. Such systems, called stiff differential equations, may introduce severe computational difficulties, [28-29] requiring the use of an implicit integration scheme. In practice the justification of this engineering simplification lies in the fact that "it works". However, it happens in some cases that it leads to results not only unprecise but even qualitatively incorrect. Therefore it becomes necessary to verify the validity of such an approximation.

Consider a physical system described by a nonlinear singularly perturbed system

$$\dot{x} = f(x,y) \qquad x(t_0) = x_0 \tag{3-1a}$$

$$\varepsilon\dot{y} = g(x,y,\varepsilon) \qquad y(t_0) = y_0 \tag{3-1b}$$

where ε is small positive parameter. An approximate model is defined by ignoring parasitics, that is setting $\varepsilon = 0$ and removing initial condition $y(t_0) = y_0$ to obtain

$$\dot{x} = f(x,y) \qquad x(t_0) = x_0 \tag{3-2a}$$

$$0 = g(x,y,0) \tag{3-2b}$$

The justification for such an approximation on a finite interval of time was first studied by Tihonov [30], corrected later on by Hoppenstead [31], and was extended to the semi-infinite interval of time [32]. There has been some other work in this area (eg. [33], [34], [35]) which basically treat the same problem with different settings and generalizations. However, the result of Tihonov-Hoppenstead, [30], [32] remains as a fundamental result, so called initial value theorem, in the singular perturbation theory. In [30], [32] a set of sufficient conditions is given under which, as $\varepsilon \to 0^+$, the solution of (3-1) converges to the solution of (3-2). This convergence of course cannot happen over the whole interval, finite or infinite, due to discrepancy of initial conditions of (3-1) and (3-2). Therefore, a boundary layer correction should be provided to take care of the initial finite jump.

In this chapter we show that our stability result in chapter 2 provides, as a by product, an initial value theorem result. That is our stability requirements guarantee the uniform convergence of the solutions of (3-1) to the solutions of (3-2) with a boundary layer correction. Our results unlike the results of [30], [32], [35] are not local, that is a set of initial conditions is given so that for trajectories originating from initial points in this set convergence property hold.

Therefore, in a sense it generalizes the result of Tihonov-Hoppensteadt, [30], [32] and Habets [35].

2. Problem Formulation and Main Results

Consider an initial-value problem of the form

$$\dot{x} = f(x,y) \qquad x \in B_x^1 \subset R^n \qquad x(t_0) = x_0 \tag{3-3a}$$

$$\varepsilon\dot{y} = g(x,y,\varepsilon) \qquad y \in B_y \subset R^m \qquad y(t_0) = y_0 \tag{3-3b}$$

where f and g are assumed to be continuous functions and the origin ($x = 0$, $y = 0$) is the unique equilibrium point in $B_x x B_y$. Our purpose is to investigate the behavior of the solution of (3-3), denoted by ($x_\varepsilon(t)$, $y_\varepsilon(t)$), as $\varepsilon \to 0^+$ for $t_0 \leq t < \infty$. In studying the behavior of the solution of (3-3) two associated systems, so called reduced and boundary layer systems, are defined. The reduced system is obtained by formally setting $\varepsilon = 0$, and removing the initial condition $y(t_0) = y_0$ in (3-3). This gives

$$\dot{x} = f(x,y) \qquad x(t_0) = x_0 \tag{3-4a}$$

$$0 = g(x,y,0) \tag{3-4b}$$

Assuming that in B_x and B_y (3-4b) has a unique continuous root $y = h(x)$, the reduced system is rewritten as

$$\dot{x} = f(x,h(x)) \triangleq f_r(x) \qquad x(t_0) = x_0 \tag{3-5}$$

Moreover we assume (35) has a solution on $[t_0,\infty)$ which we denote by $x_s(t)$. A boundary-layer system is obtained by stretching the time scale through transformation $\varepsilon = \dfrac{t - t_0}{\varepsilon}$ and then setting $\varepsilon = 0$ in the result.

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[1]The smybol B_x indicates a closed sphere centered at $x = 0$, B_y is defined in the same way.

This gives

$$\frac{dy}{d\tau} = g(x, y(\tau), 0) \qquad y(0) = y_0 \tag{3-6}$$

where x is treated as a parameter which takes value in B_x. We also assume that the solution of (3-6) for $x = x_0$ exists on $[t_0,\infty)$ and is denoted by $y_f(\tau,x_0)$.

We assume all the conditions of our stability result in chapter 2, namely I, II and III, hold with comparison functions $\psi(\cdot)$ and $\phi(\cdot)$ which belong to class $\mathcal{K}$ functions. Moreover we suppose that $L_R = \{x\epsilon B_x | V(x) \leq v_0\}$ is in the domain of attraction of the reduced system and $L_B(x) = \{y\epsilon B_y | W(x,y) \leq w_0\}$ is in the domain of attraction of the boundary-layer system, where from the conditions of our stability theorem the existence of w_0 independent of x is guaranteed. Then the set

$$L = \{(x,y)\ B_x\ x\epsilon B_y |\quad \nu(x,y) \leq \min((1-d)v_0 + d\ w_0)\} \tag{3-7}$$

is in the domain of attraction of the full system (3-3) where $\nu(x,y)$ is the composite Lyapunov function defined in chapter 1.

Our first result is given by the following theorem.

Theorem 1: Suppose $(x_0, y_0)\epsilon L$. Then, as $\varepsilon \to 0^+$, $x_\varepsilon(t)$ converges uniformly to $x_s(t)$ on $[t_0,\infty)$ and $y_\varepsilon(t)$ converges uniformly to $h(x_s(t))$ on all closed subset of $[t_0,\infty)$.

Proof: The proof of this theorem consists of three steps. In the first step we use the Lyapunov function W to define a "tube" in $[t_0,\infty)$ x $B_x x B_y$ which is invariant with respect to the solution of (3-3) for sufficiently small ε. In the next step we show that if $(x_0, y_0)\epsilon L$ then for sufficiently small ε the solution of (3-3) intersects the "tube" defined in the first step at an arbitrary time $t_1 > t_0$. Finally in the last step we show convergence of $x_\varepsilon(t)$ and $y_\varepsilon(t)$ to $x_s(t)$ and $h(x_s(t))$ respectively.

Step 1;

Lemma 1: Given a positive number $\widetilde{w}_0 < w_0$ there exist ε_0^* such that for any $\varepsilon < \varepsilon_0^*$ the tube described by;

$$\Gamma_{\widetilde{w}_0} = \{(x,y)\ B_x \times B_y \mid W(x,y) \leq \widetilde{w}_0 \leq w_0\}$$

is invarant with respect to the solution of (3-3), that is, any solution $(x_\varepsilon(t),\ y_\varepsilon(t))$ which meets $\Gamma_{\widetilde{w}_0}$ cannot leave it there after.

Proof: Since our stability conditions hold, there exist ε^* so that for $\varepsilon < \varepsilon^*$ we can assume $x_\varepsilon(t)$ is inside a ball, that is there exists $r > 0$ such that $||x_\varepsilon(t)|| < r$. Next we consider the boundary of the set

$$\partial\Gamma_{\widetilde{w}_0} = \{(x,y)\in B_x \times B_y \mid W(x,y) = \widetilde{w}_0\}.$$

On the boundary $\partial\Gamma_{\widetilde{w}_0}$ we have

$$\begin{aligned}
\dot{W} &= (\nabla_x W(x,y))^T f(x,y) + (\nabla_y W(x,y))^T(\tfrac{1}{\varepsilon} g(x,y,\varepsilon)) \\
&\leq c_1\ \phi^2(||y-h(x)||) + c_2\ \psi(||x||)\ (||y-h(x)||) - \frac{\alpha_2}{\varepsilon}\ \phi^2(||y-h(x)||) \\
&\quad + K_1\ \phi^2(||y-h(x)||) + K_2\ \psi(||x||)\ (||y-h(x)||) \\
&= (c_1 + K_1 - \frac{\alpha_2}{\varepsilon})\phi^2(||y-h(x)||) + (c_2+K_2)\psi(||x||)\ (||y-h(x)||)
\end{aligned}$$

Let us define

$$\ell = \inf\ \{||y - h(x)|| \mid W(x,y) = \widetilde{w}_0\}$$

notice that $\ell > o$. Next we define

$$\varepsilon_1^* = \frac{\alpha_2}{c_1 + K_1 + \frac{1}{\phi(\ell)}(c_2\ \psi(r) + K_2\ \psi(r))}, \tag{3-8}$$

$$\varepsilon_0^* = \min(\varepsilon^*, \varepsilon_1^*). \tag{3-9}$$

Now for any $\varepsilon < \varepsilon_0^*$ we have

$$\dot{W} \leq -\frac{(c_2+K_2)\psi(r)}{\phi(\ell)} \phi^2(\|y-h(x)\|) + (c_2+K_2)\ (\|x\|)\ (\|y-h(x)\|)$$

$$\leq -(c_2+K_2)\psi(r)\phi(\|y-h(x)\|) + (c_2+K_2)\psi(\|x\|)\phi(\|y-h(x)\|) \leq 0$$

and the result follows Q.E.D.

Step 2:

In this step we want to show that for sufficiently small ε the solution of (3-3) in fact meets the tube $\Gamma_{\tilde{W}_0}$. To show this it is sufficient to show that for some arbitrary time t_1 close to t_0, we can make $\|y_\varepsilon(t_1) - h(x\ (t_1))\|$ arbitrarily small by choosing ε small enough. We need the following lemma;

Lemma 2: Let $\eta > 0$ and $t_1^* > t_0$, then for sufficiently small ε we have

$$\|y_\varepsilon(t_1) - h(x\ (t_1))\| \leq \eta \quad \text{for some } t_0 < t_1 < t_1^*.$$

Proof: We start by applying a time transformation $\tau = \dfrac{t-t_0}{\varepsilon}$ to (3-3) and we get

$$\frac{dx}{d\tau} = \varepsilon f(x,y) \qquad \hat{x}_\varepsilon(0) = x_0 \tag{3-10a}$$

$$\frac{dy}{d\tau} = g(x,y,\varepsilon) \qquad \hat{y}_\varepsilon(0) = y_0 \tag{3-10b}$$

where $\hat{x}_\varepsilon(\tau)$ and $\hat{y}_\varepsilon(\tau)$ denote the solution of (3-10). First observe that $\hat{x}_\varepsilon(\tau) = x_\varepsilon(t)$ and $\hat{y}_\varepsilon(\tau) = y_\varepsilon(t)$. Next we consider the boundary layer system at x_0, namely,

$$\frac{dy}{d\tau} = g(x_0,\ y(\tau),\ 0) \qquad y_f(0,\ x_0) = y_0 \tag{3-11}$$

By our assumption in stability theorem, (3-11) is uniformly asymptotically stable. Therefore there exist $T(\eta)$ such that

$$|y_f(\tau, x_0) - h(x_0)| \leq \eta/3 \qquad \eta > T(\eta). \tag{3-12}$$

(since $y_0 \epsilon L_B(x_0)$). Now we fix τ_1 so that $\tau_1 > T(\eta)$. From continuous dependence of solutions of differential equations on parameters [36, page 24, theorem 3.4], it follows that, for sufficiently small ε, $x_\varepsilon(\tau)$ and $y_\varepsilon(\tau)$ exist and are continuous with respect to ε. Thus there exists ε_2^* such that for $\varepsilon < \varepsilon_2^*$ we have:

$$\|\hat{y}_\varepsilon(\tau_1) - y_f(\tau_1, x_0)\| \leq \eta/3 \tag{3-13}$$

Now observe that $\tau_1 = \dfrac{t_1 - t_0}{\varepsilon}$, so given that τ_1 is fixed we can always choose ε sufficiently small so that t_1 corresponding to τ_1 be arbitrarily close to t_0. Namely let t_1^* be given then there exist ε_3^* such that for $\varepsilon < \varepsilon_3^*$ we have $t_0 < t_1 < t_1^*$. From continuity of h, it follows that there exist ε_4^* so that for $\varepsilon < \varepsilon_4^*$ we have,

$$\|h(x_\varepsilon(t_1)) - h(x_0)\| \leq \eta/3 \tag{3-14}$$

Let $\varepsilon_5^* = \min(\varepsilon_0^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*)$, then for $\varepsilon < \varepsilon_5^*$ we have

$$\begin{aligned}\|y_\varepsilon(t_1) - h(x_\varepsilon(t_1)\| &\leq \|y_\varepsilon(t_1) - h(x_0)\| + \|h(x_\varepsilon(t_1)) - h(x_0)\| \\ &\leq \|y_\varepsilon(t_1) - y_f(\tau_1, x_0)\| + \|y_f(\tau_1, x_0) - h(x_0)\| \\ &\quad + \|h(x\ (t_1) - h(x_0)\| \\ &\leq \eta/3 + \eta/3 + \eta/3 = \eta\end{aligned}$$

where the last step followed from (3-12), (3-13) and (3-14) and this conclude the proof of lemma 2.

Our conclusion from steps 1 and 2 is summarized in the following lemma.

Lemma 3: Let $\eta > 0$ and $t_1^* > t_0$ be given. Then there exist ε_5^* so that for $\varepsilon < \varepsilon_5^*$

$$\|y_\varepsilon(t) - h(x_\varepsilon(t))\| \le \eta \quad \text{for } t_1^* < t < \infty$$

Proof: It follows from lemma 1 and lemma 2 by choosing $\tilde{w}_0$ sufficiently small.

Step 3:

In this step we are at the position to prove convergence result. First, since the reduced system is uniformly asymptotically stable, there exist $\hat{T}(\eta)$ such that

$$\|x_s(t)\| \le \eta/2 \qquad \text{for } \hat{T}(\eta) < t < \infty$$

Also uniform asymptotic stability of (3-3) implies that there exist $\tilde{T}(\eta)$ such that

$$\| x \ (t)\| \le \eta/2 \qquad \text{for } \tilde{T}(\eta) < t < \infty$$

Let

$$T'(\eta) = \max \{\hat{T}(\eta), \tilde{T}(\eta)\},$$

then we have

$$\|x_\varepsilon(t) - x_s(t)\| < \eta \quad \text{for } T'(\eta) < t < \infty \tag{3-15}$$

Next we observe that $x = x_\varepsilon(t)$ satisfies

$$\dot{x} = f(x, y_\varepsilon(t)), \quad x(t_0 + \varepsilon\tau_1) = x_0 + \delta(\varepsilon) \tag{3-16}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and τ_1 is as chosen in step 2. By a well-known theorem on continuous dependence of solutions on parameters and on initial conditions, [36, page 24, lemma 3.1.] and in view of the result of step 2 and continuity of f, it follows that $x_\varepsilon(t), \to x_s(t)$, as $\varepsilon \to 0^+$ uniformly on $[t_0, T'(\eta)]$. So in view of (3-15) it follows that $x_\varepsilon(t) \to x_s(t)$, as $\varepsilon \to 0$ uniformly on $[t_0, \infty)$. Having established uniform convergence of $x_\varepsilon(t)$ to $x_s(t)$ on $[t_0, \infty)$, then it follows from step (2) that $y_\varepsilon(t) \to h(x_s(t))$, as $\varepsilon \to 0^+$ uniformly on $[t_1, \infty)$, and this conclude the proof of Theorem 1.

Next we state the result which extend the convergence properties of the solution of (3-3) to the whole interval.

Theorem 2: Suppose all the conditions of Theorem 1 hold. Then $y_\varepsilon(t) \to h(x_s(t)) + y_f(\tau, x_0) - h(x_0)$, as $\varepsilon \to 0^+$, uniformly on $[t_0, \infty)$.

Proof: Let $\eta' > 0$ be given, since (3-11) is uniformly asymptotical stable, there exist $T_1(\eta')$ so that

$$||y_f(\tau, x_0) - h(x_0)|| \leq \eta'/2 \text{ for } \tau > T_1(\eta') \tag{3-17}$$

Now we fix $\tau_1 > T_1(\eta')$ and consider

$$\frac{d\hat{y}}{d\tau} = g\ (\hat{x}_\varepsilon(\tau), y, \varepsilon) \qquad \hat{y}_\varepsilon(0) = y_0 \tag{3-18}$$

Notice that $\hat{y}_\varepsilon(\tau)$ satisfies (3-18). By continuous dependence of the solution on parameters we have $\hat{y}_\varepsilon(\tau) \to y_f(\tau, x_0)$, as $\varepsilon \to 0^+$, uniformly on $[0, \tau_1]$. Furthermore $h(x_s(\varepsilon\tau + t_0)) \to h(x_0)$ as $\varepsilon \to 0^+$ uniformly on $[0, \tau_1]$. So we conclude that there exist ε_9^* so that for $\varepsilon < \varepsilon_9^*$ we have

$$||\hat{y}_\varepsilon(\tau) - y_f(\tau, x_0)|| \leq \eta'/2 \qquad \text{for } 0 \leq \tau \leq \tau_1 \tag{3-19}$$

$$||h(x_s(\varepsilon\tau + t_0)) - h(x_0)|| \leq \eta'/2 \quad \text{for } 0 \leq \tau \leq \tau_1 \tag{3-20}$$

From (3-19) and (3-20) we get

$$||\hat{y}_\varepsilon(\tau) - h(x_s(t_o + \varepsilon\tau)) - y_f(\tau, x_o) + h(x_o)|| \leq \eta' \quad \text{for } 0 \leq \tau \leq \tau_1 \tag{3-21}$$

Having established (3-21), we proceed by observing that from Theorem 1 and (3-21) there exist ε^*_{10} such that for any $\varepsilon < \varepsilon^*_{10}$ we have

$$||\hat{y}_\varepsilon(\tau) - h(\hat{x}_\varepsilon(\tau)) - y_f(\tau, x_o) + h(x_o)|| \leq \frac{3}{2}\eta' \text{ for } 0 \leq \tau \leq \tau_1 \tag{3-22}$$

Next from (3-17) and (3-22) it follows that there exist ε^*_{11} such that for $\varepsilon < \varepsilon^*_{11}$ we have

$$||\hat{y}_\varepsilon(\tau_1) - h(\hat{x}_\varepsilon(\tau_1))|| \leq 2\eta'. \tag{3-23}$$

By choosing η' small enough we can force the point $(\hat{x}_\varepsilon(\tau_1), \hat{y}_\varepsilon(\tau_1))$ to be inside the tube $\Gamma_{\tilde{w}_o}$ with arbitrary $\tilde{w}_o$. Now using Lemma 1 we conclude that for sufficiently small ε, the trajectory $(\hat{x}_\varepsilon(\tau), \hat{y}_\varepsilon(\tau))$ never leaves the tube thereafter. That is, given η there exist ε^*_{12} so that for $\varepsilon < \varepsilon^*_{12}$

$$||\hat{y}_\varepsilon(\tau) - h(\hat{x}_\varepsilon(\tau))|| \leq \eta/2 \text{ for } \tau > \tau_1 \tag{3-24}$$

Finally from (3-17), (3-21), (3-24), and Theorem 1 we conclude that there exist ε^*_{13} such that for $\varepsilon < \varepsilon^*_{13}$ we have

$$||y_\varepsilon(t) - h(x_s(t)) - y_f(\tau, x_o) + h(x_o)|| \leq \eta \qquad \text{for } t_o \leq t < \infty,$$

and this concludes the proof of Theorem 2.

CHAPTER IV

STABILIZATION AND REGULATION-COMPOSITE CONTROL

1. Introduction

One of the most interesting results in the control literature on singular perturbations is the two time scale design of linear regulator problems presented by Chow and Kokotovic [2]. That design introduced the concept of composite control for singularly perturbed systems. Design of a stabilizing or near optimal controller can be broken down into the design of feeback controllers for two separate subsystems, the so called slow and fast subsystems. A composite control is formed by simply adding these two controllers. The advantages of this technique are well known [1]. Stabilizing and near optimal stabilizing feedback designs for a class of nonlinear systems which are linear in the fast state and control input has been already studied by Chow and Kokotovic in [37], [38] and [39]. The contribution of this paper is two fold. First, we extend the two-stage feedback design of Chow and Kokotovic [37] to a wider class of nonlinear systems where nonlinearity in the fast state and control input is permitted. For this result we use a new stability criterion for nonlinear singularly perturbed systems presented in chapter 2. Second, we broaden the context of nonlinear regulator problems for singularly perturbed systems by adopting an approach which is different from that of [37], [38] and [39]. We don't necessarily require optimal controllers for slow subsystems as it is the case in [39].

We start off with a slow controller which can be optimal or near optimal with respect to the slow regulator cost function. Then we design a stabilizing controller for the fast subsystem and form the composite control. The value of the cost function when the composite control is applied to the full order system is studied and shown to be close to the slow regulator cost function. This result, which we call "near performance," is obtained for different cases corresponding to different stability requirements on the slow and fast subsystems. Our results generalize Chow and Kokotovic's result [39] and cover it as a special case. Dropping the optimality requirements of the slow controller is significant because finding optimal control for nonlinear systems is a prohibitively difficult task, while there are ways to generate near optimal controllers. Another feature of our result is giving explicit expressions for the upper bounds on the singular perturbation parameter, ε, under which the near performance results hold. Thus, the near performance results are not restricted to $\varepsilon \to 0$ as is the case with the near optimal result of [39].

The organization of this chapter is as follows. In section 2 we give the composite control design procedure and show its stabilizing property. In section 3 we apply the composite control to the regulator problem. It is shown that the composite control yields a bounded cost; then near performance results are given for three cases. First, we consider the case when both the closed loop slow and fast subsystems are asymptotically stable. Second, the stability requirement on the closed loop fast subsystem is strengthened to exponential stability. Finally, both the slow and fast closed loop subsystems are required to be exponentially stable.

2. Composite Control

Consider the nonlinear singularly perturbed system

$$\dot{x} = f(x,y,u) \qquad x \in B_x \subset R^n, \qquad u \in R^r \tag{4-1a}$$[1]

$$\varepsilon\dot{y} = g(x,y,u,\varepsilon), \qquad y \in B_y \subset R^m, \tag{4-1b}$$

where $\varepsilon > 0$ is a small singular perturbation parameter. We obtain a slow subsystem by formally setting $\varepsilon = 0$ in (4-1) to get

$$\dot{x} = f(x,y,u_s) \tag{4-2a}$$

$$0 = g(x,y,u_s,0) \tag{4-2b}$$

where the subscript, s, for the control input signifies control for the slow subsystem which we refer to as the slow control. Assuming that (4-2b) has a unique root $y = h(x,u_s)$ in the region of our interest, (4-2) can be rewritten as

$$\dot{x} = f(x,h(x,u_s),u_s) \tag{4-3}$$

To derive the fast subsystem or the boundary layer system we assume that the control input to the slow subsystem, u_s, is known as a feedback function of x. We define the fast subsystem as

$$\frac{dy}{d\tau} = g(x,y,u_s + u_f,0) \tag{4-4}$$

where $\tau = t/\varepsilon$ is a stretched time scale. The vector $x \in B_x$ is treated as a fixed unknown parameter and u_f is the control input, which we refer

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[1]The symbol B_x indicates a closed sphere centered at x = 0, B_y is defined in the same way.

to as the fast control. The decomposition of (4-1) into two lower order subsystems is exactly as in [37]. However, due to the non-linearity of (4-1) in the fast state and control input, the fast subsystem (4-4) depends on u_s. It is straight forward to show that if g is linear in y and u, the effect of u_s on the fast subsystem cancels out by shifting equilibrium to y = 0. As it will be seen later, the dependence of the fast subsystem on u_s makes the design procedure sequential in nature.

Design Procedure:

Step 1: Design a slow control $u_s = M(x)$ for the slow subsystem such that the closed loop slow subsystem has a unique asymptotically stable equilibrium point at x = 0 in B_x, and there is a Lyapunov function $V: R^n \to R+$ such that for all $x \in B_x$;

$$(\nabla_x V(x))^T f(x,h(x,M(x)),M(x)) \leq - \alpha_1 \psi^2(x), \qquad \alpha_1 > 0$$

where $\psi(x)$ is a scalar-valued function of x that vanishes at x = 0 and is different from zero for all other $x \in B_x$.

Step 2: With the knowledge of $u_s = M(x)$ design a fast control $u_f = \Gamma(x,y)$ such that

(i) $\Gamma(x,y)$ vanishes at $y = h(x,M(x))$, namely $\Gamma(x,h(x,M(x))) = 0$.

(ii $g(x,y,M(x) + \Gamma(x,y),0) = 0$ has a unique root, $y = H(x)$, in the region of our interest $B_x \times B_y$.

(iii) The closed loop fast subsystem is asymptotically stable uniformly in x and there is a Lyapunov function $W: R^n x R^m \to R_+$ such that for all $x \in B_x$ and $y \in B_y$

$$(\nabla_y W(x,y))^T g(x,y,M(x) + \Gamma(x,y),0) \leq - \alpha_2 \phi^2(y - h(x,M(x)), \alpha_2 > 0,$$

where $\phi(y - h(x,M(x))$ is a scalar-valued function of $(y - h(x,M(x))) \in R^m$ that vanishes at $y = h(x,M(x))$ and is different from zero at all other $x \in B_x$ and $y \in B_y$.

Step 3: Verify the following conditions on the interaction of the slow and fast states.

(a) $(\nabla_x W(x,y))^T f(x,y,M(x) + \Gamma(x,y)) \leq c_1\, \phi^2(y - h(x,M(x))) + c_2\, \psi(x)\, \phi(y - h(x,M(x)))$;

(b) $(\nabla_x V(x))^T [f(x,y,M(x) + \Gamma(x,y)) - f(x,h(x,M(x)), M(x))] \leq \beta_1\, \psi(x)\, \phi(y - h(x,M(x)))$;

(c) $(\nabla_y W(x,y))^T [g(x,y,M(x) + \Gamma(x,y), \varepsilon) - g(x,y,M(x) + \Gamma(x,y),0)] \leq \varepsilon k_1\, \phi^2(y - h(x,M(x))) + \varepsilon k_2\, \psi(x)\, \phi(y - h(x,M(x)))$;

where the constants c_1, c_2, β_1, k_1 and k_2 are nonnegative and the inequalities should hold for all $x \in B_x$ and $y \in B_y$. We refer to the conditions of this step as the "Interaction Conditions."

Step 4: Form the composite control as

$$u_c = M(x) + \Gamma(x,y) \tag{4-5}$$

The composite control u_c is stabilizing as shown by the following theorem.

Theorem 1: Let d be a positive number such that $0 < d < 1$, and let $\varepsilon^*(d)$ be the positive number given by

$$\varepsilon^*(d) = \frac{\alpha_1\, \alpha_2}{\alpha_1 \gamma + [\beta_1(1 - d) + \beta_2 d]^2 / 4d(1 - d)} \tag{4-6}$$

where $\beta_2 = k_2 + c_2$, and $\gamma = k_1 + c_1$. Then, for all $\varepsilon < \varepsilon^*(d)$, the origin $(x = 0,\ y = 0)$ is an asymptotically stable equilibrium point of the closed loop system

$$\dot{x} = f(x,y,u_c) \tag{4-7a}$$

$$\varepsilon\dot{y} = g(x,y,u_c,\varepsilon) \tag{4-7b}$$

Moreover

$$\nu(x,y) = (1 - d)V(x) + dW(x,y)$$

is a Lyapunov function for this closed loop system.

Proof: The slow subsystem of the closed loop system (4-7) is given by

$$\dot{x} = f(x,y,u_c)$$

$$0 = g(x,y,u_c,0) \tag{4-8}$$

This is the same as the closed loop slow subsystem

$$\dot{x} = f(x,h(x,u_s),u_s). \tag{4-9}$$

This follows simply by observing that $g(x,y,u_c,0) = 0$ has a unique root $y = H(x)$. But $y = h(x,M(x))$ is also a root of this equation. Hence $H(x) = h(x,M(x))$ and the equivalence of (4-8) and (4-9) follows. This observation is a slight generalization of the invariance property of composite control shown by Suzuki [40]. With this observation Theorem 1 follows by applying the stability criterion of chapter 2.

The stability requirements of the design procedure have been discussed in detail in chapter 2. Roughly speaking, the basic task is to design controllers for the slow and fast subsystems so that the corresponding closed loop subsystems have quadratic type Lyapunov functions. For an asymptically stable system $\dot{x} = f(x)$, the function $V(x)$ is said to be a quadratic type Lyapunov function, [10], if $(\nabla_x V)^T f(x) \leq -\alpha\psi^2(x)$ and $|\nabla_x V(x)| \leq \psi(x)$ for some scalar comparison function $\psi(\cdot)$. The gradient requirement, however, does not appear explicitly in our conditions and, as it will be seen later in Example 1, it could be violated. The interaction conditions in step 3 determine the permissible interaction between the slow and fast states, and they are basically smoothness requirements. One implicit feature of the design procedure is that slow and fast controllers should be designed to preserve the slow and fast nature of the system, or in other words, to keep what is slow slow and what is fast fast. For example, a high-gain feedback slow controller that might force some of the slow variables to be fast should not be permitted. This feature, in a sense, restricts the design procedure to situations where the design objective is compatible with the slow-fast nature of the open-loop system. If this implicit requirement is not satisfied we will, typically, end up with designs for which the upper bound $\varepsilon^*(d)$ is smaller than the physical value of ε which determines the ratio between the slow and fast motion in the open loop systems.

Finally the freedom in choosing the parameter d can be employed to improve estimates of the upper bound on the singular perturbation parameter ε, to improve estimates of the domain of attraction, or to achieve a compromise between the two kinds of estimates. For further discussions related to the choice of d the readers are referred to chapter 2.

Corollary 1: If we assume that f, g, and h are continuous and ψ and ϕ belong to class $\mathcal{K}$ function, then the closed loop system resulted from applying composite control, namely (4-7), satisfies the requirements of Theorems 1 and 2 of chapter 3. Therefore the uniform convergence property of the trajectories of the full system (4-7) to those of the slow subsystem (4-3) and the fast subsystem (4-4), as $\varepsilon \to 0^+$, hold.

The design procedure is illustrated by the following example.

Example 1:

Consider:

$$\dot{x} = xy^3 \qquad x \in B_x = [-1, 1]$$

$$\varepsilon\dot{y} = y + u \qquad y \in B_y = [-1/2, 1/2]$$

It is desired to design a feedback control law to stabilize the system at the equilibrium point $x = 0$, $y = 0$. Notice that the desired stabilization cannot be achieved by linearization at the equilibrium point because of an uncontrollable pole at the origin. We go through the design procedure as follows.

Step 1: The slow subsystem is given by:

$$\dot{x} = -xu_s^3$$

We chose $u_s = x^{4/3}$. The closed loop slow subsystem is $\dot{x} = -x^5$ and with the Lyapunov function $V(x) = (1/6)x^6$ we get $\psi(x) = |x|^5$ and $\alpha_1 = 1$. Observe that $V(x)$ satisfies $|\nabla_x V| = \psi(x)$, so it is quadratic-type in the sense of [10].

Step 2: The fast subsystem is defined by:

$$\frac{dy}{d\tau} = y + u_s + u_f = y + x^{4/3} + u_f$$

We choose $u_f = -3(y + x^{4/3})$. The closed loop fast subsystem is $\frac{dy}{d\tau} = -2(y + x^{4/3})$. The choice $W = (1/2)(y + x^{4/3})^2$ yields $\phi = |y + x^{4/3}|$ and $\alpha_2 = 2$.

Step 3: It is straight forward to verify that the interaction conditions hold with $\beta_1 = 7/4$, $\beta_2 = 4/3$, and $\gamma = 7/3$.

Step 4: The composite control is formed as

$$u_c = u_s + u_f = x^{4/3} - 3(y + x^{4/3}) = -3y - 2x^{4/3}$$

Now the choice d = 21/37 yields $\varepsilon^*(21/37) = 3/7$, and the system considered is asymptotically stable for all $\varepsilon < 3/7$.

In this example we could have chosen $V = (1/4)x^4$ which yields $\psi(x) = x^4$ and $\alpha_1 = 1$. This choice of Lyapunov function doesn't satisfy the condition $|\nabla_x V(x)| < \psi(x)$, but the interaction conditions are satisfied with the same constants, namely $\beta_1 = 7/4$, $\beta_2 = 4/3$, and $\gamma = 7/3$. This remark shows that our stability criterion may be applicable even if the requirement $|\nabla_x V(x)| \leq \psi(x)$ is not satisfied.

3. Regulator Problem

This section is concerned with establishing a "near performance" property of composite control. Suppose for solving a regulator problem of a singularly perturbed system we first ignore the fast part, that is we set $\varepsilon = 0$, and solve the resulting slow regulator problem to obtain an optimal solution. Then, a stabilizing fast control is designed and a composite control is formed. The question can now be posed whether, as $\varepsilon \to 0$, the cost of applying this composite control to the singularly perturbed system tends to the cost of the slow regulator problem. To investigate this question consider the regulator problem

$$\dot{x} = f(x,y,u), \qquad x(0) = x_0,\ u \in R^r \tag{4-10a}$$

$$\varepsilon\dot{y} = g(x,y,u,\varepsilon) \qquad y(0) = y_0 \tag{4-10b}$$

$$J = \int_0^\infty L(x,y,u)dt; \tag{4-10c}$$

where L is a positive definite function with respect to its arguments. A slow subproblem is obtained by formally setting $\varepsilon = 0$ in (4-10), and removing the initial condition $y(0) = y_0$. Then (4-10) becomes

$$\dot{x} = f(x,y,u), \qquad x(0) = x_0 \tag{4-11a}$$

$$0 = g(x,y,u,0) \tag{4-11b}$$

$$J = \int_0^\infty L(x,y,u) \tag{4-11c}$$

As in Section 2 we assume that $g(x,y,u,0) = 0$ has a unique root $y = h(x,u)$ in $B_x x B_y$. With this assumption (4-11) can be rewritten as:

$$\dot{x} = f(x,h(x,u_s),u_s), \qquad x(0) = x_0 \tag{4-12a}$$

$$J_s = \int_0^\infty L(x,h(x,u_s),u_s)dt \tag{4-12b}$$

Let us assume that there exists a feedback control, u_s, which yields a bounded cost J_s for all initial conditions in a certain permissible region $\tilde{B}_x$ (that is a region that would guarantee that the trajectory of the closed loop subsystem remains in B_x). Such u_s could be an optimal control or a near-optimal control obtained through some approximation technique. With the knowledge of u_s we define the fast subsystem as in section 2 and consider the stabilization problem of finding u_f to stabilize

$$\frac{dy}{d\tau} = g(x,y,u_s + u_f,0). \tag{4-13}$$

Our task is to show a near performance property of the composite control, $u_c = u_s + u_f$, in the sense that as $\varepsilon \to 0$ the cost of full order system (4-10) with u_c converges to J_s, the cost when $\varepsilon = 0$. We have classified the results of this section into three parts depending on the stability requirements on the slow and fast subsystems.

Part I

Assume that:

A1: The slow subproblem has a solution $u_s = M(x)$ which yields a bounded cost $J_s(x_0)$ for all initial conditions $x_0 \in B_x$. Moreover $J_s(x)$ satisfies

$-\alpha_0\ \psi^2(x) \leq (\nabla_x J_s(x))^T f(x,h(x,M(x),M(x)) \leq -\alpha_1\ \psi^2(x)$ where α_0, α_1 are positive constants and $\psi(\cdot)$ is a scalar-valued positive-definite function of x that vanishes at x = 0.

A2: For the fast subproblem there exists a fast feedback control, $u_f = \Gamma(x,y)$, which satisfies all the requirements of step 2 of section 2 with $\phi(\cdot)$ being positive-definite.

A3: The interaction conditions as stated in step 3 of section 2 are satisfied, with inequality (b) strengthened by replacing the left-hand side by its magnitude.

Assumptions A1 through A3 are essentially the same as in section 2. Notice that $J_s(x)$ is a Lyapunov function for the system (4-11a), (4-11b), and it satisfies

$$(\nabla_x J_s(x))^T f(x,h(x,M(x)),M(x)) = -L(x,h(x,M(x)),M(x)), \quad (4\text{-}14)$$

and it plays the role of V(x) in section 2.

Now we form the composite control $u_c = u_s + u_f$. From Theorem 1 it follows that u_c is stabilizing, that is, the closed loop system has an asymptotically stable equilibrium point at the origin.

The asymptotic stability of the equilibrium point is not sufficient to guarantee that integral cost of the type (4-10c) will be bounded (for a counter example refer to [39]). So we need to show that the application of the composite control, u_c, yields a bounded cost. Denote the cost by $J_{uc}(x,y)$ where x,y are taken as initial conditions. To show the boundedness of J_{uc} we need the following assumption.

A4: The function L satisfies the following condition for all $(x,y)\epsilon B_y \times B_y$

$$|L(x,y,M(x) + \Gamma(x,y)) - L(x,h(x,M(x)),M(x))| \leq \delta_1\ \phi^2(y - h(x,M(x))) + \delta_2\ \psi(x)\ \phi(y - h(x,M(x)))$$

:

where δ_1 and δ_2 are nonnegative constants. This assumption expresses the permissible contribution of the fast state to the function L.

Lemma 1: Suppose assumptions A1-A4 hold. Let d and e be positive numbers such that $0 < d < 1$ and $e > \frac{\alpha_0}{(1 - d)\alpha_1}$. Then there exist $\varepsilon^*(d,e) < \varepsilon^*(d)$ such that for any $\varepsilon < \varepsilon^*(e,d)$ we have

$$J_{uc}(x_0,y_0) \leq e((1 - d)J_s(x_0) + dW(x_0,y_0)), \quad \forall(x_0,y_0)\epsilon D,$$

where D is the set of all initial conditions which give rise to asymptotically stable trajectories that are restricted to $B_x \times B_y$. For example, D can be taken as the domain of attraction determined by the interior of a closed Lyapunov surface in $B_x \times B_y$.

Proof: We consider;

$$\dot{x} = f(x,y,u_c) \tag{4-15a}$$

$$\varepsilon\dot{y} = g(x,y,u_c,\varepsilon) \tag{4-15b}$$

$$J_{uc}(x,y) = \int_t^\infty L(\tilde{x},\tilde{y},u_c)d\sigma,$$

where $\tilde{x},\tilde{y}$ denote the trajectories of (4-15a), (4-15b) with initial point (t,x,y), that is $\tilde{x}(t) = x$, $\tilde{y}(t) = y$, and we let σ be the running time. First we show that for all $(x,y)\epsilon B_x \times B_y$

$$L(x,y,u_c) + e\,\dot{\nu}(x,y) < 0 \tag{4-16}$$

where $\dot{\nu}$ denotes the derivative of composite Lyapunov function $\nu = (1 - d)J_s(x) + dW(x,y)$, along the trajectories of (4-15). In Appendix 1 it is shown that

$$L(x,y,M(x) + \Gamma(x,y)) + e\,\dot{\nu} \leq 0$$

$$
- \begin{pmatrix} \psi(x) \\ \phi(y - h(x,M(x))) \end{pmatrix}^T \mathbf{T}_1 \begin{pmatrix} \psi(x) \\ \phi(y - h(x,M(x))) \end{pmatrix}
$$

where

$$
\mathbf{T}_1 = \begin{pmatrix} e(1-d)\alpha_1 - \alpha_0 & -e(1-d)\frac{\beta_1}{2} - ed\frac{\beta_2}{2} - \frac{\delta_2}{2} \\ -e(1-d)\frac{\beta_1}{2} - ed\frac{\beta_2}{2} - \frac{\delta_2}{2} & ed(\frac{\alpha_2}{\varepsilon} - \gamma) - \delta_1 \end{pmatrix}
$$

Set

$$
\alpha'_1 = e\,\alpha_1 - \frac{\alpha_0}{1-d},\ \alpha'_2 = e\,\alpha_2,\ \ \beta'_1 = e\,\beta_1 + \frac{\delta_2}{1-d},
$$

$$
\beta'_2 = e\,\beta_2,\ \gamma' = e\,\gamma + \frac{\delta_1}{d},\ \text{and}
$$

$$
\varepsilon^*(e,d) = \frac{\alpha'_1\,\alpha'_2}{\alpha'_1\,\gamma' + [\beta'_1(1-d) + \beta'_2 d]^2/4d(1-d)} \tag{4-17}
$$

Notice that $\varepsilon^*(e,d) < \varepsilon^*(d)$ (compare (4-17) with (4-6)). It is straight forward to show that, for any $\varepsilon < \varepsilon^*(e,d)$, $\mathbf{T}_1$ is positive definite and therefore (4-16) holds. Integrating (4-16) from t to ∞, we obtain;

$$
J_{uc}(x,y) + e[\nu(x(\infty),y(\infty)) - \nu(x,y)] \leq 0
$$

Now since $\varepsilon < \varepsilon^*(d)$ the trajectories of (4-15) are asymptotically stable, that is $x(\infty) = 0$, $y(\infty) = 0$ and the result follows.

Remark 1: Notice that $\varepsilon^*(d,e) \to \varepsilon^*(d)$ as $e \to \infty$ and $\varepsilon^*(d,e) \to 0$ as $e \to \frac{\alpha_0}{(1-d)\,\alpha_1}$, which shows that we can have $\varepsilon^*(d,e)$ arbitrarily close to $\varepsilon^*(d)$ and still have bounded cost.

Having established that u_c yields a bounded cost we are in a position to consider the "Near Performance" property of the composite control. Our procedure to show near performance is motivated by [41]. We start by defining an auxiliary cost function, J_1, for (4-15a) and (4-15b) as follows

$$J_1(x,y) = \int_t^\infty \left(L(\tilde{x},\tilde{y},u_c) + \varepsilon \hat{s}_1 \frac{dW(\tilde{x},\tilde{y})}{d\sigma}\right) d\sigma \tag{4-18}$$

where $\hat{s}_1$ is an arbitrary positive number. It is easy to show that for any $\varepsilon < \varepsilon^*(e,d)$ we have;

$$J_1(x,y) = J_{uc}(x,y) - \varepsilon \hat{s}_1 W(x,y) \tag{4-19}$$

Now, define $Q(x,y)$ as

$$Q(x,y) \triangleq qJ_s(s) - J_1(x,y); \tag{4-20}$$

where q is an arbitrary positive number such that $q > 1$ and J_s is the cost of the slow subproblem (4-12) with the initial point (t,x). We want to show that $\forall q > 1$, $Q(x,y)$ is positive definite. To do this first observe that $J_1(x,y)$ satisfies

$$(\nabla_x J_1)^T f_{uc} + (\nabla_y J_1)^T \frac{g_{uc}}{\varepsilon} + L_{uc} + \varepsilon \hat{s}_1 \frac{dW}{dt} = 0, \tag{4-21}$$

where $f_{uc} \triangleq f(x,y,u_c)$, $g_{uc} \triangleq g(x,y,u_c,\varepsilon)$, $L_{uc} \triangleq L(x,y,u_c)$, and for the sake of brevity we have dropped the arguments of $J_1(x,y)$ and $W(x,y)$. Substituting (4-20) in (4-21) we get;

$$q(\nabla_x J_s)^T f_{uc} - (\nabla_x Q)^T f_{uc} - (\nabla_y Q)^T \frac{g_{uc}}{\varepsilon} + L_{uc} + \varepsilon \hat{s}_1 \frac{dW}{dt} = 0, \tag{4-22}$$

Using (4-14), (4-22) can be rewritten as

$$(\nabla_x Q)^T f_{uc} + (\nabla_y Q)^T \frac{g_{uc}}{\varepsilon} + qL_s - L_{uc} - \varepsilon \hat{s}_1 \frac{dW}{dt} - q(\nabla_x J_s)^T (f_{uc} - f_s) = 0, \tag{4-23}$$

where $L_s \triangleq L(x,h(x,M(x)), M(x))$, and $f_s \triangleq f(x,h(x,M(x)),M(x))$. Now we consider another auxiliary cost function, J_2, for the system (4-15a), (4-15b), given by;

$$J_2(x,y) = \int_t^\infty \{qL_s - L_{uc} - \varepsilon\hat{S}_1 \frac{dW}{d\sigma} - q(\nabla_x J_s)^T(f_{uc} - f_s)\}d\sigma \tag{4-24}$$

Lemma 2: For any $\varepsilon < \varepsilon^*(e,d)$, J_2 converges. Moreover $J_2(x,y) = Q(x,y)$ $\forall(x,y)\in D$.

Proof: We have;

$$J_2(x,y) = -\int_t^\infty (L_{uc} + \varepsilon\hat{S}_1 \frac{dW}{d\sigma})d\sigma + q\int_t^\infty (L_s - (\nabla_x J_s)^T(f_{uc} - f_s))d\sigma$$

or

$$J_2(x,y) = -J_1(x,y) + q\int_t^\infty (L_s - (\nabla_x J_s)^T(f_{uc} - f_s))d\sigma \tag{4-25}$$

Taking derivative of $J_s(x)$ along the trajectories of the full order system (4-15a), (4-15b) gives

$$\begin{aligned}\frac{d}{dt}(J_s(x)) &= (\nabla_x J_s)^T f_{uc} \\ &= (\nabla_x J_s)^T(f_{uc} - f_s + f_s) \\ &= -L_s + (\nabla_x J_s)^T(f_{uc} - f_s),\end{aligned} \tag{4-26}$$

where in the last step we have used (4-14). Substituting (4-26) in (4-25) we get

$$J_2(x,y) = -J_1(x,y) - q\int_t^\infty \frac{d}{d\sigma}(J_s(\tilde{x}))d\sigma$$

Since (4-15) is asymptotically stable it follows that

$$J_2(x,y) = -J_1(x,y) + qJ_s(x) \tag{4-27}$$

or

$$J_2(x,y) = Q(x,y). \qquad \text{Q.E.D.}$$

If we establish that the integrand of J_2 is positive definite, then we have shown that Q(x,y) is positive definite. This is done in the following lemma.

Lemma 3: For any $q > 1$ and ε sufficiently small we can always choose $\hat{s}_1$ such that $J_2(x,y)$ be positive definite.

Proof: In Appendix 2 we have shown that for any $(x,y) \in B_x \times B_y$:

$$qL_s - L_{uc} - \varepsilon\hat{s}_1 \frac{dW}{dt} - q(\nabla_x J_s)^T(f_{uc} - f_s) \geq \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \mathbf{T}_2 \begin{pmatrix} \psi \\ \phi \end{pmatrix},$$

where $\mathbf{T}_2$ is given by;

$$\mathbf{T}_2 = \begin{pmatrix} (q-1)\alpha_1 & -(1/2)(\delta_2 + \varepsilon\hat{s}_1 \beta_2 + q\beta_1) \\ -(1/2)(\delta_2 + \varepsilon\hat{s}_1 \beta_2 + q\beta_1) & \hat{s}_1 \alpha_2 - \delta_1 - \varepsilon\hat{s}_1 \gamma \end{pmatrix} \tag{4-28}$$

By inspection we observe that, for sufficiently small ε and for $q > 1$, $\hat{s}_1$ can always be chosen to make $\mathbf{T}_2$ and subsequently J_2, positive definite.

Positive definiteness of Q(x,y) implies that $J_1 < qJ_s$, or using (4-19), $J_{uc} < qJ_s + \varepsilon\hat{s}_1 W(x,y)$. In order to get the sharpest bound on J_{uc} we should choose q very close to one. However as q approaches one the required $\hat{s}_1$, according to lemma 3, grows and becomes unbounded. To overcome this problem and get the sharpest bound on J_{uc}, we set

$q = 1 + s_o \varepsilon^{1/2}$ and $\hat{s}_1 = \varepsilon^{-1/2} s_1$, where s_o, and s_1 are positive numbers. To establish our result corresponding to these choices of s_o and s_1 we need to define some parameters. Let;

$$a = (1/4)\ (s_o\beta_1 + s_1\ \beta_2)^2 + s_o\ s_1\ \alpha_1\gamma,$$

$$b = (1/2)\ (\delta_2 + \beta_1)\ (s_o\ \beta_1 + s_1\ \beta_2) + s_o\ \alpha_1\ \delta_1$$

$$c = s_o s_1\ \alpha_1\ \alpha_2 - (1/4)\ (\delta_2 + \beta_1)^2.$$

Then the upper bound on J_{uc} is summarized in the following lemma.

Lemma 4: Suppose assumptions A1 through A4 hold. Let s_o and s_1 be arbitrary positive numbers such that $s_o s_1 > \dfrac{(\delta_2 + \beta_1)^2}{4\ \alpha_1\ \alpha_2}$. Then there exists $\varepsilon^*(s_o,s_1)$ given by

$$\varepsilon^*(s_o,s_1) = \left(\frac{b - \sqrt{b^2 + 4ac}}{2a}\right)^2 \tag{4-29}$$

such that for any $\varepsilon < \min(\varepsilon^*(s_o,s_1),\ \varepsilon^*(e,d))$, and for all initial conditions $(x_o,y_o)\epsilon D$ we have;

$$J_{uc}(x_o,y_o) \leq J_s(x_o) + \varepsilon^{1/2}(s_o J_s(x_o) + s_1 W(x_o,y_o)) \tag{4-30}$$

Now we use the same machinery in order to get a lower bound on J_{uc}. We start off by defining an auxiliary function, J_3, for the system (4-15a), (4-15b) as follows

$$J_3(x,y) = \int_t^{\infty} (L(\tilde{x},\tilde{y},u_c) - \varepsilon\tilde{s}_1\ \frac{dW(\tilde{x},\tilde{y})}{d\sigma})d\sigma, \tag{4-31}$$

where $\tilde{s}_1$ is an arbitrary positive number. This cost function plays the role of J_1 in finding upper bound on J_{uc}. For any $\varepsilon < \varepsilon^*(e,d)$ we have

$$J_3(x,y) = J_{uc}(x,y) + \varepsilon\tilde{s}_1 W(x,y). \tag{4-32}$$

Define P(x,y) as

$$P(x,y) \triangleq J_3(x,y) - \hat{q}J_s(x), \tag{4-33}$$

where $\hat{q}$ is a positive number such that $\hat{q} < 1$. As before the next step would be to show that P(x,y) is positive definite. Observe that $J_3(x,y)$ satisfies

$$(\nabla_x J_3)^T f_{uc} + (\nabla_y J_3)^T \frac{g_{uc}}{\varepsilon} + L_{uc} - \varepsilon\tilde{s}_1 \frac{dW}{dt} = 0$$

Substituting (4-33) in above we get:

$$(\nabla_x P)^T f_{uc} + (\nabla_y P)^T \frac{g_{uc}}{\varepsilon} + \hat{q}(\nabla_x J_s)^T(f_{uc} - f_s)$$
$$+ L_{uc} - \hat{q}L_s - \varepsilon\tilde{s}_1 \frac{dW}{dt} = 0, \tag{4-34}$$

where we have dropped the argument of P(x,y). We define another auxiliary cost function, J_4, for the system (4-15a) and (4-15b) which plays the role of J_2 in the derivation of the upper bound on J_{uc}, and is given by;

$$J_4(x,y) = \int_t^{\infty} \{L_{uc} - \varepsilon\tilde{s}_1 \frac{dW}{d\sigma} - \hat{q}L_s + \hat{q}(\nabla_x J_s)^T(f_{uc} - f_s)\}d\sigma \tag{4-35}$$

<u>Lemma 5</u>: For any $\varepsilon < \varepsilon^*(e,d)$, J_4 converges.
Moreover $J_4(x,y) = P(x,y)\ \forall (x,y)\in D$.

<u>Proof</u>: Similar to the proof of Lemma 2.

The next step is to show that the integrand of J_4 is positive definite and therefore J_4 is positive definite. This is done in the following lemma.

Lemma 6: For any $0 < \hat{q} < 1$ and for sufficiently small ε we can always choose $\tilde{s}_1$ such that $J_4(x,y)$ is positive definite.

Proof: As in Appendix 2, it can be shown that for any $(x,y) \in B_x \times B_y$

$$L_{uc} - \varepsilon\tilde{s}_1 \frac{dW}{dt} - \hat{q}L_s + \hat{q}(\nabla_x J_s)^T (f_{uc} - f_s) \geq \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \mathbf{T}_3 \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

where $\mathbf{T}_3$ is given by

$$\mathbf{T}_3 = \begin{pmatrix} (1 - \hat{q})\, \alpha_1 & -(1/2)\,(\delta_2 + \varepsilon\tilde{s}_1\, \beta_2 + q\, \beta_1) \\ -(1/2)\,(\delta_2 + \varepsilon\tilde{s}_1\, \beta_2 + \hat{q}\, \beta_1) & \tilde{s}_1\, \alpha_2 - \delta_1 - \varepsilon\tilde{s}_1\gamma \end{pmatrix}$$

It is obvious that for any $0 < \hat{q} < 1$ and for sufficiently small ε, we can choose $\tilde{s}_1$ so that $\mathbf{T}_3$ be positive definite and the result follows. Q.E.D.

Now for the same reason that was stated in the derivation of upper bound on J_{uc} we set $\hat{q} = 1 - \varepsilon^{1/2}s_0$ and $\tilde{s}_1 = \varepsilon^{-1/2}s_1$. The lower bound on J_{uc} is given by the following lemma.

Lemma 7: Suppose assumptions A1 through A4 hold. Let s_0, s_1 and $\varepsilon^*(s_0,s_1)$ be the same as in Lemma 4. Then for any $\varepsilon < \min(\varepsilon^*(s_0,s_1), \varepsilon^*(e,d))$, and for all initial conditions $(x_0,y_0) \in D$ the following holds

$$J_{uc}(x_0,y_0) \geq J_s(x_0) - \varepsilon^{1/2}(s_0 J_s(x_0) + s_1 W(x_0,y_0)).$$

Proof: With the above choices of q and $\hat{q}$, the only difference between $\mathbf{T}_2$ and $\mathbf{T}_3$ is that in the outer diagonal element the term $-\varepsilon^{1/2}s_0\beta_1/2$ in $\mathbf{T}_2$ is replaced by $\varepsilon^{1/2}s_0\beta_1/2$ in $\mathbf{T}_3$. Thus $\det(\mathbf{T}_3) \geq \det(\mathbf{T}_2)$ and for all $\varepsilon < \varepsilon^*(s_0,s_1)$, $\mathbf{T}_3$ is positive definite. The rest of the proof is exactly the same as in Lemma 4.

Combining Lemmas 4 and 7 gives Theorem 2, the result of this part.

Theorem 2: Suppose assumptions A1 through A4 hold. Let d, e, s_0, and s_1 be arbitrary positive numbers such that

$$0 < d < 1, e > \frac{\alpha_0}{(1 - d)\,\alpha_1}, \quad \text{and} \quad s_0 s_1 > \frac{(\delta_2 + \beta_1)^2}{4\,\alpha_1\,\alpha_2}.$$

Then for any $\varepsilon < \min(\varepsilon^*(e,d), \varepsilon^*(s_0,s_1))$ and for all initial conditions $(x_0,y_0) \epsilon D$ the following inequality holds

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \leq \varepsilon^{1/2}(s_0 J_s(x_0) + s_1 W(x_0,y_0)).$$

Moreover $\varepsilon^*(e,d)$ and $\varepsilon^*(s_0,s_1)$ are given by expressions (4-17) and (4-29) respectively.

Remark 2: As we stated before, by appropriately choosing e, $\varepsilon^*(e,d)$ can be made arbitrarily close to $\varepsilon^*(d)$. This shows that in Theorem 2 we can ignore e and $\varepsilon^*(e,d)$, and focus on parameters d, s_0, and s_1. By doing so the result of Theorem 2 holds for any $\varepsilon < \min(\varepsilon^*(s_0,s_1), \varepsilon^*(d))$.

Remark 3: The choice of the parameter d is usually made based on stability considerations and does not involve the cost function. The choice of the parameters s_0 and s_1, on the other hand, depends on the cost function since both s_0 and s_1 appear in the bounds on J_{uc}. Generally, the choice of s_0 and s_1 will be guided by two requirements. The first one is to get a large $\varepsilon^*(s_0,s_1)$ and the second one is to get a sharp bound on J_{uc}. The two requirements are usually contradicting and a compromise should be sought.

Part II:

In this part we consider a "Near Performance" property of the composite control when the closed loop fast subsystem is exponentially stable. It turns out that in this case the closeness of J_{uc} and J_s are of order of ε unlike the result in Part I which was order of $\varepsilon^{1/2}$. We start by stating the extra assumptions required for this case.

<u>A5</u>: The Lyapunov function for the closed loop fast subsystem, W(x,y), satisfies

$$\xi_1\phi^2(y - h(x,M(x))) \leq W(x,y) \leq \xi_2\phi^2(y - h(x,M(x)))$$

where ξ_1 and ξ_2 are positive constants.

<u>A6</u>:

(i) $\psi(\cdot)$ is differentiable and satisfies;

$$|\nabla_x \psi(x)| \leq \lambda < \infty \qquad \forall x \epsilon B_x$$

where λ is some positive constant.

(ii) $|f(x,y,M(x) + \Gamma(x,y))| \leq C_3\, \phi(y - h(x,M(x)) + C_4\, \psi(x).$

where C_3 and C_4 are positive constants.

Assumption A5 implies exponential stability of the fast subsystem. Assumption A6 restricts f and ψ further over what has been required by assumptions A1-A4. However, for a large class of problems assumption A6 will not add extra restrictions over the interaction conditions A5.

The machinery used to prove the result of this part is similar to that one used in Part I, although there are critical changes in auxiliary cost functions used in the proof. We start by defining auxiliary cost function J_5 for the system (4-15a) and (4-15b) as follows

$$J_5 = \int_t^\infty \{L(\tilde{x},\tilde{y},u_c) + s_3\,\frac{dW(\tilde{x},\tilde{y})}{d\sigma} + \varepsilon H\,\frac{d}{d\sigma}\,(\psi(\tilde{x})\,\,(W(\tilde{x},\tilde{y}))^{1/2})\}d\sigma, \tag{4-37}$$

where s_3 is an arbitrary positive number and H is a fixed constant to be chosen later. It is straight forward to show that for any $\varepsilon < \varepsilon^*(e,d)$ we have:

$$J_5(x,y) = J_{uc}(x,y) - \varepsilon s_3 W(x,y) - \varepsilon H\, \psi(x)\, (W(x,y))^{1/2}, \quad \forall (x,y)\in D$$

Define $\tilde{Q}(x,y)$ as

$$\tilde{Q}(x,y) \triangleq qJ_s(x) - J_5(x,y), \qquad q > 1 \tag{4-38}$$

Repeating the derivations of Part I, it can be shown that for any $\varepsilon < \varepsilon^*(e,d)$ and any $(x,y)\in D$ we have:

$$\tilde{Q}(x,y) = \int_t^\infty \{qL_s - L_{uc} - \varepsilon s_3 \frac{dW}{d\sigma} - \varepsilon H \frac{d}{d\sigma}(\psi W^{1/2}) - q(\nabla_x J_s)^T (f_{uc} - f_s)\} d\sigma$$

Moreover for any $(x,y) \in B_x \times B_y$

$$qL_s - L_{uc} - \varepsilon s_3 \frac{dW}{dt} - \varepsilon H \frac{d}{dt}(\psi W^{1/2}) - q(\nabla_x J_s)^T (f_{uc} - f_s) \geq \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \mathbf{T}_4 \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{4-39}$$

and $\mathbf{T}_4$ is given by

$$\mathbf{T}_4 = \begin{pmatrix} (q-1)\,\alpha_1 - \dfrac{\varepsilon\, \beta_2 H}{2\sqrt{\xi_1}} & -0.5G \\ -0.5G & s_3\, \alpha_2 - \delta_1 - \varepsilon s_3 \gamma - \varepsilon H\, \lambda C_3 \sqrt{\xi_2} \end{pmatrix}$$

where

$$G = \varepsilon \left(s_3\, \beta_2 + \frac{\gamma H}{2\sqrt{\xi_2}} + H\lambda \sqrt{\xi_2}\, C_4 \right) + (q-1)\, \beta_1 + \left(\delta_2 + \beta_1 - (1/2) \frac{\alpha_2 H}{\sqrt{\xi_2}} \right).$$

Notice that if we choose

$$H = \frac{2\sqrt{\xi_2}\,(\delta_2 + \beta_1)}{\alpha_2} \tag{4-40}$$

we get asymptotically, as $\varepsilon \to 0$, sharper estimates on J_{uc} in comparison with those obtained in Part I. In fact this has been the motivation for adding the term $\varepsilon H \frac{d}{dt}(\psi(x)W^{1/2}(x,y))$ to the integrand of J_5. So we set $q = 1 + \varepsilon s_2$, where s_2 is an arbitrary positive number. Taking H as in (4-40) we define the following parameters:

$$\tilde{\alpha}_1 = s_2\,\alpha_1 - \frac{\beta_2 H}{2\sqrt{\xi_1}}, \qquad \tilde{\beta}_1 = s_2\,\beta_1 + \frac{\gamma H}{2\sqrt{\xi_2}} + \lambda H C_4\sqrt{\xi_2}$$

$$\tilde{\beta}_2 = \beta_2, \qquad \tilde{\alpha}_2 = \alpha_2 - \frac{\delta_1}{s_3}, \qquad \tilde{\gamma} = \gamma + \frac{H\lambda C_3\sqrt{\xi_2}}{s_3}$$

Then it can be shown that for any $s_2 > \frac{\beta_2 H}{2\sqrt{\xi_1}\,\alpha_1}$ and $s_3 > \frac{\delta_1}{\alpha_2}$ there exists $\varepsilon^*(s_2,s_3)$ given by;

$$\varepsilon^*(s_2,s_3) = \frac{\tilde{\alpha}_1\,\tilde{\alpha}_2}{\tilde{\alpha}_1\tilde{\gamma} + [\tilde{\beta}_1 + s_3\,\tilde{\beta}_2]^2/4s_3} \tag{4-41}$$

such that for any $\varepsilon < \min(\varepsilon^*(s_2,s_3),\ \varepsilon^*(e,d))$ we have;

$$J_{uc}(x,y) \le J_s(x) + \varepsilon(s_2 J_s(x) + s_3 W(x,y) + H\,\psi(x)W^{1/2}(x,y)), \quad \forall (x,y)\in D$$

We can invoke the same machinery to get the lower bound on J_{uc}, which we are not going to repeat here. The result of this part can be summarized in the following theorem.

Theorem 3: Suppose assumptions A1 through A6 hold. Let d, e, s_2, and s_3 be arbitrary positive numbers which satisfy

$$0 < d < 1, \; e > \frac{\alpha_0}{(1 - d)\,\alpha_1}, \; s_2 > \frac{\beta_2 H}{2\sqrt{\xi_1}\,\alpha_1}, \; s_3 > \frac{\delta_1}{\alpha_2}.$$

Then there exist $\varepsilon^*(e,d)$ and $\varepsilon^*(s_2,s_3)$ such that for any $\varepsilon < \min(\varepsilon^*(s_2,s_3), \varepsilon^*(e,d))$ and for all initial conditions $(x_0,y_0)\in D$ we have;

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \leq \varepsilon(s_2 J_s(x_0) + s_3 W(x_0,y_0) + H\,\psi(x_0)W^{1/2}(x_0,y_0))$$

Moreover $\varepsilon^*(e,d)$ and $\varepsilon^*(s_2,s_3)$ are given by expressions (4-17) and (4-41), respectively.

The remarks 2 and 3 of Part I apply to this part as well. Moreover we have the following remark.

Remark 4: The choice of H in (4-40) was made to be independent of ε, however, we can choose H to set off-diagonal terms of matrix $\mathbf{T}_4$ equal to zero. This choice of H, denoted by H' is given by

$$H' = (\varepsilon s_2\,\beta_1 + \delta_2 + \beta_1 + \varepsilon s_3\,\beta_2)\;x$$

$$\left(\frac{\alpha_2}{2\sqrt{\xi_2}} - \frac{\varepsilon\gamma}{2\sqrt{\xi_2}} - \lambda C_4\sqrt{\xi_2}\right)^{-1}$$

which is dependent on ε. It is straight forward to show that for

$$s'_2 > \frac{\beta_2 H'}{2\sqrt{\xi_1}\,\alpha_1} \text{ and for any } s'_3 > (\delta_1 + \varepsilon^*(d,e)\,H'\lambda C_3\sqrt{\xi_2})/(\alpha_2 - \varepsilon^*(d,e)\gamma)$$

the matrix $\mathbf{T}_4$ is positive definite. Recalling that $\varepsilon^*(d,e)$ can be made arbitrarily close to $\varepsilon^*(d)$, the result of Theorem 3 with the above choice of s'_2, s'_3, and H' holds for any $\varepsilon < \varepsilon^*(d)$. Notice that H', s'_2, and s'_3 are greater than H, s_2 and s_3 as given in (4-40), which implies that with the above choices we have improved the upper bound on ε but for sufficiently small ε the choices of H, s_2 and s_3 used in Theorem 3 yield a sharper bound on J_{uc}.

Part III:

This part is concerned with the case when both closed loop slow and fast subsystems are exponentially stable. The result of this part is similar to that obtained in Part II except that exponential stability of the closed loop slow subsystem will allow us to drop assumption A6. The exponential stability requirement on the slow subsystem is stated by the following assumption.

A7: The cost function for the slow regulator problem satisfies

$$\mu_1 \, \psi^2(x) \le J_s(x) \le \mu_2 \, \psi^2(x)$$

where μ_1 and μ_2 are positive constants.

The procedure to get the result of this part is almost the same as in Part II. The basic difference is that the term $\varepsilon H \frac{d}{dt}(\psi(x)W^{1/2}(x,y))$ in the integrand of J_5 is replaced by $\varepsilon\tilde{H} \frac{d}{dt} (J_s^{1/2}(x)W^{1/2}(x,y))$, where $\tilde{H}$ will be defined later. Denoting the free parameter s_3 in Part II by s_5 for this part, we define $\tilde{Q}(x,y)$ as in (4-38). Then it can be shown that inequality (4-39) is satisfied with $\mathbf{T}_4$ replaced by $\mathbf{T}_5$ which is given by

$$\mathbf{T}_5 = \begin{pmatrix} (q-1)\,\alpha_1 - \dfrac{\varepsilon\beta_2 \, \tilde{H}\sqrt{\mu_2}}{2\sqrt{\xi_1}} & A/2 \\ A/2 & s_5\,\alpha_2 - \delta_1 - \varepsilon s_5 \gamma - \dfrac{\varepsilon\beta_1\tilde{H}\sqrt{\xi_2}}{2\sqrt{\mu_1}} \end{pmatrix}$$

where

$$A = \varepsilon s_5 \, \beta_2 + (q-1)\,\beta_1 + (1/2)\,\varepsilon\tilde{H}\left(\frac{\gamma\sqrt{\mu_1}}{\sqrt{\xi_2}} - \frac{\alpha_1\sqrt{\xi_1}}{\sqrt{\mu_2}}\right)$$

and $\tilde{H}$ is taken as

$$\tilde{H} = \frac{2(\delta_2 + \beta_1)\sqrt{\xi_2}}{\alpha_2\sqrt{\mu_1}}$$

By setting $q = 1 + \varepsilon s_4$, where s_4 is an arbitrary positive number, the upper bound on J_{uc} can be obtained. Using the machinery of Part I the lower bound on J_{uc} is similarly obtained. To state the result of this part we define the following parameters

$$\hat{\alpha}_1 = s_4\,\alpha_1 - \frac{\beta_2\sqrt{\mu_2}}{2\sqrt{\xi_1}}\tilde{H},\quad \hat{\beta}_1 = s_4\,\beta_1 + \frac{\gamma\tilde{H}\sqrt{\mu_1}}{2\sqrt{\xi_2}} - \frac{\alpha_1\tilde{H}\sqrt{\xi_1}}{2\sqrt{\mu_2}}$$

$$\hat{\beta}_2 = \beta_2,\ \hat{\gamma} = \gamma + \frac{\beta_1\tilde{H}\sqrt{\xi_2}}{2s_5\sqrt{\mu_1}},\quad \hat{\alpha}_2 = \alpha_2 - \frac{\delta_1}{s_5}$$

and take $\varepsilon^*(s_4,s_5)$ as

$$\varepsilon^*(s_4,s_5) = \frac{\hat{\alpha}_1\,\hat{\alpha}_2}{\hat{\alpha}_1\hat{\gamma} + [\hat{\beta}_1 + \hat{\beta}_2 s_5]^2/4s_5} \tag{4-42}$$

<u>Theorem 4</u>: Suppose assumptions A1 through A5 and A7 hold. Let d, e, s_4, and s_5 be arbitrary positive numbers which satisfy

$$0 < d < 1,\ e > \frac{\alpha_0}{(1-d)\,\alpha_2},\ s_4 > \frac{\beta_2\tilde{H}\sqrt{\mu_2}}{2\alpha_1\sqrt{\xi_1}},\ \text{and } s_5 > \frac{\delta_1}{\alpha_2}$$

Then there exist $\varepsilon^*(e,d)$ and $\varepsilon^*(s_4,s_5)$ such that for any $\varepsilon < \min(\varepsilon^*(s_4,s_5),\ \varepsilon^*(e,d))$ and for all initial conditons $(x_o,y_o)\in D$ we have

$$|J_{uc}(x_o,y_o) - J_s(x_o)| \leq \varepsilon(s_4 J_s(x_o) + s_5 W(x_o,y_o) + \tilde{H}J_s^{1/2}(x_o)W^{1/2}(x_o,y_o)).$$

Moreover $\varepsilon^*(e,d)$ and $\varepsilon^*(s_4,s_5)$ are given by expressions (4-17) and (4-42), respectively.

The result obtained in Parts I, II, and III also satisfy following corollary.

Corollary 2: Suppose f, g, and h are continuous and ψ, ϕ belong to class $\mathcal{K}$ functions. Then the closed loop system (4-15) satisfy the conditions of theorems 1 and 2 in chapter 3. Therefore the uniform convergence property of the trajectories of the full regulator (4-15) to the trajectories of the slow regulator (4-12) under the boundary layer correction holds.

The "Near Performance" results are illustrated by the following examples.

Example 2: In this example we consider the class of singularly perturbed systems studied by Chow and Kokotovic [39], namely

$$\dot{x} = a_1(x) + A_1(x)y + B_1(x)u, \quad x(0) = x_o, \qquad u \in R^r$$

$$\varepsilon\dot{y} = a_2(x) + A_2(x)y + B_2(x)u, \quad y(0) = y_o$$

$$J = \int_t^{\infty} [P(x) + s^T(x)y + y^T M(x)y + u^T R(x)\ ]dt$$

where a_1, a_2, A_1, A_2^{-1}, B_1, B_2, P, s, M, and R are continuous with respect to x in B_x. Moreover, $P + s^T y + y^T My$ is a positive definite function of its arguments x and y in $B_x \times B_y$ and R(x) and M(x) are positive definite matrices for all $x \in B_x$. The slow subsystem is given by

$$\dot{x} = a_o(x) + B_o(x)u_s, \qquad x(0) = x_o$$

$$J_s = \int_t^{\infty} [P_o(x) + 2s_o^T(x)u_s + u_s^T R_o(x)u_s]dt$$

where

$$a_o = a_1 - A_1A_2^{-1}a_2$$

$$B_o = B_1 - A_1A_2^{-1}B_2$$

$$P_o = P - s^TA_2^{-1}a_2 + a_2^T(A_2^T)^{-1}MA_2^{-1}a_2$$

$$s_o = B_2^T(A_2^T)^{-1}(MA_2^{-1}a_2 - (1/2)s)$$

$$R_o = R + B_2^T(A_2^T)^{-1}MA_2^{-1}B_2$$

We make the following assumptions.

I - There exists a twice differential function $J_s^*(x)$ which satisfies the Hamilton-Jacobi equation

$$0 = (P_o - s_o^TR_o^{-1}s_o) + (\nabla_xJ_s^*)^T(a_o - B_oR_o^{-1}s_o)$$
$$-(1/4)\ (\nabla_xJ_s^*)^TB_oR_o^{-1}B_o(\nabla_xJ_s^*), \quad J_s^*(0) = 0$$

This assumption implies that

$$u_s = -R_o^{-1}(s_o + (1/2)B_o^T(\nabla_xJ_s^*))$$

is the minimizing control for the slow subsystem and J_s^* is the corresponding optimal cost.

II - For all $x \in B_x$ the following inequalities hold

(i) $\quad -\alpha_o\ \psi^2(x) < (\nabla_xJ_s^*)^T\bar{a}_o < -\alpha_1\ \psi^2(x), \quad \alpha_o, \alpha_1 > 0$

(ii) $\quad |\bar{a}_o| \leq \alpha_3\ \psi(x) \qquad \alpha_3 > 0$

(iii) $\quad |s_o| \leq \alpha_4\ \psi(x) \qquad \alpha_4 > 0$

where $\bar{a}_o = a_o(x) + B_o(x)u_s = a_o - B_oR_o^{-1}(s_o + (1/2)B_o^T(\nabla_x J_s^*))$, and $\psi(\cdot)$ is a positive-definite scalar-valued function which vanishes at $x = 0$

III - The pair (A_2, B_2) is stabilizable uniformly in x in the sense that there exist $K_F(x)$ such that min $\text{Re}\lambda(A_{22}(x) + B_2(x)K_F(x)) < -\sigma < 0$.

With above assumptions it can be shown that the requirements A1 through A6 of Theorem 3 hold. This example extends the result of [39] in three directions. First, the fast control could be any stabilizing control and not necessarily the solution of the fast optimal control problem as in [39]; although the optimizing control is still the most natural candidate. Second, upper bound on ε and on the performance criterion are provided. Third, the assumptions are weaker than those of [39] (see Example 4). This can be seen by observing that $\psi(x)$ can be taken as $|\nabla_x J_s|$. We should, however, mention that the requirement $|s_o| < \alpha_4|\nabla_x J_s|$ is not explicitly required in [39] although it is implicit in the Hamilton-Jacobi equation and inequalities (i) and (ii).

Example 3: This example is borrowed from Chow and Kokotovic [39]. Consider

$$\dot{x} = -(3/4)x^3 + y \qquad B_x = [-1/2, 1/2]$$

$$\varepsilon\dot{y} = -y + u$$

$$J = \int_0^{\infty} (x^6 + (3/4)y^2 + (1/4)u^2)dt$$

The slow subproblem is

$$\dot{x} = (3/4)x^3 + u_s$$

$$J_s = \int_0^\infty (x^6 + u_s^2)dt$$

and its optimal solution is $u_s = -x^3/2$, which yields the optimal cost $J_s(x) = x^4/4$. This cost function satisfies assumption A_1 with $\psi(x) = |x|^3$ and $\alpha_0 = \alpha_1 = 5/4$. The next step is to stabilize the fast subsystem, which is given by:

$$\frac{dy}{d\tau} = -y + u_s + u_f = -y - x^3/2 + u_f$$

The open loop fast subsystem is asymptotically stable and u_f could be taken zero but to be able to compare with the result of Chow and Kokotovic [39], we use their choice of u_f, namely, $u_f = -(y + x^3/2)$. The composite control is given by

$$u_c = u_s + u_f = -x^3 - y$$

With the choice of $W(x,y) = (1/2)(y + x^3/2)^2$, assumption A2 is satisfied with $\phi = |y + (1/2)x^3|$ and $\alpha_2 = 2$. It is straight forward to show that the interaction conditions, A3, are also satisfied with $\beta_1 = 1$, $\beta_2 = 15/32$, and $\gamma = 3/8$. Now we can apply Theorem 1. The largest upper bound $\varepsilon^*(d)$ provided by Theorem 1 is $\varepsilon^*(d) = 8/3$ which is the result of the choice $d = 32/47$. However, in order to improve the estimate of the domain of attraction we choose $d = 1/4$ which yields $\varepsilon^*(1/4) = 1.699$ and Theorem 1 guarantees that for any $\varepsilon < 1.699$ the closed loop system with composite control u_c is asymptotically stable, and the corresponding composite Lyapunov function is given by

$$\nu(x,y) = (3/16)x^4 + (1/8)(y + (1/2)x^3)^2$$

Taking into consideration that $x \in B_x = [-1/2, 1/2]$, an estimate of the domain of attraction is given by

$$D = \{x^4 + (2/3)(y + (1/2)x^3)^2 < 1/16\}$$

These estimates are less conservative than those obtained in [39]. In [39] the asymptotic stability is guaranteed for $\varepsilon < 480/881$ and the domain of attraction is obtained by

$$\tilde{D} = \{x^4 + (y + (1/2)x^3)^2 < 1/16\}$$

Now we apply the result of section 2 to obtain bounds on J_{uc} and determine the range of ε for which these bounds hold. Since the fast subsystem is exponentially stable, bounds can be obtained by applying the results of either Part I or Part II. We apply both parts.

To apply Theorem 2 of Part I, we observe that the assumption A4 is satisfied with $\delta_1 = 1$, $\delta_2 = 1/2$. Choosing $s_0 = 1$ and $s_1 = 4$, we get $\varepsilon^*(s_0,s_1) = 1.3794$. By Theorem 2 we conclude that $\forall \varepsilon < 1.3794$ and for all initial conditions $(x_0,y_0) \in D$ we have

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \leq \varepsilon^{1/2}(x_0{}^4/4 + 2(y_0 + (1/2)x_0{}^3)^2) \quad (4\text{-}43)$$

To apply Theorem 3 of Part II we observe that assumptions A5 and A6 are satisfied with $\xi_1 = \xi_2 = \frac{1}{2}$, $\lambda = 3/4$, $C_3 = 1$ and $C_4 = 5/4$. We choose $s_2 = 1$ and $s_3 = 4$ which yields $\varepsilon^*(s_2,s_3) = 1.1277$. From Theorem 3 we conclude that for any $\varepsilon < 1.1277$ and all initial conditions $(x_0,y_0) \in D$ we have

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \leq \varepsilon[x_0{}^4/4 + 2(y_0 + (1/2)x_0{}^3)^2$$

$$+ 0.75|x_0|^3 \cdot |y_0 + (1/2)x_0{}^3|] \quad (4\text{-}44)$$

We finish up this example by pointing out that, for sufficiently small ε, (4-44) is a sharper bound than (4-43). However, for values of ε not so small the comparison between (4-44) and (4-43) is not definite and both should be considered.

Example 4: This example is from Chow and Kokotovic [38]. Consider

$$\dot{x} = xy \qquad B_x = [-1,1]$$

$$\varepsilon\dot{y} = -y + u$$

$$J = \int_0^\infty (x^4 + y^2/2 + u^2/2)dt$$

The slow subproblem is given by

$$\dot{x} = xu_s$$

$$J_s = \int_0^\infty (x^4 + u_s^2)dt$$

Solving for the optimal control of the slow subproblem we obtain $u_s = -x^2$ and $J_s(s) = x^2$. So assumption A1 is satisfied with $\psi(x) = x^2$ and $\alpha_1 = 2$. The fast subsystem is defined as

$$\frac{dy}{d\tau} = -y + u_s + u_f = -y - x^2 + u_f$$

We choose the same feedback for the fast subsystem as in [38], namely, $u_f = -(\sqrt{2} - 1)(y + x^2)$. With the choice of $W = (1/2)(y + x^2)^2$ the assumption A2 is satisfied with $\phi = |y + x^2|$ and $\alpha_2 = \sqrt{2}$. Moreover the interaction conditon, A3, holds with $\beta_1 = \beta_2 = \gamma = 2$. Next we choose $d = 1/2$, then from (4-6) we get $\varepsilon^*(d) = \frac{\sqrt{2}}{4}$. From Theorem 1 it follows that for all $\varepsilon < \frac{\sqrt{2}}{4}$ the composite control

$$u_c = u_s + u_f = -(\sqrt{2} - 1)y - \sqrt{2}\,x^2$$

is stabilizing. Moreover the estimate of the domain of attraction corresponding to this choice of d is given by

$$D = \{x^2 + (1/2)(y + x^2)^2 \le 1\}.$$

The fast subsystem in this example is exponentially stable, so as in Example 2, we apply the result of both Parts I and II. To apply the result of Part I, first we observe that assumption A4 holds with $\delta_1 = \delta_2 = 2 - \sqrt{2}$. We choose $s_0 = 1$ and $s_1 = 1$ which yields $\varepsilon^*(s_0,s_1) = 0.0233$. From Theorem 2 we conclude that for any $\varepsilon < 0.0233$ and for all initial conditons $(x_0,y_0) \in D$ we have

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \le \varepsilon^{1/2}(x_0^2 + (1/2)(y_0 + x_0^2)^2)$$

To apply the result of Part II we observe that assumptions A5 and A6 hold with $\lambda = 2$, $C_3 = 1$, $C_4 = 1$ and $\xi_1 = \xi_2 = 1/2$. We choose $s_2 = s_3 = 4$ and from (4-41) it follows that $\varepsilon^*(s_2,s_3) = 0.118$. From Theorem 3 we conclude that $\forall\varepsilon < 0.118$ and for all initial conditions $(x_0,y_0) \in D$ we have

$$|J_{uc}(x_0,y_0) - J_s(x_0)| \le \varepsilon(4x_0^2 + 2(y_0 + x_0^2)^2 + (4 - \sqrt{2})x_0^2|y_0 + x_0^2|)$$

Finally it is pointed out that the cost function $J_s(x)$ is not quadratic-type but it satisfies all of our requirements with $\psi(x) \neq |\nabla_x J_s|$. This problem belongs to the class of problems considered in [39], but it doesn't satisfy the conditions of [39], which is mainly due to the fact that $\psi(x)$ is not equal to $|\nabla_x J_s|$ which is an implicit requirement in [39].

Appendix 1

From chapter 2 we have;

$$\dot{\nu} \leq - \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \tilde{\mathbf{T}} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{A-1}$$

where

$$\tilde{\mathbf{T}} = \begin{pmatrix} (1 - d)\,\alpha_1 & -(1 - d)\,\beta_1/2 - d\,\beta_2/2 \\ -(1 - d)\,\beta_1/2 - d\,\beta_2/2 & d(\frac{\alpha_2}{\varepsilon} - \gamma) \end{pmatrix} \tag{A-2}$$

Also we can write

$$\begin{aligned} L(x,y,M(x) + \Gamma(x,y)) + e\dot{\nu} = {} & L(x,y,M(x) + \Gamma(x,y)) \\ & - L(x,h(x,M(x)),M(x)) + \\ & L(x,h(x,M(x)),M(x)) + e\dot{\nu} \end{aligned}$$

Using (A-1) and assumptions A1 and A4 we have

$$L(x,y,M(x) + \Gamma(x,y)) + e\dot{\nu} \leq \delta_1\,\phi^2 + \delta_2\psi\phi - e \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \tilde{\mathbf{T}} \begin{pmatrix} \psi \\ \phi \end{pmatrix} + \alpha_0\psi^2$$

$$= - \begin{pmatrix} \psi \\ \phi \end{pmatrix} \quad \mathbf{T}_1 \begin{pmatrix} \psi \\ \phi \end{pmatrix} \qquad \text{Q.E.D.}$$

Appendix 2

From assumptions A1 and (4-14) it follows that

$$L_s > \alpha_1 \psi^2$$

Also taking derivative of W along trajectories of (4-15) and using assumptions A2 and A3 we will have

$$-\frac{dW}{dt} \geq \frac{\alpha_2}{\varepsilon} \phi_2 - K_1\phi^2 - K_2\phi\psi - C_1\phi^2 - C_2\psi\phi$$

$$= \frac{\alpha_2}{\varepsilon} \phi^2 - \gamma\phi^2 - \beta_2\phi\psi$$

Now we have

$$qL_s - L_{uc} - \varepsilon\hat{s}_1 \frac{dW}{dt} - q(\nabla_x J_s)^T(f_{uc} - f_s)$$

$$= (q - 1)L_s + L_s - L_{uc} - \varepsilon\hat{s}_1 \frac{dW}{dt} - q(\nabla_x J_s)^T(f_{uc} - f_s)$$

$$\geq (q - 1)\ \alpha_1\psi^2 - \delta_1\phi^2 - \delta_2\psi\phi + \hat{s}_1\ \alpha_2\phi^2 - \varepsilon\hat{s}_1\gamma\phi^2$$

$$- \varepsilon\hat{s}_1\ \beta_2\phi\psi - q\ \beta_1\psi\phi$$

$$= \begin{pmatrix} \psi \\ \phi \end{pmatrix}^T \dot{T}_2 \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

Q.E.D.

CHAPTER V

DECENTRALIZED CONTROL, USING LOCAL HIGH-GAIN STATE FEEDBACK

1. Introduction

Decentralized stabilization of the nonlinear interconnected system

$$\dot{x}_i = f_i(x_i) + B_i(g_i(x_1,\ldots,x_N) + u_i), \qquad i = 1,2,\ldots,N$$

is considered. This class of systems includes as a special case the one studied by Davison [42]. It also includes the linear systems studied by Ikeda and Siljak [43], Yamakami and Geromel [44], Young [45], Sezer and Huseyin [46] and Bradshaw [47]. In [3] we showed that the use of local high-gain state feedback control laws can stabilize this system provided that N lower-order isolated subsystems are stabilizable. That result was a conceptual one in the sense that it was shown to hold for sufficiently large feedback gains. In this chapter we address the important question of deterimining how large the gains should be in order for the closed-loop system to possess certain stability properties. In particular, we determine the gains needed so that the closed-loop system is asymptotically stable, the gains needed so that it is asymptotically stable with a certain domain of attraction or the gains needed so that it is exponentially stable with a certain degree of exponential stability.

This chapter is organized as follows. In Section 2 we state the problem and motivate the structure of high-gain feedback control. In

Section 3 the main asymptotic stability result is given and the choice of the feedback gains is discussed. Section 4 contains an exponential stability result and Section 5 summarizes the stabilization algorithm. In Section 6 two examples are used to illustrate the procedure and compare our results with some of the previous ones.

2. Problem Statement

Consider the nonlinear interconnected system

$$\dot{x}_i = f_i(x_i) + B_i(g_i(x) + u_i), \qquad i = 1,2,\ldots,N \tag{5-1}$$

where $x_i \epsilon R^{n_i}$, $u_i \epsilon R^{r_i}$ and $x^T = (x_1^T, x_2^T, \ldots, x_N^T)$. It is assumed that $x_i = 0$ is an equilibrium point of the ith isolated free subsystem

$$\dot{x}_i = f_i(x_i) \qquad i = 1,2,\ldots,N \tag{5-2}$$

and that $x = 0$ is the unique point in the region of interest where $g_i(x)$ vanishes. It is desired to design a decentralized state feedback control law in the form

$$u_i = h_i(x_i) \tag{5-3}$$

such that $h_i(0) = 0$ and $x = 0$ is an asymptotically stable equilibrium point of the closed-loop system

$$\dot{x}_i = f_i(x_i) + B_i(g_i(x) + h_i(x_i)). \tag{5-4}$$

It is assumed that the nonlinear functions f_i, g_i and h_i satisfy the conditions for existence and uniqueness of the solution of (5-4) and that the matrix B_i has full rank. In designing the control law (5-3) we follow the philosophy adopted in [42-47]; namely, u_i is designed to stabilize the ith isolated subsystem with a large stability margin so that the interconnected system (5-4) remains asymptotically stable in spite of arbitrary but bounded interconnections. The main drawback of this philosophy is neglecting any stabilizing effect of the interconnections and treating all of them as if they were destabilizing,

leading naturally to conservative results. On the other hand, adopting this philosophy alleviates the need for thorough modeling of the interconnections and results in feedback control laws that are robust with respect to variations in those interconnections. Achieving large stability margins for the isolated subsystems requires the use of high gain feedback. We start by assuming high-gain feedback in the form

$$u_i = k_i h_i(x_i) \tag{5-5}$$

where k_i is a positive scalar parameter. It is assumed that all the coefficients of $h_i(x_i)$ are of order one so that the order of magnitude of the feedback function is determined by the order of k_i which can be chosen sufficiently large. To see clearly the effect of applying the control (5-5), we follow [48] in using the state transformation

$$\begin{pmatrix} y_i \\ z_i \end{pmatrix} = T_i x_i \tag{5-6}$$

where $y_i \epsilon R^{m_i}$, $z_i \epsilon R^{r_i}$, $m_i = n_i - r_i$ and T_i satisfies

$$T_i B_i = \begin{pmatrix} 0 \\ G_i \end{pmatrix} \tag{5-7}$$

with G_i being an r_i x r_i nonsingular matrix. The ith isolated subsystem is transformed into

$$\begin{aligned} \dot{y}_i &= \phi_i(y_i, z_i) \\ \dot{z}_i &= \pi_i(y_i, z_i) + G_i u_i, \end{aligned} \tag{5-8}$$

where ϕ_i and π_i are functions of f_i and T_i. Applying the ith control law (5-5) to (5-8) and writing $k_i = 1/\varepsilon_i$ yields

$$\begin{aligned} \dot{y}_i &= \phi_i(y_i, z_i) \\ \varepsilon_i \dot{z}_i &= \varepsilon_i \pi_i(y_i, z_i) + G_i \tilde{h}_i(y_i, z_i) \end{aligned} \tag{5-9}$$

where $\tilde{h}_i(y_i, z_i) = h_i \left(T_i^{-1}\begin{pmatrix} y_i \\ z_i \end{pmatrix}\right)$.

Equation (5-9) resembles a singularly perturbed system. Therefore, stability of (5-9), when ε_i is sufficiently small, can be investigated using singular perturbation techniques presented in chapter 2. We define a reduced-order subsystem and a boundary-layer subsystem. The reduced-order subsystem is obtained by setting $\varepsilon_i = 0$ in (5-9) yielding

$$\begin{aligned} \dot{y}_i &= \phi_i(y_i, z_i) \\ 0 &= \tilde{h}_i(y_i, z_i) \end{aligned} \tag{5-10}$$

where we have used the nonsingularity of G_i. It is crucial now to solve the second equation of (5-10) to get z_i as a function of y_i. We simplify this process by restricting $\tilde{h}_i$ to the form

$$\tilde{h}_i(y_i, z_i) = M_i(z_i - \eta_i(y_i)) \tag{5-11}$$

where M_i is an r_i x r_i nonsingular matrix. With (5-11) the reduced-order subsystem is given by

$$\dot{y}_i = \phi_i(y_i, \eta_i(y_i)) \tag{5-12}$$

and the boundary-layer subsystem is given by

$$\varepsilon_i \dot{z}_i = G_i M_i z_i \tag{5-13}$$

Since asympotic stability of both the reduced-order and boundary-layer subsystems is required for the asymptotic stability of the singularly perturbed system when ε_i is sufficiently small (chapter 2), $\eta_i(y_i)$ should be chosen to stabilize (5-12) and M_i should be chosen to satisfy

$$\mathrm{Re}\lambda(G_i M_i) < 0 \tag{5-14}$$

We notice that (5-14) can be always achieved since G_i is nonsingular. The ith control law is given by

$$u_i = \frac{1}{\varepsilon_i} M_i\left(z_i - \eta_i(y_i)\right). \tag{5-15}$$

when (5-15) is applied to the interconnected system (5-1), the closed-loop system is given by

$$\dot{y}_i = \phi_i(y_i,\ z_i),$$
$$\varepsilon_i \dot{z}_i = \varepsilon_i \pi_i(y_i,\ z_i) + \varepsilon_i G_i \theta_i(y,\ z) + G_i M_i\left(z_i - \eta_i(y_i)\right), \tag{5-16}$$

where θ_i is a function of g_i, G_i and T_i, and y, z are given by $y^T = (y_1^T,\ldots,y_N^T)$, $z^T = (z_1^T,\ldots,z_N^T)$. Conditions for the asymptotic stability of (5-16) will be derived in Section 3. The proof of the main result in Section 3 utilizes the two-time-scale decomposition and the composite Lyapunov function techniques [cf. chapter 2, 8 - 9].

3. Main Result

Let $S_{y_i} \subset R^{m_i}$ be a closed set that contains the origin. Assume that the following conditions hold* for every $i = 1,\ldots,N$.

(I) There exists a function $\eta_i(y_i)$ and a Lyapunov function $V_i(y_i)$ such that $\forall y_i \in S_i$

(a) $[\nabla_{y_i} V_i(y_i)]^T \phi_i(y_i,\ \eta_i(y_i)) \leq -k_{i1}\psi_i^2(y_i)$,

(b) $\|\nabla_{y_i} V_i(y_i)\| \leq k_{i2}\psi_i(y_i)$,

where $\psi_i(\cdot)$ is a positive-definite function. Let c_i be a positive number such that

$$L_i \triangleq \left\{y_i | V_i(y_i) \leq c_i\right\} \subset S_{y_i}.$$

---

*We use c_i, d_i, $\overline{d}_i$, k_{ij} and $k_{\ell mn}$ to denote positive scalars.

This condition implies the existence of a stabilizing feedback control law such that the closed-loop reduced-order subsystem (5-12) is asymptotically stable, has a quadratic-type Lyapunov function and has L_i as an estimate of its domain of attraction.

(II) The functions $\phi_i(y_i, z_i)$, $\pi_i(y_i, z_i)$ satisfy Lipschitz conditions in z_i, i.e.,

(a) $\|\phi_i(y_i, z_i) - \phi_i(y_i, \bar{z}_i)\| \leq k_{i3}\|z_i - \bar{z}_i\|$,

(b) $\|\pi_i(y_i, z_i) - \pi_i(y_i, \bar{z}_i)\| \leq k_{i4}\|z_i - \bar{z}_i\|$,

$\forall y_i, z_i \in S_i$, where $S_i = \left\{ y_i \in S_{y_i} \text{ and } z_i \in R^{r_i} \,\middle|\, \|z_i - \eta_i(y_i)\|^2 \leq \xi_i \right\}$.

(III) The functions $\pi_i(y_i, \eta_i(y_i))$, $\phi_i(y_i, \eta_i(y_i))$ and $\eta_i(y_i)$ satisfy the following smoothness requirements $\forall y_i \in S_{y_i}$;

(a) $\|\pi_i(y_i, \eta_i(y_i)\| \leq k_{i5}\psi_i(y_i)$,

(b) $\|[\nabla_{y_i}\eta_i(y_i)]^T \phi_i(y_i, \eta_i(y_i))\| \leq k_{i6}\psi_i(y_i)$,

(c) $\|\nabla_{y_i}\eta_i(y_i)\| \leq k_{i7}$.

(IV) The functions $\theta_i(y, z)$ and $\eta(y)$ satisfy the following interconnection constraints $\forall y, z \in S$ where $\eta^T(y) = (\eta_1^T(y), \ldots, \eta_N^T(y))$, and $S = S_1 x, \ldots x S_N$.

(a) $\|G_i\theta_i(y, \eta(y))\| \leq \sum_{j=1}^{N} k_{i8j}\psi_j(y_j)$,

(b) $\|G_i\theta_i(y, z) - G_i\theta_i(y, \bar{z})\| \leq \sum_{j=1}^{N} k_{i9j}\|z_j - \bar{z}_j\|$.

From (I-b) and (II-a) it follows that

(II-a)' $(\nabla_{y_i}V_i(y_i))^T(\phi_i(y_i, z_i) - \phi_i(y_i, \bar{z}_i)) \leq e_{ii}\psi_i(y_i)\,\|z_i - \bar{z}_i\|$;

also from (II-a) and (III-c) it follows that

(III-c)' $(\nabla_{y_i}\eta_i(y_i))^T(\phi_i(y_i, z_i) - \phi_i(y_i, \bar{z}_i)) \leq K_{i10}\,\|z_i - \bar{z}_i\|$, where e_{ii} and K_{i10} can be taken as $K_{i2}\,K_{i3}$ and $K_{i3}\,K_{i7}$, respectively. However, direct evaluation of e_{ii} and K_{i10} by verifying inequalities (II-a)' and (III-c)' rather than inequalities (I-b), (II-a) and (III-c) will lead to less conservative results, since (II-a)' and (III-c)' are the inequalities that will be used later. Before we proceed we need to define some parameters and matrices. Let the symmetric matrix P_i be the positive definite solution of the Lyapunov equation

$$P_i(G_iM_i) + (G_iM_i)^T P_i = -I_i \qquad (5\text{-}17)$$

where I_i is the $r_i \times r_i$ identity matrix and M_i has been chosen to satisfy (5-14). We denote $\|P_i\|$ by p_i and the minimum eigenvalues of P_i by $\hat{p}_i$. Let $t_{ii} = -2(k_{i5} + k_{i6} + k_{i8i})$, $t_{ij} = -2k_{i8j}$,

$$\gamma_{ii} = 2(k_{i10} + K_{i4} + k_{i9i}),\ \gamma_{ij} = 2k_{i9j},$$

$$q_{ii} = \frac{1}{p_i \varepsilon_i} - \gamma_{ii},\ q_{ij} = -\gamma_{ij},$$

and define the matrices Q, T, K and E as

$$Q = [q_{ij}],\ T = [t_{ij}],\ K = \mathrm{diag}(k_{i1}),\ E = \mathrm{diag}(-e_{ii}).$$

Taking $\bar{d}_i$, d_i, $i = 1,\ldots,N$, as arbitrary positive numbers, we define the matrices

$$D = \mathrm{diag}(d_i),\ \bar{D} = \mathrm{diag}(\bar{d}_i),\ \tilde{Q} = \tfrac{1}{2}[DQ + Q^T D]$$

$$V = \begin{pmatrix} K & E \\ T & Q \end{pmatrix},\ \Gamma = \tfrac{1}{2}(\bar{D}E + DT),$$

$$R = \frac{1}{2}\left[\begin{pmatrix} \bar{D} & 0 \\ 0 & D \end{pmatrix} V + V^T \begin{pmatrix} \bar{D} & 0 \\ 0 & D \end{pmatrix}\right] = \begin{pmatrix} \bar{D}K & \Gamma^T \\ \Gamma & \tilde{Q} \end{pmatrix}$$

Our stability criterion is stated in the following theorem.

<u>Theorem 1</u>: If assumptions (I)-(IV) hold and if the matrices $\tilde{Q}$ and $(\bar{D}K - \Gamma^T \tilde{Q}^{-1} \Gamma)$ are positive-definite, then the origin $(y = 0,\ z = 0)$ is an asymptotically stable equilibrium point of the closed-loop system

(5-16). Moreover, the set L,

$$L \triangleq \left\{y, z \,\middle|\, \nu(y, z) \leq \min(\min_i \bar{d}_i c_i, \min_i \frac{d_i \hat{p}_i \xi_i}{p_i})\right\}$$

where

$$\nu(y, z) = \sum_{i=1}^{N} \bar{d}_i V_i(y_i) + \sum_{i=1}^{N} \frac{d_i}{p_i} (z_i - \eta_i(y_i))^T P_i (z_i - \eta_i(y_i)) \tag{5-18}$$

is included in the domain of attraction.

<u>Proof</u>: Consider the composite Lyapunov function $\nu(y, z)$. It is shown in Appendix A that the derivative of ν along the trajectory of (5-16) is bounded by

$$\dot{\nu}(y, z) \leq \begin{pmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y_N)\| \end{pmatrix}^T R \begin{pmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y_N)\| \end{pmatrix} \forall y, z \in S \tag{5-19}$$

It is sufficient to show that R is positive definite, which is equivalent to the positive-definiteness of both $\tilde{Q}$ and $(\bar{D}K - \Gamma^T \tilde{Q}^{-1} \Gamma)$. Observing that $L \subset S$ completes the proof.

We discuss some aspects of Theorem 1. First, it is always possible to choose $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ in such a way that the matrices $\tilde{Q}$ and $(\bar{D}K - \Gamma^T \tilde{Q}^{-1} \Gamma)$ are positive-definite. This can be seen by observing that the diagonal elements of $\tilde{Q}$ can be made as large as desired by using sufficiently small ε_i's. Second, for the class of systems considered in this chapter Theorem 1 is more general than our earlier result [3] because it provides an asymptotic stability rather than exponential stability criterion. It also allows for a wider class of interconnections. Third, the positive parameters $d_1, \ldots, d_N$, $\bar{d}_1, \ldots, \bar{d}_N$, which are used as weights in the composite

Lyapunov function (5-18), are arbitrary. The freedom in choosing these parameters may be used to establish certain stability properties for the closed-loop system (5-16). Finally, we notice that the choice of identity matrix in solving the Lyapunov equation (5-17) has been made in order to reduce the upper bound on $\dot{v}$[49].

The design of the feedback control law (5-15) to meet the conditions of Theorem 1 may follow one of two possible approaches. In the first approach the parameters d_i and $\bar{d}_i$ are chosen first in such a way that the estimate of domain of attraction, L, contains a certain desired set; then the parameters ε_i are chosen to make $\tilde{Q}$ and $(\bar{D}K - \Gamma^T\tilde{Q}^{-1}\Gamma)$ positive-definite. One interesting choice of d_i and $\bar{d}_i$ is the one that provides the largest possible estimate of domain of attraction. To find out that choice, let $w_i = \bar{d}_i c_i$, $w_{i+N} = \frac{d_i p_i \hat{\xi}_i}{p_i}$ $i = 1,\ldots,N$, $w_0 = \min w_i$ and $\tilde{w}_i = w_i/w_0$. The set L can be rewritten as

$$L(\tilde{w}) = \left\{ y,z \left| \sum_{i=1}^{N} (\tilde{w}_i/c_i)V_i(y_i) + \sum_{i=1}^{N} \left(\frac{\tilde{w}_{i+N}p_i}{\hat{p}_i\xi_i}\right) \frac{1}{p_i} (z_i - \eta_i(y_i))^T P_i (z_i - \eta_i(y_i)) \leq 1 \right. \right\}.$$

Since $\tilde{w}_i \geq 1$ for $i = 1,\ldots,2N$ with $\tilde{w}_j = 1$ for some j, it is obvious that $L(\tilde{w})$ is largest when $\tilde{w}_i = 1 \; \forall i$. In summary we have the following observation.

<u>Observation 1</u>: For the largest estimate of the domain of attraction L^*, that can be obtained using Theorem 1, choose $\bar{d}_i = 1/c_i$ and $d_i = p_i/\hat{p}_i\xi_i$. Then, L^* is given by

$$L^* = \left\{ y,z \left| \sum_{i=1}^{N} \frac{1}{c_i} V_i(y_i) + \sum_{i=1}^{N} \frac{1}{\hat{p}_i\xi_i} (z_i - \eta_i(y_i))^T P_i (z_i - \eta_i(y_i)) \leq 1 \right. \right\}.$$

Moreover, if conditions II and IV hold globally in z, then choose ξ_i as large as needed to extend L^* in the direction of the z_i-axis.

The second approach to meet the conditions of Theorem 1 is to start by choosing the largest possible parameters ε_i for which our results guarantee that the closed-loop system (5-16) is asymptotically stable, and then choose the parameters d_i and $\overline{d}_i$ to meet the conditions of the theorem. This approach is interesting because it provides the smallest possible feedback gains within the framework of our results. To see how to achieve that, recall that the matrix R in (5-19) can be written as

$$R = \frac{1}{2}\left[\begin{pmatrix} \overline{D} & 0 \\ 0 & D \end{pmatrix} V + V^T \begin{pmatrix} \overline{D} & 0 \\ 0 & D \end{pmatrix}\right]$$

where

$$V = \begin{pmatrix} K & E \\ T & Q \end{pmatrix}.$$

Observing that the off-diagonal elements of V are non-negative, it follows that there exists d_i and $\overline{d}_i$ such that R is positive-definite if and only if V is an M-matrix [10]. Moreover, V is an M-matrix if and only if both Q and $(K - EQ^{-1}T)$ are M-matrices [50]. Since ε_i's appear only on the diagonal elements of Q, choosing ε_i's sufficiently small will make both Q and $(K - EQ^{-1}T)$ M-matrices. Thus, we have the following observation.

<u>Observation 2</u>: The largest numbers ε_i's for which our results guarantee that the closed-loop system (5-16) is asymptotically stable are those numbers just enough to make both Q and $(K - EQ^{-1}T)$ M-matrices.

We notice that choosing ε_i's as in Observation 2 guarantees that the origin will be an asymptotically stable equilibrium point of (5-16), but does not guarantee that L, the estimate of the domain of attraction, will include any specified set.

We can have explicit expression for ε_i's in each case by using a diagonal dominance approach to fulfill the requirement on corresponding matrices. However this is achieved at the cost of getting conservative results for ε_i's and therefore higher gains.

4. Exponential Stability

In this section we give an exponential stability result for the closed-loop system (5-16).

Theorem 2: Suppose assumptions (I-IV) of Theorem 1 hold. Besides, let $V_i(y_i)$ satisfy

(V) $\quad \sigma_{i2}\psi_i^2(y_i) \leq V_i(y_i) \leq \sigma_{i1}\psi_i^2(y_i), \quad i = 1,\dots,N,$

and ψ_i's be functions of class $\mathcal{K}$, then the conclusions of Theorem 1 hold with exponential stability replacing asymptotic stability. Moreover, let

$$\psi_i(y_i) = \|y_i\| \quad i = 1,\dots,N \text{ and } \Sigma = \mathrm{diag}(\sigma_{i1}),$$

then any positive number α, such that:

$$\alpha < \overset{*}{\alpha} \triangleq (1/2)(\min_i (K_{i1}/\sigma_{i1})) \tag{5-20}$$

is a degree of exponential stability of the closed-loop system (5-16) if

$$R_1 \triangleq \begin{pmatrix} \overline{D}K - 2\alpha\overline{D}\Sigma & \Gamma^T \\ \Gamma & \tilde{Q} - 2\alpha D \end{pmatrix} \text{ is positive semi definite.}$$

Proof: From Theorem 1 and assumption (V) it follows that the comparison functions of the Lyapunov function (5-18) and its derivative along the trajectory of (5-16), are of the same order of magnitude which implies exponential stability [9]. Furthermore, one can majorize the derivative of (5-18) along (5-16) as

$$\dot{v}(y,z) \leq - \begin{bmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y\ )\| \end{bmatrix}' R_1 \begin{bmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y\ )\| \end{bmatrix} -2\alpha v(y, z), \tag{5-21}$$

which in fact is a simple manipulation of (5-19). In order for the closed-loop system (5-16) to be exponentially stable with degree of exponential stability α, (i.e. $\dot{V} \leq -2\alpha V$), it is enough to require that R_1 be positive semi-definite and this completes the proof of Theorem 2.

Similar to Theorem 1 we notice that matrix R_1 can be always made positive semi-definite by choosing ε_i's sufficiently small. Also we notice that Theorem 2 is more general than our earlier exponential stability result [3], since it provides an estimate of the degree of exponential stability. The choice of parameters d_i, $\overline{d}_i$ and ε_i to meet the condition of Theorem 2 follows the same guidelines discussed in Section 3. Without repeating the discussions we mention that Observation 1 still holds, while in Observation 2 the matrices Q and $(K - EQ^{-1}T)$ are replaced by $(Q - 2\alpha I)$ and $(K - 2\alpha\Sigma - E(Q - 2\alpha I)^{-1}T)$, respectively. In fact, the requirement that these matrices be M-matrices can be relaxed since Theorem 2 requires only the positive-semidefiniteness of R_1. However, stating the relaxed conditions requires involved notation and definitions. Interested readers are referred to [51].

5. Algorithm

The stabilization of system (5-1) by means of decentralized state feedback control law can be performed according to the following algorithm.

Step (1): Apply local transformation (5-6) to transform (5-1) into

$$\dot{y}_i = \phi_i(y_i, z_i)$$

$$\dot{z}_i = \pi_i(y_i, z_i) + G_i\theta_i(y, z) + G_i u_i \qquad i = 1 \ldots N.$$

Step (2): Find $\eta_i(y_i)$ such that $y_i = \phi_i(y_i, \eta_i(y_i))$ is asymptotically stable. Find also a quadratic-type Lyapunov function $V_i(y_i)$ and a comparison function $\psi_i(y_i)$ and compute the numbers. c_i, k_{i1}, e_{ii}, k_{i6}, and k_{i10}.

Step (3): Choose M_i so that Re $\lambda[G_iM_i] < 0$, and compute P_i, p_i, $\hat{p}_i$.

Step (4): Find the interconnection bounds k_{i4}, k_{i5}, k_{i8j} and k_{i9j}.

Step (5): Choose $\varepsilon_1, \ldots, \varepsilon_N$ to meet the conditions of Theorem 1 or 2. This choice may be direct as in Observation 2 or indirect by first choosing d_i, $\bar{d}_i$'s and then choosing ε_i's.

Step (6): Use the decentralized state feedback control law

$$u_i = \frac{1}{\varepsilon_i} G_iM_i(z_i - \eta_i(y_i)) \qquad i = 1 \ldots, N.$$

The decentralized feedback law proposed above is a robust feedback control law, namely we have infinite gain-increase margins and by choosing ε_i's appropriately we can have arbitrary gain reduction. Moreover the proposed feedback law is robust in the structural sense, namely it maintains stability in the presence of changes in the interconnections or disconnection of any subsystem from the others.

Another aspect of the proposed algorithm is that stabilization of subsystems is done using lower-order models, namely we stabilize reduced-order subsystems of order $(n_i - r_i)$.

6. Examples

Two examples will be considered. One linear and the other nonlinear.

Example 1: We consider the linear system studied in [46], where the overall system consists of two linearly interconnected subystems, and it is represented by:

$$S_1 \begin{cases} \dot{x}_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left\{ u_1 + L_{12}\tilde{y}_2 \right\} \\ \tilde{y}_1 = (1 \quad 1)x_1 \end{cases}$$

$$S_2 \begin{cases} \dot{x}_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left\{ u_2 + L_{21}\tilde{y}_1 \right\} \\ \tilde{y}_2 = (1 \quad 1 \quad 1)x_2 \end{cases}$$

where $x_1 = (x_{11}, x_{12})^T$, $x_2 = (x_{21}, x_{22}, x_{23})^T$, $|L_{12}| \leq 1, |L_{21}| \leq 1$.

Step (1): This interconnected system is already in the appropriate form and we don't need local transformation. Let us write this example in our notation by defining $y_1 = x_{11}$, $z_1 = x_{12}$, $y_2 = (x_{21}, x_{22})^T$, $z_2 = x_{23}$.

$$S_1 \begin{cases} \dot{y}_1 = z_1 \\ \dot{z}_1 = -y_1 - z_1 + L_{12}(1 \quad 1)y_2 + L_{12}z_2 + u_1 \end{cases}$$

$$S_2 \begin{cases} \dot{y}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} z_2 \\ \dot{z}_2 = (0 \quad 1)y_2 + L_{21}y_1 + L_{21}z_1 + u_2 \end{cases}$$

Step (2): We choose $\eta_1(y_1) = -y_1$, $\eta_2(y_2) = -(1 \;\; 1)y_2$ and $V_1(y_1) = 1/2\, y_1^2$,
$V_2(y_2) = y_2^T \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{pmatrix} y_2$, then we have $\psi_1(y_1) = \|y_1\|$, $\psi_2(y_2) = \|y_2\|$,
$k_{i1} = k_{i6} = 1 \quad i = 1, 2$, $e_{11} = 1$, $e_{22} = \sqrt{5}$, $k_{i10} = 1 \quad i = 1, 2$.

Step (3): We choose $M_i = -1 \quad i = 1, 2$, then we have, $p_1 = p_2 = 1/2$.

Step (4): In this step we compute interconnection bounds and they are:

$k_{14} = 1$, $k_{24} = 0$, $k_{15} = 0$, $k_{25} = 1$, $k_{i8j} = 0$, $i = 1,2$, $j = 1,2$, $k_{191} = 0$, $k_{192} = 1$, $k_{291} = 1$, $k_{292} = 0$.

Step (5): To choose ε_i's in order to get lowest possible gains, using Observation (2) we should choose ε_i's so that matrices Q and $(K - EQ^{-1}T)$ are M-matrices where:

$$Q = \begin{pmatrix} \frac{2}{\varepsilon_1} - 4 & -2 \\ -2 & \frac{2}{\varepsilon_2} - 2 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$E = \begin{pmatrix} -1 & 0 \\ 0 & -2.236 \end{pmatrix}, \qquad T = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}.$$

We observe that both matrices Q and $(K - EQ^{-1}T)$ are M-matrice for the choice $\varepsilon_1 = .2711$, $\varepsilon_2 = .1442$.

Step (6): The feedback law is given by:

$u_1 = -k_1 x_1 = -(3.69 \;\; 3.69)x_1$

$u_2 = -k_2 x_2 = -(6.93 \;\; 6.93 \;\; 6.93)x_2$

At this point let us compare this result with those proposed in [42], [46], [43], and [44]. For this purpose let us use norm infinity, i.e. $\|\cdot\|_\infty$, as a measure of the largeness of the feedback gains.

Our method produces a stabilizing feedback law with, $\|k_1\|_\infty = 3.69$, $\|k_2\|_\infty = 6.93$.

The method proposed in [42] produces gains,

$k_1 = (\rho^4 \quad 2\rho^2)$, $k_2 = (\rho^3 \quad 3\rho^2 \quad 3\rho)$, $\rho > 486$ and $\|k_1\|_\infty = (486)^4$,

$\|k_2\|_\infty = (486)^3$.

In [46] the proposed gains are:

$k_1 = (3 \quad 4)$, $k_2 = (32 \quad 45 \quad 13)$, and $\|k_1\|_\infty = 4$, $\|k_2\|_\infty = 45$.

The method suggested in [43] produces following gains:

$k_1 = (1.46 \quad 2.22)$, $k_2 = (2.12 \quad 6.78 \quad 4.98)$, $\|k_1\|_\infty = 2.22$, $\|k_2\|_\infty = 6.78$.

Finally in [44] the proposed gains are:

$k_1 = (.41 \quad .68)$, $k_2 = (1.00 \quad 3.36 \quad 2.78)$, $\|k_1\|_\infty = .68$, $\|k_2\|_\infty = 3.36$.

The above numerical figures show that our feedback gains are seven orders of magnitude less than Davison's method [42], and, more or less, of the same order of magnitude as the methods of Sezer and Huseyin [46], Ikeda and Siljak [43], and Yamakami and Geromel [44]. In fact, our gains are lower than those of [46] but higher than those of [43] and [44]. It is not our point here to evaluate our method versus those of [43, 44, 46] since those methods were developed for linear systems using techniques limited to linear systems (e.g., Riccati equations), so it would not be a surprise if they work with linear systems better than our method. We are, however, pleased with the obvious improvement over Davison's method since it is the only method that is applicable to a class of nonlinear systems.

<u>Example 2</u>: We consider the Large Space Telescope, L.S.T. example provided by Siljak and Vukcevic [52]. L.S.T. is described by a nonlinear interconnection of three subsystems and it is represented by:

S_i, $\dot{x}_i = A_i x_i + b_i(u_i + h_i(x)) \quad i = 1,2,3$

where $x_i = (x_{i1}, x_{i2})^T$ and $A_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad i = 1,2,3,$

$b_1 = \begin{pmatrix} 0 \\ 85.62 \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ 13.69 \end{pmatrix}$, $b_3 = \begin{pmatrix} 0 \\ 13.21 \end{pmatrix}$,

$h_1(x) = -.0026\, x_{22}\, x_{32}$, $h_2(x) = .064\, x_{12} x_{32}$, $h_3(x) = -.061\, x_{12}\, x_{22}$

Let us apply our algorithm to this example.

Step (1): This example is already in appropriate form and we don't need to use any local transformation. In order to write L.S.T. in our notation let, $y_1 = x_{11}$, $y_2 = x_{21}$, $y_3 = x_{31}$, $z_1 = x_{12}$, $z_2 = x_{22}$, $z_3 = x_{32}$ then L.S.T. can be written as

$$S_1 \begin{cases} y_1 = z_1 \\ z_1 = -\alpha_1 z_2 z_3 + 85.62\, u_1 \end{cases}$$

$$S_2 \begin{cases} \dot{y}_2 = z_2 \\ \dot{z}_2 = +\alpha_2 z_1 z_3 + 13.69\, u_2 \end{cases}$$

$$S_3 \begin{cases} \dot{y}_3 = z_3 \\ \dot{z}_3 = -\alpha_3 z_1 z_2 + 13.21\, u_3 \end{cases}$$

where $\alpha_1 = .2221$, $\alpha_2 = .8754$, $\alpha_3 = .8112$.

Step (2): We choose $\eta_i(y_i) = -y_i \quad i = 1,2,3$, $V_i(y_i) = (1/2)y_i^2 \quad i = 1,2,3$.

Then we have;

$\psi_i(y_i) = |y_i| \quad i = 1,2,3$, $k_{i1} = k_{i6} = k_{i10} = e_{ii} = 1$.

Step (3): We choose $M_i = -1$ so $P_1 = 1/171.24$, $P_2 = 1/27.38$, $P_3 = 1/26.42$.

Step (4): Let us assume $|y_i| \le \mu_i$, $|z_i| \le \sigma_i \quad i = 1,2,3$ then we have

$k_{i4} = k_{i5} = 0 \quad i = 1,2,3$, $k_{182} = \alpha_1 \mu_3$, $k_{181} = k_{183} = k_{281} = k_{282} = k_{382}$

$= k_{383} = 0$, $k_{283} = \alpha_2\mu_1$, $k_{381} = \alpha_3\mu_2$, $k_{191} = k_{292} = k_{393} = k_{192} = \alpha_1\sigma_3$, $k_{193} = \alpha_1\sigma_2$, $k_{291} = \alpha_2\sigma_3$, $k_{293} = \alpha_2\sigma_1$, $k_{391} = \alpha_3\sigma_2$, $k_{392} = \alpha_3\sigma_1$.

Step (5): We have

$$K = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} -2 & -2\alpha_1\mu_3 & 0 \\ 0 & -2 & -2\alpha_2\mu_1 \\ -2\alpha_3\mu_2 & 0 & -2 \end{pmatrix},$$

$$Q = \begin{bmatrix} \frac{2x85.62}{\varepsilon_1} - 2 & -2\alpha_1\sigma_3 & -2\alpha_1\sigma_2 \\ -2\alpha_2\sigma_3 & \frac{2x13.69}{\varepsilon_2} - 2 & -2\alpha_2\sigma_1 \\ -2\alpha_3\sigma_2 & -2\alpha_3\sigma_1 & \frac{2x13.21}{\varepsilon_3} - 2 \end{bmatrix}.$$

According to Observation 2, the lowest possible gains are achieved by choosing the largest possible ε_i's, referred to as ε_i^*'s, which makes both Q and $(K - EQ^{-1}T)$ M-matrices. But, the off-diagonal elements of T and Q depend on σ_i's and μ_i's, which means that ε_i^*'s will depend on σ_i's and μ_i's. It can be verified that the smaller σ_i's and μ_i's the larger the ε_i^*'s. Moreover, for any choice of the σ_i's and μ_i's, ε_i^*'s must satisfy;

$\varepsilon_1^* < 42.7115$, $\varepsilon_2^* < 6.8390$, $\varepsilon_3^* < 6.6491$.

We have chosen $\sigma_1 = \sigma_2 = \sigma_3 = .1892$, $\mu_1 = \mu_2 = \mu_3 = .075$ and the ε_i^*'s corresponding to this choice are $\varepsilon_1^* < 40.4789$, $\varepsilon_2^* < 5.1678$, $\varepsilon_3^* < 5.5761$.

Step (6): Let us choose $\varepsilon_1 = 40.4$, $\varepsilon_2 = 5$, $\varepsilon_3 = 5$, then the decentralized feedback law is given by

$u_1 = -.0247(x_{11} + x_{12})$

$u_2 = -.2000(x_{21} + x_{22})$

$u_3 = -.2000(x_{31} + x_{32})$,

The estimate of the domain of attraction corresponding to this feedback law is given by

$$L = \left\{ x_1, x_2, x_3 \,\middle|\, 511.10\, x_{11}^2 + 177.78\,(x_{21}^2 + x_{31}^2) + 505.28\,(x_{11} + x_{12})^2 + 128.20\,(x_{21} + x_{22})^2 + 138.34\,(x_{31} + x_{32})^2 \leq 1 \right\}.$$

To get a feeling for the interaction between the choice of the feedback gains and the estimate of the domain of attraction, let us fix σ_i and μ_i at their previously chosen values, namely $\sigma_i = .1892$, $\mu_i = .075$, and find the gains corresponding to the largest estimate of the domain of attraction. From Observation 1, the largest estimate of the domain of attraction is given by;

$$L^* = \left\{ x_1, x_2, x_3 \,\middle|\, \sum_{i=1}^{3}(177.78\, x_{i1}^2) + \sum_{i=1}^{3}(76.78\,(x_{i1} + x_{i2})^2) \leq 1 \right\}$$

and from Theorem 1, the corresponding feedback law is given by:

$u_1 = -.0299\,(x_{11} + x_{12})$

$u_2 = -.2331\,(x_{21} + x_{22})$

$u_3 = -.2277\,(x_{31} + x_{32})$,

As we can see we have achieved this estimate, L^*, at the cost of higher gains.

To be able to compare our result with [52], we explore this example further with higher value for σ_i and μ_i. Suppose $\sigma_i = 4$, $\mu_i = 1$, $i = 1,2,3$, then the highest upper bound on ε_i's are: $\varepsilon_i^* < 23.3214$, $\varepsilon_2^* < 1.18906$, $\varepsilon_3^* < 1.23808$. Let us choose $\varepsilon_1 = 23.3$, $\varepsilon_3 = 1.189$, $\varepsilon_3 = 1.238$, then the corresponding decentralized feedback law is given by:

$u_1 = -.04291\ (x_{11} + x_{12})$

$u_2 = -.84104\ (x_{21} + x_{22})$

$u_3 = -.80775\ (x_{31} + x_{32}).$

The estimate of the domain of attraction in this case is given by

$$L = \left\{x_1, x_2, x_3 \middle| 1.8361\ x_{11}^2 + x_{21}^2 + x_{31}^2 + 1.5684\ (x_{11} + x_{12})^2 + .4341\ (x_{21} + x_{22})^2 + .4684\ (x_{31} + x_{32})^2 \leq 1\right\}.$$

Again from Observation 1 the best estimate of domain of attraction, L^*, for this choice of σ_i and μ_i is given by:

$$L^* = \left\{x_1, x_2, x_3 \middle| x_{11}^2 + x_{21}^2 + x_{31}^2 + \frac{1}{9}\ (x_{11} + x_{12})^2 + \frac{1}{9}\ (x_{21} + x_{22})^2 + \frac{1}{9}\ (x_{31} + x_{32})^2 \leq 1\right\},$$

and the decentralized feedback law corresponding to, L^*, is

$u_1 = -.0949\ (x_{11} + x_{12})$

$u_2 = -.8923\ (x_{21} + x_{22})$

$u_3 = -.8560\ (x_{31} + x_{32}),$

and it shows that we have achieved the largest estimate of the domain of attraction, L^*, at the cost of higher gains.

Finally, we recall that the estimate of the domain of attraction in [52] is

$$L = \left\{x_1, x_2, x_3 \middle| 4.8\|x_1\| + 13.76\|x_2\| + 4.30\|x_3\| \leq 1.34\right\},$$

and the feedback gains that yield L are given by

$u_1 = -(2.82\ x_{11} + .3988\ x_{12})$

$u_2 = -(17.64\ x_{21} + 2.5\ x_{22})$

$u_3 = -(18.28\ x_{31} + 2.58\ x_{32})$

Obviously, our method results in smaller gains and a larger estimate of the domain of attraction compared with the method of [52].

Appendix

We have,

$$\dot{v}(y, z) = \sum_{i=1}^{N} \bar{d}_i \left[\nabla_{y_i} V_i(y_i)\right]^T \phi_i(y_i, z_i) + \sum_{i=1}^{N} \Big\{ \frac{d_i}{p_i}(z_i - \eta_i(y_i))^T P_i[\pi_i(y_i, z_i)$$

$$+ G_i\theta_i(y, z) + \frac{1}{\varepsilon_i} G_i M_i(z_i - \eta_i(y_i)) - \left[\nabla_{y_i} \eta_i(y_i)\right]^T \phi_i(y_i, z_i)] +$$

$$\frac{d_i}{p_i}[\pi_i(y_i, z_i) + G_i\theta_i(y, z) + \frac{1}{\varepsilon_i} G_i M_i(z_i - \eta_i(y_i)) -$$

$$\left[\nabla_{y_i} \eta_i(y_i)\right]^T \phi_i(y_i, z_i)]^T P_i(z_i - \eta_i(y_i))] \Big\}$$

$$\leq -\sum_{i=1}^{N} \bar{d}_i k_{i1} \psi_i^2(y_i) + \sum_{i=1}^{N} \bar{d}_i e_{ii} \|z_i - \eta_i(y_i)\| \psi_i(y_i)$$

$$- \sum_{i=1}^{N} \frac{d_i}{\varepsilon_i p_i} \|z_i - \eta_i(y_i)\|^2 + \sum_{i=1}^{N} 2d_i k_{i5} \psi_i(y_i) \|z_i - \eta_i(y_i)\|$$

$$+ \sum_{i=1}^{N} 2d_i \|z_i - \eta_i(y_i)\| \sum_{j=1}^{N} k_{i8j} \psi_j(y_j) + \sum_{i=1}^{N} d_i k_{i6} \psi_i(y_i) \|z_i - \eta_i(y_i)\|$$

$$+ \sum_{i=1}^{N} 2d_i k_{i4} \|z_i - \eta_i(y_i)\|^2 + \sum_{i=1}^{N} 2d_i \|z_i - \eta_i(y_i)\| \sum_{i=1}^{N} k_{i9j} \|z_j - \eta_j(y_i)\|$$

$$+ \sum_{i=1}^{N} 2d_i k_{i10} \|z_i - \eta_i(y_i)\|^2$$

$$= \begin{bmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y_N)\| \end{bmatrix}^T R \begin{bmatrix} \psi_1(y_1) \\ \vdots \\ \psi_N(y_N) \\ \|z_1 - \eta_1(y_1)\| \\ \vdots \\ \|z_N - \eta_N(y_N)\| \end{bmatrix}$$

CHAPTER VI

DECENTRALIZED CONTROL, USING LOCAL HIGH-GAIN DYNAMIC OUTPUT FEEDBACK

1. Introduction

Decentralized control methods for the stabilization of an interconnected system using only local feedback have developed in two distinct directions. In the first approach (e.g., [53]-[56]) a decentralized control is designed using a complete model of the interconnected system (i.e., a model which describes the local subsystems as well as the interconnections). In the second approach (e.g., [42], [43] and chapter 5) a decentralized control is designed using only models of the local subsystems whereas interconnections are characterized by bounds on their magnitudes. The latter approach invariably employs local high-gain feedback either implicity [42], [43] or explicitly as in chapter 5. Comparing the two approaches for the case of linear systems shows that the first approach, which uses more information, can stabilize a wider class of interconnected systems compared with the second approach. On the other hand, the second approach, which uses only detailed models of the local subsystems, can tolerate modeling errors and nonlinearities in the interconnections. While the use of local output feedback and dynamic compensators is typical in methods of the first approach, until recently methods of the second approach have used local state feedback. Since it is common that the entire state of each subsystem would not be accessible for feedback control, there is a definite interest in extending the methods of the second approach to the output feedback case. Recently Huseyin et. al. [57] have devised an interesting decentralized feedback control strategy in which each isolated subsystem, which is assumed to be single input-single output, is stabilized using a dynamic compensator. While designing the local compensators, the gains of the loops of the interconnected system are weakened, by an implicit use of local high-gain feedback, so that stability is retained in the presence of arbitrary interconnections satisfying the prescribed bounds. Motivated by the work of [57] and by an explicit use of local high-gain feedback as in our earlier state feedback work in chapter 5, we propose a decentralized output feedback control strategy which applies to invertible multi-input multi-output isolated subsystems. The use of explicit high-gain feedback leads to a clear understanding of the structural properties of the system and yields a simple decentralized control scheme.

2. Problem Statement

We consider the interconnected system Σ composed of N linear time invariant subsystems Σ_i described by

$$\Sigma_i; \ \dot{\tilde{x}}_i = \hat{A}_i\tilde{x}_i + \hat{B}_i(u_i + \hat{H}_i(\tilde{x})) + \hat{M}_i(y) \qquad (6\text{-}1a)$$

$$y_i = \hat{C}_i\tilde{x}_i \qquad i=1,\ldots,N \qquad (6\text{-}1b)$$

where $\tilde{x}_i \in R^{n_i}$ is the state, $u_i \in R^{m_i}$ is the control input $y_i \in R^{\ell_i}$ is the measured output of Σ_i, $\tilde{x} = (\tilde{x}_1^T,\ldots,\tilde{x}_N^T)^T$, and $y = (y_1^T,\ldots,y_N^T)^T$. The matrices $\hat{A}_i$, $\hat{B}_i$ and $\hat{C}_i$ are constant and of appropriate dimensions, and the mappings $\hat{H}_i$ and $\hat{M}_i$ satisfy the following smoothness conditions:

$$||\hat{M}_i(y)|| \leq \sum_{j=1}^{N} \hat{\beta}_{ij} \; ||y_j|| \qquad i=1,\ldots N \tag{6-2}$$

$$||\hat{H}_i(x)|| \leq \sum_{j=1}^{N} \hat{\gamma}_{ij} \; ||x_j||. \qquad i=1,\ldots N \tag{6-3}$$

We derive our results for the case of square subsystems, that is $\ell_i = m_i$ $(i=1,\ldots,N)$. However in the case of non-square subsystems, (i.e., $\ell_i \neq m_i$), they can be cast in the former case through a square-down procedure which we will discuss later. Therefore in the following we assume $\ell_i = m_i$ $(i=1,\ldots,N)$. Furthermore without loss of generality it is assumed that $\hat{C}_i$ and $\hat{B}_i$ are of full rank.

The only requirement for deriving our results is that each isolated subsystem be invertible and has all its transmission zeros in the open left-half plane. We have chosen to derive our result for three different cases depending on properties of infinite zeros of the triple $(\hat{C}_i, \hat{A}_i, \hat{B}_i)$. The first two cases are special forms of the last case. Developing results for these special cases is helpful for understanding and clarity. Before starting our analysis for each case we state the common assumption for all cases, that is:

<u>A1</u> - The isolated subsystem $\tilde{\Sigma}_i$ defined by

$$\tilde{\Sigma}_i; \; \dot{\tilde{x}}_i = \hat{A}_i x_i + \hat{B}_i u_i$$

$$y_i = \hat{C}_i \tilde{x}_i$$

has all its transmission zeros in the open left-half plane. (λ is a transmission zero of $\tilde{\Sigma}_i$ if the rank of $\begin{pmatrix} \hat{A}_i - I^*_{n_i} & \hat{B}_i \\ \hat{C}_i & 0 \end{pmatrix}$ is strictly less than $(n_i + \min(\ell_i, m_i))$, [58].)

$^*I_r$ denotes an rxr identity matrix.

3. Case 1

This case is characterized by the requirement that the infinite zeros of the triple $(\hat{C}_i, \hat{A}_i, \hat{B}_i)$ are of the first order or equivalently we assume;

A2 - $\hat{C}_i\hat{B}_i$ is nonsingular for $i=1,\ldots,N$.

This assumption obviously implies invertibility of $\tilde{\Sigma}_i$. Following chapter 5 we consider a local transformation $x_i = T_i\tilde{x}_i$ where T_i satisfies

$$T_i\hat{B}_i = \begin{pmatrix} 0 \\ B_i \end{pmatrix} \tag{6-4}$$

where $B_i \in R^{m_i x m_i}$ is nonsingular. The freedom in choosing T_i can be exploited to choose T_i such that

$$\hat{C}_i T_i^{-1} = (0 \quad , \quad I_{m_i}). \tag{6-5}$$

Performing the local transformation $x_i = T_i\tilde{x}_i$ brings Σ_i into the following form

$$\begin{aligned} \Sigma_i;\ \dot{x}_{i1} &= A_{i1}x_{i1} + A_{i2}x_{i2} + M_{i1}(y) \\ \dot{x}_{i2} &= A_{i3}x_{i1} + A_{i4}x_{i2} + B_i(u_i + H_i(x)) + M_{i2}(y) \\ y_i &= x_{i2} \qquad\qquad i=1,\ldots N \end{aligned}$$

where all the matrices A_{i1}, A_{i2}, A_{i3}, A_{i4} and mappings M_{i1}, M_{i2}, H_i are obtained in an obvious way. Notice that (6-2) and (6-3) in the new states can be rewritten as

$$||M_{i1}(y)|| \leq \sum_{j=1}^{N} \beta_{ij1}||y_j|| , \qquad i=1,\ldots N \tag{6-6a}$$

$$\|M_{i2}(y)\| \leq \sum_{j=1}^{N} \beta_{ij2} \|y_j\| \qquad i=1,\dots,N \qquad (6\text{-}6b)$$

$$\|H_i(x)\| \leq \sum_{j=1}^{N} (\gamma_{ij1} \|x_{j1}\| + \gamma_{ij2} \|x_{j2}\|) \; i=1\dots,N \qquad (6\text{-}6c)$$

where β_{ij1}, β_{ij1} and γ_{ij2} are non-negative constants.

Assumption A1 implies that the matrix A_{i1} is a stable matrix, (i.e., all its eigenvalues are in the open left half plane), since the eigenvalues of A_{i1} are the transmission zeros of the triple $(\hat{C}_i, \hat{A}_i, \hat{B}_i)$ [45]. Now we consider a local static high gain output feedback;

$$u_i = \frac{1}{\varepsilon_i} F_i y_i \qquad (6\text{-}7)$$

where ε_i is a small positive number to be specified later, and F_i is chosen so that the matrix $B_i F_i$ is stable. Since B_i is nonsingular we can always choose F_i such that the matrix $B_i F_i$ has desired eigenvalues. The decentralized control law (6-7) is stabilizing for sufficiently small ε_i. This is shown in the following theorem.

Theorem 1: Suppose assumptions A1 and A2 are satisfied. Then there exist ε_i^*, (i=1,...,N) such that for any $\varepsilon_i \leq \varepsilon_i^*$ the decentralized control law (6-7) is stabilizing, that is the closed loop system Σ under (6-7) is asymptotically stable.

Proof: The closed loop system is in the form of multi parameter singularly perturbed

$$\dot{x}_{i1} = A_{i1}x_{i1} + A_{i2}x_{i2} + M_{i1}(y) \qquad (6\text{-}8a)$$

$$\varepsilon_i \dot{x}_{i2} = B_i F_i x_{i2} + \varepsilon_i h_i(x) \qquad (6\text{-}8b)$$

$$y_i = x_{i2} \qquad (6\text{-}8c)$$

where

$$h_i(x) = A_{i3}x_{i1} + A_{i4}x_{i2} + B_iH_i(x) + M_{i2}(y).$$

To show the stability of (6-8) we make a decomposition based on the time-scale structure of (6-8). We consider (6-8) as interconnection of 2N subsystems

$$\dot{x}_{i1} = A_{i1}x_{i1} \qquad i=1,\ldots,N,$$

$$\varepsilon_i\dot{x}_{i2} = B_iF_ix_{i2} \qquad i=1,\ldots,N.$$

Based on this decomposition we can choose a composite Lyapunov function and verify the stability of (6-8). We start by observing that assumption A1 guarantees the existence of a positive definite matrix P_i satisfying the Lyapunov equation.

$$P_iA_{i1} + A_{i1}^TP_i = -2I_{(m_i - n_i)} \tag{6-9}$$

Also in the same manner since B_iF_i is a stable matrix, there exists a positive definite matrix Q_i which satisfies

$$Q_iB_iF_i + (B_iF_i)^TQ_i = -2I_{m_i} \tag{6-10}$$

Next we observe that from (6-6b) and (6-6c) it follows that:

$$\|h_i(x)\| \leq \sum_{j=1}^{N} (\xi_{ij1}\|x_{j1}\| + \xi_{ij2}\|x_{j2}\|),$$

where ξ_{ij1} and ξ_{ij2} are non-negative constants.

Now we form a composite Lyapunov function

$$\nu(x) = \sum_{i=1}^{N} (1/2)(d_ix_{i1}^TP_ix_{i1} + \bar{d}_ix_{i2}^TQ_ix_{i2}) \tag{6-11}$$

where d_i and $\bar{d}_i$, $(i=1,\ldots,N)$, are arbitrary positive numbers. Let

$$e_{ii} = -\|P_iA_{i2}\| - \beta_{ii1}\|P_i\|,\quad e_{ij} = -\beta_{ij1}\|P_i\| \qquad i\neq j,$$

$$s_{ii} = \frac{1}{\varepsilon_i} - \xi_{ii2}\|Q_i\|,\quad s_{ij} = -\xi_{ij2}\|Q_i\| \qquad i\neq j,$$

$$r_{ij} = -\xi_{ij1}\|Q_i\|, \qquad i,j=1,\ldots,N.$$

and define the matrices E, S, R, D and $\bar{D}$ as

$$E = [e_{ij}],\ S = [s_{ij}],\ R = [r_{ij}],$$

$$D = \mathrm{diag}[d_i],\quad \bar{D} = \mathrm{diag}[\bar{d}_i].$$

In Appendix A it is shown that the derivative of the composite Lyapunov function $\nu(x)$ along the trajectories of (6-8) is bounded by

$$\dot{\nu}(x) \leq - \begin{bmatrix} \|x_{11}\| \\ \vdots \\ \|x_{N1}\| \\ \|x_{12}\| \\ \vdots \\ \|x_{N2}\| \end{bmatrix}^T \mathbf{T} \begin{bmatrix} \|x_{11}\| \\ \vdots \\ \|x_{N1}\| \\ \|x_{12}\| \\ \vdots \\ \|x_{N2}\| \end{bmatrix} \qquad (6\text{-}12)$$

where

$$\mathbf{T} = 1/2 \left[\begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix} \mathbf{V} + \mathbf{V}^T \begin{pmatrix} D & 0 \\ 0 & \bar{D} \end{pmatrix} \right], \qquad (6\text{-}13)$$

$$\mathbf{V} = \begin{pmatrix} I_N & E \\ R & S \end{pmatrix}.$$

Notice that positive definiteness of $\mathbf{T}$ implies asymptotic stability of (6-8). It can be shown that for sufficiently small ε_i, $(i=1,\ldots,N)$ the matrix $\mathbf{T}$ is positive definite. In fact, we can choose ε_i^*, $(i=1,\ldots N)$, just small enough to make the matrix $\mathbf{T}$ positive definite. Then we observe that for any $\varepsilon_i < \varepsilon_i^*$, $\mathbf{T}$ will remain positive definite and this concludes our proof.

Remark 1: The arbitrariness of the parameters d_i and $\bar{d}_i$, $(i=1,\ldots,N)$ can be employed to obtain the smallest possible gains, $\frac{1}{\varepsilon_i}$, by simply choosing ε_i^*'s small enough to make matrix $\mathbf{V}$ M-matrix [10].

4. Case 2

In this case we assume that isolated subsystems are uniform rank systems, [59], that is the following assumption holds:

A3 - For each isolated subsystem $\tilde{\Sigma}_i$ there exists a positive integer q_i such that the Markov parameters of $\tilde{\Sigma}_i$ satisfy

$$\hat{C}_i\hat{A}_i^k\hat{B}_i = 0 \qquad k = 0,\ldots,(q_i - 2)$$

where

$$\hat{C}_i\hat{A}_i^{q_i-1}\hat{B}_i \text{ is nonsingular.}$$

This assumption is a sufficient condition for invertibility of $\tilde{\Sigma}_i$, [60]. In [61] it is shown that the freedom in choosing the local transformation $\hat{x}_i = T_i\tilde{x}_i$, where T_i satisfies (6-4), can be exploited to transform the subsystem Σ_i into the canonical form

$$\Sigma_i;\ \dot{\hat{x}}_{io} = A_{io}\hat{x}_{io} + A_{i1}\hat{x}_{i1} + \tilde{M}_{io}(y) \tag{6-14a}$$

$$\dot{\hat{x}}_{ij} = \hat{x}_{ij+1} + \tilde{M}_{ij}(y) \quad j=1,\ldots,(q_i - 1) \tag{6-14b}$$

$$\dot{\hat{x}}_{iq_i} = \sum_{j=0}^{q_i} D_{ij}\hat{x}_{ij} + B_i(u_i + \tilde{H}_i(\hat{x})) + \tilde{M}_{iq_i}(y) \tag{6-14c}$$

$$y_i = \hat{x}_{i1} \qquad i=1,\dots,N \tag{6-14d}$$

where $\hat{x}_{io}$ is an $r_i \overset{\Delta}{=} q_i m_i$ dimensional and $\hat{x}_{ij}$, $(j=1,\dots,q_i)$, are each m_i dimensional vectors, and the mappings $\tilde{M}_{ij}$, $(j=0,\dots,q_i)$ and $\tilde{H}_i$ are obtained in an obvious way. To motivate our design and analysis let us first ignore the interconnections and assume that successive derivatives of y_i can be measured exactly. Then we have all the measurements of the states $\hat{x}_{i1},\dots,\hat{x}_{iq_i}$. Now upon availability of these states, and in view of the canonical structure of the isolated subsystems we consider a repeated application of high-gain feedback in a manner similar to case 1. By doing so we convert all the states $\hat{x}_{i1},\dots,\hat{x}_{iq_i}$, $(i = 1,\dots N)$, into fast states. Now in the presence of interconnections we observe that state or output coupling that appears on the R.H.S. of the fast state equations can be tolerated if we have provided enough gain as in case 1. Moreover the coupling that appears on the R.H.S. of the slow state equation would depends solely on the fast states and can be tolerated as in case 1. So we can retain the stability of the system in the presence of interconnections. Unlike case 1, the high-gain output feedback in this case is dynamic since it involves derivatives of the local output y_i. To realize such a control law we use local observer-based controllers. We start off by preparing the system for high-gain feedback. To do that we scale the states and control variable according to

$$x_{io} = \hat{x}_{io},\ x_{ij} = \mu_i^{j-1}\hat{x}_{ij},\ j=1,\dots,q_i,\ \mu_i = \varepsilon_i^{\frac{1}{q_i}}$$

$$u_i = \frac{1}{\varepsilon_i} v_i$$

where ε_i's are small positive numbers to be chosen later and v_i is the new control variable.

In order to maintain the boundedness of the state coupling $\tilde{H}_i(x)$ we restrict the choices of the small parameters $\varepsilon_1,\ldots,\varepsilon_N$ to those choices satisfying

$$\varepsilon_i \leq 1, \qquad i=1,\ldots,N \tag{6-15a}$$

$$\frac{\varepsilon_i^{\frac{q_i-1}{q_i}}}{\varepsilon_j^{\frac{q_j-1}{q_j}}} \leq m_{ij} < \infty, \; i,j=1,\ldots,N \tag{6-15b}$$

for some numbers m_{ij}.

After scaling (6-14) is given by

$$\dot{x}_{io} = A_{io}x_{io} + \bar{A}_{i1}x_{if} + \bar{M}_{io}(y) \tag{6-16a}$$

$$\mu_i\dot{x}_{if} = \mu_i^{q_i}\bar{A}_{i2}x_{io} + \mu_i\bar{A}_{i3}x_{if} + A_{if}x_{if}$$

$$+ B_{if}v_i + \mu_i B_{if}\bar{H}_i(x_o,x_f) + \mu_i\bar{M}_{i1}(y) \tag{6-16b}$$

$$y_i = C_{if}x_{if}, \tag{6-16c}$$

where

$$x_{if} = (x_{i1}^T,x_{i2}^T,\ldots,x_{iq_i}^T),\; x_o = (x_{10}^T,\ldots,x_{NO}^T)^T,$$

$$x_f = (x_{1f}^T,\ldots,x_{Nf}^T)^T,$$

$$\bar{A}_{i1} = (A_{i1},0,\ldots,0),\; \bar{A}_{i2} = (0,0\ldots,D_{io}^T)^T$$

$$B_{if} = (0,0\ldots,B_i^T),\; C_{if} = (I_{m_i},0,\ldots,0)$$

$$A_{if} = \begin{pmatrix} 0 & I_{(q_i-1)m_i} \\ 0 & 0 \end{pmatrix}, \qquad A_{i3} = \begin{pmatrix} 0 \\ \\ \Gamma_i \end{pmatrix},$$

$$\Gamma_i = \left(\mu_i^{q_i-1} D_{i1},\ \mu_i^{q_i-2} D_{i2}, \ldots, D_{iq_i}\right),$$

and the mappings $\overline{M}_{io}$, $\overline{M}_{i1}$ and $\overline{H}_i$ are given in Appendix B. In Appendix B it is also shown that the smoothness conditions 2 and 3, in view of assumption (6-15), can be rewritten as,

$$||\overline{M}_{io}(y)|| \le \sum_{j=1}^{N} \beta'_{ij1} ||x_{j1}||, \qquad i=1,\ldots,N \tag{6-17a}$$

$$||\overline{M}_{i1}(y)|| \le \sum_{j=1}^{N} \beta'_{ij2} ||x_{j1}||, \qquad i=1,\ldots,N \tag{6-17b}$$

$$||\overline{H}_i(x_o,x_f)|| \le \sum_{j=1}^{N} \left(\mu_i^{q_i-1} \gamma_{ijo} ||x_{jo}|| + \gamma_{ijf} ||x_{jf}||\right), \quad i=1,\ldots,N \tag{6-17c}$$

where β'_{ij1}, β'_{ij2}, γ_{ijo} and γ_{ijf} are non negative constants.

Next notice that the system (6-16) is in a singularly perturbed form and a decomposition based on the time-scale structure of (6-16) leads us to consider (6-16) as interconnection of 2N subsystems

$$\dot{x}_{io} = A_{io}x_{io}, \qquad i=1,\ldots,N \tag{6-18a}$$

$$\mu_i x_{if} = A_{if}x_{if} + B_{if}v_i \tag{6-18b}$$

$$y_i = C_{if}x_{if}, \qquad i=1,\ldots,N \tag{6-18c}$$

the stability of subsystems (6-18a) are implied by assumption A1 since the eigenvalues of A_{io} are the transmission zeros of the triple $(\hat{C}_i, \hat{A}_i, \hat{B}_i)$ [61]. So our immediate task is to stabilize subsystem (6-18b). Before dealing with this stabilization problem we need the following lemma.

<u>Lemma 1</u>: The pair (A_{if}, B_{if}) is controllable and the pair (A_{if}, C_{if}) is observable.

<u>Proof</u>: This is a consequence of A_{if}, B_{if}, C_{if} having a special canonical structure and B_i being nonsingular.

To stabilize the subsystems (6-18b) and (6-18c) we use an observer-based controller. First a state feedback control $v_i = F_i x_{if}$ is designed such that $(A_{if} + B_{if}F_i)$ has all its eigenvalues in the open left half plane. Then a local observer is considered which is given by

$$\mu_i \dot{\hat{x}}_{if} = A_{if}\hat{x}_{if} + K_{if}(y_i - C_{if}\hat{x}_{if}) + B_{if}v_i, \tag{6-19}$$

where K_{if} is the observer gain which is chosen so that the matrix $(A_{if} - K_{if}C_{if})$ is a stable matrix.

Now we can apply $v_i = F_i\hat{x}_{if}$ to (6-18b) and (6-19). The resulting closed loop local subsystem is asymptotically stable. Having stable subsystems we can proceed to obtain a composite Lyapunov function and establish stability of Σ for sufficiently small ε_i. The result is stated in the following theorem.

<u>Theorem 2</u>: Suppose assumptions A1 and A3 hold. Then there exist ε_i^*, $(i=1,\ldots,N)$, such that for any $\varepsilon_i \leq \varepsilon_i^*$ which satisfies (6-15) the decentralized observer-based controller

$$\mu_i \dot{\hat{x}}_{if} = A_{if}\hat{x}_{if} + K_{if}(y_i - C_{if}\hat{x}_{if}) + B_{if}F_i\hat{x}_{if} \tag{6-20a}$$

$$u_i = \frac{1}{\varepsilon_i} F_i\hat{x}_{if} \tag{6-20b}$$

is stabilizing that is the closed loop system Σ and (6-20) is asymptotically stable.

<u>Proof</u>: Setting $e_i = \hat{x}_{if} - x_{if}$ as the observation error, the closed loop system (6-16) and (6-20) can be rewritten as

$$\dot{x}_{io} = A_{io}x_{io} + \ell_i(x_f) \tag{6-21a}$$

.
.

$$\mu_i \begin{pmatrix} \dot{x}_{if} \\ \\ \dot{e}_i \end{pmatrix} = \begin{pmatrix} (A_{if} + B_{if}F_i) & B_{if}F_i \\ \\ 0 & (A_{if} - K_{if}C_{if}) \end{pmatrix} \begin{pmatrix} x_{if} \\ \\ e_i \end{pmatrix}$$

$$+ \quad \mu_i \begin{pmatrix} g_i(x_0,x_f) \\ \\ -g_i(x_o,x_f) \end{pmatrix} \tag{6-21b}$$

$$y_i = C_{if}x_{if}, \qquad i=1,\ldots,N \tag{6-21c}$$

where

$$\ell_i(x_f) = \bar{A}_{i1}x_{i1} + \bar{M}_{io}(y)$$

$$g_i(x_o,x_f) = B_{if}\bar{H}_i(x_o,x_f) + \bar{M}_{i1}(y) + \bar{A}_{i3}x_{if} + \mu_i^{q_i-1}\bar{A}_{i2}x_{io}.$$

Moreover in view of (6-17) it follows that

$$||\ell_i(x_f)|| \leq \sum_{j=1}^{N} k_{ij}\, ||x_{j1}||$$

$$||g_i(x_o,x_f)|| \leq \sum_{j=1}^{N} (\gamma'_{ijf}||x_{if}|| + \mu_i^{q_i\ -1}\gamma'_{ijo}||x_{jo}||)$$

A decomposition based on the time-scale structure of (6-21) will lead us to consider (6-21) as an interconnection of 2N subsystems

$$\dot{x}_{io} = A_{io}x_{io}, \qquad i=1,\ldots,N \tag{6-22a}$$

$$\mu_i \begin{pmatrix} x_{if} \\ \\ e_i \end{pmatrix} = A^c_{if} \begin{pmatrix} x_{if} \\ \\ e_i \end{pmatrix}, \quad i=1,\ldots,N \tag{6-22b}$$

where

$$A^c_{if} = \begin{pmatrix} (A_{if} + B_{if}F_i) & B_{if}F_i \\ \\ 0 & (A_{if} - K_{if}C_{if}) \end{pmatrix}$$

Let $\hat{P}_i$ and $\hat{Q}_i$ be positive definite matrices which satisfy the following Lyapunov equations

$$\hat{P}_i A_{io} + A^T_{io}\hat{P}_i = -2I_{n_i - q_i m_i} \tag{6-23a}$$

$$Q_i A^c_{if} + (A^c_{if})^T Q_i = -2I_{2q_i m_i} \tag{6-23b}$$

Notice that due to the fact that A_{io}, and A^c_{if} are stable, (the latter one is stable since $(A_{if} + B_{if}F_i)$ and $(A_{if} - K_{if}C_{if})$ are stable), $\hat{P}_i$ and $\hat{Q}_i$ exist and are unique. Now we are at the position to form a composite Lyapunov function

$$\nu_1(x) = \sum_{i=1}^{N} (1/2)\left[d_i x^T_{io}\hat{P}_i x_{io} + \bar{d}_i \begin{pmatrix} x_{if} \\ e_i \end{pmatrix}^T \hat{Q}_i \begin{pmatrix} x_{if} \\ e_i \end{pmatrix}\right], \tag{6-24}$$

where as usual d_i and $\bar{d}_i$, $(i=1,\ldots,N)$ are arbitrary positive numbers. From this point on the proof proceeds in the same way as in case 1.

<u>Remark</u>: Our result for case 2 required that the choices of $\varepsilon_1,\ldots,\varepsilon_N$ satisfy (6-15), which wasn't required in case 1.

5. Case 3

This case is the most general case and covers cases 1 and 2 as special cases. It is characterized by the following assumption.

A4 - Each isolated subsystem $\tilde{\Sigma}_i$ is invertible.

Recently Sannuti [62] has shown that under the assumption A4 one can choose local transformation T_i which satisfies (6-4) and brings the isolated subsystem $\tilde{\Sigma}_i$ to a canonical form which is appropriate for high-gain feedback analysis. To perform this local transofrmation, from [62], we observe that for each isolated subsystem there exists an integer $p_i \leq n_i$ and indices σ_{ij} and r_{ij}, $j=0,\ldots,p_i$ $\sigma_{io} = m_i$, $r_{io} = 0$ and $\sigma_{ip_i} = 0$, so that the transformed system Σ_i is in the following canonical form

$$\Sigma_i;\ \dot{\hat{x}}_{io} = A_{io}\hat{x}_{io} + A_{i1}\hat{x}_{i1} + M_{io}(y) \tag{6-25a}$$

$$\dot{\hat{x}}_{ijb} = E_{ij}\hat{x}_{i1} + \hat{x}_{ij+1} + \tilde{M}_{ijb}(y), \quad j=1,\ldots,p_i-1 \tag{6-25b}$$

$$\dot{\hat{x}}_{ija} = \sum_{\ell=0}^{N} D_{ij\ell}\hat{x}_{i\ell} + B_{ij}u_i + H_{ij}(\hat{x}) + M_{ijo}(y), \quad j=1,\ldots,p_i \tag{6-25c}$$

$$y_i = N_i\hat{x}_{i1} \tag{6-25d}$$

where;

$$\hat{x}_{ij} = (\hat{x}^T_{ija}, \hat{x}^T_{ijb})^T,\ \hat{x}_{ija} = \hat{x}_{ijo},\ \hat{x}_{ijb} = (\hat{x}^T_{ij1},\ldots,\hat{x}^T_{ijp_i-j})^T$$

$$\hat{x}_i = (\hat{x}^T_{io}, \hat{x}^T_{i1},\ldots,\hat{x}_{ip_i})^T,$$

the matrices N_i and $B_i = (B^T_{i1},\ldots,B_{ip_i})^T$ are nonsingular, and the mappings $\tilde{M}_{ijb}$, $j=1,\ldots,p_{i-1}$, M_{ijo}, H_{ij}, $j=1,\ldots,p_i$, and M_{io} are obtained , in an obvious way. The variables $\hat{x}_{io}$, $\hat{x}_{ip_ia}$, $\hat{x}_{ija}$, $\hat{x}_{ijb}$ and $\hat{x}_{ij\ell}$ are of dimension n_{i1}, r_{ip_i}, r_{ij}, σ_{ij} and $r_{ij+\ell}$ respectively ($j=1,\ldots,p_{i-1}$, $\ell = 1,\ldots,p_{i-j}$). Furthermore

$$\sigma_{ij} \triangleq \sigma_{ij-1} - r_{ij} = \sum_{\ell=j+1}^{p_i} r_{i\ell},\ d_{if} \triangleq \sum_{j=0}^{p_i-1} \sigma_{ij},\ r_{ip_i} = \sigma_{ip_i-1}, \tag{6-26a}$$

$$n_{i1} = n_i - d_{if}. \tag{6-26b}$$

For further details on this canonical form and the transformation T_i, interested readers are referred to [62]. Now observe that the terms $D_{ij\ell}\hat{x}_{i\ell}$ in (6-25c) can be included in the state coupling function $H_{ij}(\hat{x})$ and this inclusion doesn't violate the smoothness requirement (6-3). So we set

$$\tilde{H}_{ij}(\hat{x}) = H_{ij}(\hat{x}) + \sum_{\ell=0}^{p_i} D_{ij\ell}\, x_\ell.$$

Furthermore in our analysis we also consider the term $E_{ij}\hat{x}_{i1}$ in (6-25b) as an interconnection term rather than part of an isolated subsystem, that is we set

$$M_{ijb}(y) = E_{ij}\hat{x}_{i1} + \tilde{M}_{ijb}(y) = E_{ij}N_i^{-1}y_i + M_{ijb}(y)$$

Finally observing that $m_i = \sum_{j=1}^{p_i} r_{ij}$ and the fact that B_i is non-singular we can define a new input vector

$$\hat{u}_i = B_i u_i,$$

and partition the new input vector as

$$\hat{u}_i = (\hat{u}_{i1}^T,\ldots,\hat{u}_{ij}^T,\ldots,\hat{u}_{ip_i}^T)^T,$$

where $\hat{u}_{ij} \in R^{r_{ij}}$. In view of above considerations we can rewrite (6-25) as follows.

$$\Sigma_i; \ \dot{\hat{x}}_{io} = A_{io}\hat{x}_{io} + A_{i1}\hat{x}_{i1} + M_{io}(y) \tag{6-27a}$$

$$\dot{\hat{x}}_{ijb} = \hat{x}_{ij+1} + M_{ijb}(y), \qquad j=1,\dots,p_i-1 \tag{6-27b}$$

$$\dot{\hat{x}}_{ija} = \hat{u}_{ij} + \tilde{H}_{ij}(\hat{x}) + M_{ijo}(y), \qquad j=1,\dots,p_i \tag{6-27c}$$

$$y_i = \hat{x}_{i1}, \tag{6-27d}$$

where, since N_i is nonsingular, without loss of generality, we have taken N_i as an identity matrix. Now we perform a regrouping of the states of Σ_i which is an essential part of our analysis. First we partition $M_{ijb}(y)$ as

$$M_{ijb}(y) = (M^T_{ij1}(y), M^T_{ij2}(y), \dots, M^T_{ijp_i-j}(y))^T$$

which is compatible with the partitioning of $\hat{x}_{ijb}$. Then we start by picking an arbitrary component of $\hat{x}_{i1b}$, say $\hat{x}_{i1j}$, and we have

$$\dot{\hat{x}}_{i1j} = \hat{x}_{i2j-1} + M_{i1j}(y) \tag{6-28}$$

Next we choose the state which appears on the right-hand side of (6-28), namely $\hat{x}_{i2j-1}$, and we have

$$\dot{\hat{x}}_{i2j-1} = \hat{x}_{i3j-2} + M_{i2j-1}(y)$$

and the next step will be to consider $\hat{x}_{i3j-2}$ and so on, until we get to $\hat{x}_{ij+1\ o}$ which satisfies

$$\dot{\hat{x}}_{ij+1\ o} = \hat{u}_{ij} + \tilde{H}_{ij}(\hat{x}) + M_{ij+1\ o}(y)$$

and we stop. Then we group the states x_{i1j}, $x_{i2j-1}, \ldots, x_{ij+1\,o}$. By performing this regrouping algorithm for every component of x_{i1b}, we can rewrite (6-27) as follows

$$\Sigma_i; \quad \dot{\hat{x}}_{io} = A_{io}\hat{x}_{io} + A_{i1}\hat{x}_{i1} + M_{io}(y) \tag{6-29a}$$

$$\dot{\hat{x}}_{i1j} = \hat{x}_{i2j-1} + M_{i1j}(y) \tag{6-29b}$$

$$\hat{x}_{i2j-1} = \hat{x}_{i3j-2} + M_{i2j-1}(y) \tag{6-29c}$$

$$\vdots$$

$$\dot{\hat{x}}_{ij+1\,o} = \hat{u}_{ij+1} + \tilde{H}_{ij+1}(\hat{x}) + M_{ij+1\,o}(y) \tag{6-29d}$$

$$y_{ij} = \hat{x}_{i1j}, \qquad j=0,\ldots,p_i-1, \tag{6-29e}$$

where y_{ij} are components of y_i which are partitioned compatible with partitioning of x_{i1}. To write (6-29) in compact form let us denote

$$\hat{x}_{ifj} = (\hat{x}^T_{i1j-1},\ldots,\hat{x}^T_{i2j-2},\ldots,\hat{x}^T_{ijo})^T, \quad j=1,\ldots,p_i$$

$$\hat{x}_o = (\hat{x}^T_{1o},\ldots,\hat{x}^T_{No})^T, \; \hat{x}_f = (\ldots,\hat{x}^T_{1fj},\ldots,\hat{x}^T_{ifj},\ldots,)^T$$

$$A_{ifj} = \left[\begin{array}{c:cccc} & I_{r_{ij}} & & & O \\ & & \ddots & & \\ & & & I_{r_{ij}} & \\ & & & & \ddots \\ O & O & & & I_{r_{ij}} \\ \hdashline O & O & & & O \end{array}\right] \begin{array}{l} \updownarrow (j-1)r_{ij} \\ \\ \updownarrow r_{ij} \end{array}$$

$$B_{ifj} = \begin{pmatrix} \mathbf{O} \\ \hline I_{r_{ij}} \end{pmatrix} \begin{matrix} \updownarrow (j-1)r_{ij} \\ \updownarrow r_{ij} \end{matrix},$$

$$C_{ifj} = \underbrace{\begin{pmatrix} I_{r_{ij}} & \vdots & \mathbf{O} \end{pmatrix}}_{\substack{\longleftrightarrow \quad \longleftrightarrow \\ r_{ij} \quad (j-1)r_{ij}}}$$

$$M_{ifj} = (M^T_{i1j-1}(y),\ M^T_{i2j-2}(y),\ldots,M^T_{ijo}(y))^T.$$

Using above definitions we have the desired canonical form for Σ_i given by

$$\Sigma_i; \quad \dot{\hat{x}}_{io} = A_{io}\hat{x}_{io} + A_{i1}\hat{x}_{i1} + M_{io}(y) \tag{6-30a}$$

$$\dot{\hat{x}}_{ifj} = A_{ifj}\hat{x}_{ifj} + B_{ifj}(\hat{u}_{ij} + \tilde{H}_{ij}(\hat{x}_o,\hat{x}_f)) + M_{ifj}(y) \tag{6-30b}$$

$$y_{ij} = C_{ifj}\hat{x}_{ifj} \qquad j=1,\ldots,p_i \tag{6-30c}$$

Now it is obvious that our problem in Case 3 has been casted into a formulation somewhat similar to the one used in Case 2. To prepare (6-30) for high-gain we use the same scaling scheme we used in Case 2, that is, we let

$$x_{ifj} = \begin{bmatrix} 1 & & & & \mathbf{O} \\ & \mu_{ij} & & & \\ & & \mu^2_{ij} & & \\ & & & \ddots & \\ \mathbf{O} & & & & \mu^{j-1}_{ij} \end{bmatrix} \hat{x}_{ifj},\ x_{io} = \hat{x}_{io},$$

$$\hat{u}_{ij} = \frac{1}{\varepsilon_{ij}} v_{ij}, \quad \mu_{ij} = \varepsilon_{ij}^{\frac{1}{j}}, \qquad j=1,\ldots,p_i$$

where ε_{ij}, $j=1,\ldots,p_i$, $i=1,\ldots,N$ are small positive numbers to be chosen later. As in the Case 2 to maintain the boundedness of the state coupling $H_{ij}(x_o,x_f)$, where x_o and x_f are the scaled vector x_o, x_f, we require that the small parameters ε_{ij}, $j=1,\ldots,p_i$, $i=1,\ldots,N$, satisfy a condition similar to (6-15), which is

$$\varepsilon_{ij} \leq 1 \tag{6-31a}$$

$$\frac{\varepsilon_{ij}^{\frac{j-1}{j}}}{\varepsilon_{r\ell}^{\frac{\ell-1}{\ell}}} \leq m_{irj\ell} < \infty, \; j=1,\ldots,p_i, \ell=1,\ldots,p_r, i,r=1,\ldots,N. \tag{6-31b}$$

After scaling (6-3) is given by

$$\Sigma_i; \qquad \dot{x}_{io} = A_{io}x_{io} + A_{i1}x_{i1} + \overline{M}_{io}(y) \tag{6-32a}$$

$$\mu_{ij}\dot{x}_{ifj} = A_{ifj}x_{ifj} + B_{ifj}(v_{ij} + \mu_{ij}\overline{H}_{ij}(x_o,x_f)) + \mu_{ij}\overline{M}_{ifj}(y) \tag{6-32b}$$

$$y_{ij} = C_{ifj}x_{ifj} \qquad j=1,\ldots,p_i \tag{6-32c}$$

where the mappings $\overline{M}_{io}$, $\overline{M}_{ifj}$, and $\overline{H}_{ij}$ are appropriately defined. Similar to the case 2 it can be shown that

$$||\overline{M}_{io}(y)|| \leq \sum_{\ell=1}^{N} \sum_{s=1}^{N} \lambda_{i\ell s} ||y_{\ell s}|| \qquad i=1,\ldots,N \tag{6-33a}$$

$$||\overline{M}_{ifj}(y)|| \leq \sum_{\ell=1}^{N} \sum_{s=1}^{N} \delta_{ij\ell s} ||y_{\ell s}|| \qquad i=1,\ldots,N \tag{6-33b}$$

$$||H_{ij}(x_o,x_f)|| \leq \sum_{\ell=1}^{N} \eta_{ij\ell}||x_{\ell o}|| + \sum_{\ell=1}^{N} \sum_{s=1}^{p_i} \delta_{ij\ell s}||x_{\ell fs}|| \qquad (6\text{-}33c)$$

Next we observe that (6-32) is in a singularly perturbed form and a decomposition based on the time-scale structure of (6-32) will suggest to consider (6-32) as interconnection of $(p_i+1)N$ subsystems

$$\dot{x}_{io} = A_{io}x_{io} \qquad i=1,\ldots,N \qquad (6\text{-}34a)$$

$$\mu_{ij}\dot{x}_{ifj} = A_{ifj}x_{ifj} + B_{ifj}v_{ij} \qquad (6\text{-}34b)$$

$$y_{ij} = C_{ifj}x_{ifj} \qquad j=1,\ldots,p_i \quad i=1,\ldots,N \qquad (6\text{-}34c)$$

The stability of subsystems (6-34a) is implied by the assumption A1, since the eigenvalues of A_{io} are the transmission zeros of the triple $(\hat{C}_i, \hat{A}_i, \hat{B}_i)$, [62]. To stabilize subsystems (6-34b), (6-34c) we need the following lemma.

Lemma 2: The pair (A_{ifj}, B_{ifj}) is controllable and the pair (A_{ifj}, C_{ifj}) is observable.

Proof: Follows from the special canonical structure of the triple $(C_{ifj}, A_{ifj}, B_{ifj})$.

As in Case 2 to stabilize (6-34b), (6-34c) we use an observer-based controller. First a state feedback $v_{ij} = F_{ij}x_{ifj}$ is designed such that $(A_{ifj} + B_{ifj}F_{ij})$ has all its eigenvalues in the open left-half plane. Then a local observer is considered which is given by

$$\mu_{ij}\dot{\hat{x}}_{ifj} = A_{ifj}\hat{x}_{ifj} + K_{ifj}(y_{ij} - C_{ifj}\hat{x}_{ifj}) + B_{ifj}v_{ij} \qquad (6\text{-}35)$$

where K_{ifj} is the observer gain, which is chosen so that the matrix $(A_{ifj} - K_{ifj}C_{ifj})$ is a stable matrix. The application of the control law $v_{ij} = F_{ij}\hat{x}_{ifj}$ to (6-34b) and (6-35) results in an asymptotically stable closed loop subsystem. Having stable subsystems we can proceed to obtain a composite Lyapunov function and establish stability for sufficiently small ε_{ij}. The main result is stated in the following theorem.

Theorem 3: Suppose A1 and A4 hold then there exists ε^*_{ij} (i=1,...,N) such that for any $\varepsilon_{ij} \leq \varepsilon^*_{ij}$ which satisfy (6-31) the decentralized observer-based controller;

$$\mu_{ij}\dot{\hat{x}}_{ifj} = A_{ifj}\hat{x}_{ifj} + K_{ifj}(y_{ij} - C_{ifj}\hat{x}_{ifj}) + B_{ifj}F_{if}\hat{x}_{ifj} \quad (6\text{-}36a)$$

$$u_{ij} = \frac{1}{\varepsilon_{ij}} F_{ij}\hat{x}_{ifj} \qquad j=1,\ldots,p_i,\ i=1,\ldots,N \quad (6\text{-}36b)$$

is stabilizing, that is the closed loop system Σ and (6-36) is asymptotically stable.

Proof: Similar to proof of Theorem 2.

Remark: Notice that for each subsystem, Σ_i, p_i observers are designed. These observers are in different time-scales. This decomposition of the local observer based on its time scale structure is an interesting feature of our design. Furthermore the remark in previous cases regarding minimizing the gains of each subsystem is applicable to this case as well.

As we stated earlier our results are based on the assumption that each isolated subsystem is a square system. However, our result can be extended to the case of non-square isolated subsystems, by performing a square-down procedure, [63], [64], [65] for all subsystems. That is we should find a pre-compensator (Post-Compensator) matrix $\hat{G}_i$ such that the squared-down subsystem $(\hat{C}_i, \hat{A}_i, \hat{B}_i\hat{G}_i)$, $((\hat{G}_i\hat{C}_i, \hat{A}_i, \hat{B}_i))$, has all its transmission zeros in the open left half plane. From this point on we can proceed with our investigation as we did before.

Appendix A

We have

$$\dot{v}(x) = 1/2 \sum_{i=1}^{N} d_i x_{i1}^T P_i (A_{i1} x_{i1} + A_{i2} x_{i2} + M_{i1}(y)) +$$

$$d_i (x_{i1}^T A_{i1}^T + x_{i2}^T A_{i2}^T + M_{i1}^T(y)) P_i x_{i1} +$$

$$\bar{d}_i x_{i2}^T Q_i (\frac{1}{\varepsilon_i} B_i F_i x_{i2} + h_i(x)) + \bar{d}_i (\frac{1}{\varepsilon_i} x_{i2}^T (B_i F_i)^T + h_i^T(x)) Q_i x_{i2}$$

$$\leq \sum_{i=1}^{N} d_i \Big(-\|x_{i1}\|^2 + \|P_i A_{i2}\| \cdot \|x_{i1}\| \cdot \|x_{i2}\| +$$

$$\sum_{j=1}^{N} \beta_{ij1} \|P_i\| \cdot \|x_{i1}\| \cdot \|x_{j2}\| \Big) + \bar{d}_i \Big(-\frac{1}{\varepsilon_i} \|x_{i2}\| +$$

$$\sum_{j=1}^{N} \xi_{ij1} \|Q_i\| \cdot \|x_{i2}\| \cdot \|x_{j1}\| + \sum_{j=1}^{N} \xi_{ij2} \|Q_i\| \cdot \|x_{i2}\| \cdot \|x_{j2}\| \Big)$$

$$= - \begin{bmatrix} \|x_{11}\| \\ \vdots \\ \|x_{N1}\| \\ \|x_{12}\| \\ \vdots \\ \|x_{N2}\| \end{bmatrix}^T \mathbf{T} \begin{bmatrix} \|x_{11}\| \\ \vdots \\ \|x_{N1}\| \\ \|x_{12}\| \\ \vdots \\ \|x_{N2}\| \end{bmatrix}$$

Q.E.D.

Appendix B

We have

$$\overline{M}_{io}(y) = \widetilde{M}_{io}(y),$$

$$\overline{M}_{i1}(y) = (\widetilde{M}_{i1}^T(y),\ \mu_i\widetilde{M}_{i2}^T(y),\ldots,\mu_i^{j-1}\widetilde{M}_{ij}^T(y),\ldots,\mu_i^{q_i-1}\widetilde{M}_{iq_i}^T(y))^T$$

Now in view of Condition (2), it follows that (6-17a) and (6-17b) hold. New we observe that

$$||\overline{H}_i(x_o,x_f)|| = \mu_i^{q_i-1}\widetilde{H}_i(\hat{x}). \tag{B-1}$$

The smoothness requirement (3) implies that

$$||\widetilde{H}_i(\hat{x})|| \leq \sum_{r=1}^{N}\sum_{j=0}^{q_r} \gamma_{irj}\,||\hat{x}_{rj}||, \tag{B-2}$$

where all γ_{irj} are non negative constants. Now in terms of scaled state variables (B-2) can be written as

$$||\widetilde{H}_i(\hat{x})|| \leq \sum_{r=1}^{N}\sum_{j=1}^{q_r} \frac{1}{\mu_r^{j-1}}\gamma_{irj}x_{rj} + \sum_{r=1}^{N}\gamma_{iro}\,||x_{ro}|| \tag{B-3}$$

Using (B-3) in (B-1) we obtain

$$||\overline{H}_i(x_o,x_f)|| \leq \sum_{r=1}^{N}\sum_{j=1}^{q_r} \frac{\mu_i^{q_i-1}}{\mu_r^{j-1}}\,||x_{rj}|| + \sum_{r=1}^{N}\mu_i^{q_i-1}\gamma_{iro}\,||x_{ro}||$$

Now in view of Condition (15), (6-17c) follows.

Q.E.D.

CHAPTER VII

CONCLUSION

In part 1 of this thesis the analysis and the design of nonlinear singularly perturbed systems has been investigated. The results of part 1 consist of stability analysis, initial-value problem, stabilization and regulation. These results cover the related existing results in the literature as special cases and generalize them in the following directions:

1. Generalization to a general class of nonlinear singularly perturbed systems.
2. Providing an explicit upper bound on the perturbation parameter ε, for each result, under which that result is valid. This is very significant in engineering practice.
3. Providing a degree of freedom for the designer to exploit according to his design objective, such as including a certain set in the domain of attraction or getting the largest upper bound on ε, and so on.
4. The generalization in the regulator problem has significant features beside those stated in 1, 2 and 3. First, explicit upper and lower bounds on $\|J_{uc} - J_s\|$ are given. Second, the solution of slow regulator is not required to be optimal. These features have great practical and theoretical implications.

One interesting feature of the results of part 1 is the uniformity of the requirements, that is to say, to be able to perform a decomposition based on the time-scale structure of a system for analysis (e.g. stability) or design (e.g. stabilization, regulation), we have only required an "Interaction Condition" which sets permissible interactions between slow and fast dynamics, and the rest of requirements in each case are the natural consequences of the specific problem area under study. This "Interaction Condition" basically is dictated by the strength of the stability of slow and fast subsystems. For example if the slow and fast subsystems are exponentially stable then the "Interaction Condition" is satisfied by requiring that f and g be continuously differentiable. On the other hand if the slow and fast subsystems are asymptotically stable then they set a certain permissible interaction between slow and fast dynamics through their comparison functions.

It is the author's speculation that most of the control related results in singular perturbation can be stated in one or two theorems and the major requirement would be the "Interaction Condition".

Finally we mention that the results obtained in part 1 lay the ground and provide the basic tools for solving a host of control related problems.

In part 2 we consider the problem of designing a robust decentralized control law using high-gain state or output feedback. Although the results of part 2 are stated in the context of decentralized control, they can be extended to the design of robust control using state or output feedback for systems with modeling errors and uncertainlies.

Decentralized high-gain state feedback proposed in chapter 5 is the first attempt for the class of large scale systems with nonlinear multi-input subsystems and nonlinear state coupling through input matrix, which covers the existing results in the literature as special cases. We should mention that the capability of high-gain goes far beyond this result and the author forsees a greater role for high-gain state feedback in the problem areas of decentralized control and distrubance rejection, in the sense of broadening the permissible interconnections or disturbances.

In chapter 6 a decentralized high-gain output feedback is designed for a large-scale system with multi variable linear isolated subsystems and nonlinear state coupling through input matrix and arbitrary nonlinear output coupling. This result generalize the only existing work in the literature [57] which was for single input single output isolated subsystems. The result of chapter 6 indicates the crucial role of transmission zeros in the design of robust output feedback. There are open problems to be investigated in high-gain output feedback. For example, a very natural one would be the extension of the results of chapter 6 to the class of large scale systems with nonlinear isolated subsystems.

REFERENCES

[1] Kokotovic, P. V., R. E. O'Malley Jr. and P. Sannuti, "Singular Perturbation and Order Reduction in Control Theory-An Overview," Automatica, Vol. 12, pp. 123-132.

[2] Chow, J.H. and P. V. Kokotovic, "A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Mode," IEEE Trans. Automat. Contr., Vol. AC-21, pp. 701-705, 1976.

[3] Khalil, H. K. and A. Saberi, "Decentralized Stabilization of Nonlinear Interconnected Systems Using Local State Feedback," IEEE Trans. Automat. Contr., AC-27, February 1982, pp. 265-268.

[4] Habets, P., "Stabilite Asymptotique pour des Problems de Perturbations Singulieres," Centro Internazionalle Matematico Estimo, Ed. Cremonese, 1974.

[5] Grujic, L. T., "Uniform Asymptotic Stability of Nonlinear Singularly Perturbed and Large Scale Systems," Int. J. Control, 1981, Vol. 33, No. 3, pp. 481-504.

[6] Chow, J. H., "Asymptotic Stability of a Class of Nonlinear Singularly Perturbed Systems," J. Franklin Inst., 1978, Vol. 305, No. 5, pp. 275-281.

[7] Chow, J. H., and P. V. Kokotovic, "A Two-Stage Lyapunov-Bellman Feedback Design of a Class of Nonlinear Systems," IEEE Transaction on Automatic Control, Vol. AC-26, No. 3, pp. 656-663.

[8] Siljak, D. D., Large-Scale Dynamic Systems: Stability and Structure, North-Holland, New York, 1978.

[9] Michel, A. N. and R. K. Miller, Qualitative Analysis of Large Scale Dynamical Systems. Academic Press, New York, 1977.

[10] Araki, M., "Stability of Large-Scale Nonlinear Systems-Quadratic-Order Theory of Composite-System Method Using M-Matrices," IEEE, Transactions of Automatic Control, AC-23, pp. 129-142.

[11] Khalil, H. K., "Asymptotic Stability of Nonlinear Multiparameter Singularly Perturbed Systems," Automatica, Vol. 17, No. 6, pp. 797-804, 1981.

[12] Zien, L., "An Upper Bound for the Singular Parameters in a Stable, Singularly Perturbed System," J. Franklin Inst., Vol. 295, No. 5, 1973.

[13] Javid, S. H., "Uniform Asymptotic Stability of Linear Time-Varying Singularly Perturbed Systems," J. Franklin Inst., Vol. 305, No. 1, 1978.

[14] Siddiqee, M. W., "Transient Stability of an A.C. Generator By Lyapunov's Direct Method," Int. J. Control, 1968, Vol. 8, No. 2, pp. 131-144.

[15] Araki, M., M. Saeki, and B. Kondo, "Application of a New Stability Criterion of Composite Systems to Multimachine Power Systems," IEEE Transaction on Automatic Control, Vol. AC-25, No. 3, June 1980.

[16] Hahn, W., "Stability of Motion," New York, Springer, 1967.

[17] Chua, L., and D. Green, "A Qualitative Analysis of the Behavior of Dynamic Nonlinear Networks: Stability of Autonomous Networks," IEEE Transaction on Circuits and Systems, Vol. CAS-23, No. 6, June 1976.

[18] Hahn, W., "Uber Stabilitätserhaltende Abbildungen und Ljapunovsche Funktionen," J. Für die reine und angewandtle Mathematik, Vol. 228, pp. 189-192, 1967.

[19] Wilde, R. R. and P. V. Kokotovic, "Stability of Singularly Perturbed Systems and Networks with Parasitics," IEEE Transaction Automatic Control, April 1972, pp. 245-246.

[20] Fouad, A. A., "Stability Theory-Criteria for Transient Stability," Proc. Eng. Foundation Conf. on System Eng. for Power: Status and Prospect, Henniker, New Hampshire, Aug. 1975, pp. 421-450.

[21] Pai, M. A., POWER SYSTEM STABILITY ANALYSIS BY THE DIRECT METHOD OF LYAPUNOV, North-Holland Publishing Company, 1981.

[22] Willems, J. L., Comment on "Transient Stability of an A.C. Generator by Lyapunov's Direct Method," Int. J. Control, 1969, Vol. 10, No. 1, pp. 113-116.

[23] Sasaki, H., "An Approximation of Field Flux Decay Into Transient Stability Analysis of Multimachine Power Systems By the Second Method of Lyapunov," IEEE Transaction on Power Apparatus and Systems, Vol. PAS-98, No. 2, March/April, 1979.

[24] Willems, J. L., "The computation of Finite Stability Regions By Means of Open Lyapunov Surfaces," Int. J. Control, 1969, Vol. 10, No. 5, pp. 537-544.

[25] Anderson, P. M., and A. A. Fouad, Power System Control and Stability, The Iowa State University Press, 1977.

[26] Demello, F. P. and C. Concordia, "Concepts of Synchronous Machine Stability as Affected by Excitation Control," IEEE Transactions on Power Apparatus and Systems, Vol. 88, No. 4, April 1969.

[27] Krasovskii, N. N., Stability of Motion, Stanford University Press, 1963.

[28] Gear, C.W., "Numerical Integration of Stiff Ordinary Differential Equations," Department of Computer Science, University of Illinois, Urbana, Rep. 221, Jan. 20, 1967.

[29] Sandberg, I.W. and H. Schickman, "Numerical Integration of Systems of Stiff Nonlinear Differentiable Equations," Bell Syst. Tech. J., Vol. 41, 1968, pp. 511-527.

[30] Tihonov, A.N., "Systems of Differential Equations Containing a Small Parameter Multiplying the Derivative," Mat. Sb. N.S., (31), 73, pp. 575-585. (In Russia)

[31] Hoppensteadt, F., "Stability in Systems with Parameters," J. Math. Anal. Appl., Vol. 18, 1967, pp. 128-134.

[32] Hoppensteadt, F., "Singular Perturbations on the Infinite Interval, Trans. Amer. Math. Soc., 123 (1966), pp. 521-535.

[33] Chang, K.W. and W.A. Coppel, "Singular Perturbations of Initial Value Problems Over a Finite Interval," Arch. Rational Mech. Anal., 32, (1969), pp. 268-280.

[34] Levin, J.J. and N. Levinson, "Singular Perturbations of Non-Linear Systems of Differential Equations and an Associated Bounday Layer Equations," J. Rational Mech. Anal., 13, (1954), pp. 247-270.

[35] Habets, P., "A Consistency Theory of Singular Perturbations of Differential Equations," SIAM J. Appl. Math., Vol. 26, No. 1, Jan. 1974.

[36] Hale, J.K., "Ordinary Differential Equations," New York, 1980.

[37] Chow, J.H. and P.V. Kokotovic, "Two-Time-Scale Feedback Design of a Class of Nonlinear Systems," IEEE Trans. Automat. Contr., Vol. AC-23, pp. 438-443, 1978.

[38] Chow, J.H. and P.K. Kokotovic, "Near-Optimal Feedback Stabilization of a Cla-s of Nonlinear Singularly Perturbed Systems," SIAM J. Contr. Optimiz., Vol. 16, pp. 756-770, 1978.

[39] Chow, J.H., and P.V. Kokotovic, "A Two-Stage Lyapunov-Bellman Feedback Design of a Class of Nonlinear Systems," IEEE Trans. on Automat. Contr., Vol. AC-26, pp. 656-663, 1981.

[40] Suzuki, M., "Composite Control for Singularly Perturbed Systems," IEEE Trans. Automat. Contr., Vol. AC-26, pp. 505-507, 1981.

[41] Laub, A.J. and F.N. Bailey, "Suboptimality Bounds and Stability in the Control of Nonlinear Dynamic Systems," IEEE Trans. Automat. Contr., Vol. AC-23, pp. 396-399, 1978.

[42] Davison, E.J., "The Decentralized Stabilization and Control of a Class of Unknown Nonlinear Time-Varying Systems," Automatica, Vol. 10, No. 3, 1974, pp. 309-316.

[43] Ikeda, M. and D.D. Siljak, "Decentralized Stabilization of Linear Large Scale Systems," IEEE Transactions of Automatic Control, Vol. 25, February 1980, pp. 106-107.

[44] Yamakami A. and J.C. Geromel, "Decentralized Stabilization and Minimum Norm Optimization of Linear Systems," I.F.A.C. 8th Triennial World Congress, 1981. Kyoto, Japan.

[45] Young, K.D., "Analysis and Synthesis of High Gain and Variable Structure Feedback Systems," Ph.D. Dissertation, University of Illinois, Urbana, Illinois, 1978.

[46] Sezer, M.E. and O. Huseyin, "Stabilization of Linear Time-Invariant Interconnected Systems, Using Local State Feedback," IEEE Transactions of Systems, Man and Cybernetics, Vol. SMC-8, Oct. 1978, pp. 751-756.

[47] Bradshaw, A., "High-Gain Decentralized Control of Interconnected Dynamical Systems," Electronic Letters, Vol. 16, March 1980, pp. 216-217.

[48] Young, K.D., P.V. Kokotovic, and V.I. Utkin, "A Singular Perturbation Analysis of High-Gain Feed-Back Systems," IEEE Transactions of Automatic Control, Vol. AC-22, 1977, pp. 931-938.

[49] Patel, R.V. and M. Toda, "Qualitative Measures of Robustness For Multivariable Systems," Joint Automatic Control, 1980, Vol. II.

[50] Ando, T., "Inequalities for M-Matrices," Linear and Multilinear Algebra, Vol. 8, 1980, pp. 291-316.

[51] Berman, A., PLPS: Matrices with Nonpositive Off-Diagonal Entries," Linear Algebra and its Applications, 1978, Vol. 21, pp. 233-244.

[52] Siljak, D.D. and M. B. Vukcevic, "Decentrally Stabilizable Linear and Bilinear Large-Scale Systems," Int. J. Control, 1977. Vol. 26, No. 2, pp. 289-305.

[53] Wang, S.H. and E.J. Davison, "On the Stabilization of Multivariable Decentralized Systems," IEEE Trans., 1973, AC-18, pp. 473-478.

[54] Corfmat, J.P. and A.S. Morse, "Decentralized Control of Linear Multivariable Systems," Automatica, 1976, 12, pp. 479-495.

[55] Saeks, R., "On the Decentralized Control of Interconnected Dynamical Systems," IEEE Trans., 1979, AC-24, pp. 269-271.

[56] Chan, W.S. and C.A. Desoer, "Eigenvalue Assignment and Stabililization of Interconnected System Using Local Feedbacks," IEEE Trans., AC-24, pp. 312-317.

[57] Huseyin, O., M.E. Sezer and D.D. Siljak, "Robust Decentralized Control Using Output Feedback," IEE Proc., Vol. 129, Pt. D, No. 6, Nov. 1982.

[58] Davison, E.J. and S.H. Wang, "Properties and Calculation of Transmission Zeros of Linear Multivariable Systems," Automatica, Vol. 10, pp. 643-658, 1974.

[59] Owens, D.H., "Dynamic Transformations and the Calculations of Multivariable Root-Loci," Int. J. Control, 1978, Vol. 28, No. 3, pp. 333-343.

[60] Dorato, P., "On the Inverse of Linear Dynamical Systems," IEEE Trans. Syst. Sci. & Cybern., SSC-5, 43, 1969.

[61] Sannuti, P., "Direct Singular Perturbation Analysis of High-Gain and Cheap Control Problems," Automatic, Vol. 19, No. 1, pp. 41-51, 1983.

[62] Sannuti, P. and H. Wason, "A Singular Perturbation Canonical Form of Invertible Systems: Determination of Multivariable Root-Loci," Int. J. Control, 1983, Vol. 37, No. 6, 1259-1286.

[63] Kouvaritakis, B. and A.G.J. Macfarlane, "Geometric Approach to Analysis and Synthesis of System Zeros Part 2. Non-Square Systems," INt. J. Control, 1976, Vol. 23, No. 2, pp. 167-181.

[64] Vardulakis, Antonis I.G., "Zero Placement and the "Squaring Down" Problem: A Polynomial Matrix Approach," INt. J. Control, 1980, Vol. 31, No. 5, 821-832.

[65] Araki, M., "A Note on the Calculation and Placing of Finite Zeros," Int. J. Control, 1983, Vol. 37, No. 4, pp. 873-878.