# THE NATURE OF THE FORCE FUNCTION IN SELF-EXCITED VIBRATIONS

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William John Ruby

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This is to certify that the

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# THE NATURE OF THE FORCE FUNCTION
# IN SELF-EXCITED VIBRATIONS

By

William John Ruby

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## I. Statement of Problem

The object of this paper is to determine the nature of the driving force causing self-excited vibrations. It has been assumed, that the curve, showing amplitude plotted against time, is available.

## II. Introduction

A self-excited vibration is a phenomenon in which the force furnishing the energy to cause oscilation is dependent upon the motion of the system under consideration, in contrast to the usual forced vibration which does not depend on the motion.

In general, this subject has two natural groupings:

(1) systems where this form of vibration is necessary for the unit to carry on its intended function, and,

(2) systems where a self-excited vibration inhibits the intended action.

Some typical systems where self-excited vibrations may occur are: vibration of transmission lines coated with sleet, airplane-wing flutter, nosing of locomotives, and some cases of hunting of generators. All bowed string or blown musical instruments, the doorbell, the automobile horn, power-driven tuning fork, and many toys function because of self-excited vibrations.

The characteristic, common to all systems subject to this type of vibration, is instability. A motion is said to be unstable when the driving force tends to increase the amplitude of the existing motion. A system is not necessarily unstable for all frequencies or conditions. There are usually certain ranges of frequencies for which the motion is stable and other ranges for which the motion is unstable, for any given system.

It is desirable to control the self-excited vibration in many systems. There are two approaches leading to the control of this form of vibration. They are:

(1) the elimination of the instability of the unit; and,

(2) the reduction of the vibration by introducing sufficient damping in the existing system.

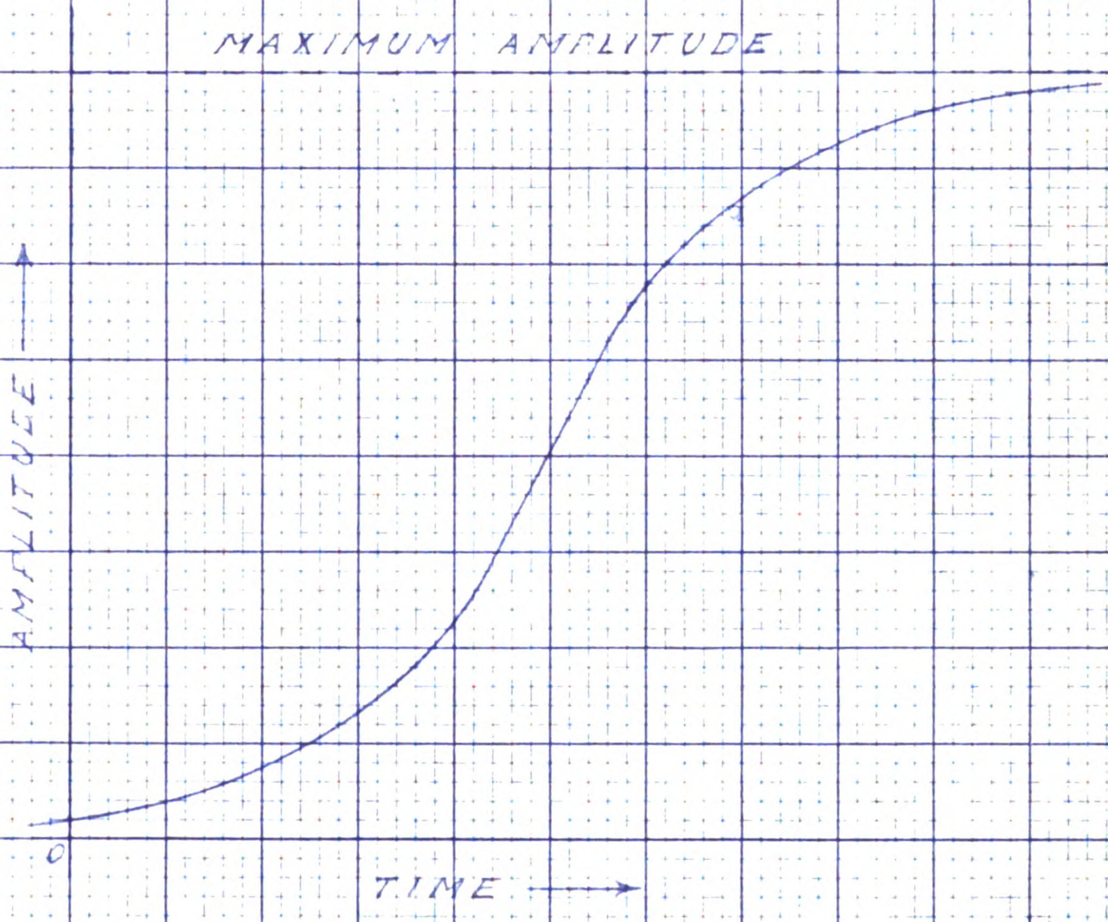
Both of these cannot, in general, be applied to all systems. When it is possible to eliminate the instability and retain the usefulness of a system, the problem is usually quite easily solved. If the second expedient is to be utilized, it would be a distinct advantage to have some expression for the force to be reduced by damping.

## III. Conversion of Data

Preliminary data. The information which was assumed to be available, at the outset of the problem, was the curve showing amplitude plotted against time. This curve varies for each particular case, however, the general shape varies only slightly. A typical build-up curve, shown in Figure I, has time plotted as abscissa and amplitude as the ordinate.

FIGURE I

BUILDING-UP CHARACTERISTIC
OF SELF-EXCITED VIBRATION



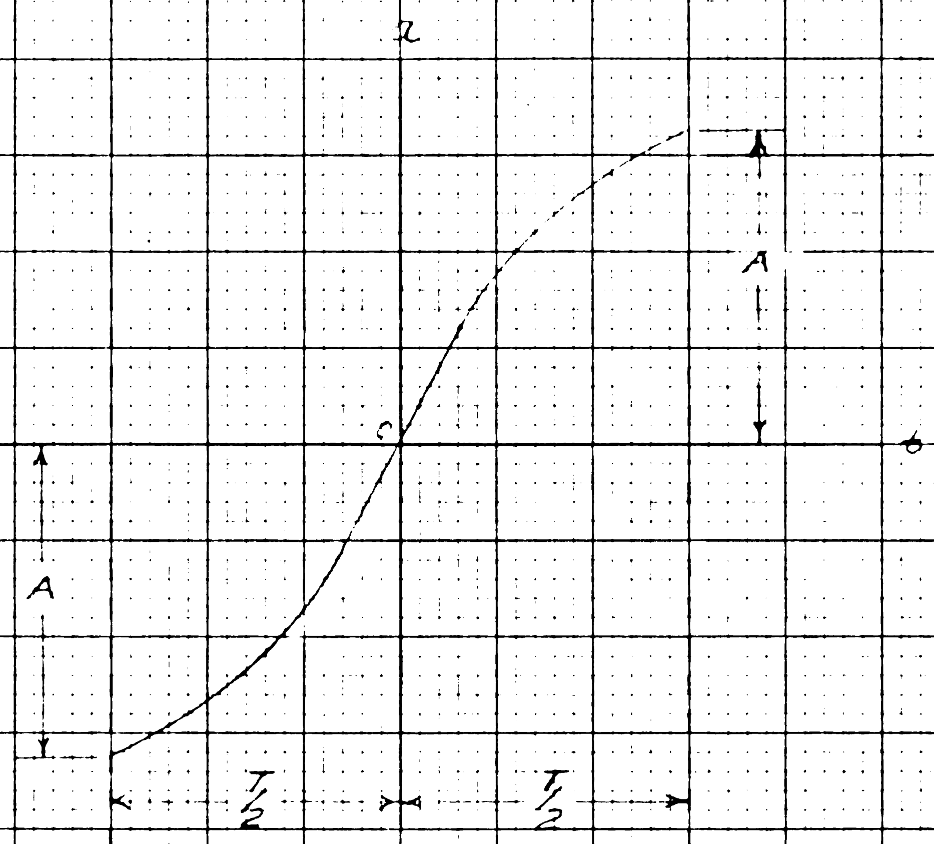
Since the exciting force in a self-excited vibration depends on the motion, it would take an indefinite length of time for the vibration to start and reach, say, 10 per cent of its maximum value. It is because of this characteristic that some random disturbance must cause the initial motion of the system. As can be seen from Figure I, it takes an infinite length of time for the amplitude to reach a maximum, however, 90 per cent of this maximum can be reached in a short period of time. The time lapse between 10 per cent and 90 per cent of the maximum amplitude will vary from a few seconds to an hour or more, depending on the system under consideration.

Idealized case. For the work in this paper, the portion of the build-up curve between 10 per cent and 90 per cent of the maximum amplitude was used due to the uncertainty of both extremities. The origin was placed on the curve, mid-way between the ordinates representing 10 per cent and 90 per cent of maximum amplitude respectively, as shown in Figure II.

In order to apply a general development to a particular case, it is convenient to use expressions of a non-dimensional nature as much as possible throughout the development. It is with this in mind that the choice of abscissa and ordinate in Figure III was made. The assumption has been made that this ideal curve is symmetrical with respect to the origin.

FIGURE II

PORTION OF FIGURE I BETWEEN 15% AND 90% OF MAXIMUM AMPLITUDE

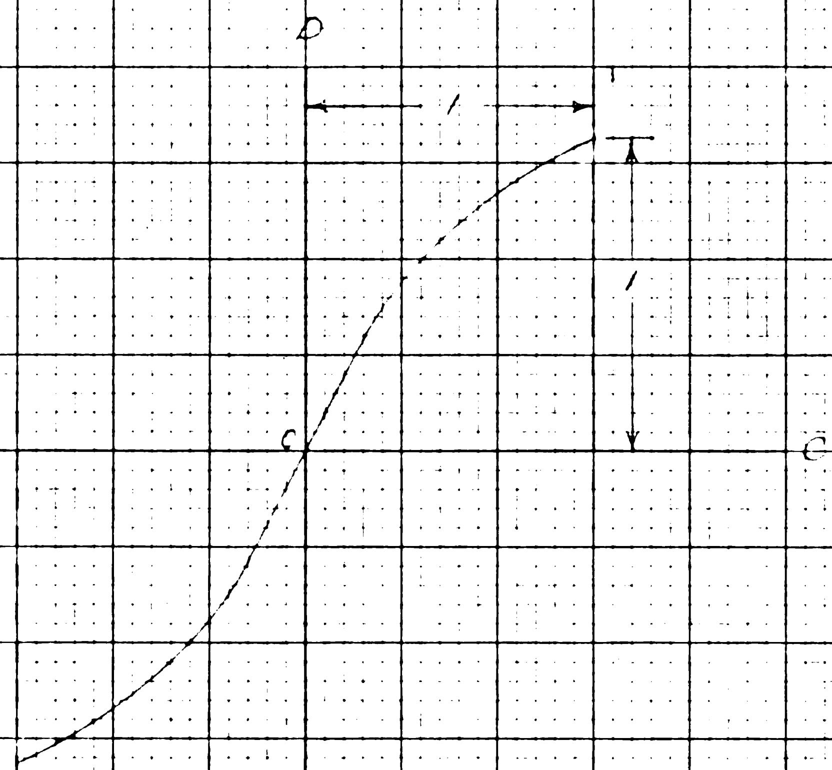



A = 40% MAXIMUM AMPLITUDE

T = TOTAL TIME BEING CONSIDERED

FIGURE III

NON-DIMENSIONAL BUILD-UP CURVE




$D = \frac{a}{A}$, A NON-DIMENSIONAL QUANTITY

$C = \frac{2t}{T}$, A NON-DIMENSIONAL QUANTITY

## IV. Development of Function

Two assumptions are needed before proceeding. They are:

(1) that the natural damping force of the system during the period under consideration is small compared to the driving force and thus can be neglected; and,

(2) that the motion is of a sinusoidal nature.

With these assumptions it seems reasonable to assume a sine series to represent a curve of the shape shown in Figure II. Thus,

$$a = \sum_{1}^{\infty} b_n \sin \frac{n\pi t}{T}, \qquad 1.$$

where $a$ = amplitude, $b_n$ = coefficient of the series, $t$ = time, $T$ = total time interval under consideration. From the fact that $a = DA$ and, $t = \frac{CT}{2}$,[1] equation 1 gives,

$$D = \frac{1}{A} \sum_{1}^{\infty} b_n \sin \frac{n\pi c}{2}. \qquad 2.$$

It can be readily shown,[2] that for a system undergoing a periodic motion, the change in energy per cycle, $U$, is,

$$U = \frac{k}{2} a_2^2 - \frac{k}{2} a_1^2, \qquad 3.$$

where $a_1$ and $a_2$ are the amplitudes at the beginning and end of the cycle, respectively, and k is a constant depending on the physical properties of the system.[3] Again using $a = DA$, equation 3 can be rewriten as,

$$\frac{2U}{kA^2} = D_2^2 - D_1^2 . \qquad 4.$$

By assuming that the system will require a considerable number of cycles to build up, the following approximation can be made,

$$a_2 = a_1 + \lambda' \frac{2\pi}{\omega} . \qquad 5.$$

Here, $\lambda'$ is the slope of the build-up curve, half way between successive points of maximum displacement, and $\frac{2\pi}{\omega}$ represents the time for one cycle. To change equation 5 to non-dimensional form, it is necessary to determine $\lambda'$. First,

$$\lambda' = \frac{da}{dt} = \frac{da}{dD} \times \frac{dD}{dC} \times \frac{dC}{dt} .$$

Then, since $a = DA$, $\frac{da}{dD} = A$, $C = \frac{2t}{T}$, $\frac{dC}{dt} = \frac{2}{T}$ ;

$$\lambda' = \frac{2A}{T} \frac{dD}{dC} . \qquad 6.$$

Now letting the slope, $\frac{dD}{dC}$, on the non-dimensional curve in Figure III be equal to $\lambda$ , equation 6 becomes,

$$\lambda' = \frac{2A\lambda}{T}. \qquad 6a.$$

Substituting for $\lambda'$, its value from equation 6a and $a = DA$, in equation 5 gives,

$$D_2 = D_1 + \frac{4\pi\lambda}{\omega T}. \qquad 7.$$

When the right hand side of equation 4 is factored and the value of $D_1$, from equation 7, substituted in the second factor, it gives,

$$\frac{2U}{kA^2} = (D_2 + D_1)\ (D_2 - D_1) = (D_2 + D_1)\ \left(\frac{4\pi\lambda}{\omega T}\right). \qquad 8.$$

Further, $D_2 + D_1 = 2D$, where D is the ordinate half way between $D_2$ and $D_1$. This is also the point where the slope $\lambda$ is to be evaluated. Therefore, from equation 8,

$$U = \frac{4\pi kA^2 D\lambda}{\omega T}. \qquad 9.$$

The two variables, D and $\lambda$, can each be expressed as a series,

$$D = \frac{1}{A} \sum_{1}^{\infty} b_n \sin \frac{n\pi C}{2},$$

$$\lambda = \frac{dD}{dC} = \frac{1}{A} \sum_{1}^{\infty} b_n \frac{n\pi}{2} \cos \frac{n\pi C}{2}.$$

Using these series in equation 9 gives,

$$U = \frac{2\pi^2 k}{\omega T} \sum_1^\infty b_n \sin \frac{n\pi C}{2} \sum_1^\infty b_n\, n \cos \frac{n\pi C}{2} . \qquad 10.$$

The energy transfered per cycle,[4] through the driving force F, to the system under consideration is ,

$$U = \int^{cycle} F \frac{dy}{dt}\, dt , \qquad 11.$$

where y is displacement as shown in Figure IV, and can be expressed

as, $$y = a \sin \omega t . \qquad 12.$$

Using the expression,

$$a = \sum_1^\infty b_n \sin \frac{n\pi t}{T} \text{ gives,}$$

$$y = \left[\sum_1^\infty b_n \sin \frac{n\pi t}{T}\right] \sin \omega t . \qquad 12a.$$

The derivative of y with respect to t yields,

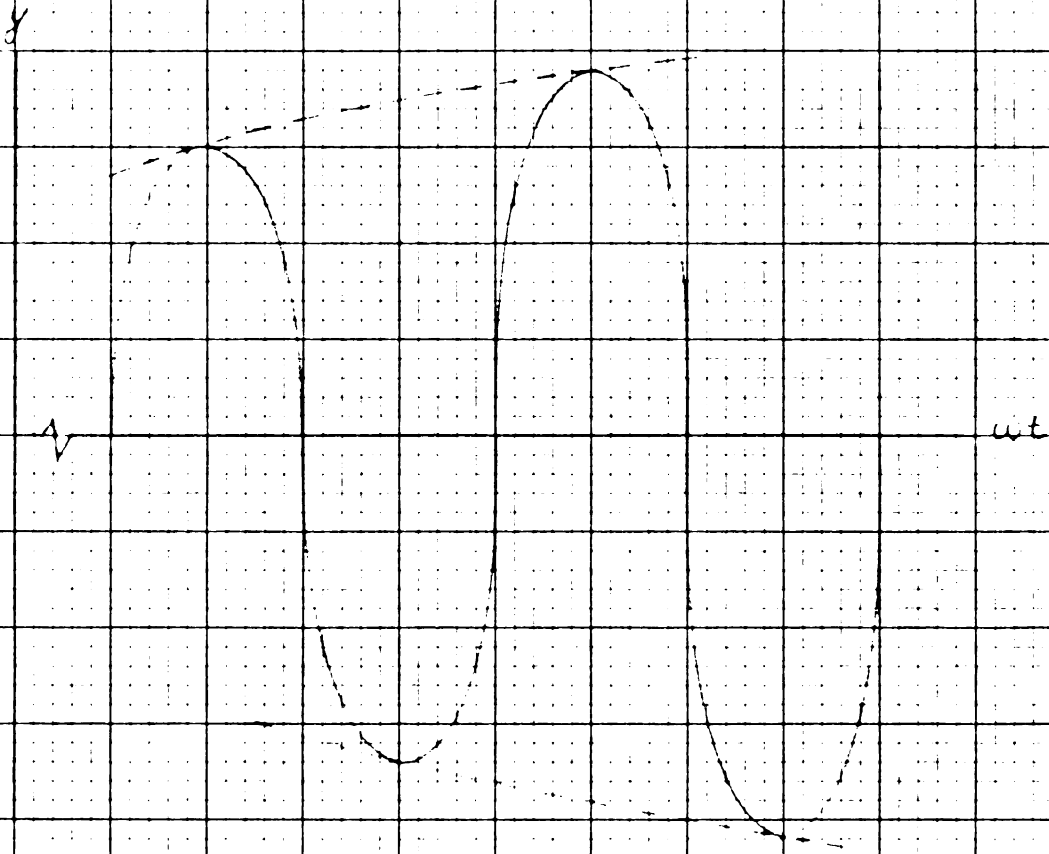
$$\frac{dy}{dt} = \omega \left[\sum_1^\infty b_n \sin \frac{n\pi t}{T}\right] \cos \omega t + \left[\sum_1^\infty b_n \frac{n\pi}{T} \cos \frac{n\pi t}{T}\right] \sin \omega t .$$

Rewriting equation 11, using the above expression for $\frac{dy}{dt}$ gives,

$$U = \int^{cycle} F \left\{ \omega \left[\sum_1^\infty b_n \sin \frac{n\pi t}{T}\right] \cos \omega t + \left[\sum_1^\infty b_n \frac{n\pi}{T} \cos \frac{n\pi t}{T}\right] \sin \omega t \right\} dt . \qquad 13.$$

FIGURE IV

PERIODIC VARIATION OF DISPLACEMENT




PORTION OF BUILD-UP CURVE
SHOWN DASHED

To convert equation 13 to non-dimensional form, use is made of $t = \frac{CT}{2}$, $dt = \frac{T}{2}\,dC$; and the resulting expression is,

$$U = \int^{cycle} F\left\{\omega\left[\sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\right]\cos\frac{\omega TC}{2} + \left[\sum_{1}^{\infty} b_n \frac{n\pi}{T}\cos\frac{n\pi C}{2}\right]\sin\frac{\omega TC}{2}\right\}\frac{T}{2}\,dC \,. \qquad 14.$$

The two expressions for U can now be equated; however, if the derivative of each is taken with respect to C, the integral will be eliminated. This gives,

$$\frac{\pi^3 k}{\omega T}\left[\left(\sum_{1}^{\infty} b_n\, n\cos\frac{n\pi C}{2}\right)^2 - \sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\sum_{1}^{\infty} b_n\, n^2 \sin\frac{n\pi C}{2}\right]$$

$$= F\left[\omega\left(\sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\right)\cos\frac{\omega TC}{2} + \left(\sum_{1}^{\infty} b_n \frac{n\pi}{T}\cos\frac{n\pi C}{2}\right)\sin\frac{\omega TC}{2}\right]\frac{T}{2}\,. \qquad 15.$$

Solving equation 15 for F gives,

$$F = \frac{2\pi^2 k}{\omega T}\left\{\frac{\left[\left(\sum_{1}^{\infty} b_n\, n\cos\frac{n\pi C}{2}\right)^2 - \sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\sum_{1}^{\infty} b_n\, n^2\sin\frac{n\pi C}{2}\right]}{\left[\frac{\omega T}{\pi}\left(\sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\right)\cos\frac{\omega TC}{2} + \left(\sum_{1}^{\infty} b_n\, n\cos\frac{n\pi C}{2}\right)\sin\frac{\omega TC}{2}\right]}\right\} \qquad 16.$$

Going back to equation 2 which states,

$$D = \frac{1}{A}\sum_{1}^{\infty} b_n \sin\frac{n\pi C}{2}\,,$$

The $\frac{1}{A}$ can be placed within the summation, thus,

$$D = \sum_{1}^{\infty} \frac{b_n}{A} \sin \frac{n\pi C}{2} .$$

Then let $\frac{b_n}{A} = b_n'$, which yields,

$$D = \sum_{1}^{\infty} b_n' \sin \frac{n\pi C}{2} .$$

When evaluating the series constants, by harmonic analysis, either $b_n$ or $b_n'$ can be found, depending on whether it is desirable to use dimensional or non-dimensional form.

If the $b_n$ in equation 16 is replaced by $b_n' A$ it becomes,

$$\frac{F}{kA} = \frac{2\pi^2}{\omega T} \left\{ \frac{\left[ \left( \sum_{1}^{\infty} b_n' \, n \cos \frac{n\pi C}{2} \right)^2 - \sum_{1}^{\infty} b_n' \sin \frac{n\pi C}{2} \sum_{1}^{\infty} b_n' \, n^2 \sin \frac{n\pi C}{2} \right]}{\left[ \frac{\omega T}{\pi} \left( \sum_{1}^{\infty} b_n' \sin \frac{n\pi C}{2} \right) \cos \frac{\omega TC}{2} + \left( \sum_{1}^{\infty} b_n' \, n \cos \frac{n\pi C}{2} \right) \sin \frac{\omega TC}{2} \right]} \right\} \qquad 16a.$$

This is the expression for the non-dimensional force F'.

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1. See Figure III for definitions of C, D, and A.
2. See any complete textbook on vibrations.
3. Essentially the familiar spring constant used in theory of vibrations.
4. See any complete textbook on vibrations.

## V. Comments and Conclusions

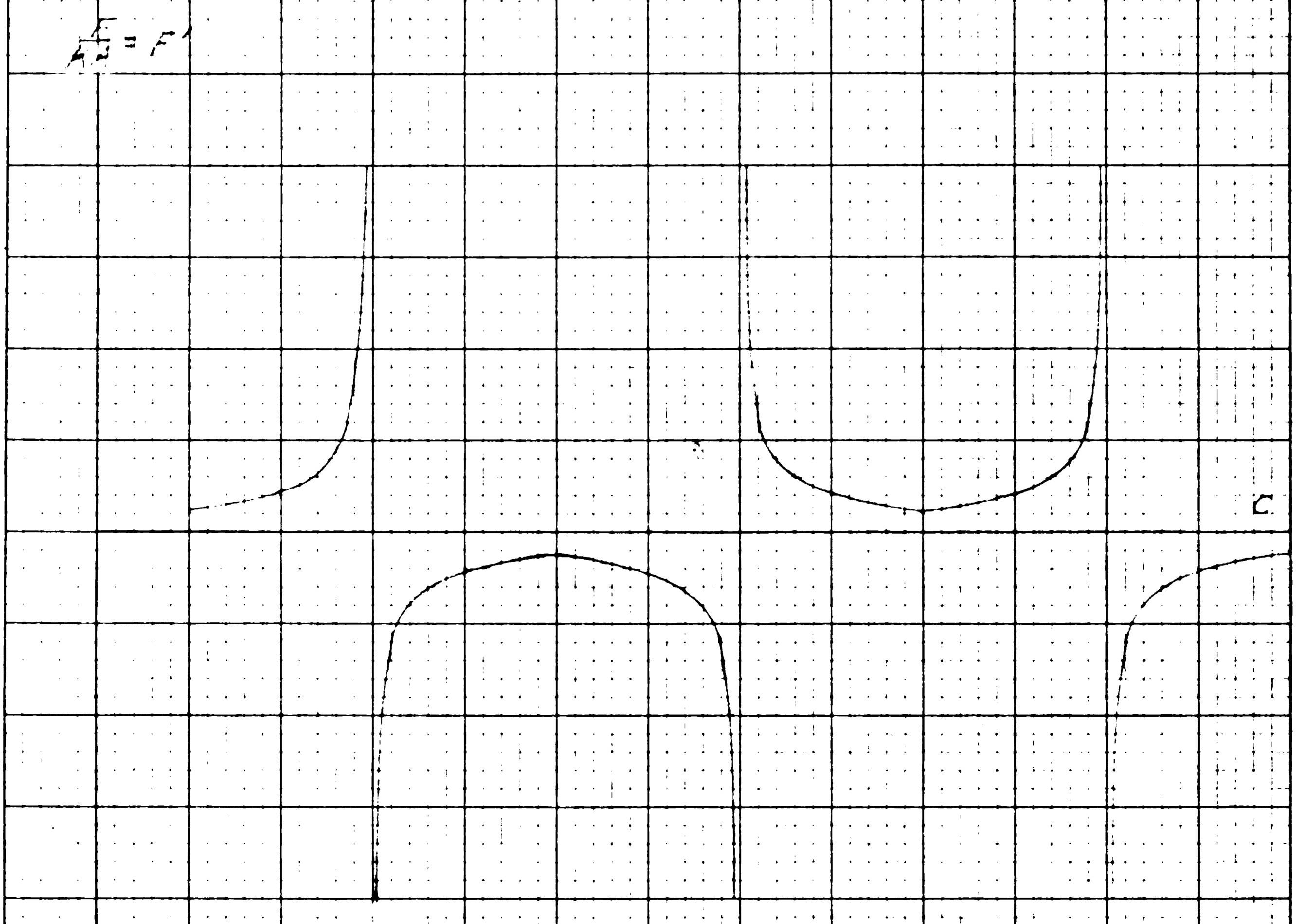
The non-dimensional force function $F'$, expressed in equation 16a, depends upon the frequency of vibration, $\omega$, the build-up time, T, and the various constants which are known or can easily be found for a particular system. The general shape of this function has been plotted in Figure V, using only the terms for $n = 1$.

The conclusions are as follows:

1. The shape of the curve for the non-dimensional force function seems to be reasonable from the solutions of analogous problems.
2. The function is periodic.
3. The frequency of the force is the same as that of the motion of the system and its higher integral harmonics.
4. When the displacement of the system, y, is a minimum, the force function is also a minimum in magnitude and vice versa.
5. The expression 16a for $F'$ could be used to advantage in the calculation of a damping unit for a system subject to self-excited vibrations.
6. Since there is nothing in the function representing the condition of the source of the energy increase, it could be assumed to influence the force function through the frequency, $\omega$, and possibly the time, T.

# FIGURE V

## GENERAL SHAPE OF FORCE FUNCTION PLOTTED FOR A CYCLE AND A HALF



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