SURFACES ASSOCIATED WITH A SPACE CURVE

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SURFACES ASSOCIATED WITH A SPACE CURVE

by

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SURFACES ASSOCIATED WITH A SPACE CURVE

Introduction

It is the purpose of this paper to discuss some of the metric differential properties of a torus.

Let us consider a curve $\mathcal{C}$, a point $\mathcal{P}$ on $\mathcal{C}$, a local trihedron of $\mathcal{C}$ at $\mathcal{P}$, and any point $\mathcal{Q}$. The local trihedron is defined by the tangent line, principal normal, and binormal to $\mathcal{C}$ at $\mathcal{P}$, wherein these lines are referred to as ξ-, η-, and ζ-axes in a manner similar to the x-, y-, and z-axes of a fixed coördinate system. Let $\mathcal{P}$ have coördinates (x,y,z) in a fixed system of coördinates and coördinates (0,0,0) in the local coördinate system of $\mathcal{C}$ at $\mathcal{P}$; similarly let $\mathcal{Q}$ have coördinates (X,Y,Z) in a fixed system and coördinates (ξ, η, ζ) in the local system of $\mathcal{C}$ at $\mathcal{P}$.

The projection of the line segment joining $\mathcal{P}$ and $\mathcal{Q}$ on the x-axis can be represented analytically in two ways,

$$\text{proj. } PQ_x = X - x \quad ,$$
$$\text{proj. } PQ_x = \xi\alpha + \eta l + \zeta\lambda \ .$$

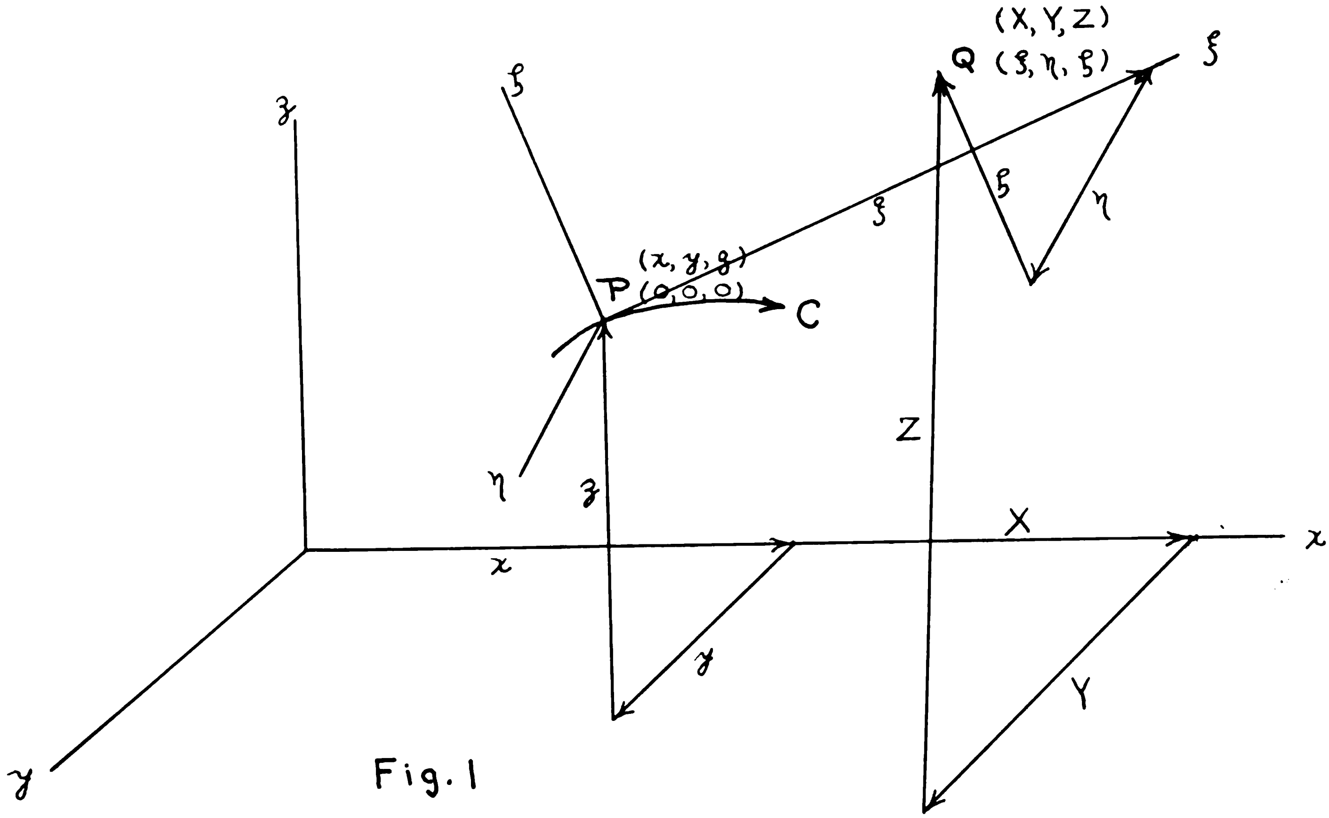

Fig. 1

Hence

$$X - x = \xi\alpha + \eta l + \zeta\lambda .$$

Similarly

$$Y - y = \xi\beta + \eta m + \zeta\mu ,$$

and

$$Z - z = \xi\gamma + \eta n + \zeta\nu .$$

wherein α, β, γ ; l, m, n ; λ, μ, ν are the cosines of the angles made respectively by the tangent line, principal normal, and binormal to C at P with the lines through P parallel respectively to the x-, y-, and z-axes; that is they are the direction cosines with respect to the fixed coördinate system.

Hence the equations of transformation between the fixed coördinates (X,Y,Z) of Q and the local coördinates

(ξ, η, ζ) of Q referred to the local coördinate system at P with fixed coordinates (x, y, z) of C are*

$$(1) \qquad \begin{cases} X = x + \alpha\xi + l\eta + \lambda\zeta , \\ Y = y + \beta\xi + m\eta + \mu\zeta , \\ Z = z + \gamma\xi + n\eta + \nu\zeta . \end{cases}$$

For example consider a circle C' of radius a and lying in the normal plane with center at the point P on C. The local coördinates (ξ, η, ζ) of a point Q on C' are

$$(2) \qquad \begin{cases} \xi = 0 , \\ \eta = a \cos u , \\ \zeta = a \sin u . \end{cases}$$

wherein u is the angle from the principal normal to the line through P and the origin.

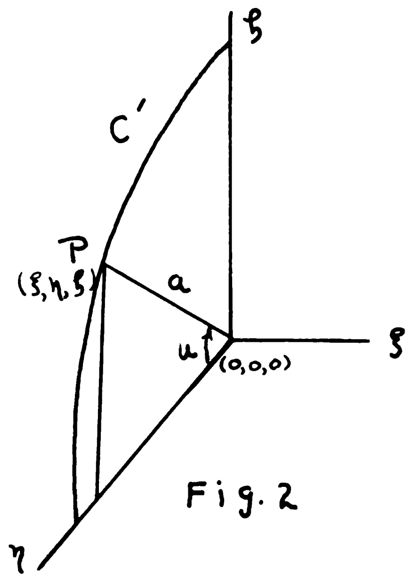

Fig. 2

The fixed coördinates (X, Y, Z) of P are

$$(3) \qquad \begin{cases} X = x + al \cos u + a\lambda \sin u , \\ Y = y + am \cos u + a\mu \sin u , \\ Z = z + an \cos u + a\nu \sin u . \end{cases}$$

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*V. G. Grove, Metric Differential Geometry of Curves and Surfaces, Notes prepared for use in Michigan State College, p. 36, eq. (74). Hereinafter referred to as Grove, Geometry.

The Torus Defined

If in equations (2) and (3) above C is a circle of radius $b > a$, with a and b both positive and finite, the locus of the circle C' is a torus. Let v be the arc length of the circle C measured from a fixed point on C.

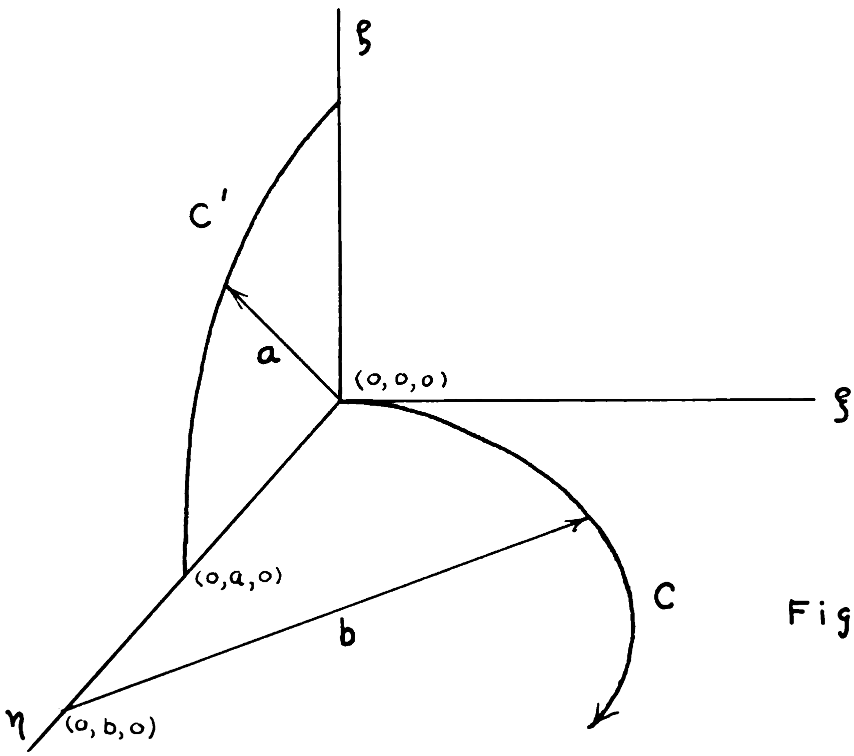

Fig. 3

For the circle C, the radius of curvature ρ is b. Also being a plane curve the torsion $\frac{1}{\tau}$ is zero. Grouping these last restrictions and observations together for convenience we have

$$(4) \qquad \begin{cases} 0 < a < b < \infty, \quad \rho = b, \quad \frac{1}{\tau} = 0, \\ v = \text{arc length on } C. \end{cases}$$

As a matter of notation, it will greatly shorten the number of equations if we represent only the X values; corresponding values of Y and Z are found by permitting the following permutations:

$$x \to y\,,\quad l \to m\,,\quad \lambda \to \mu \qquad \text{for } X \to Y\,,$$

and

$$x \to z\,,\quad l \to n\,,\quad \lambda \to \nu \qquad \text{for } X \to Z\,.$$

Preliminary Computations

Several computations can be made which will prove useful in discussing some of the properties of the torus.

By means of the Frenet-Serret formulas*

$$(5)\qquad \begin{cases} \alpha' = \dfrac{l}{\rho}\,, & \beta' = \dfrac{m}{\rho}\,, & \gamma' = \dfrac{n}{\rho}\,, \\ l' = -\left(\dfrac{\alpha}{\rho} + \dfrac{\lambda}{\tau}\right), & m' = -\left(\dfrac{\beta}{\rho} + \dfrac{\mu}{\tau}\right), & n' = -\left(\dfrac{\gamma}{\rho} + \dfrac{\nu}{\tau}\right), \\ \lambda' = \dfrac{l}{\tau}\,, & \mu' = \dfrac{m}{\tau}\,, & \nu' = \dfrac{n}{\tau}\,. \end{cases}$$

wherein primes denote differentiation with respect to the arc length, and the relationships expressed in equations (3) and (4) we readily find the following partial derivatives of the coördinate X for the torus:

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* L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, New York, Ginn and Company, 1909, p. 17, eq. (50). Hereinafter referred to as Eisenhart, Geometry.

$$
(6)\qquad \begin{cases}
X = x + a l \cos u + a \lambda \sin u, \\
X_u = -a l \sin u + a \lambda \cos u, \\
X_v = \alpha \left(1 - \frac{a}{b} \cos u\right), \\
X_{uu} = -a l \cos u - a \lambda \sin u, \\
X_{uv} = \frac{a}{b} \alpha \sin u, \\
X_{vv} = \frac{l}{b}\left(1 - \frac{a}{b} \cos u\right).
\end{cases}
$$

Also certain fundamental coefficients defined as follows will be used: *

$$E = X_u^2 + Y_u^2 + Z_u^2 ,$$

$$F = X_u X_v + Y_u Y_v + Z_u Z_v ,$$

$$G = X_v^2 + Y_v^2 + Z_v^2 ,$$

$$H^2 = EG - F^2 , \quad **$$

$$L = \frac{1}{H} \begin{vmatrix} X_{uu} & Y_{uu} & Z_{uu} \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix} , \qquad ****$$

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* E. P. Lane, Metric Differential Geometry of Curves and Surfaces, Chicago, The University of Chicago Press, 1938, p.74 eq.(3.3). Hereinafter referred to as Lane, Geometry.

** Lane, Geometry, p. 75 eq. (3.6)

*** Grove, Geometry, p.75 eq.(130), with Lane, p. 123.

$$M = \frac{1}{H}\begin{vmatrix} X_{uv} & Y_{uv} & Z_{uv} \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix},$$

$$N = \frac{1}{H}\begin{vmatrix} X_{vv} & Y_{vv} & Z_{vv} \\ X_u & Y_u & Z_u \\ X_v & Y_v & Z_v \end{vmatrix},$$

Using the notation suggested at the end of the last section these relationships become

$$(7)\quad \begin{cases} E = \Sigma X_u^2, \quad F = \Sigma X_u X_v, \quad G = \Sigma X_v^2, \\ L = \frac{1}{H}\begin{vmatrix} X_{uu} \\ X_u \\ X_v \end{vmatrix}, \quad M = \frac{1}{H}\begin{vmatrix} X_{uv} \\ X_u \\ X_v \end{vmatrix}, \quad N = \frac{1}{H}\begin{vmatrix} X_{vv} \\ X_u \\ X_v \end{vmatrix}. \end{cases}$$

It will further be noticed by comparing equations (4) with (7) that HL, HM, and HN are all of the form Q where

$$(8)\quad Q \equiv \begin{vmatrix} a_1\alpha + b_1 l + c_1\lambda & a_1\beta + b_1 m + c_1\mu & a_1\gamma + b_1 n + c_1\nu \\ a_2\alpha + b_2 l + c_2\lambda & a_2\beta + b_2 m + c_2\mu & a_2\gamma + b_2 n + c_2\nu \\ a_3\alpha + b_3 l + c_3\lambda & a_3\beta + b_3 m + c_3\mu & a_3\gamma + b_3 n + c_3\nu \end{vmatrix}.$$

But Q can be written in the form of a product, namely

$$Q = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}.$$

However the second determinant of the product*

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = +1 .$$

Hence

$$(9) \qquad Q = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} .$$

We may therefore for ease in computation state the following introductory

THEOREM. Any determinant of the form Q expressed in (8) can be evaluated as a determinant of the coefficients of α, l, λ.

The evaluation of these Q determinants agrees favorably with the X value notation already adopted and expressed in equations (6) and (7). Making use of the introductory theorem just stated and the relationships expressed in equations (6) and (7), we may derive the following fundamental coefficients for the torus:

$$(10) \qquad \begin{cases} E = a^2 , & L = -a , \\ F = 0 , & M = 0 , \\ G = \left(1 - \frac{a}{b}\cos u\right)^2 , & N = \frac{\cos u}{b}\left(1 - \frac{a}{b}\cos u\right) , \\ & H = a\left(1 - \frac{a}{b}\cos u\right) . \end{cases}$$

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*Lane, Geometry, p. 18.

Some Properties of the Torus

Theorem 1. The parametric net on the torus is an orthogonal net.

The parametric net is formed by the families of lines $u = \text{constant}$ and $v = \text{constant}$. The curves $u = \text{constant}$ are the circles whose planes are parallel to the osculating plane of C. The curves $v = \text{constant}$ are the generating circles C'.

Proof: A necessary and sufficient condition that the parametric net on a surface be orthogonal is that $F = 0$.* By equations (10), this condition is satisfied for the torus.

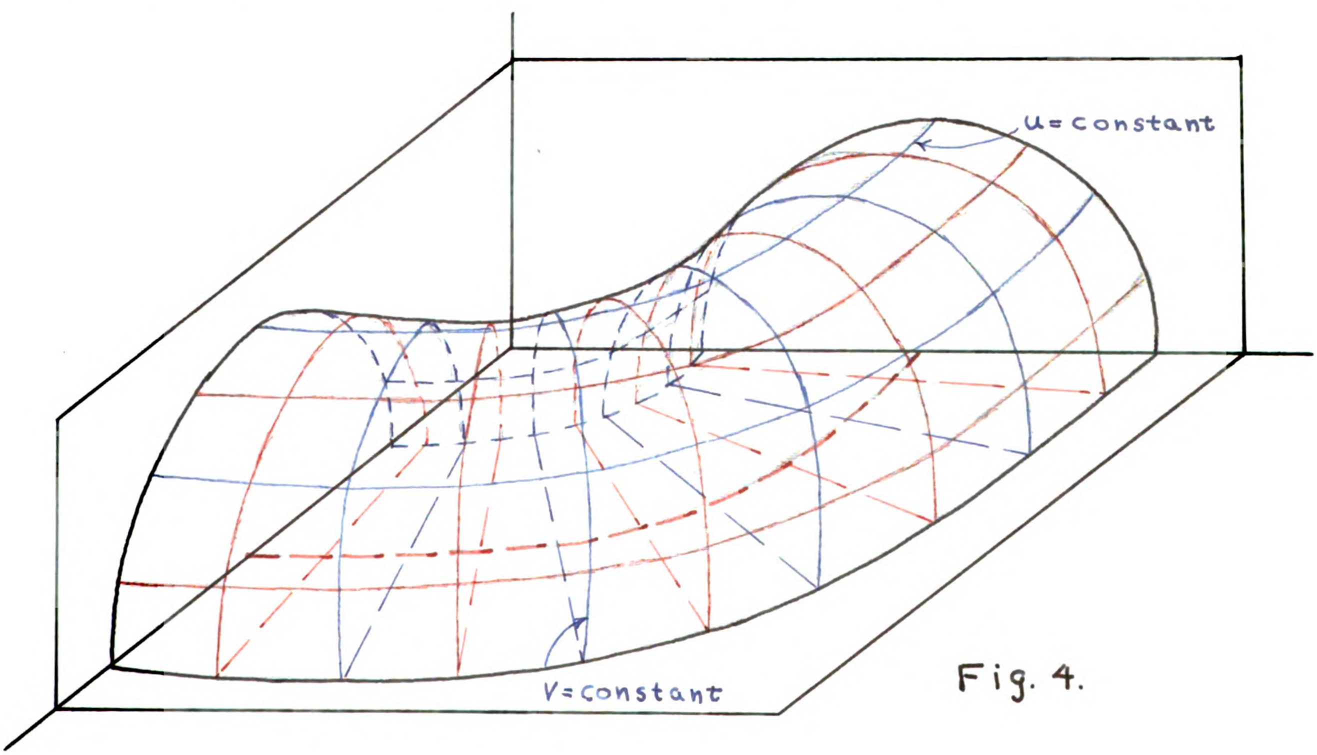

Fig. 4.

* Lane, Geometry, p. 116.

Theorem 2. The parametric net on the torus constitutes a conjugate net.

Proof: A necessary and sufficient condition that a parametric net be a conjugate net is that $M = 0$.* By equations (10), this condition is met for the torus.

By definition a net of curves on a surface is a conjugate net in case the tangents of the curves of one family of the net at the points of each fixed curve of the other family form a developable surface.** On the torus, for a fixed v, the tangents of the curves u = constant form a circular cylinder. For a fixed u, the tangents of the curves v = constant form a circular cone which degenerates to a plane for $u = \left|\frac{\pi}{2}\right|$ and a circular cylinder for $u = 0$.

Theorem 3. The asymptotic curves on a torus can be found by quadratures.

Proof: The curvilinear differential equation of the asymptotic curves on a surface is given by:***

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0 .$$

Making use of the relations in equations (10), the differentail equation for the torus is

$$(11) \qquad -a\,du^2 + \frac{\cos u}{b}\left(1 - \frac{a}{b}\cos u\right)dv^2 = 0 ,$$

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* Lane, Geormetry, p. 138 Th. 3.

** Ibid., p. 135 Def. 1.

*** Ibid., p. 123 Eq. (4.6).

from which we find

(12) $$V = \pm b\sqrt{a}\int_0^u \frac{du}{\sqrt{\cos u\,(b-a\cos u)}} .$$

Equation (12) can be shown to be an elliptic integral* by use of the following substitutions:

Let

$$u = \cos^{-1} x ,$$

then

$$V = \mp b\int_1^x \frac{dx}{\sqrt{1-x^2}\sqrt{\frac{b^2}{4a^2}-\left(x-\frac{b}{2a}\right)^2}} ;$$

let

$$x - \frac{b}{2a} = \frac{b}{2a}\sin\theta ,$$

then

$$V = \mp b\int_{\sin^{-1}\left(\frac{2a}{b}-1\right)}^{\sin^{-1}\left(\frac{2ax}{b}-1\right)} \frac{d\theta}{\sqrt{1-\frac{b^2}{4a^2}(1+\sin\theta)^2}} ;$$

and let

$$\theta = \frac{\pi}{2} - \phi ,$$

then

(13) $$V = \pm b\int_{\frac{\pi}{2}-\sin^{-1}\left(\frac{2a}{b}-1\right)}^{\frac{\pi}{2}-\sin^{-1}\left(\frac{2ax}{b}-1\right)} \frac{d\phi}{\sqrt{1-\frac{b^2}{4a^2}\cos^4\frac{\phi}{2}}} .$$

For convenience we shall call the curves u = constant parallels of latitude, and v =constant meridians.

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* Harris Hancock, <u>Lectures on the Theory of Elliptic Functions</u>, New York, John Wiley & sons, 1910, p. 187.

Theorem 4. The parametric curves are the lines of curvature.

Proof: The lines of curvature on a surface not a plane or a sphere are parametric if and only if $F = 0$, and $M = 0$.* That is, the lines of curvature are parametric from equations (10). But the lines of curvature form the <u>only</u> orthogonal conjugate net on a surface.** It follows then from Theorems 1 and 2 that the parametric curves are the lines of curvature.

This theorem follows readily from the differential equation of the lines of curvature. For any surface this differential equation is***

$$(EM - FL)du^2 + (EN - GL)du\,dv + (FN - GM)dv^2 = 0.$$

which becomes for the torus by equations (10)

(14) $$a\left(1 - \frac{a}{b}\cos u\right)du\,dv = 0 .$$

The solutions of equation (14) are seen to be u = constant and v = constant.

Theorem 5. The points $|u| > \frac{\pi}{2}$ are elliptic points; the points $|u| < \frac{\pi}{2}$ are hyperbolic points; the points $|u| = \frac{\pi}{2}$ are parabolic points.

P Proof: When the lines of curvature are parametric on a surface, a condition which is true for the torus by Theorem 4, the principal normal curvatures $\frac{1}{R_1}, \frac{1}{R_2}$ at a point on the surface are given by the formulas****

(15) $$\frac{1}{R_1} = \frac{L}{E} , \qquad \frac{1}{R_2} = \frac{N}{G} .$$

(15) $$\frac{1}{R_1} = \frac{L}{E}\ , \qquad \frac{1}{R_2} = \frac{N}{G}\ .$$

By equations (10), the principal normal curvatures for the torus are

(16) $$\frac{1}{R_1} = -\frac{1}{a}\ , \qquad \frac{1}{R_2} = \frac{\cos u}{b - a\cos u}\ .$$

The Gaussian curvature $K \equiv \frac{1}{R_1}\cdot\frac{1}{R_2} > 0$ defines an elliptic point, $K < 0$ a hyperbolic point, and $K = 0$ a parabolic point. For the torus

(17) $$K = -\frac{\cos u}{a(b - a\cos u)}\ .$$

Recalling that $b > a$ from relationships (4), it is readily seen that for $|u| > \frac{\pi}{2}$, $K > 0$ since the factor $(b - a\cos u) > 0$; also for $|u| < \frac{\pi}{2}$, $K < 0$: and for $|U| = \frac{\pi}{2}$, $K = 0$.

It may be recalled that in a region on a surface in which all points are elliptic, the asymptotic curves are imaginary; and where all points are hyperbolic, the

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* Lane, Geometry, p. 145 Th. 4.

** Ibid, p. 145 Th. 3.

*** Ibid, p. 143 Eq. (7.5).

**** Ibid, p. 160 Eq. (2.12).

asymptotic curves are real.* For the parabolic points, The curvilinear differential equation of the asymptotic curves, equation (11), vanishes since du = 0. From these remarks we might add the following corollary to theorem 3:

Corollary: The asymptotic curves are imaginary for $|u| > \frac{\pi}{2}$, and real for $|u| < \frac{\pi}{2}$.

Theorem 6. The parametric net on the torus is isothermally orthogonal.

Proof: Necessary and sufficient conditions that a parametric net be isothermally orthogonal are**

$$(18) \qquad F = 0, \quad \text{and} \quad \left(\log \frac{E}{G}\right)_{uv} = 0 .$$

From equations (10), F = 0, also it will be observed that $\frac{E}{G}$ = a function of u <u>only</u>, hence $\left(\log \frac{E}{G}\right)_{uv} = 0$.

Theorem 7. The torus is an isothermic surface.

Proof: By definition an isothermic surface is a surface on which the lines of curvature form an isothermally orthogonal net.*** Theorem 7 follows immediately from the statements of theorems 4 and 6.

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* Lane, <u>Geometry</u>, p.162.

** Ibid., p. 120 Eq. (3.21).

*** Ibid., p. 145 Ex. 6.

Theorem 8. The parametric net on a torus is isothermally conjugate.

Proof: Necessary and sufficient conditions that the parametric net be isothermally conjugate are*

(19) $$M = 0, \quad \text{and} \quad \left(\log \frac{L}{N}\right)_{uv} = 0 .$$

From equations (10) we see readily that $M = 0$, and that since $\frac{L}{N}$ = a function of u <u>only</u>, $\left(\log \frac{L}{N}\right)_{uv} = 0$.

Theorem 9. The minimal curves on the torus can be found by quadratures.

Proof: The curvilinear differential equation for the minimal curves on a surface is**

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0 .$$

By equations (10), the differential equation for the torus becomes

(20) $$a^2 du^2 + \left(1 - \frac{a}{b}\cos u\right)^2 dv^2 = 0,$$

from which we obtain

(21) $$v = \pm\, a b i \int_0^u \frac{du}{b - a\cos u} .$$

Performing the indicated integration we find that

(22) $$v = \pm \frac{abi}{\sqrt{b^2 - a^2}} \tan^{-1}\left[\frac{\sqrt{b^2-a^2}\,\sin u}{b\cos u - a}\right].$$

It is seen that the minimal curves on the torus are imaginary, which is true of all real surfaces. It may also be remarked that minimal curves on a surface are geodesics.***

If the sum of the principal normal curvatures is zero, the surface is a minimal surface.**** By equations (16) we have

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2a\cos u - b}{a(b - a\cos u)} \,. \tag{23}$$

If $\cos u = \frac{b}{2a}$, then

$$\frac{1}{R_1} + \frac{1}{R_2} = 0 \,.$$

A minimal surface is characterized by the following equation*****

$$EN + 2FM + GL = 0.$$

By equations (10), the equation above becomes for the torus

$$a\left(1 - \frac{a}{b}\cos u\right)\left(\frac{2a}{b}\cos u - 1\right) = 0 \,. \tag{24}$$

Lane, Geometry, p. 141 Eq. (6.16).

** Ibid., p. 110 Eq. (2.4).

*** Ibid., p. 150 Ex. 7.

**** Eisenhart, Geometry, p. 129; where mean curvature is $1/R - 1/R$.

***** Lane, Geometry, p. 163 Eq. (3.3).

If $a = 0$, the torus degenerates to the generating circle in the osculating plane. If the second factor is set equal to zero, then $\cos u = \frac{b}{a}$ which is impossible by relations (4); that is $b > a$ which would make $\cos u > 1$ for this factor. The third factor equated to zero yields

$$\cos u = \frac{b}{2a} .$$

Thus we have the following

Theorem 10. In the neighborhood for those points for which $\cos u = \frac{b}{2a}$ the torus resembles a minimal surface.

The torus could never be a minimal surface, since the only surface of revolution not a plane that is a minimal surface is the catenoid.*

If we take the partial derivatives of the functions E, F, and G defined in equations (10), we find that

$$(25) \qquad \begin{cases} E_u = F_u = E_v = F_v = G_v = 0 , \\ G_u = 2\left(1 - \frac{a}{b}\cos u\right)\left(\frac{a}{b}\sin u\right) . \end{cases}$$

Hence the Christoffel three-index symbols of the second kind for the first fundamental form**

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2H^2}\left(GE_u + FE_v - 2FF_u\right) , & \Gamma^2_{11} &= \frac{1}{2H^2}\left(-FE_u - EE_v + 2EF_u\right) \\ \Gamma^1_{12} &= \frac{1}{2H^2}\left(GE_v - FG_u\right) , & \Gamma^2_{12} &= \frac{1}{2H^2}\left(EG_u - FE_v\right) , \\ \Gamma^1_{22} &= \frac{1}{2H^2}\left(-FG_v - GG_u + 2GF_v\right) , & \Gamma^2_{22} &= \frac{1}{2H^2}\left(EG_v + FG_u - 2FF_v\right) \end{aligned}$$

* Lane, Geometry, p. 165 Ex. 8.

** Ibid., P.132 Eq. (5.4).

become

$$(26)\qquad \begin{cases} \Gamma^1_{11} = 0, & \Gamma^1_{12} = 0, \quad \Gamma^1_{22} = -\dfrac{G_u}{2E}, \\ \Gamma^2_{11} = 0, & \Gamma^2_{12} = -\dfrac{G_u}{G}, \quad \Gamma^2_{22} = 0 . \end{cases}$$

Assuming the parameter along a geodesic curve on the torus to be u, the curvilinear differential equation of the geodesics is*

$$v'' = -\Gamma^2_{11} - (2\Gamma^2_{12} - \Gamma^1_{11})v' + (2\Gamma^1_{12} - \Gamma^2_{22})v'^2 + \Gamma^1_{22} v'^3,$$

which becomes by equations (26)

$$(27)\qquad \frac{d^2v}{du^2} = \frac{2G_u}{G}\frac{dv}{du} - \frac{G_u}{2E}\left(\frac{dv}{du}\right)^3 .$$

In order to find solutions for equation (27) we shall use the following substitutions:

Let

$$p = \frac{dv}{du},$$

then equation (27) becomes

$$-2p^{-3}\frac{dp}{du} + 4\frac{G_u}{G}p^{-2} = \frac{G_u}{E};$$

let now

$$p^{-2} = z,$$

and then we obtain

$$\frac{dz}{du} + 4\frac{G_u}{G}z = \frac{G_u}{E} .$$

This last equation has the solution

$$z = \frac{G^5 + 5EC}{5EG^4},$$

where C is an arbitrary constant.

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*Lane, Geometry, p. 147 Eq. (8.4).

Retracing our substitutions, equation (27) may be written

(28) $$\frac{dv}{\sqrt{5E}} = \pm \frac{G^2 du}{\sqrt{G^5 + 5EC}} .$$

Theorem 11. The geodesic curves on the torus can be found by quadratures.

Proof: In view of the above discussion, the geodesic curves v, which are solutions of equation (28) are

(29) $$v = \pm \sqrt{5E} \int_0^u \frac{G^2 du}{\sqrt{G^5 + 5EC}} .$$

It has been remarked already (under Theorem 9) that the minimal curves on a surface are geodesics. Hence the curves

(22) $$v = \pm \frac{abi}{\sqrt{b^2 - a^2}} \tan^{-1}\left[\frac{\sqrt{b^2-a^2} \sin u}{b \cos u - a}\right] ,$$

are solutions of equation (29).

Maps on the Plane

The element of arc length is represented by*

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

which becomes by equations (10) for the torus

(30) $$ds^2 = a^2 du^2 + (1 - \frac{a}{b} \cos u)^2 dv^2 .$$

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* Grove, Geometry, p. 69 Eq. (119).

If we make a transformation $\bar{u} = \bar{u}(u)$ and $\bar{v} = v$, wherein

(31) $$d\bar{u} = \frac{ab}{b - a\cos u}\,du ,$$

the square of the differential of arc length equation (30) becomes

(32) $$ds^2 = \left(1 - \frac{a}{b}\cos u\right)^2 (d\bar{u}^2 + d\bar{v}^2) .$$

From the transformation equation (31) we obtain

(33) $$\bar{u} = \frac{ab}{\sqrt{b^2 - a^2}} \tan^{-1} \frac{\sqrt{b^2 - a^2}\,\sin u}{b\cos u - a} .$$

Finally if we let $\bar{u} = y$ and $\bar{v} = x$, we may state the following

Theorem 13. The transformation $y = \frac{ab}{\sqrt{b^2 - a^2}} \tan^{-1} \frac{\sqrt{b^2 - a^2}\,\sin u}{b\cos u - a}$ and $x = v$ represents a conformal mapping of the torus on the xy-plane, wherein the parallels of latitude map into lines parallel to the x-axis and the meridians map into lines parallel to the y-axis.

Proof: A transformation between two surfaces is conformal if and only if the minimal curves on the two surfaces correspond.*

For a plane, the minimal curves are given by the differential equation

(34) $$dx^2 + dy^2 = 0 .$$

The minimal curves for the xy-plane are thus

(35) $$x = \pm i y .$$

But from the transformation expressed in theorem 13, the minimal curves on the xy-plane are

(36) $$x = \pm i \frac{ab}{\sqrt{b^2 - a^2}} \tan^{-1} \frac{\sqrt{b^2 - a^2} \sin u}{b \cos u - a} .$$

Finally comparing expression (36) with equation (22), we see that by the given transformation the minimal curves on the torus and the plane are identical.

Also by observing the expression for the transformation it is readily seen that for u =constant, y = constant; and for v = constant, x = constant.

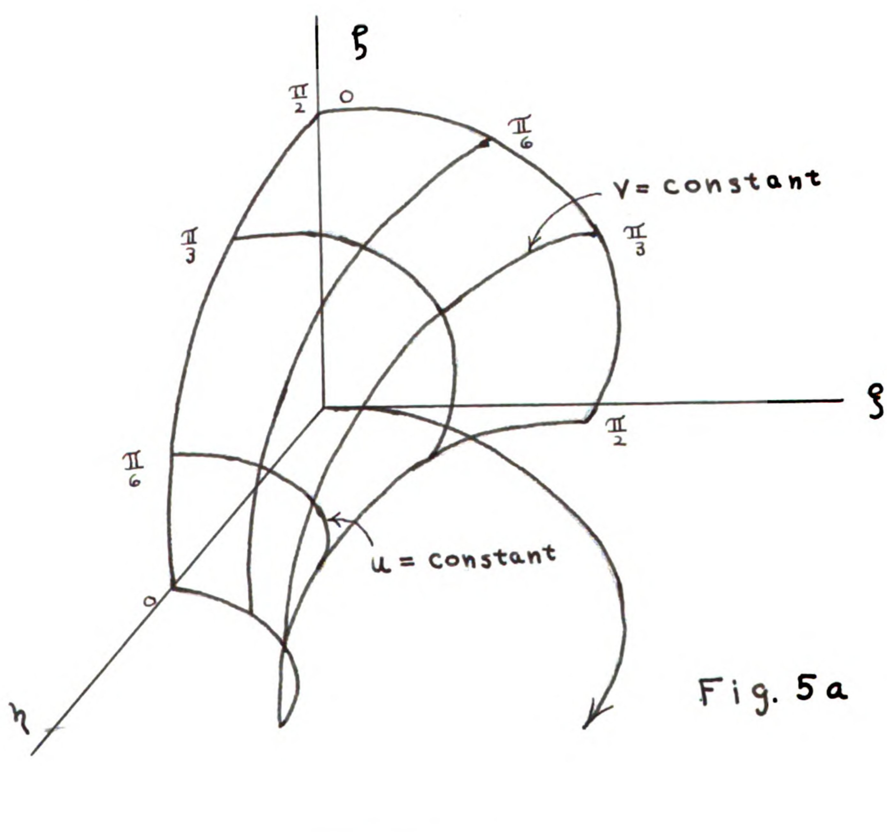

Fig. 5a

* Lane, Geometry, p. 193 Th. 2.

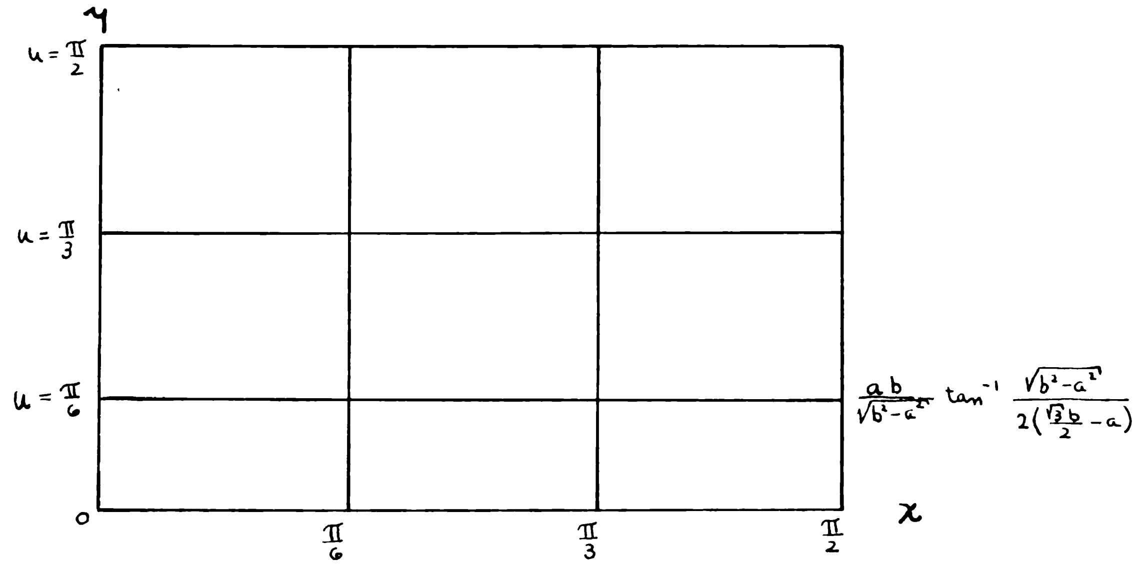

Fig. 5b

For some other possible mappings of the torus on the xy-plane, we recall that the transformation $x = x(u,v)$, $y = y(u,v)$, between a surface referred to parameters u, v and the xy-plane is an equiareal map of the surface upon the xy-plane in case the functions x, y are solutions of one or the other of the two partial differential equations*

(37) $$X_uY_v - X_vY_u = \pm H .$$

Using this relationship and those of equations (10) we have for the torus

(38) $$x_u y_v - x_v y_u = \pm a\left(1 - \frac{a}{b}\cos u\right).$$

Let $x = v$, and $y = f(u)$, and choose the negative sign before H; then equation (38) becomes

(39) $$a\left(1 - \frac{a}{b}\cos u\right) = f'(u) .$$

Hence

(40) $$f(u) = a\left(u - \frac{a}{b}\sin u\right).$$

* Lane, Geometry, p. 203 Th. 2.

Theorem 15. The transformation $x = v$, $y = a\left(u - \frac{a}{b}\sin u\right)$ represents an equiareal mapping of the torus on the xy-plane.

Proof: The transformation (40) satisfies the partial differential equation (38); therefore the transformation expressed in the theorem represents an equiareal map of the torus on the xy-plane.

This transformation also maps the meridians into lines parallel to the y-axis and the parallels of latitude into lines parallel to the x-axis.

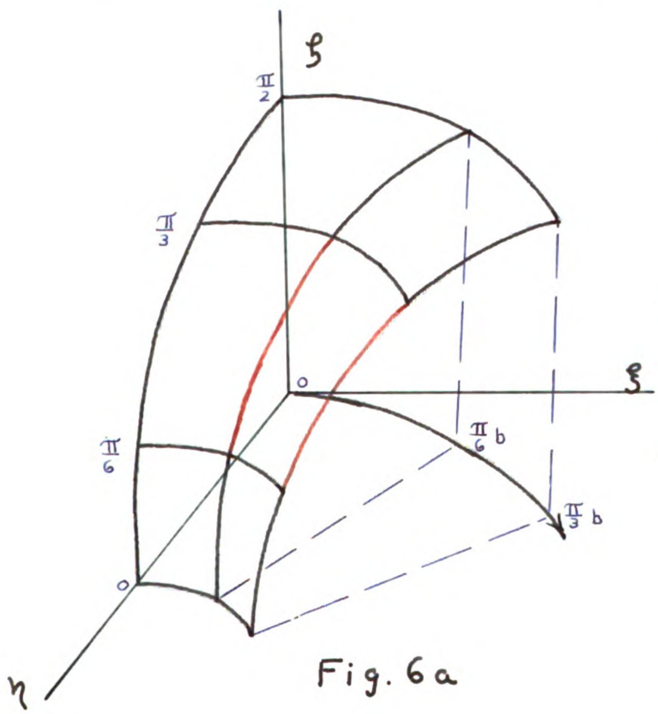

Fig. 6a

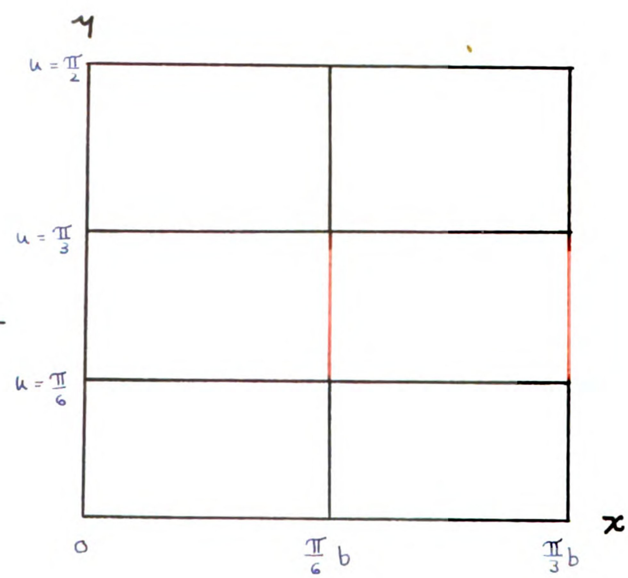

Fig. 6b

The map on the xy-plane extends from $x = 0$ to $x = 2\pi b$ and $y = 0$ to $y = 2\pi a$. Thus the surface area of the torus is represented on the plane by the area of the rectangle which has for its dimensions $2\pi a$ and $2\pi b$, or $4\pi^2 ab$.

Comparing this method of finding the surface area of the torus with the method of the calculas, the advantage of the equiareal map just described is seen.

Let the torus be generated by revolving the circle

$$(x-b)^2 + y^2 = a^2$$

about the y-axis.

Then the total surface S is expressed

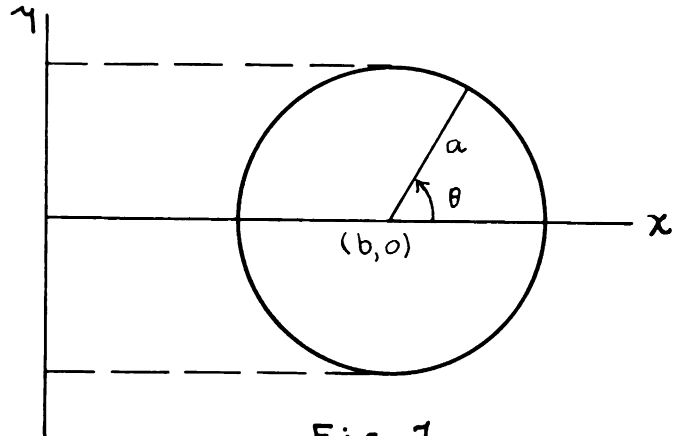

Fig. 7

(41)
$$S = 2\pi \int_{\theta=0}^{\theta=2\pi} x \, ds \ ,$$

wherein $x = b + a\cos\theta$, and $ds = a\,d\theta$.

$$S = 2\pi \int_0^{2\pi} (b + a\cos\theta) a \, d\theta \ = 2\pi a \Big[b\theta + a\sin\theta \Big]_0^{2\pi} \ ,$$

therefore

$$S = 4\pi^2 ab.$$

Another equiareal map may be found as follows: let $y = u$ and $x = vf(u)$, then equation (38) becomes

(42)
$$a\left(1 - \frac{a}{b}\cos u\right) = x_v \ .$$

It should be noticed from equations (10) that X_v is a function of u only; $x_v = f(u)$ has as a solution $x = vf(u)$, or by equations (10)

(43)
$$x = av\left(1 - \frac{a}{b}\cos u\right).$$

This map is not as convenient a representation as the other equiareal map already discussed as seen pictorially.

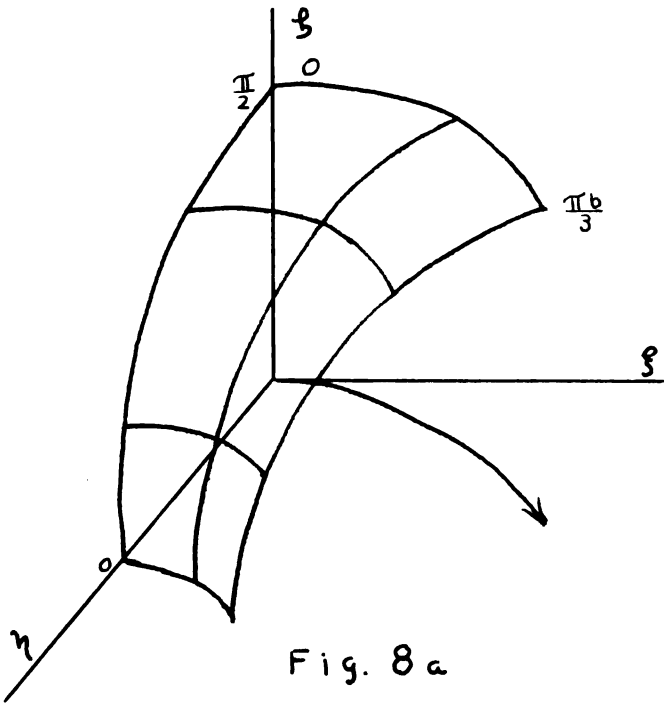

Fig. 8a

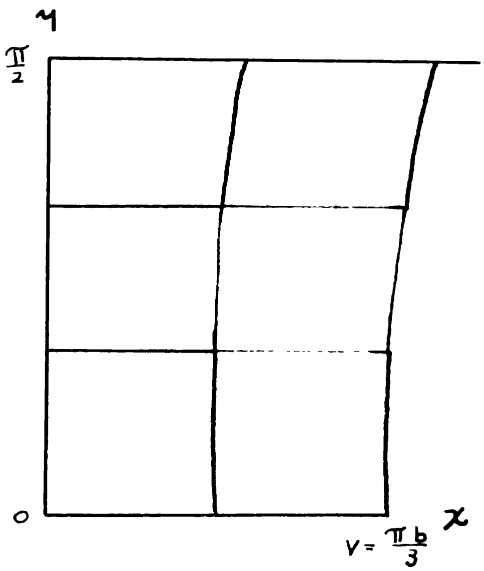

Fig. 8b

For the entire surface, the map appears as follows:

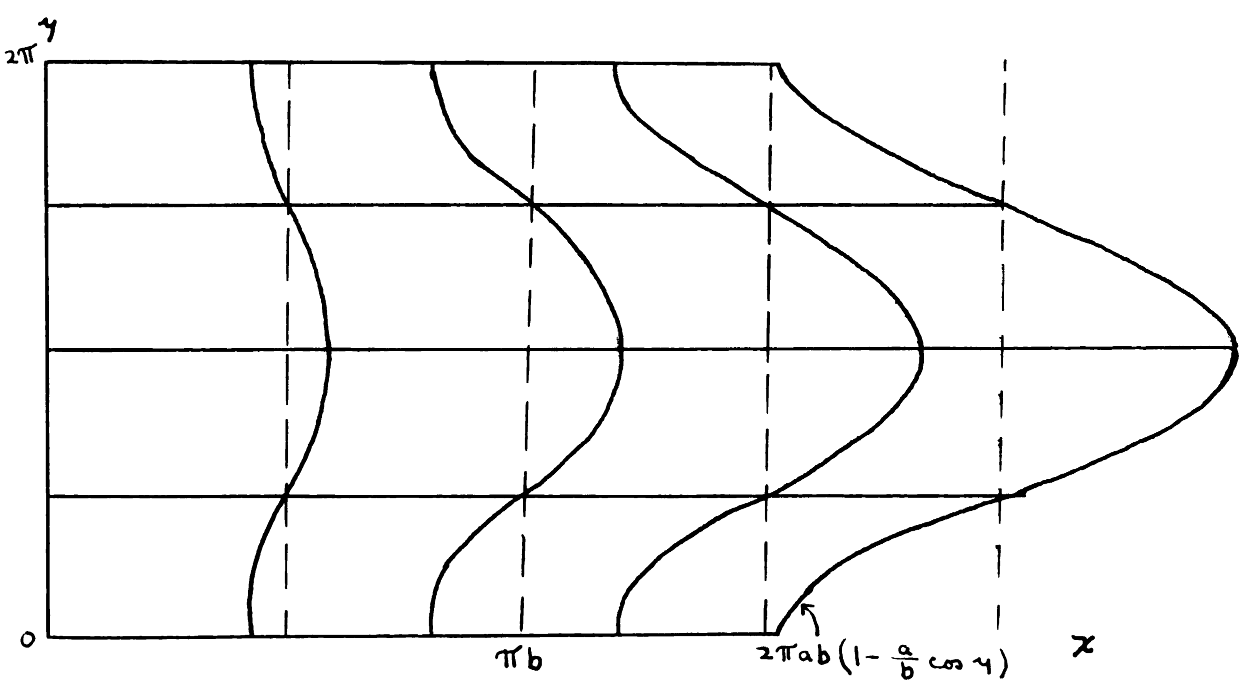

Fig. 8c

For this equiareal map the x ranges in value from zero to $2\pi ab\left(1-\frac{a}{b}\cos y\right)$, and y from zero to 2π. As a check, the area under the curve

$$x = 2\pi ab\left(1-\frac{a}{b}\cos y\right), \tag{44}$$

represents the area of the torus.

$$A = \int_0^{2\pi} 2\pi a\left(1-\frac{a}{b}\cos y\right)dy = 2\pi ab\left[y-\frac{a}{b}\sin y\right]_0^{2\pi},$$

therefore

$$A = 4\pi^2 ab\ .$$

The Normal Congruence

Let us now choose a line normal to the circle in the normal plane C' with coördinates of a point on the line $(\underline{X},\underline{Y},\underline{Z})$ and having direction cosines A, B, and C. Let D be the distance from the point (X,Y,Z) to the point $(\underline{X},\underline{Y},\underline{Z})$.

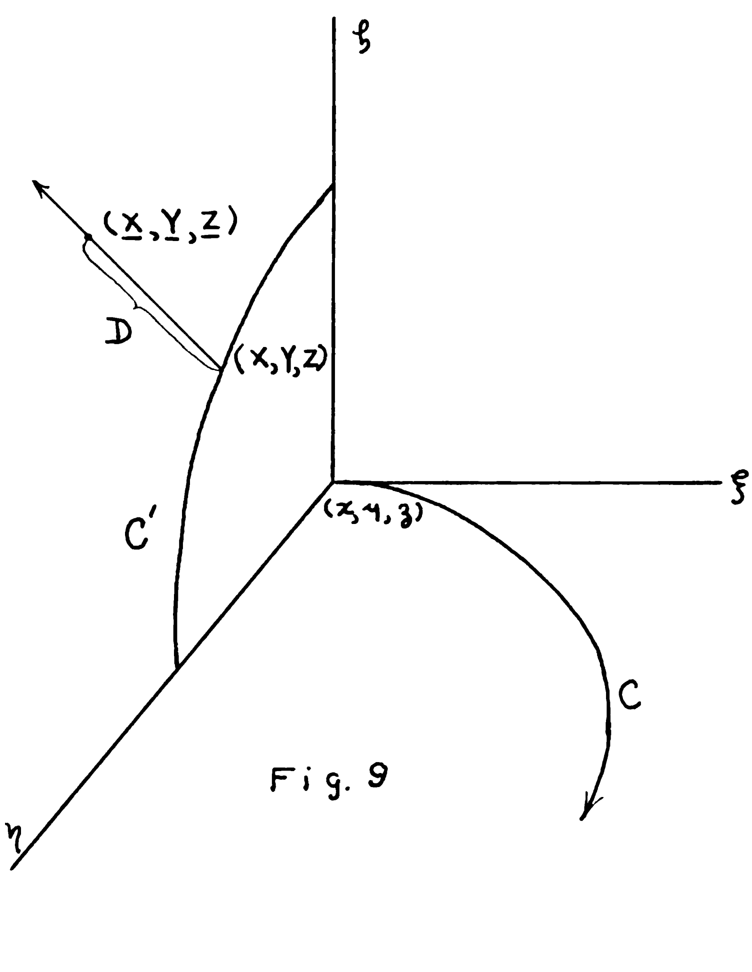

Fig. 9

Hence we have

(45) $\underline{X} = X + DA$,

and similar expressions for $\underline{Y}$, and $\underline{Z}$.

The direction cosines of the normal to a curve are given by the relationships*

(46) $$A = \frac{J_1}{H}, \quad B = \frac{J_2}{H}, \quad C = \frac{J_3}{H},$$

wherein

$$J_1 = y_u z_v - y_v z_u ,$$

and J_2 and J_3 are similar expressions using the notation at the top of page 5.

Combining the relationships expressed in equations (3) with those in equations (45) and (46), with some algebraic simplification we have as a result

(47) $$A = (\beta n - \gamma m)\sin u + (\gamma\mu - \beta\nu)\cos u ,$$

and similar expressions for B and C by permitting

$$\beta \to \gamma,\ \gamma \to \alpha,\ \mu \to \nu,\ \nu \to \lambda,\ m \to n,\ n \to \ell;\ A \to B,$$

and

$$\beta \to \alpha,\ \gamma \to \beta,\ \mu \to \lambda,\ \nu \to \mu,\ m \to \ell,\ n \to m;\ A \to C.$$

But in the determinant

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \ell & m & n \\ \lambda & \mu & \nu \end{vmatrix} = 1,$$

each element is equal to its own cofactor.** Hence,

(48) $$A = \lambda \sin u + \ell \cos u .$$

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* Lane, Geometry, p. 79 Eq. (4.8), and p. 64 Eq. (1.4).

** Eisenhart, Geometry, p. 13 Eq. (40).

Therefore we have combining equations (48) with equations (45) and (3),

$$\underline{X} = x + (D + A)(\lambda \sin u + l \cos u). \tag{49}$$

But for D any constant value, the surfaces for which the coördinates of a movable point are ($\underline{X}, \underline{Y}, \underline{Z}$) and (X,Y,Z) respectively are related by

$$\underline{x} = x - At$$

where t is a constant. We may recall that the parametric equations of a surface $\underline{S}$ parallel to a surface S can be written in the form $\underline{X} = x - At$, $\underline{y} = y - Bt$, $\underline{z} = z - Ct$ in which t is the constant algebraic distance from a point P(x,y,z) of S to the corresponding point $\underline{P}(\underline{x},\underline{y},\underline{z})$ of $\underline{S}$ and A, B, C are the direction cosines of the normal of S at P.*

From equations (49), the surface for which a point $P(\underline{X},\underline{Y},\underline{Z})$ is represented is seen also to be a torus since it is of the form of equations (6). We may now state the following

Theorem 16. A surface parallel to a torus is also a torus.

An interesting geometrical interpretation for the principal radii of normal curvature may be developed.

Let us construct a line in the normal plane perpendicular to the osculating plane at a distance b from

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* Lane, Geometry, p. 208 Th. 1.

(x,y,z), that is the polar line. Also for any angle u, construct the normal to the surface of the torus and extend it to meet the polar line. The normal line also passes through the center of the local coördinate system, that is through the point (x,y,z). To show this we make use of equations (49); comparing the definition of the torus, equations (6) we see that

$$X = x + aA \quad ,$$

or

$$\frac{X - x}{A} = \frac{Y - y}{B} = \frac{Z - z}{C} = +a \quad ,$$

which is the equation of a line with direction cosines A, B, and C through the point (x,y,z). Call the line segment on the polar line intercepted by the principal normal and the normal line T; and the line segment on the normal line intercepted by the torus and the polar line Q.

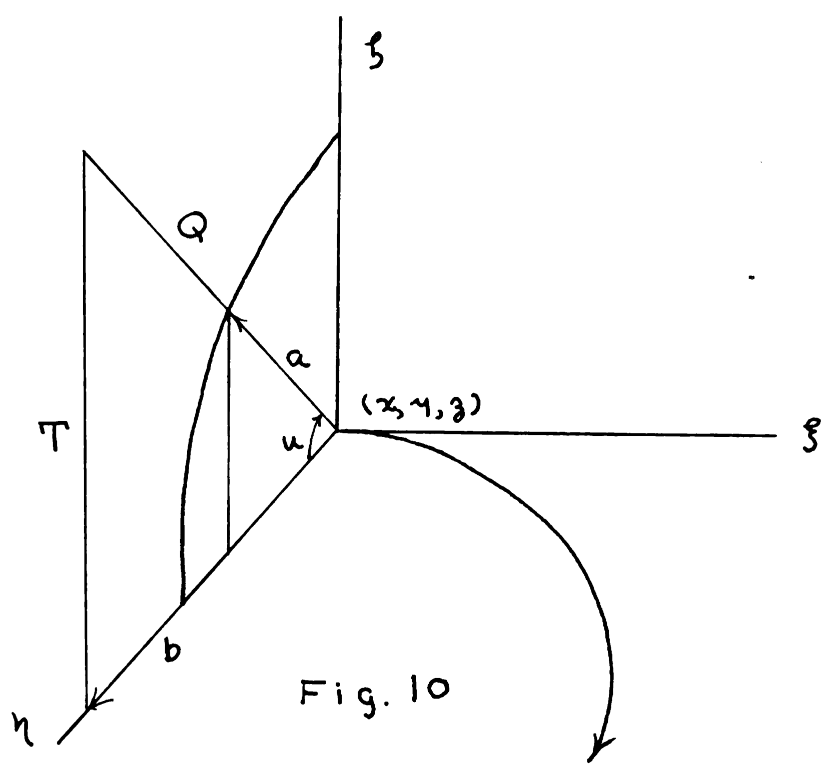

Fig. 10

From the figure, T is given by

$$T = b \tan u \ .$$

Squaring, and making use of trigonometric relationships we have

$$T^2 = b^2 \tan^2 u = b^2(\sec^2 u - 1)$$

$$= \frac{b^2}{\cos^2 u} - b^2$$

(50) $$= \left(\frac{b - a \cos u}{\cos u} + a\right)^2 - b^2 \ .$$

Also by the Pythagorean Theorem we see the following relationship:

(51) $$T^2 = (Q + a)^2 - b^2 \ .$$

Equating equations (50) and (51)

$$-b^2 + \left(\frac{b - a\cos u}{\cos u} + a\right)^2 = (Q + a)^2 - b^2 \ ,$$

we see that

(52) $$Q = \frac{b - a\cos u}{\cos u} \ .$$

Now comparing equation (52) and our figure with equations (16) we have

$$Q = R_2 \quad \text{and} \quad a = -R_1 .$$

The last relationship with our construction gives us a means of constructing the principal radii of normal curvature for all u except $|u| = \frac{\pi}{2}$, at which points R_2 is undefined.

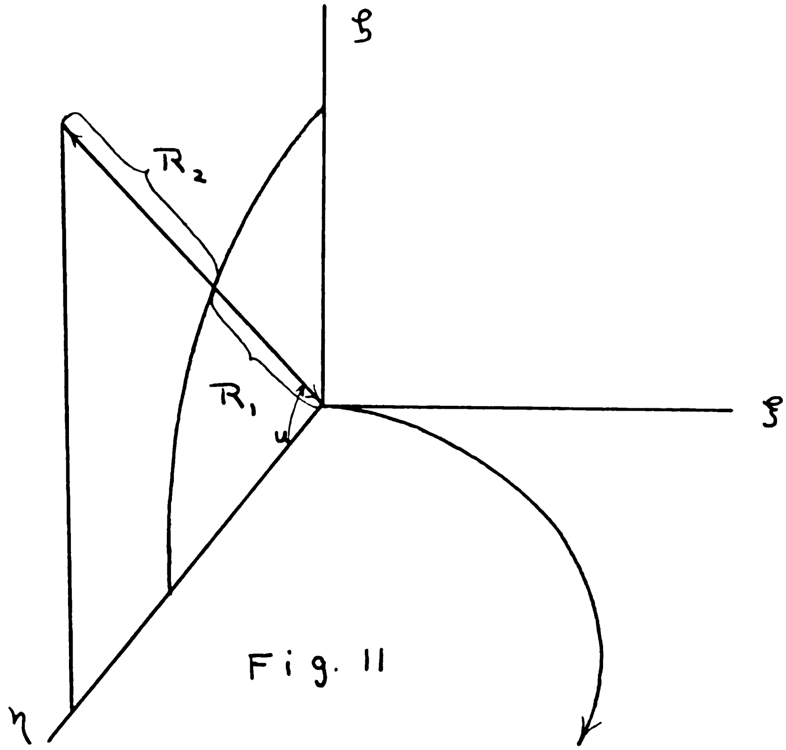

Fig. 11

Loxodromes on the Torus

Corresponding to the loxodromes on the sphere, we may ask the question what are the equations of the curves which cross the meridians of the torus at a constant angle?

If we let θ be the constant angle, then*

$$(53)\qquad \begin{cases} \cos\theta = \dfrac{1}{ds\,ds_1}\left[E\,du\,du_1 + F(du\,dv_1 + du_1\,dv) + G\,dv\,dv_1\right], \\ \sin\theta = -\dfrac{H}{ds\,ds_1}\left(du\,dv_1 - du_1\,dv\right) \quad , \end{cases}$$

wherein ds and ds are elements of arc length expressed as preceding equation (30).

Lane, Geometry, p. 116 Eq. (3.4).

If the curve C_1 is the u-curve at the point P, then v_1 = constant or $dv_1 = 0$; that is v_1 corresponds to the meridians. Using this restriction, the angle θ may also be defined by the equation*

(54) $$\tan\theta = \frac{H\,dv}{E\,du + F\,dv}\,.$$

Using the relations of equations (10), equation (54) becomes for the torus

(55) $$\cot\theta\,dv = \frac{ab}{b - a\cos u}\,du\,,$$

which has for a solution

(56) $$v\cot\theta = \frac{ab}{\sqrt{b^2-a^2}}\tan^{-1}\frac{\sqrt{b^2-a^2}\,\sin u}{b\cos u - a} + C\,.$$

These curves are of the form $av - \phi$ wherein a is a constant and ϕ an angle which varies, and thus spiral about the torus.

The equation of the loxodromes (56) may be written in the form

(57) $$v\cot\theta = \frac{ab}{\sqrt{b^2-a^2}}\cos^{-1}\frac{b\cos u - a}{b - a\cos u} + C.$$

It is seen that there is no restriction on the angle

$$\cos^{-1}\frac{b\cos u - a}{b - a\cos u}$$

other than $a > -b$ which is certainly true for all a and b by relations (4).

Eisenhart, Geometry, p. 76 Eq. (24).

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