SOME INTERPOLATION FORMULAS IN TWO VARIABLES

Thesis for the Degree of M. A.
MICHIGAN STATE COLLEGE
Frank Saidel
1941

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SOME INTERPOLATION FORMULAS IN TWO VARIABLES

by

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A THESIS

Submitted to the Graduate School of Michigan
State College of Agriculture and Applied
Science in partial fulfilment of the
requirements for the degree of

MASTER OF ARTS

Department of Mathematics

1941

ACKNOWLEDGMENT

To Doctor William Dowell Baten whose encouragement and many suggestions have made this thesis possible.

CONTENTS

1. Introduction 1

2. Euler's Polynomials In Two Variables 5

3. Extension of Tschebyscheff's Formula of Mechanical Quadrature To a Cubature Formula 17

4. Cubature Formulas Involving Differences 25

Bibliography 31

SOME INTERPOLATION FORMULAS IN TWO VARIABLES

1. INTRODUCTION

Properties of two classes of polynomials in one variable, which play an important part in the finite calculus, namely the polynomials of Bernoulli and the polynomials of Euler, have been developed.[1]

One of the objects of this thesis is to extend the polynomials of Euler to two variables and develop some important properties of these polynomials. An example is given showing the use of these polynomials in evaluating a double sum.

A cubature formula for approximating the value of double integrals is obtained by extending Tschebyscheff's formula of mechanical quadrature[2] in one variable to two variables. A remainder term is found and an example illustrating the use of the formula is given.

Finally by using Newton's interpolation formula with divided differences of functions of two variables[3], results are obtained from which a variety of cubature form-

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1 L. M. Milne-Thomson, The Calculus of Finite Differences, London, Macmillan and Company, Limited, 1933, pp. 124 - 150. Hereafter referred to as Milne-Thomson.
2 Ibid, p. 177.
3 J. F. Steffensen, Interpolation, Baltimore, The Williams and Wilkins Company, 1927, p. 205. Hereafter referred to as Steffensen.

ulas may be deduced. Remainder terms are given as well as an illustrative example.

Notation will play an important part in simplifying and condensing the work of developing the aforementioned formulas.

In one variable the operators Δ and ∇ are defined as follows;

(1) $$\Delta f(x) = f(x+1) - f(x),$$

(2) $$\nabla f(x) = \tfrac{1}{2}[f(x+1) + f(x)].$$

For two variables the operator $\Delta_x \Delta_y$ is defined as follows;

(3) $$\begin{aligned}\Delta_x \Delta_y f(x,y) &= \Delta_x [f(x,y+1) - f(x,y)] \\ &= \Delta_y [f(x+1,y) - f(x,y)] \\ &= f(x+1,y+1) - f(x,y+1) - f(x+1,y) + f(x,y),\end{aligned}$$

from which it may be concluded that the symbols Δ_x and Δ_y are commutative. Also, the operator $\nabla_x \nabla_y$ is defined so that

(4) $$\begin{aligned}\nabla_x \nabla_y f(x,y) &= \nabla_x \left\{\tfrac{1}{2}[f(x,y+1) + f(x,y)]\right\} \\ &= \nabla_y \left\{\tfrac{1}{2}[f(x+1,y) + f(x,y)]\right\} \\ &= \tfrac{1}{4}[f(x+1,y+1) + f(x,y+1) + f(x+1,y) + f(x,y)],\end{aligned}$$

which shows that ∇_x and ∇_y are likewise commutative.

Divided differences are used in Newton's interpolation formula. For one variable these are defined as follows;

$$f(a_0,a_1) = \frac{f(a_0) - f(a_1)}{a_0 - a_1}$$

$$f(a_0,a_1,a_2) = \frac{f(a_0,a_1) - f(a_1,a_2)}{a_0 - a_2}$$

and in general

$$f(a_0,a_1,\cdots,a_n) = \frac{f(a_0,\cdots,a_{n-1}) - f(a_1,\cdots,a_n)}{a_0 - a_n}$$

where this last expression is an n-th order divided difference of f(x) with respect to the arguments $a_0, a_1, \cdots, a_n$.

Similarly, for two variables,

$$f(a_0,a_1;b_0) = \frac{f(a_0,b_0) - f(a_1,b_0)}{a_0 - a_1}$$

$$f(a_0,a_1;b_0,b_1) = \frac{f(a_0,a_1;b_0) - f(a_0,a_1;b_1)}{b_0 - b_1}$$

$$= \frac{f(a_0,b_0) - f(a_1,b_0) - f(a_0,b_1) + f(a_1,b_1)}{(a_0 - a_1)(b_0 - b_1)}$$

and in general

$$f(a_0,a_1,\cdots,a_n;b_0) = \frac{f(a_0,\cdots,a_{n-1};b_0) - f(a_1,\cdots,a_n;b_0)}{a_0 - a_n}$$

and

$$f(a_0,\cdots,a_n;b_0,\cdots,b_n) = \frac{f(a_0,\cdots,a_{n-1};b_0,\cdots,b_{n-1}) - f(a_0\cdots a_{n-1};b_1,\cdots b_n)}{(a_0 - a_n)(b_0 - b_n)}$$

$$+ \frac{-f(a_1\cdots,a_n;b_0,\cdots b_{n-1}) + f(a_1,\cdots,a_n;b_1,\cdots,b_n)}{(a_0 - a_n)(b_0 - b_n)}$$

An important formula[1] for placing limits on the remainder in Newton's formula is

$$f(x, a_0, \cdots, a_n; y, b_0, \cdots, b_m) = \frac{D_\xi^{n+1} D_\eta^{m+1} f(\xi, \eta)}{(n+1)!\,(m+1)!}$$

where D_ξ and D_η denote partial differentiation, and where $a_0 \leqq \xi \leqq a_n$, $b_0 \leqq \eta \leqq b_m$.

Important also will be the following Theorem of Mean Value[2] for integrals;

> Let f(x) and g(x) be integrable functions of which f(x) is continuous in the closed interval $a \leqq x \leqq b$, where g(x) does not change sign in the interval. There exists, then, at least one point ξ inside the interval such that
>
> $$\int_a^b f(x)\,g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$$
>
> where $a < \xi < b$.

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1 Steffensen, p. 205.
2 Ibid, p. 3.

2. EULER'S POLYNOMIALS IN TWO VARIABLES

Defining E polynomials $E_{v,u}^{(n,m)}(x,y)$ of degree v in x and u in y and order n in x and m in y by the relations

$$(5) \qquad D_x^i\, D_y^j\, E_{v,u}^{(n,m)}(x,y) = v^{(i)}\, u^{(j)}\, E_{v-i,u-j}^{(n,m)}(x,y)$$

and

$$(6) \qquad \overset{m}{\nabla}_y\, \overset{n}{\nabla}_x\, E_{v,u}^{(n,m)}(x,y) = x^v y^u$$

where $v^{(i)} = v(v-1)\cdots(v-i+1)$, it may be shown by the method suggested by Baten[1] that these E polynomials for two independent variables x and y are the products of Euler polynomials for the single variables.

By Taylor's Theorem

$$\begin{aligned} E_{v,u}^{(n,m)}(x+h,y+k) &= E_{v,u}^{(n,m)}(x,y) + h D_x E_{v,u}^{(n,m)}(x,y) + k D_y E_{v,u}^{(n,m)}(x,y) \\ &\quad + \frac{1}{2!}\Big[h^2 D_x^2 E_{v,u}^{(n,m)}(x,y) + 2hk D_x D_y E_{v,u}^{(n,m)}(x,y) \\ &\quad + k^2 D_y^2 E_{v,u}^{(n,m)}(x,y)\Big] + \cdots \\ &= \sum_{j=0}^{u} \sum_{i=0}^{v} \frac{1}{i!}\frac{1}{j!} h^i k^j D_x^i D_y^j E_{v,u}^{(n,m)}(x,y). \end{aligned}$$

Substituting from (5) for $D_x^i\, D_y^j\, E_{v,u}^{(n,m)}(x,y)$ the above becomes

$$(7) \qquad E_{v,u}^{(n,m)}(x+h,y+k) = \sum_{j=0}^{u} \sum_{i=0}^{v} \binom{v}{i}\binom{u}{j} h^i k^j E_{v-i,u-j}^{(n,m)}(x,y).$$

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1 W. D. Baten, A Remainder for the Euler-MacLaurin Summation Formula in Two Independent Variables, American Journal of Mathematics, Vol. LIV, No. 2, April, 1932.

Substituting from (7) in (4) with h and k equal to (1,1), (1,0), (0,1) and (0,0) and by (6)

$$\nabla_y \nabla_x E^{(n,m)}_{v,u}(x,y) = \frac{1}{4}\Big[\sum_{j=0}^{u}\sum_{i=0}^{v}\binom{v}{i}\binom{u}{j}E^{(n,m)}_{v-i,u-j}(x,y) + \sum_{i=0}^{v}\binom{v}{i}E^{(n,m)}_{v-i,u}(x,y) + \sum_{j=0}^{u}\binom{u}{j}E^{(n,m)}_{v,u-j}(x,y) + E^{(n,m)}_{v,u}(x,y)\Big]$$

Letting $n = m = 1$ gives $\nabla_y \nabla_x E_{v,u}(x,y) = x^v y^u$ so that

$$\sum_{j=0}^{u}\sum_{i=0}^{v}\binom{v}{i}\binom{u}{j}E_{v-i,u-j}(x,y) + \sum_{i=0}^{v}\binom{v}{i}E_{v-i,u}(x,y) + \sum_{j=0}^{u}\binom{u}{j}E_{v,u-j}(x,y) + E_{v,u}(x,y) = 4x^v y^u$$

where $E_{v,u}(x,y) = E^{(1,1)}_{v,u}(x,y)$

For various values of v and u these polynomials become

$E_{0,0}(x,y) = 1$

$E_{1,0}(x,y) = x - \frac{1}{2}$, $\quad E_{0,1}(x,y) = y - \frac{1}{2}$

$E_{1,1}(x,y) = (x - \frac{1}{2})(y - \frac{1}{2})$

$E_{2,0}(x,y) = x(x - 1)$, $\quad E_{0,2}(x,y) = y(y - 1)$

$E_{2,1}(x,y) = x(x - 1)(y - \frac{1}{2})$,

$E_{1,2}(x,y) = y(y - 1)(x - \frac{1}{2})$

$E_{2,2}(x,y) = xy(x - 1)(y - 1)$

$E_{3,0}(x,y) = (x - \frac{1}{2})(x^2 - x - \frac{1}{2})$

$E_{0,3}(x,y) = (y - \frac{1}{2})(y^2 - y - \frac{1}{2})$

$E_{3,1}(x,y) = (x - \frac{1}{2})(x^2 - x - \frac{1}{2})(y - \frac{1}{2})$

$E_{1,3}(x,y) = (y - \frac{1}{2})(y^2 - y - \frac{1}{2})(x - \frac{1}{2})$

$E_{3,2}(x,y) = (x - \frac{1}{2})(x^2 - x - \frac{1}{2})y(y - 1)$

$E_{2,3}(x,y) = (y - \frac{1}{2})(y^2 - y - \frac{1}{2})x(x - 1)$

$E_{3,3}(x,y) = (x - \frac{1}{2})(x^2 - x - \frac{1}{2})(y - \frac{1}{2})(y^2 - y - \frac{1}{2})$

Etc.

From the above poynomials it is seen that

$$E_{p,q}(x,y) = E_{p,0}(x,y)\ E_{0,q}(x,y)$$

where p and q run from 0 to 3. This is evident for any p and q. This shows that these E polynomials for two variables x and y are the products of Euler polynomials for the single variables. $E_{p,0}(x,y)$ is an Euler polynomial in the single variable x and $E_{0,q}(x,y)$ is an Euler polynomial in the single variable y.

It is possible to reach this same conclusion by another method. This will now be done since many interesting properties of the E polynomials may be brought out in the procedure.

Define φ polynomials $\varphi^{(n,m)}_{v,u}(x,y)$ of degree v in x and u in y and order n in x and m in y by the relation

$$(8)\qquad f_{n,m}(t,w)e^{G(xt+yw)+g(t,w)} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi^{(n,m)}_{v,u}(x,y)$$

where $f_{n,m}(t,w)$, $G(xt+yw)$ and $g(t,w)$ are such that for a certain range of x and y the expansion on the right exists as a uniformly convergent series in t and w and where $G(xt+yw)=0$ for $x=y=0$ and $G[t(x+r)+w(y+s)] = G(xt+wy)+G(tr+ws)$.

Putting $x = y = 0$ in (8) gives

(9) $$f_{n,m}(t,w)e^{g(t,w)} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}$$

where $\varphi_{v,u}^{(n,m)} = \varphi_{v,u}^{(n,m)}(0,0)$ is called a φ number of order n,m.

In (8) let $x = x + r$ and $y = y + s$ giving

(10) $$\sum_{u=0}^{\infty}\sum_{0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}(x+r,y+s) = f_{n,m}(t,w)e^{G[t(x+r)+w(y+s)]+g(t,w)}$$

$$= e^{G(xt+yw)}f_{n,m}(t,w)e^{G(tr+ws)+g(t,w)}$$

$$= e^{G(xt+yw)}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}(r,s)$$

Now let $G(xt+yw) = xt+yw$ so that (10) becomes

(11) $$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}(x+r,y+s) = e^{xt+yw}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}(r,s)$$

and (8) becomes

(12) $$f_{n,m}(t,w)e^{xt+yw+g(t,w)} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\varphi_{v,u}^{(n,m)}(x,y)$$

Substituting $1 + (xt+yw) + \frac{(xt+yw)^2}{2!} + \cdots$ for e^{xt+yw} in the right member of (11) and equating coefficients of $t^v w^u$ gives

$$\varphi_{v,u}^{(n,m)}(x+r,y+s) = \varphi_{v,u}^{(n,m)}(r,s) + \binom{v}{1}x\varphi_{v-1,u}^{(n,m)}(r,s)$$
$$+\binom{u}{1}y\varphi_{v,u-1}^{(n,m)}(r,s) + \binom{v}{2}x^2\varphi_{v-2,u}^{(n,m)}(r,s)$$

$$+\binom{v}{1}\binom{u}{1}xy\,\varphi^{(n,m)}_{v-1,u-1}(r,s)+\binom{u}{2}y^2\varphi^{(n,m)}_{v,u-2}(r,s)$$

$$+\cdots\cdot+\binom{v}{v}\binom{u}{u}x^v y^u\varphi^{(n,m)}_{0,0}(r,s)$$

Letting $r = s = 0$ the above becomes

(13) $$\varphi^{(n,m)}_{v,u}(x,y)=\varphi^{(n,m)}_{v,u}+\binom{v}{1}x\,\varphi^{(n,m)}_{v-1,u}+\binom{u}{1}y\,\varphi^{(n,m)}_{v,u-1}$$

$$+\binom{v}{2}x^2\varphi^{(n,m)}_{v-2,u}+\binom{v}{1}\binom{u}{1}xy\,\varphi^{(n,m)}_{v-1,u-1}$$

$$+\binom{u}{2}y^2\varphi^{(n,m)}_{v,u-2}+\cdots+\binom{v}{v}\binom{u}{u}x^v y^u\varphi^{(n,m)}_{0,0}$$

This shows that unless $\varphi^{(n,m)}_{0,0}=0$, $\varphi^{(n,m)}_{v,u}(x,y)$ is of degree v in x and u in y.

It is possible to rewrite (13) in the symbolic expression

(14) $$\varphi^{(n,m)}_{v,u}(x,y)\doteq\left[\varphi^{(n)}+x\right]^v\left[\varphi^{(m)}+y\right]^u$$

where, after expansion of the right hand member, each superscript of $\varphi^{(n,m)}$, including zero, is to be replaced by the corresponding subscript. For example,

$$\left[\varphi^{(n)}\right]^v\left[\varphi^{(m)}\right]^u\doteq\varphi^{(n,m)}_{v,u}$$

Differentiating both sides of (12) partially with respect to x and y gives

(15) $$\frac{\partial^2\varphi^{(n,m)}_{v,u}(x,y)}{\partial x\,\partial y}=\binom{v}{1}\binom{u}{1}\varphi^{(n,m)}_{v-1,u-1}+2\binom{v}{2}\binom{u}{1}x\,\varphi^{(n,m)}_{v-2,u-1}$$

$$+ 2\binom{v}{1}\binom{u}{2} y \varphi^{(n,m)}_{v-1,u-2} + 3\binom{v}{3}\binom{u}{1} x^2 \varphi^{(n,m)}_{v-3,u-1}$$

$$+ \cdots\cdots + vu\binom{v}{v}\binom{u}{u} x^{v-1} y^{u-1} \varphi^{(n,m)}_{0,0}$$

which by (13) is equal to $vu\,\varphi^{(n,m)}_{v-1,u-1}(x,y)$. This could be written as

$$D_x D_y \varphi^{(n,m)}_{v,u}(x,y) = vu\,\varphi^{(n,m)}_{v-1,u-1}(x,y) \tag{16}$$

and by repeating this process (16) would become

$$D_x^i D_y^j \varphi^{(n,m)}_{v,u}(x,y) = v^{(i)} u^{(j)} \varphi^{(n,m)}_{v-i,u-j}(x,y) \tag{17}$$

which is seen to be similar to (5), one of the definitions in the first method.

From (12)

$$\Delta_y \Delta_x f_{n,m}(t,\omega) e^{xt+y\omega+g(t,\omega)} = \Delta_y \Delta_x \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{\omega^u}{u!}\varphi^{(n,m)}_{v,u}(x,y)$$

and using (3) the left hand side becomes

$$(e^{t+\omega} - e^t - e^\omega + 1) f_{n,m}(t,\omega) e^{xt+y\omega+g(t,\omega)}$$

giving

$$(e^t-1)(e^\omega-1) f_{n,m}(t,\omega) e^{xt+y\omega+g(t,\omega)} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{\omega^u}{u!}\Delta_y\Delta_x \varphi^{(n,m)}_{v,u}(x,y) \tag{18}$$

Also from (12)

$$\nabla_y \nabla_x f_{n,m}(t,\omega) e^{xt+y\omega+g(t,\omega)} = \nabla_y \nabla_x \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{\omega^u}{u!}\varphi^{(n,m)}_{v,u}(x,y)$$

and using (4) the left hand side becomes

$$\frac{1}{4}(e^{t+\omega} + e^t + e^\omega + 1) f_{n,m}(t,\omega) e^{xt+y\omega+g(t,\omega)}$$

giving

(19) $$\frac{e^t+1}{2}\cdot\frac{e^w+1}{2}f_{n,m}(t,w)\,e^{xt+yw+g(t,w)}=\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\nabla_y\nabla_x\varphi_{v,u}^{(n,m)}(x,y)$$

Formula (19) suggests that a simple class of φ polynomials should arise if in (12)

$$f_{n,m}(t,w)=\frac{2^{n+m}}{(e^t+1)^n(e^w+1)^m}$$

where n and m are any integers, positive, negative or zero. The polynomials which arise in this way shall be called N polynomials defined by

(20) $$\frac{2^{n+m}}{(e^t+1)^n(e^w+1)^m}e^{xt+yw+g(t,w)}=\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}N_{v,u}^{(n,m)}(x,y)$$

Since N polynomials are φ polynomials, (19) gives

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}\nabla_y\nabla_x N_{v,u}^{(n,m)}(x,y)=\frac{e^t+1}{2}\,\frac{e^w+1}{2}\,\frac{2^{n+m}}{(e^t+1)^n(e^w+1)^m}e^{xt+yw+g(t,w)}$$

$$=\frac{2^{n-1+m-1}}{(e^t+1)^{n-1}(e^w+1)^{m-1}}e^{xt+yw+g(t,w)}$$

$$=\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^v}{v!}\frac{w^u}{u!}N_{v,u}^{(n-1,m-1)}(x,y)$$

and equating coefficients of $t^v w^u$ this results in

(21) $$\nabla_y\nabla_x N_{v,u}^{(n,m)}(x,y)=N_{v,u}^{(n-1,m-1)}(x,y)$$

But $\nabla_y\nabla_x N_{v,u}^{(n,m)}(x,y)=\frac{1}{4}\left[N_{v,u}^{(n,m)}(x+1,y+1)+N_{v,u}^{(n,m)}(x+1,y)+N_{v,u}^{(n,m)}(x,y+1)+N_{v,u}^{(n,m)}(x,y)\right.$

Then, $4N_{v,u}^{(n-1,m-1)}(x,y)=N_{v,u}^{(n,m)}(x+1,y+1)+N_{v,u}^{(n,m)}(x+1,y)+N_{v,u}^{(n,m)}(x,y+1)+N_{v,u}^{(n,m)}(x,y)$

Letting $x=0$, $y=0$ gives the recurrence relation for N numbers,

$$4N^{(n-1,m-1)}_{v,u} = N^{(n,m)}_{v,u}(1,1) + N^{(n,m)}_{v,u}(1,0) + N^{(n,m)}_{v,u}(0,1) + N^{(n,m)}_{v,u}$$

The simplest N polynomials are obtained by putting $g(t,w) = 0$, $n = 0$, $m = 0$ in (20). This gives

$$e^{xt+yw} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!} N^{(0,0)}_{v,u}(x,y)$$

and equating coefficients of $t^v w^u$ after expanding the left hand side shows that

$$N^{(0,0)}_{v,u}(x,y) = x^v y^u$$

These simplest N polynomials shall be regarded as Euler's polynomials for two variables of order zero, denoted by $E^{(0,0)}_{v,u}(x,y)$. E polynomials of order n,m are now defined by

(22) $$\frac{2^{n+m}}{(e^t+1)^n(e^w+1)^m} e^{xt+yw} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!} E^{(n,m)}_{v,u}(x,y)$$

Following Nörlund's notation for Euler's polynomials[1], let

(23) $$E^{(n,m)}_{v,u}(0,0) = 2^{-(v+u)} c^{(n,m)}_{v,u}$$

The generating function for the c numbers is therefore

(24) $$\frac{2^{n+m}}{(e^t+1)^n(e^w+1)^m} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!}\frac{1}{2^{v+u}} c^{(n,m)}_{v,u}$$

The values of $2^{v+u}E^{(n,m)}_{v,u}(\frac{n}{2},\frac{m}{2})$ are called E numbers, $E^{(n,m)}_{v,u}$ of order n,m;

(25) $$E^{(n,m)}_{v,u} = 2^{v+u}E^{(n,m)}_{v,u}\left(\frac{n}{2},\frac{m}{2}\right)$$

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1 Milne-Thomson, p. 143.

Some fundamental properties of E polynomials may be listed by noting that E polynomials are N polynomials and therefore φ polynomials. Hence they have the properties of φ polynomials as well as of N polynomials.

$$E^{(n,m)}_{v,u}(x,y) \doteq \left[E^{(n)}(0)+x\right]^{v}\left[E^{(m)}(0)+y\right]^{u} \qquad \text{from (14)}$$

(26) $$E^{(n,m)}_{v,u}(x,y) \doteq \left[\tfrac{1}{2}C^{(n)}+x\right]^{v}\left[\tfrac{1}{2}C^{(m)}+y\right]^{u}$$

(27) $$D^{i}_{x}\,D^{j}_{y}\,E^{(n,m)}_{v,u}(x,y) = v^{(i)}u^{(j)}E^{(n,m)}_{v-i,u-j}(x,y) \qquad \text{from (17)}$$

(28) $$\nabla_{y}\nabla_{x}E^{(n,m)}_{v,u}(x,y) = E^{(n-1,m-1)}_{v,u}(x,y) \qquad \text{from (21)}$$

By repeated application of (28)

$$\overset{m}{\nabla}_{y}\overset{n}{\nabla}_{x}E^{(n,m)}_{v,u}(x,y) = E^{(0,0)}_{v,u}(x,y)$$

It follows that

(29) $$\overset{m}{\nabla}_{y}\overset{n}{\nabla}_{x}E^{(n,m)}_{v,u}(x,y) = x^{v}y^{u}$$

since $$E^{(0,0)}_{v,u}(x,y) = x^{v}y^{u}$$

Notice that properties (27) and (29) are exactly the same as the definitions (5) and (6) upon which the first method depended.

An interesting theorem will now be proved showing that E numbers in which either subscript is odd, are all zero. This is the complementary argument theorem for two variables.

The arguments x and n - x, y and m - y, are called complementary. Replacing x by n - x, y by m - y in (22) gives

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\frac{t^{v}}{v!}\,\frac{w^{u}}{u!}\,E^{(n,m)}_{v,u}(n-x,\,m-y) = \frac{2^{n+m}}{(e^{t}+1)^{n}(e^{w}+1)^{m}}\,e^{(n-x)t+(m-y)w}$$

$$= \frac{2^{n+m}\,e^{-xt-yw}}{(e^{-t}+1)^{n}(e^{-w}+1)^{m}}$$

$$= \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \frac{(-t)^v}{v!} \frac{(-w)^u}{u!} E_{v,u}^{(n,m)}(x,y)$$

Equating coefficients of $t^v w^u$ shows that

$$E_{v,u}^{(n,m)}(n-x, m-y) = (-1)^{v+u} E_{v,u}^{(n,m)}(x,y). \tag{30}$$

Equation (30) is the complementary argument theorem for two variables and it is true for any N polynomial in whose generating function g(t,w) is an even function.

Letting $x = 0$, $y = 0$ and $v = 2r$, $u = 2s$ in (30)

$$E_{2r,2s}^{(n,m)}(n,m) = E_{2r,2s}^{(n,m)}(0,0)$$

$$= 2^{-(2r+2s)} C_{2r,2s}^{(n,m)} .$$

Thus $E_{2r,2s}^{(n,m)} - 2^{-(2r+2s)} C_{2r,2s}^{(n,m)}$ has zeros at $x = 0$, $y = 0$ and $x = n$, $y = m$. Putting $x = \frac{n}{2}$, $y = \frac{m}{2}$ and $v = 2r + 1$, $u = 2s$ in (30)

$$E_{2r+1,2s}^{(n,m)}\left(\frac{n}{2}, \frac{m}{2}\right) = (-1)^{2r+1+2s} E_{2r+1,2s}^{(n,m)}\left(\frac{n}{2}, \frac{m}{2}\right)$$

$$E_{2r+1,2s}^{(n,m)}\left(\frac{n}{2}, \frac{m}{2}\right) = 0.$$

Putting $x = \frac{n}{2}$, $y = \frac{m}{2}$ and $v = 2r$, $u = 2s + 1$ in (30)

$$E_{2r,2s+1}^{(n,m)}\left(\frac{n}{2}, \frac{m}{2}\right) = 0,$$

Thus, E numbers in which either of the subscripts is odd, are all zero.

E polynomials of the first order may be obtained by placing $n = m = 1$ in (22). Then,

$$\frac{2^2}{(e^t+1)(e^w+1)} e^{xt+yw} = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \frac{t^v}{v!} \frac{w^u}{u!} E_{v,u}(x,y) \tag{31}$$

and letting $x = y = 0$ gives

$$\frac{2^2}{(e^t+1)(e^w+1)} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!} 2^{-(v+u)} c_{v,u}.$$

Letting $x = y = \frac{1}{2}$ shows that

$$\frac{2^2}{(e^t+1)(e^w+1)} e^{\frac{1}{2}t+\frac{1}{2}w} = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!} E_{v,u}\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!}\frac{1}{2^{v+u}} E_{v,u}.$$

From (31)

$$e^{xt+yw} = \frac{(e^t+1)(e^w+1)}{4}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{t^v}{v!}\frac{w^u}{u!} E_{v,u}(x,y).$$

Expanding the exponentials on both sides and equating coefficients of $t^p w^q$, $p = 0,1,2,\cdots$, $q = 0,1,2,\cdots$ gives the same polynomials as those found by the first method.

To find the E numbers, let $x = y = \frac{1}{2}$ in the first order E polynomials which shows that

$E_{0,0} = 1$

$E_{2,0} = -1,$ $\quad E_{0,2} = -1$

$E_{2,2} = 1$

and it could further be shown that

$E_{4,0} = 5,$ $\quad E_{0,4} = 5$

$E_{4,2} = -5,$ $\quad E_{2,4} = -5$

$E_{4,4} = 25$

$E_{6,0} = -61,$ $\quad E_{0,6} = -61$

$E_{6,2} = 61, \qquad E_{2,6} = 61$

$E_{6,4} = -305, \qquad E_{4,6} = -305$

$E_{6,6} = 3721$

An example will now be taken up to show how E polynomials could be used in evaluating a double sum. To evaluate

$$\sum_{q=1}^{m} \sum_{p=1}^{n} (-1)^{p+q} p^{v} q^{u}$$

make use of (29) for n = m = 1, which gives

$$\nabla_y \nabla_x E_{v,u}(x,y) = x^v y^u.$$

It follows that

$$\sum_{q=1}^{m} \sum_{p=1}^{n} (-1)^{p+q} p^{v} q^{u} = \sum_{q=1}^{m} \sum_{p=1}^{n} (-1)^{p+q} \nabla_q \nabla_p E_{v,u}(p,q)$$

$$= \frac{1}{4}\left[E_{v,u}(1,1) + (-1)^{n+1} E_{v,u}(n+1,1) + (-1)^{m+1} E_{v,u}(1,m+1) + (-1)^{n+m} E_{v,u}(n+1,m+1)\right].$$

In particular to evaluate

$$\sum_{q=1}^{5} \sum_{p=1}^{8} (-1)^{p+q} p^{6} q^{4}$$

$$\sum_{q=1}^{5} \sum_{p=1}^{8} (-1)^{p+q} p^{6} q^{4} = \frac{1}{4}\left[E_{6,4}(1,1) - E_{6,4}(9,1) + E_{6,4}(1,6) - E_{6,4}(9,6)\right].$$

It may be shown that

$$E_{6,4}(x,y) = x(x-1)(x^4-2x^3-2x^2+3x+3)y(y-1)(y^2-y-1)$$

so that $\quad E_{6,4}(1,1) = E_{6,4}(9,1) = E_{6,4}(1,6) = 0$

and $\quad E_{6,4}(9,6) = 311,383,440.$

Then $\quad \displaystyle\sum_{q=1}^{5} \sum_{p=1}^{8} (-1)^{p+q} p^{6} q^{4} = -77,845,860.$

By actual calculation, it is seen that this is the true value of the double sum.

3. EXTENSION OF TSCHEBYSCHEFF'S FORMULA OF MECHANICAL QUADRATURE TO A CUBATURE FORMULA

Let F(x,y) be a given function, and E(x,y) an arbitrary function which is assumed to have continuous derivatives in x and y up to and including the (n+1)th. Points $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ are sought such that

$$\int_{-1}^{1}\int_{-1}^{1} F(x,y)\,E(x,y)\,dx\,dy = hk\sum_{j=1}^{n}\sum_{i=1}^{n} E(x_i, y_j) + R_{n,n}, \tag{32}$$

where hk and the points $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ are independent of the particular function E(x,y) and where the remainder term $R_{n,n}$ depends upon the above points and $(xD_x + yD_y)^{n+1}E(x,y)$, where

$$D_x^u D_y^v E(x,y) = E_{u,v}(x,y) = \frac{\partial^{u+v}}{\partial x^u \partial y^v} E(x,y).$$

By Maclaurin's Theorem

$$E(x,y) = E(0,0) + (xD_x + yD_y)E(0,0) + \frac{(xD_x+yD_y)^2}{2!}E(0,0)$$
$$+ \cdots + \frac{(xD_x+yD_y)^n}{n!}E(0,0) + \frac{(xD_x+yD_y)^{n+1}}{(n+1)!}E(a,b)$$

where $0 \leqq a \leqq x,\ 0 \leqq b \leqq y.$

Consider from (32)

$$R_{n,n} = \int_{-1}^{1}\int_{-1}^{1} F(x,y)\,E(x,y)\,dx\,dy - hk\sum_{j=1}^{n}\sum_{i=1}^{n} E(x_i, y_j).$$

Substituting for E(x,y) from Maclaurin's expansion gives

$$R_{n,n} = \int_{-1}^{1}\int_{-1}^{1} F(x,y)\Big[E(0,0) + (xD_x + yD_y)E(0,0) + \frac{(xD_x+yD_y)^2}{2!}E(0,0)$$
$$+ \cdots + \frac{(xD_x+yD_y)^n}{n!}E(0,0) + \frac{(xD_x+yD_y)^{n+1}}{(n+1)!}E(a,b)\Big]dx\,dy$$

$$- hk \sum_{j=1}^{n} \sum_{i=1}^{n} \Big[E(0,0) + (x_i D_x + y_j D_y) E(0,0) + \frac{(x_i D_x + y_j D_y)^2}{2!} E(0,0)$$

$$+ \cdots + \frac{(x_i D_x + y_j D_y)^n}{n!} E(0,0) + \frac{(x_i D_x + y_j D_y)^{n+1}}{(n+1)!} E(a_i, b_j) \Big].$$

Letting $T_{u,v} = \int_{-1}^{1}\int_{-1}^{1} \frac{x^u}{u!} \frac{y^v}{v!} F(x,y)\,dx\,dy$ above gives

$$R_{n,n} = T_{0,0} E_{0,0}(0,0) + T_{1,0} E_{1,0}(0,0) + T_{0,1} E_{0,1}(0,0) + T_{2,0} E_{2,0}(0,0)$$

$$+ T_{1,1} E_{1,1}(0,0) + T_{0,2} E_{0,2}(0,0) + \cdots + T_{n,0} E_{n,0}(0,0)$$

$$+ T_{n-1,1} E_{n-1,1}(0,0) + \cdots + T_{1,n-1} E_{1,n-1}(0,0) + T_{0,n} E_{0,n}(0,0)$$

$$+ \int_{-1}^{1}\int_{-1}^{1} \frac{(x D_x + y D_y)^{n+1}}{(n+1)!} E(a,b) F(x,y)\,dx\,dy$$

$$- n^2 hk E_{0,0}(0,0) - nhk E_{1,0}(0,0) \sum_{i=1}^{n} x_i - nhk E_{0,1}(0,0) \sum_{j=1}^{n} y_j$$

$$- \cdots - nhk \frac{E_{0,n}(0,0)}{n!} \sum_{j=1}^{n} y_j^n - hk \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{(x_i D_x + y_j D_y)^{n+1}}{(n+1)!} E(a_i, b_j),$$

where $0 \leqq a_i \leqq x_i$, $0 \leqq b_j \leqq y_j$ for $i, j = 1, 2, \cdots, n$.

The terms containing $E_{0,0}(0,0), E_{1,0}(0,0), \cdots,$ $E_{0,n}(0,0)$ can be made to disappear by taking

$$n^2 hk = T_{0,0}$$

$$nhk \sum_{i=1}^{n} x_i = T_{1,0}$$

$$nhk \sum_{j=1}^{n} y_j = T_{0,1}$$

$$nhk \sum_{i=1}^{n} x_i^2 = 2!\, T_{2,0}$$

$$2hk \sum_{j=1}^{n} \sum_{i=1}^{n} x_i y_j = 2!\, T_{1,1}$$

$$n\,hk\sum_{j=1}^{n} y_j^2 = 2!\,T_{0,2}$$

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$$n\,hk\sum_{i=1}^{n} x_i^n = n!\,T_{n,0}$$

$$\binom{n}{1} hk \sum_{j=1}^{n}\sum_{i=1}^{n} x_i^{n-1} y_j = n!\,T_{n-1,1}$$

- - - - - - - - - - - - - -

$$\binom{n}{1} hk \sum_{j=1}^{n}\sum_{i=1}^{n} x_i y_j^{n-1} = n!\,T_{1,n-1}$$

$$n\,hk\sum_{j=1}^{n} y_j^n = n!\,T_{0,n}$$

Since $\sum_{j=1}^{n}\sum_{i=1}^{n} x_i^{\alpha} y_j^{\beta} = \sum_{i=1}^{n} x_i^{\alpha} \sum_{j=1}^{n} y_j^{\beta}$ there are only 2n + 1 independent equations above. These are

$$n^2 hk = T_{0,0}$$

$$n\,hk\sum_{i=1}^{n} x_i = T_{1,0} \qquad n\,hk\sum_{j=1}^{n} y_j = T_{0,1}$$

$$n\,hk\sum_{i=1}^{n} x_i^2 = 2!\,T_{2,0} \qquad n\,hk\sum_{j=1}^{n} y_j^2 = 2!\,T_{0,2}$$

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$$n\,hk\sum_{i=1}^{n} x_i^n = n!\,T_{n,0} \qquad n\,hk\sum_{j=1}^{n} y_j^n = n!\,T_{0,n}$$

The 2n + 1 numbers $hk, x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ may now be determined since they are expressed above in terms of the moments, $T_{u,v}$. Then (32) constitutes Tschebyscheff's formula for two variables, the remainder term being

$$(33) \qquad R_{n,n} = \int_{-1}^{1}\int_{-1}^{1} \frac{(xD_x+yD_y)^{n+1}}{(n+1)!} E(a,b)\, F(x,y)\, dx\, dy$$

$$- k k \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{(x_iD_x+y_jD_y)^{n+1}}{(n+1)!} E(a_i,b_j)$$

which vanishes when $G(x,y)$ is a polynomial of degree n at most since then $(xD_x+yD_y)^{n+1}G(x,y)=0$. In this case formula (32) is exact, that is, there is no remainder term. In general, (33) will not vanish. In this case it is not a practical form for the remainder since each of the binomial expansions in the formula contains $n+2$ terms and the total number of terms in the remainder is $(n^2+1)(n+2)$.

The points $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ may be obtained by making some assumptions as to the nature of $F(x,y)$, $E(x,y)$ and introducing a function of z and w, $f(z,w)$, so that the problem reverts to the method of Tschebyscheff for one variable[1].

The procedure is as follows;

Let $F(x,y) = G(x)\, H(y)$, a product of two given functions.

Let $\qquad f(z,w) = r(z)\, s(w)$

where $\qquad r(z) = (z-x_1)(z-x_2)(z-x_3)\cdots(z-x_n)$

and $\qquad s(w) = (w-y_1)(w-y_2)(w-y_3)\cdots(w-y_n)$

so that $\qquad x_1, x_2, \cdots, x_n \qquad$ are the roots of $r(z)=0$

and $\qquad y_1, y_2, \cdots, y_n \qquad$ are the roots of $s(w)=0$.

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1 Milne-Thomson, pp. 178-180.

Take $\quad E(x,y) = g(x)\,h(y)$

where $\quad g(x) = (z - x)^{-1}$

and $\quad h(y) = (w - y)^{-1}$

It follows then, that

$$\int_{-1}^{1}\int_{-1}^{1} F(x,y)E(x,y)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} G(x)H(y)g(x)h(y)\,dx\,dy$$

$$= \int_{-1}^{1} \frac{G(x)}{z-x}\,dx \int_{-1}^{1} \frac{H(y)}{w-y}\,dy .$$

It has been shown[1] that

$$\int_{-1}^{1} \frac{G(x)}{z-x}\,dx = k\,\frac{r'(z)}{r(z)} + \frac{c_1}{z^{n+2}} + \frac{c_2}{z^{n+3}} + \cdots$$

and
$$\int_{-1}^{1} \frac{H(y)}{w-y}\,dy = k\,\frac{s'(z)}{s(z)} + \frac{d_1}{w^{n+2}} + \frac{d_2}{w^{n+3}} + \cdots$$

where $c_1, c_2, \cdots$ and $d_1, d_2, \cdots$ are independent of z and w respectively, so that

$$\int_{-1}^{1}\int_{-1}^{1} \frac{F(x,y)}{(z-x)\,(w-y)}\,dx\,dy = \left[k\,\frac{r'(z)}{r(z)} + \frac{c_1}{z^{n+2}} + \frac{c_2}{z^{n+3}} + \cdots\right]$$

$$\left[k\,\frac{s'(w)}{s(w)} + \frac{d_1}{w^{n+2}} + \frac{d_2}{w^{n+3}} + \cdots\right]$$

Integrating both sides with respect to z and w gives

$$\int_{-1}^{1}\int_{-1}^{1} F(x,y)\log(z-x)\log(w-y)\,dx\,dy = \left[k\log\frac{r(z)}{C} - \frac{c_1}{(n+1)z^{n+1}} - \frac{c_2}{(n+2)z^{n+2}} - \cdots\right]$$

$$\left[k\log\frac{s(w)}{D} - \frac{d_1}{(n+1)w^{n+1}} - \frac{d_2}{(n+2)w^{n+2}} - \cdots\right]$$

where C and D are constants. It follows that

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1 Milne-Thomson, p.178.

$$\int_{-1}^{1} G(x)\log(z-x)\,dx \int_{-1}^{1} H(y)\log(w-y)\,dy = \left[k\log\frac{r(z)}{c} - \frac{c_1}{(n+1)z^{n+1}} - \cdots\right]$$

$$\left[h\log\frac{s(w)}{\theta} - \frac{d_1}{(n+1)w^{n+1}} - \cdots\right].$$

Since this is now a function of z times a function of w equal to a function of z times a function of w, the functions of the like variables may be equated giving

(34) $$\int_{-1}^{1} G(x)\log(z-x)\,dx = k\log\frac{r(z)}{c} - \frac{c_1}{(n+1)z^{n+1}} - \cdots$$

and

(35) $$\int_{-1}^{1} H(y)\log(w-y)\,dy = h\log\frac{s(w)}{\theta} - \frac{d_1}{(n+1)w^{n+1}} - \cdots$$

This throws the problem of determining the points directly upon the case of one variable where it is shown that for[1]

$n = 2$,	$-x_1 = x_2 = .57735027$	
$n = 3$,	$-x_1 = x_3 = .70710678$,	$x_2 = 0$
$n = 4$,	$-x_1 = x_4 = .79465447$	
	$-x_2 = x_3 = .18759247$	
$n = 5$,	$-x_1 = x_5 = .83249749$	
	$-x_2 = x_4 = .37454141$,	$x_3 = 0$,

where G(x) has been taken equal to one. The same values, of course, apply to the y's for corresponding values of n and again, H(y) has been taken equal to one.

To determine hk, proceed as follows;

$$n^2 hk = \int_{-1}^{1}\int_{-1}^{1} F(x,y)\,dx\,dy$$

but $F(x,y) = G(x)\,H(y) = 1$ so

1 Milne-Thomson, p. 180.

$$n^2 h k = \int_{-1}^{1}\int_{-1}^{1} dx\,dy$$

and

$$h k = \frac{4}{n^2}$$

An example will show how formula (32) is used. Suppose it is desired to evaluate

$$\int_1^3\int_1^3 \frac{x}{x^2+y^2}\,dx\,dy .$$

Here $E(x,y) = \frac{x}{x^2+y^2}$, $F(x,y)=1$. A transformation of variables must first be made that will change the limits of integration so that they will be from -1 to 1.

Let x = p + 2, y = q + 2. Then

$$\int_1^3\int_1^3 \frac{x}{x^2+y^2}\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \frac{p+2}{(p+2)^2+(q+2)^2}\,dp\,dq$$

where $E(p,q) = \frac{p+2}{(p+2)^2+(q+2)^2}$.

Taking n = 3, the result is, exclusive of the remainder,

$$\int_{-1}^{1}\int_{-1}^{1} \frac{p+2}{(p+2)^2+(q+2)^2}\,dp\,dq = \frac{4}{9}\left[.38673+.22796+.14365+.35264+.25000+.17655+.30079+.23897+.18470\right]$$
$$= 1.00533.$$

Taking n = 4, the result is, exclusive of the remainder,

$$\int_{-1}^{1}\int_{-1}^{1} \frac{p+2}{(p+2)^2+(q+2)^2}\,dp\,dq = \frac{1}{4}\left[.41483+.25441+.19321+.13012+.38256+.27588+.22457+.16335+.35067+.27106+.22856+.17368+.30170+.25189+.22187+.17891\right]$$
$$= 1.00432.$$

However, the integration may be performed directly in this case, giving

$$\int_1^3\int_1^3 \frac{x}{x^2+y^2}\,dx\,dy = \frac{3}{2}\log 18 - 2\log 10 + \frac{1}{2}\log 2 + 2\pi - 6\tan^{-1}\frac{1}{3} - 2\tan^{-1}3$$

$$= 1.00382.$$

This shows that by increasing n, more accurate results may be obtained.

4. CUBATURE FORMULAS INVOLVING DIFFERENCES

From Newton's interpolation formula with divided differences for a function of two variables[1], f(x,y)

$$f(x,y) = \sum_{v=0}^{n} \sum_{u=0}^{m} X_v \, y_u \, f_{vu} + R, \tag{36}$$

where the following notation has been used;

$$X_v = (x-a_0)\cdots(x-a_{v-1})$$

$$y_u = (y-b_0)\cdots(y-b_{u-1})$$

$$X_0 = y_0 = 1$$

$$f_{vu} = f(a_0 \cdots a_v;\, b_0 \cdots b_u)$$

$$R = \frac{X_{n+1}}{(n+1)!} D_\xi^{n+1} f(\xi, y) + \frac{y_{m+1}}{(m+1)!} D_\eta^{m+1} f(x,\eta) - \frac{X_{n+1}}{(n+1)!}\frac{y_{m+1}}{(m+1)!} D_\alpha^{n+1} D_\beta^{m+1} f(\alpha,\beta)$$

Assume f(x,y) to have continuous derivatives in x and y up to and including the (n + 1)th in x and the (m + 1)th in y in the region being considered, and

$$a_0 \leqq x, \xi, \alpha \leqq a_n\,, \quad b_0 \leqq y, \eta, \beta \leqq b_m$$

Let a and a + w be numbers such that in the interval $a < x < a + w$, the product $(x-a_0)\cdots(x-a_n) = X_{n+1}$ has no zeros. In this interval the product has a constant sign.

Let b and b + d be numbers such that in the interval $b < y < b + d$, the product $(y-b_0)\cdots(y-b_m) = y_{m+1}$ has no zeros. In this interval the product has a constant sign.

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1 Steffensen, p. 205.

Therefore in the rectangle whose vertices are

(37) (a,b), $(a+w,b)$, $(a,b+d)$, $(a+w,b+d)$

the product $(x-a_0)\cdots(x-a_n)(y-b_0)\cdots(y-b_m) = x_{n+1}y_{m+1}$ has no zeros. In this rectangle the product has a constant sign.

Hence, applying the Theorem of Mean Value

$$\int_b^{b+d}\int_a^{a+w} R\,dx\,dy = \int_b^{b+d}\int_a^{a+w} \frac{x_{n+1}}{(n+1)!} D_\xi^{n+1} f(\xi,y)\,dx\,dy + \int_b^{b+d}\int_a^{a+w} \frac{y_{m+1}}{(m+1)!} D_\eta^{m+1} f(x,\eta)\,dx\,dy$$

$$- \int_b^{b+d}\int_a^{a+w} \frac{x_{n+1}}{(n+1)!}\frac{y_{m+1}}{(m+1)!} D_\alpha^{n+1} D_\beta^{m+1} f(\alpha,\beta)\,dx\,dy$$

$$= d\,\frac{D_{\xi_1}^{n+1} f(\xi_1,\eta_1)}{(n+1)!}\int_a^{a+w} x_{n+1}\,dx + w\,\frac{D_{\eta_2}^{m+1} f(\xi_2,\eta_2)}{(m+1)!}\int_b^{b+d} y_{m+1}\,dy$$

$$- \frac{D_{\alpha_1}^{n+1} D_{\beta_1}^{m+1} f(\alpha_1,\beta_1)}{(n+1)!\,(m+1)!}\int_a^{a+w} x_{n+1}\,dx \int_b^{b+d} y_{m+1}\,dy$$

where min. $\{a_v\}, a, a+w \leqq \xi_1, \xi_2, \alpha_1 \leqq$ max. $\{a_v\}, a, a+w$

and min. $\{b_u\}, b, b+d \leqq \eta_1, \eta_2, \beta_1 \leqq$ max. $\{b_u\}, b, b+d$

Integrating both sides of (36) with respect to x and y over the rectangle of (37)

$$\int_b^{b+d}\int_a^{a+w} f(x,y)\,dx\,dy = \int_b^{b+d}\int_a^{a+w} \sum_{v=0}^{n}\sum_{u=0}^{m} x_v y_u f_{vu}\,dx\,dy + \int_b^{b+d}\int_a^{a+w} R\,dx\,dy$$

$$= wd\sum_{v=0}^{n}\sum_{u=0}^{m} A_{v,u} f_{vu} + dw A_{n+1,0}\,\frac{D_{\xi_1}^{n+1} f(\xi_1,\eta_1)}{(n+1)!}$$

$$+ wd A_{0,m+1}\,\frac{D_{\eta_2}^{m+1} f(\xi_2,\eta_2)}{(m+1)!} - wd A_{n+1,0} A_{0,m+1}\,\frac{D_{\alpha_1}^{n+1} D_{\beta_1}^{m+1} f(\alpha_1,\beta_1)}{(n+1)!\,(m+1)!}$$

Dividing by wd

$$(38)\quad \frac{1}{wd}\int_{\ell}^{\ell+d}\int_{a}^{a+w} f(x,y)\,dx\,dy = \sum_{\nu=0}^{n}\sum_{\mu=0}^{m} A_{\nu,\mu} f_{\nu\mu} + A_{n+1,0}\frac{D_{\xi_1}^{n+1} f(\xi_1,\eta_1)}{(n+1)!}$$

$$+ A_{0,m+1}\frac{D_{\eta_2}^{m+1} f(\xi_2,\eta_2)}{(m+1)!} - A_{n+1,0}A_{0,m+1}\frac{D_{\alpha_1}^{n+1} D_{\beta_1}^{m+1} f(\alpha_1,\beta_1)}{(n+1)!\,(m+1)!}$$

where

$$(39)\qquad A_{\nu,\mu} = \frac{1}{wd}\int_{\ell}^{\ell+d}\int_{a}^{a+w} X_\nu Y_\mu \,dx\,dy.$$

From this result a variety of cubature formulas might be deduced by assigning suitable values to $\{a_\nu\}$ and $\{b_\mu\}$.

It is noticed that in (38) the values of $\{a_\nu\}$ and $\{b_\mu\}$ all lie outside the rectangle of (37) whereas in the extension of Tschebyscheff's formula the points lie inside the rectangle over which the integral is taken.

The same problem that was used to illustrate the formula derived in 3. will not serve as an illustration here since the remainder terms in (38) are large if the smallest points are near one.

Consider, instead,

$$\int_3^5\int_3^5 \frac{x}{x^2+y^2}\,dx\,dy.$$

Let $a_0 = b_0 = 2.8$

$a_1 = b_1 = 2.9$

$a_2 = b_2 = 5.1$

$a_3 = b_3 = 5.2$

Then n = m = 3, a = b = 3, w = d = 2.

To find the values of f_{vu} first set up a function-table giving the values of the function at the points (a_n, b_m).

x	y = 2.8	2.9	5.1	5.2
2.8	.17857	.17231	.08272	.08028
2.9	.17846	.17241	.08425	.08181
5.1	.15066	.14817	.09804	.09614
5.2	.14908	.14669	.09802	.09615

Table I

Next set up a difference-table[1] in x in which the numbers in the first, second, etc., column are obtained by forming the divided differences of the values in the first, second, etc., column of the function-table.

$f(a_0, b_0)$ .17857	$f(a_0, b_1)$ .17231	$f(a_0, b_2)$ .08272	$f(a_0, b_3)$ .08028
$f(a_0,a_1; b_0)$ -.00110	$f(a_0,a_1; b_1)$ .00100	$f(a_0,a_1; b_2)$ .01530	$f(a_0,a_1; b_3)$ .01530
$f(a_0,a_1,a_2; b_0)$ -.00502	$f(a_0,a_1,a_2; b_1)$ -.00523	$f(a_0,a_1,a_2; b_2)$ -.00393	$f(a_0,a_1,a_2; b_3)$ -.00382
$f(a_0,a_1,a_2,a_3; b_0)$ .00152	$f(a_0,a_1,a_2,a_3; b_1)$ .00150	$f(a_0,a_1,a_2,a_3; b_2)$ .00047	$f(a_0,a_1,a_2,a_3; b_3)$ .00043

Table II

1 Steffensen, p. 217.

Finally set up a difference-table[1] in x and y by forming the divided differences of the values in the first, second, etc., row of the difference-table in x.

$f(a_0,b_0)$ .17857	$f(a_0;b_0,b_1)$ -.06260	$f(a_0;b_0,b_1,b_2)$ .00951	$f(a_0;b_0,b_1,b_2,b_3)$ -.00100
$f(a_0,a_1;b_0)$ -.00110	$f(a_0,a_1;b_0,b_1)$ .02100	$f(a_0,a_1;b_0,b_1,b_2)$ -.00630	$f(a_0,a_1;b_0,b_1,b_2,b_3)$ .00145
$f(a_0,a_1,a_2;b_0)$ -.00502	$f(a_0,a_1,a_2;b_0,b_1)$ -.00210	$f(a_0,a_1,a_2;b_0,b_1,b_2)$ .00117	$f(a_0,a_1,a_2;b_0,b_1,b_2,b_3)$ -.00040
$f(a_0,a_1,a_2,a_3;b_0)$ .00152	$f(a_0,a_1,a_2,a_3;b_0,b_1)$ -.00020	$f(a_0,a_1,a_2,a_3;b_0,b_1,b_2)$ -.00012	$f(a_0,a_1,a_2,a_3;b_0,b_1,b_2,b_3)$ .00006

Table III

Table III gives the values of f_{vu}. Evaluate $A_{v,u}$ from (39) and then from (38) apart from the remainder,

$$\int_3^5\int_3^5 \frac{x}{x^2+y^2}\,dx\,dy = 4[.17857 - .07512 + .01572 + .00105 - .00132 + .03024 - .01250 - .00183 - .00830 - .00417 + .00320 + .00070 - .00160 + .00025 + .00021 + .00007$$

$$\simeq .50068.$$

The following inequality which has been shown[2] to hold for the function in question may be used to find an upper bound on the remainder.

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1 Steffensen, p. 217.
2 Ibid, p. 229.

$$\left| D_x^{2\nu} D_y^{2\mu} \right| \leqq \frac{(2\nu+2\mu)!}{\rho^{2\nu+2\mu+1}}$$

where $\rho = \sqrt{x^2+y^2}$. For the points being used, take $\rho = 2.8\sqrt{2}$. Then the absolute value of the remainder term is found to be

$$\leqq 4\left[2\,\frac{2.1181}{2}\,\frac{\frac{4!}{(2.8\sqrt{2})^5}}{4!} + \left(\frac{2.1181}{2}\right)^2 \frac{\frac{8!}{(2.8\sqrt{2})^9}}{4!\;4!} \right]$$

$$= .00968.$$

By actual integration

$$\int_3^5\!\!\int_3^5 \frac{x}{x^2+y^2}\,dx\,dy = .50002,$$

so the result obtained is satisfactory.

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