SOME PROPERTIES OF THE CUT-SET AND THE CUT-SET MATRIX

Thesis for the Degree of M. S.

MICHIGAN STATE UNIVERSITY

Vichit Sirikul

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THESIS

This is to certify that the

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By

Vichit Sirikul

AN ABSTRACT

Submitted to the School of Graduate Studies of Michigan State University of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Department of Electrical Engineering

1956

Approved Myril B. Reed

Vichit Sirikul

ABSTRACT

This thesis is a study of a subgraph of a connected graph known as a cut-set. The cut-set is defined here on the basis of a segregation of the vertices of a graph into two, mutually exclusive, non-empty sets. The elements of the graph, each with a vertex in each vertex set, is called the cut-set. The vertex-segregation definition turns out to be more useful than the formerly employed one of defining a cut-set as the set of elements which, on removal, separate a graph into two parts.

A cut-set matrix is formulated--one row for each element in a cut-set. A fundamental cut-set matrix, $\mathcal{C}_T$, is defined based on the elements of a tree. The properties of this matrix, $\mathcal{C}_T$, are deduced in terms of rank, criteria for forming, relation to incidence and to circuit matrices, and coverage of a graph.

Some interrelations of v-functions and i-functions are formulated in terms of the fundamental cut-set matrix, $\mathcal{C}_T$

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Dedicated to

Field Marshal P. Pibulsonggram

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GENERAL AIMS

The purpose of this thesis is to present: (1) Some characteristics of the cut-set sub-graph of linear graphs; (2) the properties of the cut-set matrix; and (3) the relations between the cut-set matrix and the vertex and circuit matrices of the graph. This work has been carried on from the paper On Topology and Network Theory by Myril B. Reed and Sundaram Seshu, Proceedings of the University of Illinois Symposium on Circuit Analysis, May 1955. This thesis is based on the assumptions from that work, and in some parts of this thesis, it is necessary to abstract from the Reed-Seshu paper some properties of graphs, vertex and circuit matrices in order to clarify the relation between them and the cut-set matrix.

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INTRODUCTION

A study of electrical networks in general is based on the assumptions that the network consists of a finite set of network elements connected in the form of a graph. The network elements can be classified according to their electrical characteristics as resistive, inductive, capacitive and mutually inductive elements. Vacuum tube, transistor and other electric devices are also considered network elements. Measuring devices are used to observe the electrical phenomena in two ways of measurement. A series or current measurement is used to observe the current functions of the elements by placing the meter in series with the element. The parallel or voltage measurement is used to observe the voltage functions of the elements by placing the meter in parallel with the element.

A finite set of interconnected network elements, together with their orientations, forms the graph of the network. The measurements not only vary with the types of elements, but also with the geometric pattern or the graph formed by these elements. The mathematical relations of the functional representations of series and parallel measurements for a network can be put into equations as:

1. Vertex equation[1] $a_a \mathscr{I}_e(t) = 0$

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[1] Reed and Seshu, On Topology and Network Theory, Proceedings of the University of Illinois Symposium on Circuit Analysis, May 1955, pp. 3, 21.

where $\mathcal{A}_a$ is the vertex matrix of the corresponding graph,

$\mathcal{J}_e(t)$ is the matrix of current functions of the elements of the graph.

2. Circuit equation[1] $\mathcal{B}_a \mathcal{V}_e(t) = 0$

where $\mathcal{B}_a$ is the circuit matrix of the graph,

$\mathcal{V}_e(t)$ is the matrix of the voltage functions of the elements of the graph.

The vertex and circuit equations are known as Kirchhoff's current and voltage equations. The method of obtaining solutions for the network problem is to form the graph equations and solve by mathematical procedures for the unknown variables from the given information. The study of certain topological characteristics of the graph such as the vertex matrix, the circuit matrix and the cut-set matrix becomes an important part of this method.

The topological characteristics, derivable from the properties of the vertex and circuit matrices of the graph are presented by Reed and Seshu.[2] The cut-set subgraph, first presented by Foster,[3] is examined in this thesis with the intention of establishing how effective as a basis of network study the cut-set matrix may be.

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1 Reed and Seshu, op. cit., p. 22.

2 Ibid.

3 Foster, R. M., Geometric Circuits of Electric Networks, BSTJ No. B-653.

I. THE CUT-SET

1. Definition of the cut-set:

First place the vertices of a graph into two arbitrary, mutually exclusive, non-empty sets A and B. A cut-set of the graph is the set of elements with one vertex in vertex-set A and the other vertex in vertex-set B.

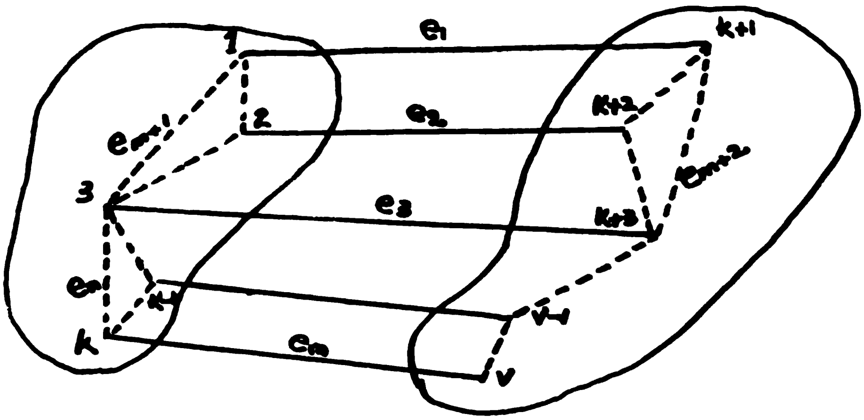

Fig. 1.

Suppose a graph G has v vertices 1, 2, 3,......v, and these vertices are placed in two arbitrary mutually exclusive sets: as vertices 1, 2,......k, in set A and the remaining vertices k+1, k+2,.....v-1, v, in set B. The set of elements e_1, e_2, e_3,.....e_m, (see Fig. 1) which have one vertex in set A and the other vertex in set B is a cut-set of the graph, and the elements e_1, e_2, e_3,.....e_m, are the cut-set elements. The elements with both vertices in only one set such as e_{m+1}, e_{m+2},.....e_n, are not cut-set elements of this cut-set, but they may be in other cut-sets depending on how the vertices are placed in the two arbitrary sets, A and B. The number of ways of forming two mutually exclusive, non-empty sets of a graph increases very greatly with the number of vertices, v. Therefore, the number of cut-sets increases likewise.

2. <u>The properties of a cut-set</u>:

From consideration of the nature of a graph, the following properties of cut-sets appear evident:

a. A cut-set of a graph separates the vertices of the graph into two mutually exclusive sets. This property follows from the definition.

b. A cut-set element is a path joining the two separate sets A and B of vertices of the graph.

c. A cut-set contains at least one element.

d. A connected graph contains at least one cut-set.

e. No proper sub-set of a cut-set is a cut-set.

f. The elements incident to any vertex is a cut-set.

g. Every non-circuit element is a cut-set.

h. A circuit element is not a cut-set.

i. A set of elements which separates the graph into two parts by removal of that set of elements is a cut-set.

3. Cut-set orientation.

Just as it is necessary to assign an orienting mark to elements, paths, and circuits of a graph, so it is necessary to assign an orienting pattern to a cut-set. Either of the arrows of Fig. 2 serve this purpose.

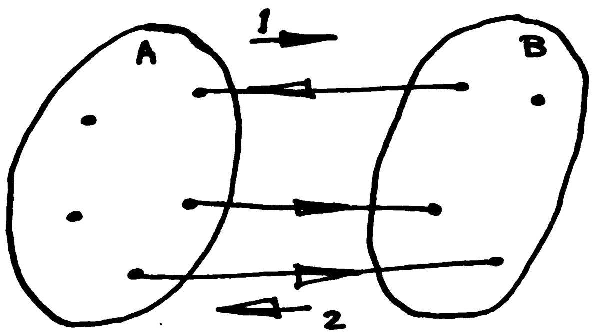

Fig. 2. Cut-set orientation

Unless otherwise specified, because of its arbitrariness, the cut-set orienting arrow alignment is specified, hereafter in this discussion, as pointing from the A to the B set of vertices in the sense of Fig. 3.

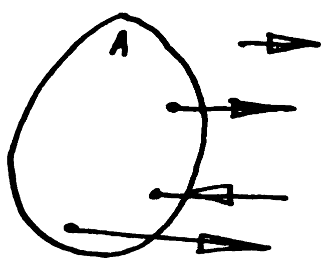

Fig. 3. General cut-set orientation

4. The cut-set of the tree.

The following brief listing of the properties of trees serves as a basis from which to study cut-sets and the matrices associated with them.

A tree is by definition a connected subgraph of a connected graph containing all the vertices of the graph and containing no circuit.[1]

Some basic properties of trees are:

4-a. Every connected graph has at least one tree.[2]

4-b. An element of a tree is named a branch.

4-c. An element of the complement of a tree is named a chord.

4-d. Every tree has one more vertex than branches.[3]

4-e. A connected graph of e elements and v vertices contains v-1 branches and e-v+1 chords.[4]

4-f. There exists one and only one path in a tree between any two vertices of a tree.[4]

4-g. Each branch specifies a cut-set.[5]

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1 Corresponds to the "complete tree" by Cauer, Theore der linearen Wechselstromschaltungen, Akademische Verlagsgesellschaft, Leipzig, 1941.

2 Koenig, Theore der endlichen und unendlichen graphen, Chelsea Publishing Company, New York, 1950, p. 52.

3 Ibid., p. 51.

4 Reed and Seshu, op. cit., p. 6.

5 Unpublished class notes by Myril B. Reed.

To illustrate the definition, consider Fig. 4. The tree is shown by the solid elements and the chords by the dashed elements. Corresponding to branch, b_i, the chords ch_1, ch_2, ch_3, ch_4, ch_5, with this branch form one of the tree cut-sets.

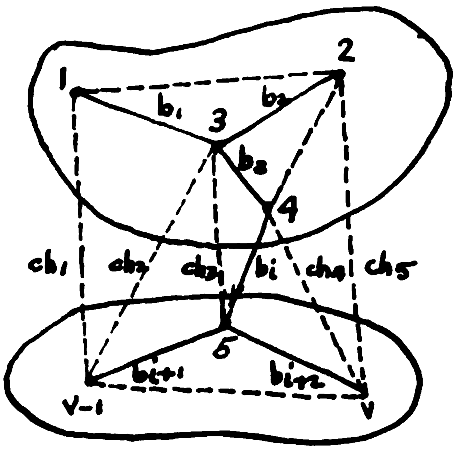

Fig. 4. Shows a cut-set c_i of the tree.

Because of the fundamental character for this thesis of the statement that each branch specifies a cut-set, the following brief review without proof of the ideas involved is given. In the first place: The vertices of a connected graph may be placed in two mutually exclusive, non-empty sets A and B corresponding to each branch by:

(a) Placing in vertex set A all vertices which have a path-in-tree to one of the vertices of the branch, none of these paths-in-tree are to contain the branch;

(b) Placing in vertex set B all other vertices of the tree.

As a consequence of this last given property of a branch: each branch defines a cut-set. It is specified here that the orienting mark of a branch specifies the orientation of the cut-set, i.e., the orienting arrow for the cut-set (Fig. 3) is aligned with the orienting arrow of the defining branch.

Some properties of the cut-sets of a tree

Some properties of the cut-sets of a tree which are useful for the following developments are:

1. A cut-set of a tree contains one and only one tree branch, all other elements of the cut-set as a consequence are chords.[1]
2. A tree contains v-1 branches and so specifies v-1 cut-sets.
3. A branch of a tree is a cut-set of a graph which is the tree itself.

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[1] Unpublished class notes of Myril B. Reed, Michigan State University.

II. THE CUT-SET MATRIX

1. The cut-set matrix of the graph.

A matrix can be formed, corresponding to each cut-set or collection of cut-sets, as follows:

Definition: The cut-set matrix $\mathcal{C}_a = [c_{ij}]$ is the rectangular array containing c rows one corresponding to each cut-set of the graph and e columns corresponding to e elements of the graph. The entries of the matrix are defined by:

$c_{ij} = 1$ If the element e_j is in the cut-set c_i and the orientation of the element coincides with that of the cut-set.

$c_{ij} = -1$ If the element e_j is in the cut-set c_i and the orientation of the element and of the cut-set are opposite.

$c_{ij} = 0$ If the element e_j is not in the cut-set c_i.

The shape of the cut-set matrix of the graph will be n x e, where n is the number of cut-sets:

$$\mathcal{C}_a = \left[\begin{array}{ll} \text{Columns} & = \overbrace{\text{element of graph}} \\ \text{rows} & = \text{all possible cut-sets of the graph} \end{array} \right]$$

2. The tree cut-set matrix.

A tree cut-set matrix, $\mathcal{C}_T$, may be formed by considering only the v-1 rows of $\mathcal{C}_a$ which correspond to the tree.

Theorem 1: The tree cut-set matrix, $\mathcal{C}_T$, of the graph containing v vertices has the rank of v-1.

Proof: Since there are v-1 rows in any matrix, $\mathcal{C}_T$, corresponding to a v-vertex e-element connected graph, $\mathcal{C}_T$ is (v-1) x e. Hence, the rank of $\mathcal{C}_T$ matrix can not exceed v-1.

Order the rows and columns of the tree cut-set matrix $\mathcal{C}_T$ such that the first v-1 columns (right to left) and the rows (top to bottom) correspond to the same ordering of the defining branches. The cut-set orientation is the same as the branch orientation which defines the cut-set. The first square sub-matrix of $\mathcal{C}_T$ of v-1 rows and columns has the diagonal entries, c_{11}, c_{22},.....c_{ii}, equal to +1 and all other entries zero. The leading (v-1)x(v-1) sub-matrix of $\mathcal{C}_T$ matrix can thus always be made the square unit matrix of rank v-1, i.e.,

$$
\mathcal{C}_T = \begin{array}{c} \\ \\ c_1 \\ c_2 \\ c_3 \\ \cdot\cdot \\ c_{v-1} \end{array}
\begin{array}{c}
\overbrace{\begin{array}{ccccc} b_1 & b_2 & b_3 & \cdots\cdots & b_{v-1} \end{array}}^{\text{branches}} \qquad \overbrace{\begin{array}{cccc} e_1 & e_2 & \cdots & e \end{array}}^{\text{chords}} \\
\left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & \cdots\cdots & 0 & x & x & \cdots & x \\
0 & 1 & 0 & \cdots\cdots & 0 & x & x & \cdots & x \\
0 & 0 & 1 & \cdots\cdots & 0 & x & x & \cdots & x \\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots\cdots & 1 & x & x & \cdots & x
\end{array}\right]
\end{array}
$$

Therefore, the rank of tree cut-set matrix will not be less than v-1.

Then, the rank of the tree cut-set matrix $\mathcal{C}_T$ is v-1.

Corollary 1:1: Removal of any one of the tree cut-sets reduces the rank of the matrix, $\mathcal{C}_T$, by one.

Proof: This can be easily seen by removing any one tree cut-set, the number of rows of $\mathcal{C}_T$ will be reduced from v-1 to v-2, and the rank of the matrix will not exceed and will not be less than v-2. Hence, the rank of the $\mathcal{C}_T$ matrix is reduced by one.

3. Relation between $\mathcal{C}_a$ and $\mathcal{B}_a$ matrices.

The circuit matrix[1] associated with each graph contains one row corresponding to each possible circuit and one column corresponding to each element of the graph. The k_i entry of the circuit matrix is 1 if element α_i is in the circuit c_k with the same orientation, -1 if the element α_i is in the circuit c_k with the opposite orientation, and 0 if the element α_i is not in the circuit c_k.

Lemma 1.2: An even number and only an even number of elements of any cut-set are contained in any circuit. See Fig. 5. These elements occur in pairs.

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[1] Reed and Seshu, op. cit., p. 11.

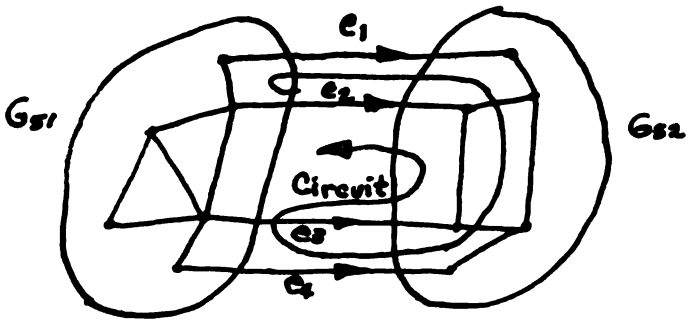


Fig. 5. Circuit formed by cut-set elements

Proof: Consider the forming of the sequential labeling which defines a circuit.[1] Any vertex of the circuit may be taken as the starting point of the labeling. Consider a vertex in the A set. Incorporating any one cut-set element into the circuit introduces a vertex of the B set into the circuit. A second cut-set element must then be contained in the circuit in order that it be possible to complete the defining labeling, i.e., include another, or the starting vertex, of the A set in the circuit. The foregoing pattern is general. The cut-set elements contained in circuits occur in pairs. Hence, the lemma.

A fundamental relation exists between the cut-set matrix, $\mathcal{C}_a$, and the circuit matrix, $\mathcal{B}_a$, in the form indicated by the following theorem.

Lemma 2.2: If the columns of $\mathcal{C}_a$ and $\mathcal{B}_a$ are arranged corresponding to the same element ordering and if X is any column of $\mathcal{B}_a'$, then

$$\mathcal{C}_a X = 0$$

---

[1] Reed and Seshu, op. cit., p. 5.

<u>Proof</u>: For a connected graph G, write the circuit matrix $\mathcal{B}_a$ and the cut-set matrix $\mathcal{C}_a$ with columns corresponding to the same order of elements of the graph. Consider an arbitrary column of $\mathcal{B}_a'$, X, corresponding to the i-circuit. The j row of $\mathcal{B}_a'$ corresponds to the element α_j also the r^{th} row of $\mathcal{C}_a$ corresponds to r^{th} cut-set of the graph G.

(1) If the element α_j is not in the cut-set c_r the entry $c_{rj} = 0$, then

(a) If the element α_j is also not in the i-circuit, the entry $x_i = 0$, therefore,

$$c_{rj} \cdot x_i = 0 \times 0 = 0$$

(b) If the element α_j is in the i-circuit, the entry x_i will be non-zero and equal to ± 1, so $c_{rj} \cdot x_i = 0 \times (\pm 1) = 0$

(2) If the element α_j is in the cut-set c_r, the entry c_{rj} of $\mathcal{C}_a$ will be non-zero and equal to ± 1, then

(a) If the element α_j is not in the i-circuit, the entry x_i is zero, and

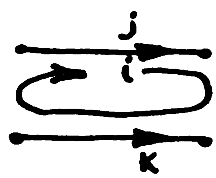
$$c_{rj} \cdot x_i = (\pm 1) \times 0 = 0$$

(b) If the element α_j is in the i-circuit, the entry x_i will be non-zero and equal to ± 1.

Of the preceding four conditions, 1-a, 1-b, 2-a, and 2-b, only 2-b introduces other than zero into the product.

According to Lemma 1.2, the elements of a graph common to a circuit and a cut-set occur in pairs. It is sufficient therefore to consider all possible cases for one pair of elements. The following tabulation exhibits the four possible element, cut-set and circuit combinations:

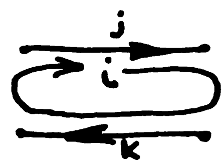
I



	j	k
r cut-set:	± 1	± 1
j circuit:	$+1$	-1

$$(\pm 1)(+1)+(\pm 1)(-1)=0$$

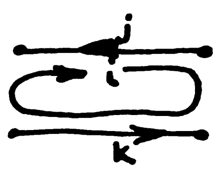
II



	j	k
r cut-set:	± 1	∓ 1
j circuit:	$+1$	$+1$

$$(\pm 1)(+1)+(\mp 1)(+1)=0$$

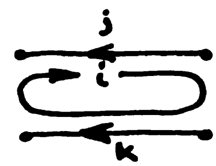
III



	j	k
r cut-set:	∓ 1	± 1
j circuit:	-1	-1

$$(\mp 1)(-1)+(\pm 1)(-1)=0$$

IV



	j	k
r cut-set:	∓ 1	∓ 1
j circuit:	-1	$+1$

$$(\mp 1)(-1)+(\mp 1)(+1)=0$$

The theorem, therefore, follows.

<u>Theorem 2</u>: If the columns of $\mathcal{C}_a$ and $\mathcal{B}_a$ are arranged corresponding to the same element order and $\mathcal{C}_a'$ and $\mathcal{B}_a'$ are the transposes of $\mathcal{C}_a$ and $\mathcal{B}_a$, then

$$\mathcal{C}_a \mathcal{B}_a' = 0$$

$$\mathcal{B}_a \mathcal{C}_a' = 0$$

<u>Proof</u> Since $\mathcal{C}_a X = 0$ from Lemma 1.2 where X is any column of $\mathcal{B}_a'$, at once

$$\mathcal{C}_a \mathcal{B}_a' = 0$$

then $\left[\mathcal{C}_a \mathcal{B}_a'\right]' = \mathcal{B}_a \mathcal{C}_a' = 0$

<u>Corollary 1.2</u>: There exists a linear relationship among the columns of the matrix $\mathcal{B}_a$ which corresponds to the elements of the cut-set.

<u>Proof</u>: Since $\mathcal{B}_a \mathcal{Y} = 0$, for $\mathcal{Y}$ any column of $\mathcal{C}_a$, the columns of $\mathcal{B}_a$ are linearly related.

4. <u>Rank of the cut-set matrix $\mathcal{C}_a$ of connected graph</u>.

The rank of the cut-set matrix $\mathcal{C}_a$ of the graph can be found as follows:

<u>Lemma 1.3</u>: The rank of the cut-set matrix $\mathcal{C}_a$ of a connected graph is at least v-1.

<u>Proof</u>: Every connected graph has a tree and the cut-set of this tree is a sub-set of all cut-sets of the graph. Therefore, the tree cut-set matrix $\mathcal{C}_T$ is a sub-matrix of the cut-set matrix $\mathcal{C}_a$. Since $\mathcal{C}_T$ has the rank of v-1 by Theorem 1, the rank of the cut-set matrix $\mathcal{C}_a$ is not less than v-1.

<u>Lemma 2.3</u>: The rank of the cut-set matrix $\mathcal{C}_a$ of a connected graph can not exceed v-1.

Proof: Let r_b represent the rank of $\mathcal{B}_a$, r_c represent the rank of $\mathcal{C}_a'$, and e the number of columns of $\mathcal{B}_a$ and rows of $\mathcal{C}_a'$. If, as shown by Theorem 2, $\mathcal{B}_a' \mathcal{C}_a' = 0$ then[1]

$$r_b + r_c \leq e$$

or $$r_c \leq e - r_b$$

But,[2] $$r_b = e - v + 1$$

hence $$r_c \leq v - 1$$

and so the Lemma.

Theorem 3: The rank of the cut-set matrix $\mathcal{C}_a$ of a connected graph is v-1.

Proof: This theorem follows from Lemma 1.3 and Lemma 2.3 above. For the rank of $\mathcal{C}_a$ cannot exceed nor be less than v-1, hence it is v-1.

Theorem 4: Every set of v-1 cut-sets such that the matrix $\mathcal{C}$ of these cut-sets has a rank of v-1 includes every element of the graph.

Proof: Let the v-1 rows of $\mathcal{C}$ be brought to the first (v-1) row position of $\mathcal{C}_a$ and rearrange the columns so as to make the leading square sub-matrix of order v-1 non-singular. This is certainly possible because $\mathcal{C}$ has a rank of v-1.

It is sufficient to show that a column of $\mathcal{C}_a$ in which the entries of the first v-1 rows are zeros will not exist

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[1] Hohn, Franz E., Linear Transformations and Matrices, University of Illinois, class notes.

[2] Reed and Seshu, op. cit., p. 14.

unless all entries of this column are zeros. This proposition may be proved by contradiction. Assume that there exists a k^{th} column of $\mathcal{C}_a$ such that the entries of this column in the first v-1 rows are zeros but all other entries of this column are not zero. The k^{th} column will not be one of the first v-1 columns since the leading v-1 square matrix is non-singular. The k^{th} column may be interchanged with the v^{th} column. Now, there is a non-zero entry of this column in the r^{th} row where r is greater than v-1. The r^{th} may be interchanged with v^{th} row. Then, the leading square sub-matrix of $\mathcal{C}_a$ of order v becomes non-singular. Hence, $\mathcal{C}_a$ has a rank of at least v which contradicts theorem 3 above that $\mathcal{C}_a$ has a rank v-1.

Therefore, a non-zero column of $\mathcal{C}_a$ with zero entries in the first v-1 rows will not exist unless all entries of this column are zeros. Hence, every cut-set element appears in the first v-1 rows of $\mathcal{C}_a$ which is the proposition of the theorem.

5. Criteria for choosing v-1 cut-sets to establish a cut-set matrix of rank v-1.

The number of cut-sets for any graph may be very large and in all but the simplest graphs the cut-set pattern is complex. The set of cut-sets, from which $\mathcal{C}$, a matrix of v-1 rows and of rank v-1, may be established can be specified by any one of the following criteria:

Criterion 1: Form the tree cut-sets according to section I, paragraph 4. The tree cut-set matrix $\mathcal{C}_T$ of the graph has a rank of v-1 by theorem 1 and the cut-set matrix $\mathcal{C}_T$ of rank v-1 covers all elements of the graph by theorem 4.

Example 1: A connected graph G contains 5 vertices v_1, v_2, v_3, v_4, v_5 and 9 elements e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, and e_9. Choose as a tree the $v-1=4$ elements, e_1, e_2, e_3, and e_4 shown in Fig. 6.

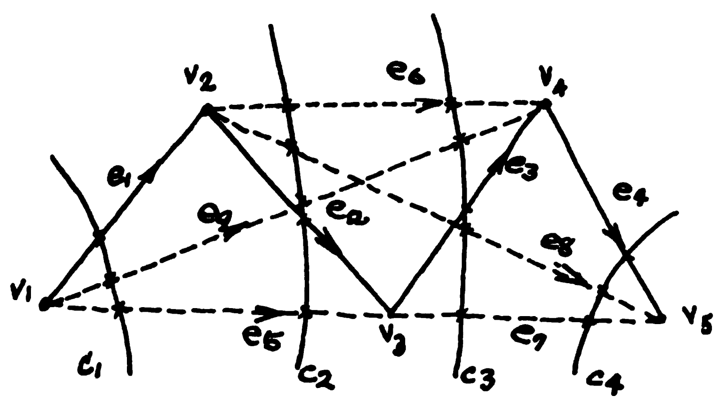

Fig. 6. Four tree cut-sets of graph G.

The cut-set: c_1 contains branch e_1 and chords e_5, e_9

c_2 contains branch e_2 and chords e_5, e_6, e_8, e_9

c_3 contains branch e_3 and chords e_6, e_7, e_8, e_9

c_4 contains branch e_4 and chords e_7, e_8

By choosing the cut-set orientation to coincide with that of the defining branch and placing the columns corresponding to the branches in the leading positions, the cut-set matrix $\mathcal{C}_T$ of the graph of Fig. 6 becomes

$$\mathcal{C}_T = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \overbrace{\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}}^{\text{branches}} \;\; \overbrace{\begin{array}{ccccc} e_5 & e_6 & e_7 & e_8 & e_9 \end{array}}^{\text{chords}} \\ \left|\begin{array}{cccc:ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right| \end{array}$$

The cut-set matrix can be partitioned corresponding to the branches ($\mathcal{C}_{T11}$) and the chords ($\mathcal{C}_{T12}$), i.e.,

$$\mathcal{C}_T = [\mathcal{C}_{T11} \; \mathcal{C}_{T12}] = [\mathcal{U} \; \mathcal{C}_{T12}]$$

Since $\mathcal{U}$ is unit matrix of rank $v-1=4$, the rank of $\mathcal{C}_T$ is $v-1=4$ and covers all elements of the graph by theorem 4.

By ordering the entries in $\mathcal{C}_T$, as in this example, a $(v-1) \times (v-1)$ unit matrix can always be located in the leading position. Hence, the rank of $\mathcal{C}_T$ is $v-1$.

<u>Criterion 2</u>: Choose a tree of the graph and order the cut-sets such that the first cut-set is a tree cut-set.

Each succeeding cut-set must include one new branch and some or all of the chords and may include some or all of branches and chords in the preceding cut-sets. The cut-set orientation may be made to coincide with that of the new branch.

The cut-set matrix $\mathcal{C}$ formed by this criterion has a rank of v-l and fits the condition specified. This can be shown by rearranging the columns of the cut-set matrix $\mathcal{C}$ so that the i^{th} column corresponds to the new branch included in i^{th} cut-set, the leading square matrix of order v-l is triangular with +l's on the main diagonal. Hence, the $\mathcal{C}$ matrix has a rank of v-l and will include all elements of the graph.

Example 2. Consider the connected graph G of Fig. 6 with 5 vertices and 9 elements. Choose a tree with v-l = 4 branches as e_1, e_2, e_3 and e_4, the remaining elements e_5, e_6, e_7, e_8 and e_9 are the chords of the graph G. Form the cut-set c_1, c_2, c_3 and c_4 such that:

c_1 is a tree cut-set which contains only one branch e_1 and chords e_5, e_9; the c_1 orientation is made to coincide with that of e_1;

c_2 is a succeeding cut-set which contains a new branch e_2 and includes branch e_1 and chords e_6, e_8, the orientation of c_2 is made to coincide with that of e_2;

c_3 contains a new branch e_3, includes branch e_1 and chords e_5, e_6, e_7 and e_8, the c_3 orientation is made to coincide with that of e_3 (element e_9 is not in this cut-set);

c_4 contains a new branch e_4, includes branch e_3, and chords e_6, e_9, the c_4 orientation is made to coincide with that of e_4.

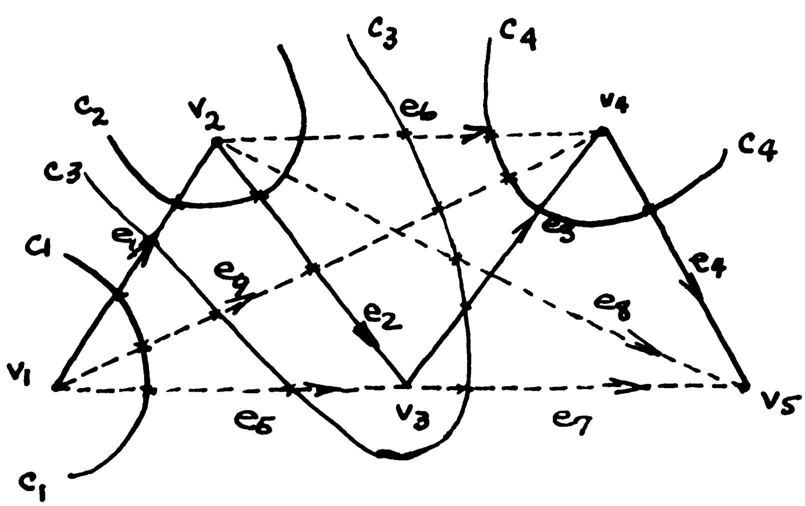

Fig. 7. Shows the cut-set pattern formed by criterion 2.

The cut-set pattern by this criteria is shown in Fig. 7 and the cut-set matrix $\mathcal{C}$ of the graph is:

$$\mathcal{C} = \begin{array}{c} \\ \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \overbrace{\begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array}}^{\text{branches}} \overbrace{\begin{array}{ccccc} e_5 & e_6 & e_7 & e_8 & e_9 \end{array}}^{\text{chords}} \\ \left|\begin{array}{cccc:ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \end{array}\right| \end{array}$$

The rank of this cut-set matrix $\mathcal{C}$ is v-1 and this cut-set matrix $\mathcal{C}$ covers all elements of the graph.

Criterion 3: Form the cut-sets according to the definition in section I, paragraph 1, by placing one and only one vertex in set A and the other v-1 vertices of the graph in set B. There will be v cut-sets for the graph of v vertices formed by this criteria. Any set of v-1 of these cut-sets will form the cut-set matrix, $\mathcal{C}_v$, of rank v-1.

Proof: Define the cut-set orientation as from set A to set B. The elements of any one of these cut-sets are the element incident to that vertex. The c_i^{th} row of the cut-set matrix $\mathcal{C}_v$ which corresponds to vertex v_i is the same as the i^{th} row which corresponds to the v_i vertex of the incidence matrix $\mathcal{A}_a$. Therefore, the cut-set matrix $\mathcal{C}_v$ formed by this criteria is the same as the vertex matrix $\mathcal{A}_a$ of the same graph. Since vertex matrix $\mathcal{A}_a$ has a rank of v-1[1] and includes all the elements of the graph, the cut-set matrix $\mathcal{C}_v$ of the graph will have a rank of v-1 and covers all the elements of the graph. The cut-set matrix formed by this criteria may be named as "Vertex cut-set matrix $\mathcal{C}_v$" of the graph.

Corollary: Any incidence matrix, $\mathcal{A}_a$, is a sub-matrix of the cut-set matrix $\mathcal{C}_a$.

Proof: From the manner of constructing $\mathcal{C}_v$, $\mathcal{C}_v = \mathcal{A}_a$, where $\mathcal{A}_a$ is the incidence matrix of the graph.

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[1] Reed and Seshu, op. cit., p. 8.

<u>Example 3</u>. Consider the same connected graph G as in Example 1 and 2 as reproduced in Fig. 8. The vertex cut-sets c_1, c_2, c_3, c_4 and c_5 are formed corresponding to vertices v_1, v_2, v_3, v_4 and v_5 of the graph and the cut-set orientation coincides with the elements oriented away from v_1, v_2, v_3, v_4 and v_5 respectively. The vertex cut-set pattern is shown in Fig. 8.

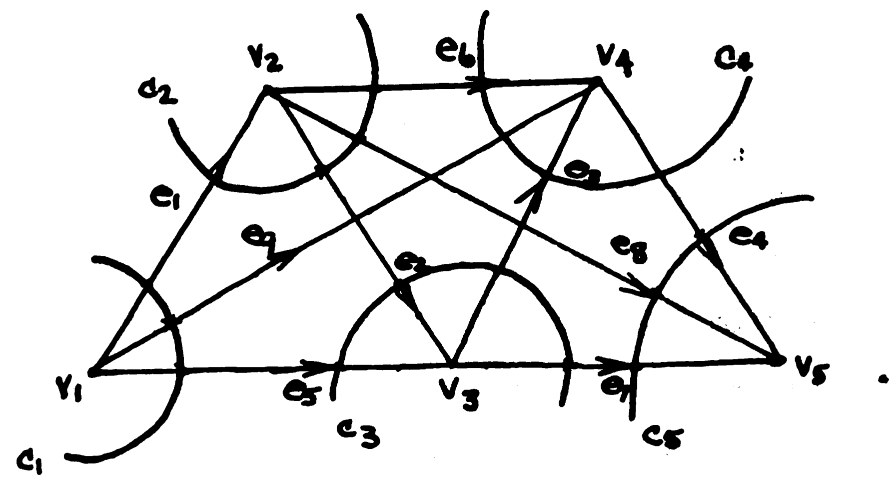

Fig. 8. Vertex cut-sets.

The vertex cut-set matrix $\mathscr{C}_v$ is:

$$\mathscr{C}_v = \begin{array}{c} \\ c_1 = v_1 \\ c_2 = v_2 \\ c_3 = v_3 \\ c_4 = v_4 \\ c_5 = v_5 \end{array} \overbrace{\begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array}}^{\text{all elements of graph}} $$

$$\begin{array}{c|ccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \\ \hline c_1 = v_1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ c_2 = v_2 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ c_3 = v_3 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ c_4 = v_4 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & -1 \\ c_5 = v_5 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \end{array}$$

Any vertex cut-set matrix of v-1 rows (4 rows) will be of the rank $v-1=4$ and includes all elements of the graph.

Criterion 4: Form the cut-set in such an order that the first cut-set is a vertex cut-set of the graph. Each succeeding cut-set is formed by taking a new vertex and may include some or all vertices used in forming the preceding cut-sets. The cut-set orientation may be made to coincide with that of the element directed away from the new vertex.

Proof: Arrange the leading columns of the cut-set matrix $\mathcal{C}_4$ formed by this criteria, in correspondence with a "new" element introduced by each succeeding cut-set. The leading square matrix of order v-1 is triangular with +1's on the main daigonal. Hence, the cut-set matrix $\mathcal{C}_4$ has a rank of v-1 and includes all elements of the graph.

Example 4. For the connected graph of Fig. 9, the cut-set matrix $\mathcal{C}_4$ may be formed according to criterion 4 by taking the first cut-set c_1 as a vertex cut-set corresponding to vertex v_1 and oriented away from v_1 with element e_1 corresponding to the 1^{st} column. Form c_2 at vertex v_2 with orientation away from v_2, make the column corresponding to element e_8 the 2^{nd} column. Form c_3 from v_3 and v_2. Orient c_3 away from v_3-v_2 and make column 3 correspond to element e_3. Form the cut-set c_4 from v_4 and v_3 oriented away from v_4-v_3. Locate the column corresponding to element e_4 as the 4^{th} column. The cut-set pattern is shown in Fig. 9.

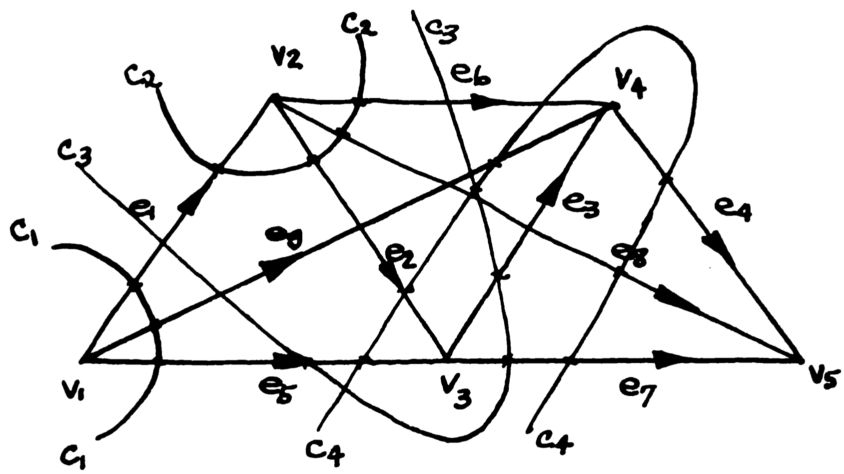

Fig. 9. Cut-sets formed by criterion 4.

The cut-set matrix $\mathcal{C}_4$ is:

$$\mathcal{C}_4 = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_8 & e_3 & e_4 & e_2 & e_5 & e_6 & e_7 & e_9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & -1 \end{bmatrix} \end{array}$$

This cut-set matrix $\mathcal{C}_4$ has a rank of v-1 and includes all elements of the graph.

<u>Theorem 6</u>: Let the matrix $\mathcal{C}$ be cut-set matrix of rank v-1. Then a square sub-matrix of order v-1 of $\mathcal{C}$ is non-singular if and only if the columns of this sub-matrix correspond to a set of branches.

<u>Proof</u>: (1) Consider the columns of a sub-matrix of $\mathcal{C}$ which correspond to the branches of some tree T. Then there exists a set of tree cut-sets for this tree.

Let these cut-sets be arranged so that:

$$\mathcal{C}_T = [\,\mathcal{U}\ \mathcal{C}_{T12}\,]$$

Let the columns of $\mathcal{C}$ be arranged in the same order as those of $\mathcal{C}_T$, then

$$\mathcal{C} = [\,\mathcal{C}_{11}\ \mathcal{C}_{12}\,]$$

where the columns of $\mathcal{C}_{11}$ correspond to the branches.

Now the rows of $\mathcal{C}_T$ are a basis for the set of all cut-sets of the tree (T-cut-set) and so are the rows of $\mathcal{C}$ (rank v-1).

Hence, there exists a non-singular transformation matrix $\mathcal{D}$ so that: $\mathcal{D}\mathcal{C}_T = \mathcal{C}$

from which $\mathcal{D}[\,\mathcal{U}\,\mathcal{C}_{T12}\,] = [\,\mathcal{C}_{11}\,\mathcal{C}_{12}\,]$

so that $\mathcal{D} = \mathcal{C}_{11}$ is non-singular. Hence, the sub-matrix of $\mathcal{C}$ corresponding to a tree is non-singular.

(2) If the sub-matrix $\mathcal{C}_{11}$ does not correspond to the v-1 branches of some tree, at least one column other than those of $\mathcal{C}_{11}$ corresponds to a branch. There is some row of $\mathcal{C}_a$ then with v-1 zero entries but not all zero entries. Place this row in the v^{th} position with the v-1 zeros in the first v-1 columns and any non-zero entry in the v^{th} column. Then $\mathcal{C}_a$ has a rank of v which contradicts the known property of rank v-1 of $\mathcal{C}_a$. Hence, if $\mathcal{C}_{11}$ is non-singular, it corresponds to a tree.

Corollary 1.5: If $\mathcal{C}_T$ is the matrix of the tree cut-sets for any tree of a connected graph G arranged so that

$$\mathcal{C}_T = [\mathcal{U}\ \mathcal{C}_{T12}]$$

and if $\mathcal{C}$ is the matrix of rank v-1 of any set of v-1 tree cut-sets of graph G with columns arranged in the same order as $\mathcal{C}_T$, then

$$\mathcal{C} = \mathcal{C}_{11}\mathcal{C}_T$$

and

$$\mathcal{C}_T = \mathcal{C}_{11}^{-1}\mathcal{C}$$

where $\mathcal{C} = [\mathcal{C}_{11}\ \mathcal{C}_{12}]$ and $\mathcal{C}_{11}$ is square.

III. CUT-SET MATRIX USE IN NET-WORK THEORY

1. Relation between $\mathcal{C}_T$ and $\mathcal{B}_c$ matrices.

Theorem 6: Let $\mathcal{C}_T$ represent the tree cut-set matrix and $\mathcal{B}_c$ the c-circuit matrix for any one tree of connected graph G. Let $\mathcal{C}_T$ and $\mathcal{B}_c$ be arranged so that they are partitioned corresponding to chords and branches of the tree as

$$\mathcal{C}_T = [\underset{\text{chords}}{\mathcal{C}_{T11}} \quad \underset{\text{tree}}{\mathcal{U}_T}]$$

and

$$\mathcal{B}_c = [\underset{\text{chords}}{\mathcal{U}_c} \quad \underset{\text{tree}}{\mathcal{B}_{c12}}]$$

where $\mathcal{U}_T$ and $\mathcal{U}_c$ are v-1 and e-v+1 order unit matrices respectively. Then, we have

$$\mathcal{C}_{T11} = -\mathcal{B}'_{c12}$$

$$\mathcal{B}_{c12} = -\mathcal{C}'_{T11}.$$

and

$$\mathcal{C}_T = [-\mathcal{B}'_{c12} \quad \mathcal{U}_T]$$

$$\mathcal{B}_c = [\mathcal{U}_c \quad -\mathcal{C}'_{T11}]$$

Proof: The relations

$$\mathcal{C}_T \mathcal{B}'_c = 0 \qquad \text{...............(1)}$$

$$\mathcal{B}_c \mathcal{C}'_T = 0 \qquad \text{...............(2)}$$

are valid by Theorem 2 since $\mathcal{C}_T$ and $\mathcal{B}_c$ are sub-matrices of $\mathcal{C}_a$ and $\mathcal{B}_a$.

Therefore

$$\mathcal{C}_T \mathcal{B}'_c = [\mathcal{C}_{T11} \quad \mathcal{U}_T] \begin{bmatrix} \mathcal{U}'_c \\ \mathcal{B}'_{c12} \end{bmatrix} = 0$$

from which

$$\mathcal{C}_{T11}\,\mathcal{U}_c' + \mathcal{U}_T\,\mathcal{B}_{c12}' = 0$$

Since $$\mathcal{U}' = \mathcal{U}$$

then $$\mathcal{C}_{T11} + \mathcal{B}_{c12}' = 0$$

and so $$\mathcal{C}_{T11} = -\mathcal{B}_{c12}'$$

By taking the transpose of both sides or by substituting $\mathcal{C}_T$ and $\mathcal{B}_c$ in equation (2) there results

$$\mathcal{B}_{c12} = -\mathcal{C}_{T11}'$$

The substitution of this relation into $\mathcal{C}_T$ and $\mathcal{B}_c$ produces

$$\mathcal{C}_T = \left[-\mathcal{B}_{c12}' \quad \mathcal{U}_T\right]$$

and

$$\mathcal{B}_c = \left[\mathcal{U}_c \quad -\mathcal{C}_{T11}\right]$$

2. <u>Relation between $\mathcal{C}_T$ and $\mathcal{A}_a$ matrices</u>.

<u>Theorem 7</u>: The vertex matrix, $\mathcal{A}_a$, of a connected graph G can always be arranged and partitioned so that

$$\mathcal{A}_a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $\mathcal{A}_{12}$ is of order (v-1) x (v-1) and is non-singular. There exists a set of tree cut-sets in G for which the matrix of tree cut-sets is given by

$$\mathcal{C}_T = \left[\mathcal{C}_{T11} \quad \mathcal{U}_T\right] = \left[\mathcal{A}_{12}^{-1}\,\mathcal{A}_{11} \quad \mathcal{U}_T\right]$$

where $\mathcal{A}_{12}$ is the unit (or identity) matrix of order v-1.

<u>Proof</u>: Since $\mathcal{A}_a$ has a rank v-1 the columns and rows of $\mathcal{A}_a$ can be permuted so that the non-singular sub-matrix of order v-1 appears at the upper right hand corner so that

$$\mathcal{A}_a = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{a1} & \mathcal{A}_{a2} \end{bmatrix}$$

and $\mathcal{A}_{12}$ of order v-1 is non-singular. There exists a set of c-circuits for which the c-circuit matrix,[1] $\mathcal{B}_c$, is given by

$$\mathcal{B}_c = \left[\mathcal{U}_c \;\; \mathcal{B}_{c12}\right] = \left[\mathcal{U}_c \;\; -\mathcal{A}_{11}' \mathcal{A}_{12}^{-1'}\right]$$

where

$$\mathcal{B}_{c12} = -\mathcal{A}_{11}' \mathcal{A}_{12}^{-1'}$$

Since, Theorem 6 gives

$$\mathcal{C}_{T11} = -\mathcal{B}_{c12}'$$

then

$$\mathcal{C}_{T11} = \mathcal{A}_{12}^{-1} \mathcal{A}_{11}$$

and

$$\mathcal{C}_T = \left[\mathcal{C}_{T11} \;\; \mathcal{U}_T\right] = \left[\mathcal{A}_{12}^{-1} \mathcal{A}_{11} \;\; \mathcal{U}_T\right]$$

<u>Corollary 1.7</u>: $\mathcal{A} = \mathcal{A}_{12} \mathcal{C}_T$

and $\mathcal{C}_T = \mathcal{A}_{12}^{-1} \mathcal{A}$

<u>Proof</u>: Since $\mathcal{A}_{12}^{-1}$ is of order v-1 and non-singular, from Theorem 7:

$$\mathcal{A}_{12} \times \mathcal{C}_T = \mathcal{A}_{12} \left[\mathcal{A}_{12}^{-1} \mathcal{A}_{11} \;\; \mathcal{U}_T\right]$$

$$= \left[\mathcal{A}_{11} \;\; \mathcal{A}_{12}\right] = \mathcal{A}$$

Therefore,

$$\mathcal{A}_{12} \mathcal{C}_T = \mathcal{A}$$

and

$$\mathcal{C}_T = \mathcal{A}_{12}^{-1} \mathcal{A}$$

---

[1] Reed and Seshu, <u>op</u>. <u>cit</u>., p. 18.

From the result given by this last corollary:

$\mathcal{C}_T$ and the incidence matrix of the tree, $\mathcal{A}_{12} \rightarrow \mathcal{A}$

$\mathcal{A}$ and the incidence matrix of the tree, $\mathcal{A}_{12} \rightarrow \mathcal{C}_T$

<u>Example 5</u>. Consider the graph of Fig. 10.

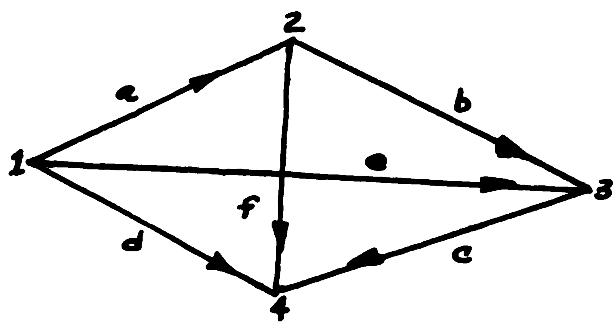

Fig. 10.

The incidence matrix is:

$$\mathcal{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

$$\mathcal{A}_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \text{, and } \mathcal{A}_{12}^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Then:

$$\mathcal{C}_T = \mathcal{A}_{12}^{-1} \mathcal{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then $$\mathcal{C}_T = [\mathcal{C}_{T11} \; \mathcal{U}_T]$$

The vertex matrix $\mathcal{A}$ can also be determined from the tree cut-set matrix, $\mathcal{C}_T$, and the tree incidence matrix in accordance with $\mathcal{A} = \mathcal{A}_{12} \mathcal{C}_T$.

3. Current-function and voltage-function relationships.

Two of the fundamental postulates of electrical network theory take the mathematical form of vertex and circuit equation.[1] The vertex equations are

$$\mathcal{A} \mathcal{I}_e(t) = 0 \qquad \text{..............(A)}$$

where the matrix $\mathcal{I}_e(t)$ represents the functions i(t) associated with the network elements, and the circuit equations are

$$\mathcal{B} \mathcal{V}_e(t) = 0 \qquad \text{..............(B)}$$

where the matrix $\mathcal{V}_e(t)$ represents the functions v(t) associated with the network elements.

A variety of interrelations among the current functions, i(t), and among the voltage functions, v(t), can be expressed in terms of $\mathcal{C}_T$ and its sub-matrices.

Theorem 8: If $\mathcal{C}_T$ is the tree cut-set matrix of the graph G and $\mathcal{I}_e(t)$ is the matrix of current functions, i(t), associated each element of the graph, then

$$\mathcal{C}_T \mathcal{I}_e(t) = 0$$

---

[1] Reed and Seshu, op. cit., pp. 21, 22.

<u>Proof</u>: Let $\mathcal{C}_T$ be a tree cut-set matrix of graph G, and $\mathcal{I}_e(t)$ the matrix of current functions, i(t), associated with each element of the graph.

Since $\mathcal{C}_T = \mathcal{A}_{12}^{-1}\mathcal{A}$ by corollary 1.7,

then $\mathcal{C}_T \, \mathcal{I}_e(t) = \mathcal{A}_{12}^{-1}\mathcal{A}\,\mathcal{I}_e(t)$

$$= \mathcal{A}_{12}^{-1}\left[\mathcal{A}\,\mathcal{I}_e(t)\right]$$

But by the fundamental postulate

$$\mathcal{A}\,\mathcal{I}_e(t) = 0$$

Therefore: $\mathcal{C}_T\,\mathcal{I}_e(t) = \mathcal{A}_{12}^{-1} \times 0 = 0$

<u>Theorem 9</u>: If $\mathcal{C}_T$ is a tree cut-set matrix, $\mathcal{V}_e(t)$ and $\mathcal{V}_b(t)$ are the matrices of voltage functions associated with all elements of a graph and the branches of the tree, then

$$\mathcal{C}_T\,\mathcal{V}_e(t) = \left[\mathcal{B}_{c12}' \cdot \mathcal{B}_{c12} + \mathcal{U}_T\right] \cdot \mathcal{V}_b(t)$$

<u>Proof</u>: Let $\mathcal{C}_T$ be a tree cut-set matrix of graph G, $\mathcal{V}_e(t)$ the matrix of voltage functions associated with elements of G and let $\mathcal{B}_c$ be the circuit matrix of the same tree of the graph. Partitioning $\mathcal{C}_T$ and $\mathcal{B}_c$ corresponding to chords and branches, $\mathcal{V}_e(t)$ can be partitioned corresponding to chords and branches as

$$\mathcal{V}_e(t) = \begin{bmatrix} \mathcal{V}_c(t) \\ \mathcal{V}_b(t) \end{bmatrix}$$

Then $$\mathcal{C}_T \mathcal{V}_e(t) = \left[\mathcal{C}_{T11} \; \mathcal{U}_T\right] \begin{bmatrix} \mathcal{V}_c(t) \\ \mathcal{V}_b(t) \end{bmatrix}$$

$$= \mathcal{C}_{T11} \mathcal{V}_c(t) + \mathcal{V}_b(t)$$

Since, by Theorem 8, $\mathcal{C}_{T11} = -\mathcal{B}'_{c12}$

by direct substitution

$$\mathcal{C}_T \mathcal{V}_e(t) = -\mathcal{B}'_{c12} \cdot \mathcal{V}_c(t) + \mathcal{V}_b(t)$$

From fundamental postulate (B),

$$\mathcal{B}_c \mathcal{V}_e(t) = 0$$

and so $$\left[\mathcal{U}_c \; \mathcal{B}_{c12}\right] \begin{bmatrix} \mathcal{V}_c(t) \\ \mathcal{V}_b(t) \end{bmatrix} = 0$$

$$\mathcal{V}_c(t) + \mathcal{B}_{c12} \mathcal{V}_b(t) = 0$$

$$\mathcal{V}_c(t) = -\mathcal{B}_{c12} \mathcal{V}_b(t)$$

Using this value of

$$\mathcal{C}_T \mathcal{V}_e(t) = -\mathcal{B}'_{c12}\left[\mathcal{B}_{c12} \cdot \mathcal{V}_b(t)\right] + \mathcal{V}_b(t)$$

$$\mathcal{C}_T \mathcal{V}_e(t) = \left[\mathcal{B}'_{c12} \cdot \mathcal{B}_{c12} + \mathcal{U}_T\right] \cdot \mathcal{V}_b(t)$$

<u>Theorem 10</u>: The functions of $\mathcal{V}_c(t)$ associated with the chords of a tree of a connected graph G, can be expressed in terms of the functions of $\mathcal{V}_b(t)$ associated with the branches of the same tree.

Proof: Let $\mathcal{B}_e$ and $\mathcal{C}_T$ be c-circuit and tree cut-set matrices of a tree of a connected graph G. Partition $\mathcal{B}_e$ and $\mathcal{C}_T$ to correspond to chords and branches as

$$\mathcal{B}_c = \left[\mathcal{U}_c \ \mathcal{B}_{c12}\right]$$
$$\mathcal{C}_T = \left[\mathcal{C}_{T11} \ \mathcal{U}_T\right]$$

Let $\mathcal{V}_c(t)$ and $\mathcal{V}_b(t)$ represent the voltage functions associated with chords and branches of the tree respectively, then

$$\mathcal{V}_e(t) = \begin{bmatrix} \mathcal{V}_c(t) \\ \mathcal{V}_b(t) \end{bmatrix}$$

Since, by fundamental postulate (B),

$$\mathcal{B}_c \mathcal{V}_e(t) = 0$$

$$\left[\mathcal{U}_c \ \mathcal{B}_{c12}\right] \begin{bmatrix} \mathcal{V}_c(t) \\ \mathcal{V}_b(t) \end{bmatrix} = 0$$

and
$$\mathcal{V}_c(t) + \mathcal{B}_{c12} \mathcal{V}_b(t) = 0$$

By Theorem 6
$$\mathcal{B}_{c12} = -\mathcal{C}'_{T11}$$

therefore,
$$\mathcal{V}_c(t) - \mathcal{C}'_{T11} \mathcal{V}_b(t) = 0$$

$$\mathcal{V}_c(t) = \mathcal{C}'_{T11} \mathcal{V}_b(t)$$

Corollary 1.10: The voltage functions $\mathcal{V}_e(t)$ associated with all elements of the graph can be expressed in terms of the voltage functions $\mathcal{V}_b(t)$ associated with branches of any tree of the graph as

$$\mathcal{V}_e(t) = \mathcal{C}'_T \mathcal{V}_b(t)$$

<u>Proof</u>: Since

$$v_e(t) = \begin{bmatrix} v_c(t) \\ v_b(t) \end{bmatrix}$$

on substituting the value of $v_c(t)$ in terms of $v_b(t)$ from the theorem, there results

$$v_e(t) = \begin{bmatrix} \mathcal{C}'_{T11} v_b(t) \\ v_b(t) \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{C}'_{T11} \\ \mathcal{U} \end{bmatrix} v_b(t)$$

But

$$\begin{bmatrix} \mathcal{C}'_{T11} \\ \mathcal{U} \end{bmatrix} = \mathcal{C}'_T$$

and so

$$v_e(t) = \mathcal{C}'_T v_b(t)$$

<u>Example 6</u>. Consider the graph given in Fig. 11 and suppose the voltage functions associated with the branches of a tree of G are known as v_a, v_b and v_c which are associated with branches a, b and c respectively.

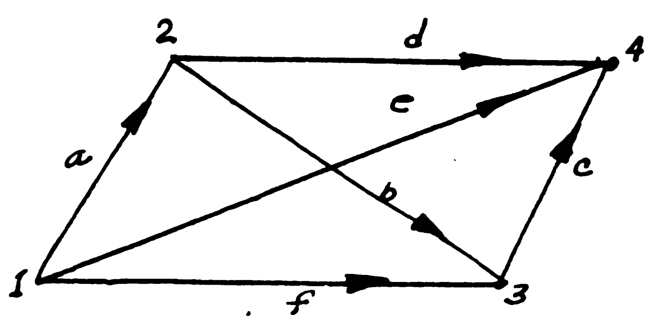

Fig. 11.

The problem is to find the voltage functions associated with all elements of the graph

$$\mathcal{V}_e'(t) = \left[v_a(t)\ v_b(t)\ v_c(t)\ v_d(t)\ v_e(t)\ v_f(t) \right]$$

First write the tree cut-set matrix for this tree with elements arranged in the same order as in

$$\mathcal{C}_T = \begin{array}{c} \\ c_a \\ c_b \\ c_c \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The matrix $\mathcal{V}_b(t)$ is in the form of single column as

$$\mathcal{V}_b(t) = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}$$

Then the $\mathcal{V}_e(t)$ matrix can be found by

$$\mathcal{V}_e(t) = \mathcal{C}_T' \mathcal{V}_b(t)$$

which in detail is

$$\mathcal{V}_e(t) = \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \\ v_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} v_a \\ v_b \\ v_c \\ v_b + v_c \\ v_a + v_b + v_c \\ v_a + v_b \end{bmatrix}$$

CONCLUSION

The cut-set and cut-set matrix properties studied here are useful in solving net-work problems. For instance, the node transformation equation[1]

$$\mathcal{V}_e(t) = \mathcal{A}' \mathcal{V}_n(t)$$

where $\mathcal{V}_n(t)$ are arbitrary functions, obscures the fact that the v(t) of a tree determine all the v(t), whereas in terms of the cut-set matrix

$$\mathcal{V}_e(t) = \mathcal{C}_T' \mathcal{V}_b(t)$$

in which $\mathcal{V}_b(t)$ is the matrix of voltage functions corresponding to a tree. Furthermore, the cut-set matrix seems to be more general for the vertex matrix is always a sub-matrix of the cut-set matrix. Also the circuit matrix can be found from matrix $\mathcal{C}$. It is here suggested that the cut-set system be considered as the fundamental i-function postulate in place of the vertex equations of Kirchhoff, i.e.,

$$\mathcal{C}\mathcal{I}_e(t) = 0$$

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1 Reed and Seshu, op. cit., p. 26.

BIBLIOGRAPHY

Cauer, W., "Theorie der linearen Wechselstromschaltungen" Akademische Verlagsgeselschaft, Leipzig, 1941.

Foster, R. M., "Geometric Circuits of Electric Networks" BSTJ No. B-653.

Koenig, D. "Theorie der endlichen und unendlichen graphen," Chelsea Publishing Company, New York, 1950.

Le Corbeiller, P., "Matrix Analysis of Electric Networks", Harvard Monographs in Applied Science, Number 1, Harvard University Press, 1950.

Reed, Myril B. and Sundaram Seshu, "On Topology and Network Theory". Proceedings of the University of Illinois Symposium on Circuit Analysis, May 1955.