# CONCEPTUALIZING VECTORS IN COLLEGE GEOMETRY: A NEW FRAMEWORK FOR ANALYSIS OF STUDENT APPROACHES AND DIFFICULTIES

by

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**ABSTRACT**

CONCEPTUALIZING VECTORS IN COLLEGE GEOMETRY: A NEW FRAMEWORK FOR ANALYSIS OF STUDENT APPROACHES AND DIFFICULTIES

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This dissertation documents a new way of conceptualizing vectors in college mathematics, especially in geometry. First, I will introduce three problems to show the complexity and subtlety of the construct of vectors with the classical vector representations. These highlight the need for a new framework that: (1) differentiates abstraction from a physical embodiment, (2) intertwines the representational perspective and the cognitive perspective on vectors, and (3) reveals cognitive development on geometric representations of vectors. These needs ground the development of the framework that permits a layered view of the construct of vectors. The framework comprises three layers of progressive refinements: (1) a layer that describes a global distinction between physical vectors and mathematical vectors by the difference between a physical embodiment and abstraction, (2) a layer that recounts the difference between the representational perspective and the cognitive perspective on vectors as the difference between ontological perspective and epistemological perspective, and (3) a layer that identifies ontological and epistemological obstacles in terms of transitions towards abstraction. Data was gathered from four empirical studies with ninety-eight total students to find evidence of the three major transition points in the new framework: physical to mathematical coming from the first layer, geometric to symbolic and analytic to synthetic from the second layer, and the prevalence of the analytic approach over the synthetic approach while developing abstraction enlightened by the third layer. Limitations of the framework on the distinction of physical and mathematical vectors, directions of cognitive development, and repetition of transitions suggest further refinements of the framework and the implications for teaching and learning of vectors.

To my family

## ACKNOWLEDGEMENTS

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## TABLE OF CONTENTS

## LIST OF TABLES

## LIST OF FIGURES

## PREFACE

During the summer 2009 semester, I had an opportunity to co-teach with my advisor, a college geometry course for future high school teachers. We developed and taught lessons of classical Euclidean Plane Geometry by suggesting to our students to choose freely from the methods of exploration among (1) coordinate geometry, (2) synthetic geometry, and (3) vector geometry. For example, here is Triangle Midpoints Theorem that can be explored with all three methods above.

**Triangle Midpoint Theorem** The line segment connecting the midpoints of two sides of a triangle is parallel to the third side and is congruent to one half of the third side.

Students were asked to prove the theorem using coordinates, similar triangles synthetically, or vectors. Most students felt comfortable locating a vertex of a triangle at the origin on a coordinate plane. They tried to use coordinates for the remaining two vertices and midpoints to figure out the slopes and the lengths of the two line segments connecting points. See Figure 0.1.

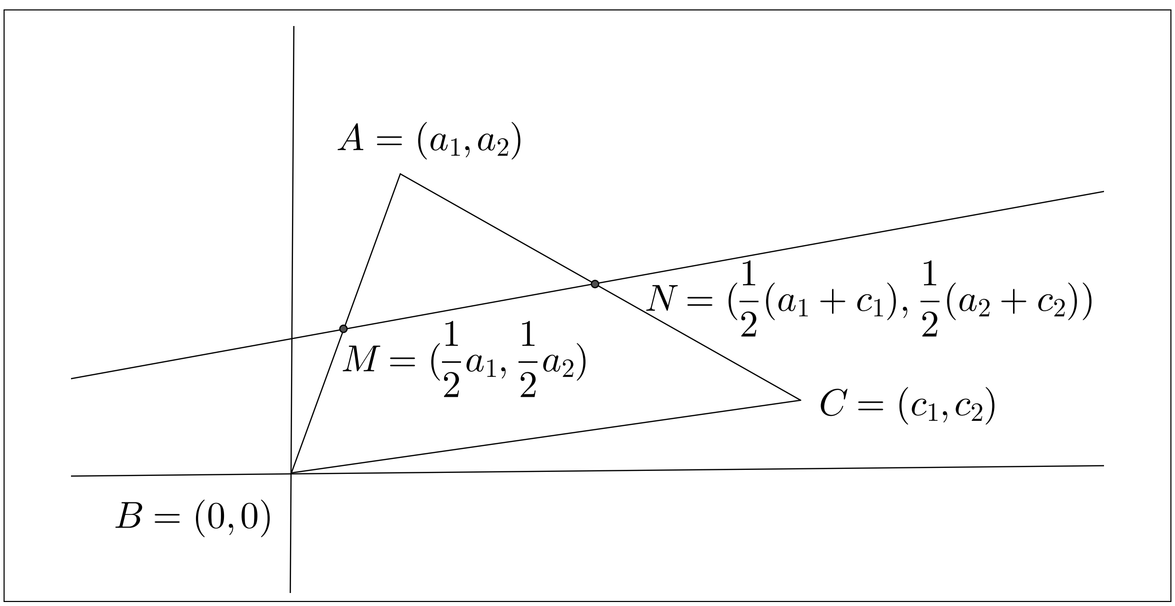



Figure 0.1: Triangle Midpoints Theorem with Analytic Geometry

When students thought synthetically, they also felt comfortable in using triangle similarity. One could show that $\triangle AMN \sim \triangle ABC$, and $\angle AMN \cong \angle ABC$. This implied $\overline{MN} \parallel \overline{BC}$ and $\overline{BC} = 2\overline{MN}$

by a scale factor. This is a typical approach in the majority of classical Euclidean Plane Geometry textbooks; so some students tried this approach with the help of books and other resources. See Figure 0.2.

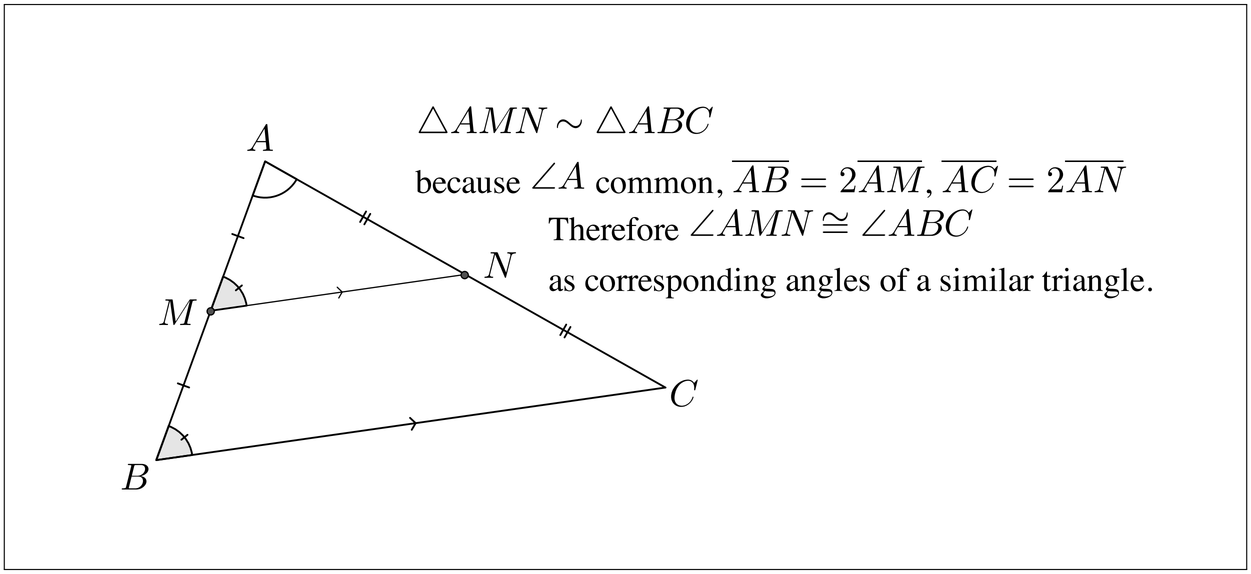



Figure 0.2: Triangle Midpoints Theorem with Synthetic Geometry

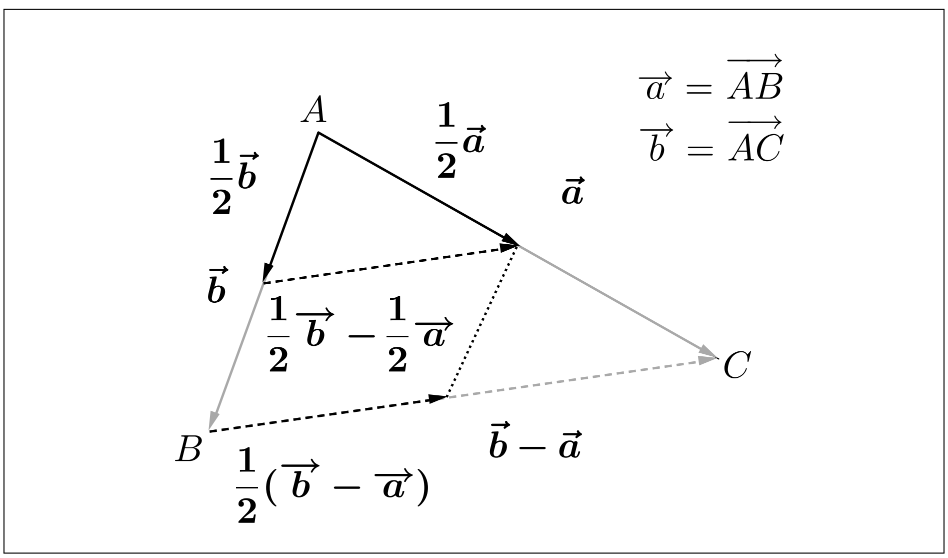



Figure 0.3: Triangle Midpoints Theorem with Vector Geometry

However, students were very reluctant to use vectors in the proof. By setting up $\overrightarrow{a}$ and $\overrightarrow{b}$ on each side of a triangle, we can easily find $\frac{1}{2}\overrightarrow{a}$, $\frac{1}{2}\overrightarrow{b}$, $\frac{1}{2}\overrightarrow{a}-\frac{1}{2}\overrightarrow{b}$, and $\overrightarrow{a}-\overrightarrow{b}$. With a simple symbolic calculation using scalar distribution, we have that $\frac{1}{2}(\overrightarrow{b}-\overrightarrow{a})=\frac{1}{2}\overrightarrow{b}-\frac{1}{2}\overrightarrow{a}$. By comparing this result with the the triangle figure, we can see the result implies both geometric meaning of 'parallel' and 'the same length' simultaneously. (Figure 0.3). I believed that the proof of the triangle midpoints theorem was one of the easiest applications of vectors.

Contrary to my belief, students felt this vector proof was much more difficult than coordinate or synthetic geometric proofs. They drew arrow representations of vectors on the triangle, labeled them with symbols such as $\overrightarrow{AB}$, and tried to translate arrow representations to coordinate representations using coordinates of the initial and terminal points of vectors. Their symbols were mainly for naming, not for manipulating. What they really tried was measuring slopes of arrows using coordinates of the initial and terminal points. Again, they apparently just went back to coordinate geometry! They could draw arrows and calculate 2-tuples by subtracting numbers component wise, but they did not know how to handle arrows with symbolic operations nor how those 2-tuples differ from the coordinates of points.

Although my observation was just anecdotal evidence from my personal teaching experience, I saw the students' ideas of vectors were very scattered and isolated. Students felt uncomfortable with connecting and converting various representations of vectors to each other, and were reluctant to apply them to geometric problems. This brought me to think again about conceptualizing vectors in mathematics. If there were a new way of talking about various vectors coherently in mathematics, it would be very beneficial to both teachers and students. In this dissertation, I tried to re-conceptualize vectors in mathematics, not just by juxtaposing them with different uses or looks, but by refining the construct of vectors more carefully.

# Chapter 1

# BACKGROUND, RATIONALE, AND PURPOSE

This dissertation documents a new way of conceptualizing vectors in college mathematics, especially in geometry. In this chapter, I will introduce three problems to describe the complexity and subtlety of the construct of vectors with the classical vector representations[1]. These motivate the necessity of the new framework that permits: (1) differentiating abstraction from physical embodiment, (2) intertwining the representational perspective and the cognitive perspective on vectors, and (3) revealing cognitive development on geometric representations. These needs guided me to deliver a new framework for conceptualizing the construct of vectors.

By building a new framework, I explored a complex construct of vectors in mathematics with respect to mathematical abstraction, multiple representations, and cognitive development. The primary goal of the framework is to give a new way to discuss the complexity of vectors, both conceptual and pedagogical, that students may grapple with in order to understand vectors in geometry effectively. This study will be propitious for students to grasp flexibility on the use of various vectors.

## 1.1 Complexity of the Construct of Vectors

The section contains three problems illustrating the complexity and subtlety of vectors that cannot be identified well with the classical vector representations. These motivate the necessity of a new framework that provides a better and/or different way to describe the construct of vectors.

[1] These are vector representations frequently used in the textbooks such as arrows, coordinate forms, etc.

### 1.1.1 Vector as a Translation

When representing a geometric translation with a vector in an arrow form, the complexity of vector representations comes out in a cognitive sense. This complexity can be described as a difference between physical vectors based on physical embodiment and mathematical vectors regarding on mathematical abstraction. Differentiating mathematical abstraction from physical embodiment is essential to understand the complexity and subtlety of vectors in mathematics.

Students' experiences with the concept of vectors vary over physics and mathematics, but relatively few studies have focused on an analytic, detailed explanation or examination of learning of vectors in mathematics compared to physics. Recent critical studies focus on physics education (Aguirre and Erickson, 1984; Aguirre, 1988; Hestenes et al., 1992; Heller and Huffman, 1995; Knight, 1995; Savinainen and Scott, 2002; Nguyen and Meltzer, 2003; Flores et al., 2004; Coelho, 2010). These studies covered only algebraic aspects of vectors with simple addition/subtraction operations and focused only on the interrelationship among physical quantities such as displacement, velocity, acceleration, and force, not vectors themselves. This physical embodiment of vectors helps students understand vectors initially, but soon block the progression to advanced and abstract understanding of vectors.

For example, in Figure 1.1 vector $\overrightarrow{f}$ looks far away from the triangle and nothing to do with the translation. Vector $\overrightarrow{d}$ pushes the triangle. Vector $\overrightarrow{e}$ pulls the image of the triangle, not the preimage. Vector $\overrightarrow{a}$ gives an idea of moving vertices on a triangle. Vector $\overrightarrow{c}$ is penetrating, not translating the triangle. However, all the vectors represent the same translation of a triangle even though their locations are quite scattered and different. This difference between physical meaning and mathematical meaning is from the fact that the motion in mathematics is quite different from that in physical or realistic contexts. Freudenthal (1983) described the distinction between physical motion and mathematical motion. Physical motion is something that occurs to an object within space or plane within time, but mathematical motion should be differentiated from physical motion in three ways: from the limited object to the total space (plane), from within space (plane) to on space (plane), from within time to at one blow. When talking about teaching and learning of vectors

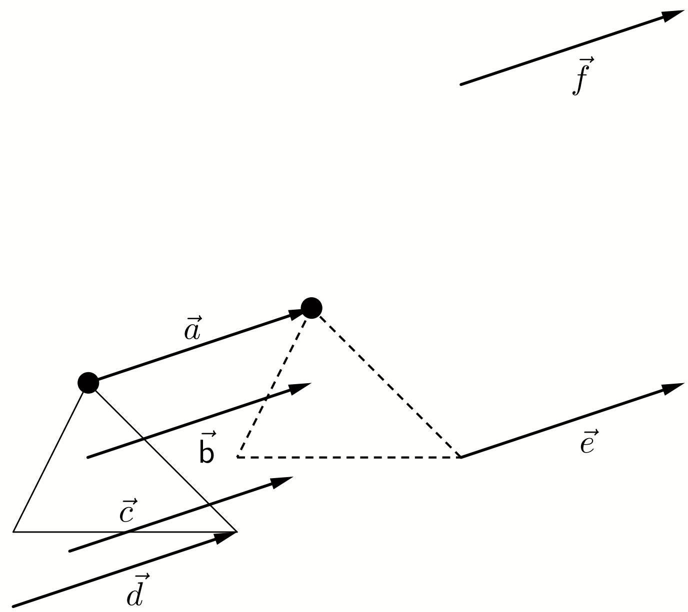



Figure 1.1: Vectors representing a translation

in mathematics, this distinction is sometimes not regarded as a sharp contrast by the researchers. Watson et al. (2003) showed that by focusing on the effect rather than the specific actions involved, several highly sophisticated concepts such as equivalent vectors, vector sums could be taught better. This study was based on the three worlds of mathematics framework developed in Tall (2004b,a, 2008). See Figure 1.2.

The 'action-effect' approach lays the groundwork for students' cognitive development on vectors. However, to judge from the description of Freudenthal (1983) on the distinction between physical motion and mathematical motion, this approach has a limitation. As mentioned in Tall (2004b), this approach does not call attention to a fact that the equivalent vectors cannot be explained by limiting oneself in conceptual-embodied world or proceptual-symbolic world of mathematics.

In the case of vectors, both a totally intuitive sense with physical contexts and a totally abstract sense with mathematics are entailed in the equivalence relation. The equivalent vector concept is not just related with the physical world but also related with the formal definition that lives in the axiomatic-formal world of mathematics. In Figure 1.1, all the vectors are equivalent just as all the

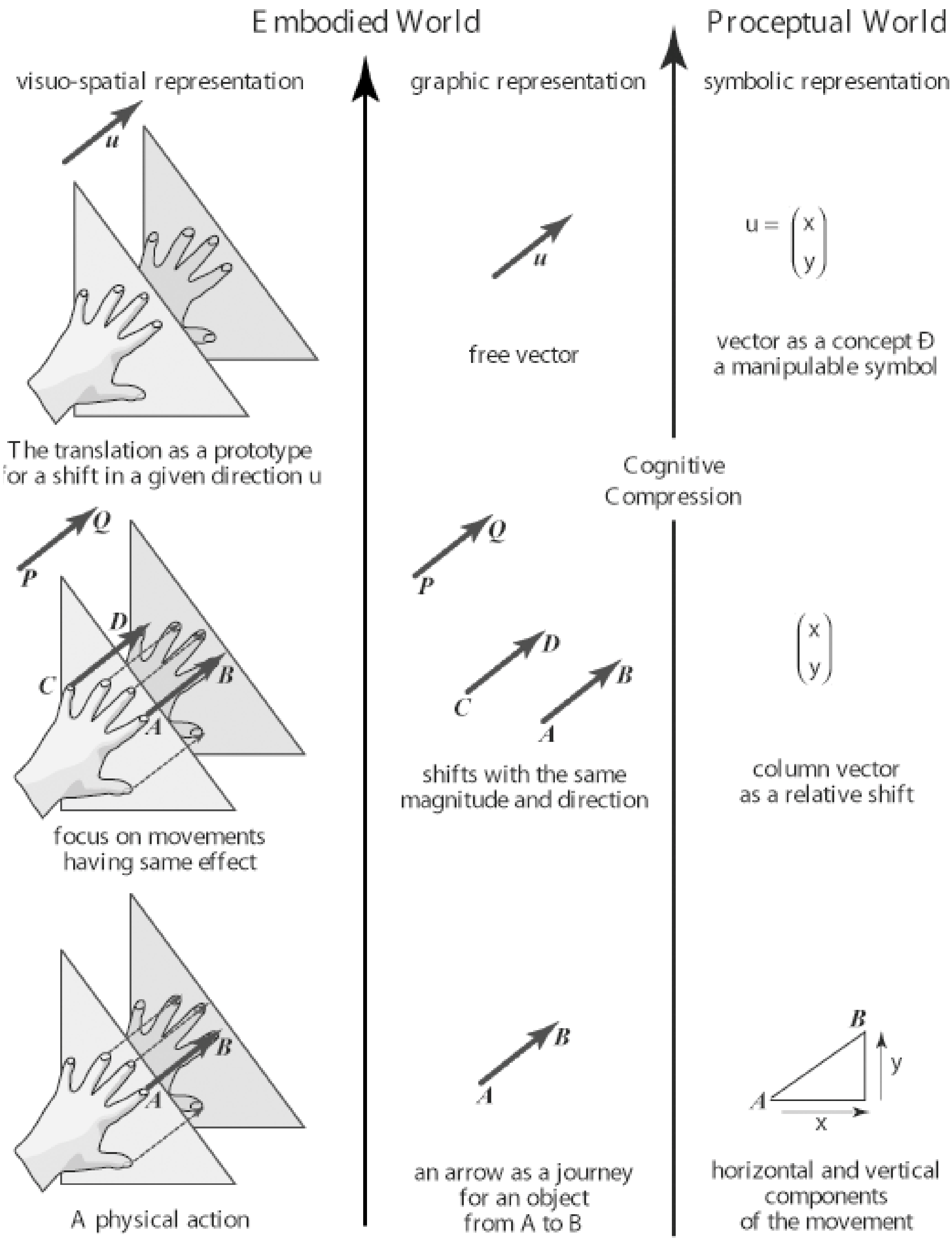



Figure 1.2: Action-Effect Approach (Watson, 2004)

vectors represent the same translation. However, their equivalence is not clearly explained by the translation 'action' with the given triangle and the same 'effect' those vectors would bring. Equivalent vectors can be explained by getting students into meta-cognitive level of the three worlds of mathematics or at least by letting them understand the different use of the similar looking vectors in different contexts and situations.

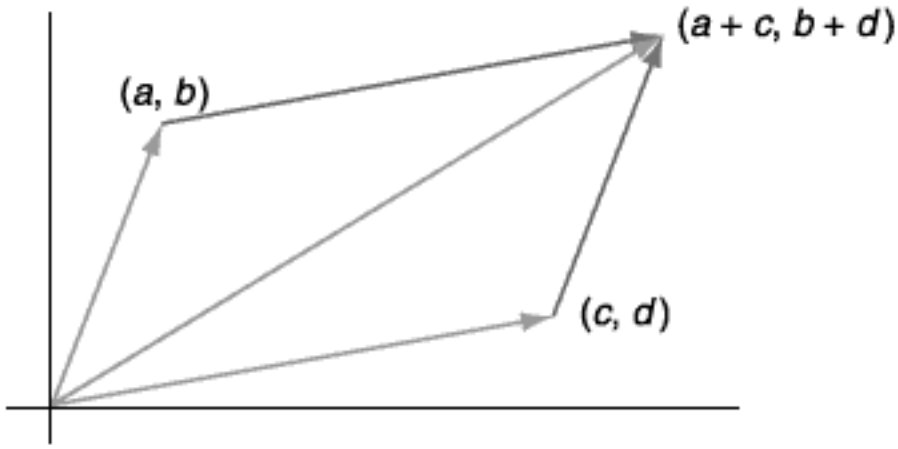
For a better observation and description of this complexity and subtlety of vectors, we need a new framework that integrates and balances these intuitive understanding and abstract understanding of vectors in college mathematics. We want a new framework that (1) refines the notion of vector representations that distinguish mathematical vectors from physical vectors, and (2) considers the student's cognitive development in terms of various vector representations.

While talking about vectors as geometric translations, we saw that the cognitive elements play an important role when a vector representation is fixed such as an arrow form. What if we have a different form of a representation such as a coordinate form? How does the cognitive elements work in this case? This question motivated me to think about two different perspectives on the construct of vectors: representational and cognitive.

### 1.1.2 Vector as a Point, Point as a Vector

Teaching and learning multiple representations of vectors need to be treated carefully due to the complexity of the construct of vectors. When dealing with multiple representations, we need to deliberate two different perspectives: representational and cognitive simultaneously. As we can see in the following example, the complexity of vectors does not allow students an easier translation/conversion from one representation to the other. A combined view of the representational perspective and the cognitive perspective helps us understand this translation/conversion from one representation to the other.

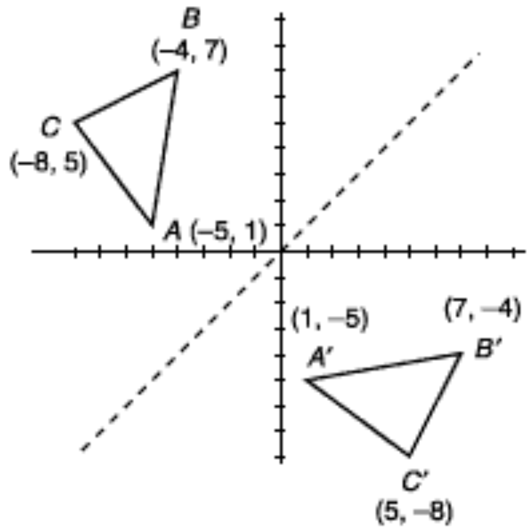
In Figure 1.3, the vector sum in arrow forms is usually accompanied by 2-tuple of numbers. However, it is not clear if this 2-tuple represents the terminal point of the arrow or the arrow itself. When geometry meets linear algebra, this problem of multiple representations gets more complicated as in Figure 1.4. The term 'vectors', 'points (vertices)', and 'arrows' are not easily distinguishable in these two examples. Similar difficulty in distinguishing 'vectors-as-arrows' and 'vectors-as-points' is introduced by Hillel (2002) as well.




$$(a,b)+(c,d)=(a+c,b+d)$$

Figure 1.3: Determine the sum of two vectors. (NCTM, 2000)

According to Hillel, undergraduate linear algebra courses generally include three modes of description for basic objects and operations. They are (a) the abstract mode using the language and concepts of the general formalized theory; (b) the algebraic mode using the language and concepts of the more specific theory of $\mathbb{R}^n$; (c) the geometric mode using the language and concept of 2- and 3-spaces. He distinguished 'vectors-as-arrows' and 'vectors-as-points' in terms of modes of description, and criticized that in practice, most instructors tended to shift back and forth between the arrow and point depiction of vectors 'implicitly' and 'unconsciously'. Hillel (2002) offered unique insight into this problem by putting emphasis on translations/conversions between representations with the modes of description. However, his argument did not show enough clarity on the subtlety of vectors in a sense that the difference of modes of description cannot capture this 'implicit' and 'unconscious' changes. 'Vectors-as-arrows' and 'vectors-as-points' are not just a problem of modes of description, but also a problem of how students think about representations. The study of modes of thinking by Sierpinska (2002) reveals an expansion of the discussion. Sierpinska (2002) classified students' reasoning in linear algebra courses into three modes: Synthetic-geometric,




Consider a triangle $ABC$ with vertices $A = (-5, 1)$, $B = (-4, 7)$, and $C = (-8, 5)$.

Reflect the triangle over the line $y = x$ to obtain the triangle $A'B'C'$ as shown.

Determine a matrix $M$ such that $MA = A'$, $MB = B'$, and $MC = C'$, where the points are represented as vectors.

Explore the properties of the matrix $M$.

Find a matrix $M$ so that $MA = A'$, $MB = B'$, $MC = C'$ where **the points are represented as vectors** (NCTM, 2000).

Figure 1.4: Points are represented as vectors.

analytic-arithmetic, and analytic-structural modes. Sierpinska focused more on students' thinking and reasoning about epistemology.

These classifications of vectors motivated us to consider multiple perspectives on vectors: representational and cognitive. In the similar context, one observation that Hillel (2002) made about 'vectors-as-arrows' and 'vectors-as-points' brought up a refreshing idea. By emphasizing pedagogical and psychological advantages related with physics on 'vectors-as-arrows', he has displayed student's possible preference of one representation to the other in a specific setting or for a purpose. However, no indication is provided as to how this preference can change if the setting is different, such as in synthetic geometry other than linear algebra. Studies on multiple representations of vectors are few and focused on the views from linear algebra (Dorier, 2002; Harel, 1989; Dorier and Sierpinska, 2001). For this reason, we need more holistic way covering both representational and cognitive perspectives to talk about vectors in mathematics.

Studies on multiple representations in mathematics education have promoted the importance of understanding multiple representations and translations/conversions among them. Research in the mathematics education community posits that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (Janvier, 1987; Kaput, 1987a). Keller and Hirsch (1998) emphasized the importance of multiple representations by describing potential benefits related to the use of representations. Those benefits are "providing multiple concretizations of a concept, selectively emphasizing and de-emphasizing different aspects of complex concepts, and facilitating cognitive linking of representations." (Keller and Hirsch, 1998, p.1). In this context, several standards documents have advocated K-12 curricula that emphasize mathematical connections among representations (NCTM, 1989, 2000; CCSSM, 2010). They suggest that students use graphical, numerical, and algebraic representations to investigate concepts, problems, and express results. However, these discussions are very focused on the way to talk about functions, not vectors, and on the way to discuss connections between representations as separate entities from representations themselves. The construct of vectors is more complex than functions, so that graphical, numerical, and algebraic representations are not enough to describe this complexity (Pavlopoulou, 1993 as cited in Artigue et al. 2002). One cannot fully use the benefits of multiple representations until sufficient understanding and sophistication are developed.

The observations bring a need for a unified, inclusive, and multidimensional framework. This is not about listing all the possible contexts, different perspectives, and juxtaposing levels of sophistication, but about a new way to discuss a combined view of representational and cognitive perspectives on the complexity of the construct of vectors.

### 1.1.3 Vector Sum Geometrically

When talking about vectors and the interplay between the representational and the cognitive perspectives on them, the problem stems not only from the multiple representations but also from the translations/conversions among them. What the classical representations cannot provide from the complexity of vectors is the cognitive development of geometric representations. Specifically, translations/conversions in terms of the cognitive perspective are portrayed in considerable detail as cognitive development theories for symbolic representations. The vector sum problem that we discuss here shows a cognitive obstacle in geometric representations of vectors. This brings us a deliberation of a framework that can recognize and reveal cognitive development on geometric representations of vectors.

During the last two decades, the critical problems of translations between and within representations have been discussed, and the importance of moving among multiple representations and connecting them is also discussed in several studies. Pavlopoulou's research (as cited in Artigue et al., 2002) studied translations/conversions from one representation to the other (Duval, 2006). However, it is very restricted to certain forms of vectors: graphical, table, and symbolic representations (registers). As it has been noted earlier, this translation/conversion of representations is not just among representations but also on cognitive development.

Consider a vector sum $\overrightarrow{u_1}+\overrightarrow{u_2}+\ldots+\overrightarrow{u_n}$ or a combination of a sum and a difference $\overrightarrow{u_1}+\overrightarrow{u_2}+\ldots+\overrightarrow{u_n}-\overrightarrow{v}$ in Figure 1.5. We only have arrow forms of vectors, not coordinate $n$-tuple forms. However, there could be several different ways to approach this calculation, because arrow forms show a process of a sum as well as a sum as an object. The interpretation of the middle dots is also a possible obstacle to calculating and proceeding the sum.

Figure 1.6 illustrates these two different approaches of a sum briefly. Addition and subtraction are binary operations. Students can figure out $\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}+\overrightarrow{d}$ as a repetitive binary operation process with the triangle method. Students are also able to figure out the sum by connecting the initial point of $\overrightarrow{a}$ to the terminal point of $\overrightarrow{d}$ focusing the structure without thinking of the intermediate processes. The latter sometimes involves the idea of a particle moving along the

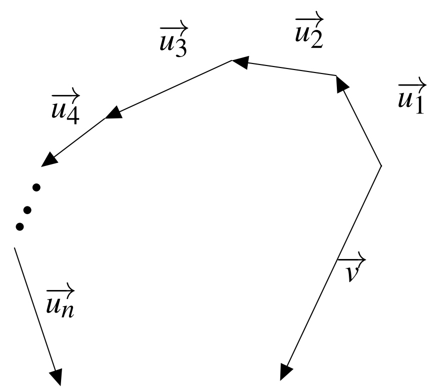
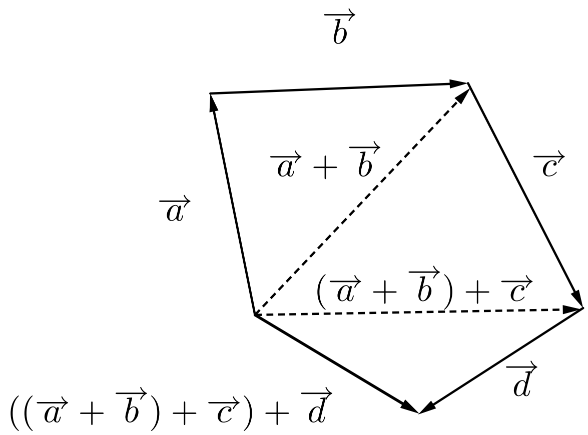


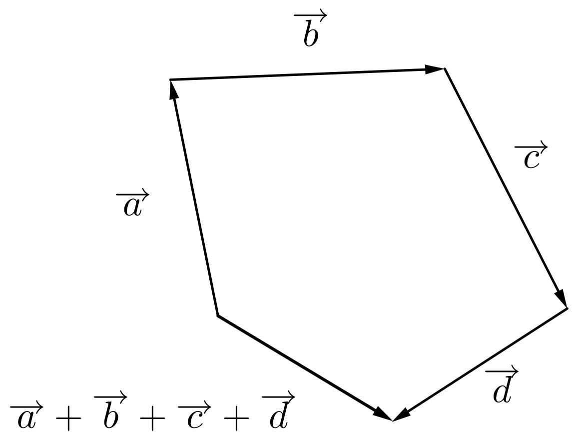

Figure 1.5: Vector sum and difference

arrows.

Two studies (Sfard and Thompson, 1994; Yerushalmy, 1997) articulated these differences of procedural thinking and structural thinking. They are based on the assumption that students' ability to understand mathematical concepts depends on their ability to make translations among several modes of representations. Tall et al. (1999) analyzed several theories, which describe students' transitions from viewing mathematical ideas operationally, or as processes, to viewing them structurally, or as objects. These transitions are referred to as "encapsulation" by Dubinsky (1991) and "reification" by Sfard (1991). The Action-Process-Object-Schema theory (Asiala et al., 1996) and reification theory (Sfard and Linchevski, 1994) are based on the duality of the mathematical concepts and on the assumption that process conception precedes object conception. Sfard (1991) calls process conception operational outlook and object conception structural view. However, these studies are restricted to discussions of symbolic representations. In terms of geometric representations, Figure 1.7 describes interesting observation about process-object duality (Sfard, 1991; Gray and Tall, 1993, 2001; Forster, 2000). When students calculate a vector sum with the tip to toe triangle method, the process view of a single vector is described as shifting or moving a particle. As




(a) Perform Operations




(b) Represent the Structure

Figure 1.6: Vector Additions

a result of the sum, students can put the resulting vector on the appropriate position, and make the sum itself as an object in the structure of a triangle. On the other hand, in parallelogram method, vectors are objects and moving the object with equivalence relation to draw a parallelogram is the process of sum. This example shows that process and object are preceders and succeedors of each other and shows further need for cognitive development as a part of the framework.

The work of Sfard (1991) views both operational and structural conceptions as important in mathematical understanding. A structural conception enables recognition (at a glance) and manipulation as a whole; an operational conception is grounded in actions, processes, and algorithms. However, this idea of encapsulation or reification, not just in symbolic modes of representation

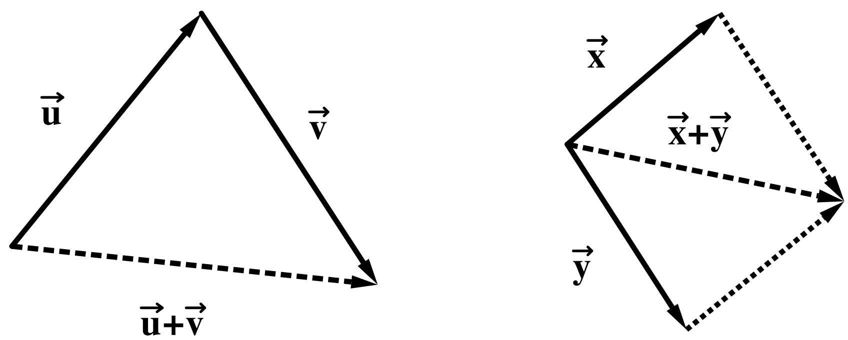



Figure 1.7: Sum of two vectors

but also in geometric modes of representation has not been studied (Meissner, 2001b,a; Meissner et al., 2006).

The problem of identifying cognitive development in geometric representations of vectors poses the need for a new framework that can show the representational and the cognitive obstacles more clearly in terms of the transitions towards mathematical abstraction. [2] This new framework brings the complexity of vectors to the surface so that one can capture the whole picture of encapsulation or reification both happening in a symbolic way and a geometric way simultaneously in the construct of vectors.

## 1.2 Needs for a New Framework

The foregoing three problems on vectors and related observations have demonstrated the need for: (1) differentiating mathematical abstraction from a physical embodiment, (2) intertwining the representational and the cognitive perspectives, and (3) revealing cognitive development on geometric representations of the new framework. From those needs, I tried to deliberate a framework that (1) affords an opportunity to examine the complexity of mathematical vectors as emerging from physical vectors, (2) includes ontological and epistemological perspectives of vectors, (3) identi-

[2] Representational obstacles are things that prevent students to translate/convert one form of representations to the other. Cognitive obstacles are things that avoid students to proceed cognitive development such as encapsulation or reification.

fies pivotal juncture points/transitions towards mathematical abstraction in the construct of vectors, and (4) permits cognitive development focused on geometric representations.

In preparation for the development of a new framework, I observed the way vector concepts were described and vector representations were presented in high school textbooks, college textbooks and various research studies. I also observed and listened to the way that mathematics education researchers, mathematicians, graduate students in mathematics and mathematics education, and undergraduate students describe the vector concepts and present vector representations.

In Chapter 2, I seek to : (1) construct a configuration of vectors as a new framework based on the process of refinement grounded by the needs in Chapter 1, (2) describe three layers of progressive refinements that are comprised in the framework, and (3) summarize how this newly built framework accomplishes the needs that comes from Chapter 1.

In Chapter 3, I describe the method and the research focus to gather student data that would show evidence of the three major transition points: physical to mathematical, geometric to symbolic, and analytic to synthetic coming from each layer of the framework, and the prevalence of the analytic approach to the synthetic approach while developing mathematical abstraction. In particular, I describe the design and construction of survey and interview questions to establish these three layers in the configuration more concretely.

In Chapter 4, I provide evidence of the focal points that I hypothesized in Chapter 2 after careful analyses of the four consecutive empirical studies. Included from student data are examples of three transition points and the prevalence of the analytic approach over the synthetic approach introduced in Chapter 3.

Finally in Chapter 5, I document the extent of my framework and summarize significant evidence away from the framework and suggesting the limitations of my configuration. I also comment on the implications of this study for teaching and learning, and the type of future research that could be fruitful in helping students engage in learning of vectors.

**Chapter 2**

# NEW FRAMEWORK FOR THE CONSTRUCT OF VECTORS

In Chapter 1, we saw the complexity and subtlety of the construct of vectors that cannot be fully appreciated by understanding the classical representations of vectors. To clarify this complexity especially in geometry, we needed a new framework that is unified, holistic, inclusive, and multidimensional. In this chapter, I will present a configuration based on series of progressive refinements of the construct of vectors. The configuration is comprised of three layers of refinements: (1) a layer that describes a global distinction between physical vectors and mathematical vectors as the difference between physical embodiment and mathematical abstraction, (2) a layer that expresses the difference between the representational perspective and the cognitive perspective on vectors as the difference between the ontological perspective and the epistemological perspective, and (3) a layer that identifies the ontological and the epistemological obstacles in terms of transitions towards mathematical abstraction. I will also summarize how these layers in the newly built framework accomplish the needs that come from Chapter 1. With this summary, I will set my goals for the empirical studies in the next chapter as well.

## 2.1 Configuration and its Role

Figure 2.1 is the configuration of vectors as a result of progressive refinements of the complex construct of vectors described in the remainder of this chapter.[1]

I arranged all the major representations of vectors used in school and college textbooks on the configuration together in a plane with two axes: ontological and epistemological axes. This arrangement of representations in each cell is my subjective idea for familiarizing the students and readers with the configuration of vectors. The way I arranged all the vector representations

[1]It is important that the reader consider this configuration figure (Figure 2.1) with the remainder of Chapter 2 and thus presented first.

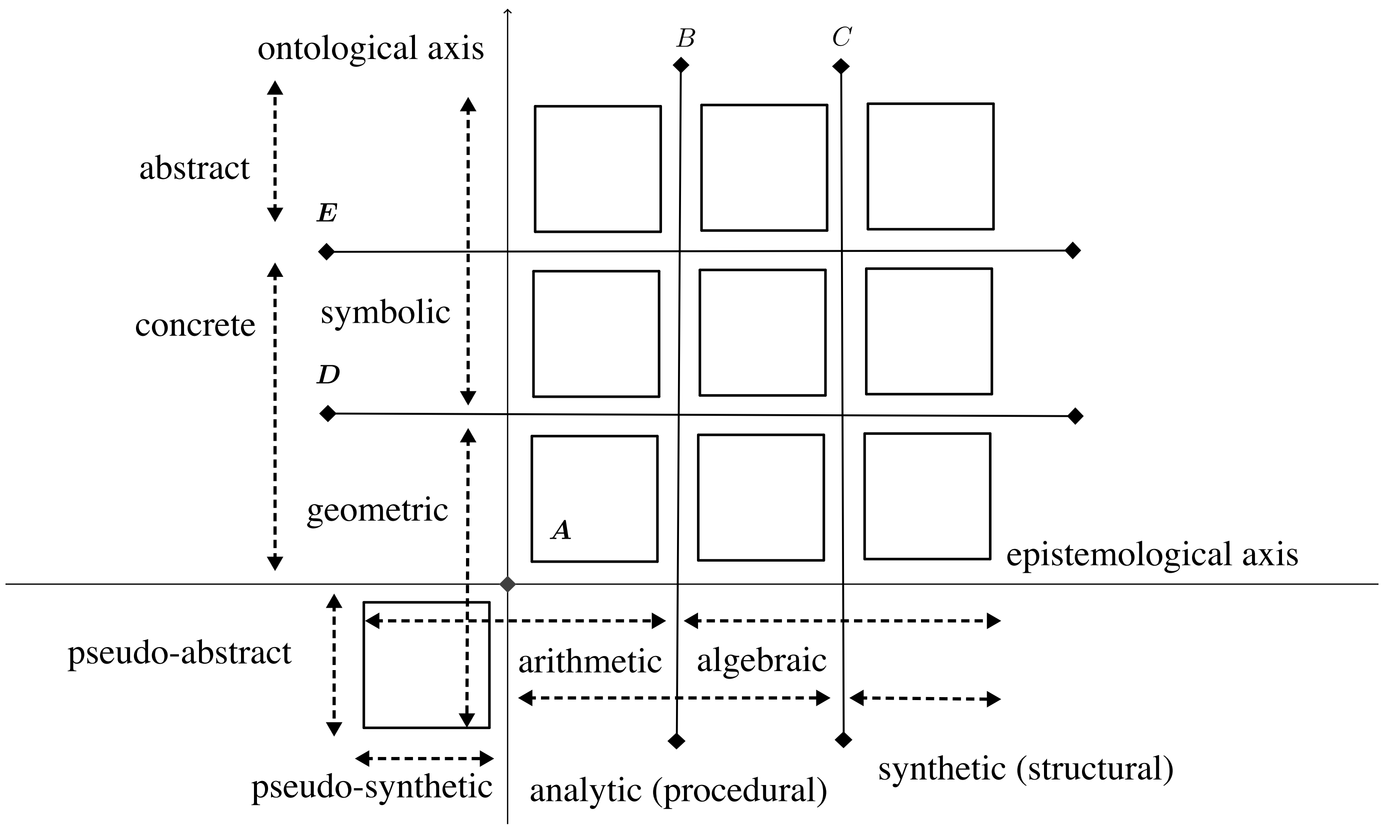



Figure 2.1: The Configuration of Vectors

in the appropriate cells is based on the three worlds of mathematics studies that also motivate the progressive refinements of the construct of vectors.

Tall and Watson placed the notion of vector in the three worlds of mathematics: conceptual-embodied world, proceptual-symbolic world, and axiomatic-formal world (Watson and Tall, 2002; Watson et al., 2003; Watson, 2004; Tall, 2004b). This placement gave an idea of the direction of cognitive development and a hierarchy of vectors.

This hierarchy and the non-congruent translations/conversions from Pavlopoulou's work (as cited in Artigue et al., 2002) show a structure built upon two different developmental directions: ontological (representational) and epistemological (cognitive).

Pavlopoulou studied translations/conversions among representations for vectors with graphical, table, and symbolic representations. The study showed that in the restricted setting of having three representations in two and three dimensions, those conversions were not easily accomplished by students. This study also showed that there were easier directions for translations/conversions.

In the newly built configuration, I used the term, ‘transitions’, instead of ‘conversions’ or ‘translations’. The reason is because I believe that Pavlopoulou’s study shows certain levels of sophistication separated by the representational and the cognitive obstacles in translating/converting representations. Sometimes it is not possible to translate/convert one representation to the other. For example, not all graphical representations of a function such as stochastic graphs or graphs from realistic data can be represented by symbols in algebraic equations. I also used the term, ‘geometric representations’, instead of ‘graphical representations’ to avoid the interpretation of the word ‘graphical’ from a graph with the coordinates similar to the case of a function. Vectors in geometry are not always accompanied by specific coordinate systems. The term geometric vector representations include both ‘graphical’ representations with the coordinate systems and non-‘graphical’ but still iconic/pictorial representations without the coordinate systems. Table representations in Pavlopoulou’s study are also classified as ‘analytic’ symbolic representations compared to classical symbolic representations as ‘synthetic’ symbolic representations.

Initially, I tried to build up a hierarchical model that could describe various vector representations along with student conceptualization, or that could predict student thinking or learning trajectories. However, as I combined the theories of mathematical abstraction, multiple representations, and cognitive development, I found that constructing a hierarchical model was not simple because vectors are a very complex construct and have nuances in various categorizations such as examples in Chapter 1. Now, I characterize this new framework as “a map of the territory, a tool of a certain grain size that we, as teachers, researchers and curriculum developers, can yield as we organize our thinking about teaching and learning the concept...” (Zandieh, 2000, p.103). This new role of the framework suggests locations of where to look to see the differences in students’ use of vectors.

## 2.2 Construction of the Configuration

To construct a framework, I revisited the needs that I identified earlier in Chapter 1. Those needs grounded this development of the framework that allows a layered view to see the complex construct of vectors. Three layers of progressive refinements will be introduced sequentially. They have different scales of focus from aggregate and global dealing with the difference of physical and mathematical vectors (first layer) to individual and local dealing with the representational and the cognitive obstacles (third layer).

### 2.2.1 First Layer towards Mathematical Abstraction

The first layer describes a global distinction between physical vectors and mathematical vectors as the difference between physical embodiment and mathematical abstraction. See Figure 2.2. As we have seen in Chapter 1, the difference between physical motion and mathematical motion (Freudenthal, 1983), and the idea of the vector equivalence relation with 'action-effect' approach (Watson et al., 2003; Watson, 2004) show the difference between physical embodiment and mathematical abstraction. Roughly speaking, in physical embodiment, not only the vectors and all the physical settings around the vectors, but also the interaction between the vectors and the physical settings are important. In mathematical abstraction, vectors are not objects attached to any physical settings. Assuming a difference of mathematical vectors and physical vectors, I first set up two categories: mathematical vectors and physical vectors. Direction of intended movement is the direction of the development towards mathematical abstraction wanted from students.

The first layer positions the focus of my study on the understanding of the construct of vectors from a mathematical point of view. Even though this layer is for differentiating mathematical abstraction from physical embodiment, I do not assume the empty intersection of these two categories. The non-empty intersection will be discussed further in Chapter 5. For now I call it as a transition 'point' from physical vectors to mathematical vectors to emphasize other non-intersecting parts under the assumption of the direction of intended movement. This transition

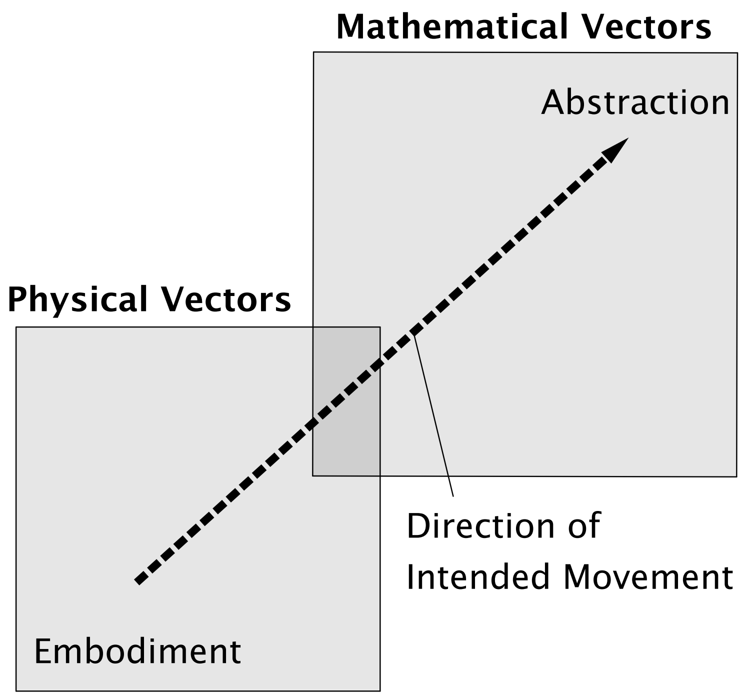



Figure 2.2: First Layer

point serves as the origin of the configuration.

For the next step, I tried to consider multiple perspectives on the direction of intended movement. I could think of two different perspectives of the development: representational and cognitive. These two perspectives come from the ideas of the meaning of multiple representations (Kaput, 1987a; Lesh et al., 1987; Goldenberg, 1995; Johnson, 1998) and Modes of Description (Hillel, 2002) vs. Modes of Thinking (Sierpinska, 2002). With the assumption of direction, I placed these two perspectives as ontological axis and epistemological axis on the plane of the configuration.

### 2.2.2 Second Layer for Multiple Perspectives

The second layer describes the difference between the representational and the cognitive perspectives on vectors as the difference between the ontological and the epistemological perspectives. See Figure 2.3.

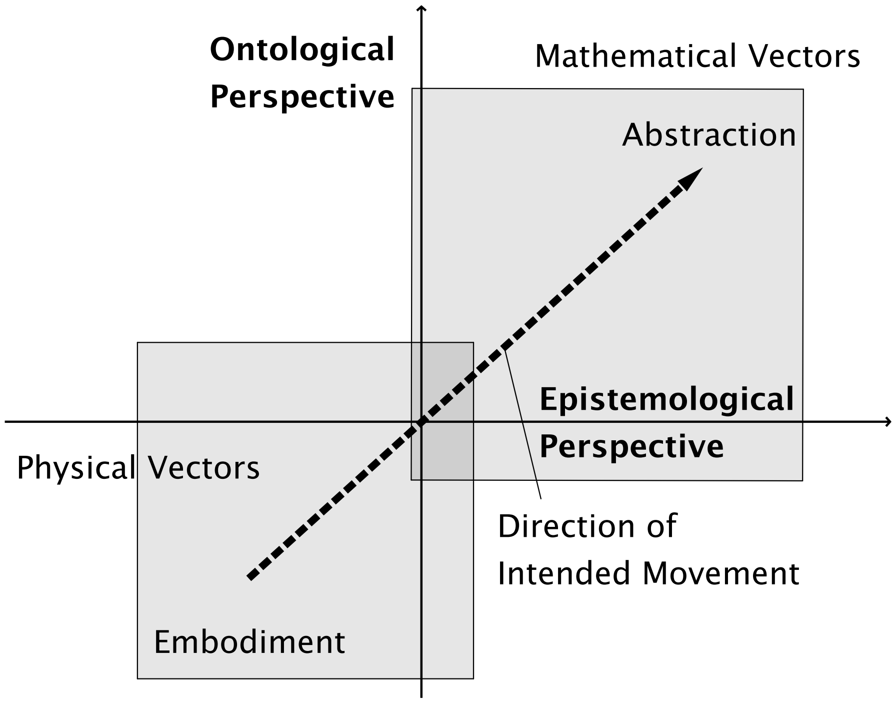



Figure 2.3: Second Layer

The origin represents an important transition from physical vectors to mathematical vectors. With this refinement, we can see that ontologically, the origin is a shift of a view from vectors as representations of physical quantities with physical units to vectors as representations of mathematical objects, i.e., a loss of physical appearance such as units, physical directions attached to physical objects. We can also see that epistemologically, this origin is a shift related with understanding of mathematical equivalence relations, i.e., a loss of physical embodiment context such as a translation as a displacement of an object, etc. Now, the first quadrant of the configuration is for mathematical vectors and the third quadrant is the realm of physical vectors. The focus of the rest of this dissertation will be on the first quadrant of this layer.

The direction of each axis implies not only one intended movement towards mathematical abstraction, but also changes in two different perspectives: a representational change and a cognitive change. To talk about translations/conversions from one representation to the other, these two changes are important.

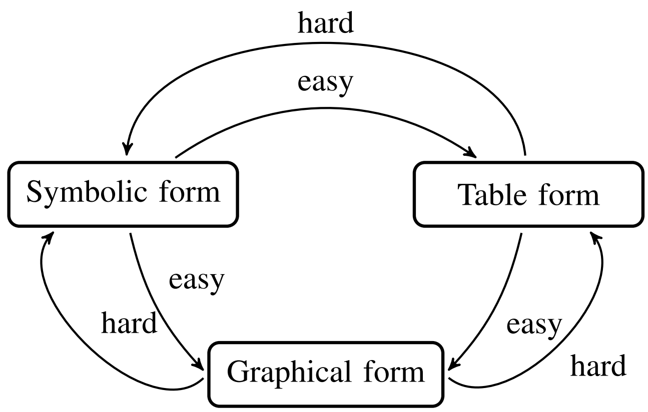



Figure 2.4: Conversion among Registers of Vector Representation

Many studies on the multiple representations of functions compare three different representations of functions such as graphical, table, and symbolic representations (Kaput, 1987b,a; Lesh et al., 1987; Janvier et al., 1993; Goldenberg, 1995). Similarly, Pavlopoulou's work (as cited in Artigue et al., 2002) distinguishes between three registers of semiotic representation: the graphical register (arrows), the table register (columns of coordinates), and the symbolic writing register (axiomatic theory of vector spaces). They talked about translations/conversions only in terms of the representational change.

From Chapter 1, we also saw that Hillel (2002) and Sierpinska (2002) talked about conversions from one representation to the other without any hierarchy of sophistication. Pavlopoulou's work (as cited in Artigue et al., 2002) covered this missing hierarchy of Hillel(2002) and Sierpinska(2002) as an application and verification of Duval (2006)'s theory of semiotic representation in the context of teaching linear algebra. Pavlopoulou shows that in a restricted setting such as three registers in two- or three-dimensions, certain conversions are not easily achieved by students (Figure 2.4).

In the light of above studies, I found needs for a framework that covers both representational perspective and cognitive perspective, and a framework that assumes a certain type of a hierarchy despite Pavlopoulou's the graphical register was not the full substitute for the geometric representations in the configuration.

As we have already seen in Chapter 1, this type of the hierarchical structure and the existence of the levels of sophistication cannot be understood just by the representational perspective or by the cognitive perspective. To talk about vectors using both perspectives together, I put both axes on the second layer as a refinement of the first layer. The ontological axis is a directed dimension that covers changes in existence or being. It is related with explicit change of representations or notions. Geometric or symbolic representations of vectors are matters of an 'ontological difference'. Similarly, concrete vs. abstract characteristics are also related with the ontological aspect of vectors. The epistemological axis is a directed dimension that covers changes in knowledge or thinking. It is related with understanding of representations. The difference between arithmetic thinking (using numbers) and algebraic thinking (using letters), the difference between analytic thinking (procedural) and synthetic thinking (structural) can be described with this axis. Using an ontological axis and an epistemological axis provides a way to locate and see a vector as a multifaceted mathematical entity in college mathematics and help illustrate a combined view of the representational and the cognitive perspective on vectors.

While talking about the need for a new framework with a combined view of the representational and the cognitive perspectives in Chapter 1, I pointed out the observation of student's possible preference of one representation to the other in a specific setting or for a purpose (Hillel, 2002). I also mentioned another possible refinement of the construct of vectors. In the case of vectors, this observation of the preference can imply more refined layer to see the complexity and subtlety of the construct of vectors.

### 2.2.3 Third Layer for Transitions towards Abstraction

The third layer identifies the representational and the cognitive obstacles in terms of transitions towards abstraction. The prevalence mentioned in the previous discussions could uncover certain difficulties in understanding the construct of vectors. Those difficulties could be realized as obstacles with respect to the ontological and the epistemological perspectives. See Figure 2.5. In the domain of mathematical vectors, each axis was hypothesized to have two important transitions that can be identified in the configuration. On the epistemological axis, there are (1) one from arithmetic to algebraic, and (2) one from analytic (procedural) to synthetic (structural). On the ontological axis, there are (1) one from geometric to symbolic, and (2) one from concrete to abstract. Among those four transitions, we will only focus on two transitions: analytic to synthetic and geometric to symbolic in the third layer.

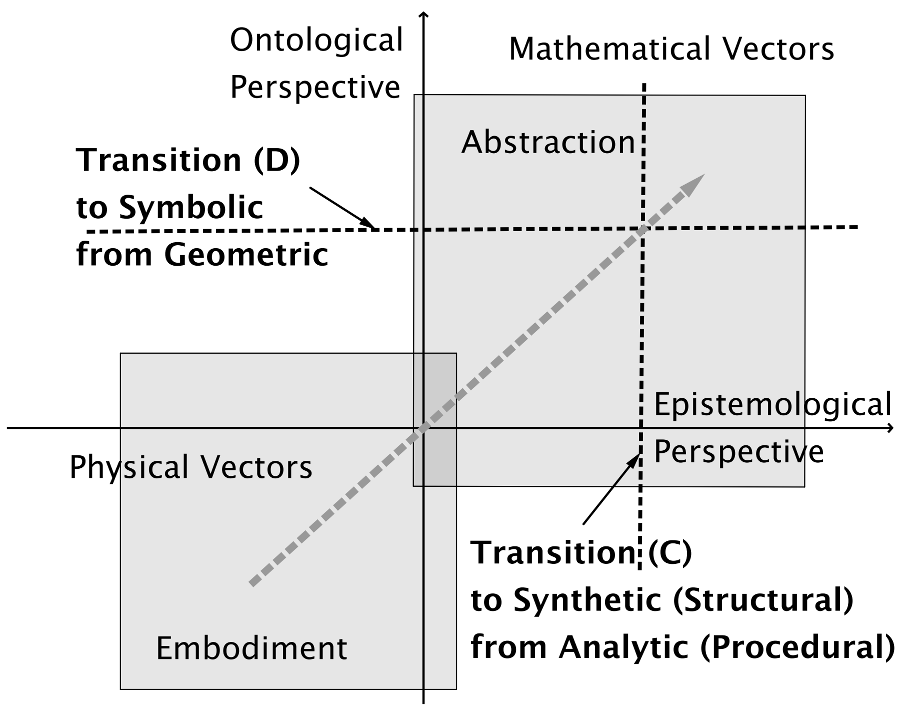



Figure 2.5: Third Layer with two major transitions

The main transition for the discussion of vectors in the mathematical domain is Transition (A) from physical to mathematical vectors. Along with progressive refinements, the difference between physical vectors and mathematical vectors is getting clearer with additional layers.

From the first and second layers, we can see what I will call five transition points in the configuration.

**Transition (A) from physical to mathematical**: Transition (A) is a transition from physical understanding of a motion to mathematical understanding of a motion. Understanding of the vector equivalence relation in arrow forms is critical in this transition. Without understanding of mathematical equivalence relation, one cannot understand the concept of free vectors or mathematical position vectors. Those vectors are very important for the ontological shift from a geometric representation to a symbolic representation. Transition (A) can be also the transition from vectors as the representations of physical quantities that have both physical directions and magnitudes to vectors as mathematical objects. In physics, vectors are attached to physical objects to give information related with physical quantities affecting these physical objects. However in mathematics, the focus is shifted from the physical objects to the vectors themselves. The vectors do not carry physical units or directions any more.

Next, I wish to define the following two transitions along the epistemological axis.

**Transition (B) from arithmetic to algebraic**: Transition (B) is a transition from thinking with numerical values to thinking with variables. This transition is shown up early in elementary and middle school mathematics. Even though it is shown up very early, it is important to see this transition is critical to college students when they confront variable thinking in higher-level mathematics with the context of vectors (Usiskin, 1988).

**Transition (C) from analytic (procedural) to synthetic (structural)**: Transition (C) is a transition from analytic thinking to synthetic thinking. It also includes a transition from procedural thinking to structural thinking in cognitive development such as reification or process-object encapsulation. In the original research of reification or process-object encapsulation (Sfard, 1991), this transition is introduced as an ontological shift with more focus on new names and new symbols. However, in the context of vectors, it is not just a change of names or representations but also a change of thinking. For example, a geometric vector (an arrow) can be studied both analytically and synthetically. Switching back and forth on the ways of thinking and investigating with geometric vectors are epistemological shifts. See Figure 2.6.

The following two transitions are along the ontological axis.

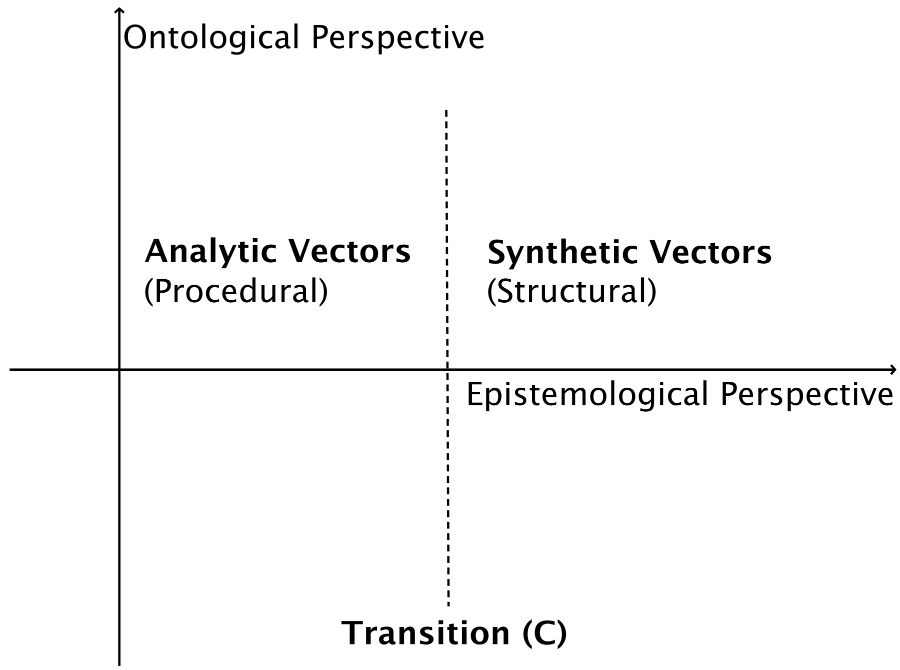



Figure 2.6: Transition (C) from analytic (procedural) to synthetic (structural)

**Transition (D) from geometric to symbolic**: This transition is a transition of explicit representations from geometric to symbolic. While this transition often entails a simple change of appearance of vectors, it can give rise to complex issues for vectors. In the high school level, with standard basis vectors, the coordinates for the terminal points of position vectors will automatically be the coordinate forms of vectors or become column vector forms. However, in the undergraduate mathematics, this transition includes understanding the difference between a vector space and an Euclidean coordinate space. The coordinates of the points will not directly correspond to the representations but the component-wise coefficients of basis vectors on linear combination of basis vectors. This can be one of the most important and hardest transitions in the undergraduate mathematics. See Figure 2.7.

**Transition (E) from concrete to abstract**: Transition (E) is a transition from tangible dimensions such as 2D and 3D to intangible dimension such as 4D and above. Transition (E) is somewhat easier than the rest when restricted to symbolic extensions from 2D or 3D to arbitrary dimension (i.e., appending one more component on the representations). However, it can be also a complicated

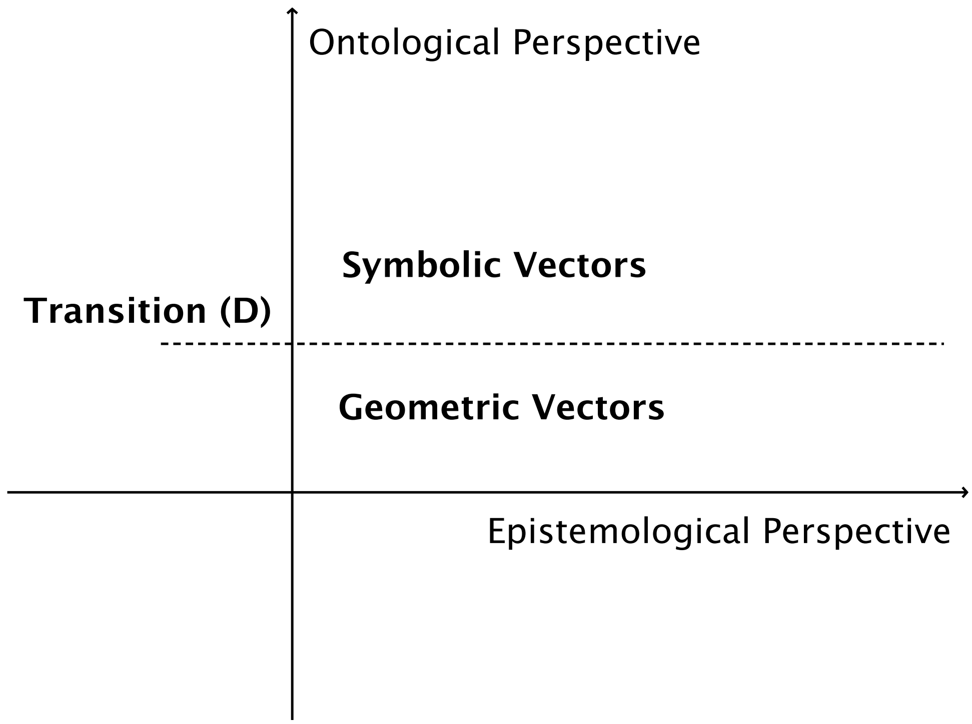



Figure 2.7: Transition (D) from geometric to symbolic

transition in the sense that a definition and the consequence of the definition can be switched. For example, in 2D or 3D, both geometrical and symbolic representations assume that the magnitudes of two vectors determine the inner product of two vectors ($\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\theta$). In 4D or above, the inner product determines the magnitude of a vector. As such it is a big transition from concrete concepts to abstract concepts of vectors.

A detail descriptions and extended discussions about the configuration are in an earlier work (Kwon, 2011). I also want to let readers know that I postponed the empirical studies on the transition (B) and (E). This is because the transition (B) from arithmetic to algebraic occurs early in elementary and middle school mathematics (Usiskin, 1988), and transition (E) from concrete to abstract seems a simple extension of the number of entries once transition (B) is done. In this dissertation, we only deal with the three major transitions (A), (C), and (D).

## 2.3 Features of the Configuration

This new framework has important features that drew the distinction between itself and other frameworks. These features will be revisited and hypothesized as research focus in the Chapter 3.

First, the configuration shows the difference between physical vectors and mathematical vectors emerged from the first layer. It also illustrates interplay between the ontological and epistemological perspectives in the construct of vectors. These multiple perspectives enable us to talk about the difference between analytic (procedural) and synthetic (structural) thinking of vectors, and the difference between geometric and symbolic representations of vectors through the second layer. The configuration also shows the analytic approach and the synthetic approach towards mathematical abstraction described by the second layer. In terms of the configuration, I can describe the analytic approach as a trend of changing explicit representations along the ontological axis quickly from geometric representations such as arrows, to symbolic representations such as coordinate/column vector forms. Epistemological development is postponed and following symbolically after the ontological change. This analytic approach in the configuration can be illustrated as an upside down 'L' shape route (Γ) towards abstraction. The synthetic approach is defined as the trend of changing the views/thinking of a geometric representation while maintaining arrow forms, from analytic (procedural) to synthetic (structural) first. Ontological development is postponed until the achievement of the change in epistemological perspectives such as from analytic (procedural) to synthetic (structural). This approach in the configuration marked as a reversed 'L' shape route (⅃) towards abstraction. [2] See Figure 2.8.

The transition points on the configuration are identified as representational and cognitive obstacles by the third layer.[3] We can see levels in each ontological and epistemological perspective,

[2]Analytic/synthetic distinction in Transition (C) is characterized by a focus between procedure and structure. This is not directly related with the analytic approach and synthetic approach defined here.

[3]As we have mentioned in section 1.1, representational obstacles are things that prevent students to translate/convert one form of representations to the other. Cognitive obstacles are things

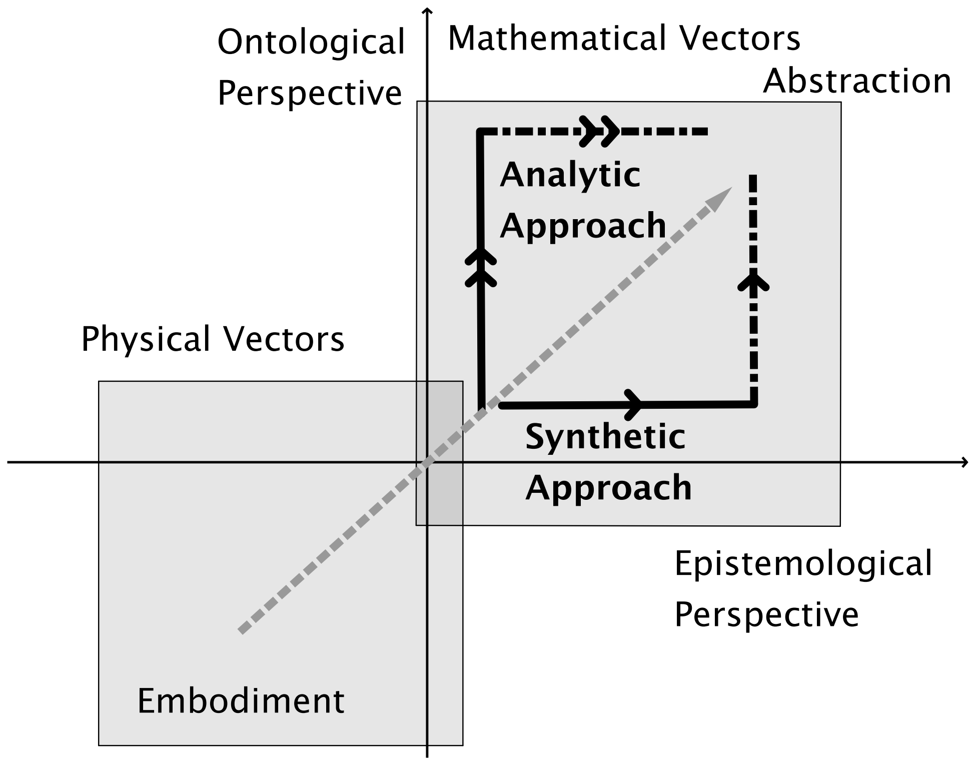



Figure 2.8: Analytic and Synthetic Approaches from the Third Layer

especially cognitive development along with Action-Process-Object-Schema and/or process-object duality in geometric representations emphasized in the configuration. Such systematizing transitions among representations of vectors with the ontological and epistemological perspectives is an important feature of this configuration that was not realized before in other frameworks.

In the next chapter, I will discuss the method for gathering evidence from student data to see if this layered view of the construct of vectors as a result of progressive refinements is reasonable. Questions and interviews will be designed to investigate student work to find evidence of these features of the configuration. Those are (1) Transition (A) from physical to mathematical, (2) Transition (C) from analytic (procedural) to synthetic (structural) and Transition (D) from geometric to symbolic, (3) the prevalence of the analytic approach to the synthetic approach while

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that avoid students to proceed cognitive development such as encapsulation or reification.

developing abstraction, and (4) process-object duality in geometric representations.

# Chapter 3

# EMPIRICAL STUDIES FOR THE NEW FRAMEWORK

Four empirical studies were used to gather data of ninety-eight students to show evidence of three major transition points in the new framework: Transition (A) from physical to mathematical coming from the first layer, the prevalence of the analytic approach to the synthetic approach while developing abstraction enlightened by the second and the third layers, Transition (C) from analytic to synthetic; and Transition (D) from geometric to symbolic coming from the second and third layers, and process-object duality in geometric representations. The design and construction of the surveys as well as interviews reflect the emphasis of my research on gathering evidence in student work for confirmation of those focuses.

## 3.1 Research Focuses

This series of empirical studies is highly exploratory and formative. The focus is on evidence in student data of the following features that are hypothesized by the configuration of the construct of vectors:

(1) Transition (A) from physical to mathematical in terms of the configuration: Student data will show evidence of the possible differentiating mathematical abstraction from a physical embodiment that was described as a global difference between physical and mathematical vectors in the first layer of refinement. This data will also show evidence of the interplay between the ontological and epistemological perspectives to help distinguish mathematical vectors from physical vectors when thinking of the second layer added to the first.

(2) Transition (C) from analytic (procedural) to synthetic (structural) and Transition (D) from geometric to symbolic: Student data will show evidence of the difference between the representational and the cognitive perspectives, and how those different views combined and

interplayed as the ontological and epistemological perspectives of the construct of vectors in the second layer. The student data also will show evidence of the epistemological obstacle (C) and the ontological obstacle (D), and confirm those obstacles as transition points towards mathematical abstraction in terms of the second layer together with the third layers. Process-object duality on geometric representations also will be identified from student data.

(3) The prevalence of the analytic approach to the synthetic approach: Student data will show evidence of the prevalence of the analytic approach to the synthetic approach as described in the extended discussion of the second and the third layers while developing mathematical abstraction. This will also serve as a reasonable clue that of the progression of refinements to identify obstacles in transitions/conversions on the construct of vectors.

The purpose in collecting data from four different surveys and interviews is (1) to determine if transition points are evident in student work, (2) to identify the developmental evidence in student work suggested by the configuration, and (3) to examine likely dominance of the analytic approach to the synthetic approach towards mathematical abstraction implied by the configuration.

## 3.2 Method and Participants

Four surveys and interviews were carried out to gather evidence on the important features suggested by the configuration. The results reported in Chapter 4 are a synthesis of the data gathered with the focus on the three major transitions (A); (C); and (D), and the prevalence of the analytic approach to the synthetic approach. Multiple administrations were used to: (1) test appropriate survey questionnaire and interview process, (2) gather deeper knowledge of student background and idea on vectors, and (3) modify surveys and interviews in order to avoid any confusion derived from the questions. All data were collected from students located in the midwest public university.

The first empirical study on vectors was done during the beginning of the 2010 Fall Semester. Five pre-service secondary mathematics teachers from the senior level ‘Reflection and Inquiry in

Teaching Practice' course participated in the survey, and three of them were interviewed after finishing the surveys.

The second empirical study was conducted during the 2010 Fall Semester. A task-based survey developed and revised based on the first study was given. This survey asked for student backgrounds on vectors in addition to the survey questions. Twenty-nine students from the senior level 'Capstone in Mathematics' course for pre-service secondary mathematics teachers participated. After the survey, four students were selected for follow up interviews to probe deeper connections of the configuration and the actual student work.

The third study was carried out in the 2011 Spring Semester. Fifty-eight pre-service elementary teachers were examined. These pre-service elementary school teachers were taking the 'Elementary Geometry For Teachers' course. The main focus of the course was geometry and measurement needed for K-8 teaching. The textbook was 'Mathematics for Elementary Teachers' by Sybilla Beckmann (Beckmann, 2005). This course consisted of four big topics: Geometry (of 2D, 3D objects), Geometry of Motion and Change, Measurement, More about area and volume. Students in this course already studied the first three sections of Geometry of Motion and Change that covered various geometric transformations such as translations, reflections, rotations, and glide reflections.

The fourth study was carried out in the 2011 Summer Semester. Six pre-service secondary teachers from 'College Geometry' course were participated. No interviews were conducted.

After administering each survey (I and II), students were selected for interviews based upon their response. The selected students signed up for an one-hour block of time for their interviews on the day and time that was most convenient for them. Interviews were held in a neutral location away from the students' classrooms and were audio-recorded for further analysis. Transcribed interviews were coded and analyzed in order to find evidence in student work. Descriptive statistics were used for the second study and the third study on which a sufficient number of participants were available.

## 3.3 Design and Construction of Survey and Interview

The following Table 3.1 shows the overview of my design of the survey questions. Each question is connected to the focal transition (A), (C), and (D). Survey III is an expansion of ‘Translation’ question in Survey I, II and IV. Survey III and Survey IV also have additional questions not included in this table that are off the focus of the study. Because Transition (A) has two different perspectives: epistemological and ontological perspectives derived from the second layer, I distinguished the questions that examine the epistemological and ontological perspective each by marking $\rightarrow$ and $\uparrow$. By marking Γ, and L, I also differentiated the analytic approach related questions from the synthetic approach related questions. This table also has an indicator V’ for the evolution of the original question V. The detail evolution is in the following section.

| | Transitions | | | | | Surveys | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Question Name** | **A** | **B** | **C** | **D** | **E** | **I** | **II** | **III** | **IV** |
| Translation | X(→) | | | | | V | V’ | | V’ |
| Translation of Polygon | X(→) | | | | | | | V | V |
| Geometric Translation | X(→) | | | | | | | V | V |
| Rainy Day | X(→, ↑) | | | | | V | | | |
| Force | X(→) | | | | | | V | | V |
| Robot Arm | X(↑) | | | X(Γ) | | V | V | | |
| Origin | | | | X(Γ) | | V | V | | V |
| Basis | | | | X(Γ) | | V | V | | |
| Rotation | | | X(⅃) | | | V | V | | |
| Polygon | | | X(⅃) | X(Γ) | | V | V | | V |
| Very Long Sum | | | X(⅃) | | | V | V | | V’ |
| Cube | | | X(⅃) | X(Γ) | | V | V’ | | V’ |
| △ Midpoints | | | X(⅃) | X(Γ) | | V | V | | V |
| Associativity | | | X(Γ) | X(⅃) | X(⅃) | V | V | | V |
| Point and Vector | | | X(⅃) | X(Γ) | | V | V’ | | V’ |

Table 3.1: Map from questions to configuration

For the in-depth discussion of the research focus in the next chapter, I chose the following questions from Table 3.1 and synthesized the results. See Table 3.2.

Interviews were conducted mainly by repeating the survey questions and asking further about

| Key Features | Related Layers | Transitions | Questions |
|---|---|---|---|
| Physical vs. Mathematical | I | A | Translation, (Translation of Polygon), (Geometric Translation), Rainy Day, Robot Arm |
| Epistemological Diff. & Ontological Diff. | II | A, C<br>A, D | Translation, Polygon<br>Rainy Day, Robot Arm |
| Epistemological Obst. & Ontological Obst. | III | (A), C<br>(A), D | Polygon, Very Long Sum<br>Origin, Robot Arm |
| Process-Object Duality in Geometric Representation | III | C | Very Long Sum |
| Prevalence of Analytic Approach | II & III | C, D | Cube, △ Midpoints, Associativity |

Table 3.2: Questions for Layered View of Configuration

Which vector representations do you want to use to prove the theorem, coordinate/column vector forms such as $(a,b,c)$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or generic vector forms such as $\overrightarrow{AB}$, $\overrightarrow{u}$?

(b) Proceed your proof with the preferred vector representation.

(c) Why have you preferred this representation and not another one?

(d) Prove with the other vector representation that you did not use if you can.

(e) What difficulties are you faced with, for each method? Namely, what forced you to abandon a representation, if anything like that has happened, or what has prevented you in reaching a final proof?

Table 3.3: Example of Interview Questions

what the students were thinking. Because questions for examining the prevalence of an approach to the other in the surveys were only asked to students to choose the representations, I asked students to proceed and finish the proof in the interview sessions. See Table 3.3. This helped me to see the deeper thought process of students.

## 3.4 Evolution of Survey Questions

Due to the complexity of the vector construct, a single question can never achieve any clear view of a single facet. By grouping them with respect to progressive layers of refinements and transition points related with each layers, I can discuss the construct of vectors in a succinct manner while still reflecting the complexity and subtlety. Originally, questions were intended to explore all five transitions in the configuration. However, it was hard to develop appropriate questions to explore the transition (B) from arithmetic to algebraic vectors. College level mathematics usually assumes students who finished this developmental transition earlier in their elementary and middle school mathematics. In case of symbolic vectors, this transition simply means changes from numbers to letters in entries. Inquiry for Transition (E) was also postponed because the main focus of this study was vectors in geometry, and the transition was regarded as the one related with symbols in linear algebra. Further discussion about Transition (E) will be on Chapter 5 as a limitation.

The first study was intended to test appropriate survey questionnaire for investigating the whole configuration with focus on the three transitions. The follow up interviews informed modifying survey questions so that questions would more effectively capture student work. This portion of the study had two goals. One goal was to look into evidence of three important transitions (A), (C) and (D). The other goal was to identify evidence of two different approaches: analytic and synthetic approaches towards mathematical abstraction in student work. Locating student work in the related transitions in the configuration as evidence of developmental obstacles and exploring different approaches from the data using descriptive statistics were also concerns of this study. See Table 3.2.

In the second study, background information such as college major, high school courses related with vectors, college courses related with vectors, etc. was gathered to examine any correlation between questions and student background information. Twelve vector questions were reviewed and revised based on the first study. ‘Translation’, ‘Cube’, and ‘Point and Vector’ questions were modified for a better understanding of the student work and for avoiding confusion while the students attempted surveys. For example, here is the original cube diagonal and its modification. The first survey asked the students to prove that the two vectors shown in the cube were perpendicular. Some students see two vectors in the cube as they were literally intersecting and perpendicular to each other. To avoid this confusion, I changed the figure and made the two vectors not intersect. See Figure 3.1.

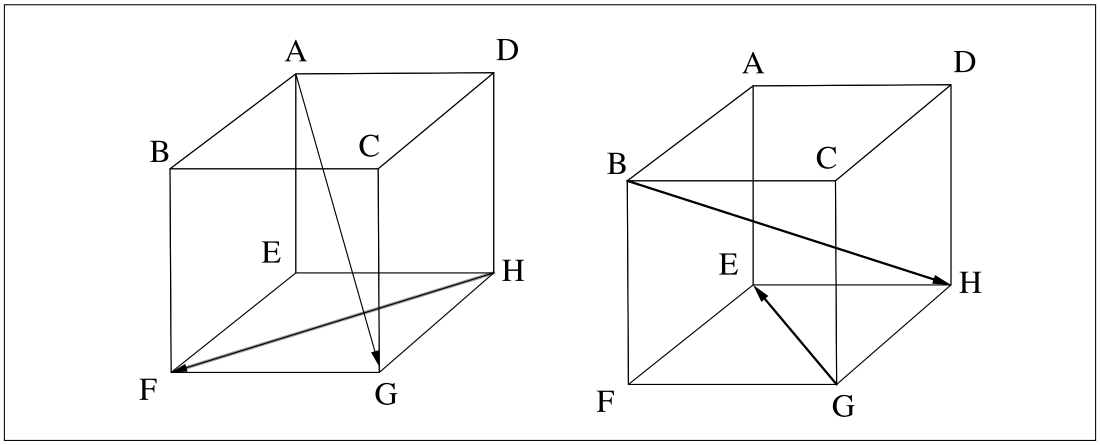



Figure 3.1: Modified Cube Figure in Survey II

‘Rainy Day’ question was dropped after survey I, because most responses utilize physics knowledge more than mathematics knowledge. While developing three layers of refinements, I emphasized mathematical vectors over physical vectors. By dropping ‘Rainy Day’ question, I made students concentrate on mathematical vectors with their responses.

‘Force’ question was introduced for the first time in survey II to set up a different physical situation other than a motion-related situation. This gave additional examples of epistemological translation (A) other than ‘Translation’ that only covers a motion.

The third study focused on a single specific topic of geometry: a vector as a geometric translation. The first and second studies showed enough evidence of student use of physical vectors in mathematical context. The difference between physical vectors and mathematical vectors in the configuration was evident from the student responses. However I sought further refinement in the data with a population where the differences are most likely to exist. The focus of this study was how representations were related with cognitive development. By refining the notion of vector representations and by integrating different cognitive development theories, I tried to probe more about transition (A).

In the fourth study, several questions from the first, second and third studies were selected. ‘Robot Arm’ question was also dropped and some translation questions from survey III were added instead. ‘Basis’ and ‘Rotation’ questions were dropped because they were heavily dependent on Linear Algebra knowledge and provided little opportunity to distinguish or differentiate student thinking relative to framework. ‘Very Long Sum’ question was simplified to only address a sum structure based upon previous results.

Chapter 4 shows synthesis of results and analyses from the above four studies. This presents a summary of the collected data, statistical treatments of the data, an interpretation of survey data, and excerpts of the interviews dedicated to reflecting conceptions to the configuration. Mechanics of analysis and findings are also discussed.

# Chapter 4

# FINDINGS AND DISCUSSIONS

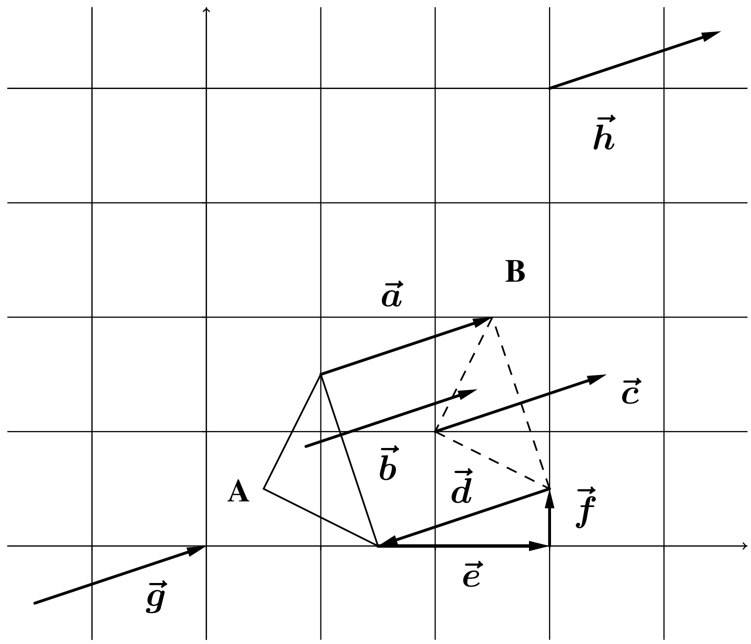
Collective results from the four empirical studies consisting of written surveys and interviews are used to look into evidence of focal points that we assumed and hypothesized in process of refinements. In summary, student data showed evidence of (1) the global difference between mathematical vectors and physical vectors, (2) interplay between the ontological and epistemological perspectives, (3) the epistemological obstacle defined as Transition (C) and the ontological obstacle defined as Transition (D), (4) the prevalence of the analytic approach to the synthetic approach, and (5) process-object duality on geometric representations. I present evidence of each following the refinement process for the construction of the configuration that was described in Chapter 2.

## 4.1 From Physical to Mathematical Vectors

From Chapter 1, we saw that it is hard to describe the complexity and subtlety of a vector as a translation only with the classical vector representations. The layered approach to the construct of vectors enabled us to differentiate mathematical abstraction from physical embodiment in Chapter 2. I hypothesized that there existed a transition from physical vectors to mathematical vectors in the configuration that could be described from both epistemological and ontological perspectives. In this section, I begin with student evidence that shows the differentiation of mathematical vectors from physical vectors is possible as I classified in the configuration.

The global difference between mathematical abstraction and physical embodiment is evident in student work. Student data showed evidence of the different interpretations between 'the same translations' and 'the equivalent vectors'. We usually assume that the concept of the vector equivalence relation in physics is the same with that in mathematics, because 'directions' and 'magnitudes' of vectors are used to verify equivalent relations in both fields. This means that the equivalent vectors are always representing the same translations and vice versa both in physics and

**Translation:** A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point $P$.




(a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

(b) List all vectors that are equivalent to $\overrightarrow{a}$.

Figure 4.1: Translation Question

mathematics. However, student work for 'Translation' question (Figure 4.1) shows the difference of the interpretation between the same translations and the equivalent vectors.

In the second study, the cluster tree diagrams (Figure 4.2 and 4.3) from hierarchical clustering with Euclidean distance are supposed to show similar categorizations assuming 'the same translations' and 'the equivalent vectors' are the same concept. The first cluster tree was made from the student responses to question (a), and the second cluster tree was made from those to (b). They show two different categorizations in Figure 4.2 and 4.3.

Even though geometric translations are not representing any physical quantities, students seem to understand them through physical embodiment. They connect geometric translations to physical

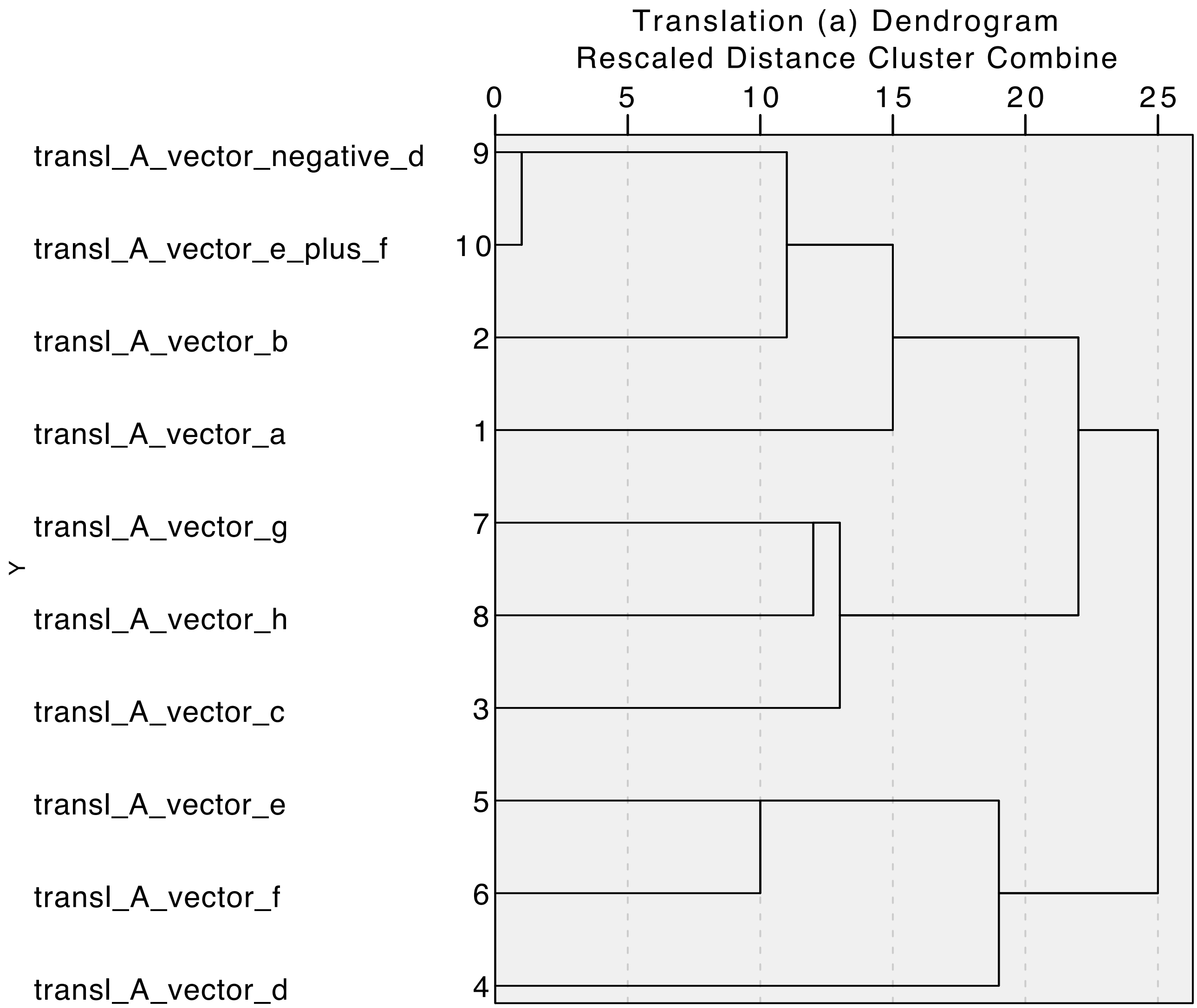



Figure 4.2: Cluster Tree of the same translations

movements of geometric objects, and restrict their knowledge on vector equivalence relation to equivalency of physical quantities.

This result of a quantitative analysis tells us that there is evidence of the difference between mathematical vectors and physical vectors hypothesized in the configuration. This difference comes from the students' ideas of the vector equivalence relation. However, it informed us only of a cognitive difference assuming the same form of the representation such as arrows. Could there be an element other than a cognitive element that differentiate mathematical vectors from physical

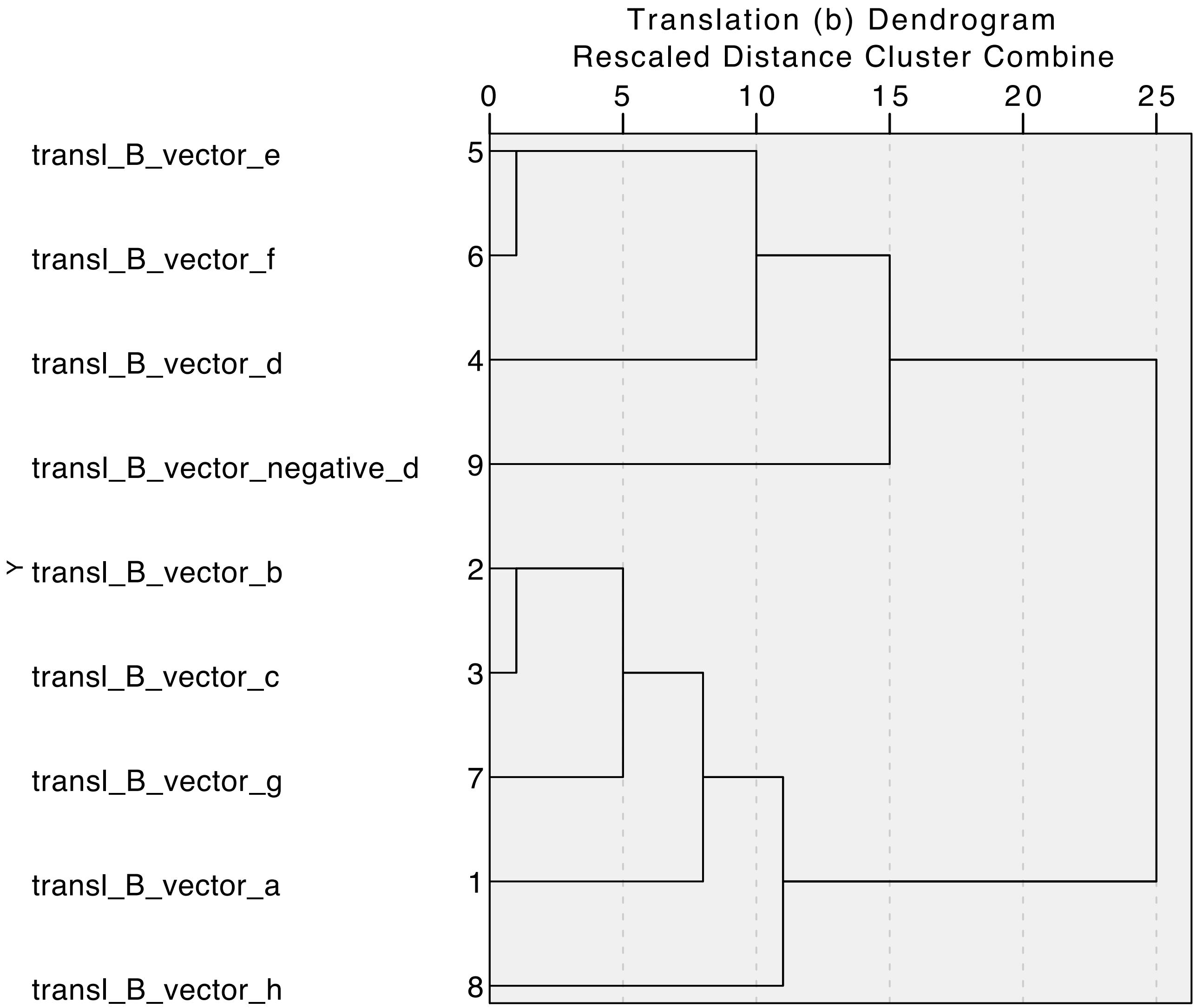



Figure 4.3: Cluster Tree of the equivalent vectors

vectors? In response to this inquiry, I introduced another layer of refinement for the construct of vectors in Chapter 2. With this refinement, we could assume there existed two different perspectives: epistemological and ontological perspectives when we discuss the complexity and subtlety of the construct of vectors. The difference between mathematical and physical vectors from the epistemological perspective (cognitive change) was evident in student work. This is a result from the quantitative analysis above. In the next section, I will seek evidence of this ontological difference (representational change from physical to mathematical) by looking at individual student

**A Rainy Day:** The rain is falling straight down on your head with speed 1 m/s. You start walking forward with speed of 2 m/s and notice the rain is now hitting your face. What is the angle that the rain now appears to make with the vertical? Draw a picture that explains your thinking.

Table 4.1: A Rainy Day Question

data.

## 4.2 Epistemological and Ontological Perspectives

Student data showed evidence of the difference between mathematical vectors and physical vectors in the previous section. However, it only showed the epistemological difference. In this section, I examine student data to see if there is an ontological difference between physical and mathematical vectors. In Chapter 2, this ontological difference between physical vectors and mathematical vectors was described and hypothesized as a change from vectors accompanied by physical units for representing physical quantities to vectors as self-standing mathematical objects. Translating physical directions and magnitudes to mathematical directions and magnitudes, on a coordinate system or on a coordinate free Euclidean (Affine) space, or possible dropping of physical units from vectors were those examples.

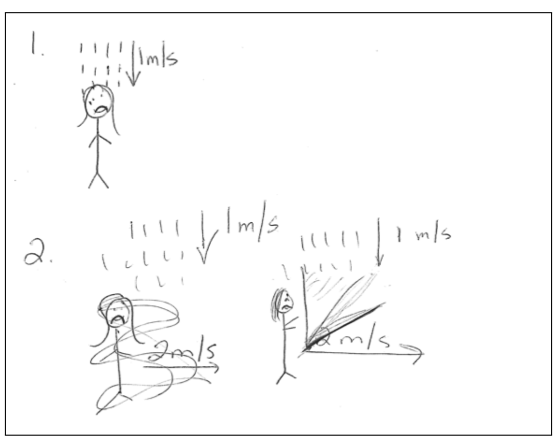



Figure 4.4: Construction of a Diagram

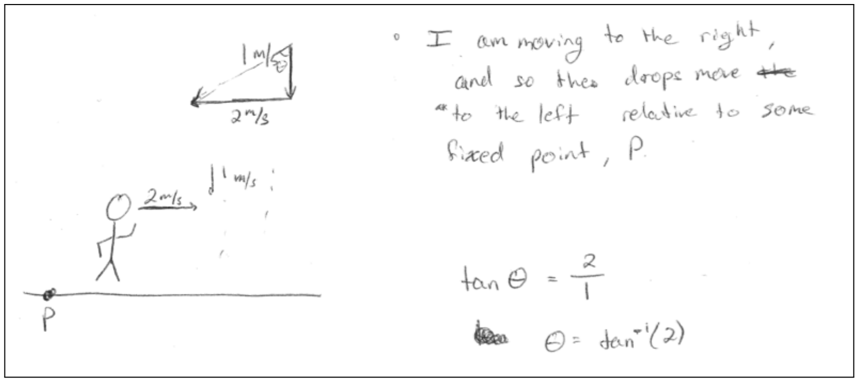



Figure 4.5: Student response based on vectors with no evidence of symbolic vector addition

Student work from the ‘Rainy Day’ question showed evidence of this ontological difference between mathematical and physical vectors. This difference was not shown as a clear cut of two opposite examples that have no intersection. Instead, it was realized as a gradual change in spectrum of responses. Because the ‘Rainy Day’ question itself was designed to see specific instances of student work, I provided representative and qualitative evidence of student work.

Figure 4.4 shows student’s work that exhibits no constructions from the vector addition. This student provided only the descriptions of moving directions and speeds in a physical situation. Each numbering of the situation shows the unfinished effort moving towards abstraction. The student in Figure 4.5 showed evidence of a different situation that the student knew the solution vector should be from the triangle vector sum. We are not sure if that is from the actual symbolic operation of the vector addition. Student work in Figure 4.6 shows evidence of another situation. This student responded with both geometric and symbolic vector sum, and showed an example of vectors as self-standing objects in mathematics by translating relative speed of raindrops from the walking speed to $\overrightarrow{v_1} + \overrightarrow{v_2}$. These three examples illustrate a gradual change in spectrum of responses from physical vectors to mathematical vectors with a loss of physical appearance attached to physical objects and with developing symbolic calculations.

In other responses to ‘Rainy Day’ question, I saw several cases regarding arrow forms as line segments and not dealing with any vector operations. Figure 4.7 shows one of those examples. The construction of a triangle figure was independent of the triangle method of vector addition and the

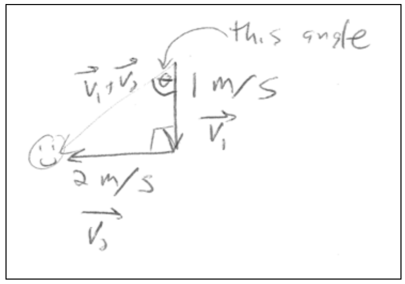



Figure 4.6: Student Response based on geometric and symbolic vector addition

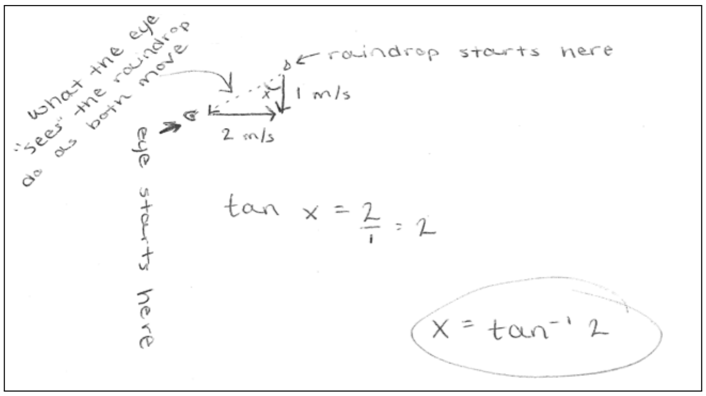



Figure 4.7: Student Response based on geometry without vector addition

student just used vectors as sides of the triangle. This example shows an epistemological change hidden on arrow forms of vectors. From the above example, we can see that when talking about the gradual ontological change in spectrum of student responses, we should consider the hidden epistemological change too. To discuss the interplay between the epistemological perspective and the ontological perspective is therefore reasonable.

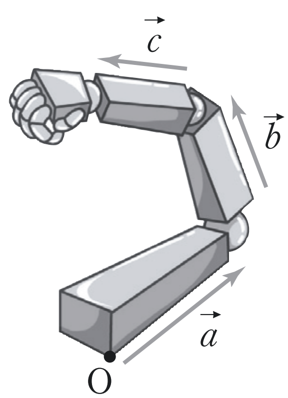
### 4.2.1 Interplay between Epistemological and Ontological Perspectives

The previous responses shows evidence of the difference between the epistemological and ontological perspectives with the difference between epistemological changes and ontological changes on the second layer of refinement. However, we also noticed that it was really hard to see evidence of the difference solely from one perspective. This is because the ontological perspective does not always assume the representational change i.e., change in explicit forms such as arrows to coordinate forms. In practice, physics uses very similar forms of representations with mathematics.

Changes within a representation with respect to ontological perspective are always hard to see, and this is why we need to put an emphasis on interplay between the epistemological and ontological perspectives. Figure 4.7 is an example that shows interplay between a cognitive change (epistemological change) and a representational change (ontological development towards mathematical abstraction).

The interplay between the ontological and epistemological perspectives is also evident in the following example. In the 'Robot Arm' question, numerical information given in the question was not consistent with the pictures if one read the picture physically (Figure 4.8).

**Robot Arm:** A robot arm has three joints as shown. The bottom part can be represented by a vector $\overrightarrow{a} = (10, 15, 20)$, the middle part can be represented by a vector $\overrightarrow{b} = (5, -5, 15)$, and the top part can be represented by $\overrightarrow{c} = (20, -5, -10)$. The units are in centimeters($cm$).




Find the vector $\overrightarrow{p}$ that tells the location of the hand.

Figure 4.8: Robot Arm Question

One of the responses with the correct 3-tuple answer said, "It's a little difficult to reconcile this diagram with the given vector values. (i.e., $\overrightarrow{c}$ seems to go up, so why does $z = -10$ ?)" It was about the ontological perspective (comparison of representations) together with the epistemological perspective (interpretation of directions and coordinate axes). In physical view, vector $\overrightarrow{c}$ is *going up* so that the $z$-value should be positive not negative assuming $x$, $y$, $z$-coordinate system of a physical space where positive $z$ represents a space above $xy$ horizontal plane. In mathematics, the coordinate system is separate from physical locations or directions. Positive directions of axes can be anywhere. This student work shows the difference between mathematical and physical vectors from the first layer view as previously discussed. This difference can be discussed as well in more detail as above with the second layer that provides multiple perspectives and their interplay.

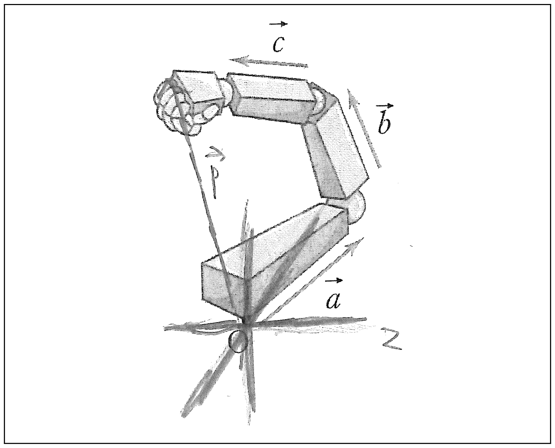



Figure 4.9: A Student Drawing of Physical Axes

The above discussions about the epistemological and ontological perspectives and their interplay on the construct of vectors are accompanied by a direction towards mathematical abstraction. The second layer of refinement presumes decomposing this direction into two perspectives as well. In the configuration, the ontological axis is a directed dimension that covers changes in existence or being. It is about explicit change of representations or notions. The epistemological axis is a directed dimension that covers changes in knowledge or thinking. It is about understanding of representations. The direction of each axis implies not only the intended movement towards mathematical abstraction, but also certain degrees of the developmental movement. We hypothesized this from the prevalence of the analytic approach to the synthetic approach as described in Chapter 3. In terms of the configuration, I defined the analytic approach as a trend of changing explicit representations along the ontological axis from geometric representations to symbolic representations quickly as possible. Because this prevalence of the analytic approach to the synthetic approach could suggest a refinement to see the complexity and subtlety of the construct of vectors, I looked

| | Analytic Approach | Synthetic Approach | No response |
|---|---|---|---|
| Cube: | **41% (12)** | 31% (9) | 28% (8) |
| △ Midpoints: | **55.5% (16)** | 34.5% (10) | 10% (3) |
| Associativity: | **72.4% (21)** | 24.1% (7) | 3.4% (1) |

Table 4.2: Prevalence Result from Survey II

into this from the student data.

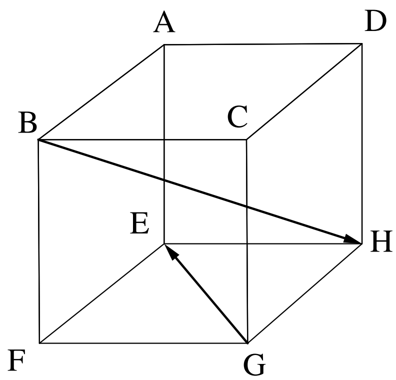
### 4.2.2 Prevalence of Analytic Approach to Synthetic Approach

Studies on prevalence are natural in a sense that we set up the developmental direction towards mathematical abstraction by differentiating mathematical vectors from physical vectors through the first layer and identified two perspectives to see the development in detail through the second layer. Moreover the result of prevalence can give a motivation of further refinement because the prevalence may be coming from harder obstacles to overcome in one developmental direction. This evidence of the prevalence in this section will motivate us to expect the next refinement in section 4.3.

I categorized students' responses into two approaches: analytic and synthetic as defined in Chapter 2. Students tended to use particular representations more and confine their understanding and using vectors in one approach rather than having flexibility of using both. This tendency was identified in the responses as the prevalence of the analytic approach to the synthetic approach. Because this prevalence is studied and regarded as a trend, and not a specific student's preference, I use the collective data of twenty-nine students rather than concentrating on specific cases. 'Cube', 'Triangle Midpoints', 'Associativity' are questions specially designed to look into the prevalence of the analytic approach to the synthetic approach. See Table 4.2. The results show that the prevalence of the analytic approach to the synthetic approach that we hypothesized in Chapter 2 is evident in student work.

In the 'Cube' question, the choice of (i) or (ii) was categorized as the choice of the analytic approach and (iii), (iv) as the synthetic approach. The result of 'Cube' question implies more

**Cube:** In the following figure, we want to show that the two vectors shown in the cube are perpendicular using the inner product also known as the dot product. The sides of the cube can be assumed to be of length 1.




(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer.

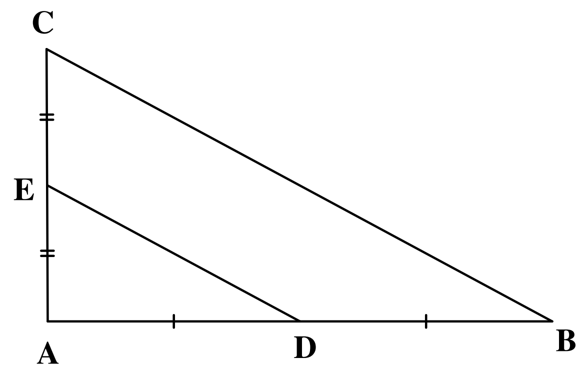
(i) $(a,b,c)$ (ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:________________________

Figure 4.10: Cube Question

students chose the analytic approach than the synthetic approach. One can argue that the cube structure itself encourages students to use the analytic approach because mutually orthogonal sides of the cube can be three axes associated with a space. However, when mutual orthogonality is not guaranteed such as in ‘Triangle Midpoints’ (Figure 4.11), the prevalence of the analytic approach to the synthetic approach still shows up.‘Triangle Midpoints’ question is actually accessible when students use the synthetic approach and the structure of vector additions. This is because positioning the origin of the coordinates onto the appropriate location of the triangle makes the question harder due to no evidence of a right angle. Therefore, it is worth using this quesion as a way to see the prevalence together with the ‘Cube’ question.

All the interview results also supported the prevalence of the analytic approach to the synthetic approach. One student described the reason as his experience on vectors in high school mathematics. He said that because the focus of the curricula was on algebra, it made the coordinate form

**Triangle Midpoints:** In the following triangle $\triangle ABC$, $AB = 2AD$ and $AC = 2AE$. We want to show that $\overline{BC}$ is parallel to $\overline{DE}$.




(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:

Figure 4.11: Triangle Midpoints Question

and the analytic approach "more recognizable." 'Associativity' gave a similar prevalence of the analytic approach to the synthetic approach. See Table 4.2.

**Associativity:** We want to prove the following property of vector addition:

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}).$$

(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer. You don't actually need to prove this.
(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) directed line segments (iv) Other:

(b) Why have you preferred this representation and not another one?

Figure 4.12: Associativity Question

In the follow up interviews from the first and the second studies, only one student out of eight was able to prove this associativity with arrow forms of vectors when asked to proceed the proof.

Even though it would be tempting to say because it is much easier to prove associativity using coordinate/column vector forms, we should be alerted from this example with possible obstacles that prevent student's flexible use of various vectors. These obstacles were hypothesized to be evident in student work through the third layer of refinement.

Another interview result describes that the prevalence against the synthetic approach is because it is not as 'concrete' as the analytic approach. This 'concreteness' was from the fact that when using analytic approach, students can utilize number substitutions in entries. An interesting observation is that this dominance is spread over all the data and not just the data from those three intended questions. This dominance is in disgreement with Hillel (2002) that geometric vectors are more concrete than symbolic vectors. This prevalence together with Hillel's argument shows that even if we have the same arrow forms of vectors, up to the setting (cognitively different settings or obstacles), the prevalence of the coordinate/column vector forms to arrow forms can change.

In this section, it was evident in student work that (1) there are the epistemological and the ontological perspectives in the construct of vectors, (2) interplay between those two is essential to talk about the construct of vectors, and (3) there is a prevalence of the analytic approach to synthetic approach that might imply certain obstacles. In terms of the configuration, we need a refinement of the construct of vectors that can see these obstacles as transitions. The third layer of refinement will identify transitions in the next section.

## 4.3 Obstacles towards Mathematical Abstraction

In Chapter 1, we saw that when talking about interplay between the representational and the cognitive perspectives on the construct of vectors, the problem stems not only from the multiple representations but also from the translations/conversions among them. We can also recall that the third layer was the layer that identifies the representational and the cognitive obstacles in terms of transitions towards mathematical abstraction in Chapter 2. In this section, I want to look for evidence from student work that shows these ontological and epistemological obstacles. These obstacles will play roles as transitions in the configuration of vectors. By doing so, we can especially see cognitive development along with Action-Process-Object-Schema and/or process-object duality in geometric representations emphasized in the configuration.

### 4.3.1 Ontological Obstacles: Transition (D) from Geometric to Symbolic

As we explored in section 4.2, it was hard to extract a clear ontological part from a combined view when we talked about the difference between mathematical vectors and physical vectors. ‘Robot Arm’ question still gives a blurry idea of the ontological part different from the epistemological part about matching up the coordinates with the axes. The coordinate axes that a student drew in Figure 4.9 were not for coordinate representations of arrow vectors, but for the coordinates of the terminal points of vectors when their initial points were located at the origin. The following quote from a student response supports that there is an obstacle related with the interpretation of physical world mathematically. It is still cognitive, but the physical settings change to mathematical settings in an explicit way.

> **R**: “You said, unsure, cannot reconcile image & vector coordinates. What was happening here?”
> **S**: “[...] Vectors are being used to describe the parts of the arm in here. [...] I was trying to see [...] if I draw the space out for three axes, where should this point be, where should the point be for this vector. I don’t know [...] not sure if the pictures are not actually representing the vector listed here or if I just couldn’t map it out right in my head or on the graph I drew out, but I just couldn’t determine [...]”
> **R**: “In the picture, the origin was the intersection of those three axes. What do you mean by coordinates? For example, $(10,15,20)$ in your case would be the coordinate of the terminal point of $\overrightarrow{a}$. What about $\overrightarrow{b}$? $(5,-5,15)$?”
> **S**: “I think that also represents the point well..though [...] At that time, I was setting all the vectors out to have their initial points be the origin, and then their end points be what was given for the vector.”

| | one correct | two correct | three correct | non correct | total |
|---|---|---|---|---|---|
| Position Vector: | 3.4% (1) | 0% (0) | 86.2% (25) | 10.3% (3) | 100% (29) |
| Free Vector: | 31% (9) | 3.4% (1) | 37.9% (11) | 27.6% (8) | 100% (29) |

Table 4.3: Origin Result from Survey II

**R**: "[...] What was the hardest part? What was the obstacle?"

**S**: "[...] coordinates are not matched up with my axes. [...]"

Taking into account the response above, we can see evidence of the ontological obstacles in: (1) interpreting and relating the location and direction of a vector without the reference of the physical world corresponding to Transition (A), and (2) identifying the locations of vectors and the directions of vectors when they are given in coordinate forms corresponding to Transition (D).

We can also identify these ontological obstacles in the responses for the 'Origin' question. 'Origin' is related with transition (D) from geometric vectors to symbolic vectors (Figure 4.13). As we expected from Chapter 2 on Transition (D), confusion between the coordinates of points and the coordinate forms of vectors still exists and is evident in student work. See Table 4.3.

Students are usually familiar with converting an arrow form of a vector to a coordinate form from their basic vector study in physics and high school mathematics. However, those conversions are limited to position vectors with $(0,0)$ as the origin in most cases. I could see from student work that Transition (D) was still a big obstacle to some college students even if they already knew how to convert one from the other. The variable location of the origin and the coordinate forms of free vectors are still not familiar to them. See Table 4.3. In terms of our discussion on the layered view of the construct of vectors, this is where the interplay comes in. Transition (D) as an ontological obstacle is evident from the data (Table 4.3), but to talk about why, we need to think from the epistemological perspective. Converting from geometric to symbolic was more successful in the case of position vectors than free vectors as expected. This is because students are more familiar with the external coordinate form of vectors when producing the coordinate forms using the difference in the coordinates for initial and terminal points. They do not feel comfortable

**Origin:** For the three different origins: $O$, $C$, and $A$, complete the table by finding the position vector of $A$ and the vector $\overrightarrow{AB}$.

| Origin | Position Vector of $A$ | $\overrightarrow{AB}$ |
|---|---|---|
| $O$ | (____,____) | (____,____) |
| $C$ | (____,____) | (____,____) |
| $A$ | (____,____) | (____,____) |

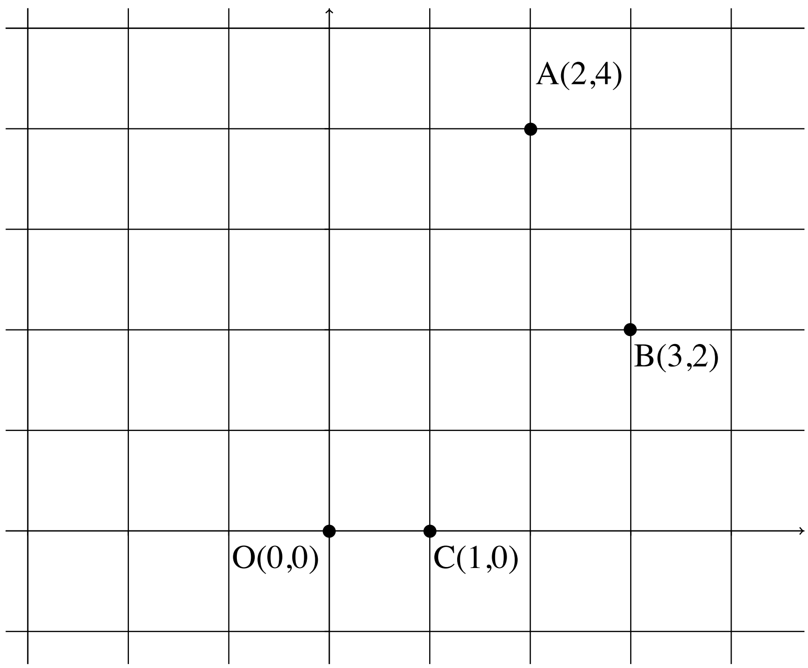



Figure 4.13: Origin Question

with the internal coordinate form for vectors that can be calculated without the coordinates for the origin, initial and terminal points, but counting the grids from the initial points to the terminal points. The latter focuses more on vectors themselves, but the former focuses more on additional information such as coordinates of points in vectors. Knowledge of vector equivalent relation and the concept of position/free vector constituted to this obstacle. This means that the epistemological perspective can give ideas of the reason to the ontological obstacles.

In Chapter 2, we hypothesized epistemological obstacles in the cognitive development towards mathematical abstraction. In the previous section, we talked about the prevalence of the analytic approach to the synthetic approach. For this reason, it is no less dubious to connect that to the obstacles along the synthetic approach, especially to cognitive obstacles.

### 4.3.2 Epistemological Obstacle: Transition (C) from Analytic to Synthetic

The existence of an epistemological obstacle is evident in student work for 'Polygon' and 'A very long sum' questions. This obstacle prevents students to continue calculating the binary vector sum. This is Transition (C) from analytic to synthetic where the obstacle is along cognitive development on geometric vectors in the configuration. For 'Polygon' question, students' responses told us that question (a), (b), and (d) based on a triangle, a parallelogram, a rectangle figure were easier for students, but question (c) with a pentagon was harder to solve than other questions (Table 4.4).

From Table 4.4, it is clear that students were not familiar with vector sum on a weird shape or a polygon with more than four sides. It is interesting to see a response that shows drawing parallelogram sum, stopping, and saying 'very unique arrangement' in the regular pentagon. This student described his experience during the interview as follows.

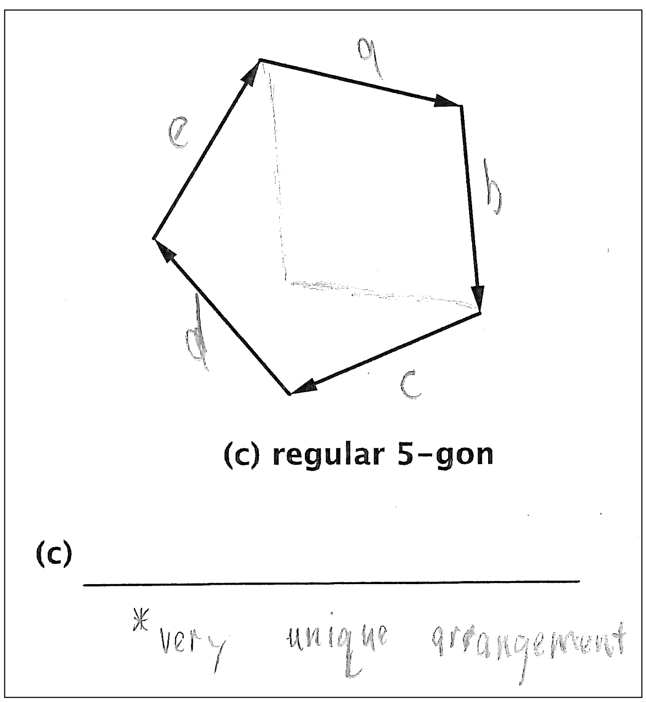
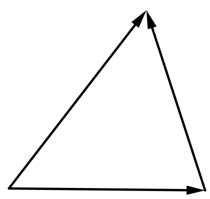
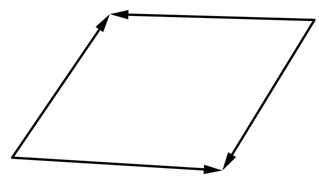



Figure 4.14: "Very unique arrangement"

**Polygons:** label each vector and write an equation of the relations.

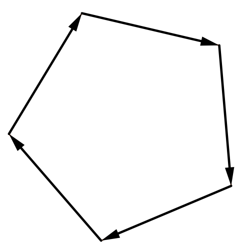


**(a) triangle**

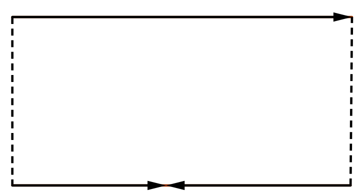


**(b) parallelogram**

**(a)** ______________________ **(b)** ______________________



**(c) regular 5-gon**



**(d) rectangle**

**(c)** ______________________ **(d)** ______________________

| question | (a) | (b) | **(c)** | (d) |
|---|---|---|---|---|
| correct | 72.4% | 65.5% | **48.3%** | 69% |
| not correct | 27.6% | 34.5% | **51.7%** | 31% |
| total | 100 % | 100 % | **100 %** | 100 % |

Table 4.4: Polygon Result from Survey II

**R**: “Why did you draw these middle line segments? What are they?”

**S**: “I think I was trying to do the vector addition and [...] I couldn’t really find, based on the method I was trying, couldn’t find the way to express the relationship together from a polygon or from a pentagon. [...] Normally I’ve never seen vectors arranged in that kind of relationship. I’ve seen them in the triangle, [...] usually in many of these, a parallelogram, a four sided figure, but nothing like this one.”

From the interview with the student above, I could see that the student drew a parallelogram to figure out the sum. Thinking the sizes and the directions of arrows compared to thinking the structure that vectors lie on can be regarded as procedural thinking, because the sum was a binary operation and we needed those information for a binary operation. It is evident that there is an obstacle that prevents students using synthetic vectors or structural thinking. What we expect from a structural thinking is also evident from an interview quote for question (c) in the ‘Polygons’ question, “If I started here, I kind of saw that...if I just kind of follow my nose around this thing, I’m going to end up with where I started. Okay. Sum of all these is zero." This student saw that sum as a whole, not separated operations. Seeing a structure as a whole, not as collection of separated operations could be evidence of a successful transition from analytic (procedural) to synthetic (structural). We can notice this more in the following example.

Student responses to ‘A Very Long Sum’ also show that there is an obstacle to continue the sum procedurally. This obstacle is visually clear in a sense that the middle dots in the figure block students to continue the binary sum operations. Only 27.6% (8) of 29 students gave a correct answer. One of students who responded with structural thinking of the vector sum in terms of *flow* with moving a nose around the figure. Usually a good interpretation of middle dots and/or skipping dots imply a transition from process to object or from procedural to structural thinking in symbolic representation ($0.9\ldots = 1$).

Proceeding from what has been observed above, Transition (C) from analytic (procedural) to synthetic (structural) is evident in student work as a cognitive obstacle. In the next discussion, we will also see that process-object duality and Action-Process-Object-Schema development in terms

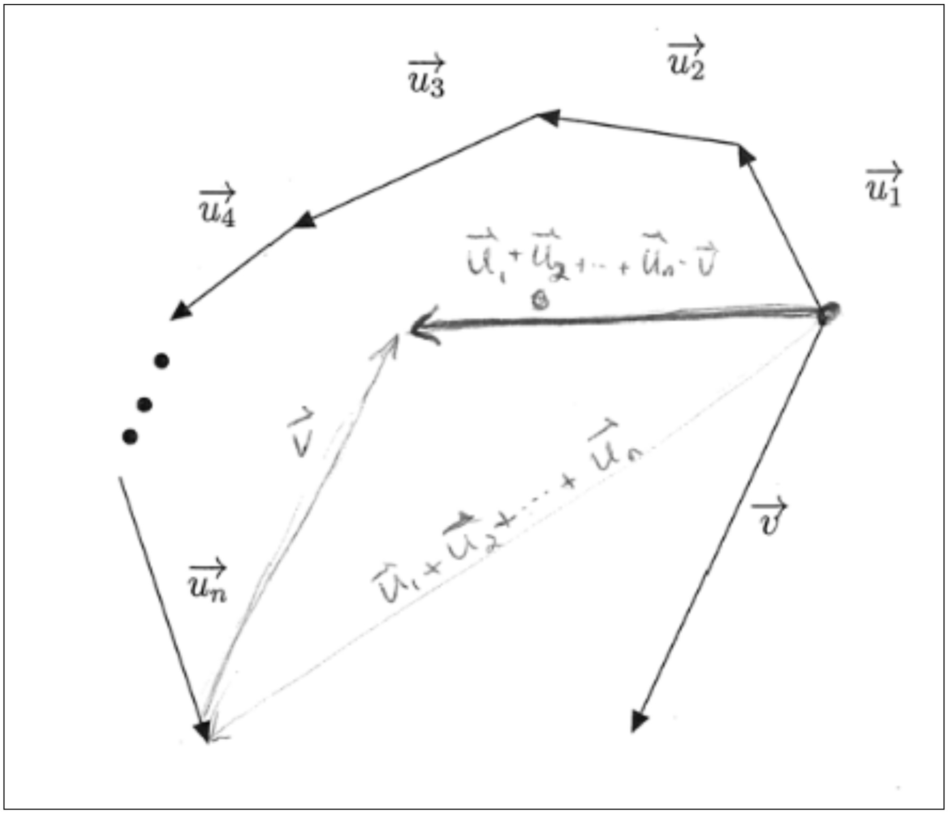



Figure 4.15: Understanding Structures of Vector Operations

of geometric representations in a similar way to symbolic representations.

### 4.3.3 Cognitive Development of Geometric Representations

So far we have seen evidence in student work for three important transition points: (A) Physical to Mathematical, (D) Geometric to Symbolic, and (C) Analytic to Synthetic with the progressive refinements of the construct of vectors. In the configuration, geometric representations and symbolic representations are displayed as parallel to each other along epistemological axis. Transition (C) from analytic (procedural) to synthetic (structural) is not just for symbolic representations but also for geometric representations. This is to reveal cognitive development in geometric representations in a similar fashion to symbolic representations. Addition of two numbers is one of the famous examples in process-object duality theory. For example, $1+3$ may be understood as adding 3 to 1. On the other hand, $3+5$ may also be understood as a sum without calculation (Sfard, 1991; Gray and

Tall, 1993, 2001). These basic examples are all about symbolic representations. How geometric representations of vectors correspond to the cognitive development of process-object duality is the main concern. From process to object, or from procedural to structural transitions in the construct of vectors are expected to be evident in student work simultaneously in both representations.

**A Very Long Sum:** Find $\overrightarrow{u_1} + \overrightarrow{u_2} + \ldots + \overrightarrow{u_n}$.

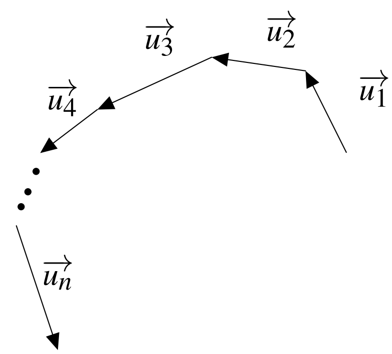



Figure 4.16: Understanding Addition Structures

Figure 4.16 shows the modified version of 'Very Long Sum'. I had two students whose responses were totally different. One student used the strategy of the structure sum with drawing an arrow as the result of the sum. The student connected two end points of the picture and gave the direction of the vector. The other did not give the answer in arrow forms. Instead, he provided the following work.

Let $u$, $u^1$ be points such that $\overrightarrow{u_1} = \overrightarrow{uu^1}$.

Let $u^1$, $u^2$ be points such that $\overrightarrow{u_2} = \overrightarrow{u^1u^2}$.

...

Let $u^{n-1}$, $u^n$ be points such that $\overrightarrow{u_n} = \overrightarrow{u^{n-1}u^n}$.

Then $\overrightarrow{u_1} + \overrightarrow{u_2} + \ldots + \overrightarrow{u_n} = \overrightarrow{uu^n}$

The student clearly noticed that the dots in the figure correspond to dots in the symbolic manipulation. To get over the obstacle of '$\cdots$' in the figure, this student used the corresponding '$\cdots$' in the symbolic representations. By changing a reduced symbol to a vector connecting two points, he continue to convert symbolic representations to geometric representations. It seemed the student already knew the fact that symbolic calculation could be done without assuming pictures, for example, a symbolic canceling such as $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{A\not{B}}+\overrightarrow{\not{B}C}=\overrightarrow{AC}$.

It seems reasonable to assume that cognitive development in both representations can help students see more coherent relationship between geometric representations and symbolic representations. This example together with 'Polygon' extends the coverage of cognitive development theories such as process-object duality from symbolic world to geometric world.

## 4.4 Summary of Findings

In this chapter, we saw that the following are evident in student work when discussing the complexity/subtlety of vectors:

(1) the difference between physical and mathematical vectors,

(2) the multiple perspectives: ontological and epistemological, and interplay between those two,

(3) the prevalence of the analytic approach to the synthetic approach,

(4) an epistemological obstacle defined as Transition (C) and an ontological obstacle defined as Transition (D),

(5) process-object duality on geometric representations of vectors.

What stands out most from these empirical studies is this new framework is very helpful when talking about the complexity and subtlety of the construct of vectors. However, these tentative conclusions with the configuration await further refinement and correction in the light of further research.

# Chapter 5

# LIMITATIONS OF THE FRAMEWORK

This chapter described unexpected evidence from the three progressive refinements of the construct of vectors. These limitations include: (1) the categorization of mathematical vectors while the first layer differentiating mathematical vectors from physical vectors, (2) the direction of the development towards abstraction while the second layer recounting the ontological and epistemological perspectives and decomposing the direction of development into two, and (3) the transitions on the configuration while the third layer identifying ontological and epistemological obstacles in terms of transitions towards abstraction. Specifically, the limitations described in this chapter are: (1) non-empty intersection between mathematical vectors and physical vectors, (2) unreasonable levels of sophistication reflected in the direction towards abstraction, and (3) problems in repetition of transitions and revered transitions in different contexts. These limitations suggest further refinement and correction of the framework and the implications for teaching and learning of vectors.

## 5.1 Categorization of Mathematical Vectors

What is mathematical and what is physical? This question stimulates serious rethinking about differentiating mathematical vectors from physical vectors done in the first refinement. The major difference was described as different understanding of the coordinate systems including axes, points on the space, and the space itself. What we call a motion was quite different in mathematics compared to a motion in physics according to Freudenthal (1983). He insisted that physical motion is something that occurs to an object within space or plane within time, but mathematical motion should be differentiated from physical motion in three ways: from the limited object to the total space (plane), from within space (plane) to on space (plane), from within time to at one blow. This difference is noteworthy for students to develop their vector understanding in mathematics. However, this difference is not clear cut in a sense that we still have a non-empty intersection between

**Translated Image:** Circle one that describes the image of a translation by an arrow in the following figure.

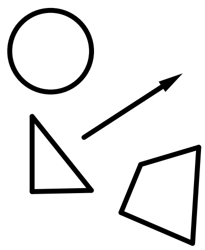
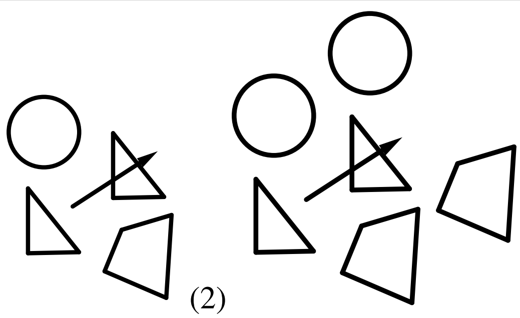


Figure 5.1: Translated Image Question



(1) (2) (3) None of the left.
. (Explain or draw your own answer.)

Figure 5.2: Translated Image Choices

mathematical vectors and physical vectors. Transition (A) in the configuration does not mean a short time jump. It can be a long time development. For example, when a question from the survey III asked about the definition of the geometric translation (Figure 5.1), the assumption behind the question was that understanding a geometric translation as ‘moving entire plane’ is ‘mathematical’ not ‘physical’. In the discussion related to the complexity and the first refinement in Chapter 1 and 2, this assumption was regarded as true. If students focused on the object, they would choose the picture that moves only a triangle as expected from a physical motion (Freudenthal, 1983).

Most of students gave the correct answer (2) in Figure 5.2 as anticipated. However, this answer (2) could originate from physical meaning of ‘moving an object’ if the students assumed the translation as ‘moving entire plane’, and ‘entire plane’ as an object. Although we have the mathematical

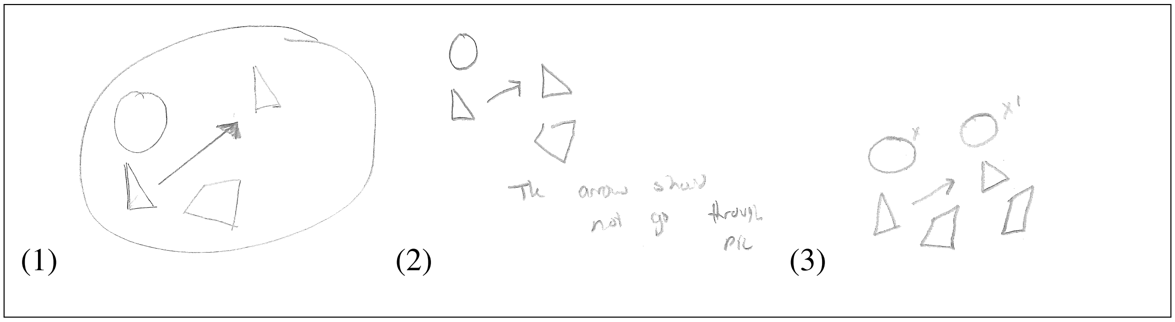



Figure 5.3: Responses to Translated Images Question

terms such as a ‘map’ or ‘generalized function’ for the rigorous definition of the translation, ‘moving entire plane’ is also usually assumed to be a correct description of a geometric translation in mathematics Beckmann (2005). Lakoff and Nunez (2001) also described the similar nature of mathematics that is inseparable from physical embodiment. In addition to the above discussion, a few more responses from the ‘Translated Image’ question showed different interpretations on a geometric translation and its representation. The response (1) in Figure 5.2 is exactly what I expected under the assumption of students’ conception of geometric translation as ‘moving an object’. However, some students did not regard an arrow as a mathematical representation reflecting both direction and magnitude (Figure 5.3). Rather, they regarded an arrow as a non-mathematical diagram or a visual symbol that represented the direction of movement. Some thought the arrow should not intersect the geometric object because geometric objects might hinder the direction.

These responses could be an example of under-developed notion of physical vectors that needs to be studied further both in mathematics education and physics education. Because we set up the two different perspectives: ontological and epistemological in the second refinement of the construct of vectors, looking into how those perspectives work in physics could be interesting as well. The question could be about physical vectors in the second, third, and forth quadrants on the configuration differentiated from mathematical vectors. See Figure 5.4.

In summary, the categorization of mathematical vectors and physical vectors while the first layer differentiating mathematical vectors from physical vectors needs more refinements and reconsideration. In the next section, I will discuss the limitation from the direction of the develop-

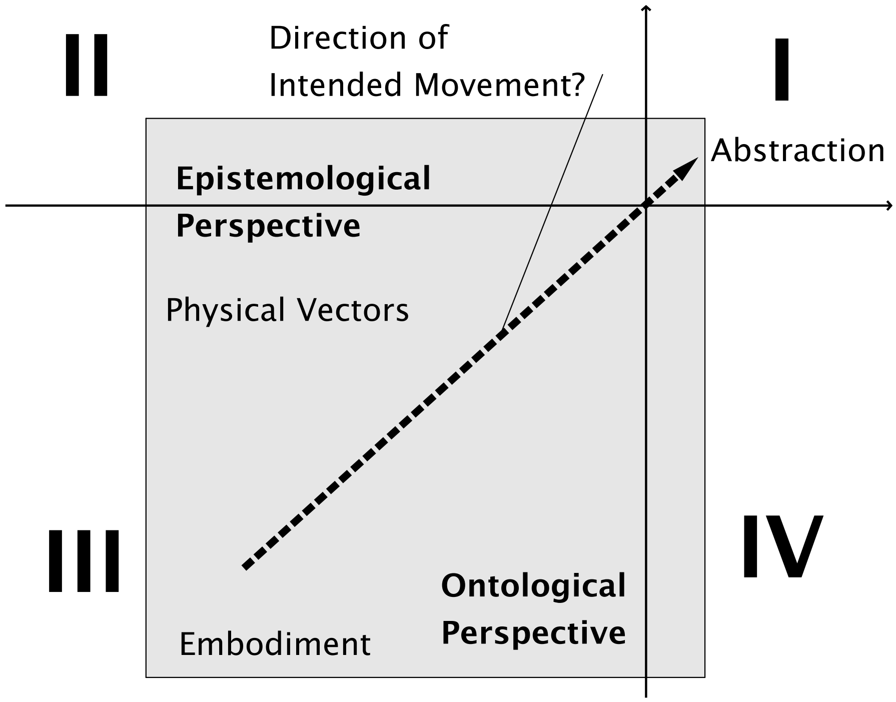
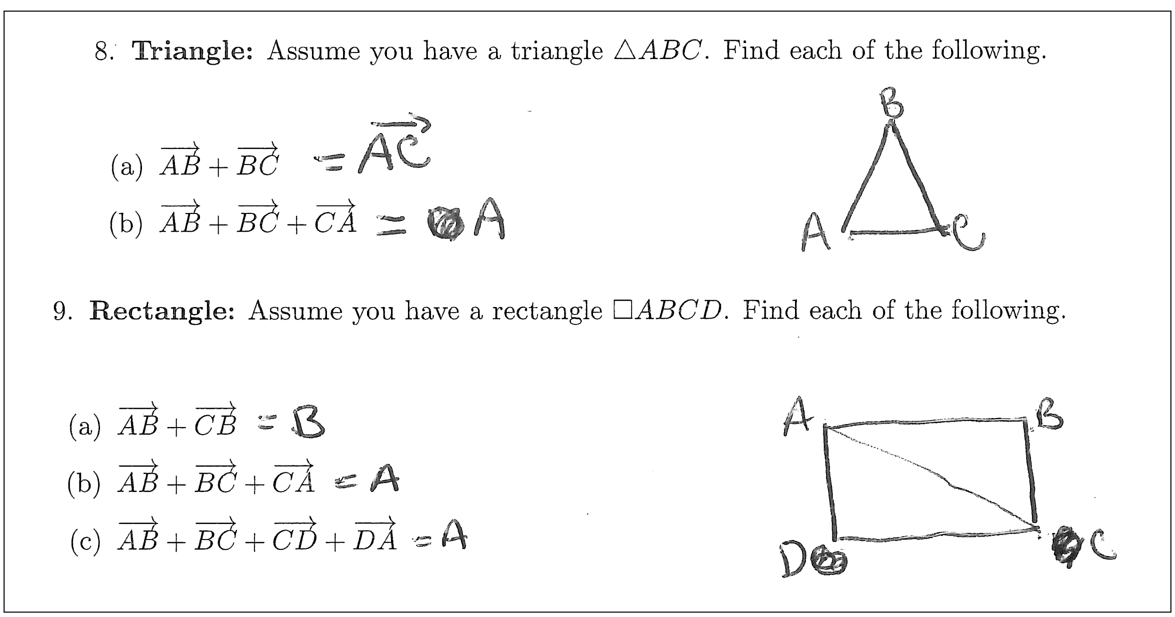



Figure 5.4: Underdeveloped Physical Vectors?

ment towards mathematical abstraction in terms of the above two perspectives about the direction.

## 5.2 Direction of the Development towards Mathematical Abstraction

The direction and the dimension of the development introduce levels of sophistication reflected in the direction towards abstraction while the second layer intertwining the ontological and epistemological perspectives and decomposing the direction into two. Moving up or moving to the right suggest more abstract and higher levels of sophistication in the configuration. However, there is no definite evidence that this direction always follows the direction that student understanding follows. For example, we saw that a structural sum is harder to achieve than a procedural binary sum. However, some students used non-mathematical, even non-physical ways of thinking such as thinking arrows as routes of displacements and solved the structure sum easily. I identified those students who interpreted vectors in arrow forms, as routes of displacements, not as representing displacements themselves. In fact, displacement vectors never specify the routes in physics. They include the information about the initial, terminal locations, and the distances between them, not the actual routes. These displacements can be of points within space (one by one), of objects within space (objects as collections of points), and of a space itself (as a collection of points) in physics as well. Depending on the attachment of objects to a vector, and students' understanding of points and spaces, students' interpretations can vary from 'route' of a displacement to a 'displacement' itself. All the above interpretations about 'displacement' are helpful in certain mathematical situations, although not all of them are with a mathematical rigor. To solve 'Very long Sum' questions, some students regarded arrows connected in a 'tip to toe' way as routes of displacements that a particle moves along. This enables students to perform the structure sum without thinking the intermediate steps of series of binary operations. However, this is not mathematically true or even not physically true. Students who uses only arrows as routes of particle movements for vector sums can misunderstand the vector sum when translated to symbolic representations. For example, there were students who marked the final destination of the consecutive displacements in connected routes as the answer for the vector sum. See Figure 5.5.


8. **Triangle:** Assume you have a triangle $\triangle ABC$. Find each of the following.

(a) $\overrightarrow{AB} + \overrightarrow{BC}$ = $\overrightarrow{AC}$

(b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ = A

9. **Rectangle:** Assume you have a rectangle $\square ABCD$. Find each of the following.

(a) $\overrightarrow{AB} + \overrightarrow{CB}$ = B

(b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ = A

(c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$ = A



Figure 5.5: Misunderstanding of Arrows

This example shows (1) a higher level thinking does not always mean mathematically rigorous,

(2) a lower level thinking can help higher level thinking when it used carefully, and (3) directions of the ontological and epistemological developments are not a reflection of strict hierarchy or levels of sophistication.

Here is another example of (3) that does not support the levels of sophistication in the configuration. Bryant (1984) introduced geometric ideas of vectors and their statistical and probabilistic analogs. In science and engineering, vectors depend heavily upon physical vectors with physical meanings such as forces, velocities, moments, momentum, etc. However, this new application of vectors emphasizes not only the efficiency or convenience of calculations but also the fundamental idea behind the concept. The formula for the correlation coefficient looks very complicated and hard to remember. However, once we regard data as vectors, the correlation coefficient is just the cosine value of the angle between two arrows representing $z-\mu_z$ and $x-\mu_x$ for two data sets $x$ and $z$. See Figure 5.6.

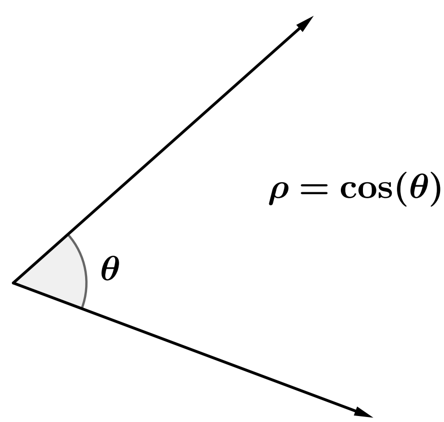



Figure 5.6: Vectors that can explain the correlation concept (Bryant,1984)

This is one of the applications of vectors that shows the significance of ambiguity and connections in mathematics independent from physical meanings (Bryant, 1984). Students can learn that arrows are not just for two- or three-dimensional vectors. However, this brings a serious reconsideration of common belief that geometric vectors are more concrete than symbolic vectors (Hillel, 2002). In Bryant (1984)'s example, two arrows connected together means any $n$-dimensional vectors. This is conflicting with what I described in the configuration that higher dimensional vectors were more abstract and ware represented only as coordinate/column vector forms. The example of associativity when we talked about the prevalence of the analytic approach to the synthetic approach also gives us a similar insight that geometric representations can be more abstract than symbolic representations. A proof using arrows gives one that works in any dimension. See Figure 5.7

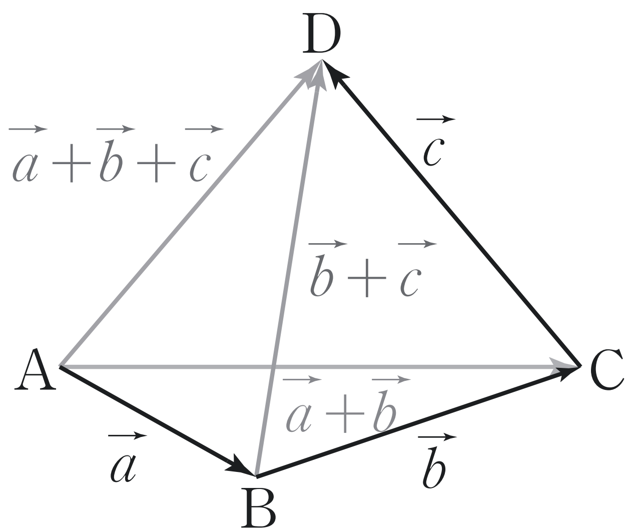



Figure 5.7: Associativity proved using arrows

These examples show that a dimension of a vector space is not actually an element that distinguishes the difference or levels of sophistication in the current form of refinements for the construct of vectors. This implies Transition (E) in the configuration could be on a different axis. Further refinements in the future would form more axes and transitions along the axes. For this possible extension of dimensions to the configuration, I rather see the new configuration as a map or snapshot of multi-dimensional construct, not as a full developmental hierarchy or a simple two-dimensional model for levels of sophistication. Further refinements should be able to detect more transitions with respect to multiple perspectives.

## 5.3 Repeated or Reversed Transitions on the Configuration

The three transitions, (A) from physical to mathematical, (C) from analytic to synthetic, and (D) from geometric to synthetic on the configuration were evident in student work when we saw ontological and epistemological obstacles in terms of transitions towards abstraction with the third layer. When assuming the direction of the development towards abstraction, we never discussed the possibility of the repetition of transitions and the reversed transitions in different contexts. In Chapter 1, one of the difficulty that students could have on the construct of vectors with the classical vector representations was confusion between vectors-as-points and vectors-as-arrows (Hillel, 2002). This confusion is mostly related to Transition (D) from geometric to symbolic vectors on the configuration as discussed earlier. However, a complete understanding of the arguments in Hillel (2002) requires the interpretation of directions of each axis on the configuration because the confusion is caused by reversing or repeating the developmental directions, that is, repeating or reversing Transition (D). For example, representations of vectors for a real vector space $\mathbb{R}^n$ used in linear algebra sometimes are not stemmed from physics or geometry. Moreover the use of Cartesian coordinates is different from that in geometry. In linear algebra, symbolic representations of vectors were developed first and the Cartesian coordinates are an aid to translate the language of symbolic representations of vectors to geometric language. 'Point' is another name of $n$-tuple of numbers, not a geometric object. However, the direction towards mathematical abstraction implicitly and initially reflected historic development of vectors in mathematics curriculum and corresponded to the student's learning trajectory. The learning trajectory of vectors introduced in high school mathematics is beginning with geometric representations of vectors. See Table 5.1.

This means Transition (D) from geometric to symbolic vectors can be reversed or repeated in different contexts such as contexts before linear algebra or after linear algebra, with or without bases, in $\mathbb{R}^n$ or in $\mathbb{E}^n$, etc. Some students can succeed in linear algebra with the textbooks reflecting this reversed transition and felt uncomfortable with geometric representations that assumed to be easier in the configuration. This means a reversed or a repeated transition in a different context could be easier than the original transition. The third layer of refinement basically follows the

| | **Vectors in Linear Algebra producing vectors-as-points, vectors-as-arrows confusion** | **Corresponding Development in the configuration emphasizing Geometry** |
|---|---|---|
| Start | Symbolic vector $(a_1, a_2)$ in $\mathbb{R}^2$ | A directed line segment in $\mathbb{E}^2$ |
| | | Position Vector in $\mathbb{E}^2$ |
| | | Terminal point $(a_1, a_2)$ |
| | | of position vector in $\mathbb{E}^2$ |
| | | Corresponding $(a_1, a_2)$ in $\mathbb{R}^2$ |
| | | with different coordinates for $\mathbb{R}^2$ |
| End | Point (dot) in coordinates for $\mathbb{R}^2$ | Point (dot) in $\mathbb{R}^2$ |
| | | with new coordinates for $\mathbb{R}^2$ |
| | | Coordinates for $\mathbb{E}^2$ and those for $\mathbb{R}^2$ |
| | | are different. |
| **Emphasis** | **Symbolic to Geometric** | **Geometric to Symbolic and then to Geometric** |

Table 5.1: Resolution of Vectors-as-Arrows vs. Vectors-as-Points Confusion

original direction of development towards abstraction. Further refinements of the construct are required to unfold these repeated or reversed transitions for more careful description such as in Table 5.1 of the construct of vectors for teaching and learning .

## 5.4 Epilogue

Collectively, the three progressive refinements that were described in this dissertation help push the understanding of vectors in an important direction: They encourage students to see vectors as a rich trove of intuitive and also abstract representations with complex connections with multiple perspectives, rather than an assemblage consisting merely of different symbols, notations and procedures. They also suggest to teachers to prepare their own refinement needed in specific curriculum and instruction of vectors. It emphasizes the APOS cycles, and domain-specific instruction of vectors such as vectors in physical world with physical motions; vectors in geometry with abstract understanding of coordinate systems, points, and motions; and vectors in linear algebra with a vector space $\mathbb{R}^n$. The teacher and student should consider both ontological and epistemological perspectives to understand the construct of vectors. They should also be careful with the learning trajectories that can go backward from the direction of cognitive development in the configuration.

In the curriculum of school mathematics, in standards, and in assessment, connecting geometry with the other content areas such as algebra and probability will be important more and more (Bryant, 1984). Study of vector representations is therefore worth continuing. I want to initiate studies on the analysis of student work with this new framework while continuing to modify and refine this work.

**APPENDIX**

## Appendix A

## VECTOR REPRESENTATION SURVEY I

1. **Translation:** A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point $P$.

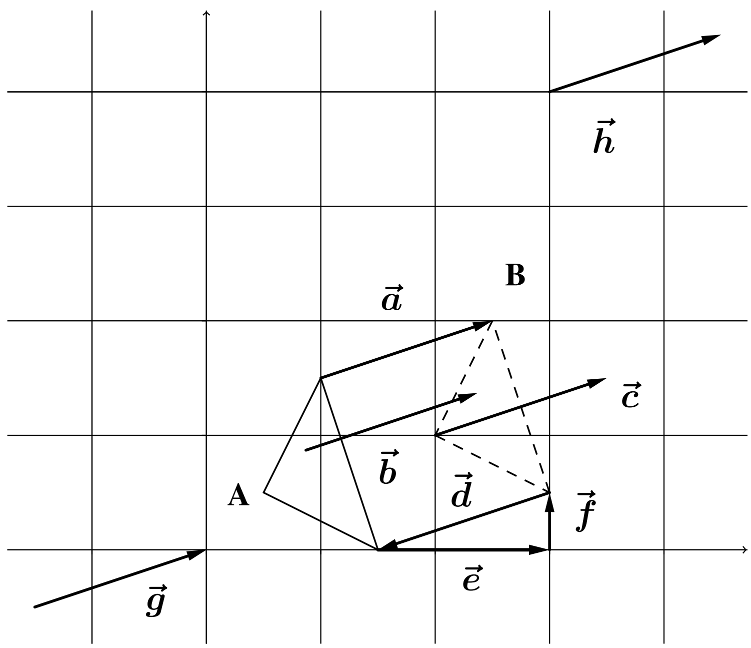



Figure A.1: Translation

   (a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

   (b) Circle all vectors that are equivalent to $\overrightarrow{\text{a}}$.

2. **A Rainy Day:** The rain is falling straight down on your head with speed 1 m/s. You start walking forward with speed of 2 m/s and notice the rain is now hitting your face. What is the angle that the rain now appears to make with the vertical? Draw a picture that explains your thinking.

3. **Robot Arm:** A robot arm has three joints as shown. The bottom part can be represented by a vector $\vec{a} = (10, 15, 20)$, the middle part can be represented by a vector $\vec{b} = (5, -5, 15)$, and the top part can be represented by $\vec{c} = (20, -5, -10)$. The units are in centimeters$(cm)$.

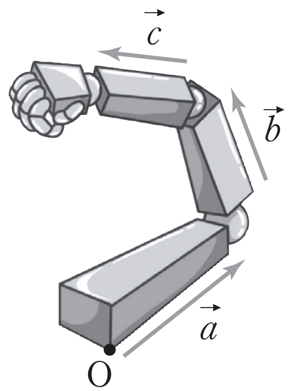



Figure A.2: Robot Arm

Find the vector $\vec{p}$ that tells the location of the hand.

4. **Origin:** For the three different origins: $O$, $C$, and $A$, complete the table by finding the position vector of $A$ and the vector $\overrightarrow{AB}$.

| Origin | Position Vector of $A$ | $\overrightarrow{AB}$ |
|---|---|---|
| $O$ | (____,____) | (____,____) |
| $C$ | (____,____) | (____,____) |
| $A$ | (____,____) | (____,____) |

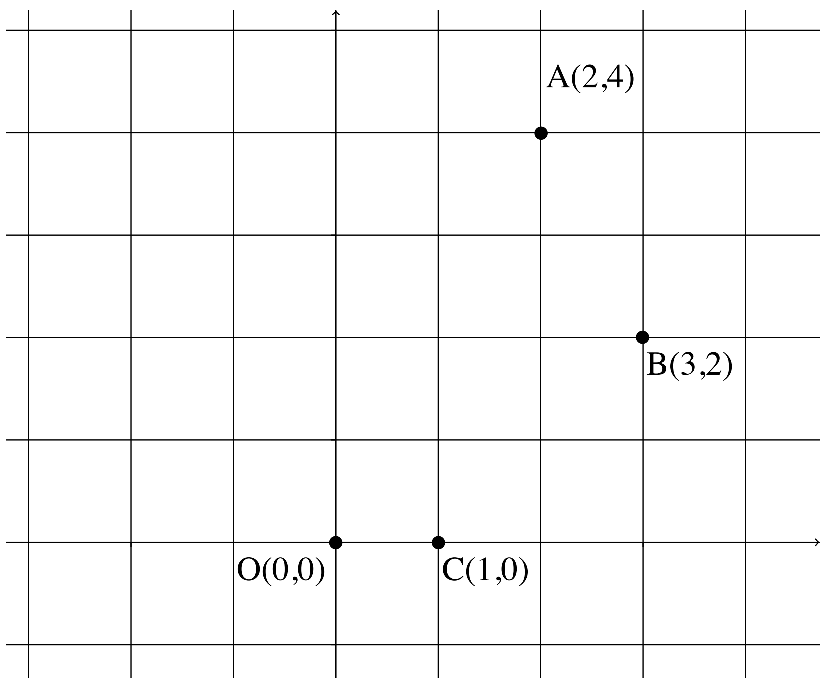



Figure A.3: Origin

5. **Non-standard Basis:** In a standard basis of $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ for $\mathbb{R}^2$, the vector $\overrightarrow{u}$ can be represented by the ordered pair $(1,2)$. The set $\{\overrightarrow{e_1}, \overrightarrow{u}\}$ also forms a basis for $\mathbb{R}^2$. Find an ordered pair $(a,b)$ for the vector $\overrightarrow{v}$ using the basis $\{\overrightarrow{e_1}, \overrightarrow{u}\}$.

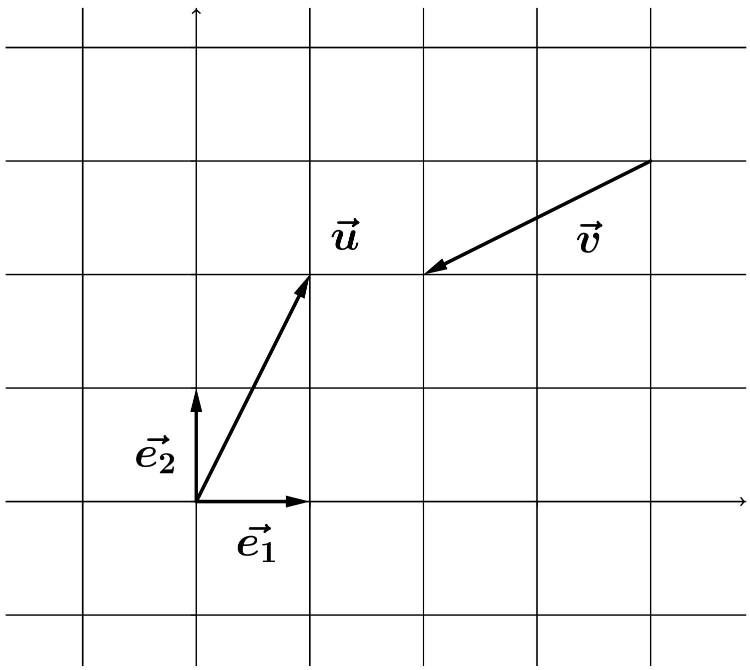



Figure A.4: Non-standard Basis

6. **Rotation:** Which, if any, of the following vectors $\overrightarrow{v}, \overrightarrow{w}, \overrightarrow{z}$ is the result of "the 90° rotation about the origin" of $\overrightarrow{u}$? You may or may not have multiple answers. Give an explanation of your thinking.

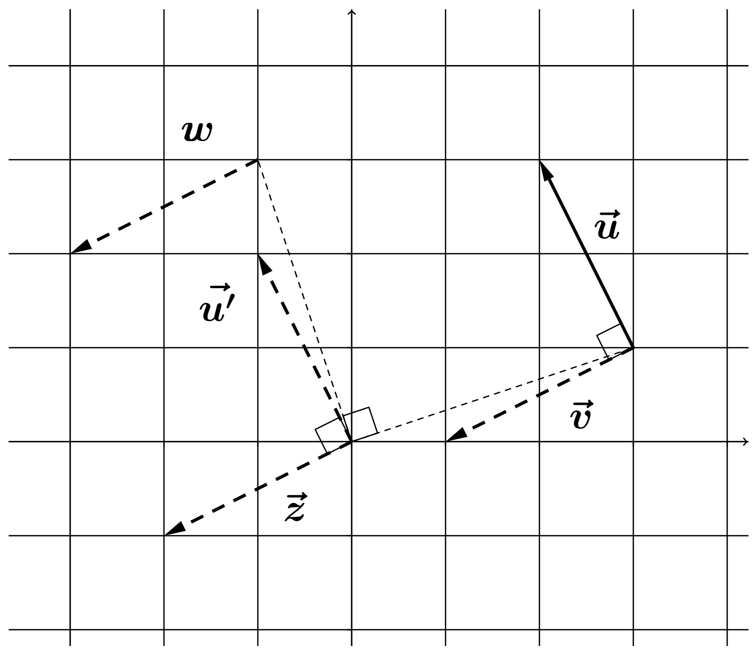



Figure A.5: Rotation

7. **Polygons:** For each polygon and drawn vectors, label each vector and write an equation that can describe the relations among them. You may reuse labels.

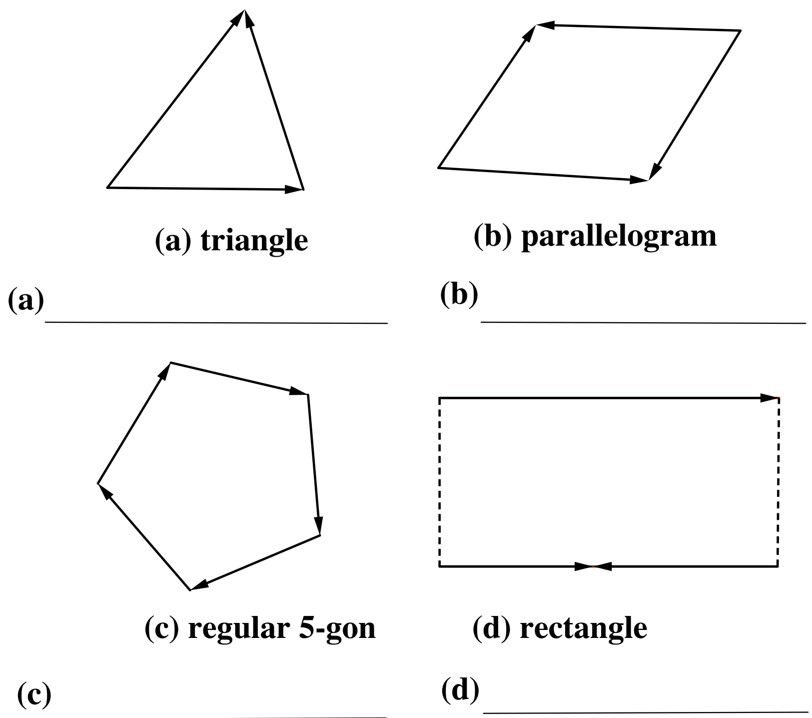



Figure A.6: Polygons

8. **A Very Long Sum:** Find $\overrightarrow{u_1} + \overrightarrow{u_2} + \ldots + \overrightarrow{u_n} - \overrightarrow{v}$.

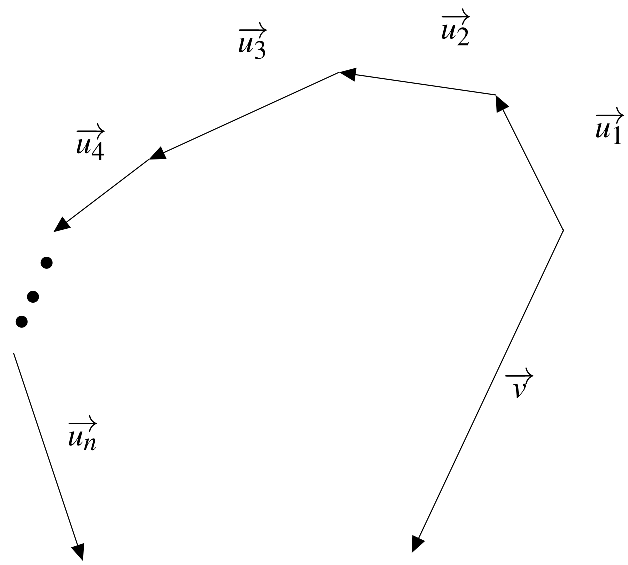



Figure A.7: A Very Long Sum

9. **Cube:** Prove that the two vectors shown in the cube are perpendicular. The sides of the cube can be assumed to be of length 1. Hint: You may wish to show something about the inner product also known as the dot product.

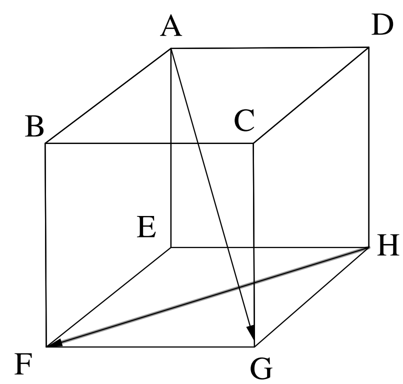



Figure A.8: Cube

10. **Triangle Midpoints:** In the following triangle $\triangle ABC$, $AB = 2AD$ and $AC = 2AE$. We want to show that $\overline{BC}$ is parallel to $\overline{DE}$.

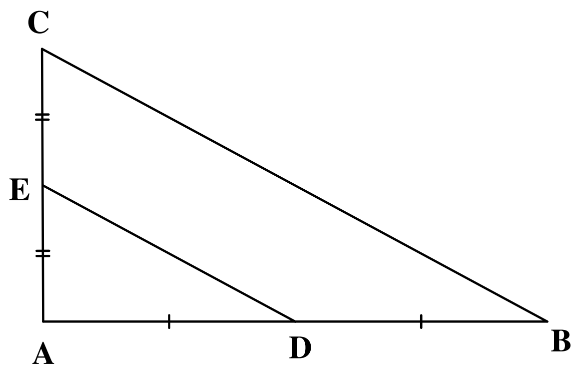



Figure A.9: Triangle Midpoints

(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:

(b) What vectors might be useful to show $\overline{BC} \parallel \overline{DE}$?

(c) What would you try to show to get that $\overline{BC} \parallel \overline{DE}$?

11. **Associativity:** We want to prove the following property of vector operation:

$$(\overrightarrow{a}+\overrightarrow{b})+\overrightarrow{c}=\overrightarrow{a}+(\overrightarrow{b}+\overrightarrow{c}).$$

Which form of vectors, do you think will be most useful? Circle your answer. You don't actually need to prove this.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) directed line segments

(iv) Other:

12. **Point/Vector:** For each case, find the midpoint $M$ of $\overline{AB}$ **using vectors**. Explain with your vector notation.

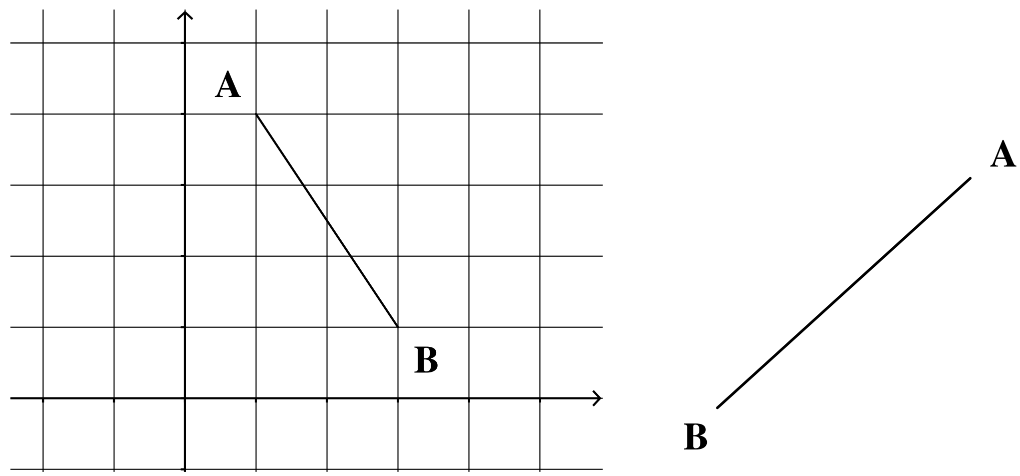



Figure A.10: Point/Vector

## Appendix B

## VECTOR REPRESENTATION SURVEY II

1. **Majors:** You are a (i) Mathematics major with secondary education focus, (ii) Mathematics major with elementary education focus, (iii) Mathematics minor, (iv) Other:________________________.

2. **High School:** In which high school courses did you see vectors? Select all that apply.

   (i) Physics (ii) Algebra I (iii) Algebra II (iv) Geometry

   (v) Trigonometry (vi) Precalculus (vii) Calculus

   (viii) Integrated Mathematics I, II, III, IV (ix) Other:________________________

3. **College:** In which college courses did you see vectors? Select all that apply.

   (i) Physics (ii) Linear Algebra (iii) Multivariable Calculus

   (iv) Higher Geometry (v) Abstract Algebra (vi) Real Analysis

   (vii) Differential Equations (viii) Complex Analysis (ix) Axiomatic Geometry

   (x) Other:________________________

4. **Concept Images:** How would you describe what a vector is to a high school student?

5. **Scale:** How important are vectors for high school mathematics?

   Unimportant Of Little Importance Moderate Important Important Very Important

6. **Scale:** How can you describe your understanding of vectors?

Very Poor    Poor    Barely Acceptable    Good    Very Good

7. **Translation:** A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point $P$.

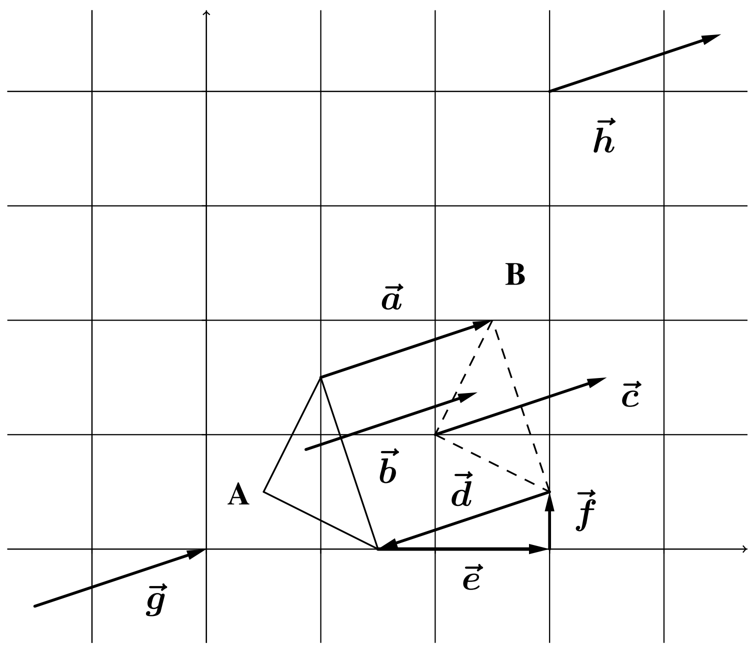



Figure B.1: Translation

(a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

(b) Circle all vectors that are equivalent to $\overrightarrow{\text{a}}$.

(a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

(b) List all vectors that are equivalent to $\overrightarrow{a}$.

8. **Force:** The following vectors represent **forces** applied on a physical object. A force 'a' pushes the tip of the block.

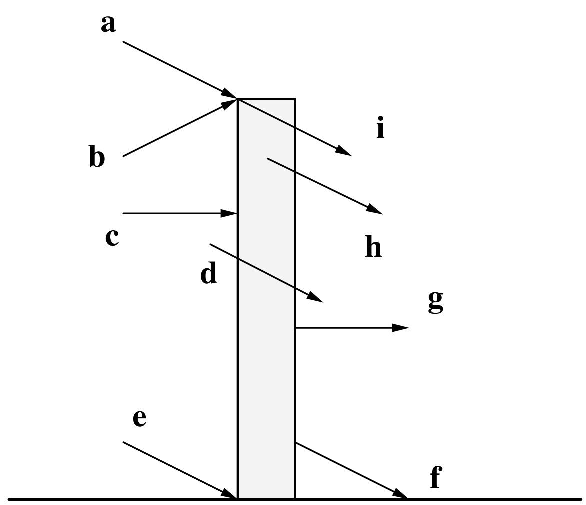



Figure B.2: Force

(a) List all that would produce the same result.

(b) List all vectors that are equivalent to 'a'.

(c) List all forces that are equivalent to 'a'.

9. **Robot Arm:** A robot arm has three joints as shown. The bottom part can be represented by a vector $\overrightarrow{a} = (10, 15, 20)$, the middle part can be represented by a vector $\overrightarrow{b} = (5, -5, 15)$, and the top part can be represented by $\overrightarrow{c} = (20, -5, -10)$. The units are in centimeters$(cm)$.

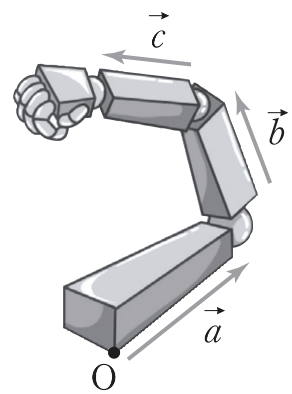



Figure B.3: Robot Arm

Find the vector $\overrightarrow{p}$ that tells the location of the hand.

10. **Origin:** For the three different origins: $O$, $C$, and $A$, complete the table by finding the position vector of $A$ and the vector $\overrightarrow{AB}$.

| Origin | Position Vector of $A$ | $\overrightarrow{AB}$ |
|---|---|---|
| $O$ | (____,____) | (____,____) |
| $C$ | (____,____) | (____,____) |
| $A$ | (____,____) | (____,____) |

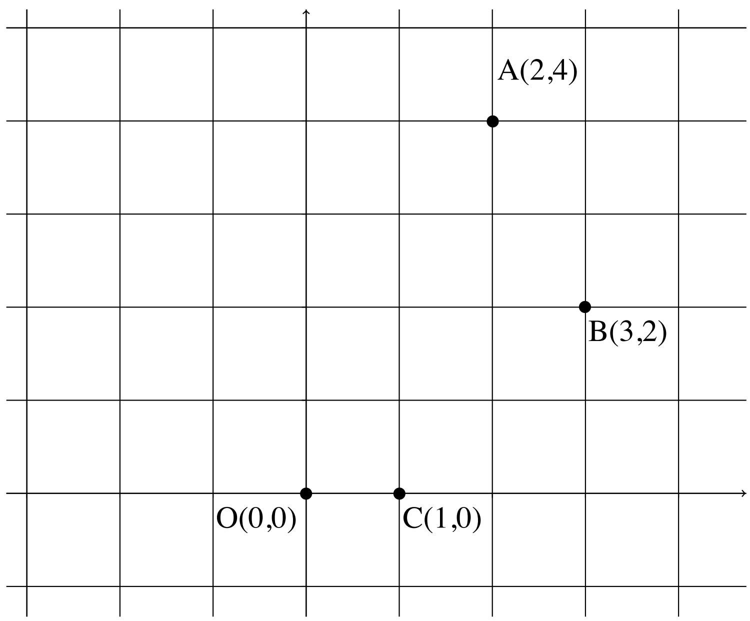



Figure B.4: Origin

11. **Non-standard Basis:** In a standard basis of $\{\overrightarrow{e_1}, \overrightarrow{e_2}\}$ for $\mathbb{R}^2$, the vector $\overrightarrow{u}$ can be represented by the ordered pair $(1,2)$. The set $\{\overrightarrow{e_1}, \overrightarrow{u}\}$ also forms a basis for $\mathbb{R}^2$. Find an ordered pair $(a,b)$ for the vector $\overrightarrow{v}$ using the basis $\{\overrightarrow{e_1}, \overrightarrow{u}\}$.

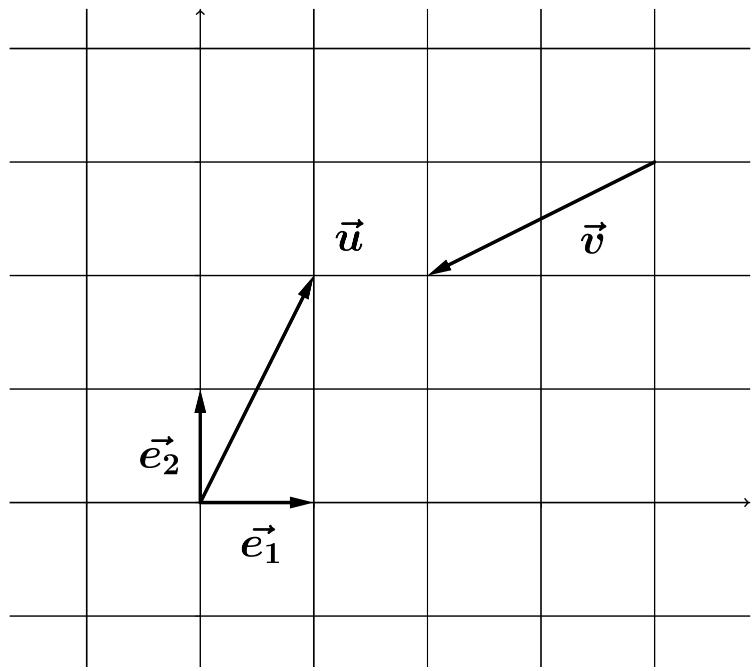



Figure B.5: Non-standard Basis

12. **Rotation:** Which, if any, of the following vectors $\overrightarrow{v}, \overrightarrow{w}, \overrightarrow{z}$ is the result of "the 90° rotation about the origin" of $\overrightarrow{u}$? You may or may not have multiple answers. Give an explanation of your thinking.

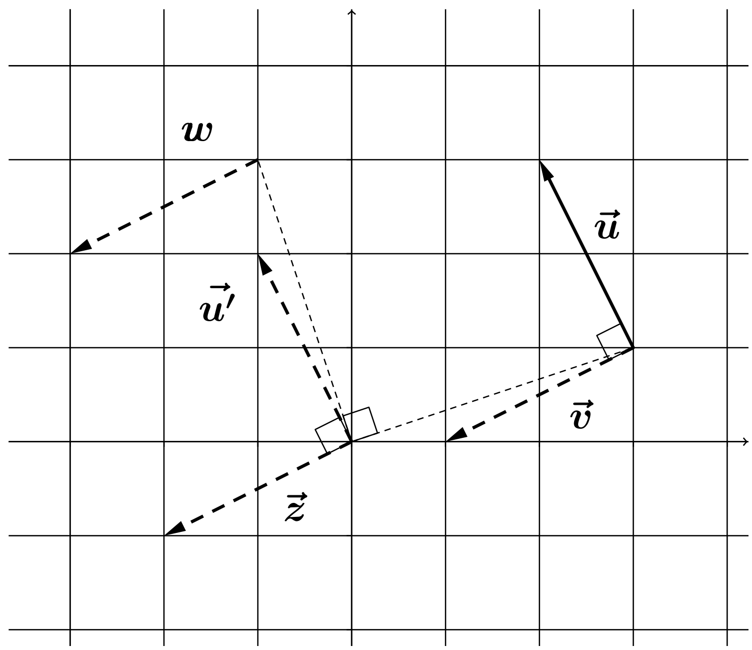



Figure B.6: Rotation

13. **Polygons:** For each polygon and drawn vectors, label each vector and write an equation that can describe the relations among them. You may reuse labels.

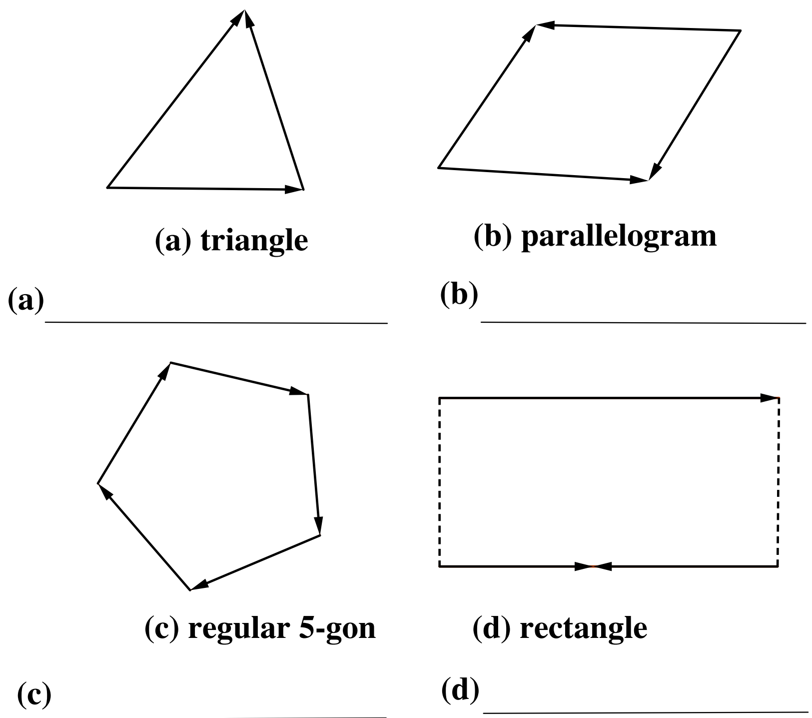



Figure B.7: Polygons

14. **A Very Long Sum:** Find $\overrightarrow{u_1} + \overrightarrow{u_2} + \ldots + \overrightarrow{u_n} - \overrightarrow{v}$.

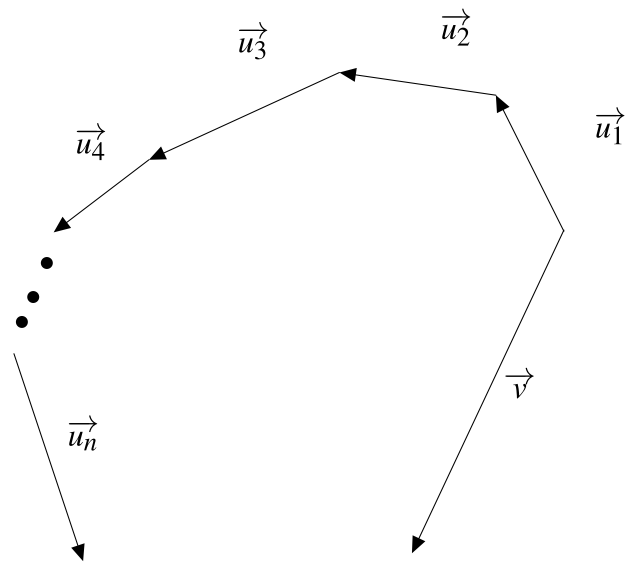



Figure B.8: A Very Long Sum

15. **Cube:** Prove that the two vectors shown in the cube are perpendicular. The sides of the cube can be assumed to be of length 1. Hint: You may wish to show something about the inner product also known as the dot product.

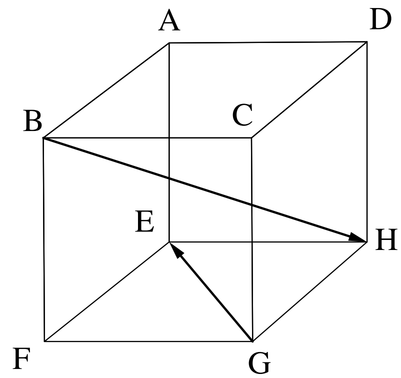



Figure B.9: Cube

16. **Triangle Midpoints:** In the following triangle $\triangle ABC$, $AB = 2AD$ and $AC = 2AE$. We want to show that $\overline{BC}$ is parallel to $\overline{DE}$.

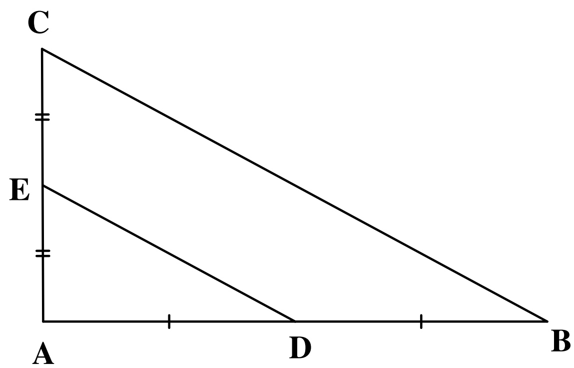



Figure B.10: Triangle Midpoints

(a) Which form of vectors, do you think will be most useful? Circle your answer.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:________________________

(b) List two vectors in any form that might be useful so that you would want to use.

(c) What would you try to show to get that $\overline{BC} \parallel \overline{DE}$?

17. **Associativity:** We want to prove the following property of vector addition:

$$(\overrightarrow{a}+\overrightarrow{b})+\overrightarrow{c}=\overrightarrow{a}+(\overrightarrow{b}+\overrightarrow{c}).$$

(a) Which form of vectors, do you think will be most useful? Circle your answer. You don't actually need to prove this.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) directed line segments

(iv) Other:________________________

(b) Why have you preferred this representation and not another one?

18. **Point/Vector:** For each case, find the midpoint $M$ of $\overline{AB}$ **using vectors**. Explain with your **vector** notation.

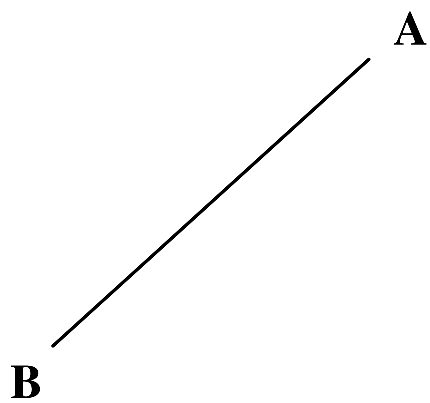



Figure B.11: Point/Vector

**Appendix C**

**VECTOR REPRESENTATION SURVEY III**

# Student Understand of Geometric Translation

## Translation from CMP

**1.** The diagram below shows a polygon and its image under a translation. Draw a line segment from each vertex of polygon to its image. Describe the relationship among the line segments you drew.

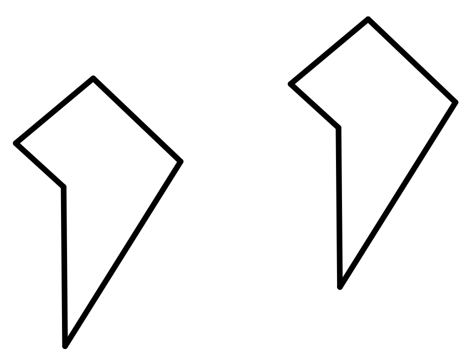


Figure C.1: Polygon Diagram

**2.** Complete this definition of a translation: A *translation* matches any two points $X$ and $Y$ on a figure to image points $X'$ and $Y'$ so that...

**3.** Ella specified a translation of a polygon by drawing an arrow. How would you interpret Ella's drawing? Perform the translation Ella specified.

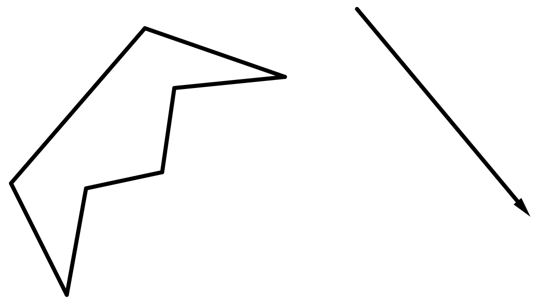


Figure C.2: Translation of Polygon

**4.** When you translate a figure, you can imagine that you are translating the entire plane and brining the figure along for the ride. When you perform a translation, are any points in the plane in the same location after the translation? That is, are there any fixed points? Explain your answer.

## Translation from Beckmann

**1.** A *translation* (or slide) of a plane by a given distance in a given direction is the end result of moving each point in the plane the given distance in the given direction. The solid arrow represents the distance and direction of the translation. **In the following figure, triangle** $B$ **is the image of triangle** $A$ **under a given translation.**

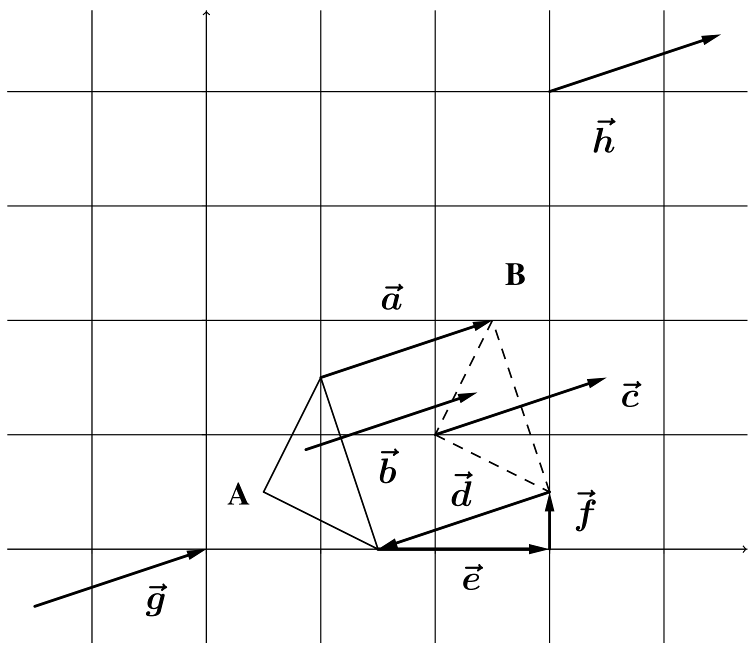
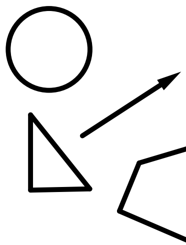



Figure C.3: Translation

(a) List all arrows that do **NOT** represent the translation that was specified above.

(b) List all arrows that represent the same translation with the arrow $\overrightarrow{a}$.

**2.** Circle one that describes the image of a translation by an arrow in the following figure.



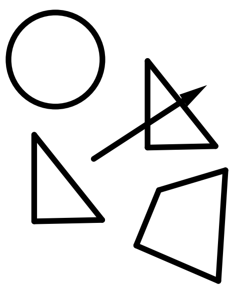
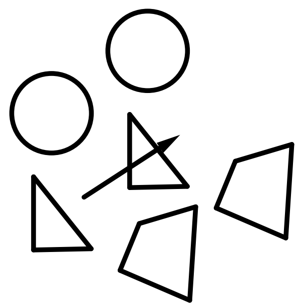

,

, None of the left.(Explain or draw your own answer.)

Figure C.4: Geometric Translation

## Appendix D

## VECTOR REPRESENTATION SURVEY IV

1. **Majors:** You are a (i) Mathematics major with secondary education focus, (ii) Mathematics major with elementary education focus, (iii) Mathematics minor, (iv) Other:________________________.

2. **High School:** In which high school courses did you see vectors? Select all that apply.
   (i) Physics (ii) Algebra I (iii) Algebra II (iv) Geometry
   (v) Trigonometry (vi) Precalculus (vii) Calculus
   (viii) Integrated Mathematics I, II, III, IV (ix) Other:________________________

3. **College:** In which college courses did you see vectors? Select all that apply.
   (i) Physics (ii) Linear Algebra (309) (iii) Multivariable Calculus (133, 234)
   (iv) Higher Geometry (330) (v) Abstract Algebra (310) (vi) Real Analysis (320)
   (vii) Differential Equations (235, 340) (viii) Complex Analysis (425) (ix) Axiomatic Geometry (432)
   (x) Other:________________________

4. **Concept Images:** How would you describe what a vector is to your future students?

5. **Scale:** How important are vectors for school mathematics?
   Unimportant Of Little Importance Moderate Important Important Very Important

6. **Scale:** How can you describe your understanding of vectors?
   Very Poor Poor Barely Acceptable Good Very Good

7. **Translation:** A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point $P$.

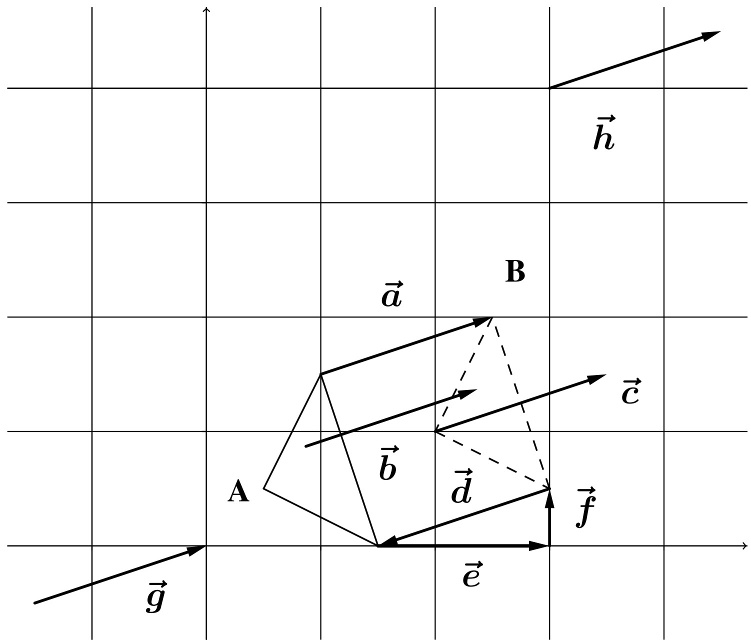



Figure D.1: Translation

(a) List all vectors that do **NOT** represent the translation that maps triangle A to triangle B in the figure.

(b) List all vectors that are equivalent to $\overrightarrow{a}$.

8. **Triangle:** Assume you have a triangle $\triangle ABC$. Find each of the following.

   (a) $\overrightarrow{AB} + \overrightarrow{BC}$

   (b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

9. **Rectangle:** Assume you have a rectangle $\square ABCD$. Find each of the following.

   (a) $\overrightarrow{AB} + \overrightarrow{CB}$

   (b) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

   (c) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$

10. **Translation of Polygon:** Ella specified a translation of a polygon by drawing an arrow. How would you interpret Ella's drawing? Perform the translation Ella specified.

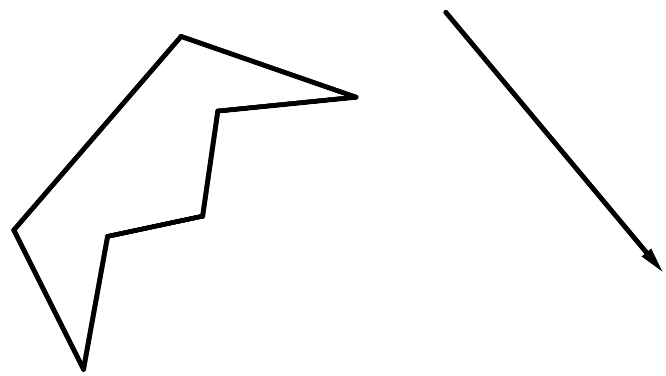
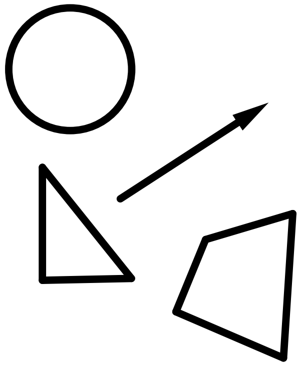
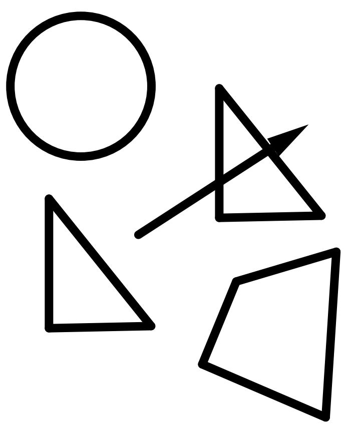


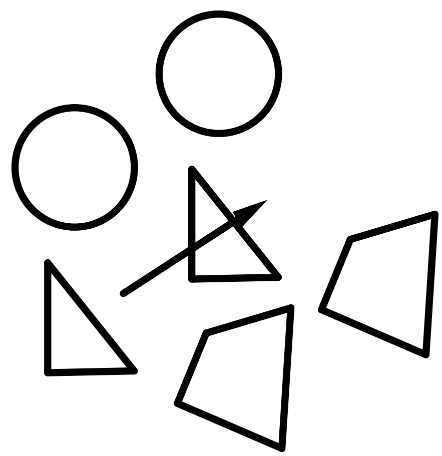
Figure D.2: Translation of Polygon

11. **Geometric Translation:** Circle one that describes the image of a translation by an arrow in the following figure.





(1) ,



(2) ,

(3) None of the above. (Explain or draw your own answer.)

Figure D.3: Geometric Translation

12. **Force:** The following vectors represent **forces** applied on a physical object. A force 'a' pushes the tip of the block.

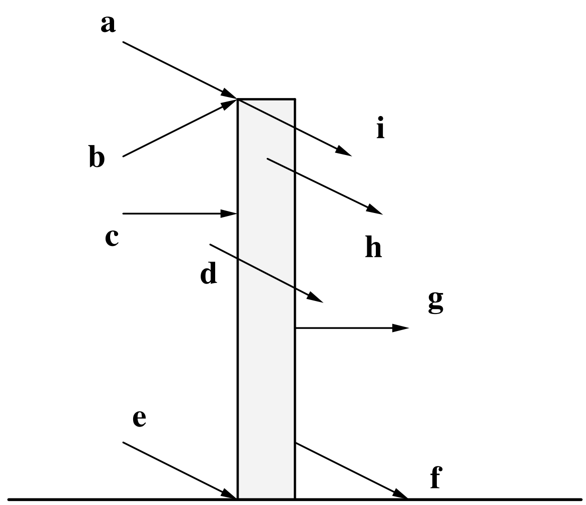



Figure D.4: Force

(a) List all vectors that are equivalent to 'a'.

(b) List all forces that are equivalent to 'a'.

13. **Origin:** For the three different origins: $O$, $C$, and $A$, complete the table by finding the position vector of $A$ and the vector $\overrightarrow{AB}$.

| Origin | Position Vector of $A$ | $\overrightarrow{AB}$ |
|---|---|---|
| $O$ | (____,____) | (____,____) |
| $C$ | (____,____) | (____,____) |
| $A$ | (____,____) | (____,____) |

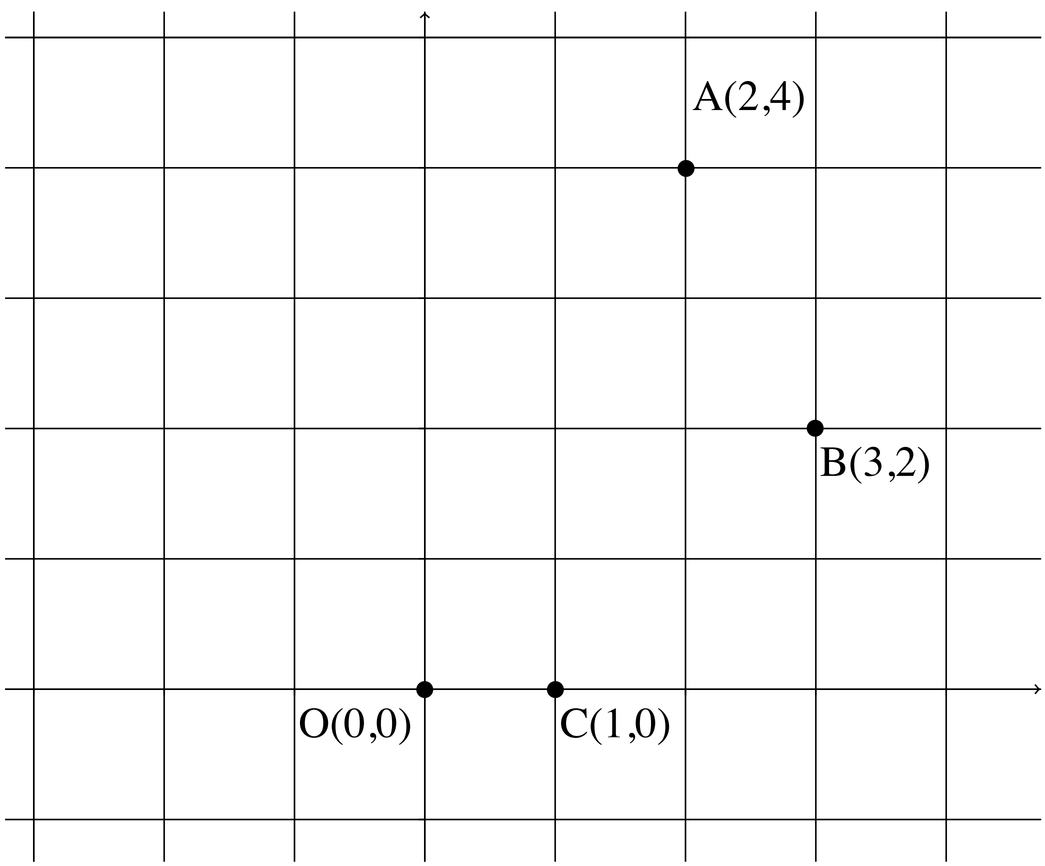



Figure D.5: Origin

14. **Polygons:** For each polygon and drawn vectors, label each vector and write an equation that can describe the relations among them. You may reuse labels.

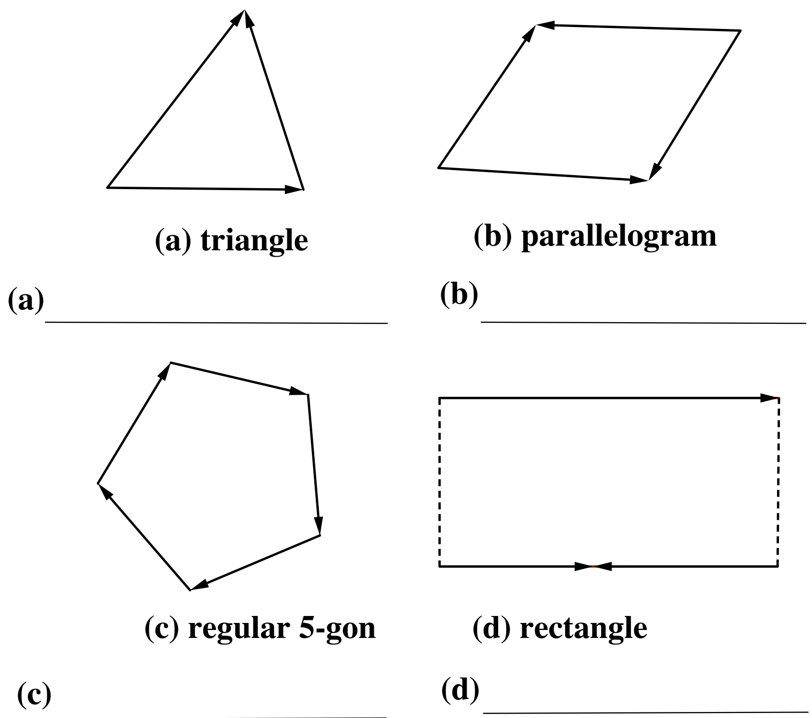



Figure D.6: Polygons

15. **A Very Long Sum:** Find $\overrightarrow{u_1} + \overrightarrow{u_2} + \ldots + \overrightarrow{u_n}$.

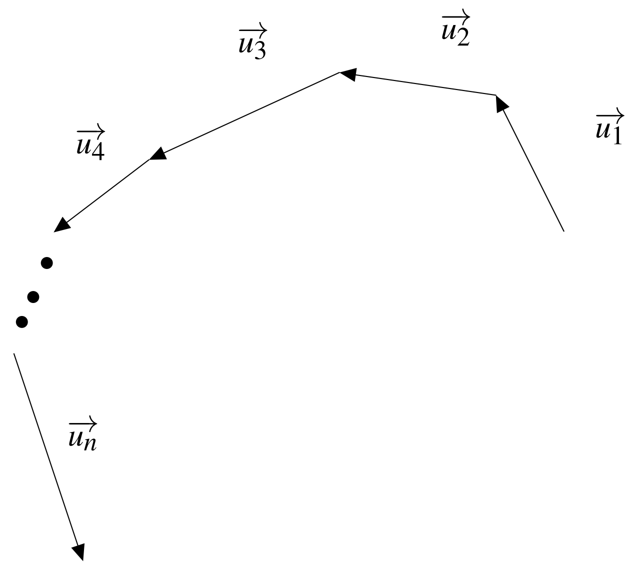



Figure D.7: A Very Long Sum

16. **Cube:** In the following figure, we want to show that the two vectors shown in the cube are perpendicular using the inner product also known as the dot product. The sides of the cube can be assumed to be of length 1.

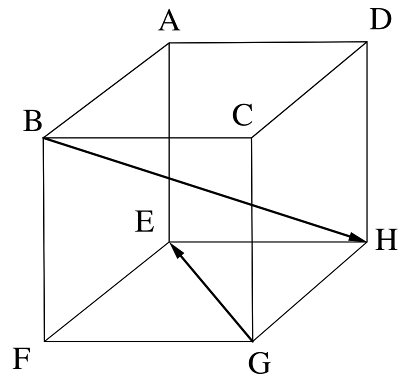



Figure D.8: Cube

(a) Which form of vectors, do you think will be most useful? Circle your answer.

(i) $(a,b,c)$ (ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:________________________

(b) Which form of vectors would be the next choice if the form you choose in (a) does not work?

17. **Triangle Midpoints:** In the following triangle $\triangle ABC$, $AB = 2AD$ and $AC = 2AE$. We want to show that $\overline{BC}$ is parallel to $\overline{DE}$.

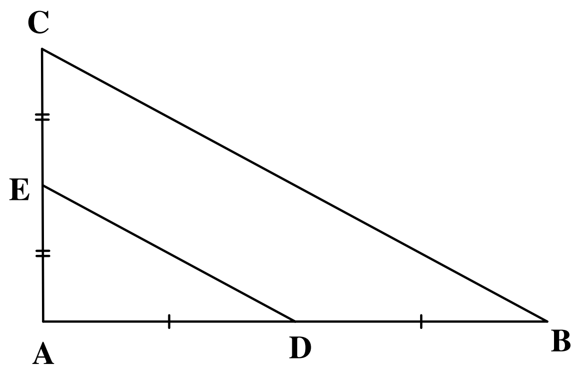



Figure D.9: Triangle Midpoints

(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer.

(i) $(a,b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:________________________

(b) Which <u>form</u> of vectors would be the next choice if the form you choose in (a) does not work?

(c) List two vectors in any form that might be useful so that you would want to use.

(d) What would you try to show to get that $\overline{BC} \parallel \overline{DE}$?

18. **Associativity:** We want to prove the following property of vector addition:

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}).$$

(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer. You don't actually need to prove this.

(i) $(a, b)$ (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ (iii) $\overrightarrow{PQ}$ (iv) $\overrightarrow{u}$ (v) Other:________________________

(b) Why have you preferred this representation and not another one?

19. **Point/Vector:** For each case, find the midpoint $M$ of $\overline{AB}$ **using vectors**. Explain with your **vector** notation.

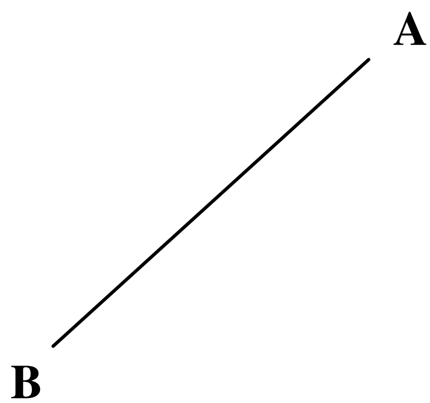



Figure D.10: Point/Vector

## Appendix E

## ADDITIONAL INTERVIEW QUESTIONS

1. **Triangle:** Assume that you have a triangle $\triangle ABC$. Find each of the following. Explain.

   (a) $\overrightarrow{AB}+\overrightarrow{BC}$

   (b) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}$

2. **Vector Space:** Only one of the following statements is true. Which one? Explain.

   (i) The vector space $V=\{0\}$ consisting of a zero vector has the basis $\{0\}$.

   (ii) The vector space $V=\{0\}$ consisting of a zero vector has the basis an empty set $\emptyset$.

   (iii) The vector space $V=\{0\}$ consisting of a zero vector has no basis.

3. **Triangle Midpoints:** In the following triangle $\triangle ABC$, $AB=2AD$ and $AC=2AE$. We want to show that $\overline{BC}$ is parallel to $\overline{DE}$.

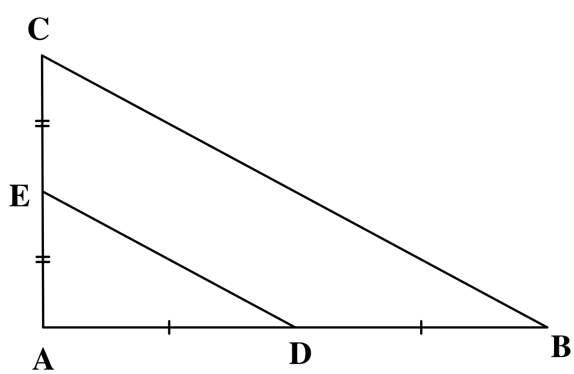



Figure E.1: Triangle Midpoints

   (a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer. • $(a,b)$

   • $\begin{pmatrix} x \\ y \end{pmatrix}$ • $\overrightarrow{PQ}$ • $\vec{u}$ • Other:________________________

(b) Using your vector form, what would you try to show to get that $\overline{BC} \parallel \overline{DE}$?

(c) Proceed your proof with the preferred vector representation.

(d) Why have you preferred this representation and not another one?

(e) Prove with the other vector representation that you did not use if you can.

(f) What difficulties are you faced with, for each method? Namely, what forced you to abandon a representation, if anything like that has happened, or what has prevented you in reaching a final proof?

4. **Associativity:** We want to prove the following property of vector operation:

$$(\overrightarrow{a} + \overrightarrow{b}) + \overrightarrow{c} = \overrightarrow{a} + (\overrightarrow{b} + \overrightarrow{c}).$$

(a) Which <u>form</u> of vectors, do you think will be most useful? Circle your answer.

- $(a,b)$
- $\begin{pmatrix} x \\ y \end{pmatrix}$
- directed line segments
- Other:________________________

(b) Proceed your proof with the preferred vector representation.

(c) Why have you preferred this representation and not another one?

(d) Prove with the other vector representation that you did not use if you can.

(e) What dimension are you working on with your vector representation?

(f) What difficulties are you faced with, for each method? Namely, what forced you to abandon a representation, if anything like that has happened, or what has prevented you in reaching a final proof?

5. **Point vs.Vector:** Prove that the midpoints of the two diagonals in the figure coincide using vectors.

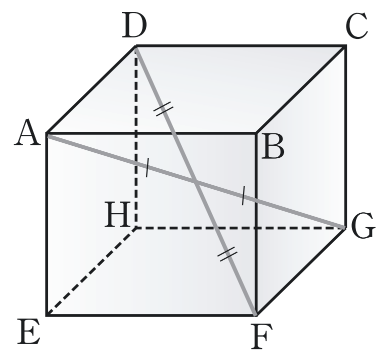



Figure E.2: Point vs. Vector

(a) Which vector representations do you want to use to prove the theorem, coordinate/column vector forms such as $(a,b,c)$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or generic vector forms such as $\overrightarrow{AB}$, $\overrightarrow{u}$?

(b) Proceed your proof with the preferred vector representation.

(c) Why have you preferred this representation and not another one?

(d) Prove with the other vector representation that you did not use if you can.

(e) What difficulties are you faced with, for each method? Namely, what forced you to abandon a representation, if anything like that has happened, or what has prevented you in reaching a final proof?

6. **Additional Questions:** Any questions from test that the researcher may want to revisit with selected students will be included in addition.

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# BIBLIOGRAPHY

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