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GEODESICS IN 3-SPACE

THESIS FOR THE DEGREE OF M. A.

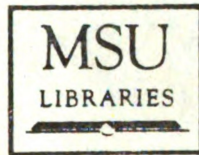
John William Zimmer

1933

Calculus of variations
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GEODESICS IN 3-SPACE

A Thesis

Submitted to the Faculty

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of

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In Partial Fulfillment of the

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John William Zimmer

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INTRODUCTION

We shall define geodesics as curves of shortest distance on a surface. That is, a geodesic is a curve on a surface such that the distance between two points measured along it is less than the distance measured along a neighboring curve joining the two points. The problem of finding geodesics is then a problem of the calculus of variation. We shall show that the definitions usually given in metric differential geometry are necessary conditions that a curve be a geodesic according to our definition. They are not always sufficient.

It is not the purpose of this paper to give all the properties of geodesics that have been discovered. In the bibliography will be found a list of papers giving these properties. We shall confine ourselves to a discussion of these properties that can be derived by the methods of the calculus of variations.

In part I the problem is set up as an ordinary problem in the calculus of variations. Necessary conditions and sufficient conditions are given for a curve to be a geodesic. The results obtained are applied to several examples, including the sphere and pseudo-sphere. In part II the problem is treated as a Lagrange problem. In part III an interesting inverse problem connected with geodesics is considered. The problem is to find

all surfaces on which the geodesics can be represented linearly in u and v . Part IV is a discussion of the isoperimetric problem of finding curves of shortest distance joining two points A and B and enclosing with a given curve, joining A and B , a given area.

I GEODESICS AS AN ORDINARY PROBLEM OF THE CALCULUS OF VARIATIONS

1. Statement of the problem. The equations of our surface S are taken in the Gaussian form

$$(1) \quad \begin{aligned} x &= x(u, v), \\ S. \quad y &= y(u, v), \\ z &= z(u, v), \end{aligned}$$

We shall assume that $x(u, v)$, $y(u, v)$ and $z(u, v)$ have as many derivatives as necessary to carry through the analysis. We shall also treat only those surfaces for which $H^2 = EG - F^2$ is everywhere different from zero.* The length of a curve joining two points on S is given by the integral

$$(2) \quad \int_{u_1}^{u_2} \sqrt{E + 2Fv' + Gv'^2} du. \quad **$$

The problem of the calculus of variations, then, is to find among all admissible arcs joining the points 1 and 2, one which minimizes the integral (2). Such curves we shall call geodesics. We shall discuss the necessary conditions and sufficient conditions that a curve be a geodesic.

* L.P. Eisenhart, Differential Geometry of Curves and Spaces, p. 71. (Hereafter referred to as Eisenhart.)

** Eisenhart, pp. 220-221.

2. Euler's equation. With the function f in the form given in (2) the Euler equation

$$(3) \quad \frac{d}{du} f_{v'} - f_v = f_{v'u} + f_{v'v} V' + f_{v'v'} V'' - f_v = 0,$$

becomes

$$(4) \quad \begin{aligned} & 2(E G - F^2) V'' + (G \frac{\partial G}{\partial u} + 3F \frac{\partial G}{\partial v} - 2F \frac{\partial G}{\partial v'} - 2 \frac{\partial F}{\partial v}) V'^3 \\ & + (3F \frac{\partial G}{\partial u} + 2E \frac{\partial G}{\partial v} + 2F \frac{\partial F}{\partial v} - G \frac{\partial E}{\partial v} - E \frac{\partial G}{\partial v} - 4F \frac{\partial F}{\partial v} - G \frac{\partial E}{\partial v}) V'^2 \\ & + (2E \frac{\partial G}{\partial u} + 2F \frac{\partial F}{\partial u} - G \frac{\partial E}{\partial u} - F \frac{\partial E}{\partial v} - 2F \frac{\partial E}{\partial v'}) V' \\ & + 2E \frac{\partial F}{\partial u} - F \frac{\partial E}{\partial u} - E \frac{\partial E}{\partial v} = 0, \end{aligned}$$

which may be written in the form

$$(5) \quad \begin{aligned} & V'' + \frac{(G \frac{\partial G}{\partial u} + F \frac{\partial G}{\partial v} - 2 \frac{\partial F}{\partial v})}{2H^2} V'^3 \\ & + \left\{ \frac{(E \frac{\partial G}{\partial v} + F \frac{\partial G}{\partial u} - 2F \frac{\partial F}{\partial v})}{2H^2} - \frac{2(G \frac{\partial E}{\partial v} - F \frac{\partial G}{\partial u})}{2H^2} \right\} V'^2 \\ & + \left\{ \frac{2(E \frac{\partial G}{\partial u} - F \frac{\partial E}{\partial v})}{2H^2} - \frac{(G \frac{\partial E}{\partial u} + F \frac{\partial E}{\partial v} - 2F \frac{\partial F}{\partial u})}{2H^2} \right\} V' \\ & + \left\{ \frac{2E \frac{\partial F}{\partial u} - F \frac{\partial E}{\partial u} - E \frac{\partial E}{\partial v}}{2H^2} \right\} = 0. \end{aligned}$$

Expressing (5) in terms of Christoffel symbols of the second kind we get

$$\begin{aligned}
 & V'' - \{^2_1, 2\} V'^3 + (\{^2_2, 2\} - \{^1_1, 2\}) V'^2 \\
 (6) \quad & + \{^2_2, 1\} V' + \{^1_1, 1\} = 0.
 \end{aligned}$$

Geodesics are defined by means of (6) in metric differential geometry.* Since we have shown that Euler's equation reduces to the form of (6) for our problem the ordinary definition of a geodesic becomes a necessary condition. In other words, a curve on the surface must satisfy (6) before it can be a geodesic. This condition, however, is not sufficient. We shall call solutions of equation (6) extremals.

3. The Weierstrass-Erdmann corner condition.** A corner is defined to be a point on the geodesic at which V' is discontinuous but has a right and left hand limit.

We shall define admissible arcs as those arcs that are continuous and have only a finite number of corners. If the geodesic has a corner (u, v) the Weierstrass-Erdmann cor-

* Eisenhart, p. 205.

** G.A. Bliss, Calculus of Variations. p. 143. (Hereafter referred to as Bliss I.)

ner condition says that

$$(7) \quad f_{V'}(u, v, V'(u-0)) = f_{V'}(u, v, V'(u+0)),$$

using $f = \sqrt{E + 2FV' + G V'^2}$, (7) becomes

$$(8) \quad \frac{F + G V'_-}{\sqrt{E + 2F V'_- + G V'^2_-}} = \frac{F + G V'_+}{\sqrt{E + 2F V'_+ + G V'^2_+}}.$$

If we square both sides of (8) and then multiply both sides by $(E + 2F V'_+ + G V'^2_+)(E + 2F V'_- + G V'^2_-)$ we get

$$(9) \quad \begin{aligned} & \{EF^2 + 2EFG V'_+ + EG^2 V'^2_+ + 2F^3 V'_- + 4GF^2 V'_+ V'_- \\ & 2FG^2 V'^2_+ V'^2_- + F^2 G V'^2_- + 2FG^2 V'_+ V'^2_- + G^2 V'^2_+ V'^2_- \} = \\ & \{ 2EFG V'_- + EG^2 V'^2_- + 2F^3 V'_+ + 4F^2 G V'_+ V'_- + 2FG^2 V'_+ V'^2_- \\ & - F^2 G V'^2_+ + 2FG^2 V'^2_+ V'_- + G V'^2_+ V'^2_- \}. \end{aligned}$$

Equation (9) reduces to

$$(10) \quad H^2(V'_+ - V'_-) \{2F + G(V'_+ + V'_-)\} = 0.$$

Equation (10) can be zero when

$$(11) \quad V'_+ = V'_-,$$

or

$$(12) \quad 2F + G(V'_+ + V'_-) = 0.$$

From (12) we find that $\sqrt{E + 2FV'_+ + G{V'_+}^2} = \sqrt{E + 2FV'_- + G{V'_-}^2}$
and $F + GV'_- = -(F + GV'_+)$. Therefore (8) becomes

$$(13) \quad \frac{-(F + GV'_+)}{\sqrt{E + 2FV'_+ + G{V'_+}^2}} = \frac{F + GV'_+}{\sqrt{E + 2FV'_+ + G{V'_+}^2}}.$$

From (12) and (13) we have that $V'_+ = V'_- = -\frac{F}{G}$. Hence we conclude that a geodesic can not have corners.

4. The necessary condition of Weierstrass.* Let

$G_{1,2}$ be a geodesic. The necessary condition of Weierstrass is that for every (u, v, v') of $G_{1,2}$ the function

$$(14) \quad E(u, v, v', V') = f(u, v, V') - f(u, v, v') - (V' - v') f_{v'}(u, v, v')$$

must be greater than or equal to zero for every $(u, v, V') \neq (u, v, v')$.

For the present problem (14) becomes

$$(15) \quad E(u, v, v', V') = \sqrt{E + 2FV' + G{V'}^2} - \sqrt{E + 2Fv' + G{v'}^2} - \frac{(V' - v')(F + Gv')}{\sqrt{E + 2Fv' + G{v'}^2}},$$

which may be written as

$$(16) \quad E(u, v, v', V') = \frac{\sqrt{E + 2FV' + G{V'}^2} \sqrt{E + 2Fv' + G{v'}^2} - [E + (V' + v')F + Gv'V']}{\sqrt{E + 2Fv' + G{v'}^2}}.$$

Since the denominator of the right hand side of (16) is the expression for ds , and is always taken to be positive, we need only to show that

* Bliss I p. 131.

$$(17) \quad \sqrt{E + 2FV' + G V'^2} \sqrt{E + 2FV' + G V'^2} \geq E + (V' - V)F + G V' V'.$$

If we write the right hand side of (17) as $\sqrt{[E + (V' - V)F + G V' V']^2}$ and express the left hand side as a single radical, we may then compare the two radicands. We get

$$(18) \quad E G V'^2 + 4 F^2 V' V' + E G V'^2 \geq \\ 2 E G V' V' + V'^2 F + 2 F^2 V' V' + V'^2 F^2.$$

Setting $H^2 = EG - F^2$ we may write (18) as

$$(19) \quad (F^2 + H^2) V'^2 + 4 F^2 V' V' + (F^2 + H^2) V'^2 \geq \\ 2(F^2 + H^2) V' V' + V'^2 F^2 + 2 F^2 V' V' + V'^2 F^2,$$

which reduces to

$$(20) \quad H^2 (V' - V) > 0.$$

We have shown that for every $V' \neq V$, $E(u, v, v'; V) > 0$ everywhere on our surface. The Weierstrass condition is, therefore, not a condition on the minimizing curve at all.

5. The necessary condition of Legendre.^{*} The Legendre condition states that if G_2 is a geodesic then at every (u, v, v') of G_2 the condition

$$(21) \quad f_{v''v''}(u, v, v') \geq 0$$

* Bliss I, p. 131.

must be satisfied.

For our problem (21) is

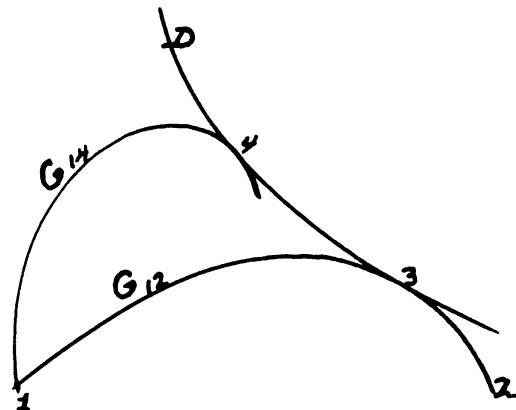
$$(22) \quad f_{v'v'} = \frac{EG - F^2}{(E + 2Fv' + Gv'^2)^{\frac{3}{2}}}$$

$$= \frac{H^2}{(E + 2Fv' + Gv'^2)^{\frac{3}{2}}}.$$

Therefore $f_{v'v'} > 0$ everywhere on the surface.

Hence Legendre's condition places no restriction on the curves that can be geodesics.

6. The necessary condition of Jacobi.* If we consider the single infinity of geodesics through the point 1, and if this family has an envelope, then the geodesic can have no contact point 3 with the envelope between 1 and 2. For, by the use of the envelope theorem the composite arc $G_{14} + D_{43} + G_{32} = G_{12}^{**}$.



But D_{43} is not a geodesic and may be replaced by an arc which is shorter than D_{43} . This would give the composite

* Bliss I, p. 132.

** Bliss I, p. 141.

arc $G_{14} + D_{43} + G_{32}$ a smaller value than the geodesic G_{12} . In every neighborhood of G_{13} there is such an arc $G_{14} + D_{43} + G_{32}$ giving a shorter distance from 1 to 2 than the geodesic. Hence the geodesic cannot be the shortest distance. This contradicts our definition of a geodesic.

To make sure that D_{43} is not a geodesic we make use of a well known property of a second order differential equation.* Euler's equation for this geodesic is

$$(23) \quad f_{v'u} + f_{v'v} v' + f_{v'v'} v'' - f_v = 0.$$

Equation (23) can be solved for v'' since $f_{v'v'} \neq 0$. If an equation of the type of (23) can be solved for v'' , then there is one and only one solution through an arbitrarily selected initial point and direction (u_3, v_3, v'_3) . Hence if D_{43} were a geodesic it would be necessary for it to coincide with G_{13} . Thus, all the geodesics through the point 1 would, by the same property, be tangent to and coincide with G_{13} . Then there would be no one-parameter family of geodesics as the theorem supposes.

If the Jacobi condition is to be of much value we will need a convenient method for determining whether the geodesic has a contact point 3, called a conjugate point of 1,

* G.A. Bliss, Princeton Colloquium Lectures.

with the envelope of the family of geodesics through the point 1. From the calculus we know that if we have a one-parameter family of curves $V = V(u, \alpha)$ the contact point of any one of these curves with the envelope of the family is a root of $V_\alpha(u, \alpha)$ when α_0 is the value of α defining that particular member of the family.*

It is not always easy to determine a 1-parameter family of extremals through the point 1 even if we do have the 2-parameter family of solutions of Euler's equation. For that reason we need a method of finding the points conjugate to 1 from the original 2-parameter family of solutions

$$V = V(u, a, b).$$

If we were to write the 2-parameter family of extremals as a 1-parameter family it would be necessary to choose the two functions $a(\alpha)$ and $b(\alpha)$ such that

$$(24) \quad V(u, \alpha) = V(u, a(\alpha), b(\alpha))$$

satisfies the equation

$$(25) \quad V_1 = V(u, a, b).$$

If we differentiate (24) and (25) with respect to α we have

$$V_\alpha = V_a(u, a, b)a' + V_b(u, a, b)b',$$

$$0 = V_a(u, a, b)a' + V_b(u, a, b)b',$$

* Wilson, Advanced Calculus, p. 136.

where a' and b' are derivatives of a and b with respect to α . When V_α vanishes the determinant of the coefficients of a' and b' must also vanish. We can now state that the points (3) conjugate to a point 1 on the geodesic are determined by the zeros for $u \neq u_1$ of the determinant

$$(26) \quad \Delta(u, u_1) = \begin{vmatrix} V_a(u, a_0, b_0) & V_b(u, a_0, b_0) \\ V_a(u_1, a_0, b_0) & V_b(u_1, a_0, b_0) \end{vmatrix}$$

when $V(u, a, b)$ is a 2-parameter family of extremals containing the particular extremal for parameter values a_0, b_0 . We shall find later that Jacobi's condition does not hold for all extremals.

7. Sufficiency conditions. In this section we shall give conditions that insure that our minimizing arc is a geodesic. To accomplish this let us first define a field. A field is a region $\mathcal{F}$ of (u, v) -space which has associated with it a 1-parameter family of extremal arcs all of which intersect a fixed curve $\mathcal{D}$ and which have the further property that through each point (u, v) of $\mathcal{F}$ there passes one and but one extremal of the family.*

By applying the fundamental sufficiency theorem of the calculus of variations we have the following theorem:**

*Bliss I, p. 132.

**Bliss I, p. 133.

Theorem. If $G_{1,2}$ is an extremal of a field $\mathcal{F}$ then $G_{1,2}$ is shorter than any other admissible arc $C_{1,2}$ in $\mathcal{F}$ joining the points 1 and 2. That is, $G_{1,2}$ is a geodesic.

This follows since $E(u, v, v'; V') > 0$ everywhere on our surface.

Theorem. If a curve $G_{1,2}$ is a solution of Euler's equation and has no conjugate point (3) between 1 and 2 nor at 2, then $G_{1,2}$ is a geodesic.

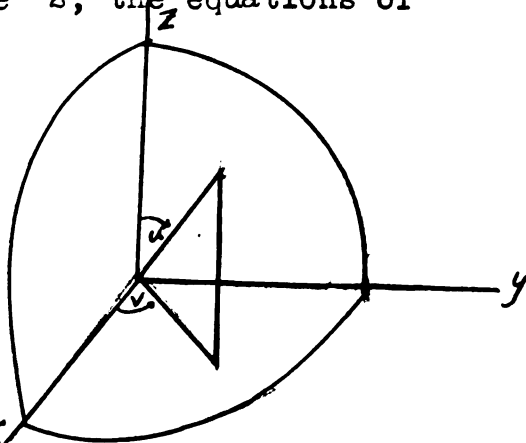
This theorem is a consequence of the ordinary sufficiency theorem for a strong relative minimum in the calculus of variations since $E(u, v, v'; V') > 0$ and $f_{v'v'}(u, v, v') > 0$ everywhere on the surface.

8. Geodesics on a sphere. If we choose the (u, v) coordinate system as indicated in figure 2, the equations of the sphere are

$$\begin{aligned} x &= a \sin u \cos v, \\ (27) \quad y &= a \sin u \sin v, \\ z &= a \cos u. \end{aligned}$$

The linear element is

$$(28) \quad ds^2 = a^2 du^2 + a^2 \sin^2 u dv^2$$



The problem is to find among all admissible arcs

$$V = V(u), \quad u_1 \leq u \leq u_2$$

one which minimizes the integral

$$\int_{u_1}^{u_2} a \sqrt{1 + \sin^2 u \, v'^2} \, du.$$

Euler's equation is

$$(29) \quad \frac{a \sin^2 u \, v'}{\sqrt{1 + \sin^2 u \, v'^2}} = C_1.$$

Solving (29) for v' we get

$$(30) \quad v' = \frac{C_1}{\sin u \sqrt{a^2 \sin^2 u - C_1^2}},$$

which gives, by a quadrature

$$(31) \quad v = \arccos \frac{K}{\tan u} + C,$$

$$\text{when } K = \frac{\sqrt{1 - C_1^2}}{C_1}.$$

With the values of $\sin v$ and $\cos v$ computed from (31), and with the aid of (27), we can write the equations of the geodesic as

$$(32) \quad \begin{aligned} x &= aK \cos u \cos C - a \cos u \sin C \sqrt{\tan^2 u - K^2}, \\ y &= aK \cos u \sin C + a \cos u \cos C \sqrt{\tan^2 u - K^2}, \\ z &= a \cos u. \end{aligned}$$

If in (32) we multiply x by $\cos C$, y by $\sin C$, z by $-K$ and add we have

$$(33) \quad x \cos C + y \sin C + (-K)z = 0.$$

Equation (33) is of the form

$$Ax + By + Cz = 0,$$

which is the equation of a plane through the center of the sphere. The intersection of such a plane with a sphere is a great circle. Hence, the geodesics on a sphere are great circles.

Suppose we have two points 1 and 2 on a sphere. The great circles through 1 have the point at the other extremity of the diameter through 1 as an envelope. Therefore to obtain the shortest distance we must use a portion of the arc less than a semi-circle.

For a sphere the determinant (26) is

$$(34) \quad \begin{vmatrix} \frac{1}{(1-c_1^2)\sqrt{1-c_1^2\sin^2 u}} & 1 \\ \frac{1}{(1-c_2^2)\sqrt{1-c_2^2\sin^2 u}} & 1 \end{vmatrix}.$$

The zeros of (34) are $u = u, \pm \pi/2$. This checks with the above statement.

9. An illustrative example. Given the surface of revolution

$$(35) \quad \begin{aligned} x &= u \cos v, \\ y &= u \sin v, \\ z &= \frac{1}{2} [u \sqrt{u^2-1} - \log(u + \sqrt{u^2-1})], \end{aligned}$$

to find the geodesics. The linear element is

$$(36) \quad ds^2 = u^2 du^2 + u^2 dv^2.$$

The problem is to minimize the integral

$$(37) \quad \int_{u_1}^{u_2} u \sqrt{1 + v'^2} du.$$

Euler's equation is

$$(38) \quad \frac{v'u}{\sqrt{1 + v'^2}} = C.$$

Solving (38) for v' and then integrating, we have

$$(39) \quad v = \log C_1 (u + \sqrt{u^2 - C^2}).$$

The determinant (26) is

$$(40) \quad -\frac{C}{C_1} \begin{vmatrix} \frac{1}{u \sqrt{u^2 - C^2} + (u^2 + C^2)} & , & 1 \\ \frac{1}{u_1 \sqrt{u_1^2 - C^2} + (u_1^2 + C^2)} & , & 1 \end{vmatrix}.$$

The zeros of (40) are $u = u_1$. Therefore, there are no conjugate points.

Some of the geodesics are shown in figure 4, page 17.

10. An illustrative example; the pseudo-sphere.

The equations of our surface are

$$(41) \quad \begin{aligned} x &= u \cos v, \\ y &= u \sin v, \\ z &= \sqrt{1 - u^2} - \log \frac{1 + \sqrt{1 - u^2}}{u}, \end{aligned}$$

The linear element is

$$(42) \quad ds^2 = \frac{1}{u^2} du^2 + u^2 dv^2.$$

The problem is to find the minimizing arcs for the integral

$$(43) \quad \int_{u_1}^{u_2} \sqrt{\frac{1}{u^2} + u^2 v'^2} du.$$

The Euler equation is

$$(44) \quad \frac{u^2 v'}{\sqrt{\frac{1}{u^2} + u^2 v'^2}} du = 0.$$

Solving (44) for v' and integrating we have

$$(45) \quad v = \frac{\sqrt{u^2 - c^2}}{cu} + c_1.$$

The determinant (26) becomes

$$(46) \quad - \begin{vmatrix} \frac{u}{c^2 \sqrt{u^2 - c^2}} & , & 1 \\ \frac{u_1}{c^2 \sqrt{u_1^2 - c^2}} & , & 1 \end{vmatrix}.$$

The zero of which is $u = u_1$. Therefore there are no conjugate points. That is, a solution of Euler's equation is always a geodesic.

Figure 5, page 19 shows the surface with some of the geodesics drawn.

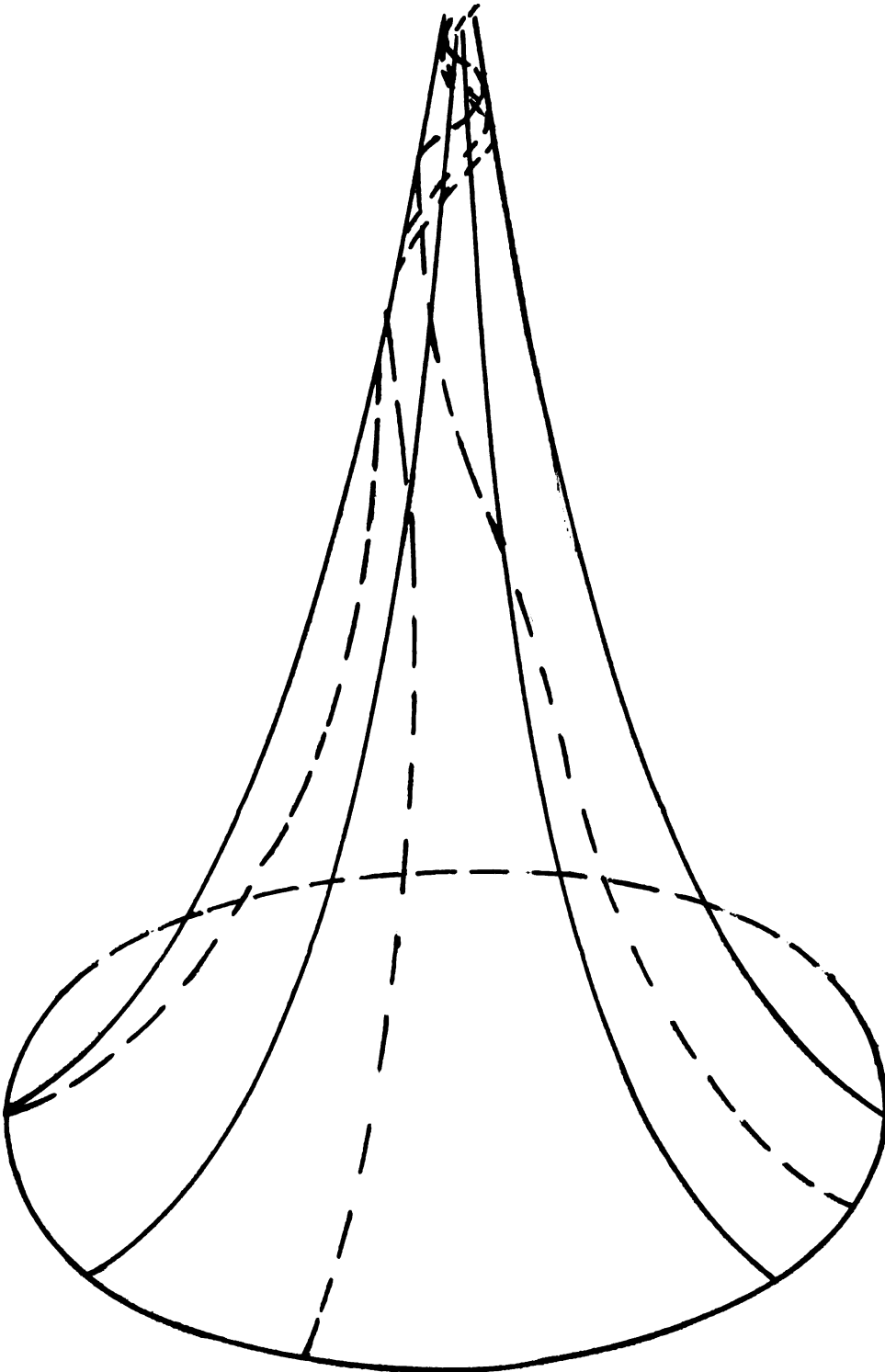


Fig 5.

II GEODESICS; A PROBLEM OF LAGRANGE

1. The Lagrange problem.* The Lagrange problem in the calculus of variations is that of finding among the arcs

$$y_i = y_i(x) \quad (x_1 \leq x \leq x_2; i = 1 \dots n),$$

satisfying a set of differential equations

$$(47) \quad \varphi_\alpha(x, y_1, \dots, y_n, y_1', \dots, y_n') \quad (\alpha = 1 \dots m < n),$$

and joining two fixed points in $(x, y_1, \dots, y_n)$ space, one which minimizes an integral of the form

$$(48) \quad \int_{x_1}^{x_2} f(x, y_1, \dots, y_n, y_1', \dots, y_n') dx.$$

2. The problem of finding geodesics. Let

$$(49) \quad S: \varphi(x, y, z) = 0,$$

be a surface on which we wish to find the geodesics. We assume that φ has continuous derivatives up to and including those of the fifth order and that

$$\varphi_x^2 + \varphi_y^2 + \varphi_z^2 \neq 0,$$

* G.A. Bliss, The Problem of Lagrange in the Calculus of Variations. American Journal of Mathematics, Vol. LII No. 4, October 1930 p. 674 (Hereafter referred to as Bliss II)

everywhere on the surface.

The form of the Lagrange problem that we are interested in is to find among the arcs

$$(50) \quad \begin{aligned} x &= x(s), \\ y &= y(s), \\ z &= z(s), \end{aligned} \quad s_1 \leq s \leq s_2$$

satisfying the equations

$$(51) \quad \begin{aligned} \varphi(x, y, z) &= 0, \\ x'^2 + y'^2 + z'^2 - 1 &= 0, \end{aligned}$$

and having end points satisfying the equations

$$(52) \quad s_1 = x_1 - \alpha_1 = y_1 - \beta_1 = z_1 - \gamma_1 = x_2 - \alpha_2 = y_2 - \beta_2 = z_2 - \gamma_2 = 0,$$

one which minimizes the integral

$$(53) \quad I = \int_{s_1}^{s_2} ds.$$

This is a problem with variable end points in (s, x, y, z) -space.* the parameter s measures distance along the arc starting at the point 1. In order to reduce this to the regular Lagrange problem we replace the first of the equations (51) by the equivalent equations

$$(54) \quad \varphi_x x' + \varphi_y y' + \varphi_z z' = 0,$$

*Bliss II, p. 689

$$(55) \quad x_1 - \alpha_1 = y_1 - \beta_1 = z_1 - \gamma_1 = x_2 - \alpha_2 = y_2 - \beta_2 = 0. *$$

We assume here that $\varphi_2 \neq 0$ at the end point 2. This does not restrict the generality as one of the $\varphi_x, \varphi_y, \varphi_z$ is different from zero at every point. We can now replace equations (53) with the equations

$$(56) \quad S_1 = x_1 - \alpha_1 = y_1 - \beta_1 = z_1 - \gamma_1 = x_2 - \alpha_2 = y_2 - \beta_2 = 0.$$

3. The equations of the geodesics. As a first necessary condition that an arc be a geodesic, we have the following theorem:

Theorem. For every geodesic G_2 there exists a set of constants C_1, C_2, C_3 and a function

$$(57) \quad F = 1 + \lambda_1(S) (\varphi_x x' + \varphi_y y' + \varphi_z z') + \frac{\lambda_2(S)}{2} (x'^2 + y'^2 + z'^2 - 1),$$

such that

$$(58) \quad \begin{aligned} F_{x'} &= \int_{S_1}^S F_x ds + C_1, \\ F_{y'} &= \int_{S_1}^S F_y ds + C_2, \\ F_{z'} &= \int_{S_1}^{S_2} F_z ds + C_3, \end{aligned}$$

* Bliss II, p. 703

are satisfied at every point of G_2 . The functions (λ_1, λ_2) are not both identically zero, are unique and continuous except possibly at values of s defining corners of G_2 .

This theorem is the first necessary condition of the calculus of variations problem. In order to write the function F in the form (57) it is necessary that the minimizing arc be normal. In section 4 we shall show that every minimizing arc is normal.

For the function F of (57) the equations (58) become

$$\begin{aligned}
 \lambda_1 \varphi_x + \lambda_2 x' &= \int_{s_1}^s \lambda_1 (\varphi_{xx} x' + \varphi_{xy} y' + \varphi_{xz} z') ds + C_1, \\
 \lambda_1 \varphi_y + \lambda_2 y' &= \int_{s_1}^s \lambda_1 (\varphi_{xy} x' + \varphi_{yy} y' + \varphi_{yz} z') ds + C_2, \\
 \lambda_1 \varphi_z + \lambda_2 z' &= \int_{s_1}^s \lambda_1 (\varphi_{xz} x' + \varphi_{yz} y' + \varphi_{zz} z') ds + C_3.
 \end{aligned}
 \tag{59}$$

By an integration by parts we can write (59) as

$$\begin{aligned}
 \lambda_2 x' &= \int_{s_1}^s -\lambda_1' \varphi_x ds + \bar{C}_1, \\
 \lambda_2 y' &= \int_{s_1}^s -\lambda_1' \varphi_y ds + \bar{C}_2, \\
 \lambda_2 z' &= \int_{s_1}^s -\lambda_1' \varphi_z ds + \bar{C}_3.
 \end{aligned}
 \tag{60}$$

From equations (60) we can state the theorem:

Theorem. A geodesic can not have corners.

From (60) we have that $\lambda_2 x'$, $\lambda_2 y'$, $\lambda_2 z'$ must be continuous on S_1, S_2 . Later we show that $\lambda_2 \neq 0$ and there-

fore x', y', z' must be continuous.

If we now take the derivative of both sides of (60) with respect to S we get the Euler-Lagrange multiplier rule.

At every point of G_{12} the equations

$$\begin{aligned}
 \varphi(x, y, z) &= 0, \\
 x'' &= \mu \varphi_x, \\
 y'' &= \mu \varphi_y, \\
 z'' &= \mu \varphi_z,
 \end{aligned}
 \tag{60}$$

must be satisfied. This proves the theorem:

Theorem. At every point of a geodesic the principle normal of the geodesic must coincide with the surface normal.

The derivatives x'', y'', z'' and the factor $\mu(s)$ are continuous on S_1, S_2 since the determinant^{*}

$$\begin{aligned}
 (61) \quad & \begin{vmatrix} \lambda_2 & 0 & 0 & \varphi_x & 2x' \\ 0 & \lambda_2 & 0 & \varphi_y & 2y' \\ 0 & 0 & \lambda_2 & \varphi_z & 2z' \\ \varphi_x & \varphi_y & \varphi_z & 0 & 0 \\ 2x' & 2y' & 2z' & 0 & 0 \end{vmatrix} \\
 &= 4(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) \neq 0.
 \end{aligned}$$

* Bliss II, p. 684.

4. The transversality condition.* The transversality condition for our problem is that the determinant

$$\begin{vmatrix} -F(x_1) + \lambda_2, & -\lambda_1 \varphi_x - \lambda_2 x', & -\lambda_1 \varphi_y - \lambda_2 y', & -\lambda_1 \varphi_z - \lambda_2 z', & F(x_2) - \lambda_2, & -\lambda_1 \varphi_x + \lambda_2 x', & -\lambda_1 \varphi_y + \lambda_2 y' \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

equals zero. This determinant reduces to

$$\lambda_2 = F(x_2) = 1.$$

5. Normality of the extremals.** Every extremal arc of our problem is normal. The condition for normality is that there exists six sets of values of $(\bar{f}_1, \bar{f}_2, \eta_1(s), \eta_2(s), \eta_3(s))$ with $\bar{f}_1, \bar{f}_2$ arbitrary and η_1, η_2, η_3 satisfying the equations

$$(62) \quad \frac{d}{ds} (\varphi_x \eta_1 + \varphi_y \eta_2 + \varphi_z \eta_3) = 0, \\ x' \eta_1' + y' \eta_2' + z' \eta_3' = 0,$$

and making the determinant

* Bliss II, p. 693.

** Bliss II, p. 693 and p. 704.

$$\begin{vmatrix}
f_{11} & f_{12} & f_{13} & \text{-----} & f_{16} \\
\eta_{11}(s_1) & \eta_{12}(s_1) & \eta_{13}(s_1) & \text{-----} & \eta_{16}(s_1) \\
\eta_{21}(s_1) & \eta_{22}(s_1) & \eta_{23}(s_1) & \text{-----} & \eta_{26}(s_1) \\
\eta_{31}(s_1) & \eta_{32}(s_2) & \eta_{33}(s_1) & \text{-----} & \eta_{36}(s_1) \\
x'_2 f_{21} + \eta_{11}(s_2) & x'_2 f_{22} + \eta_{12}(s_2) & x'_2 f_{23} + \eta_{13}(s_2) & \text{-----} & x'_2 f_{26} + \eta_{16}(s_2) \\
y'_2 f_{21} + \eta_{21}(s_2) & y'_2 f_{22} + \eta_{22}(s_2) & y'_2 f_{23} + \eta_{23}(s_2) & \text{-----} & y'_2 f_{26} + \eta_{26}(s_2)
\end{vmatrix},$$

different from zero. In (62) the arguments are the x, y, z, x', y', z' of the extremal arc. The equation (62) can be written

$$\begin{aligned}
& \varphi_x \eta'_1 + \varphi_y \eta'_2 + \varphi_z \eta'_3 + (\varphi_{xx} x' + \varphi_{xy} y' + \varphi_{xz} z') \eta_1 \\
& + (\varphi_{yx} x' + \varphi_{yy} y' + \varphi_{yz} z') \eta_2 + (\varphi_{zx} x' + \varphi_{zy} y' + \varphi_{zz} z') \eta_3 = 0, \\
& x' \eta'_1 + y' \eta'_2 + z' \eta'_3 = 0.
\end{aligned}$$

The matrix

$$\begin{vmatrix}
\varphi_x & \varphi_y & \varphi_z \\
x' & y' & z'
\end{vmatrix},$$

is of rank 2 at every point of the extremal since the directions $\varphi_x : \varphi_y : \varphi_z$ and $x' : y' : z'$ are orthogonal. Take another function $\psi(x, y, z, x', y', z')$ such that

$$\begin{vmatrix}
\varphi_x & \varphi_y & \varphi_z \\
x' & y' & z' \\
\psi_x & \psi_y & \psi_z
\end{vmatrix},$$

is of rank 3 at every point. Consider the set of equations

$$\begin{aligned} & \varphi_x \eta'_1 + \varphi_y \eta'_2 + \varphi_z \eta'_3 + (\varphi_{xx} x' + \varphi_{xy} y' + \varphi_{xz} z') \eta_1 \\ & + (\varphi_{yx} x' + \varphi_{yy} y' + \varphi_{yz} z') \eta_2 + (\varphi_{zx} x' + \varphi_{zy} y' + \varphi_{zz} z') \eta_3 = 0. \end{aligned}$$

$$x' \eta'_1 + y' \eta'_2 + z' \eta'_3 = 0,$$

$$\psi_x \eta'_1 + \psi_y \eta'_2 + \psi_z \eta'_3 + \psi_x \eta_1 + \psi_y \eta_2 + \psi_z \eta_3 = \xi(s).$$

Let $\eta_1(s)$, $\eta_2(s)$, $\eta_3(s)$, $\eta_1(s_2)$ be arbitrary and determine $\xi(s)$ such that the above equations are satisfied. Since the sets $\eta_{1\sigma}$, $\eta_{2\sigma}$, $\eta_{3\sigma}(s)$, $\eta_{2\sigma}(s_2)$, $\eta_{3\sigma}(s_2)$, $\eta_{1\sigma}(s_2)$ are arbitrary we can certainly choose them so that the determinant is different from zero if $y(s_2) \neq 0$. If $y'(s_2) = 0$ and $x'(s_2) \neq 0$ we can vary the above proof. If $y'(s_2) = x'(s_2) = 0$, then $z'(s_2) = 1$. Since $\varphi_x x' + \varphi_y y' + \varphi_z z' = 0$ on the surface we would have $\varphi_z = 0$ at the point on the extremal determined by $s = s_2$. This contradicts our hypothesis that $\varphi_z \neq 0$ at the point 2.

6. Analogue of the Weierstrass condition. The function for this problem is

$$\begin{aligned} E = & 1 + \lambda_1 (\varphi_x X' + \varphi_y Y' + \varphi_z Z') + \frac{1}{2} (X'^2 + Y'^2 + Z'^2) \\ & - \lambda_1 (\varphi_x x' + \varphi_y y' + \varphi_z z') - \frac{1}{2} (x'^2 + y'^2 + z'^2) \\ & - (X' - x')(\lambda_1 \varphi_x + x') - (Y' - y')(\lambda_1 \varphi_y + y') \\ & - (Z' - z')(\lambda_1 \varphi_z + z'). \end{aligned}$$

This reduces to

$$E = \frac{1}{2} \{ (x' - X')^2 + (y' - Y')^2 + (z' - Z')^2 \}.$$

Hence we have that $E(x, y, z, x', y', z', X', Y', Z', \lambda) > 0$ for every $(x', y', z') \neq (0, 0, 0)$ everywhere on the surface.

7. The Glebsch condition. The expression

$$(63) \quad F_{x'x'} \pi_1 \pi_1 + F_{y'y'} \pi_2 \pi_2 + F_{z'z'} \pi_3 \pi_3 + 2 F_{x'y'} \pi_1 \pi_2 + 2 F_{x'z'} \pi_1 \pi_3 + 2 F_{y'z'} \pi_2 \pi_3,$$

becomes

$$\pi_1^2 + \pi_2^2 + \pi_3^2.$$

Therefore we have the expression (63) greater than zero for all sets $(\pi_1, \pi_2, \pi_3) \neq (0, 0, 0)$ everywhere on the surface.

The conditions given in section 6 and in this section are stronger than the ordinary conditions. The ones stated here are usually denoted by II' and III' respectively. They are of importance in proving the sufficiency conditions.

8. The necessary condition of Mayer. The necessary condition of Mayer corresponds to the Jacobi condition of the ordinary problem.

Theorem. If the 1-parameter family of geodesics

through the point 1 has an envelope, the geodesic joining 1 and 2 can have no contact point 3 with the envelope between 1 and 2.

This follows from the Mayer condition of the Lagrange problem and the fact that the determinant (61) is everywhere different from zero on the surface.*

9. Determination of conjugate points. A point 3 where the extremal has contact with the envelope D is called a conjugate point of 1. In order to apply the theorem of this section it is necessary to have a method of determining the conjugate points.

Let

$$X = X(S, a_1, a_2, a_3, b_1, b_2, b_3),$$

$$Y = Y(S, a_1, a_2, a_3, b_1, b_2, b_3),$$

$$Z = Z(S, a_1, a_2, a_3, b_1, b_2, b_3),$$

be a 6-parameter family of extremals containing $G_{1,2}$ for $a_{i,0} = b_{i,0}$ ($i = 1, 2, 3$). Suppose also that the determinant

* Bliss II, p. 722.

$x_{a_1}(s, a, b)$	$x_{a_2}(s, a, b)$	$x_{a_3}(s, a, b)$	$x_b(s, a, b)$	$x_{b_2}(s, a, b)$	$x_{b_3}(s, a, b)$
$y_{a_1}(s, a, b)$	$y_{a_2}(s, a, b)$	$y_{a_3}(s, a, b)$	$y_b(s, a, b)$	$y_{b_2}(s, a, b)$	$y_{b_3}(s, a, b)$
$z_{a_1}(s, a, b)$	$z_{a_2}(s, a, b)$	$z_{a_3}(s, a, b)$	$z_b(s, a, b)$	$z_{b_2}(s, a, b)$	$z_{b_3}(s, a, b)$
$\lambda_{a_1} p_{x_{a_1}} + x'_{a_1}$	$\lambda_{a_2} p_{x_{a_2}} + x'_{a_2}$	$\lambda_{a_3} p_{x_{a_3}} + x'_{a_3}$	$\lambda_b p_{x_b} + x'_b$	$\lambda_{b_2} p_{x_{b_2}} + x'_{b_2}$	$\lambda_{b_3} p_{x_{b_3}} + x'_{b_3}$
$\lambda_{a_1} p_{y_{a_1}} + y'_{a_1}$	$\lambda_{a_2} p_{y_{a_2}} + y'_{a_2}$	$\lambda_{a_3} p_{y_{a_3}} + y'_{a_3}$	$\lambda_b p_{y_b} + y'_b$	$\lambda_{b_2} p_{y_{b_2}} + y'_{b_2}$	$\lambda_{b_3} p_{y_{b_3}} + y'_{b_3}$
$\lambda_{a_1} p_{z_{a_1}} + z'_{a_1}$	$\lambda_{a_2} p_{z_{a_2}} + z'_{a_2}$	$\lambda_{a_3} p_{z_{a_3}} + z'_{a_3}$	$\lambda_b p_{z_b} + z'_b$	$\lambda_{b_2} p_{z_{b_2}} + z'_{b_2}$	$\lambda_{b_3} p_{z_{b_3}} + z'_{b_3}$

is different from zero at the point 1 on G_2 . Then the points 3 conjugate to 1 are determined by roots $S_3 \neq S_1$ of the function^{*}(64)

$x_{a_1}(s, a_0, b_0)$	$x_{a_2}(s, a_0, b_0)$	$x_{a_3}(s, a_0, b_0)$	$x_b(s, a_0, b_0)$	$x_{b_2}(s, a_0, b_0)$	$x_{b_3}(s, a_0, b_0)$
$y_{a_1}(s, a_0, b_0)$	$y_{a_2}(s, a_0, b_0)$	$y_{a_3}(s, a_0, b_0)$	$y_b(s, a_0, b_0)$	$y_{b_2}(s, a_0, b_0)$	$y_{b_3}(s, a_0, b_0)$
$z_{a_1}(s, a_0, b_0)$	$z_{a_2}(s, a_0, b_0)$	$z_{a_3}(s, a_0, b_0)$	$z_b(s, a_0, b_0)$	$z_{b_2}(s, a_0, b_0)$	$z_{b_3}(s, a_0, b_0)$
$x_{a_1}(s, a_0, b_0)$	$x_{a_2}(s, a_0, b_0)$	$x_{a_3}(s, a_0, b_0)$	$x_b(s, a_0, b_0)$	$x_{b_2}(s, a_0, b_0)$	$x_{b_3}(s, a_0, b_0)$
$y_{a_1}(s, a_0, b_0)$	$y_{a_2}(s, a_0, b_0)$	$y_{a_3}(s, a_0, b_0)$	$y_b(s, a_0, b_0)$	$y_{b_2}(s, a_0, b_0)$	$y_{b_3}(s, a_0, b_0)$
$z_{a_1}(s, a_0, b_0)$	$z_{a_2}(s, a_0, b_0)$	$z_{a_3}(s, a_0, b_0)$	$z_b(s, a_0, b_0)$	$z_{b_2}(s, a_0, b_0)$	$z_{b_3}(s, a_0, b_0)$

*Bliss II, p. 729.

10. Sufficient conditions for an arc to be a geodesic.

A Mayer field is a region $\mathcal{F}$ of (S, x, y, z) -space containing only interior points and having associated with it a set of functions

$$p_i (s, x, y, z) \quad , \quad l = (s, x, y, z) ,$$

with the following properties:

- (a) they have continuous first partial derivatives in $\mathcal{F}$;
- (b) the sets $(s, x, y, z, p(s, x, y, z))$ defined by the points (s, x, y, z) in $\mathcal{F}$ are all admissible;
- (c) the integral

$$I^* = \int \{ F(s, x, y, z, p, l) + (dy_i - p_i dx_i) F_{y_i}' (s, x, y, z, p, l) \} ,$$

formed with these functions is independent of the path.*

By applying the sufficiency theorem of the Lagrange problem we have the two following theorems which correspond to those of part I.**

Theorem. If C_{12} is an extremal of a field $\mathcal{F}$ then C_{12} is shorter than any other admissible arc C_{12} in $\mathcal{F}$ joining the points 1 and 2. That is, C_{12} is a geodesic.

This follows from the fact that $E(x, y, y', \lambda) > 0$ at every point of the surface.

* Bliss II, p. 730.

** Bliss II, p. 731.

Theorem. If an arc E_{12} satisfies equations (60) and has no point conjugate to 1 between 1 and 2 or at 2, then G_{12} is a geodesic.

11. Application to a sphere. The Euler-Lagrange equations for a sphere are

$$\begin{aligned}
 (65) \quad & x'' = \mu x, \\
 & y'' = \mu y, \\
 & z'' = \mu z, \\
 & x^2 + y^2 + z^2 - a^2 = 0.
 \end{aligned}$$

If we differentiate the equation of a sphere twice with respect to S and make the substitution $xx'' = \mu x^2$, $yy'' = \mu y^2$ and $zz'' = \mu z^2$ we have

$$\mu = -\frac{1}{a^2}.$$

Putting in this value of μ and solving (65) we find

$$\begin{aligned}
 (66) \quad & x = C_1 \cos \frac{S}{a} + C_2 \sin \frac{S}{a}, \\
 & y = C_3 \cos \frac{S}{a} + C_4 \sin \frac{S}{a}, \\
 & z = C_5 \cos \frac{S}{a} + C_6 \sin \frac{S}{a}.
 \end{aligned}$$

If we multiply the equations (66) by

$$\begin{aligned}
 A &= C_3 C_6 - C_4 C_5, \\
 (67) \quad B &= C_2 C_5 - C_1 C_6, \\
 C &= C_1 C_4 - C_2 C_3,
 \end{aligned}$$

respectively we find that

$$Ax + By + Cz = 0.$$

Conversely, given values of A , B , and C we can solve (67) for $C_1, C_2, C_3, C_4, C_5, C_6$. Therefore the geodesics on a sphere are among the arcs of great circles.

For the sphere (64) becomes

$$\begin{vmatrix}
 \cos \frac{S}{a} & 0 & 0 & \sin \frac{S}{a} & 0 & 0 \\
 0 & \cos \frac{S}{a} & 0 & 0 & \sin \frac{S}{a} & 0 \\
 0 & 0 & \cos \frac{S}{a} & 0 & 0 & \sin \frac{S}{a} \\
 \cos \frac{S_1}{a} & 0 & 0 & \sin \frac{S_1}{a} & 0 & 0 \\
 0 & \cos \frac{S_1}{a} & 0 & 0 & \sin \frac{S_1}{a} & 0 \\
 0 & 0 & \cos \frac{S_1}{a} & 0 & 0 & \sin \frac{S_1}{a}
 \end{vmatrix},$$

the zeros of which are $\mu = \mu, \pm 170^\circ$. Thus, as in the problem in part I, the geodesic connecting two points of a sphere is that portion of the great circle, less than a semi-circle, through the points.

12. Application to a cylinder. For a cylinder the Euler-Lagrange equations are

$$\begin{aligned}
 x'' &= \mu x, \\
 y'' &= \mu y, \\
 (68) \quad z'' &= \mu z, \\
 x^2 + y^2 - a^2 &= 0.
 \end{aligned}$$

The solutions of (68) are

$$\begin{aligned}
 x &= C_1 \cos \sqrt{1-C_3^2} \frac{s}{a} + C_4 \sin \sqrt{1-C_3^2} \frac{s}{a}, \\
 y &= C_2 \cos \sqrt{1-C_3^2} \frac{s}{a} + C_5 \sin \sqrt{1-C_3^2} \frac{s}{a}, \\
 z &= C_3 s + C_6.
 \end{aligned}$$

If for convenience we set $K = \frac{\sqrt{1-C_3^2}}{a}$ the determinant (64) becomes

$\cos KS$	0	$\frac{C_2 C_1}{a\sqrt{1-C_3^2}} \sin KS - \frac{C_2 C_1}{a\sqrt{1-C_3^2}} \cos KS$	$\sin KS$	0	0
0	$\cos KS$	$\frac{C_2 C_2}{a\sqrt{1-C_3^2}} \sin KS - \frac{C_2 C_2}{a\sqrt{1-C_3^2}} \cos KS$	0	$\sin KS$	0
0	0	S	0	0	1
$\cos KS_1$	0	$\frac{C_2 C_1}{a\sqrt{1-C_3^2}} \sin KS_1 - \frac{C_2 C_1}{a\sqrt{1-C_3^2}} \cos KS_1$	$\sin KS_1$	0	0
0	$\cos KS_1$	$\frac{C_2 C_2}{a\sqrt{1-C_3^2}} \sin KS_1 - \frac{C_2 C_2}{a\sqrt{1-C_3^2}} \cos KS_1$	0	$\sin KS_1$	0
0	0	S'	0	0	1

The zeros of which are $S = S_1 \pm \frac{a\pi}{\sqrt{1-C_3^2}}$. Hence, from (69), the geodesics joining the two points will make less than one complete turn and go in the direction of the smaller angle.

III AN INVERSE PROBLEM

We know that on a plane the geodesics are all the straight lines. Therefore in the xy -coordinate system the geodesics are represented by equations linear in x and y .

It is the purpose of this part to determine whether we can so choose a uv -parameter system on a surface such that all the geodesics will be represented by equations linear in u and v . Beltrami was the first to propose and solve this problem.* He asked if it were possible to get a one to one correspondence between a sphere and a plane such that the geodesics on a sphere will correspond to straight lines of the plane. Darboux solved the problem as an inverse problem of the calculus of variations.** The method of this paper is essentially that of Darboux.

1. The inverse problem of the calculus of variations.

The inverse problem of the calculus of variations is, having given a second order differential equation

* E. Beltrami, Risoluzione del Problema: Riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette. (Annali di Matematica pura ed applicata pubblicati da B. Tortolini, t.VII,p.185;1866)

** Darboux, Theorie Des Surfaces, Vol. 3, pp. 53-63.

$$(69) \quad v'' = \psi(u, v, v'),$$

defining a 2-parameter family of extremals, to find a function $f(u, v, v')$ such that (69) is the Euler equation for the problem of minimizing the integral

$$\int_{u_1}^{u_2} f(u, v, v') du.$$

2. The problem of Beltrami. The necessary and sufficient condition that the geodesics be represented linearly in u and v is that the equation of the geodesics be

$$(70) \quad v'' = 0.$$

We shall first find a function $f(u, v, v')$ such that the Euler equation $\frac{d}{du} f_{v'} - f_v = 0$ is $v'' = 0$. The requirement that Euler's equation reduces to (70) is

$$(71) \quad \frac{\partial^2 f}{\partial v \partial v'} v' + \frac{\partial^2 f}{\partial u \partial v'} - \frac{\partial f}{\partial v} = 0.$$

If we take the partial derivative of (71) with respect to v' and set $M = \frac{\partial^2 f}{\partial v^2}$ we get

$$(72) \quad \frac{\partial M}{\partial v'} v' + \frac{\partial M}{\partial u} = 0.$$

To solve (72) we first solve the equations *

* Wilson, Advanced Calculus, pp. 267-268.

$$(73) \quad \frac{du}{1} = \frac{dv}{v'} = \frac{dv'}{0} = \frac{dM}{0}.$$

From (73) we have

$$dv' = 0,$$

$$dM = 0,$$

$$v' du = dv,$$

which gives $M = C_1$, $v' = C_2$ and $v' - u v' = C_3$. The solution of (72) is, therefore,

$$(74) \quad M = \varphi(v'; v - u v'),$$

where φ defines an arbitrary function of v' and $v - u v'$.

Thus, in order that $v' = 0$ be the Euler equation, we must have

$$(75) \quad \frac{\partial^2 f}{\partial v'^2} = \varphi(v'; v - u v').$$

From part I we know that the geodesics are extremals for minimizing the integral

$$\int_{u_1}^{u_2} \sqrt{E + 2Fv' + Gv'^2} \, du.$$

Equation (75) then requires that

$$(76) \quad \frac{EG - F^2}{(E + 2Fv' + Gv'^2)^{3/2}} = \frac{\partial^2 f}{\partial v'^2},$$

or

$$\frac{E + 2FV' + GV'^2}{(EG - F^2)^{3/2}} = \left(\frac{\partial^2 f}{\partial V'^2} \right)^{-\frac{2}{3}}.$$

We shall not consider surfaces for which $EG - F^2 = 0$. Since $EG - F^2$ is an invariant of the surface it can not be made zero by a transformation of parameters.

Since $\left(\frac{\partial^2 f}{\partial V'^2} \right)^{-\frac{2}{3}}$ is an arbitrary function of V' , $V - uV'$ we can write (77) as

$$(78) \quad \frac{E + 2FV' + GV'^2}{(EG - F^2)^{3/2}} = \varphi(V', V - uV').$$

Since the left hand side of (78) is a quadratic in V' the right hand side must be also. This is possible if and only if φ is a quadratic in V' and $V - uV'$. φ will then assume the following form

$$AV'^2 + BV' + C(V - uV')^2 + D(V - uV') + K,$$

from which we have

$$(79) \quad (K + DV + CV^2) + (B - Du - 2CuV)V' + (A + Cu^2)V'^2 = \frac{E + 2FV' + GV'^2}{(EG - F^2)^{3/2}}.$$

Equating the coefficients of powers of V' in (79) we get

$$\frac{E}{(EG-F^2)^{1/3}} = CV^2 + DV + K,$$

$$(80) \quad \frac{F}{(EG-F^2)^{1/3}} = \frac{D}{2} - CUV - \frac{D}{2}u,$$

$$\frac{G}{(EG-F^2)^{1/3}} = A + Cu^2.$$

Computing $\frac{EG-F^2}{(EG-F^2)^{1/3}}$ from (80) we get

$$(81) \quad \frac{1}{(EG-F^2)^{1/3}} = ACV^2 + (AD+BCU)V + (CK - \frac{D^2}{4})u^2 + \frac{BD}{2}u + AK - \frac{D^2}{4}.$$

For convenience we place $\frac{1}{(EG-F^2)^{1/3}} = R$. From (80) and (81) we obtain

$$(82) \quad \begin{aligned} E &= \frac{CV^2 + DV + K}{R^2}, \\ F &= \frac{\frac{D}{2} - CUV - \frac{D}{2}u}{R^2}, \\ G &= \frac{A + Cu^2}{R^2}, \\ H^2 &= EG - F^2 = \frac{1}{R^3}. \end{aligned}$$

We can now prove the following theorem:

Theorem. The only surfaces whose geodesics can be represented by linear equations are those of constant total curvature.

The total curvature, K_t , expressed in terms of E , F and G , is given by

$$(83) \quad K_t = \frac{1}{2H} \left\{ \frac{\partial}{\partial u} \left[\frac{F}{EH} \frac{\partial E}{\partial v} - \frac{1}{H} \frac{\partial G}{\partial u} \right] + \frac{\partial}{\partial v} \left[\frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial v} - \frac{F}{EH} \frac{\partial E}{\partial u} \right] \right\}^*$$

Using (80) we get

$$\begin{aligned} \frac{F}{EH} \frac{\partial E}{\partial v} = \frac{1}{R^{\frac{1}{2}}(CV^2 + DV + K)} & \left\{ (2C^2K - \frac{CD^2}{2}) u^2 v + (2ACK + \frac{B^2C}{2} + AD^2) v \right. \\ & \left. - 2AC^2V^3 + (CDK - \frac{D^3}{4}) u^2 - 3ACDV^2 + (\frac{BD^2}{2} - 2BCK) u - \frac{B^2D}{4} + ADK \right\}, \end{aligned}$$

$$\begin{aligned} \frac{1}{H} \frac{\partial G}{\partial u} = \frac{1}{R^{\frac{1}{2}}} & \left\{ (\frac{CD^2}{2} - 2C^2K) u^3 + 2AC^2uV^2 + 2ACDuV \right. \\ & \left. - 2ABCv + (AD^2 - 2ACK - \frac{CB^2}{2}) u - ABD \right\}, \end{aligned}$$

$$\begin{aligned} \frac{2}{H} \frac{\partial F}{\partial u} = -\frac{1}{R^{\frac{3}{2}}} & \left\{ (\frac{3}{4}D^3 - 3CDK) u^2 + (\frac{3}{2}CD^2 - 6C^2K) u^2 v - 2BC^2uV^2 \right. \\ & \left. - 2BCDuV + (4BCK - \frac{3}{2}BD^2) u + 3ACDV^2 + (2ACK + AD^2 + \frac{3}{2}B^2C) v \right. \\ & \left. + 2AC^2V^3 + ADK + \frac{3}{4}B^2D \right\}, \end{aligned}$$

$$\frac{1}{H} \frac{\partial E}{\partial V} = \frac{1}{R^{\frac{1}{2}}} \left\{ (2C^2K - \frac{CD^2}{2}) u^2 V - 2AC^2V^3 - 3ACDV^3 + (CDK - \frac{D^3}{4}) u^2 \right. \\ \left. + (-2ACK - \frac{CB^2}{2} - AD^2)V + (\frac{BD^2}{2} - 2BCK)u - \frac{B^2D}{4} - ADK \right\},$$

$$\frac{F}{EH} \frac{\partial E}{\partial u} = \frac{1}{R^{\frac{1}{2}}} \left\{ 2BCuV^2 + (4C^2K - CD^2)u^2V + 2BCDuV \right. \\ \left. + (2CDK - \frac{D^3}{2})u^2 - B^2CV + (BD^2 - 2BCK)u - \frac{B^2D}{2} \right\}.$$

We find that

$$(84) \quad \frac{2}{H} \frac{\partial F}{\partial u} - \frac{1}{H} \frac{\partial E}{\partial V} - \frac{F}{EH} \frac{\partial E}{\partial u} = 0,$$

and

$$\frac{F}{EH} \frac{\partial E}{\partial V} - \frac{1}{H} \frac{\partial E}{\partial u} = \frac{1}{(CV^2 + DV + K)R^{\frac{1}{2}}} \left\{ ABC^2V^3 + \frac{3}{2}ABCDV^2 \right. \\ \left. + (3BC^2K - \frac{3}{4}BCD^2)u^2V + (BC^2 - \frac{ACD^2}{2} + 2AC^2K)uV^2 \right. \\ (85) \quad \left. + (\frac{D^4}{8} - CD^2K + 2C^2K^2)u^3 + (CB^2D - \frac{AD^3}{2} + 2ACDK)uV \right. \\ \left. + (\frac{3}{2}BCDK - \frac{3}{8}BD^3)u^2 + (-\frac{B^3C}{4} + \frac{ABD^2}{2} + ABCK)V \right. \\ \left. + (\frac{3}{8}B^2D^2 - \frac{B^2CK}{2} + \frac{AD^2K}{2} + 2ACK^2)u - \frac{B^3D}{8} + \frac{ABDK}{2} \right\} \\ = \frac{R R_u}{(CV^2 + DV + K)R^{\frac{1}{2}}}.$$

Using (83), (84) and (85) we get

$$\begin{aligned}
K_t &= \frac{1}{2H} \frac{\partial}{\partial u} \frac{R R_u}{(C v^2 + D v + K) R^{\frac{3}{2}}} \\
&= \frac{1}{2(C v^2 + D v + K)} (R_{uu} R - \frac{1}{2} R_u^2) \\
&= A C K - \frac{B^2 C}{4} - \frac{A D^2}{4} \\
&= \text{Constant.}
\end{aligned}$$

We next prove the converse theorem:

Theorem. If the total curvature of a surface is constant then the geodesics on that surface may be expressed linearly in u and v .

Let us choose our curvilinear coordinates in such a manner that $v = \text{constant}$ are geodesics and $u = \text{constant}$ are the orthogonal trajectories. When such a curvilinear coordinate system is used the linear element may be put in the form

$$(86) \quad ds^2 = du^2 + G dv^2.$$

From (83) and (86) the total curvature is $-\frac{1}{G} \frac{\partial^2 G}{\partial u^2}$.

By the hypothesis of the theorem we have

$$(87) \quad -\frac{1}{G} \frac{\partial^2 G}{\partial u^2} = K.$$

We shall consider the three cases $K=0$, $K>0$, $K<0$.

For $K = 0$ we have

$$-\frac{1}{G} \frac{\partial^2 G}{\partial u^2} = 0 ,$$

or

$$(88) \quad \frac{\partial^2 G}{\partial u^2} = 0 .$$

The solution of (88) is $G = V_1 u + V_2$ where V_1 and V_2 are any two arbitrary functions of v .

For $K > 0$ we have

$$-\frac{1}{G} \frac{\partial^2 G}{\partial u^2} = \frac{1}{a^2} ,$$

or

$$(89) \quad \frac{\partial^2 G}{\partial u^2} + \frac{G}{a^2} = 0 .$$

The solution of (89) is

$$(90) \quad G = V_2 \cos \frac{u}{a} + V_1 \sin \frac{u}{a} .$$

where V_1 and V_2 are any two arbitrary functions of v .

For $K < 0$ we have

$$-\frac{1}{G} \frac{\partial^2 G}{\partial u^2} = -\frac{1}{a^2} ,$$

or

$$(91) \quad \frac{\partial^2 G}{\partial u^2} - \frac{G}{a^2} = 0 .$$

The solution of (91) is

$$(92) \quad G = V_1 e^{\frac{u}{a}} + V_2 e^{-\frac{u}{a}}.$$

If we take $V_1 = \frac{V_1}{2} + \frac{V_2}{2}$, $V_2 = -\frac{V_1}{2} + \frac{V_2}{2}$ we can change (92) into

$$(93) \quad G = V_1 \frac{e^{\frac{u}{a}} - e^{-\frac{u}{a}}}{2} + V_2 \frac{e^{\frac{u}{a}} + e^{-\frac{u}{a}}}{2}.$$

If in (90) and (93) we replace u by au we have the three forms of the linear element

$$(94) \quad \begin{aligned} ds^2 &= du^2 + (V_1 u + V_2)^2 dv^2, \\ ds^2 &= a[du^2 + (V_1 \sin u + V_2 \cos u)^2 dv^2], \\ ds^2 &= a[du^2 + (V_1 \frac{e^{\frac{u}{a}} - e^{-\frac{u}{a}}}{2} + V_2 \frac{e^{\frac{u}{a}} + e^{-\frac{u}{a}}}{2})^2 dv^2]. \end{aligned}$$

If $V = \text{constant}$ are to be geodesics it will be necessary for G to be zero when u is zero. If $G = 0$ when $u = 0$ we have from (88), (90), and (93) $V_1 = 0$. Choosing the coordinate V in a suitable manner we may reduce V_2 to unity and write the three forms of (94) as

$$(95) \quad \begin{aligned} ds^2 &= du^2 + u^2 dv^2, \\ ds^2 &= a^2[du^2 + \sin^2 u dv^2], \\ ds^2 &= a^2[du^2 + (\frac{e^u - e^{-u}}{2})^2 dv^2]. \end{aligned}$$

We next compute the geodesics for the three forms of (95). For

$$ds^2 = du^2 + u^2 dv^2,$$

Euler's equation is

$$(96) \quad \frac{u^2 v'}{\sqrt{1 + u^2 v'^2}} = C_1.$$

Solving for v' and integrating we get

$$(97) \quad A u \cos v + B u \sin v + C = 0.$$

For

$$ds^2 = a^2 [du^2 + \sin^2 u dv^2],$$

Euler's equation is

$$(98) \quad \frac{\sin^2 u v'}{\sqrt{1 + \sin^2 u v'^2}} = C_1.$$

Solving (98) for v' and integrating we get

$$(99) \quad v = \arccos \frac{C_1}{\sqrt{1 - C_1^2}} \tan u + C.$$

If we put $K = \frac{C_1}{\sqrt{1 - C_1^2}}$ (99) reduces to

$$(100) \quad A \cos v \tanh u + B \sin v \tanh u + C = 0.$$

For

$$ds^2 = a^2 [du^2 + \sinh^2 u dv^2],$$

Euler's equation is

$$\frac{\sinh^2 u v'}{\sqrt{1 + \sinh^2 v'^2}} = C,$$

or

$$(101) \quad v' = \frac{C}{\sinh \sqrt{\sinh^2 u - C^2}}.$$

Solving (101) as we did (98) we get

$$(102) \quad A \cos v \tanh u + B \sin v \tanh u + C = 0.$$

The geodesics for the three forms are

$$A u \cos v + B u \sin v + C = 0,$$

$$(103) \quad A \cos v \tanh u + B \sin v \tanh u + C = 0,$$

$$A \cos v \tanh u + B \sin v \tanh u + C = 0,$$

respectively.

The transformations which we must make on the u and v in order that the equations of the geodesics will be linear in u and v are evidently

$$\begin{aligned} \bar{u} &= u \cos v, & \bar{v} &= u \sin v, \\ (104) \quad \bar{u} &= \cos v \tanh u, & \bar{v} &= \sin v \tanh u, \\ \bar{u} &= \cos v \tanh u, & \bar{v} &= \sin v \tanh u, \end{aligned}$$

respectively.

After making the transformation (104) the linear elements (94) take the form

$$\begin{aligned} ds^2 &= du^2 + dv^2, \\ ds^2 &= \frac{a^2 [du^2 + dv^2 + (v du - u dv)^2]}{(1 + u^2 + v^2)^2}, \\ ds^2 &= \frac{a^2 [du^2 + dv^2 - (v du - u dv)^2]}{(1 - u^2 - v^2)^2}, \end{aligned}$$

respectively.

Theorem. The only surfaces on which both the equations of the geodesics will be linear in u and v , and the parametric system orthogonal are developable surfaces.

The necessary and sufficient condition that two curves be orthogonal is that $F=0$. From (82) $F = \frac{\frac{B}{2} - Cuv - \frac{D}{2}u}{R^2}$.

Since F is to be zero for all u 's and v 's it will be necessary for $B = C = D = 0$. Since $K_2 = AC - \frac{B^2}{4} - \frac{D^2}{4}$ it follows that K_2 is zero when $F = 0$. The necessary and sufficient condition that a surface be developable is $K_2 = 0$. This proves the theorem.

If we take the sphere with the same uv -coordinates as in part I and make the transformation

$$\bar{u} = \cos v \tan u, \quad \bar{v} = \sin v \tan u,$$

we find that the equations of the sphere become

$$\begin{aligned} x &= \frac{au}{\sqrt{1+u^2+v^2}}, \\ (105) \quad y &= \frac{av}{\sqrt{1+u^2+v^2}}, \\ z &= \frac{a}{\sqrt{1+u^2+v^2}}. \end{aligned}$$

The linear element will be

$$ds^2 = \frac{a^2 [(1+v^2) du^2 - 2uv du dv + (1+u^2) dv^2]}{(1+u^2+v^2)^2}.$$

If the equation of the geodesics is computed from (105) it will take the form $V'' = 0$.

IV. AN ISOPERIMETRIC PROBLEM

In this part we consider the problem of finding on a surface the curve of shortest length joining the points 1 and 2, and enclosing with a given curve C_0 , joining 1 and 2, a fixed area. It was in the solution of this problem that Minding discovered the function to which Bonnet gave the name geodesic curvature.*

1. Statement of the isoperimetric problem. The isoperimetric problem is, given two integrals

$$I = \int_{u_1}^{u_2} f(u, v, v') du, \quad J = \int_{u_1}^{u_2} g(u, v, v') du,$$

to find among all admissible arcs, joining (1) and (2), the one which gives the integral J a constant value and minimizes the integral I .**

If E_{12} is a minimizing arc for the problem there exists constants $(\lambda_0, \lambda) \neq (0, 0)$ such that

$$(106) \quad \frac{d}{du} F_{v'} - F_v = 0$$

between corners of E_{12} , where

* Minding, Crelle, Vol. V (1830), p. 297.

** G.A. Bliss, Lectures.

$$(107) \quad F = \lambda_0 f + \lambda g.$$

If the arc E_{12} is taken to be normal there exists an unique set of multipliers $(\lambda_0, \lambda) = (1, \lambda)$. Hereafter we shall consider only arcs normal on every sub-interval. Abnormal arcs are of no importance in our problem since an abnormal arc would also minimize the integral J .

The other necessary and sufficient conditions are like those of the ordinary problem of part I with $f(u, v, v')$ replaced by $F(u, v, v', \lambda)$.

2. The problem of Minding. Given an arc C_0 joining two points 1 and 2 on a surface, to find the curve of shortest length joining 1 and 2, and enclosing with C_0 a given area.

The area will be given by $\iint H du dv$, where the integration is to be taken over the uv -region corresponding to the area.* It is possible, however, to find two functions $M(u, v)$ and $N(u, v)$ such that $H = \frac{\partial N}{\partial u} - \frac{\partial M}{\partial v}$ **. By an application of Green's theorem we have***

* Eisenhart, p. 75.

** If $H(u, v)$ is continuous in u we can take $N = \int_{u_0}^u H(u, v) du$ and $M = M(u, v)$.

*** F.S. Woods, Advanced Calculus, p. 181.

$$\iint H du dv = \iint \left(\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v} \right) du dv = \int M du + N dv.$$

The problem then is to find among all the admissible arcs which join 1 and 2 the one which minimizes the integral

$\int_{u_1}^{u_2} \sqrt{E + 2FV' + GV'^2} du$ and keeps the integral $\int_{u_1}^{u_2} (M + NV') du$ equal to a constant.

The F -function (107) is

$$F = \sqrt{E + 2FV' + GV'^2} + \lambda (M + NV'),$$

and Euler's equation is

$$(108) \quad \frac{d}{du} \left(\frac{F + GV'}{\sqrt{E + 2FV' + GV'^2}} \right) - \frac{\frac{\partial F}{\partial V} + 2V' \frac{\partial F}{\partial V} + V'^2 \frac{\partial G}{\partial V}}{2 \sqrt{E + 2FV' + GV'^2}} = \lambda H.$$

We can now state and prove the theorem of Minding:

Theorem. In order that a curve C joining two points shall be the shortest which, together with a given curve through these points, incloses a portion of the surface with a given area, it is necessary that the geodesic curvature of C be constant.

From a formula derived by Bonnet the geodesic curvature of $\varphi(u, v) = \text{constant}$ is *

* Eisenhart, p. 136.

$$(109) \quad \frac{1}{\rho_g} = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{F \frac{\partial \phi}{\partial v} - G \frac{\partial \phi}{\partial u}}{E \left(\frac{\partial \phi}{\partial v} \right)^2 - 2F \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial u} + G \left(\frac{\partial \phi}{\partial u} \right)^2} \right) + \frac{\partial}{\partial v} \left(\frac{F \frac{\partial \phi}{\partial u} - G \frac{\partial \phi}{\partial v}}{E \left(\frac{\partial \phi}{\partial v} \right)^2 - 2F \frac{\partial \phi}{\partial v} \frac{\partial \phi}{\partial u} + G \left(\frac{\partial \phi}{\partial u} \right)^2} \right) \right\}.$$

If we take ϕ in the form

$$v - \psi(u) = 0,$$

then

$$(110) \quad \frac{\partial \phi}{\partial v} = 1, \quad \frac{\partial \phi}{\partial u} = -v'.$$

Substituting (110) in (109), (109) becomes

$$(111) \quad \frac{1}{\rho_g} = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} \right) - \frac{\partial}{\partial v} \left(\frac{Fv' + E}{\sqrt{E + 2Fv' + Gv'^2}} \right) \right\}.$$

Equation (108) can be written as

$$(112) \quad \lambda = \frac{1}{H} \left\{ \frac{\partial}{\partial u} \left(\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} \right) + \frac{\partial}{\partial v} \left(\frac{F + Gv'}{\sqrt{E + 2Fv' + Gv'^2}} \right) v' - \frac{\frac{\partial E}{\partial v} + 2v' \frac{\partial F}{\partial v} + v'^2 \frac{\partial G}{\partial v}}{2 \sqrt{E + 2Fv' + Gv'^2}} \right\}.$$

Hence, comparing (109) and (112) we have the geodesic curvature constant and equal to λ .

Due to the fact that the geodesic curvature is constant, the curves representing solutions of (109) are called geodesic circles.

3. The necessary and sufficient conditions. The E -function is

$$\begin{aligned} E(u, v, v'; V', \lambda) = & \sqrt{E + 2FV' + GV'^2} + \lambda(M + NV') \\ & - \sqrt{E + 2FV' + GV'^2} - \lambda(M + NV') \\ & - (V' - v') \left\{ \frac{F + Gv'}{\sqrt{E + 2FV' + GV'^2}} + \lambda N \right\}, \end{aligned}$$

which reduces to

$$\begin{aligned} E(u, v, v'; V', \lambda) = & \sqrt{E + 2FV' + GV'^2} - \sqrt{E + 2Fv' + Gv'^2} \\ (113) \quad & - \frac{(V' - v')(F + Gv')}{\sqrt{E + 2FV' + GV'^2}}. \end{aligned}$$

The $f_{v', v'}$ of the Legendre condition is

$$(114) \quad f_{v', v'}(u, v, v'; \lambda) = \frac{H^2}{(E + 2FV' + GV'^2)^{\frac{3}{2}}}.$$

Both (113) and (114) are the exact expressions for the corresponding functions in the ordinary problem of part I. Hence, we have the same necessary and sufficient conditions for the isoperimetric problem as for the ordinary problem.

4. Application to a plane. Letting $u = x$ and $v = y$ equation (108) becomes

$$\frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} = \lambda.$$

Integrating we get

$$(115) \quad \frac{y'}{\sqrt{1+y'^2}} = \lambda x + c_1.$$

The solution of (115) is

$$(116) \quad \lambda x^2 + \lambda y^2 + 2c_1\lambda x + 2c_2\lambda y + c_1^2 + c_2^2 - 1 = 0.$$

Hence the curve of shortest length that joins two points in the plane and makes the area with another fixed curve, joining the points, constant is the arc of a circle through the two points.

To show that the curves (116) are minimizing curves we need to show that the sufficient conditions of part I are satisfied. Since the only zeros of the determinant

$$\frac{1}{\lambda^2} \left| \begin{array}{cc} \frac{\lambda x + c_1}{\sqrt{1 - (\lambda x + c_1)^2}} & , & 1 \\ \frac{\lambda x_1 + c_1}{\sqrt{1 - (\lambda x_1 + c_1)^2}} & , & 1 \end{array} \right|$$

are $\lambda = \lambda_1$, it follows that there are no points conjugate to 1. Therefore the curves (116) are minimizing curves.

5. Application to a sphere. For a sphere

$$ds^2 = a^2 \sqrt{1 + \sin^2 u \, v'^2} \, ,$$

and (108) becomes

$$(117) \quad \frac{d}{du} \left(\frac{a \sin^2 u \, v'}{\sqrt{1 + \sin^2 u \, v'^2}} \right) = \lambda a^2 \sin u \, ,$$

If we integrate both sides of (117) and set $-\lambda a = K$ we have

$$\frac{\sin^2 u \, v'}{\sqrt{1 + \sin^2 u \, v'^2}} = K \cos u + c_1 \, .$$

Solving for v' we get

$$(118) \quad V' = \frac{K \cos u + C_1}{\sin u \sqrt{\sin^2 u - [K \cos u + C_1]^2}}.$$

The solution of (118) is

$$\sqrt{1+K^2-a^2} \sin(V-C_2) - C_1 \cos u = K.$$

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