# BODY DYNAMICS OF AN AUTOMOBILE BY ELECTRIC ANALOG

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Charles Hollister Single

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This is to certify that the

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J A Strelzoff
Major professor

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BODY DYNAMICS OF AN AUTOMOBILE

BY

ELECTRIC ANALOG

By

CHARLES HOLLISTER SINGLE

A THESIS

Submitted to the School of Graduate Studies of Michigan State College of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

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THESIS

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# Preface

This thesis establishes the mathematical equations for a complex mechanical vibration problem: ~~t~~he motion of an automobile due to road irregularity. The solution is found by means of an analogous electric circuit. The results from the electric analog are then compared with photographs of the automobile in motion.

The thesis clearly shows that the use of electric analogy is a powerful method in the transient solution of difficult mechanical problems.

Particular thanks is given to the Ford Motor Co. for the cooperation given. Especially, to Mr. W. E. Burnett of the Ford Research Laboratories; and to Mr. R. W. Gaines, head of the Engine Test Department, for his helpful criticism of the validity of the mechanical system used to represent the automobile, and for his patience in collecting the data for the model tested.

The author also wishes to express gratitude to Dr. J. A. Strelzoff of the Electrical Engineering Department of Michigan State College for his guidance in the completion of the thesis, also for his reading of the manuscript.

C. H. Single

Notation - Symbols

The following notations are used in this thesis:

| Quanity | Units | Symbol |
|---|---|---|
| Force | pound | F, |
| Displacement | inch | x,y,z,L |
| Spring constant | pound/inch | K, |
| Mass | $\frac{\text{pound(second)}^2}{\text{inch}}$ | M, |
| Viscous damping (shock absorber) | $\frac{\text{Pound seconds}}{\text{inch}}$ | D, |
| Center of gravity | | C.G. |
| Radius of gyration | inch | h |
| Time | second | T |
| Voltage | volt | V |
| Charge | coulomb | Q |
| Capicatance | farad | C |
| Elastance | daraf | $\frac{1}{C} = S$ |
| Inductance | henry | $L = \frac{1}{\Gamma}$ |
| Resistance | ohm | R |
| Conductance | mho | G |
| Current | ampere | I   i |
| Integral with respect to time | | $\frac{1}{p}$ |
| First derivative with respect to time | | p |
| Second derivative with respect to time | | $p^2$ |

## INTRODUCTION

The problem considered in this thesis is an extension of the type problem covered in the graduate course in transients by the Electrical Engineering Department at Michigan State College[1].

The specific problem treated is the dynamics of an automobile body resulting from road irregularity. A general viewpoint of this motion is used; that is, translation, roll, and pitch of the car body are included. For the wheel assemblies the action of these components is considered: wheel, spring, shock absorber, and tire.

A specific road condition is used, and the resultant motion of the automobile found. In so far as possible this motion is solved for from three viewpoints: an analytic solution, a solution through electric analog, and photograph of the actual car model undergoing the same road irregularity. These solutions are then compared, and an evaluation of the electric analog method for the solution of transient problems is made.

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1 The course covered the first eight chapters of <u>Transients In Linear Systems</u> - Gardner & Barnes.

## Derivation Of Equations

The assumptions necessary to establish mathematical equations of motion for the car are to consider the components ideal. That is, the masses rigid, the springs have negligible mass and damping, and the shock absorbers have no energy storage. These assumptions are quite accurate when the actual data is considered. With these approximations, the mechanical system representing the car is:

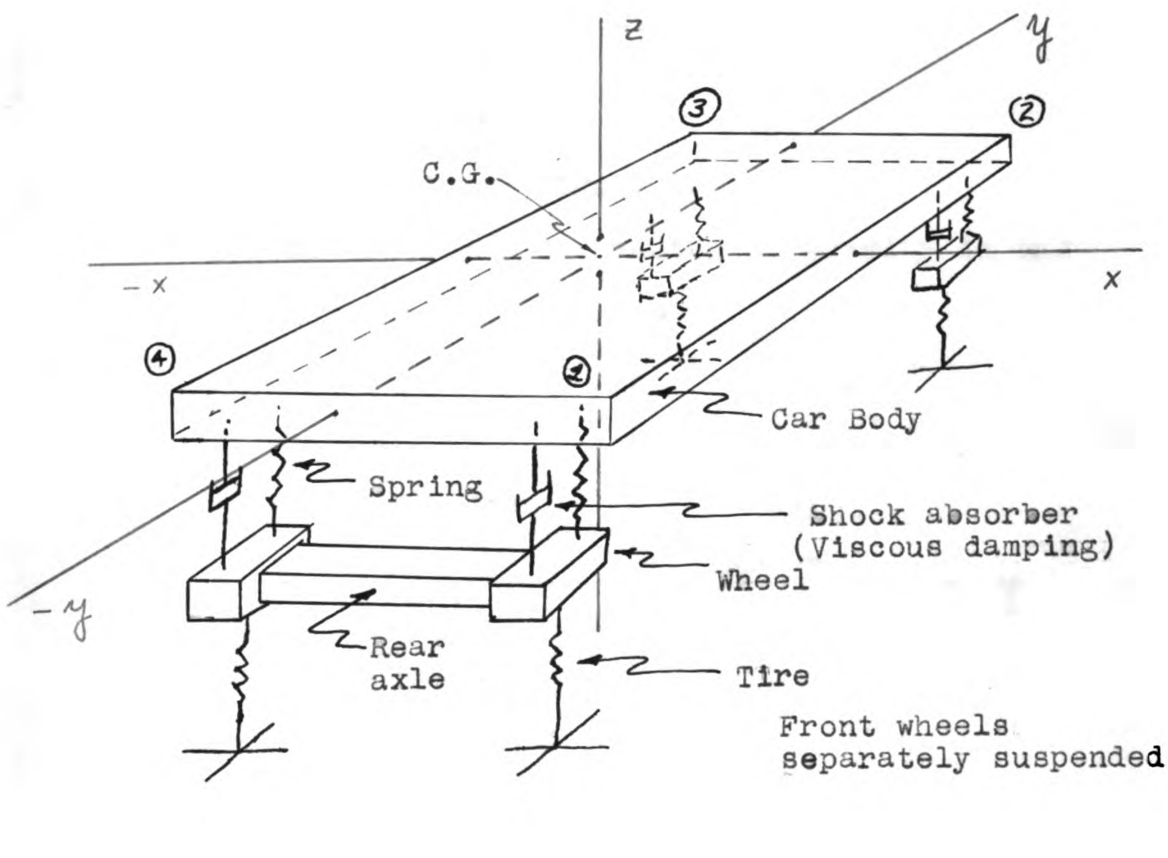



Figure 1

First, the motion of the body alone will be studied. The effects of the wheel assemblies being replaced by equivilent forces.

The general body motion can be broken into three simple motions as follows:

A uniform translation of the mass through the center of gravity;

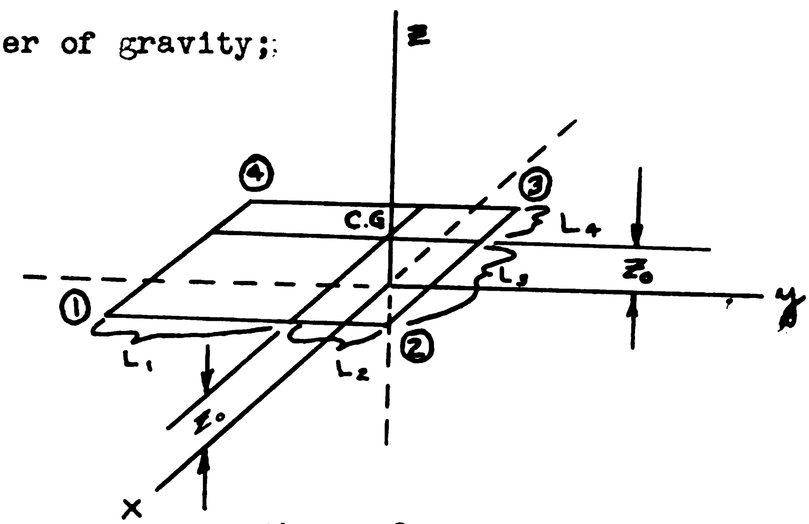



Figure 2

A rotation about the center of gravity in the x - z plane;

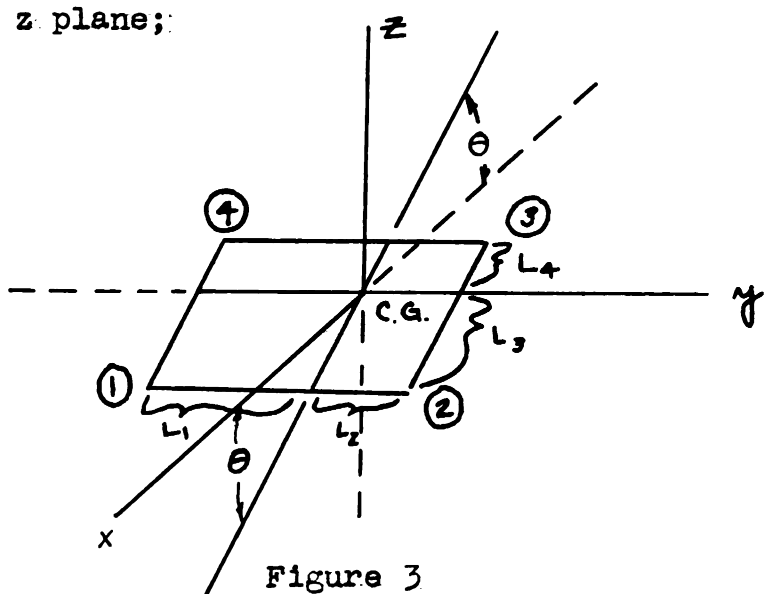



Figure 3

And, finally, a rotation in the y - z plane;

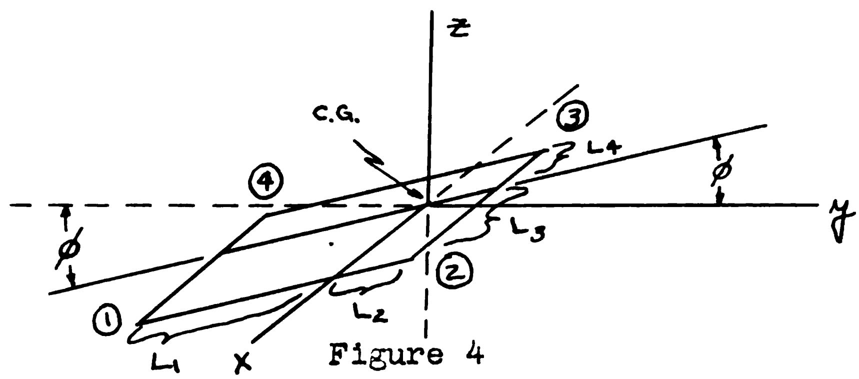



Figure 4

This gives a general movement of the car body.

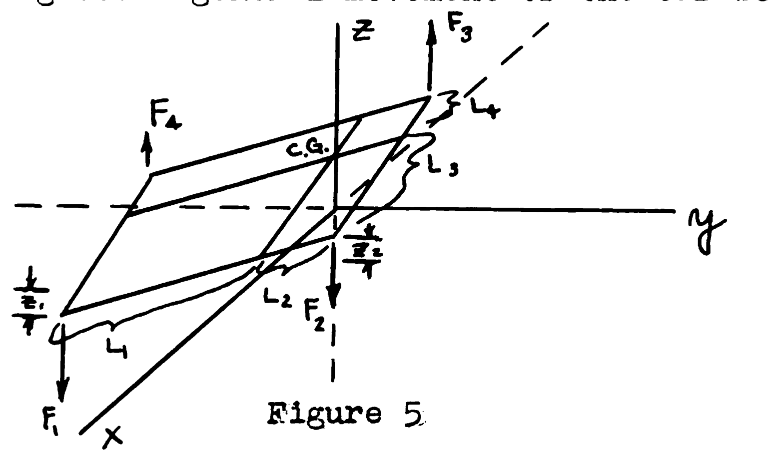



Figure 5

With the forces and displacements upward considered positive, the equations for the above general motion are:

(translation)

$$Mp^2 z_0 = -F_1 - F_2 + F_3 + F_4 \qquad (1)$$

(rotation in the x - z plane)

$$M(h_{xz})^2 \, p^2\theta = F_1L_3 + F_2L_3 + F_3L_4 + F_4L_4 \qquad (2)$$

(rotation in the y - z plane)

$$M(h_{yz})^2 \, p^2\phi = F_1L_1 - F_2L_2 + F_3L_2 - F_4L_1 \qquad (3)$$

Where $h_{xz}$ and $h_{yz}$ are the radii of gyration in their respective planes, M the mass of the automobile body; displacements, dimensions, and angles as shown in figure 5.

A more explicit description of the body motion can be formed by changing from the translation and rotations to a displacement of three specific points on the car body (in the plane of the center of gravity). These three points will be directly above the first, second, and third wheel, and noted $z_1$, $z_2$, and $z_3$. Knowing these three displacements, any other point (i.e. $z_4$) can be found, as will be shown later.

From figure 5:

$$z_1 = z_0 - L_1 \sin \phi - L_3 \sin \theta \qquad (4)$$

$$z_2 = z_0 + L_2 \sin \phi - L_3 \sin \theta \qquad (5)$$

$$z_3 = z_0 + L_2 \sin \phi + L_4 \sin \theta \qquad (6)$$

Assuming $\sin \theta \doteq \theta$ (small angles of rotation), $\sin \phi \doteq \phi$

$\phi$, and $\theta$ and $z_0$ can be expressed as functions of $z_1$, $z_2$, and $z_3$.

Thus:

$$z_0 = \frac{L_2(L_3 + L_4)z_1 + (L_1L_4 - L_2L_3)z_2 + L_3(L_1 + L_2)z_3}{(L_1 + L_2)(L_3 + L_4)} \qquad (7)$$

$$\phi = \frac{-z_1 + z_2}{(L_1 + L_2)} \qquad (8) \qquad\qquad \theta = \frac{-z_2 + z_3}{(L_3 + L_4)} \qquad (9)$$

Substituting (7), (8), and (9), in (1), (2), and (3), we find:

$$-F_1 - F_2 + F_3 = -F_4 + \frac{ML_2}{(L_1 + L_2)} p^2 z_1 + \frac{M(L_1 L_4 - L_2 L_3)}{(L_1 + L_2)(L_3 + L_4)} p^2 z_2 + \frac{ML_3}{(L_3 + L_4)} p^2 z_3 \qquad (10)$$

$$L_3F_1 + L_3F_2 + L_4F_3 = -L_4F_4 \frac{-M(h_{xz})^2}{L_3 + L_4} p^2 z_2 + \frac{M(h_{xz})^2}{L_3 + L_4} p^2 z_3 \qquad (11)$$

$$L_1F_1 - L_2F_2 + L_2F_3 = L_1F_4 - \frac{M(h_{xz})^2}{L_1 + L_2} p^2 z_1 + \frac{M(h_{yz})^2}{L_1 + L_2} p^2 z_2 \qquad (12)$$

Solving the above equationsfor $F_1$, $F_2$, and $F_3$ we find:

$$F_1 = F_4 - \frac{M\left[L_2^2 + (h_{yz})^2\right]}{(L_1 + L_2)^2} p^2 z_1 + \frac{M\left[(L_3 + L_4)(h_{yz})^2 - L(L_1L_4 - L_2L_3)\right]}{(L_1 + L_2)^2 (L_3 + L_4)} p^2 z_2 \qquad (13)$$

$$- \frac{ML_2L_3}{(L_1 + L_2)(L_3 + L_4)} p^2 z_3$$

$$F_2 = -F_4 + \frac{M\left[(L_3 + L_4)(h_{yz})^2 - L_2(L_1L_4 - L_2L_3)\right]}{(L_1 + L_2)^2 (L_3 + L_4)} p^2 z_1 \qquad (14)$$

$$- \frac{M\left[(L_1L_4 - L_2L_3)^2 + (L_1 + L_2)^2 (h_{xz})^2 + (L_3 + L_4)^2 (h_{yz})^2\right]}{(L_1 + L_2)^2 (L_3 + L_4)^2} p^2 z_2$$

$$+ \frac{M\left[(L_1 + L_2)(h_{xz})^2 - L_3(L_1L_4 - L_2L_3)\right]}{(L_1 + L_2)(L_3 + L_4)^2} p^2 z_3$$

$$F_3 = -F_4 + \frac{ML_2L_3}{(L_1 + L_2)(L_3 + L_4)} p^2z_1$$

$$+ \frac{M\left[(L_3)(L_1L_4 - L_2L_3) - (h_{xz})^2(L_1 + L_2)\right]}{(L_1 + L_2)(L_3 + L_4)^2}$$

$$+ \frac{M\left[L_3^{\ 2} + (h_{xz})^2\right]}{(L_3 + L_4)^2} \ p^2z_3$$

To simplify the notation, define:

$$M_{11} = \frac{+M\left[L_2^{\ 2} + (h_{yz})^2\right]}{(L_1 + L_2)^2};$$

$$M_{12} = \frac{+M\left[(L_3 + L_4)(h_{yz})^2 - L_2(L_1L_4 - L_2L_3)\right]}{(L_1 + L_2)^2(L_3 + L_4)};$$

$$M_{13} = \frac{ML_2L_3}{(L_1 + L_2)(L_3 + L_4)};$$

$$M_{22} = \frac{+M\left[(L_1L_4 - L_2L_3)^2 + (L_1 + L_2)^2(h_{xz})^2 + (L_3 + L_4)^2(h_{yz})^2\right]}{(L_1 + L_2)^2(L_3 + L_4)^2};$$

$$M_{23} = \frac{+M\left[(L_1 + L_2)(h_{xz})^2 - L_3(L_1L_4 - L_2L_3)\right]}{(L_1 + L_2)(L_3 + L_4)^2};$$

$$M_{33} = \frac{M\left[L_3^{\ 2} + (h_{xz})^2\right]}{(L_3 + L_4)^2}.$$

This reduces (13), (14), and (15) to:

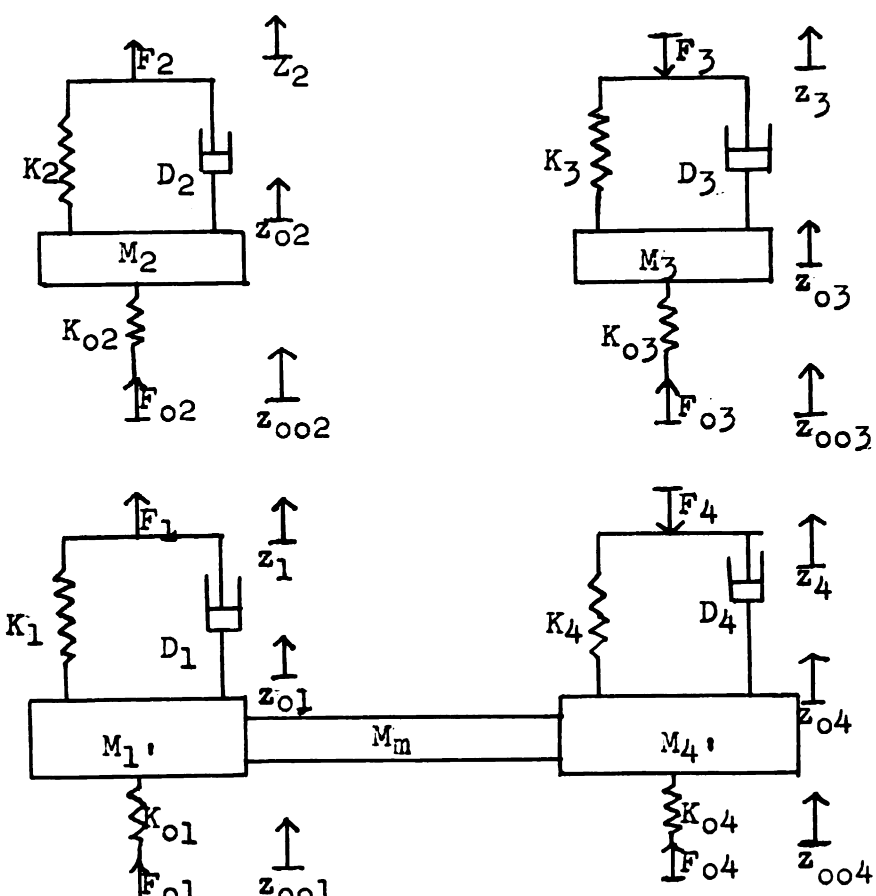
$$-F_1 + F_4 = + M_{11}p^2z_1 - M_{12}p^2z_2 + M_{13}p^2z_3 \qquad (16)$$

$$-F_2 - F_4 = - M_{12}p^2z_1 + M_{22}p^2z_2 - M_{23}p^2z_3 \qquad (17)$$

$$F_3 + F_4 = +M_{13}p^2z_1 - M_{23}p^2z_2 + M_{33}p^2z_3 \qquad (18)$$

Thus far only the automobile body has been taken into account, the effects of the wheel assemblies assumed as equivilent forces. The next step is to solve for these forces in terms of the action of the various elements of the wheel assembly. The forces, from figure 5, on the first and second corners were negative on the car body; the third and fourth positive. Therefore, the reactive force on the wheel assemblies must be of opposite signs: one and two positive; three and four negative. This can best be seen in the static case, with the car body in equilibrium.

On this basis the wheel assemblies and forces are:



The second and third are quite simple, since they are independent suspensions. The equations for the second assembly are:

$$F'_2 = (D_2p + K_2)z_2 - (D_2p + K_2)z_{02} + (0)z_{002} \qquad (19)$$

$$0 = -(D_2p + K_2)z_2 + (M_2p^2 + D_2p + K_2 + K_{02})z_{02} - (K_{02})z_{002} \qquad (20)$$

$$F'_{02} = +(0)z_2 - (K_{02})z_{02} + (K_{02})z_{002} \qquad (21)$$

In general, the force caused by the road irregularity is unknown, but the road displacement, $z_{002}$, is known. It is therefore logical to eleminate $F'_2$, and $z_{002}$ by substituting equation (20) into (19). We then have:

$$F'_2 = (D_2p + K_2)z_2 - (D_2p + K_2)z_{02} \qquad (22)$$

$$K_{02}(z_{002}) = -(D_2p + K_2)z_2 + (M_2p^2 + D_2p + K_2 + K_{02})z_{02} \qquad (23)$$

Similarly:

$$-F'_3 = (D_3p + K_3)z_3 - (D_3p + K_3)z_{03} \qquad (24)$$

$$K_{03}(z_{003}) = -(D_3p + K_3)z_3 + (M_3p^2 + D_3p + K_3 + K_{03})z_{03} \qquad (25)$$

The first and fourth wheel assemblies must be worked out in a manner similar to the car body, due to the effect of mutual mass, the axle. This follows a pattern readily found in the literature[2]. Due to symmetry in axle construction, the center of gravity for the total rear wheel mass is at the geometric center.

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2. Gardner & Barnes, Transients in Linear Systems, Page 76.

Defining the distance from the center of gravity to the wheel $L_5 = \frac{L_2 + L_3}{2}$, $M_t = M_1 + M_4 + M_m$ and the radius of gyration, h, we have the simplified system:

$$+F_1 = (D_1 p + K_1) z_1 - (D_1 p + K_1) z_{o1} + (0) z_4 + (0) z_{o4} \qquad (26)$$

$$K_{o1}(z_{oo1}) = -(D_1 p + K_1) z_1 + (D_1 p + K_1 + K_{o1} + p^2 M_1{}') z_{o1} + (0) z_4 - p^2 M_m z_{o4} \qquad (27)$$

$$-F_4 = (0) z_1 + (0) z_{o1} + (p D_4 + K_4) z_4 - (p D_4 + K_4) z_{o4} \qquad (28)$$

$$K_{o4}(z_{oo4}) = (0) z_1 - p^2 M_m z_{o1} - (p D_4 + K_4) z_4 + (D_4 p + K_4 + K_{o4} + p^2 M_4{}') z_{o4} \qquad (29)$$

Where:

$$M_1{}' = M_T \left[ \frac{L_5^2 + h^2}{4 L_5^2} \right] \qquad M_4{}' = M_T \left[ \frac{L_5^2 + h^2}{4 L_5^2} \right] \qquad M_m = M_T \left[ \frac{L_5^2 - h^2}{4 L_5^2} \right]$$

We are now ready to form the overall equations of the mechanical system. Replace $F_1$, $F_2$, $F_3$, and $F_4$ by their equivilents (equations (26), (22),(24), and (28), respectively) in the equations for the car body (16), (17), and (18). The resultant overall equations are given on page 12.

These equations of the mechanical system include four inputs: the road irregularity to each wheel, and seven unknown displacements. Four of these displacements correspond to the four wheel motions, and the remaining three are the points on the car body above the first, second, and third wheels.

## Complete Equations For Mechanical System

$$K_{o1}z_{oo1} = (M_1'p^2+D_1p+K_1+K_{o1})z_{o1} + (0)z_{o2} + (0)z_{o3} - (M_mp^2)z_{o4} - (D_1p+K_1)z_1 + (0)z_2 + (0)z_3 \tag{30}$$

$$K_{o2}z_{oo2} = +(0)z_{o1} + (M_2p^2+D_2p+K_2+K_{o2})z_{o2} + (0)z_{o3} + (0)z_{o4} + (0)z_1 - (D_2p+K_2)z_2 + (0)z_3 \tag{31}$$

$$K_{o3}z_{oo3} = +(0)z_{o1} + (0)z_{o2} + (M_3p^2+D_3p+K_3+K_{o3})z_{o3} + (0)z_{o4} + (0)z_1 + (0)z_2 - (D_3p+K_3)z_3 \tag{32}$$

$$K_{o4}z_{oo4} = -(M_mp^2)z_{o1} + (0)z_{o2} + (0)z_{o3} + (M_4'p^2+D_4p+K_4+K_{o4})z_{o4} - (D_4p+K_4)z_1 + (D_4p+K_4)z_2 - (D_4p+K_4)z_3 \tag{33}$$

$$0 = -(D_1p+K_1)z_{o1} + (0)z_{o2} + (0)z_{o3} - (D_4p+K_4)z_{o4} + \left[M_{11}p^2+(D_1+D_4)p+K_1+K_4\right]z_1 - (M_{12}p^2+D_4p+K_4)z_2 + (M_{13}+D_4p+K_4)z_3 \tag{34}$$

$$0 = +(0)z_{o1} - (D_2p+K_2)z_{o2} + (0)z_{o3} + (D_4p+K_4)z_{o4} - (M_{12}p^2+D_4p+K_4)z_1 + \left[M_{22}p^2+(D_2+D_4)p+K_2+K_4\right]z_2 - (M_{23}+D_4p+K_4)z_3 \tag{35}$$

$$0 = +(0)z_{o1} + (0)z_{o2} - (D_3p+K_3)z_{o3} - (D_4p+K_4)z_{o4} + (M_{13}+D_4p+K_4)z_1 - (M_{23}+D_4p+K_4)z_2 + \left[M_{33}p^2+(D_3+D_4)p+K_3+K_4\right]z_3 \tag{36}$$

Motion Of Other Points On The Automobile Body

These three points on the automobile body determine the plane of the body; it is a simple matter of geometry to find any other point. For example, find the displacement above the fourth wheel.

With the assumption that the car body is rigid, all points in the plane of the center of gravity must satisfy the same equation. If the position of the first corner is $x_1$, $y_1$, $z_1$; the second $x_2$, $y_2$, $z_2$, ect. equations can be formed:

$$Ax_1 + By_1 + Cz_1 + D = 0 \qquad (37)$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \qquad (38)$$

$$Ax_3 + By_3 + Cz_3 + D = 0 \qquad (39)$$

$$Ax_4 + By_4 + Cz_4 + D = 0 \qquad (40)$$

Since the positions of the first three corners are known, one can solve (37), (38), and (39), for A, B, and C, in terms of coordinate of the three corners and D. The results of this are then substituted into (40). $x_4$ and $y_4$ are equal to $-L_4$ and $-L_1$ if small angles of rotation are considered. This is equivilent to the previous assumption: $\sin\theta \doteq \theta$, $\sin\phi \doteq \phi$ modified to $\cos\theta \doteq 1$, $\cos\phi \doteq 1$.

After some manipulation it can be shown that:[3]

$$z_4 = z_1 - z_2 + z_3$$

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3 See appendix I

By graphical addition we find the displacement of the fourth corner. Any other point in the plane would add only portions of each known displacement, and could be worked out similarly.

The analytic solution of the seventh order simultaneous equation needed to describe the dynamics of the automobile is at best a tedious process. However, the model car checked used non-linear shock absorbers; that is, two different viscous damping constants were presented, depending whether the shock absorber was under compression or rebound. This one fact eliminated the completion of the analytic solution, as the problem became too involved to hope for a solution using the techniques with which the author is familiar.

## Development Of Electric Analog

A solution by electric analog, however, offers a powerful method. This approach makes use of the similarity of equations between mechanical systems and those used in electrical circuits.

Much material is available on the fundamentals of this technique[4]. There are four general methods that could be used. Since the mechanical system used was based on known forces and resultant displacements, its dual using applied velocities and resulting forces will not be considered, as displacements are the desired unknowns.

A simple mechanical circuit best illustrates the process. For the circuit of figure six the equations are:

$$F(t) = (Mp^2 + Dp + K)x(t)$$

or

$$F(t) = (Mp + D + \frac{K}{p})\, v\,(t)$$

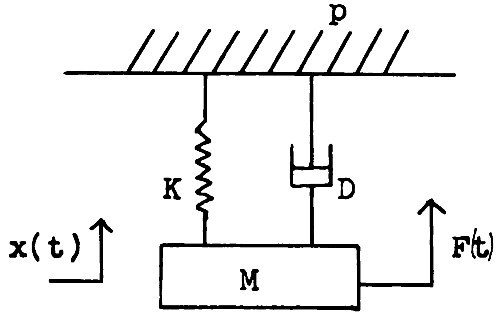


Figure 6

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4 Johnson, W. C., Mathematical and Physical Principles of Engineering Analysis, 1944
Westinghouse Engineer: March 1946, page 48
Gardner & Barnes Transients in Linear Systems

The two electrical analogues are as follows:

| F -- V | F -- I |
|---|---|

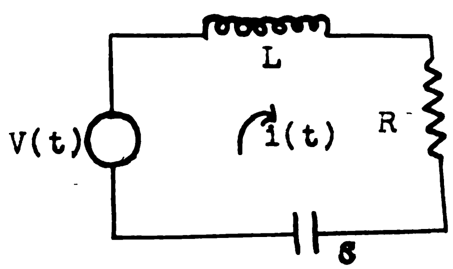



Figure 7

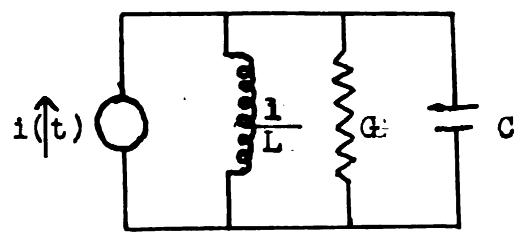



Figure 8

Where:

$$V(t) = (Lp^2 + Rp + S)q(t)$$

$$V(t) = (Lp + R + \frac{S}{p})i(t)$$

Where:

$$q(t) = (Cp^2 + Gp + \frac{1}{L})e(t)$$

$$i(t) = (Cp + G + \frac{1}{Lp})e(t)$$

| F -- V | F -- I |
|---|---|
| x -- q | X -- e(t) |
| V -- i | v -- e |
| M -- L | M -- C |
| D -- R | D -- $G = \frac{1}{R}$ |
| K -- $S = \frac{1}{C}$ | K -- $\frac{1}{L}$ |

The choice of the analogy was not arbitrary when data from the actual automobile was used. This will now be shown: equations (16), (17), and (18) for the car body envolve self masses and mutual masses between the three points of displacement. One possibility is to use the F -- V analogy.

This circuit has the correct equations:

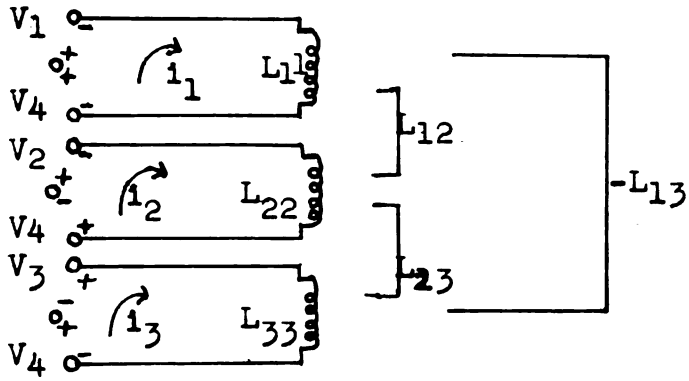



Figure 9

$$-V + V_4 = L_{11}p^2q_1 - L_{12}p^2q_2 + L_{13}p^2q_3 \tag{41}$$

$$-V_2 - V_4 = L_{12}p^2q_1 + L_{22}p^2q_2 - L_{23}p^2q_3 \tag{42}$$

$$+V_3 + V_4 = +L_{13}p^2q_1 - L_{23}p^2q_2 + L_{33}p^2q_3 \tag{43}$$

A further possibility that almost gives the same equations is to modify the circuit slightly:

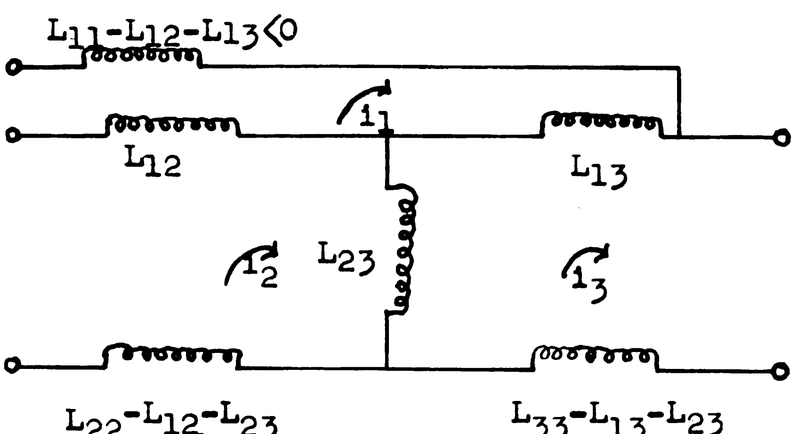



Figure 10

One trouble with this setup is that the sign of one of the terms is always incorrect. However, it will serve to illustrate another difficulty encountered with the actual car data. The mutual masses of the car body were( in one case, $L_{11}$) greater than the self mass. This required the use of a negative inductance ($L_{11} - L_{12} - L_{13} < 0$) which is impossible without using vacuum tube circuits. If a tetrode were operated in its negative resistance region, and positive inductance used in series with it, the ideal negative inductance could be approached. However, this would certainly be much more difficult than winding coils with the correct mutual inductance (figure 9).

These same problems prevent the use of the circuit with F -- I since a mutual term again has the wrong sign, and in this case a negative capacitance is needed:

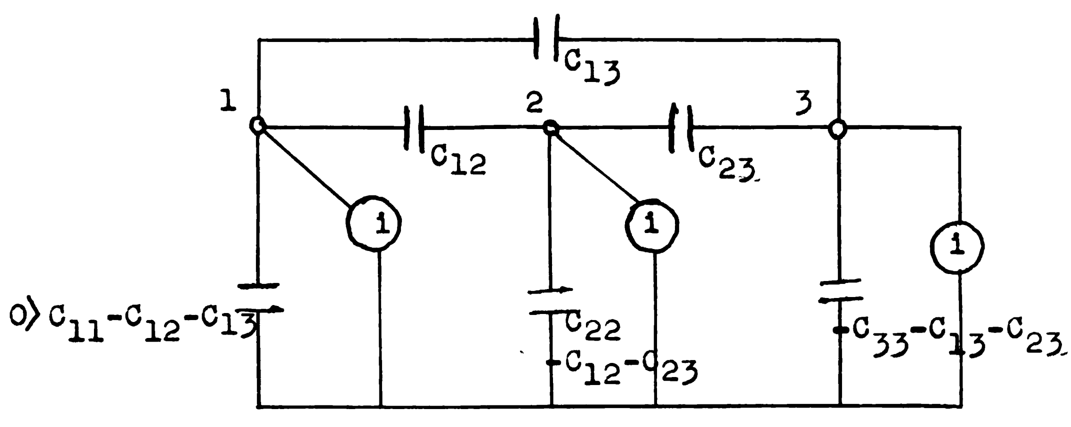



Figure 11

The sign of the mutuals are physically possible using the circuit as shown in figure 9. It can easily be seen from figure 12 that it is possible to wind the coils such that $L_{12}$ and $L_{23}$ are positive while $L_{13}$ is negative.

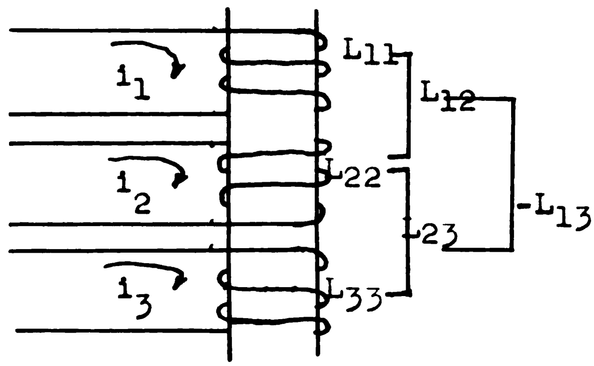



Figure 12

Applying the same process to the mechanical equations for the overall mechanical system, (equation (30) through (36) we get the analogous electric circuit:

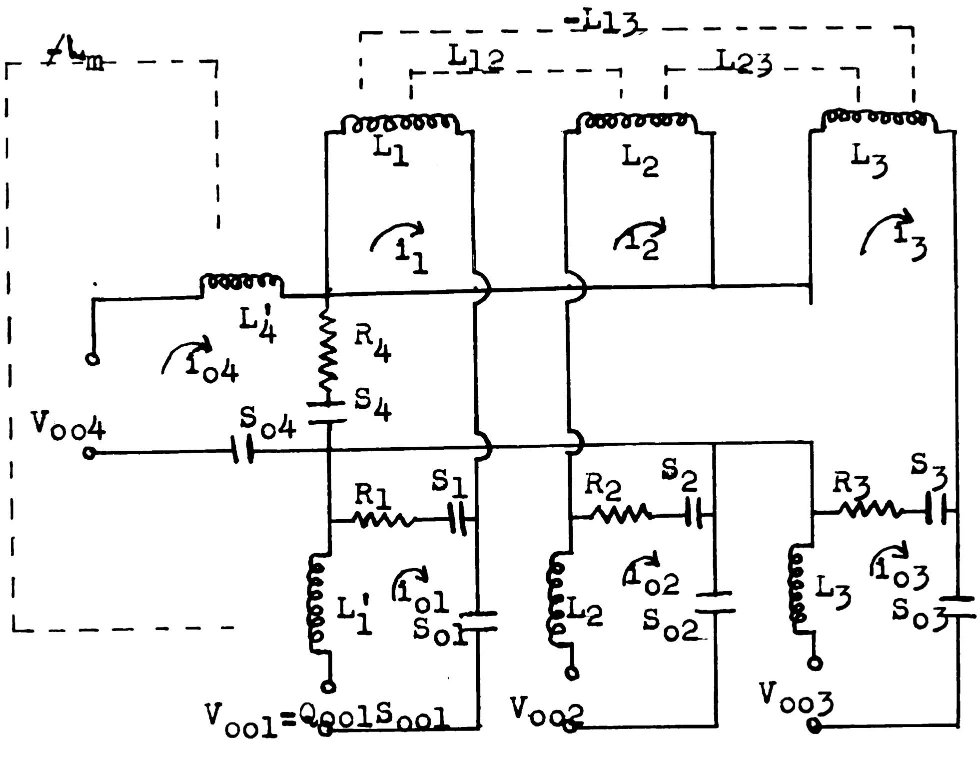



Figure 13

## Conponents For Electric Analog

Once the analogous electric circuit has been found the problem becomes: what value of the electric parameters are needed, and what charge in the electric circuit corresponds to an inch displacement. Certainly, if it were convenient, with a consistent set of units, we could let $|L| = |M|$, $|S| = |K|$, $|D| = |R|$, and one unit of change in the electric circuit would represent one unit of displacement in the mechanical system.

However, in this problem, as with many mechanical systems, an attempt to use this direct transfer of numerical data leads to an impossible electric circuit. By use of the Buckingham Pi Theorem, a practical analogous electric circuit is generally possible.*

This theorem states that if a problem is dependent upon n quanities and if these n quanities can be expressed in terms of m dimensions there will be formed n - m independent dimensionless groups from combinations of the n quanities. If the equality of the dimensionless groups is maimtained between the electrical analog and the mechanical system, much greater flexibility in the selection of electrical components is possible, and the correct transients for the mechanical problem can be found; that is, the same degree of damping will be found in both circuits.

.......................................................

* Ibid 4, page 219

In the mechanical system the n quanities are force, displacement, time, mass, viscous damping, and spring constant. These quanities can be expressed in three fundamental terms; force, displacement, and time. So there should be three dimensionless groups. One possible arrangement of these dimensionless groups or $\pi$'s is:*

$$\pi_1 = \frac{1}{T}\sqrt{\frac{M}{K}} \tag{44}$$

$$\pi_2 = \frac{D}{\sqrt{MK}} \tag{45}$$

$$\pi_3 = \frac{F}{KX} \tag{46}$$

Obviously, various other $\pi$'s can be formed; the only requirement being that they must contain all of the n quanities and be independent.** To see that the invariance of these dimensionless groups in the two systems will yield consistent results, we need only to check a simple mechanical system and its analog. This can be done with a mechanical problem previously used (figure 6). Assume the initial conditions to be zero and apply a step function force to the mass. By Laplace Transform the resultant motion will be:***

..................................................

* Appendix II

** Ibid 4, page 228.

*** Ibid 2, page 342.

$$x(t) = \frac{F}{K} + \frac{2F}{\sqrt{\frac{4MK^2 - KD^2}{M}}} \in^{-\left(\frac{D}{2M}\right)t} \qquad (47)$$

$$\text{Sin}\left[\sqrt{\frac{K}{M} - \frac{D^2}{M^2}}\ t - \phi\right]$$

Where $\phi = \arctan \sqrt{\frac{4MK - D^2}{-D}}$

Instead of using the direct analogy maintain only the $\pi$'s. Thus:

aK -- S

bM -- L

cT -- T

dD -- R

eF -- V

fX -- Q

From the $\pi$'s:

$$c = \sqrt{\frac{b}{a}} \qquad (48)$$

$$d = \sqrt{ab} \qquad (49)$$

$$\frac{e}{f} = a \qquad (50)$$

Then from the electric analog the resultant is:

$$Q(t) = fX(t) = \frac{eF}{aK} + \frac{2eF}{\sqrt{\frac{4a^2bMK^2 - aKd^2D^2}{bM}}} \qquad (51)$$

$$\in^{\frac{-dD}{2bM}ct} \sin\left[\sqrt{\frac{aK}{bM} - \frac{d^2D^2}{b^2M^2}}\ ct - \phi\right]$$

With $\phi = \arctan\sqrt{\frac{4abMK - d^2D}{-dD}}$

It can be seen that substitution of equations (48), (49), and (50) into equation (51) will reduce this equation to

(47). Thus, we do have a consistent method for transferring the mechanical quanities to practical electrical components.*

Data for a check on automobile dynamics by this electrical analog method was obtained from the Ford Motor Company. The model is a 1949 four door V-8 sedan, and the data:**

| Masses | Lb. Sec.$^2$/ in. |
|---|---|
| $M_{11}$ | 2.168 |
| $M_{22}$ | 1.531 |
| $M_{33}$ | 2.842 |
| $M_{12}$ | .5333 |
| $M_{13}$ | 1.807 |
| $M_{23}$ | .4952 |
| $M_1' = M_4'$ | .2130 |
| $M_m$ | .04677 |
| $M_2 = M_3$ | .1892 |
| Spring Constants: | $\frac{\text{Lbs.}}{\text{In.}}$ |
| $K_1 = K_4$ | 131.8 |
| $K_2 = K_3$ | 121.8 |
| $K_{o1}=K_{o2}=K_{o3}=K_{o4}$ | 935 |

* This system is different than but consistent with, one given in Westinghouse Engineer, March 1946; page 52.

** The data was not available in this form. For computation and approximation used see appendix III .

Shock Absorbers:

| | Compression | Rebound |
|---|---|---|
| | $\frac{\text{Lb. Sec.}}{\text{Ins.}}$ | |
| $D_1 = D_4$ | 2.50 | 25.0 |
| $D_2 = D_3$ | 10.5 | 15.0 |

An attempt to use the direct analog would give a very impractical electric circuit. The condensers would have to be quite large. For example, $C_2$ would be 9,210 mfd., which is extremely large. Also, coils with inductances as large as 2.842 henry cannot be wound thathave a resistance negligible with respect to the desired resistance of 2.5 ohms.

Here the use of the dimensionless groups is the answer. From (48) we see that (b) must be as large as possible with respect to (a) to keep the value of (c) large. The importance of keeping (c) large is to keep the transient frequency down below the self-resonant frequency of the inductances. Another factor requiring a large value for (c) is the actual circuit will have wiring capacitances that introduce less error at low frequencies. And yet, (49) demands that the product of (a) x (b) be as large as possible. This keeps the desired resistance large compared with the resistance of the coils, which is an unavoidable error that must be introduced. Obviously, this is a task that can end only in a compromise. The values of (a) and (b), that were used are $19.388(10)^3$ and $4.847(10)^{-3}$. This gave (c) a value of $.5(10)^{-3}$ and (d), 9.694. This value of (c) makes $.5(10)^{-3}$ seconds in the electric circuit to be equal to one second in the mechanical system. The resultant electrical parameters were:

Inductance in mh.

$L_{11}$ = 10.51

$L_{22}$ = 7.421

$L_{33}$ = 13.78

$L_{12}$ = 2.585

$L_{13}$ = 8.759

$L_{23}$ = 2.400

$L_1 = L_4$ = 1.032

$L_m$ = .2267

$L_2 = L_3$ = .9171

Condensers in mfd.

| | |
|---|---|
| $C_1 = C_4$ | .391 |
| $C_2 = C_3$ | .423 |
| $C_{on}$ | .0552 |

Resistances in Ohms

| | $i_{01}-i_1 > 0$ | $i_{01}-i_1 < 0$ |
|---|---|---|
| $R_1 = R_4$ | 24.24 | 242.4 |
| $R_2 = R_3$ | 101.8 | 143.4 |

The fact that the shock absorbers were non-linear eliminated the analytic solution, but electrically it was no problem. Ideally the circuit shown in figure 14 has a resistance $R_1$ with current to the left and $R_2$ with current to the right. Perfect rectifiers do not exist, but the desired resistance was closely approximated, using this circuit.

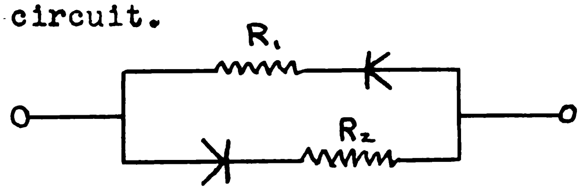



Figure 14

Vacuum tube diodes were tested, but the zero voltage current and extreme non-linearity at low voltages, plus high forward impedance, eliminated their use. A germanium crystal diode with a peak surge current of 500 milliamps was found quite adaquate, The curves on pages 27, 28, and 29, show the results of this work*. Probably, these resistances approach the ideal curve as well as do the actual mechanical shock absorbers.

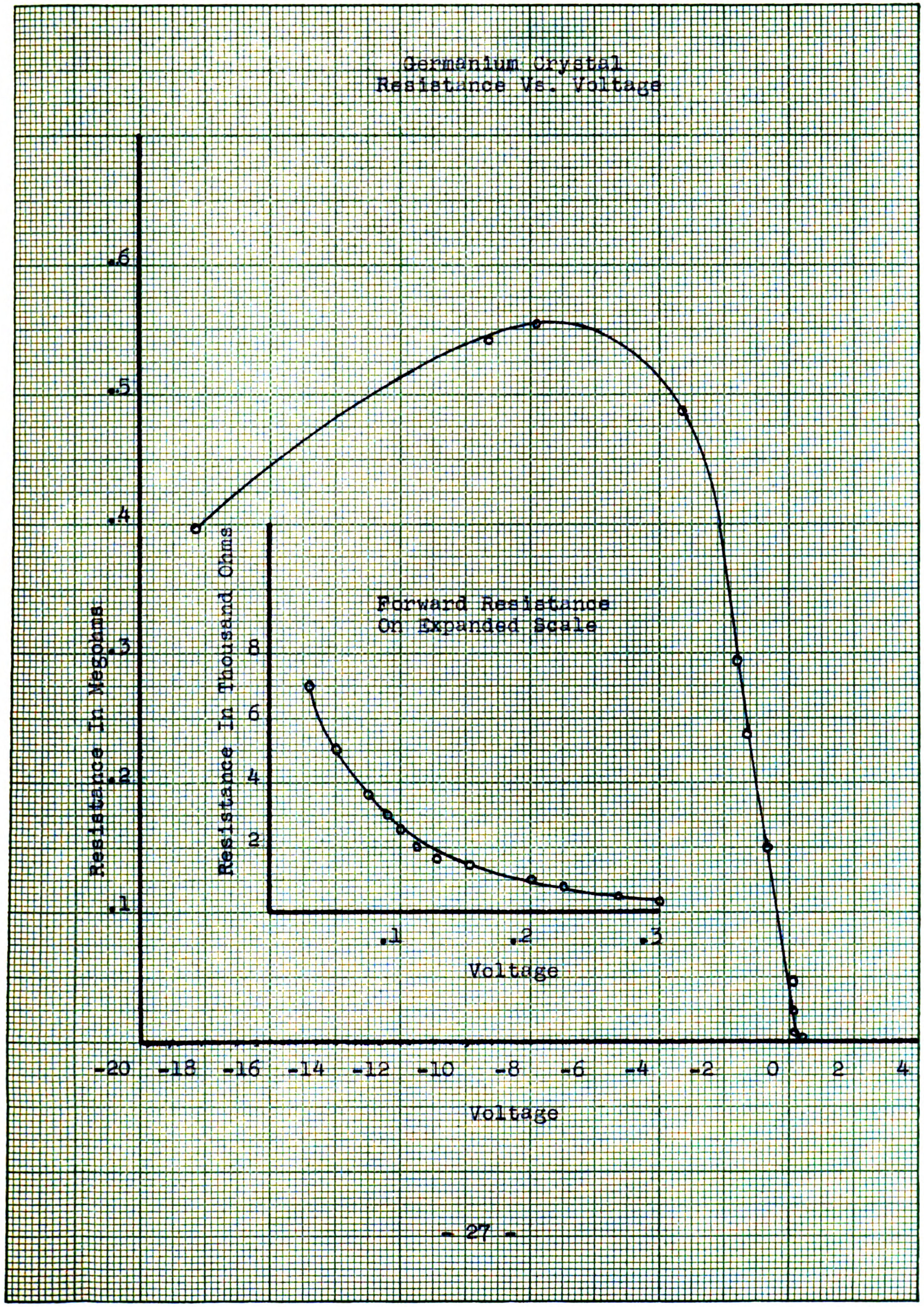
The values of the components for the analogous electric circuit of figure 13 have been determined.

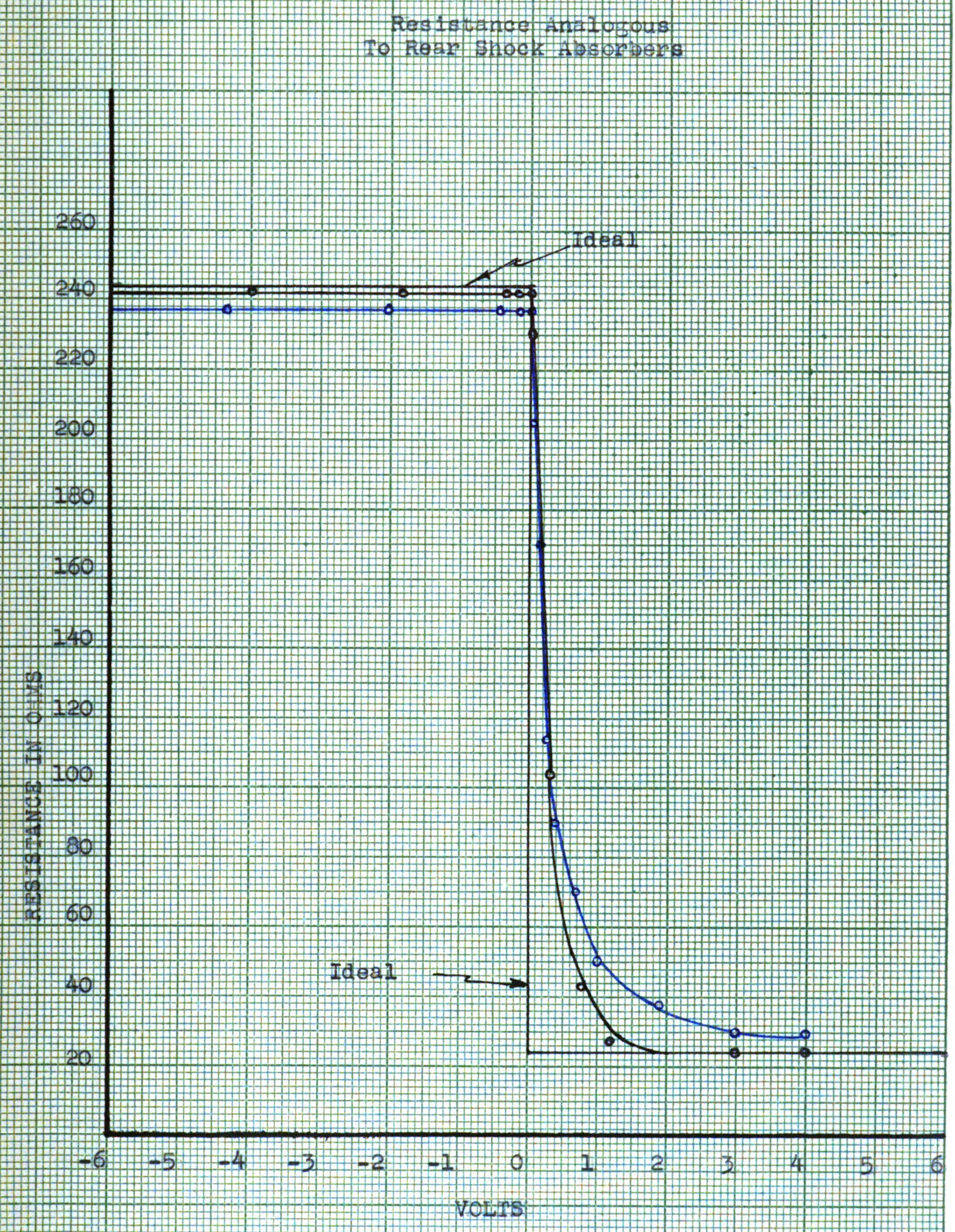
On this basis, the electric network was constructed. It is pictured on page30. From this circuit, oscillograms were taken to compare the motion indicated by the electric circuit with the actual motion of the automobile.

The response of the car to the bump was obtained by attaching lights at the desired points and taking a time exposure of the resulting motion. The scale for these photographs is found from the reference lights, spaced two feet apart, and the speed of the car. The speed of the automobile, fifteen miles per hour, was obtained only from the speedometer, and thus, is a possible source of error.

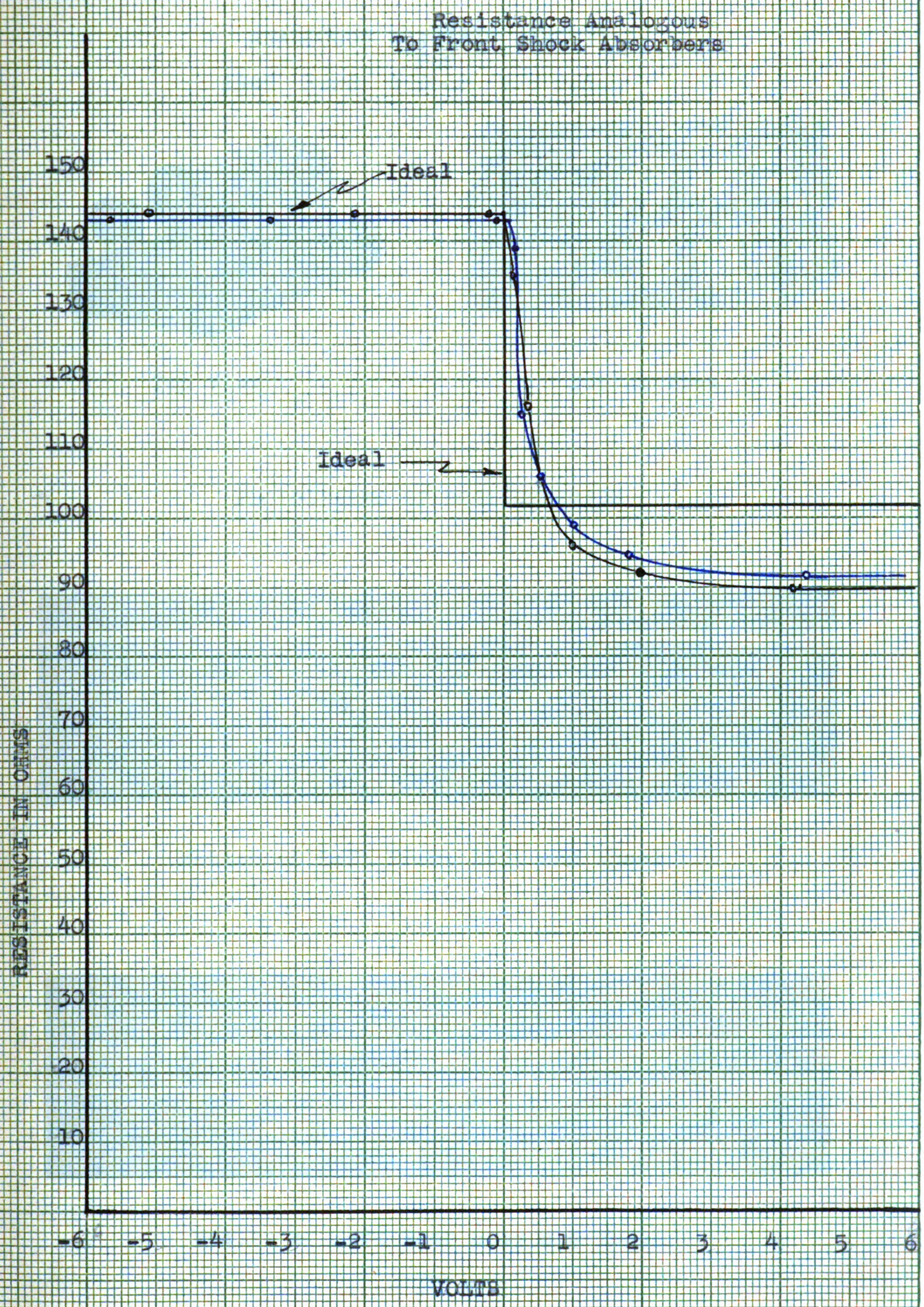
.....................................................

* The resistances and other components were tested to three significant figures on a Wheatstone Bridge. The reactances tested at 1000 cycles, which is in the midrange of the transient frequencies.


Germanium Crystal
Resistance Vs. Voltage
Resistance In Megohms
.6
.5
.4
.3
.2
.1
-20
-18
-16
-14
-12
-10
-8
-6
-4
-2
0
2
4
Voltage
Forward Resistance
On Expanded Scale
Resistance In Thousand Ohms
8
6
4
2
.1
.2
.3
Voltage



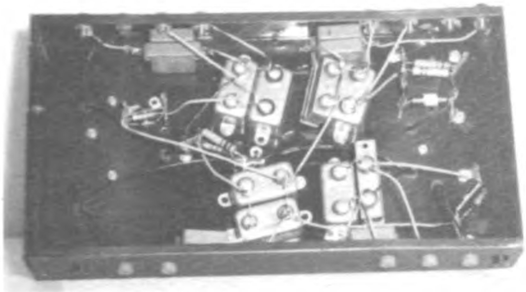
Resistance Analogous
To Rear Shock Absorbers
Ideal
Ideal
RESISTANCE IN OHMS
260
240
220
200
180
160
140
120
100
80
60
40
20
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
VOLTS



Resistance Analogous
To Front Shock Absorbers
Ideal
Ideal
RESISTANCE IN OHMS
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
VOLTS

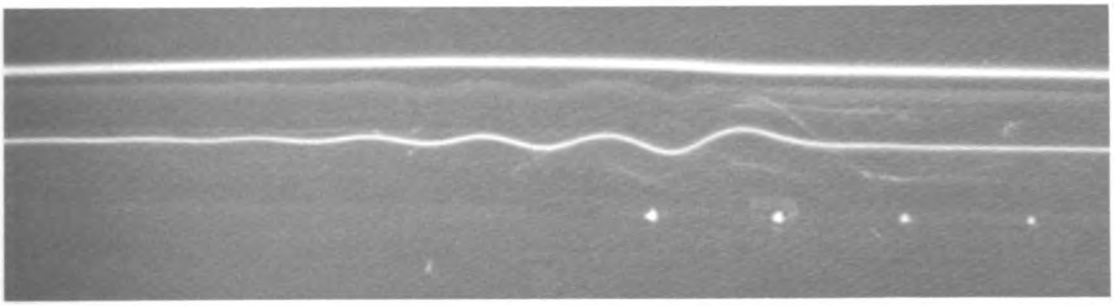
ANALOGOUS ELECTRIC CIRCUIT
for
1949 Ford 4 Door V-8 Sedan

Top View

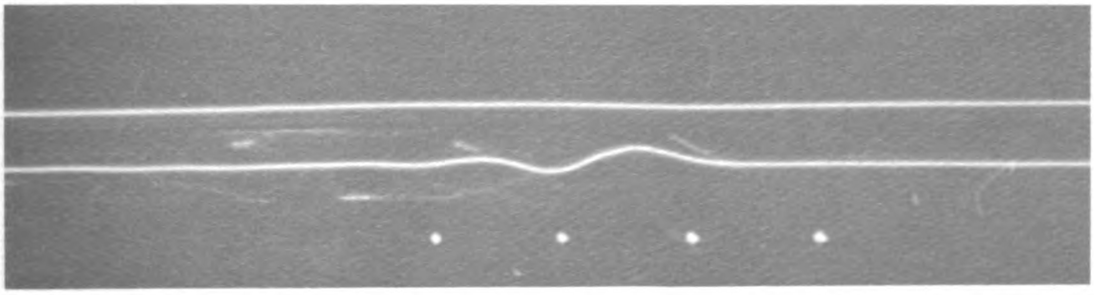


Bottom View

PHOTOGRAPHS OF BODY DYNAMICS



Motion of Left Front Body and Wheel



Motion of Left Rear Body and Wheel

## Oscillograms

In the electric circuit it is necessary to Measure the charge in the various loops. From page 22, charge is analogous to displacement, and the various displacements in the mechanical system are the unknowns.

The most convenient way to measure charge is to measure the voltage across a condenser, since voltage is directly proportional to the charge stored in the condenser. From figure 13, we see that the four loop charges, corresponding to wheel displacements, can be directly measured in this way. The body displacements must be found by adding the difference in the wheel and body displacements to the wheel displacements.

In this circuit $S_{o1} = S_{o2} = S_{o3} = S_{o4}$. Since the input voltage is also proportional to this value of elastance, all voltages except across $S_1$, $S_2$, and $S_3$ have the same ratio to the charges. It was therefore convenient to leave the oscillograms in terms of voltage, not charge. However, all voltages were referred to the common elastance, $S_{on}$. This allows easy conversion of the voltage oscillograms to mechanical motion.

The oscillograms show the motion of all points with both a positive and negative step-function "bump" applied to the left front wheel. The motion of a few points due to the same input applied to the left rear wheel is included.

With these oscillograms the motion due to square bumps can be found. This is done on pages 39 and 40 to compare with the photographs of the car passing over a similar road irregularity.

The automobile dynamics pictured on page 30 were taken as follows:

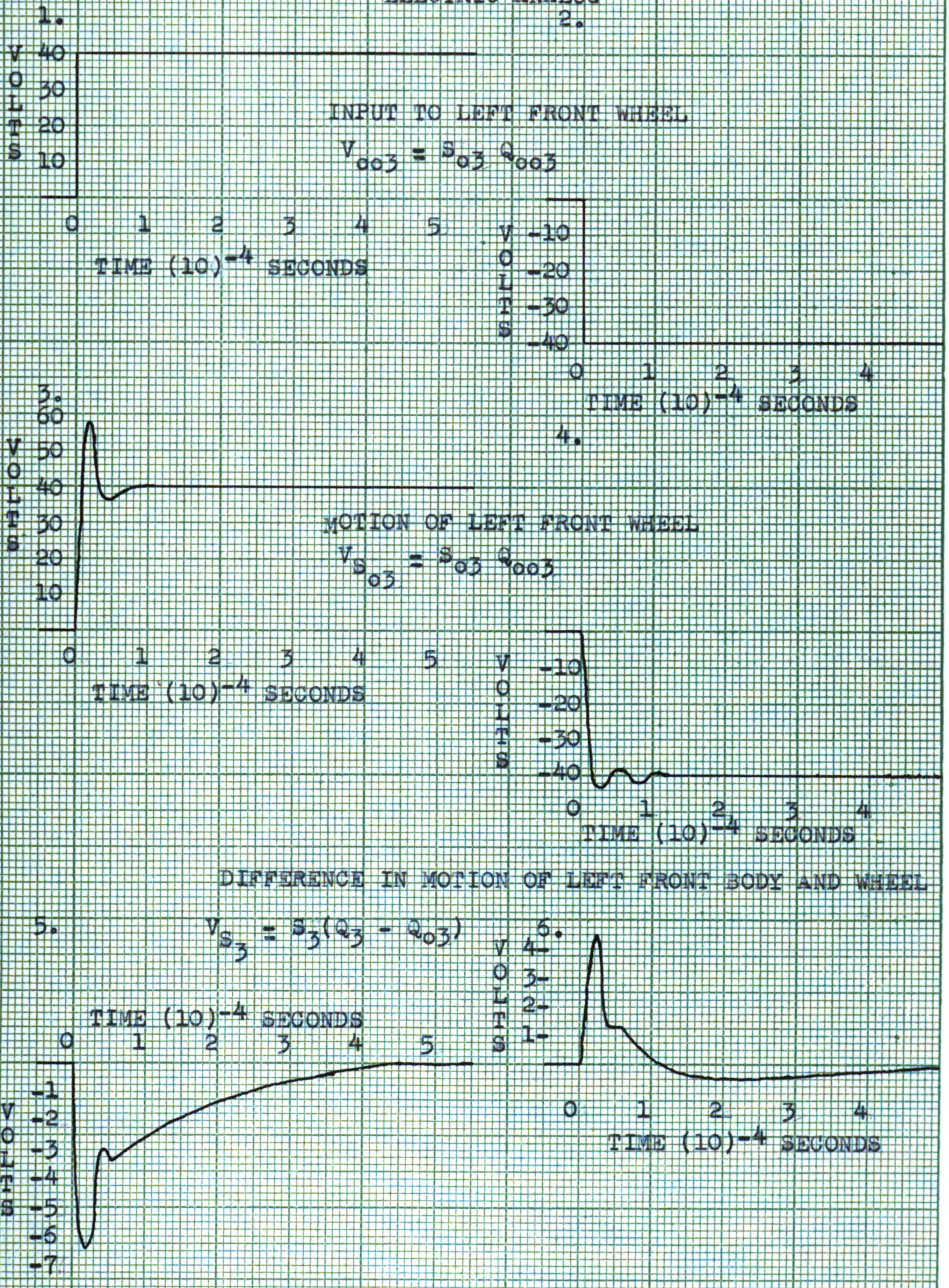
Spacing of lights: 2 feet.

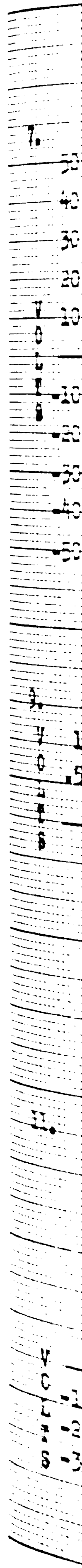
Dimensions of bump: height 2 1/2 in.
length 9 1/4 in.

Bump input to left wheels of automobile.

Thus, the reference lights were .0909 seconds apart, and the bump applied for .035 seconds. The comparison of the photograph with the combined oscillograms is given on pages 39 and 40.

## OSCILLOGRAMS FROM ELECTRIC ANALOG

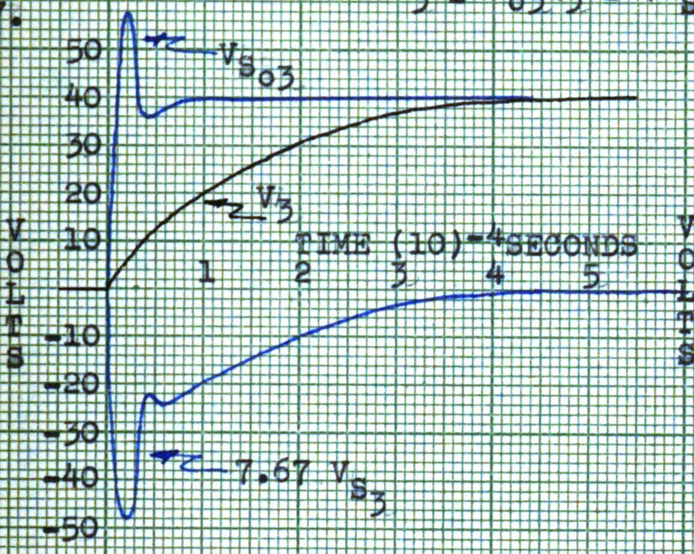



7.
VOLTS
9.
VOLTS
11.
VOLTS

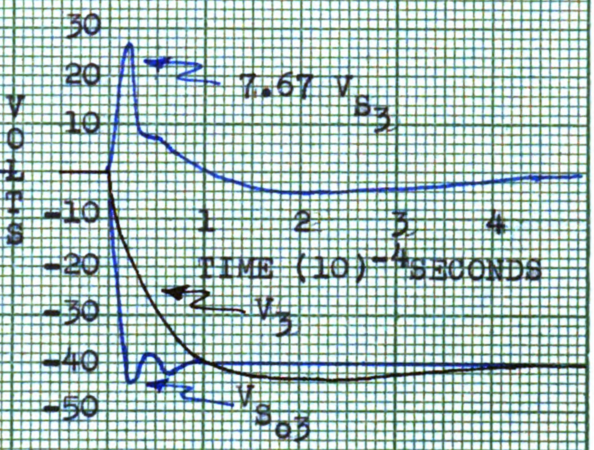
## MOTION OF LEFT FRONT BODY

$$V_3 = S_{o3}Q_3 = (V_{S_{o3}} + 7.67V_{S_3})$$

7.



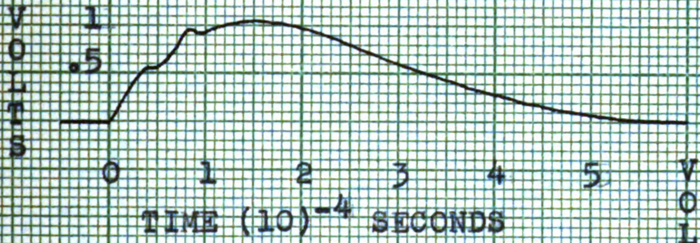

8.



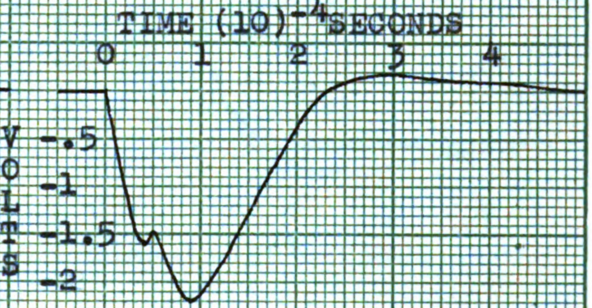

## MOTION OF LEFT REAR WHEEL

$$V_{S_{o4}} = S_{o4}\,Q_{o4}$$

9.



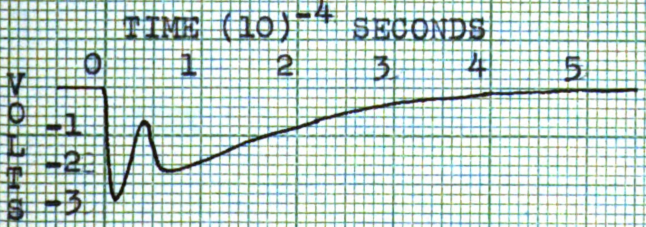

10.



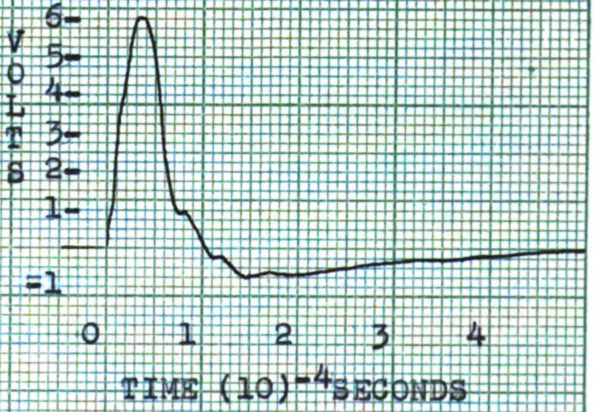

## MOTION OF RIGHT REAR WHEEL

$$V_{S_{o1}} = S_{o1}\,Q_{o1}$$

11.



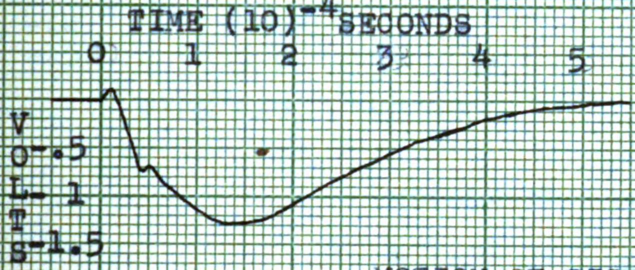

12.



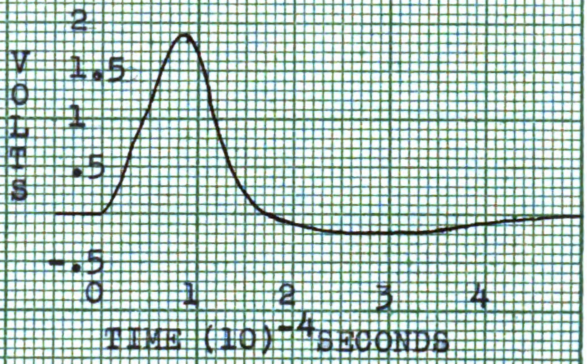
## DIFFERENCES IN MOTION OF RIGHT REAR BODY AND WHEEL

$$V_{S_1} = S_1(Q_1 - Q_{o1})$$

13.



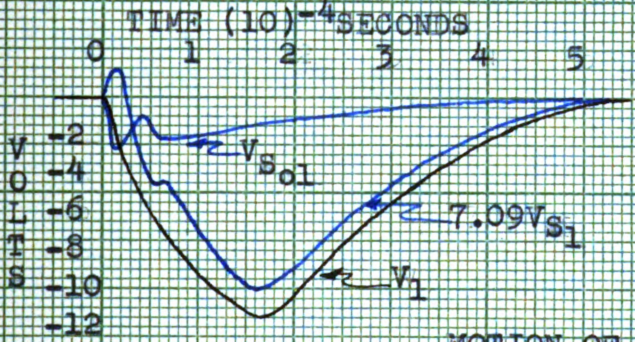

14.



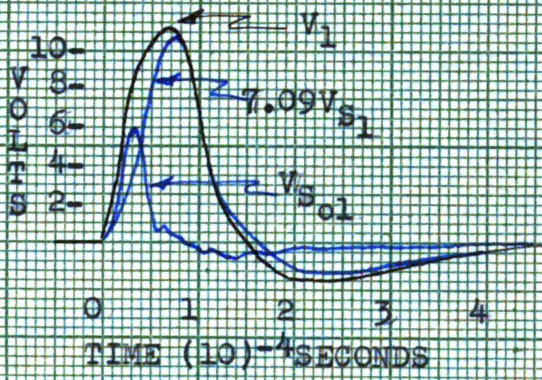

## MOTION OF RIGHT REAR BODY

$$V_1 = S_{o1}\, Q_1 = (V_{S_{o1}} + 7.09\, V_{S_1})$$

15.



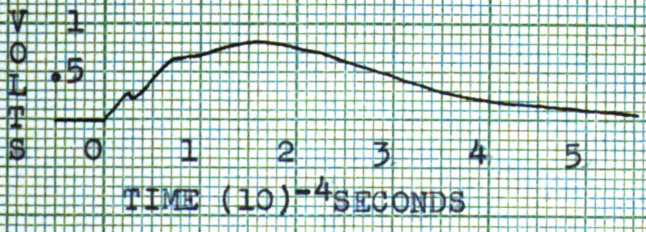

16.



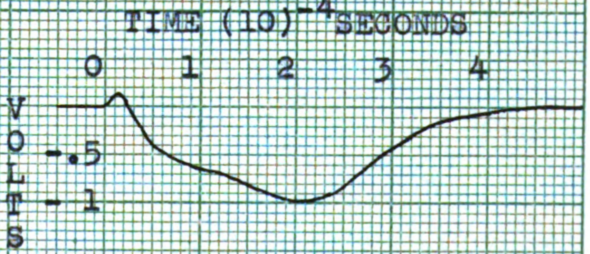

## MOTION OF RIGHT FRONT WHEEL

$$V_{S_{o2}} = S_{o2}\, Q_{o2}$$

17.



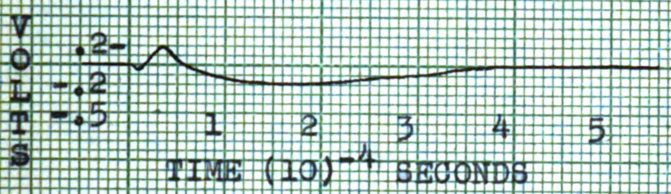

18.



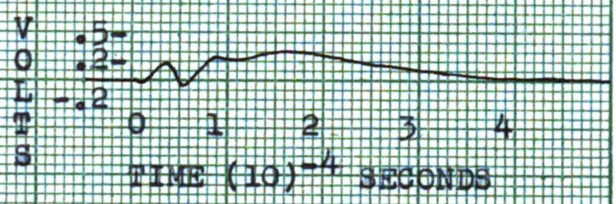
DIFFERENCE IN MOTION OF RIGHT FRONT BODY AND WHEEL

$$V_{S_2} = S_2(Q_2 - Q_{o2})$$

19.



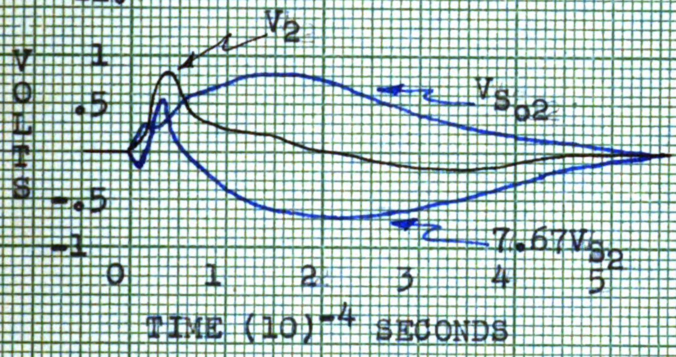

20.



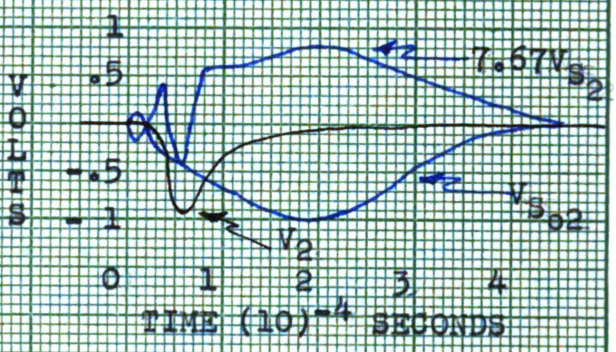

MOTION OF RIGHT FRONT BODY

$$V_2 = S_{o2}Q_2 = (V_{S_{o2}} + 7.67\, V_{S_2})$$

21.



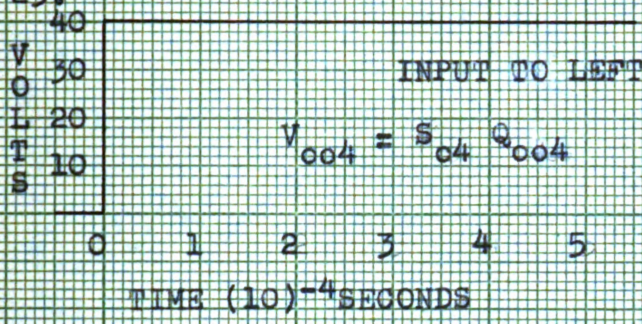

22.



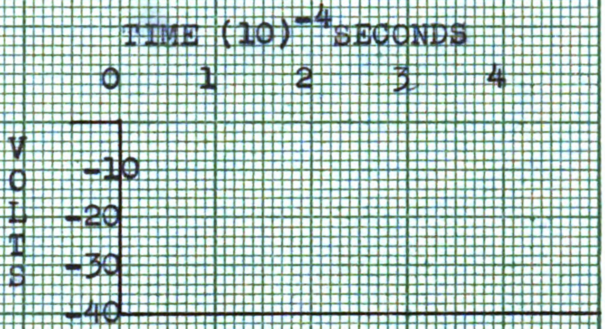
OSCILLOGRAMS FROM

ELECTRIC ANALOG

23.



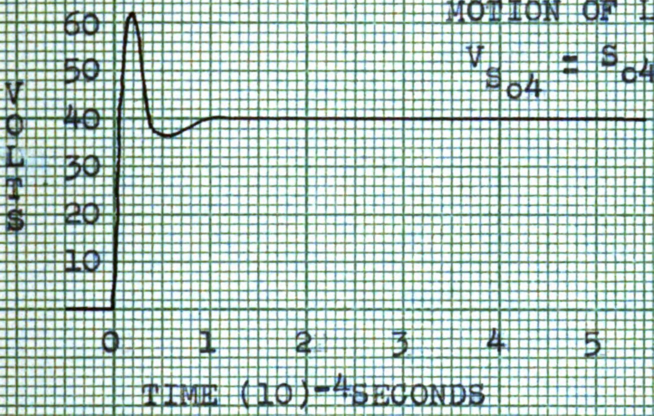

24.




INPUT TO LEFT REAR WHEEL

$V_{co4} = S_{c4}\ Q_{co4}$

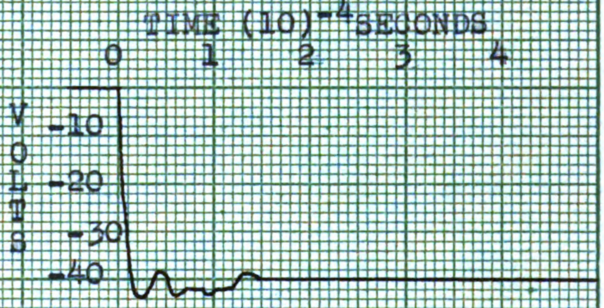
25.




MOTION OF LEFT REAR WHEEL

$V_{s_{o4}} = S_{c4}\ Q_{o4}$

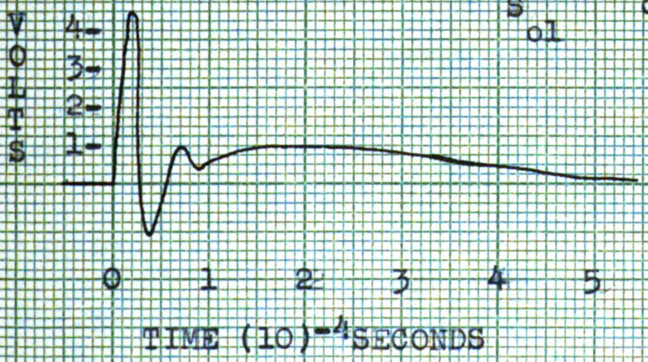
26.



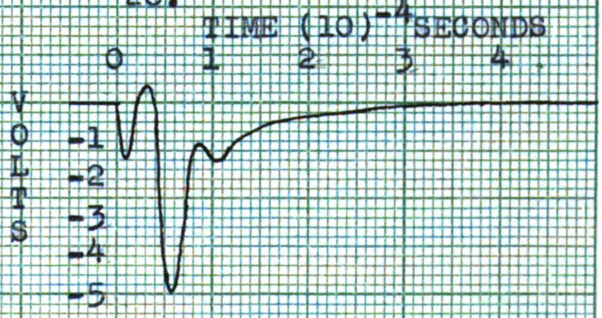

MOTION OF RIGHT REAR WHEEL

$V_{s_{o1}} = S_{o1}\ Q_{o1}$

27.



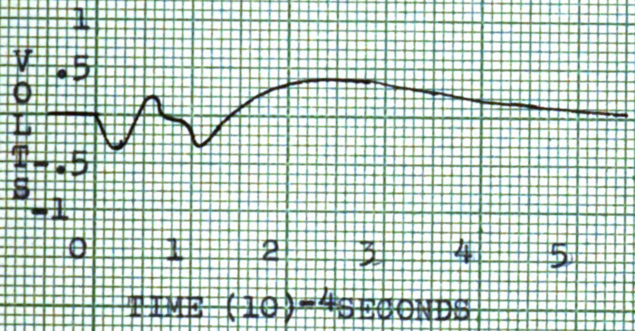

28.



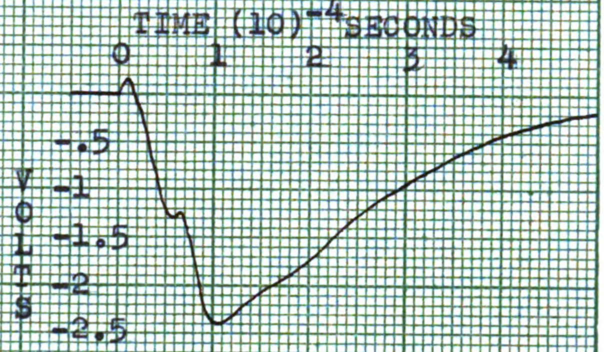
DIFFERENCE IN MOTION OF RIGHT REAR BODY AND WHEEL

$$V_{S_1} = S_1(Q_1 - Q_{o1})$$

29.



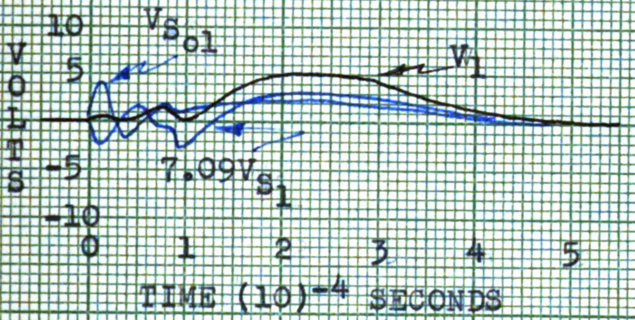

30.



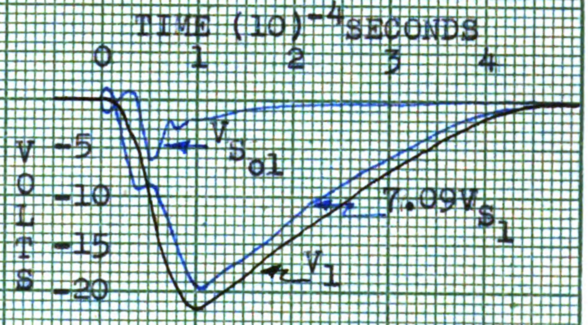

MOTION OF RIGHT REAR BODY

$$V_1 = (V_{S_{o1}} + 7.09V_{S_1}) = Q_1S_{o1}$$

31.



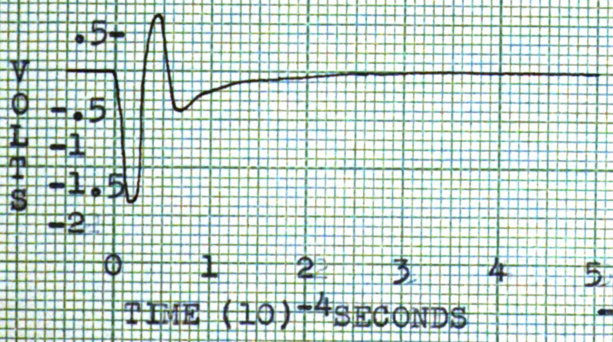

32.



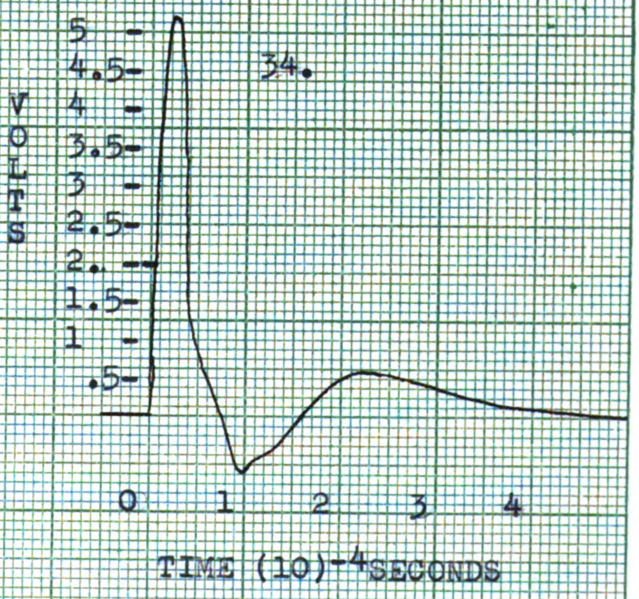

33.

MOTION OF RIGHT FRONT WHEEL

$$V_{S_{o2}} = S_{o2}Q_{o2}$$



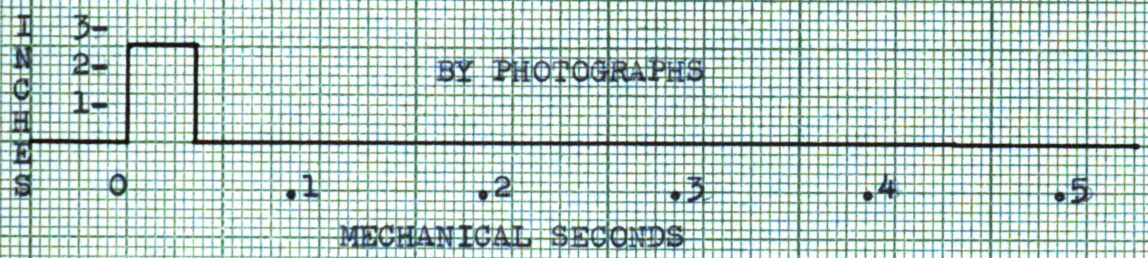

34.



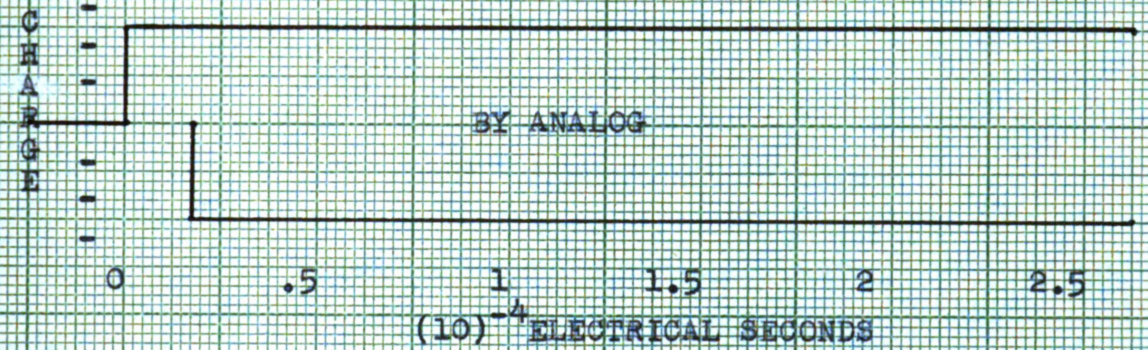
## COMPARISON OF RESULTS

### INPUT TO LEFT FRONT WHEEL

35.

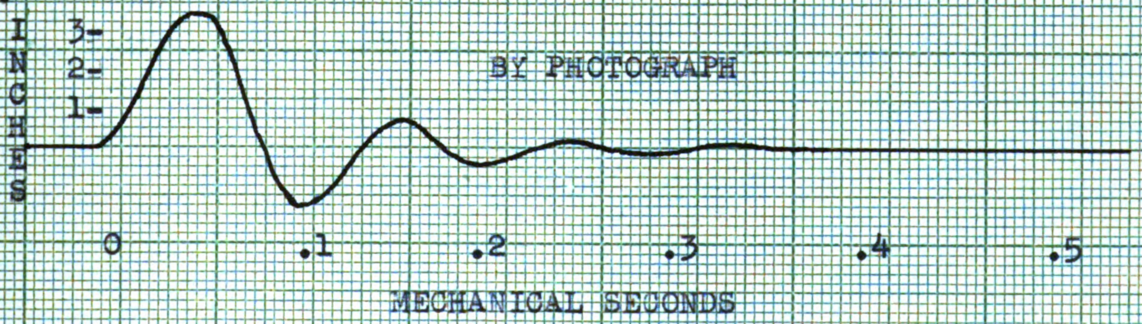



36.

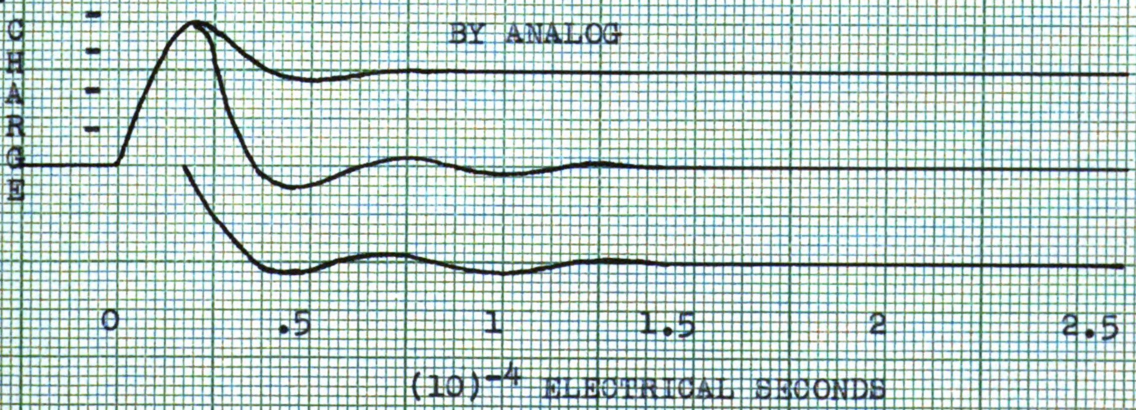



### MOTION OF LEFT FRONT WHEEL

37.

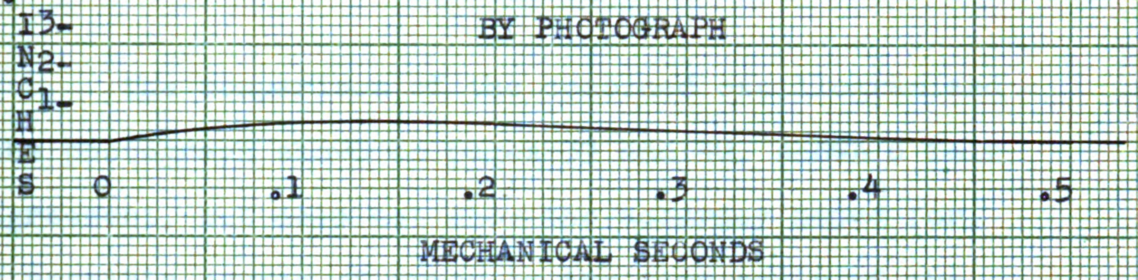



38.

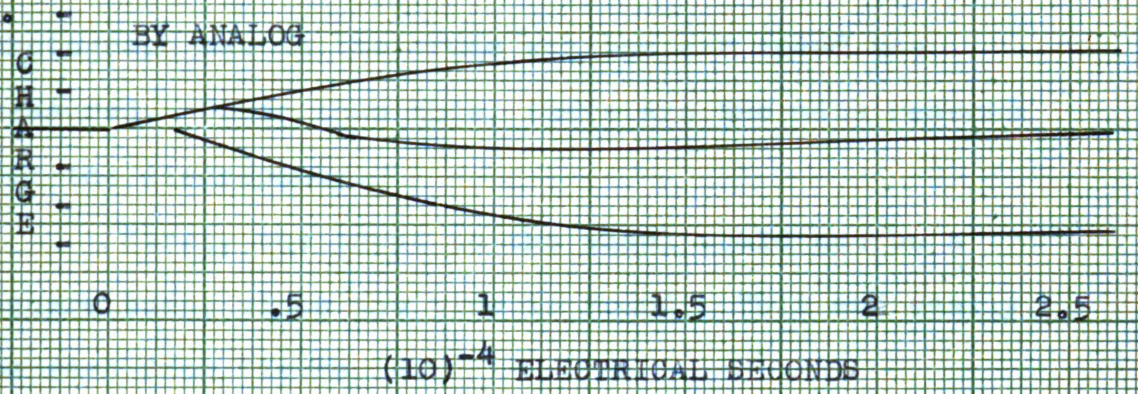


MOTION OF LEFT FRONT BODY

39.



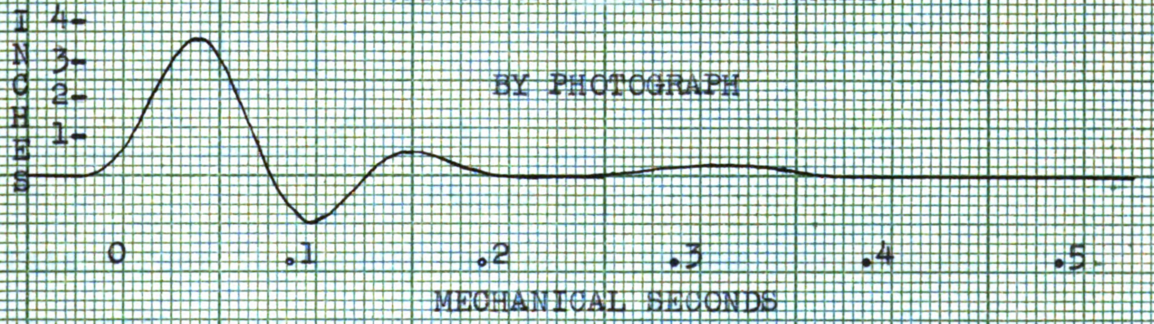

40.



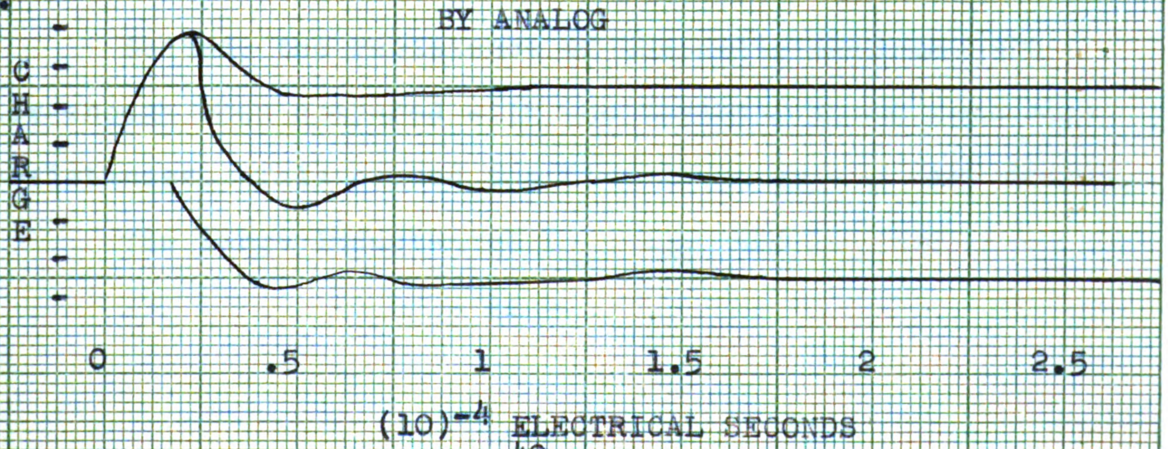

SAME INPUT TO LEFT REAR WHEEL

MOTION OF LEFT REAR WHEEL

41.




42.



## Conclusion

The comparison of the combined oscillogram and the photograph shown on pages 39 and 40, shows a fair agreement, but does not show the accuracy hoped for. A possible source of error in the time scale of the photograph was in the crude way the car speed was measured. Other than this, there are two main reasons for the discrepancy.

First, the input to the wheel could not be a square bump, since the tire is round. This can easily be seen from figure 14.

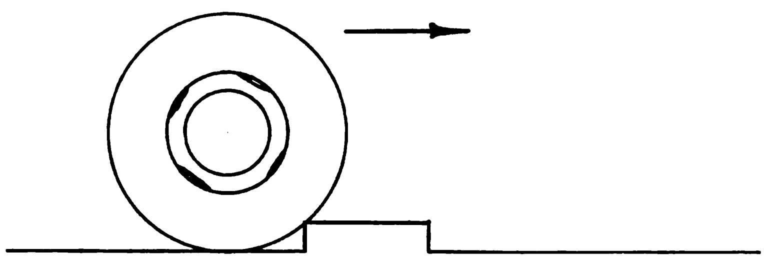


Figure 14

This effect is clearly indicated in the photograph. The wheel starts to rise before the center of the wheel reaches the bump. At higher speeds this effect would be minimized.

Second, the resistance of the coils used was not low enough when compared with the desired (shock absorber) resistances. This caused the circuit to appear over damped.

The large inductances $L_{11}$, $L_{22}$, and $L_{33}$ had resistances near 15 ohms, which is too large when compared with the desired resistances below 100 ohms. As pointed out on page 24, this was the result of a compromise*.

However, the comparison of the oscillographs gives far from completely negative results. It clearly indicates that an analogous electric circuit is a useful tool in the design of automobiles, or other vibrating machinery.

To be very usefull, all of the components would have to be variable, so that various load conditions could be considered; different spring constants and shock absorbers tested. Also, with wave shaping circuits, more general road irregularities could be tested, and a better evaluation of the automobile design made.

..........................................................

* Thefundamental difficulty was in the winding of the coils. With the multi-layer type windings used, the coil capacitance was large. This gave a low self-resonance for the coils. The large coils resonated near 45Kc.. For any accuracy at all the transcient frequencies of the circuit had to be considerably less than this. By universal type windings, or quality iron cores to improve the inductances, the accuracy of the circuit could be improved.

# APPENDIX I

## Derivation of Dimensionless Groups Used

The dimensionless groups will be combinations of the n quanities upon which the system is dependent. Thus in the mechanical system:

$$\Pi = F^a M^b D^c K^d T^e X^f$$

Expressing each of the above terms with the fundamental three; force, length, and time:

$$\Pi = (F)^a(FT^2L^{-1})^b(FTL^{-1})^c(FL^{-1})^d\, T^e\, L^f$$

The sum of the exponents of each of the fundamental terms must be zero. Therefore:

$$\underset{\text{(exponents of F)}}{a + b + c + d} = 0$$

$$\underset{\text{(exponents of L)}}{-b - c - d + f} = 0$$

$$\underset{\text{(exponents of T)}}{2b + c + e} = 0$$

This gives three simultaneous equations with six unknowns, so we can assign various values to three and attempt a solution for the remaining three.

$$\text{Let } \begin{cases} a = c = 0 \\ b = 1 \end{cases} \quad \text{yeilding:} \quad \begin{aligned} d + 0 + 0 &= -1 \\ -d + b + 0 &= 1 \\ 0 + 0 + e &= -2 \end{aligned}$$

$$\text{from which:} \quad \begin{aligned} d &= 1 \\ e &= -2 \\ f &= 0 \end{aligned}$$

$$\text{Therefore:} \quad \Pi = \frac{M}{KT^2}$$

$$\Pi_1 = \frac{1}{T}\sqrt{\frac{M}{K}}$$

Let $\begin{cases} c = 1 \\ e = f = 0 \end{cases}$ yeilding

$$\begin{aligned} a + b + d &= -1 \\ 0 - b - d &= 1 \\ 0 + 2b + 0 &= -1 \end{aligned}$$

From which:

$$\begin{aligned} a &= 0 \\ b &= -1/2 \\ d &= -1/2 \end{aligned}$$

Therefore: $\pi_2 = \dfrac{D}{\sqrt{M\ K}}$

Let $\begin{cases} a = 1 \\ c = 0 \\ e = 0 \end{cases}$ yeilding

$$\begin{aligned} b + d + 0 &= -1 \\ -b - d + f &= 0 \\ 2b + 0 + 0 &= 0 \end{aligned}$$

From which:

$$\begin{aligned} b &= 0 \\ d &= -1 \\ f &= -1 \end{aligned}$$

Therefore: $\pi_3 = \dfrac{F}{K\ X}$

# APPENDIX II

Proof That: $z_4 = z_1 - z_2 + z_3$

By using the same approximations, not $\sin \theta \doteq \theta$, but $\sin \phi = \phi$

its equivilent; $\cos \theta \doteq 1$, $\cos \phi = 1$, we reduce equations (37),(38),

and (39) to:

$$AL_3 - BL_1 + cz_1 = -D$$

$$AL_3 + BL_2 + cz_2 = -D$$

$$-AL_4 + BL_2 + cz_3 = -D$$

From which:

$$A = \frac{-D\left[(L_1 + L_2)z_2 + (L_1 + L_2)z_3\right]}{\triangle}$$

$$B = \frac{-D\left[(L_3 + L_4)z_1 - (L_3 + L_4)z_2\right]}{\triangle}$$

$$C = \frac{-D\left[(L_1 + L_2)(L_3 + L_4)\right]}{\triangle}$$

With $\triangle = L_2(L_3 + L_4)z_1 + (L_1L_4 - L_2L_3)z_2 + L_3(L_1 + L_2)z_3$

Substituting the above value for A, B, and C, into (40)

along with: $x_4 = -L_4$ and $y_4 = -L_1$ :

$$Ax_4 + By_4 + Cz_4 + D = 0$$

$$-L_1\left[(L_1 + L_2)z_2 - (L_1 + L_2)z_3\right] + L_1\left[(L_3 + L_4)z_1 - (L_3 + L_4)z_2\right]$$

$$-z_4(L_1 + L_2)(L_3 + L_4) + L_2(L_3 + L_4)z_1 + (L_1L_4 - L_2L_3)z_2$$

$$+ L_3(L_1 + L_2)z_3 = 0$$

Grouping: $(L_1 + L_2)(L_3 + L_4)\left[z_1 - z_2 + z_3 - z_4\right] = 0$

In general: $(L_1 + L_2)(L_3 + L_4) \neq 0$

Therefore: $z_4 = z_1 - z_2 + z_3$

## APPENDIX III

### Transformation of Model Data Given By The Ford Motor Company To Necessary Form

Approximations used were recommended by Mr. R. W. Gaines, Head of the Engine Test Dept. of the Ford Research Laboratories, Dearborn, Michigan.

For 1949, 4 door, V-8 sedan:

| | |
|---|---|
| Total weight | 3230 lbs. |
| Front sprung weight* | 219 lbs. |
| Rear sprung weight | 301 lbs. |

Unloaded weight distribution of fully equiped model in terms of percent of total car weight:

| | |
|---|---|
| Front wheels | 57.1% |
| Rear wheels | 42.9% |
| Left wheels | 51 % |
| Right wheels | 49 % |

Plane of center of gravity: 25.3 inches above road.

| | | |
|---|---|---|
| Front spring constant: | with tire | 108.6 lbs./in. |
| | less tire | 121.8 lbs./in. |
| Rear spring constant: | with tire | 114.5 lbs./in. |
| | less tire | 131.8 lbs./in. |

| Shock absorber constant: | rebound | compression |
|---|---|---|
| Front | 15 lb.sec./in. | 10.5 lb.sec./in. |
| Rear | 25 lb.sec./in. | 2.5 lb.sec./in. |

..................................................

* The sprung weight consisted of the weight of the wheel, wheel assembly, shock absorbers, and springs. Two-thirds of this weight was taken as wheel mass, while the remaining weight was lumped with the car body weight, since the actual wheel weight was only 46 pounds.

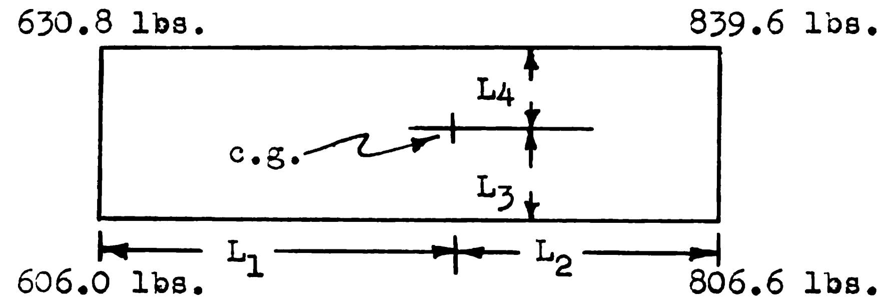
Radii of gyration: not available.

Wheel base length: 114 inches
width: 56 inches.

This data was transformed as follows:

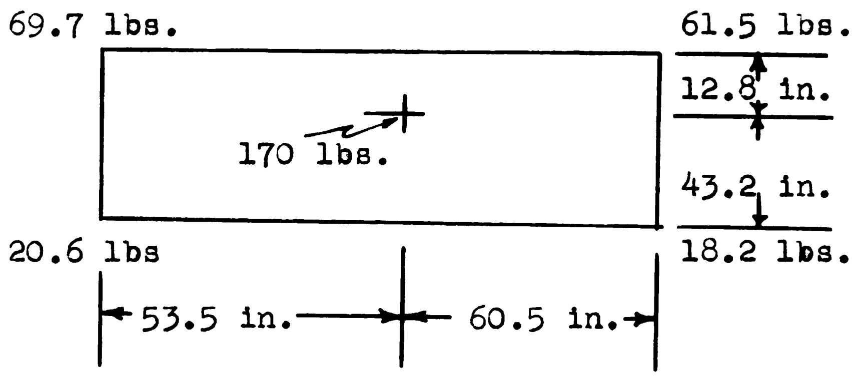
Effective car body weight = 3230 - 2/3(219 + 301) lbs.
= 2880 lbs.

By utilizing the moments involved, the center of gravity of the car body is located as follows:




with $L_1$ = 65.1 in.
$L_2$ = 48.9 in.
$L_3$ = 28.6 in.
$L_4$ = 27.4 in.

To test the car, there must be a driver added to above weight of the car.




Driver's weight: 170 lbs.

The location of the center of gravity is then modified to be:

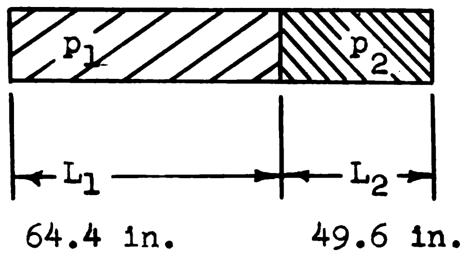
$$L_1 = 64.4 \text{ in.}$$
$$L_2 = 49.6 \text{ in.}$$
$$L_3 = 29.4 \text{ in.}$$
$$L_4 = 26.6 \text{ in.}$$

Obviously, the center of gravity is different for any other load in the automobile.

The radius of gyration in the planes was not known for the car body. This is the approximation used:

The weight was divided evenly at the center of gravity, and assumed to be uniformly distributed.

Thus for the y - z plane:




$$p_1 = \frac{2883}{2(64.4)}$$

$$p_2 = \frac{2883}{2(49.6)}$$

$$dI = dMx^2 = pdx(x^2)$$

$$I = \int_{L_1}^{o} p_1 x^2 dx + \int_{o}^{L_2} p_2 x^2 dx$$

$$I = 3.361(10)^6 \text{ lb.in.}^2$$

$$h_{yz} = \sqrt{\frac{I}{M}} = \sqrt{\frac{3.361(10)^6}{.3053(10)^4}} = 32.2 \text{ in.}$$

By a similar process: $h_{xz} = 19.6$ in.

The radius of gyration of the rear wheel assembly:

Effective weight of rear wheel assembly:

$$2/3\ (301) \doteq 200 \text{ lbs.}$$

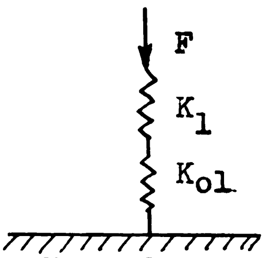
$$I_T = \left(\frac{108}{56}\right)\left[\frac{x^3}{3}\right]_{-28}^{28} + (2)(46)(28)^2$$

$$I_T = 100{,}430 \text{ lb.in.}^2$$

$$h = \sqrt{\frac{I}{M}} = 22.4 \text{ inches}$$

Using the definitions of self and mutual masses on page 7, the results of page 23 are found. The mass of the car body is the weight divided by the gravitational constant.

The spring constants were given in a form that required the separation of total spring effect of both tires and springs into its components.




$$K_T = \frac{(K_{o1})(K_1)}{(K_{o1} + K_1)}$$

Since $K_T$ and $K_1$ were given, $K_{o1}$ can easily be found. It was assumed that the tires were the same, so an average of the $K_{on}$ terms was used.

$$K_{o2} = K_{o3} = \frac{(121.8)(108.6)}{(121.8)-(108.6)} \doteq 996.5 \text{ lbs./in.}$$

$$K_{o1} = K_{o4} = \frac{(131.8)(114.5)}{(131.8)-(114.5)} = 872.3 \text{ lbs./in.}$$

$$K_{on} = \text{Average} = 935 \text{ lbs./in.}$$

The remaining data was in the desired form.

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1. Gardner, M.F., & Barnes, J.L., Transients in Linear Systems, Vol. I, 1942, New York, John Wiley & Sons, Inc.
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3. Westinghouse Engineer, March, 1946
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