MODULAR FORMS OF DIMENSION -2 FOR SUBGROUPS OF THE MODULAR GROUP

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ABSTRACT

MODULAR FORMS OF DIMENSION -2
BELONGING TO SUBGROUPS
OF THE MODULAR GROUP

by John Roderick Smart

Joseph Lehner has given a method for defining Poincaré series of dimension -2 on the modular group $\Gamma(1)$ which does not rely on the Hecke method of introducing a convergence factor. The problem considered in this thesis is the following: extend the method to congruence subgroups of the modular group; and determine when the method can be extended to arbitrary subgroups of finite index.

Let Γ be a subgroup of finite index, and let $A_j^{-1}\infty$, $j = 1, 2, \cdots, \sigma(\Gamma)$ be a complete set of inequivalent parabolic cusps. $A_j \in \Gamma(1)$. We assume $A_j^{-1}\infty = \infty$ if and only if $A_j = I$. Let $\mathcal{V}$ be an abelian character on Γ. Define $e(\kappa_j) = \mathcal{V}(A_j^{-1}U^{\lambda_j}A_j)$ where λ_j is the least positive integer such that $A_j^{-1}U^{\lambda_j}A_j \in \Gamma$ and we use the notation $e(z) = \exp[2\pi iz]$. We define for integers $\mu \neq 0$

$$(*)\ G(z,\mathcal{V},A_j,\Gamma,\mu) = \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma)}}^{\infty} \sum_{\substack{d=-\infty \\ d\in\mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

The sets of integers $\mathcal{C}(A_j, \Gamma)$ and $\mathcal{D}(c, A_j, \Gamma)$ are so defined that $V_{c,d} = (a\ b \mid c\ d)$ runs over a complete set of matrices in $A_j\Gamma$ with different lower row as c runs over $\mathcal{C}(A_j, \Gamma)$ and d runs over $\mathcal{D}(c, A_j, \Gamma)$. The double series in (*) is not aboslutely convergent, therefore, we specify that it is to be summed first on d and then on c. With this convention the functions G defined in (*) are regular in $\mathcal{H}$, the upper half plane. In order to show the proper functional equation is satisfied we must prove a Rademacher lemma.

By a lattice point for $A_j\Gamma$ we mean the lower row (c,d) of a matrix in $A_j\Gamma$. Let $\mathcal{O}(A_j, \Gamma)$ represent the set of all lattice points for $A_j\Gamma$. Furthermore, for any positive integer K let $\mathcal{O}_K(A_j, \Gamma)$ be the set of all lattice points for $A_j\Gamma$ contained in the square with sides $u = \pm K$, $v = \pm K$ in the u,v-plane. We define a class $\mathcal{M}$ of matrices such that every $V \in \mathcal{M}$ has the form $V = \pm U^{m\lambda} M U^{n\lambda}$ with $M \in \mathcal{M}$, $U^{\lambda} = (1\ \lambda \mid 0\ 1)$ and m and n integers. Then the Rademacher lemma implies:

$$(**)\quad G(z, \mathcal{V}, A_j, \Gamma, \mu) = \lim_{K \to \infty} \sum_{(c,d) \in \mathcal{O}_K(A_j, \Gamma)M} \frac{e((\mu + \kappa_j) V_{c,d} z / \lambda_j)}{\mathcal{V}(A_j^{-1} V_{c,d})(cz+d)^2}$$

$\mathcal{O}_K(A_j, \Gamma)M = \{(c\alpha + \gamma d, \beta c + \delta d) : (c,d) \in \mathcal{O}_K(A_j, \Gamma)\}$ that is we think of M as acting on the u,v-plane as an affine transformation.

Using the Lipschitz formula we derive the Fourier expansion of the functions G. This then shows that they have the proper behavior at the cusps.

All of these results required an estimate of $\mathcal{O}(|c|^{1/2+\varepsilon})$ for the Kloosterman sums corresponding to Γ and v. We use a result of Petersson's which says these sums have the proper estimate if Γ is a congruence subgroup and v is identically 1 on a principal congruence subgroup.

The problem we considered was solved in the following generality: whenever the Kloosterman sums have the proper estimate the method of Lehner can be extended to subgroups of finite index.

MODULAR FORMS OF DIMENSION -2 FOR SUBGROUPS OF THE MODULAR GROUP

By

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1. Introduction

In this section we give a description of the problem considered and the results of our research. The definitions and results required for this investigation are given in the next section.

Lehner [6] introduced the series

$$(1.1) \qquad F_\mu(\tau) = \sum_{k=0}^{\infty} \sum_{\substack{m=-\infty \\ (m,k)=1}}^{\infty} \frac{e(-(\mu-\alpha)V_{k,-m}\tau)}{\varepsilon(V_{k,-m})(-i(k\tau-m))^2}$$

where ε is a multiplier system for $\Gamma(1)$ and the dimension -2, $0 \leq \alpha < 1$, $U = (1\ 1 \mid 0\ 1)$, $V_{k,-m} = (*\ * \mid k,-m) \in \Gamma(1)$ and $e(\alpha) = \varepsilon(U)$. We use the notation

$$e(z) = \exp\,[2\pi iz].$$

Furthermore, we write matrices in one line with a bar separating rows. He proved that for $\mu = 1, 2, \ldots$ $F_\mu(\tau)$ is a modular form of dimension -2 for the multiplier system ε regular in $\mathcal{H} = \{z = x+iy: y > 0\}$. Basic to the proof is an estimation of the Kloosterman sums $A_{k,\mu}(m)$ which arise as $O(k^{1/2+\varepsilon})$. Furthermore, the results depend heavily on a generalization of a lemma due to Rademacher, which allows the rearrangement of certain conditionally convergent double series. Lehner derives the Fourier expansion of these functions at the infinite cusp. These coefficients are expressed as an infinite series of the Petersson-Rademacher type, which involve Bessel functions. The functions $F_{\mu}(\tau)$ are not identically zero since they have a pole at $\tau = i\infty$.

Our problem can now be stated:

(i) extend the methods and results to congruence subgroups of the modular group;

(ii) extend the results to arbitrary subgroups of finite index in $\Gamma(1)$.

We obtain partial solution of these problems.

Let Γ be a subgroup of $\Gamma(1)$ of finite index. Assume $-I \in \Gamma$. Let $A_j^{-1}\infty = p_j$, $j = 1, 2, \ldots \sigma$ be a set of inequivalent parabolic cusps of Γ. $A_j \in \Gamma(1)$ and $A_1 = I$. Suppose $\mathcal{V} = \mathcal{V}(\Gamma, -2)$ is a multiplier system for Γ and the dimension -2. Consider the following series:

$$(1.2)\quad G(z, \mathcal{V}, A_j, \Gamma, \mu) = \sum_{\substack{c=-\infty \\ c \in \\ \mathcal{C}(A_j, \Gamma)}}^{\infty} \sum_{\substack{d=-\infty \\ d \in \\ \mathcal{D}(c, A_j, \Gamma)}}^{\infty} \frac{e((\mu + \kappa_j) V_{c,d} z / \lambda_j)}{\mathcal{V}(A_j^{-1} V_{c,d})(cz+d)^2}$$

where μ is a non-zero integer and $V_{c,d} = (* \; * \mid c \; d) \in A_j \Gamma$. The sets $\mathcal{C}(A_j, \Gamma)$ and $\mathcal{D}(c, A_j, \Gamma)$ are so defined that as c runs over $\mathcal{C}(A_j, \Gamma)$ and d runs over $\mathcal{D}(c, A_j, \Gamma)$, $V_{c,d}$ runs over a complete set of matrices in $A_j \Gamma$ with different lower row. λ_j is a positive integer which depends upon Γ. $0 \leq \kappa_j < 1$; κ_j depends upon $\mathcal{V}$ and Γ. The series is not absolutely convergent and we shall have to specify the order in which the terms are summed.

If the series in (1.2) were absolutely convergent (which it is if 2 is replaced by $r > 2$), we could rearrange the series. We would get easily the following results:

1) $G(z, \mathcal{V}, A_j, \Gamma, \mu)$ is a modular form. That is, it satisfies the functional equation

(1.3) $$G(Vz, \mathcal{V}, A_j, \Gamma, \mu) = \mathcal{V}(V)(cz + d)^2 \, G(z, \mathcal{V}, A_j, \Gamma, \mu)$$

for each $V = (* \; * \mid c \; d) \in \Gamma$. Furthermore, it satisfies the required regularity conditions.

2) We would obtain the Fourier series coefficients given in (6.10).

3) We would obtain the inner product formula given in (9.2).

In order for us to obtain these results we must rearrange conditionally convergent series. In doing so, we must rely on:

4) a Rademacher lemma; and

5) a non-trivial estimate on the Kloosterman sums

$$(1.4)\quad W_c(n+\kappa,\mu+\kappa_j)=\sum_{d\in\mathcal{D}_c(A_j,\Gamma)}\bar{\mathcal{V}}(A_j^{-1}V_{c,d})e[(n+\kappa)d/c \quad +(\mu+\kappa_j)a/c\lambda_j],$$

namely,

$$W_c(n,\mu) = \mathcal{O}(|c|^{1/2+\varepsilon}) \qquad \varepsilon>0$$

where the constant in the $\mathcal{O}$-symbol does not depend upon n.

Our main result is this: <u>If Γ is a subgroup of finite index in $\Gamma(1)$, and if the Kloosterman sums have the required estimate then</u> 1), 2) <u>and</u> 3) <u>are obtained</u>.

In § 8 we give a series of examples of groups Γ and multipliers systems $\mathcal{V}$ for which we are able to prove the validity of the Kloosterman estimate. They include: $\Gamma = \Gamma(1)$ and all six multiplier systems $\mathcal{V}$ of dimension -2; $\Gamma = \Gamma_2$, the unique subgroup of index 2 in $\Gamma(1)$, and all nine multiplier systems; and $\Gamma_0(q)$, q a prime of the form $4m + 1$ and the multiplier system which depends upon the Legendre symbol. In § 9 we return to these examples and calculate the dimension of the space of cusp forms belonging to these systems. Here we use results of Petersson.

The proof that the G of (1.2) satisfies (1.3) depends heavily on our Rademacher-type lemma. Roughly, this lemma states that the series (1.2) can be summed over expanding parallelograms centered at the origin. This is somewhat analogous to the standard way in which one proves the convergence of the Weierstrass $\wp$-function. To some extent we follow Lehner [6] in proving our Rademacher lemma; however, since in general Γ will have more generators than $\Gamma(1)$ we need a more comprehensive lemma. M. I. Knopp has developed other extensions of the original Rademacher lemma ([5] , [5.1]); still others are in the process of publication.

For the estimate of the Kloosterman sums we rely on the researches of Petersson [11]. Petersson proves the theorem: if Γ is a congruence subgroup and $\mathcal{V}$ an abelian character on Γ which is identically 1 on some principal congruence subgroup $\Gamma(N) \subset \Gamma$, then the Kloosterman sums (1.4) have the required estimate. In § 8 we give an elaboration of his proof so that the interested reader may see the neat way in which the complicated sums (1.4) are reduced to the classical Kloosterman sums. We require the estimate of the sums (1.4) both in the proof of the regularity of the functions G of (1.2) and in the proof of the Rademacher lemma.

The above results are not all new. Petersson has obtained them in [11]; however, his method of proof is entirely different from ours. He uses the Hecke idea [4, pp. 468-476] of introducing a convergence factor $|cz + d|^{-s}$, $s > 0$ into (1.2). He then takes the limit as $s \to 0+$. Our method of rearranging conditionally convergent double series was first suggested by Rademacher [13]. In some senses our method is more natural since it follows the proofs for dimension $-r < -2$.

2. Preliminaries

In this section we give the definitions, notations and results which are needed for this investigation. No attempt has been made to give a reference for every fact which is stated, but rather, to give references for only those results which lie deeper in the theory. Almost all of the results for which no reference is given can be found in Ford's book [2].

We shall be concerned with infinite groups Γ of linear fractional transformations

$$Vz = \frac{az + b}{cz + d}$$

where a, b, c and d are rational integers and ad - bc = 1. These transformations map the upper half plane $\mathcal{H} = \{z : \operatorname{Im} z > 0\}$ onto itself in a one-to-one manner. The groups we are considering will have the further properties: (i) for every point p on the real axis there is a sequence of different substitutions $w = V_n z$ and a point z_o such that the sequence $w_n = V_n z_o$ accumulates at p; (ii) the same statement does not hold for any point $p \in \mathcal{H}$. Groups for which (ii) hold are said to be <u>discontinuous</u> in $\mathcal{H}$. Groups for which both (i) and (ii) hold are called <u>horocyclic groups</u> (they are also called Fuchsian groups of the first kind and Grenzkreisgruppen in the literature). This terminology is not used exclusively for the case in which the region of discontinuity is the upper half plane $\mathcal{H}$.

Let the letters I, U, S, V, M, A_j denote the following two-by-two matrices (written in one line with a vertical bar separating the first row of the matrix from the second)

$$
\begin{aligned}
&I = (1\ 0 \mid 0\ \ 1), && U = (1\ \ 1 \mid 0\ \ 1),\\
(2.1)\quad &V = (a\ b \mid c\ \ d), && S = (0\ {-1} \mid 1\ \ 0),\\
&M = (\alpha\ \beta \mid \gamma\ \ \delta), \text{ and} && A_j = (a_j\ b_j \mid c_j\ d_j)
\end{aligned}
$$

where all the matrices given above are real unimodular matrices. Further, for any real λ, we write

$$(2.2)\qquad U^{\lambda} = (1\,\lambda \mid 0\ 1).$$

Also, put

$$-V = (-a,\ -b \mid -c,\ -d).$$

With each of the above matrices we can associate a linear transformation, namely

$$(2.3)\qquad w = Vz = \frac{az + b}{cz + d}\ .$$

Notice that V and -V correspond to the same linear transformation w = Vz. Thus, to any group Γ of two-by-two matrices there corresponds a group $\bar{\Gamma}$ of linear fractional transformations. -I may or may not belong to Γ. However, since our interest lies in the groups of linear fractional transformations we may assume that

$-I \in \Gamma$. If it does not then we merely adjoin it to Γ without affecting $\bar{\Gamma}$. It follows that $\bar{\Gamma} \cong \Gamma / \{I, -I\}$. There should be no confusion when we let V stand for the matrix as well as the linear transformation. One uses the same terminology for Γ as we did for $\bar{\Gamma}$; namely, Γ is discontinuous or horocyclic if and only if $\bar{\Gamma}$ is.

Two points z_1 and $z_2 \in \mathcal{H}$ are said to be congruent or equivalent with respect to Γ if there is a $V \in \Gamma$ such that $Vz_1 = z_2$. A fundamental region for Γ, $R(\Gamma)$, is a subset of $\mathcal{H}$ which satisfies: (1) $R(\Gamma)$ is a non-empty open set; (2) no two distinct points of $R(\Gamma)$ are equivalent; (3) each point of $\mathcal{H}$ is equivalent to at least one point of the closure of $R(\Gamma)$. A fundamental region for Γ can be chosen so that it is bounded by circular arcs and straight lines called sides. A vertex of a fundamental region is the common end point of two sides. In our case the fixed point of a parabolic element in Γ lies on the real axis and is called a parabolic vertex or parabolic cusp. Linear fractional transformations are classified as parabolic, elliptic, hyperbolic or loxodromic. We give the same classification to their matrices.

We shall assume that a fundamental region $R(\Gamma)$ has a finite number of sides. This is a restriction on Γ. A consequence of this assumption is that the groups we are considering are finitely generated. There is a fundamental region $R(\Gamma)$ in which each parabolic cycle consists of a single vertex; we shall always choose this fundamental region. Since there are a finite number of sides there are a finite number of inequivalent parabolic cusps. Let this number be $\sigma(\Gamma)$.

The modular group, $\Gamma(1)$, consists of all matrices $V = (a\ b \mid c\ d)$, a,b,c and d rational integers and $ad - bc = 1$. The modular group is a finitely generated zonal horocyclic group discontinuous on $\mathcal{H}$. A discontinuous group of real matrices is said to be zonal if it contains a parabolic element fixing ∞. The substitutions S and U generate $\Gamma(1)$, and U is the parabolic element fixing ∞.

A fundamental region for the modular group is the set of all $z = x + iy \in \mathcal{H}$, $y > 0$, such that $-1/2 < x < 1/2$ and $|z| > 1$. Denote this fundamental region by R(1). R(1) has a finite number of sides. Rankin [14] proves that if Γ^* is a subgroup of finite index in Γ then Γ^* is horocyclic if and only if Γ is horocyclic. Further, he proves that a fundamental region for Γ^* has a finite number of sides if and only if a fundamental region for Γ has a finite number of sides. As we stated at the outset we are interested only in subgroups of the modular group. If we add the condition that the subgroup should be of finite index, then we will know that it is horocyclic and has a fundamental region with a finite number of sides.

We reiterate, Γ is a subgroup of finite index in the modular group. $R(\Gamma)$ is a fundamental region for Γ with a finite number of inequivalent parabolic cusps p_j, $j = 1, \ldots, \sigma(\Gamma)$. We may choose $A_j \in \Gamma(1)$ so that $A_j^{-1}\infty = p_j$. Then the A_j do not belong to the same coset modulo Γ. A coset decomposition for $\Gamma(1)$ modulo Γ will be

$$\Gamma(1) = \bigcup_{j=1}^{\sigma} \bigcup_{k=1}^{\lambda_j} A_j^{-1} U^k \Gamma$$

where, of course, the integers λ_j depend upon Γ. λ_j is sometimes called the <u>width</u> of $R(\Gamma)$ at p_j. λ_j is determined as the smallest non-negative integer so that $P_j = A_j^{-1} U^{\lambda_j} A_j \in \Gamma$. P_j is parabolic and fixes p_j. P_j generates the cyclic group of all transformations in Γ which fix p_j. We now remark that if V is any real two-by-two unimodular matrix and Γ is a horocyclic group, then $V\Gamma V^{-1}$ is also a horocyclic group. If, in addition, Γ possesses a parabolic element which fixes $V^{-1}\infty$ then $V\Gamma V^{-1}$ is a zonal horocyclic group. In particular for our choice of $\Gamma \subset \Gamma(1)$, we know that $A_j \Gamma A_j^{-1}$ is a zonal horocyclic group for $j = 1, \ldots, \sigma(\Gamma)$.

A very special class of subgroups of the modular group are the <u>principal</u> <u>congruence</u> <u>subgroups</u> of <u>level</u> <u>N</u> consisting of all those elements in $\Gamma(1)$ for which

$V \equiv \pm I \pmod{N}$, where the symbol $\equiv$ denotes element-wise congruence. A congruence subgroup G of level N is a subgroup of $\Gamma(1)$ such that $\Gamma(N) \subset G \subset \Gamma(1)$ and there is no smaller N for which the same statement holds. Principal congruence subgroups are normal subgroups of finite index in $\Gamma(1)$.

A <u>local</u> <u>uniformizing</u> <u>variable</u> (hereafter <u>local variable</u>) at z_0 is an analytic function $t(z)$ of z such that a complete set of incongruent points of a neighborhood of z_0 is mapped one-to-one onto a complete neighborhood of $t = 0$. At a parabolic cusp $p_j = A_j^{-1}\infty$, $t(z) = e(A_j z/\lambda_j)$ maps a parabolic sector onto a complete neighborhood of $t = 0$. A parabolic sector in our case is the intersection of a suitable circle orthogonal to the sides of the fundamental region at p_j and the fundamental region.

In the case of even integral dimension $-r$, a multiplier system v belonging to Γ and $-r$ is simply a character on Γ. That is, for M_1 and M_2 in Γ

(2.4) $$v(M_1M_2) = v(M_1)\, v(M_2),$$

and $|v(M)| = 1$ for $M \in \Gamma$. We shall need the following properties of a multiplier system $v = v(\Gamma,-2)$ for Γ and the dimension -2. If $V \in \Gamma$

(2.5) $$v(V) = v(-V), \quad v(V^{-1}) = 1/v(V) = \bar{v}(V),$$

where $\bar{v}(V)$ is the complex conjugate of $v(V)$, and, further,

$$(2.6) \qquad v(I) = v(-I) = 1.$$

The multiplier system $v(\Gamma, -2)$ induces a multiplier system on $V \Gamma V^{-1}$, $v' = v(V \Gamma V^{-1}, -2)$, defined by

$$(2.7) \qquad v'(M') = v(M), \qquad M' = VMV^{-1} \in V \Gamma V^{-1}.$$

We will be particularly interested in the case in which $V = A_j$. In this case we write $v_j = v(A_j \Gamma A_j^{-1}, -2)$. Let p_j, $j = 1, \ldots, \sigma(\Gamma)$ be a cusp of $R(\Gamma)$; we shall assume that $p_1 = \infty$ and $A_1 = I$. λ_j was defined to be the smallest positive integer so that $P_j = A_j^{-1} U^{\lambda_j} A_j \in \Gamma$. Write $\lambda_j = \lambda(A_j, \Gamma)$, then this function satisfies

$$(2.8) \qquad \lambda_j = \lambda(A_j, \Gamma) = \lambda(I, A_j \Gamma A_j^{-1}).$$

In particular $\lambda = \lambda_1 = \lambda(I, \Gamma)$. Since $|v(P_j)| = 1$, we choose κ_j so that

$$(2.9) \qquad v(P_j) = e(\kappa_j), \qquad 0 \leq \kappa_j < 1.$$

Define $\kappa_j = \kappa(A_j, \Gamma)$, then

$$(2.10) \qquad \kappa_j = \kappa(A_j, \Gamma) = \kappa(I, A_j \Gamma A_j^{-1}).$$

In particular let $\kappa = \kappa_1 = \kappa(I, \Gamma)$.

An <u>automorphic</u> <u>form</u> $F(z)$ on Γ of dimension $-r = -2$ belonging to the multiplier system v is a meromorphic function on $\mathcal{H}$ which satisfies the transformation equation

$$(2.11) \qquad F(Vz) = v(V) \quad (cz + d)^2 \quad F(z),$$

for each $V = (a\ b \mid c\ d) \in \Gamma$. One measures the behavior of F(z) by its local variable expansion. Furthermore, F(z) must be meromorphic at each cusp p_j of the fundamental region $R(\Gamma)$. A local variable for the cusp $p_j = A_j^{-1}\infty$ is given by $e(A_j z/\lambda_j)$. F(z) is meromorphic at p_j if it has an expansion

$$(2.12) \quad F(z) = (c_j z + d_j)^{-2} \sum_{n=s}^{\infty} a_n(F, A_j, \Gamma) e((n+\kappa_j)A_j z/\lambda_j)$$

where s is a finite integer. The set of everywhere regular automorphic forms of dimension -2 for Γ and v forms a complex vector space. By everywhere regular we mean regular on $\mathcal{H}^+ = \mathcal{H} \cup \mathcal{P}$, where $\mathcal{P} = \{A^{-1}\infty : A \in \Gamma(1)$ We denote this space by $\mathcal{E}(\Gamma, -2, v)$. A subspace of this is the set of all forms which vanish at each cusp of the fundamental region. This is the space of cusp forms and is denoted by $\mathcal{E}^+(\Gamma, -2, v)$.

To prove that F(z) is meromorphic at a cusp p_j it is sufficient to show that F(z) approaches a definite limit as $z \longrightarrow p_j$ through values lying entirely within the fundamental region.*

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* Lehner, The Fourier coefficients...III, Mich. Math. J., p.67.

The <u>V transform</u> of F(z) is defined to be

$$(2.13)\qquad F_V(z) = F(z) \mid V^{-1} = (-cz + a)^{-2} F(V^{-1}z),$$

where $V^{-1} = (d,-b \mid -c,a)$. It follows that $F_V(z)$ belongs to $\mathcal{C}(V\Gamma V^{-1},-2,v')$ where v' is the multiplier system on $V\Gamma V^{-1}$ induced by v on Γ. Furthermore,

$$(2.14)\qquad F_V(z) \mid V = F(z),$$

and if $A_j^{-1}\infty = p_j$ is a cusp for $R(\Gamma)$, then

$$(2.15)\qquad F_{A_j}(z) = \sum_{n=s}^{\infty} a_n(F,A_j,\Gamma)e((n + \kappa_j)z/\lambda_j)$$

where $a_n(F,A_j,\Gamma)$ is defined in (2.12).

We define the following sets of integers

$$(2.16)\qquad \begin{aligned} \mathcal{C}(A_j,\Gamma) &= \{c : \exists\, V \in A_j\Gamma,\ V = (.\ . \mid c\ .)\}, \\ \mathcal{D}(c,A_j,\Gamma) &= \{d : \exists\, V \in A_j\Gamma,\ V = (.\ . \mid c\ d)\}, \\ \mathcal{D}_c(A_j,\Gamma) &= \{d \in \mathcal{D}(c,A_j,\Gamma) : d \in [0,c\lambda]\}, \\ \mathcal{A}_c(A_j,\Gamma) &= \{a : \exists\, V \in A_j\Gamma,\ V = (a\ . \mid c\ .), \\ &\qquad a \in [0,c\lambda_j]\}, \end{aligned}$$

where $[0,c\lambda]$ is the closed interval between 0 and $c\lambda$ (note that c may be negative), $[0,c\lambda_j]$ is defined in a similar manner. The following relation is valid

$$(2.17)\qquad \mathcal{D}(c,A_j,\Gamma) = \bigcup_{q=-\infty}^{\infty} \{d + c\lambda q : d \in \mathcal{D}_c(A_j,\Gamma)\},$$

which follows from $V_{c,d} = (\cdot\ \cdot \mid c\ d) \in A_j\Gamma$, $c \in \mathcal{C}(A_j, \Gamma)$, $d \in \mathcal{D}_c(A_j, \Gamma)$ then so is $V_{c,d}U^q = (\cdot\ \cdot \mid c, d + cq\lambda) \in A_j\Gamma$.

In the course of our investigation Kloosterman sums

(2.18) $$W_c(n + \kappa, A_j, \mu + \kappa_j) = W_c(n,\mu)$$

$$= \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \bar{\nu}(A_j^{-1}V_{c,d})e\left(\frac{(n+\kappa)d}{c\lambda} + \frac{(\mu+\kappa_j)a}{c\lambda_j}\right)$$

with $V_{c,d} = (a\ b \mid c\ d) \in A_j\Gamma$ will arise. When $\Gamma = \Gamma(1)$, $A_j = I$ and $\nu = 1$, these sums are the classical Kloosterman sums. We shall assume for the present that

(2.19) $$W_c(n,\mu) = O(|c|^{1/2+\varepsilon}), \qquad \varepsilon > 0$$

for fixed μ . In sectinn 8 we shall discuss situations in which we can make such an estimate.

We include finally the Lipschitz formula [1,p.206]. If t is a complex variable for which $\mathrm{Re}(t) > 0$ and either $0 < u < 1$, $g > 0$ or $u = 1$, $g > 1$, then

(2.20) $$\frac{(2\pi)^g}{\Gamma(g)} \sum_{m=0}^{\infty} (m+u)^{g-1}\, e(it\,(m+u)) = \sum_{n=-\infty}^{\infty} e(nu)(t+ni)^{-g}$$

where $\Gamma(g)$ is the gamma function.

Any further introductory material will be dealt with in the course of the text.

3. Convergence and Regularity

In this section we prove the uniform convergence of the series introduced in (1.2) on compact subsets of $\mathcal{H}$. Thus, these functions are regular in $\mathcal{H}$. The method follows § 3 of Lehner's paper.

Consider the series

$$(3.1) \quad \sum_{\substack{c=-\infty \\ c \in \mathcal{C}(A_j, \Gamma)}}^{\infty} \; \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c, A_j, \Gamma)}}^{\infty} \frac{e((\mu + \kappa_j) V_{c,d}\, z/\lambda_j)}{\nu(A_j^{-1} V_{c,d})(cz+d)^2},$$

where $V_{c,d} = (a\ b \mid c\ d) \in A_j\Gamma$; A_j, λ_j, κ_j and the sets $\mathcal{C}(A_j, \Gamma)$ and $\mathcal{D}(c, A_j, \Gamma)$ are defined in section 2. We assume μ is a non-zero integer. The double series (3.1) does not converge absolutely and for this reason we must define in what order the summation is to be carried out. First, however, we show that while $V_{c,d}$ is not uniquely determined by the conditions $c \in \mathcal{C}(A_j, \Gamma)$, $d \in \mathcal{D}(c, A_j, \Gamma)$, that is, by its lower line $\{c,d\}$, the terms of (3.1) are determined uniquely by these conditions.

Let $V'_{c,d}$ be another matrix in $A_j\Gamma$ with lower line $\{c,d\}$, then $V_{c,d} = U^m V'_{c,d}$. However, $V_{c,d} = A_jM$, $V'_{c,d} = A_jM'$, with $M, M' \in \Gamma$. Thus $A_jM = U^m A_jM'$, and consequently $A_j^{-1} U^m A_j = MM'^{-1} \in \Gamma$. This is a transformation fixing p_j, and hence, it must equal P_j^k. In other words

$m = k\lambda_j$. This proves that $V_{c,d} = U^{k\lambda_j} V'_{c,d}$. Substituting this into the term of (3.1) determined by $\{c,d\}$, we find

$$\frac{e((\mu+\kappa_j)V_{c,d}\, z/\lambda_j)}{\nu(A_j^{-1} V_{c,d})(cz+d)^2} = \frac{e((\mu+\kappa_j)U^{k\lambda_j} V'_{c,d}\, z/\lambda_j)}{\nu(A_j^{-1}U^{\lambda_j k} V'_{c,d})(cz+d)^2}$$

$$= \frac{e((\mu+\kappa_j)(V'_{c,d}\, z+k\lambda_j)/\lambda_j)}{\nu(A_j^{-1}U^{\lambda_j k}A_j)\cdot\nu(A_j^{-1}V'_{c,d})(cz+d)^2}$$

since $U^{k\lambda_j}w = w + k\lambda_j$. From (2.9) we see that

$$\nu(A_j^{-1}U^{k\lambda_j} A_j) = \nu(P_j^k) = e(k\,\kappa_j).$$

Thus the term of (3.1) determined by $\{c,d\}$ becomes upon simplification

$$\frac{e((\mu+\kappa_j)V'_{c,d}\, z/\lambda_j)}{\nu(A^{-1}V'_{c,d})(cz+d)^2}\,.$$

Now we introduce for the $c \neq 0$ the auxiliary series

$$(3.2)\qquad H(c,z) = \sum_{\substack{d=-\infty \\ d\in\mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)V_{c,d}\, z/\lambda_j)}{\nu(A_j^{-1}V_{c,d})(cz+d)^2}$$

which for each $c \in \mathcal{C}(A_j,\Gamma)$ is nothing more than the inner sum of (3.1). In the course of our exposition we shall prove that this series converges uniformly on compact subsets of $\mathcal{H}$.

Let the series in (3.1) be understood in the sense of

$$(3.3)\quad \delta_{A_j,I}\, e((\mu+\kappa)z/\lambda) + \lim_{K\to\infty} \sum_{\substack{c=-K \\ c\in \mathcal{C}(A_j,\Gamma)}}^{-1} H(c,z) + \lim_{K\to\infty} \sum_{\substack{c=1 \\ c\in \mathcal{C}(A_j,\Gamma)}}^{K} H(c,z)$$

where

$$(3.4)\qquad \delta_{A_j,I} = \begin{cases} 2, & \text{if } A_j = A_1 = I, \\ 0, & \text{otherwise.} \end{cases}$$

If both of the limits in (3.3) exist simultaneously, we shall define the expression (3.3) to be $G(z,\nu,A_j,\Gamma,\mu)$. The objective of this section is to prove that both limits do exist uniformly for z belonging to compact subsets of $\mathcal{H}$.

If $A_j = I$ the terms of (3.1) corresponding to $c = 0$ arise from $d = \pm 1$ because c and d are relatively prime. Thus, there are just two terms and we can choose $V_{0,\pm 1} = \pm I$. This will account for the first term of (3.3) when we show that $0 \in \mathcal{C}(A_j,\Gamma)$ if and only if $A_j = I$. Suppose $0 \in \mathcal{C}(A_j,\Gamma)$; then there is a $V \in A_j\Gamma$ with $V = (a\ b \mid 0\ d)$, which implies, $a = d = \pm 1$. That is, $U^b \in A_j\Gamma$. We can write $A_j^{-1} = M^{-1}U^{-b}$, $M \in \Gamma$ and so $A_j^{-1}\infty = M^{-1}\infty$. Then $p_j = A_j^{-1}\infty$ is equivalent to ∞, because $MA_j^{-1}\infty = MM^{-1}\infty = \infty$. Because of the way we chose the A_k, this can happen only if $A_j = I$. The converse is clear.

The following estimate of $|cz + d|$ is essential.

(3.5) $\quad |cz + d| \geq |d| \sin \delta, \qquad 0 < \delta = \arg. z < \pi.$

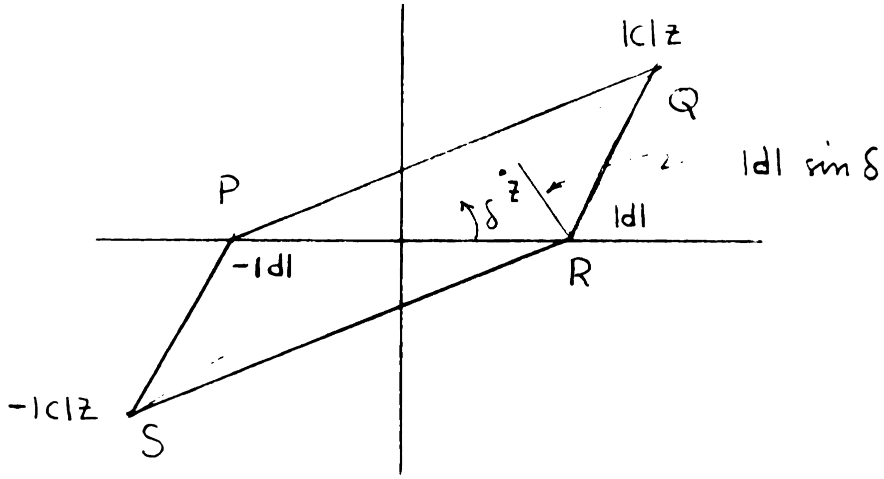

Fig. 3.1

In Fig. 3.1 we see that $|cz + d|$ is the length of one of the sides of the parallelogram PQRS with vertices $\pm |d|$, $\pm |c| z$. For, $|cz + d| = |cz - (-d)|$ is the distance from cz to -d. Thus $|cz + d|$ is the shorter side of PQRS or longer than the shorter side. $|d| \sin\delta$ is a leg of a right triangle which has the shorter side of PQRS as hypotenuse. In case z lies in the second quadrant we replace δ by $\pi - \delta$. The degenerate case is excluded since $z \in \mathcal{H}$. Also,

(3.6) $|cz + d| = \{(cx + d)^2 + c^2y^2\}^{1/2} \geq |c| y, \quad z = x + iy.$

We split H(c,z) into the sum of two series. For $c \neq 0$ we have

$$Vz = \frac{az + b}{cz + d} = \frac{a}{c} - \frac{1}{c(cz + d)} .$$

Let us define for $c \in \mathcal{C}(A_j, \Gamma)$, $c \neq 0$,

$$(3.7)\quad H_1(c,z) = \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)\,\{e[-(\mu+\kappa_j)/\lambda_j c(cz+d)]-1\}}{v(A_j^{-1}V_{c,d})(cz+d)^2}$$

and

$$(3.8)\quad H_2(c,z) = \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(c\,z+d)^2},$$

where $V_{c,d} = (a\ b \mid c\ d)$. Note that formally $H(c,z) = H_1(c,z) + H_2(c,z)$. Once convergence of H_1 and H_2 has been obtained, we will have this result. We obtain estimates of these series which involve c. Expand the second exponential in (3.7) to obtain

$$(3.9)\quad H_1(c,z) = \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \sum_{m=1}^{\infty} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)(-2\pi i)^m(\mu+\kappa_j)^m}{v(A^{-1}V_{c,d})(\lambda_j c)^m m!\,(cz+d)^{m+2}}.$$

This double series is dominated by

$$\sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \sum_{m=1}^{\infty} \frac{(2\pi)^m\,|\mu+1|^m}{|c|^m\,\lambda_j^m\,m!\,|cz+d|^{m+2}},$$

Using the estimates (3.5) and (3.6) we see for $d \neq 0$

$$|cz+d|^{-(m+2)} \leq |c|^{-(m+2)/2}y^{-(m+2)/2}|d|^{-(m+2)/2}(\sin\delta)^{-(m+2)/2},$$

and if $d = 0$

$$|cz + d|^{-(m+2)} \leq |c|^{-(m+2)} y^{-(m+2)}.$$

With these results we obtain for our dominating series

$$\sum_{m=1}^{\infty} \frac{(2\pi)^m |\mu+1|^m}{m!\, c^{2m+2} y^{2m+2}}$$

$$+ 2 \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \frac{(2\pi)^m |\mu+1|^m}{m!\, |c|^{(3m+2)/2} y^{(m+2)/2} d^{(m+2)/2} (\sin\delta)^{m+2/2}}$$

$$\leq (cy)^{-2} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi |\mu+1|}{y}\right)^m + \left\{\frac{2}{|c|^{5/2} y \sin\delta} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi |\mu+1|}{\sqrt{y \sin\delta}}\right)^m \cdot \sum_{d=1}^{\infty} \frac{1}{d^{3/2}}\right\}$$

Therefore,

$$(3.10) \quad |H_1(c,z)| \leq (cy)^{-2} \exp(2\pi |\mu+1|/y) + C\, |c|^{-5/2} (y \sin\delta)^{-1} \exp\left[\frac{2\pi |\mu+1|}{\sqrt{y \sin\delta}}\right]$$

where C is a sufficiently large constant independent of z. When we say that a series converges <u>absolutely uniformly</u> we mean that the series of absolute values converges uniformly. We have proved that $H_1(c,z)$ converges absolutely uniformly on compact subsets of $\mathcal{H}$, actually for $y \geq y_0 > 0$ and $0 < |x| \leq x_0$. Thus, $H_1(c,z)$ represents a regular function in $\mathcal{H}$.

Note that $H_2(c,z)$ of (3.8) corresponds to the missing term $m = 0$ in (3.9). The dominating series for $H_2(c,z)$ is

$$\sum_{\substack{d=-\infty \\ d\in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{1}{|cz+d|^2} .$$

Using the results (3.5) and (3.6) we find if $d \neq 0$

$$|cz+d|^{-2} \leq d^{-2}(\sin\delta)^{-2},$$

and if $d = 0$

$$|cz+d|^{-2} \leq c^{-2}\, y^{-2}.$$

Thus,

$$|H_2| \leq (cy)^{-2} + 2\sum_{d=1}^{\infty} (d \sin\delta)^{-2} ,$$

(3.11) $$|H_2(c,z)| \leqslant (cy)^{-2} + \frac{\pi^2}{3\sin^2\delta} .$$

We proved that for $|x| \leq x_0$, $y \geq y_0 > 0$ the series $H_2(c,z)$ converges absolutely uniformly. $H_2(c,z)$ is a regular function in $\mathcal{H}$. Now that we have established the convergence of H_1 and H_2 we can assert (see lines following (3.8))

(3.12) $$H(c,z) = H_1(c,z) + H_2(c,z).$$

The estimate (3.11) is not good enough for our purposes since we will want to sum on c. One first proves that the terms of H_1 and H_2 are uniquely determined by the lower line $\{c,d\}$ of $V_{c,d}$. The proof is almost identical to the case considered at the outset of this section. We make the choice of the a of $V_{c,d} = (a\ b \mid c\ d)$ unique. We saw that $V_{c,d} = U^{k\lambda_j} V'_{c,d}$, where $V'_{c,d}$ is another matrix with lower row $\{c,d\}$. That is,

$$V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & k\lambda_j \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} = \begin{pmatrix} a' + ck\lambda_j & * \\ c & d \end{pmatrix}.$$

By choosing k properly we obtain $a \in [0, c\lambda_j]$, i.e. $a \in \mathcal{A}_c(A_j, \Gamma)$. Now dividing d by $c\lambda$, $d = qc\lambda + d_1$ where $d_1 \in \mathcal{D}_c(A_j, \Gamma)$. Then

$$V_{c,d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & * \\ c & d_1 \end{pmatrix}\begin{pmatrix} 1 & q\lambda \\ 0 & 1 \end{pmatrix} = V_{c,d_1} U^{q\lambda}$$

with $c \in \mathcal{C}(A_j, \Gamma)$, $d_1 \in \mathcal{D}_c(A_j, \Gamma)$ and $a \in \mathcal{A}_c(A_j, \Gamma)$. Using these results and (2.17)

$$H_2(c,z) = \sum_{d_1 \in \mathcal{D}_c(A_j,\Gamma)} \sum_{q=-\infty}^{\infty} \frac{e((\mu+\kappa_j)a/c\lambda_j)}{v(A_j^{-1} V_{c,d_1} U^{\lambda q})(cz+d_1+c\lambda q)^2}$$

$$= \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \frac{e((\mu+\kappa_j)a/c\lambda_j)}{v(A_j^{-1} V_{c,d})} \sum_{q=-\infty}^{\infty} \frac{e(-q\kappa)}{(cz+d+c\lambda q)^2}$$

where we have used $1/v(U^{q\lambda}) = e(-q\kappa)$. The order of summation is immaterial because H_2 is absolutely convergent.

Applying the Lipschitz formula (2.20) to the inner sum of the above series, valid since g = 2, we find, whether $\kappa > 0$ or $u = \kappa = 0$,

$$\sum_{q=-\infty}^{\infty} \frac{e(-q\kappa)}{(cz+d+c\lambda q)^2} = \frac{(-i)^2}{c^2\lambda^2} \sum_{q=-\infty}^{\infty} \frac{e(q\kappa)}{(-i(z/\lambda+d/c\lambda)+qi)^2}$$

$$= \frac{(-2\pi i)^2}{c^2\lambda^2} \sum_{n=0}^{\infty} (n+\kappa) e((n+\kappa)(z/\lambda+d/c\lambda))$$

where in (2.20) we identify $t = -i(z/\lambda + d/c\lambda)$. Hence,

$$H_2(c,z) = \left\{ \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \frac{e((\mu+\kappa_j)a/c\lambda_j)}{v(A_j^{-1}V_{c,d})} \cdot \frac{(-2\pi i)^2}{c^2\lambda^2} \right\} \cdot$$

$$\cdot \left\{ \sum_{n=0}^{\infty} (n+\kappa) e\left((n+\kappa)\left(\frac{z}{\lambda}+\frac{d}{c\lambda}\right)\right) \right\}$$

$$= \frac{(-2\pi i)^2}{c^2\lambda^2} \left\{ \sum_{n=0}^{\infty} (n+\kappa) e((n+\kappa)z/\lambda) \right\} \cdot$$

$$\cdot \left\{ \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \frac{e\left[\frac{(\mu+\kappa_j)a}{c\lambda_j} + \frac{(n+\kappa)d}{c\lambda}\right]}{v(A_j^{-1}V_{c,d})} \right\}$$

$$(3.13) \quad = \left(\frac{-2\pi i}{c\lambda}\right)^2 \sum_{n=0}^{\infty} (n+\kappa) e((n+\kappa)z/\lambda) \cdot W_c(n,\mu)$$

where we have interchanged the order of summation of the finite sum and the infinite sum and have introduced the Kloosterman sum (2.18). The above interchange of summation is valid because of the absolute convergence of the double series involved (recall that $\mathcal{D}_c(A_j, \Gamma)$ is finite for fixed c and j).

At this point we derive an estimate for $H_2(c,z)$. The series (3.13) is dominated by

$$\frac{4\pi^2}{c^2\lambda^2} \sum_{n=0}^{\infty} (n+\kappa) \exp(-2\pi(n+\kappa)y/\lambda) \; |W_c(n,\mu)|.$$

Now we use the estimate (2.19) for the Kloosterman sum and the fact that $0 \leq \kappa < 1$ to obtain a dominating series

$$\frac{4\pi^2}{c^2} \sum_{n=0}^{\infty} (n+1) \exp(-2\pi\, ny/\lambda) \; C_\varepsilon \; |c|^{1/2+\varepsilon} .$$

Thus,

$$(3.14) \quad \left|H_2(c,z)\right| \leq \frac{C_\varepsilon}{|c|^{3/2-\varepsilon}} \sum_{n=0}^{\infty} (n+1) \exp(-2\pi ny/\lambda)$$

$$= C_\varepsilon \; |c|^{-3/2+\varepsilon} \; (1 - e^{-2\pi y/\lambda})^{-2}.$$

This is our desired estimate involving c.

It is now easy to see using the estimates (3.10) and (3.14) that the series

$$\sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma) \\ c\neq 0}}^{\infty} H_1(c,z) \qquad \text{and} \qquad \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma) \\ c\neq 0}}^{\infty} H_2(c,z)$$

converge absolutely uniformly for $y \geq y_0 > 0$, $|x| \leq x_0$; we proved that H_1 and H_2 are regular in $\mathcal{H}$. Hence the above sums are regular in $\mathcal{H}$. We have, moreover, from (3.12)

$$\sum_{\substack{c=1 \\ c\in\mathcal{C}(A_j,\Gamma)}}^{K} |H(c,z)| \le \sum_{\substack{c=1 \\ c\in\mathcal{C}(A_j,\Gamma)}}^{K} |H_1(c,z)| + \sum_{\substack{c=1 \\ c\in\mathcal{C}(A_j,\Gamma)}}^{K} |H_2(c,z)| ,$$

and

$$\sum_{\substack{c=-K \\ c\in\mathcal{C}(A_j,\Gamma)}}^{-1} |H(c,z)| \le \sum_{\substack{c=-K \\ c\in\mathcal{C}(A_j,\Gamma)}}^{-1} |H_1(c,z)| + \sum_{\substack{c=-K \\ c\in\mathcal{C}(A_j,\Gamma)}}^{-1} |H_2(c,z)| ;$$

therefore, the limits in (3.3) exist uniformly for $|x| \le x_0$, $y \ge y_0 > 0$. This completes the proof of the lemma.

Lemma 1: The functions $G(z,\nu,A_j,\Gamma,\mu)$ defined in (3.3) are regular in $\mathcal{H}$. Furthermore, we have the expressions

$$(3.15)\quad G(z,\nu,A_j,\Gamma,\mu) = \delta_{A_j,I}e((\mu+\kappa)z/\lambda) + \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma),c\neq 0}}^{\infty} H(c,z),$$

and

$$(3.16)\quad G(z,\nu,A_j,\Gamma,\mu) = \delta_{A_j,I}e((\mu+\kappa)a/\lambda) + \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma),c\neq 0}}^{\infty} H_1(c,z)$$

$$+ \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma),c\neq 0}}^{\infty} H_2(c,z) .$$

The three infinite series appearing in (3.15) and (3.16) are absolutely and uniformly convergent in each compact subset of $\mathcal{H}$. $H(c,z)$ is defined in (3.2), $H_1(c,z)$

in (3.7) and $H_2(c,z)$ is defined in (3.8).

For later considerations we discuss $H_1(z)$ defined by the series

$$H_1(z) = \sum_{\substack{c=-\infty \\ c \in \mathcal{C}(A_j,\Gamma) \\ c \neq 0}}^{\infty} \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)\left\{e\left[\frac{-(\mu+\kappa_j)}{c\,\lambda_j(cz+d)}\right]-1\right\}}{v(A_j^{-1}\,V_{c,d})(cz+d)^2}. \tag{3.17}$$

Expanding the second exponential we get a triple series (3.17a)

$$\sum_{\substack{c=-\infty \\ c \in \mathcal{C}(A_j,\Gamma) \\ c \neq 0}}^{\infty} \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c,A_j,\Gamma)}}^{\infty} \sum_{m=1}^{\infty} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)(-2\pi i(\mu+\kappa_j))^m}{v(A_j^{-1}\,V_{c,d})\; m!\;(c\,\lambda_j)^m\,(cz+d)^{m+2}}.$$

Using an estimation similar to the one following (3.9) we deduce easily that this triple series converges absolutely uniformly for $y \geq y_0 > 0, |x| \leq x_0$; moreover, we get for the sum of the series, $H_1(z)$, the estimate

$$|H_1(z)| \leq Cy^{-2}\exp(2\pi|\mu+1|/y) + C(y\sin\delta)^{-1}\exp\left[\frac{2\pi|\mu+1|}{\sqrt{y\sin\delta}}\right] \tag{3.18}$$

where the constants are independent of z.

4. The Rademacher Lemma

We now come to the main tool of this paper, a Rademacher type lemma, which allows us to rearrange certain conditionally convergent double series. In part we follow the method of Lehner [6, § 4]. Some preliminary investigations will be required before we can state the lemma.

By a <u>lattice point for $A_j\Gamma$</u> we shall mean an ordered pair (c,d) of integers obtained from the lower row of a matrix $V_{c,d} = (a\ b \mid c\ d) \in A_j\Gamma$. Let $\mathcal{O}(A_j, \Gamma)$ be the set of all lattice points for $A_j\Gamma$:

(4.1) $\mathcal{O}(A_j, \Gamma) = \{(c,d): \exists\ V = (a\ b \mid c\ d) \in A_j\Gamma\}$.

Let $\mathcal{S}_K = \{w = u + iv: |u| < K, |v| < K\}$ and

(4.2) $\mathcal{O}_K(A_j, \Gamma) = \mathcal{O}(A_j, \Gamma) \cap \mathcal{S}_K$.

Let $\mathcal{M}$ consist of the following matrices $M = (\alpha\ \beta \mid \gamma\ \delta) \in \Gamma$:

(4.3) $M = I$;

(4.4) $\gamma = 1, \quad 0 \leq \alpha, \delta < \lambda \quad$ and $\quad \alpha^2 + \delta^2 > 0$;

(4.5) $\gamma > 1, \quad 0 < \alpha, \delta < \gamma\lambda$.

If $M = (\alpha\ \beta \mid \gamma\ \delta) \in \Gamma$ and $\gamma \geq 1$ then the conditions in (4.4) and (4.5) imply that $\alpha \in \mathcal{a}_\gamma(I, \Gamma)$ and $\delta \in \mathcal{D}_\gamma(I, \Gamma)$. If $\gamma = 1$ then from the conditions in (4.4) we see that

α and δ are not simultaneously 0; thus, $S = (0\ -1 \mid 1\ 0)$ is never in $\mathcal{M}$. Furthermore, in the case $\gamma > 1$, $\beta > 0$ as can be seen from the conditions in (4.5) along with $\alpha\delta - \beta\gamma = 1$.

Each $V \in \Gamma$ has one of the following representations:

(4.6) $\qquad V = \pm\, U^{m\lambda} M U^{n\lambda}$ with $M \in \mathcal{M}$,

or, possibly, if $S = (0\ -1 \mid 1\ 0) \in \Gamma$

(4.7) $\qquad V = \pm\, U^{m\lambda} S U^{n\lambda}$.

The representation for each $V \in \Gamma$ is essentially unique.

To prove this result let $V = (a\ b \mid c\ d) \in \Gamma$ be an arbitrary element. We may suppose $c \geq 0$, for otherwise $-V$ has $-c > 0$. Now consider $U^{-m\lambda} V U^{-n\lambda} = (a-mc\lambda\ \ * \mid c,\ d-nc\lambda\ )$. If $c \neq 0$ there are unique choices of m and n so that $0 \leq a-mc\lambda < c\lambda$ and $0 \leq d-nc\lambda < c\lambda$. Let $M = U^{-m\lambda} V U^{-n\lambda}$. If $c = \gamma > 1$, then $\alpha = a-mc\lambda$ and $\delta = d-nc\lambda$ satisfy the inequalities of (4.5) (neither α or δ is 0 since $(\alpha, \gamma) = (\gamma, \delta) = 1$); thus, $M \in \mathcal{M}$. If $c = \gamma = 1$, either both a and d are integral multiples of λ or this is not the case. In the first instance $a - mc\lambda = 0$ and $d - nc\lambda = 0$, hence, $U^{-m\lambda} V U^{-n\lambda} = S = (0\ -1 \mid 1\ 0)$. This case leads to the representation (4.7). On the other hand, if a and d are not both multiples of λ , $M = U^{-m\lambda} V U^{-n\lambda} \in \mathcal{M}$ since the inequalities of (4.4) are satisfied. Finally, if $c = 0$ then $a = d = \pm 1$, hence, $V = \pm\, U^{k\lambda}$.

We can think of $M \in \mathcal{M}$ as acting on the u,v plane not as a linear fractional transformation, but as an affine transformation given by

(4.8) $(u,v)\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = (u',v')$, $u' = \alpha u + \gamma v$, $v' = \beta u + \delta v$.

If $\mathcal{S}$ is a set in the u,v plane let $\mathcal{S}M = \{(u,v)M: (u,v) \in \mathcal{S}\}$. In particular,

(4.9) $\mathcal{O}_K(A_j,\Gamma)M = \{(\alpha c+\gamma d, \beta c+\delta d): (c,d) \in \mathcal{O}_K(A_j,\Gamma)\}$.

The square $\mathcal{S}_K$, with sides $u = \pm K$, $v = \pm K$ is mapped by M onto a parallelogram whose sides have the equations

(4.10) $\alpha v - \beta u = \pm K$ and $\gamma v - \delta u = \pm K$.

As an example, suppose $M \in \mathcal{M}$, $\gamma - \alpha < 0$ and $\delta - \beta < 0$, then the image of $\mathcal{S}_K$, $\mathcal{S}_K M$, is given in Fig. 4.1.

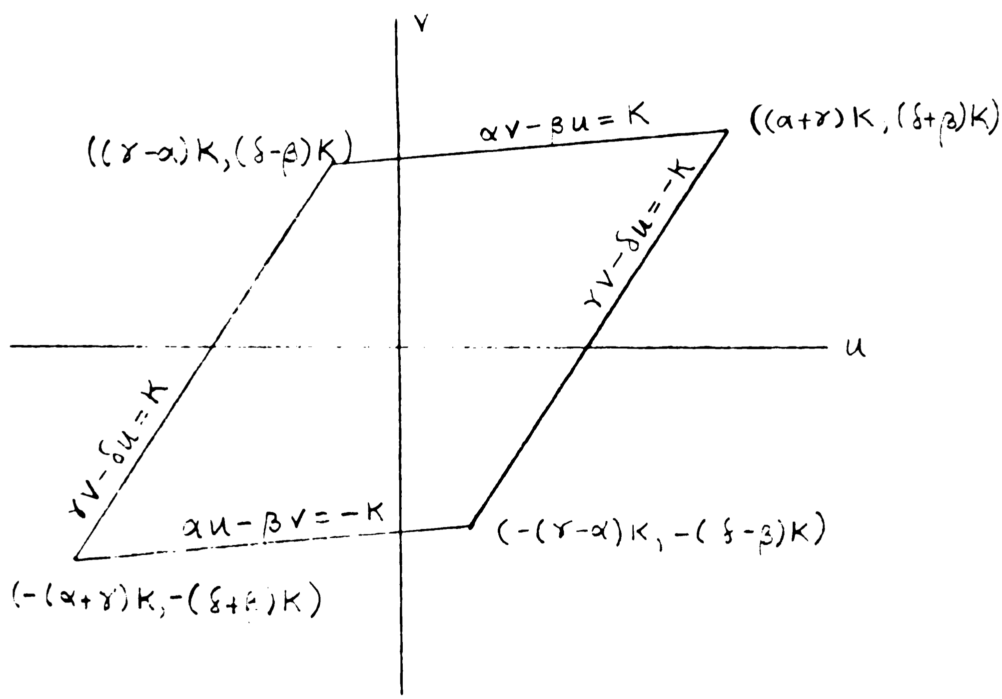

Fig. 4.1

We see that

$$(4.11) \qquad \mathcal{O}(A_j, \Gamma)M = \mathcal{O}(A_j, \Gamma)$$

for each $M \in \Gamma$. For, if $V = (a\ b \mid c\ d) \in A_j\Gamma$ then so is $VM \in A_j\Gamma$; thus, the lower line of VM is in $\mathcal{O}(A_j, \Gamma)$. This proves $\mathcal{O}(A_j, \Gamma) \subseteq \mathcal{O}(A_j, \Gamma)M$. Next, let $(c,d) \in \mathcal{O}(A_j, \Gamma)M$; in order to show that $(c,d) \in \mathcal{O}(A_j, \Gamma)$ we must show there is a $V \in A_j\Gamma$ with lower row (c,d). Since $(c,d) \in \mathcal{O}(A_j, \Gamma)M$, $(c,d) = (c',d')M$ where $(c',d') \in \mathcal{O}(A_j, \Gamma)$. Let $V' = (a'b' \mid c'd') \in A_j\Gamma$. $V'M^{-1} = (a^*b^* \mid c\ d) \in A_j\Gamma$.

If $\mathcal{S}$ and $\mathcal{T}$ are any two subsets of the plane, we have $(\mathcal{S} \cap \mathcal{T})M = \mathcal{S}M \cap \mathcal{T}M$. Thus,

$$(4.12) \qquad \mathcal{O}_K(A_j, \Gamma)M = \mathcal{S}_K M \cap \mathcal{O}(A_j, \Gamma).$$

In order to make the notation more concise we will write $\sum_{c=-M}^{K}{}_j \ \sum_{d=-\infty}^{\infty}{}_j$ in place of the more complicated summation conditions

$$\sum_{\substack{c=-M \\ c \in \mathcal{C}(A_j, \Gamma)}}^{K} \quad \sum_{\substack{d=-\infty \\ d \in \mathcal{D}(c, A_j, \Gamma)}}^{\infty}$$

If a prime (') appears on a summation symbol it means the summation variable c does not take the value 0. K and M may take on any values so long as the resulting sum has meaning.

We are now ready to state the Rademacher lemma.

Lemma 2: Let M be in the class $\mathcal{M}$ defined in (4.3), (4.4) and (4.5), then

$$\lim_{K\to\infty} \sum_{c=-K}^{K}{}_j \sum_{d=-\infty}^{\infty}{}_j \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(cz+d)^2}$$

$$(4.13) \qquad = \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)M} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(cz+d)^2}$$

where $V_{c,d} = (a\ b \mid c\ d) \in A_j\Gamma$ and $\mathcal{O}_K(A_j,\Gamma)M$ is defined in (4.9).

In terms of the notation of section 3(see (3.8)) this lemma becomes

$$(4.14) \quad \sum_{c=-\infty}^{\infty}{}'_j H_2(c,z) = \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)M}{}' \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(cz+d)^2}$$

We have shown that the series in the left member of (4.14) is absolutely uniformly convergent for $y \geq y_0 > 0$. Further, for fixed $M = (\alpha\ \beta \mid \gamma\ \delta) \in \mathcal{M}$ the series

$$\sum_{|c|\ >(\alpha+\gamma)K}{}_j H_2(c,z)$$

can be made arbitrarily small by choosing K sufficiently large. This is because $\alpha+\gamma$ is <u>definitely positive</u> by our choice of $\mathcal{M}$, (4.3), (4.4) and (4.5). Moreover, the sum

$$(4.15)\quad \sum_{c=-(\alpha+\gamma)K+1}^{(\alpha+\gamma)K-1}{}'_j H_2(c,z) = \sum_{c=-(\alpha+\gamma)K-1}^{c=(\alpha+\gamma)K-1}{}'_j \sum_{d=-\infty}^{\infty}{}_j \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{\nu(A_j^{-1}V_{c,d})(cz+d)^2}$$

is an absolutely convergent double series for fixed M $\in \mathcal{M}$ and K. This can be seen by using the estimate (3.11) obtained by taking the series of absolute values of terms of $H_2(c,z)$ and then performing the finite sum on c. We shall want to rearrange this series, but first some new notation will be introduced.

For fixed $M \in \mathcal{M}$ define the regions $\Omega_i = \Omega_i(u,v)$ $\Omega'_i = \Omega'_i(u,v)$, $i = 1,2$, as follows:

$$\Omega_1 = \{(u,v):\ (\gamma-\alpha)K \leq u < (\gamma+\alpha)K,\ \ \alpha v - \beta u \geq K\},$$
$$\Omega_2 = \{(u,v):\ -(\alpha+\gamma)K < u < (\gamma-\alpha)K,\ \ \gamma v - \delta u \geq K\},$$
$$\Omega'_1 = \{(u,v):\ -(\alpha+\gamma)K < u \leq -(\gamma-\alpha)K,\ \alpha v - \beta u \leq -K\},$$
$$\Omega'_2 = \{(u,v):\ -(\gamma-\alpha)K < u < (\alpha+\gamma)K,\ \ \gamma v - \delta u \leq -K\}.$$

The cases where $\alpha = 0$ and $M = I$ merit special attention. If $\alpha = 0$ set $\Omega_1 = \Omega'_1 = \emptyset$. If $M = I$, set $\Omega_2 = \Omega_2 = \emptyset$. The regions are given in Fig. 4.2, we have used the same choice of M as in Fig. 4.1.

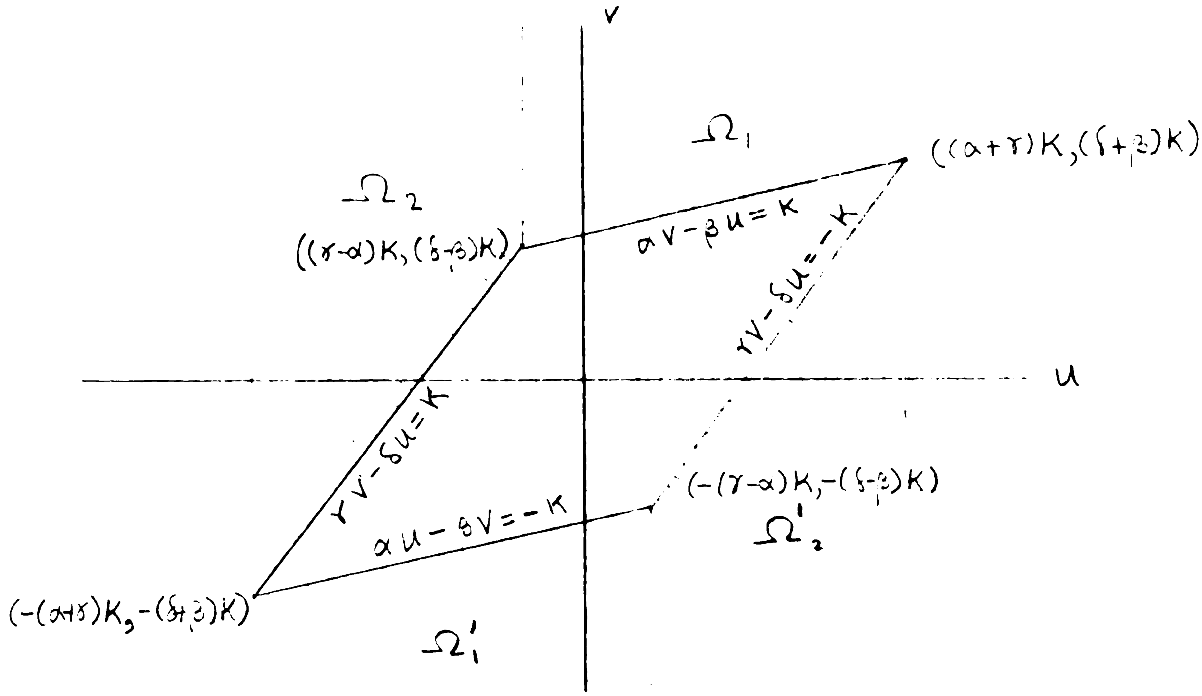

Fig. 4.2

Define

(4.16) $W_i(K) = \sum_{\Omega_i(c,d)} \frac{g(c,d)}{(cz+d)^2}$, $W_i'(K) = \sum_{\Omega_i'(c,d)} \frac{g(c,d)}{(cz+d)^2}$,

$i = 1,2$, where

(4.17) $g(c,d) = \tilde{\nu}(A_j^{-1}V_{c,d})e((\mu+\kappa_j)a/c\,\lambda_j)$

for $(c,d) \in \mathcal{O}(A_j,\Gamma)$. The summation conditions indicated in (4.16) mean summation is performed over all lattice points for $A_j\Gamma$ which lie in the region Ω_i or Ω'_i. If $\alpha = 0$ set $W_1 = W_1' = 0$. In case $M = I$ set $W_2 = W_2' = 0$. One could check directly, using the methods of section 3, that these series are absolutely uniformly convergent for $y \geq y_0 > 0$. Their convergence will come out in our development. Notice that the regions Ω_i and Ω'_i $(i=1,2)$

are symmetric with respect to the origin. This implies

(4.18) $$W_i(K) = W_i'(K) \qquad (i = 1,2).$$

Indeed, if $(c,d) \in \mathcal{O}(A_j, \Gamma)$ appears in the summation $W_i(K)$ $(i = 1,2)$, then $(-c,-d) \in \mathcal{O}(A_j, \Gamma)$ since $-I \in \Gamma$, and because of the symmetry $(-c,-d)$ will be a lattice point for $A_j\Gamma$ appearing in the sum $W_i'(K)$ $(i = 1,2)$. Furthermore, because the terms of the series depend only upon the lower row of the matrices involved we can assume that $V_{-c,-d} = -V_{c,d}$. Recall that $\mathcal{V}(-I) = \mathcal{V}(I) = 1$. Thus,

$$\frac{g(c,d)}{(cz+d)^2} = \frac{e((\mu+\kappa_j)(-a/-c\ \lambda_j)}{\mathcal{V}(A_j^{-1}V_{-c,-d}(-I))(cz+d)^2} = \frac{g(-c,-d)}{(-cz-d)^2}$$

The parallelogram of Fig. 4.2 is simply the boundary of $\mathcal{S}_K M$. Thus, because of (4.12) we can write

$$\sum_{c=-(\alpha+\gamma)K-1}^{(\alpha+\gamma)K-1}{}_j' H_2(c,z) = 2W_1(K) + 2W_2(K)$$

(4.19) $$+ \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)M}{}' \frac{e((\mu+\kappa_j)a/c\ \lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

Now write

$$\sum_{c=-\infty}^{c=\infty}{}_j' H_2(c,z) = \sum_{|c|<(\gamma+\alpha)K}{}_j' H_2(c,z) + \sum_{|c|\geq(\alpha+\gamma)K}{}_j H_2(c,z);$$

therefore, using (4.19) we get

$$\sum_{c=-\infty}^{\infty}{}'_j H_2(c,z) - \sum_{(c,d)\in \mathcal{O}_K(A_j,\Gamma)M}{}' \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{\nu(A_j^{-1}V_{c,d})(cz+d)^2}$$

$$= 2W_1(K) + 2W_2(K) + \sum_{|c|\geq(\alpha+\gamma)K}{}_j H_2(c,z) \tag{4.20}$$

Therefore, in order to prove the lemma we must prove that the three terms on the right side of (4.20) can be made small by choosing K sufficiently large. As we have already remarked this can be done for the last term. Thus, it remains to prove that $W_1(K)$ and $W_2(K)$ can be made arbitrarily small by choosing K sufficiently large. This task will occupy us for most of this section.

Following Lehner [6, p. 77] we extend the definition of g(c,d) by setting

$$g(c,d) = 0 \quad \text{for } (c,d) \notin \mathcal{O}(A_j,\Gamma), \tag{4.17a}$$

c and d rational integers. For $(c,d)\in \mathcal{O}(A_j,\Gamma)$

$$g(c,d+c\lambda) = \bar{\nu}(A_j^{-1}V_{c,d+c\lambda})\, e((\mu+\kappa_j)a/c\,\lambda_j),$$

but we may choose $V_{c,d+c} = V_{c,d}U^{\lambda}$, therefore,

$$g(c,d+c\lambda) = \bar{\nu}(A_j^{-1}V_{c,d})\,\bar{\nu}(U^{\lambda})\, e((\mu+\kappa_j)a/c\,\lambda_j)$$

$$= e(-\kappa)g(c,d).$$

We used the fact that $V_{c,d} = (a\ b \mid c\ d)$ and $V_{c,d+c} = (a\ * \mid c\ *)$. We now define a periodic function

Let

$$\varphi(c,d) = e(d\kappa/c\lambda)\, g(c,d) \tag{4.21}$$

for c and d rational integers. Since

$$\varphi(c,d+c\lambda) = e(\kappa)e(d\kappa/c\lambda)g(c,d+c\lambda) = e(d\kappa/c\lambda)g(c,d),$$

$\varphi(c,d)$ is periodic in d with period $c\lambda$. Therefore, it has a finite Fourier expansion

$$\varphi(c,d) = \sum_{k(c\lambda)} B_k e(kd/c\lambda), \quad B_k = \frac{1}{|c|\lambda} \sum_{d(c\lambda)} \varphi(c,d)e(-kd/c\lambda), \tag{4.22}$$

where in each case the sums are extended over a complete residue system modulo $c\lambda$, say 0, 1, ..., $|c|\lambda - 1$. From the first equation of (4.22) and the definition of $\varphi(c,d)$, (4.21), we see

$$g(c,d) = \sum_{k(c\lambda)} B_k\, e((k-\kappa)d/c\lambda). \tag{4.23}$$

From the second equation of (4.22) and the definitions of $\varphi(c,d)$, (4.21), and g(c,d), (4.17) and (4.17a), we obtain

$$B_k = \frac{1}{|c|\lambda} \sum_{d=0}^{|c|\lambda-1}{}_j\, \bar{\nu}(A_j^{-1}V_{c,d})e((\mu+\kappa_j)a/c\lambda_j+(-k+\kappa)d/c\lambda) \tag{4.24}$$

The summation conditions mean $d \in \mathcal{D}(c,A_j,\Gamma)$ and $0 \leq d < |c|\lambda$. However, the above sum is periodic in d with period $|c|\lambda$, thus, the finite sum in (4.24) is nothing more than the Kloosterman sum of (2.18).

$$B_k = \frac{1}{|c|\lambda} W_c(-k+\kappa, A_j, \mu+\kappa_j). \tag{4.25}$$

If we use the extended definition for $g(c,d)$ in (4.16), we can drop the conditions on the summation variables symbolized by the j on the summation sign; that is, we no longer require $c \in \mathcal{E}(A_j, \Gamma)$ and $d \in \mathcal{D}(c, A_j, \Gamma)$. Now insert for $g(c,d)$ in (4.16) the finite Fourier expansion (4.23) to obtain

$$(4.26) \qquad W_1(K) = \sum_{c=(\gamma-\alpha)K}^{(\alpha+\gamma)K-1} \; \sum_{k=0}^{|c|\lambda-1} B_k \sum_{\alpha d - \beta c \geq K} \frac{e((k-\kappa)d/c\lambda)}{(cz+d)^2}$$

$$(4.27) \qquad W_2(K) = \sum_{c=-(\gamma+\alpha)K-1}^{(\gamma-\alpha)K-1} \; \sum_{k=0}^{|c|\lambda-1} B_k \sum_{\gamma d - \delta c \geq K} \frac{e((k-\kappa)d/c\lambda)}{(cz+d)^2}$$

The dependence of the B_k on c has been suppressed by the notation; however, it is clearly present as is shown in (4.24). It is in the above form that we will make our estimate on $W_1(K)$.

In making our estimate on the inner sums of (4.26) and (4.27), we will want to identify two cases. First, we shall suppose d may take small values and c is bounded away from 0 by a multiple of K, and secondly, c may take on small values and d is bounded away from 0 by a multiple of K. We are excluding the case that both c and d can take small values. That these two situations, and only these two situations, are realized is a property of the class $\mathcal{M}$. The proof of this statement is deferred until later.

Define

$$(4.28)\qquad T(k,c,K) = \sum_{d=QK}^{\infty} \frac{e((k-\kappa)d/c\lambda)}{(cz+d)^2}$$

where $Q = Q(c)$ is defined so that QK is the lower limit of summation on the inner most sum (on d) in the equations (4.26) and (4.27) for W_1 and W_2 respectively. Notice that $Q(c)$ depends upon M. We assume that either:

$$(4.29)\qquad \begin{array}{ll} \text{(I)} & -\infty < Q(c) < +\infty \quad \text{with} \quad |c| > RK, \\ & \quad R \text{ a positive constant; or} \\ \text{(II)} & 0 < Q(c) < +\infty. \end{array}$$

These two situations are not mutually exclusive. In the proof of this lemma we always exclude $c = 0$; this comes from the fact that the summation variable c in the left member of (4.13) is not 0. Notice further, that in (I) $d \neq 0$ since $|c| > 1$ (for sufficiently large K). and $(c,d) = 1$.

We intend to make an upper estimate for $T(c,k,K)$. This estimate will be carried out in four stages listed below:

$$(4.30)\qquad \begin{array}{ll} (\text{I})_1 & \text{situation (I) with } k = 0, \\ (\text{I})_2 & \text{situation (I) with } 1 \leq k \leq |c|\lambda - 1; \\ (\text{II})_1 & \text{situation (II) with } k = 0, \\ (\text{II})_2 & \text{situation (II) with } 1 \leq k \leq |c|\lambda - 1. \end{array}$$

In order to carry out this estimation we introduce some preliminary material. Let $S_d = \sum_{t=0}^{d} e(\sigma(k-\kappa)t/c\lambda)$

with $\sigma = \pm 1$. Then $S_0 = 1$ and for $d \geq 1$

$$(4.31) \quad S_d = \sum_{t=0}^{d} e(\sigma(k-\kappa)t/c\lambda) = \frac{1-e(\sigma(k-\kappa)(d+1)/c\lambda)}{1 - e(\sigma(k-\kappa)/c\lambda)}$$

provided $k - \kappa \neq 0$. Using the inequality

$$\sin \pi x \geq \min. \{2x,\ 2 - 2x\} \qquad \text{for } 0 < x < 1,$$

we find that for k and κ in their ranges (see (4.30)) and $k - \kappa \neq 0$

$$|S_d| \leq \left\{\sin \left|\frac{(k-\kappa)\pi}{c\lambda}\right|\right\}^{-1} \leq \left\{\min[2\,|(k-\kappa)/c\lambda|\ ,2-2\,|(k-\kappa)/c\lambda|]\right\}^{-1}.$$

Thus,

$$(4.32) \quad |S_d| \leq |c|\lambda/2 \left\{|k-\kappa|^{-1} + [\,|c|\lambda - |k-\kappa|\,]^{-1}\right\}.$$

This inequality is valid for $d \geq 1$. Under the conditions on k and κ the right member of (4.32) exceeds 1, therefore, this inequality still holds if $d = 0$.

Recall the estimates (3.5) and (3.6) for $|cz + d|$. Let $\omega = \omega(z) = \min. \{\sin \delta', y\}$ where $0 < \delta' = \arg. z < \pi$. Combining these estimates

$$(4.33) \quad |cz+d| \geq |d| \sin \delta' \geq |d|\omega \text{ and } |cz+d| \geq |c|\, y \geq |c|\omega,$$

we conclude that for each η with $0 \leq \eta \leq 1$

$$(4.34) \quad |cz + d| \geq |c|^{1-\eta} |d|^{\eta}.$$

We continue with the estimation on $T(k,c,K)$.

$(I)_1$ Suppose we are in case $(I)_1$. Then with $\eta = 5/8$ we get

$$(4.35)\quad |T(0,c,K)| \le \sum_{d=QK}^{\infty} |cz+d|^{-2} \le \sum_{d=QK}^{\infty} \omega^{-2} |d|^{-5/4} \; |c|^{-3/4}$$

$$\le \sum_{d=-\infty}^{\infty} \omega^{-2} |d|^{-5/4} |c|^{-3/4} < C \; |c|^{-3/4},$$

$C = C(z)$ is a general positive constant depending on the parameters indicated.

$(I)_2$ Suppose we are in case $(I)_2$. In order to use the preliminary material uniformly we decompose the sum $T(k,c,K)$ into two series so that in each case the summation index d will be positive. If $Q(c) < 0$, write

$$T(k,c,K) = T_1(k,c,K) + T_2(k,c,K)$$

with

$$T_1 = T_1(k,c,K) = \sum_{d=QK}^{-1} (cz+d)^{-2}\, e((k-\kappa)d/c\lambda)$$

$$= \sum_{d=1}^{-QK} (cz-d)^{-2}\, e(-(k-\kappa)d/c\,\lambda)$$

and

$$T_2 = T_2(k,c,K) = \sum_{\substack{d=QK \\ d \ge 1}}^{\infty} (cz+d)^{-2}\, e((k-\kappa)d/c\,\lambda)$$

In case $Q(c) > 0$ define $T_1 = 0$. In T_1 and T_2 we replace the exponential by $S_d - S_{d-1}$. Then

$$T_1 = \sum_{d=1}^{-QK} (S_d - S_{d-1})(cz - d)^{-2}$$

with S_d defined in (4.31) for $\sigma = -1$.

Thus,

$$T_1 = \sum_{d=1}^{-QK-1} S_d[(cz-d)^{-2}-(cz-d-1)^{-2}] + S_{-QK}(cz-QK)^{-2}$$
$$-S_0(cz-1)^{-2}$$

Therefore,

$$|T_1| \le \sum_{d=1}^{-QK-1} |S_d| \{|cz-d|\,|cz-d-1|\}^{-1}\{|cz-d|^{-1} + |cz-d-1|^{-1}\}$$
$$+ |S_{-QK}|\,|cz-QK|^{-2} + |S_0|\,|cz-1|^{-2}$$

Using the estimates (4.32) and (4.34) with $\eta = 1/4, 0, 1$,

$$|T_1| \le \{|c|\lambda/2\,(|k-\kappa|^{-1} + [|c|\lambda - |k-\kappa|]^{-1})\}$$
$$\left\{\sum_{d=1}^{-QK-1} (2\omega^3\, d(d+1)^{1/4}\,|c|^{7/4})^{-1} + 2\omega^{-2}c^{-2}\right\};$$

hence,

$$(4.36)\quad |T_1| \le \{|k-\kappa|^{-1} + [|c|\lambda - |k-\kappa|]^{-1}\}\cdot C\cdot|c|^{-3/4}.$$

By similar methods an upper estimate of T_2 is made.
Let $Q' = \max\{QK, 1\}$, then with $\sigma = 1$ in the definition of S_d

$$T_2 = \sum_{d=Q'}^{\infty} S_d[(cz+d)^{-2} - (cz+d+1)^{-2}]$$
$$-S_{Q'-1}(cz+Q'+1)^{-2}.$$

We see that

$$|T_2| \le \{(|c|\lambda/2)(|k-\kappa|^{-1} + [|c|\lambda - |k-\kappa|]^{-1})\}$$
$$\left\{\sum_{d=Q'}^{\infty}[|cz+d|\,|cz+d+1|]^{-1}[|cz+d|^{-1}+|cz+d+1|^{-1}] + |cz+Q'+1|^{-2}\right\},$$

hence,

(4.37) $|T_2| < \{|k-\kappa|^{-1} + [\,|c|\lambda \quad - |k-\kappa|]^{-1}\} C\cdot |c|^{-3/4}$.

Combining the results of (4.36) and (4.37),

(4.38) $|T(k,c,K)| < C\,|c|^{-3/4}\{\,|k-\kappa|^{-1} + [\,|c|\lambda - |k-\kappa|]^{-1}\}$

for the situation $(I)_2$.

Now we turn to (II), namely $0 < Q < +\infty$.

$(II)_1$ Let the conditions of $(II)_1$ prevail. Then as in (4.35),

$$|T(0,c,K)| < \sum_{d=QK}^{\infty} d^{-5/4}\,|c|^{-3/4}\,\omega^{-2}.$$

Using an integral estimation for the series on d, we find

(4.39) $$|T(0,c,K)| < C\,|c|^{-3/4}\,K^{-1/4}.$$

$(II)_2$ Now turn to case $(II)_2$. Then as before

$$T(k,c,K) = \sum_{d=QK}^{\infty} S_d\left[(cz+d)^{-2} - (cz+d+1)^{-2}\right]$$

$$-S_{QK-1}\,(cz + QK)^{-2}$$

with $\sigma = 1$ in the definition of S_d. Making the usual upper estimate,

$$|T(k,c,K)| \leq \{|c|\lambda/2\,(\,|k-\kappa|^{-1} + [\,|c|\lambda - |k-\kappa|]^{-1})\}$$

$$\{\sum_{d=QK}^{\infty} \omega^{-3}\,d^{-5/4}\,|c|^{-7/4} + \omega^{-2}\,(QK)^{-1/4}\,|c|^{-7/4}\}.$$

Thus,

(4.40) $|T(k,c,K)| < C\,|c|^{-3/4}\,K^{-1/4}\{\,|k-\kappa|^{-1} + [\,|c|\lambda - k-\kappa|]^{-1}\}$.

We see

$$(4.41) \qquad W_i(K) = \sum_{c} \sum_{k=0}^{|c|\lambda-1} B_k \, T(k,c,K) \qquad (i=1,2)$$

provided c is summed over the proper set of integers - as dictated by (4.26) and (4.27) according as i = 1 or 2 respectively. We have essentially two estimates for T(k,c,K) corresponding to (I) and (II). Let

$$(4.42) \qquad T(c,K) = B_0 T(0,c,K) + \sum_{k=1}^{|c|\lambda-1} B_k \, T(k,c,K).$$

An estimate for B_k is obtained from the relation (4.25) and assumption (2.20), the estimate for the Kloosterman sums. We get

$$(4.43) \qquad B_k = O(|c|^{-1/2+\varepsilon}), \; k = 0, 1, \ldots, |c|\lambda - 1$$

where the constant in the O-symbol depends upon μ and ε.

From (4.43), (4.35) and (4.37), we find for (I)

$$(4.44) \qquad |T(c,K)| < C_\varepsilon \, |c|^{-5/4+\varepsilon}$$

$$+ \left\{ C_\varepsilon \, |c|^{-5/4+\varepsilon} \sum_{k=1}^{|c|\lambda-1} \left(|k-\kappa|^{-1} + [\,|c|\lambda - |k-\kappa|\,]^{-1} \right) \right\}.$$

In (II), using the estimates (4.39) and (4.40) in conjunction with (4.43), we get

$$(4.45) \qquad |T(c,K)| < C_\varepsilon \, |c|^{-5/4+\varepsilon} \, K^{-1/4}$$

$$+ C_\varepsilon \, |c|^{-5/4+\varepsilon} K^{-1/4} \sum_{k=1}^{|c|\lambda-1} \left\{ (|k-\kappa|)^{-1} + [\,|c|\lambda - |k-\kappa|\,]^{-1} \right\}$$

$C_\varepsilon = C(\varepsilon, z)$ is a general positive constant depending upon the parameters indicated. Recall c and λ are integers $c \neq 0$. Thus $|c|\lambda \geq 1$. If $|c|\lambda = 1$ the sum on k in (4.44) and (4.45) is empty and hence 0. Otherwise using an integral to estimate this finite sum,

$$\sum_{k=1}^{|c|\lambda-1} (|k-\kappa|^{-1} + [|c|\lambda - |k-\kappa|]^{-1}) = O(\log |c|) = O(|c|^{\varepsilon}).$$

Using the above result in (4.44) and (4.45) we obtain:

(4.46) in (I) $\quad T(c,K) = O(|c|^{-5/4 + 2\varepsilon})$;

(4.47) in (II) $\quad T(c,K) = O(|c|^{-5/4 + 2\varepsilon} K^{-1/4})$.

From the definition of T(c,K), (4.42), and from (4.41), we see $W_i = \sum_c T(c,K)$, $i = 1, 2$. Let us assume that we can partition the set over which c is summed into two disjoint sets, one in which (I) holds, and in the other (II) holds for the same choice of R (see (4.29)). This decomposition, if it can be effected, need not be unique. Write symbolically

(4.48) $\quad W_i(K) = \sum_c{}_{(I)} T(c,K) + \sum_c{}_{(II)} T(c,K)$,

where $\sum_c{}_{(I)}$ represents a summation over those c in our decomposition for which (I) holds. $\sum_c{}_{(II)}$ is defined by analogy. Then

$$|W_1(K)| \le \sum\nolimits_{c}^{(I)} C_\varepsilon |c|^{-5/4 + 2\varepsilon} + \sum\nolimits_{c}^{(II)} C_\varepsilon |c|^{-5/4 + 2\varepsilon} K^{-1/4}$$

$$\le C_\varepsilon \sum_{|c| > RK} |c|^{-5/4 + 2\varepsilon} + C_\varepsilon K^{-1/4} \sum_{c=-\infty}^{\infty}{}' |c|^{-5/4 + 2\varepsilon}.$$

We place the restriction that $0 < 2\varepsilon < 1/4$, then,

$$W_1(K) = O(K^{-1/4 + 2\varepsilon}) + O(K^{-1/4}) = O(K^{-1/4 + 2\varepsilon}).$$

The constant involved in the O-symbol involves only μ, ε and z. This shows that $W_1(K) \longrightarrow 0$ with $K \longrightarrow \infty$, as promised.

To complete the proof we must show how the decomposition (4.48) can be effected. This will, of course, depend upon the particular $M \in \mathcal{M}$. Certain cases must be identified and handled separately. Recall the definition of $\mathcal{M}$ given in (4.3), (4.4) and (4.5). We assumed that for $M = (\alpha\ \beta \mid \gamma\ \delta)$ all the entries are positive, further, α and δ are not simultaneously 0. We identify six cases:

1) $M = I$,

2) $M = (\alpha, -1 \mid 1\ 0)$ with $\alpha > 0$,

3) $M = (0\ \ -1 \mid 1\ \delta)$ with $\delta > 0$.

Let $M = (\alpha\ \beta \mid \gamma\ \delta)$. The remaining cases have positive entries and we identify them:

4) $\gamma - \alpha > 0, \quad \delta - \beta > 0,$

5) $\gamma - \alpha < 0, \quad \delta - \beta > 0$

and, finally,

6) $\gamma - \alpha < 0, \quad \delta - \beta < 0.$

The only other possibility $\gamma - \alpha > 0$ and $\delta - \beta < 0$ is excluded by the fact that $\alpha\delta - \beta\gamma = 1$. We determine the image of $\mathcal{S}_K$ under M in each of the six cases. From the geometry it will be clear how we can effect the decomposition (4.48).

We refer the reader to either Fig. 4.1 or Fig. 4.2.

1) Let M = I.

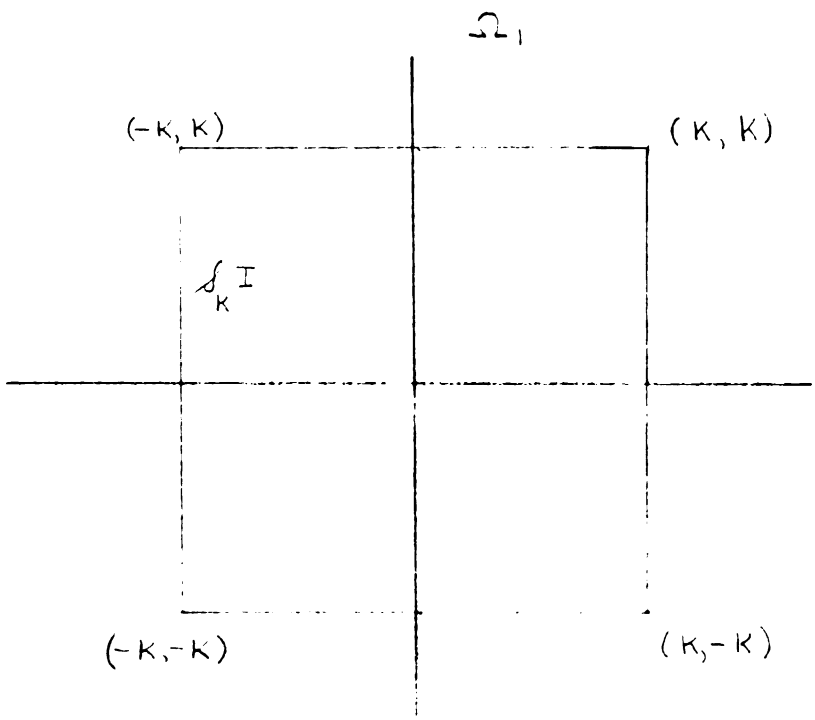

Fig. 4.3

In this case $W_2(K) = 0$. Further, in the sum for $W_1(K)$, (II) always holds with Q(c) = 1.

2) Suppose $M = (\alpha, -1 \mid 1\ 0) \in \mathcal{M}$ with $\alpha > 0$. The transformed region is given below.

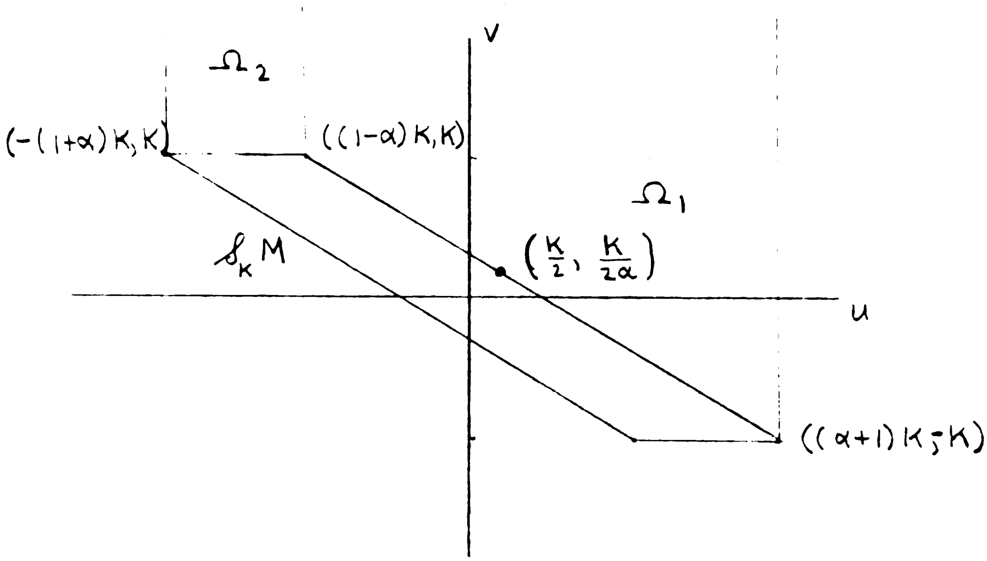

Fig. 4.4

We see that for $W_2(K)$ situation (II) prevails with $Q(c) = 1$. We handle $W_1(K)$ in the following way

$$W_1(K) = \sum_{c=[K/2]+1}^{(1+\alpha)K-1} T(c,K) + \sum_{c=(1-\alpha)K-1}^{[K/2]} T(c,K)$$

to obtain the decomposition (4.48). In the first sum above (I) holds with $R > 1/4$; in the second sum (II) is satisfied with $Q(c) \geq 1/2\alpha$.

3) Let $M = (0\ -1 \mid 1\ \delta) \in \mathcal{M}$ with $\delta > 0$.

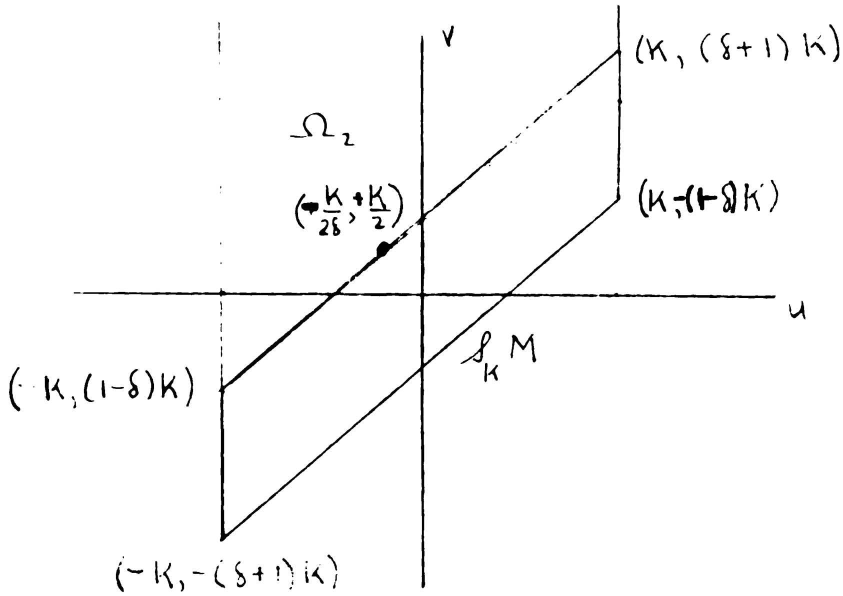

Fig. 4.5

For this choice of M, $W_1(K) = 0$. We write

$$W_2(K) = \sum_{c=-K+1}^{[-K/2\delta]} T(c,K) + \sum_{c=[-K/2\delta]+1}^{K-1} T(c,K)$$

to obtain the decomposition (4.48). In this case for (I) $R = 1/2\delta$ and for (II) $Q(c) > 1/2$.

4) Let $M = (\alpha\ \beta \mid \gamma\ \delta) \in \mathcal{M}$ with $\gamma - \alpha > 0$ and $\delta - \beta > 0$. The configuration of regions is given in Fig. 4.6.

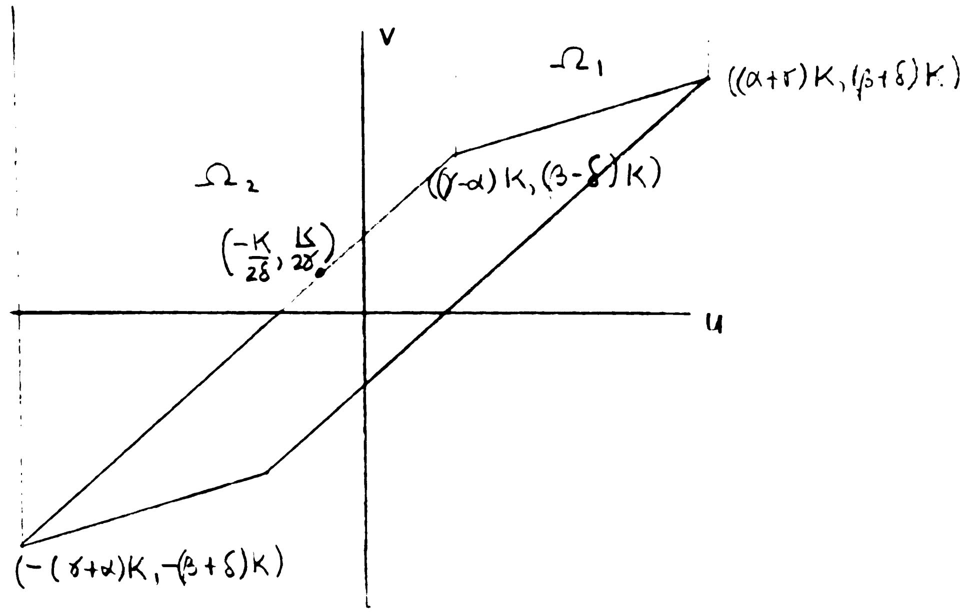

Fig. 4.6.

For $W_1(K)$ we always have (I) with $R = (\gamma - \alpha)$. However, we must decompose $W_2(K)$. Write

$$W_2(K) = \sum_{c=-(\gamma+\alpha)K+1}^{[-K/2\delta]} T(c,K) + \sum_{c=[-K/2\delta]+1}^{(\gamma-\alpha)K-1}{}' \; T(c,K)$$

to obtain (4.48).

The cases 5) and 6) are handled in a manner similar to 4). We give their configuration of regions in Fig. 4.7 and 4.8 respectively. The reader can see how these situations are handled.

5) $M = (\alpha\beta \mid \gamma\delta) \in \mathcal{M}$ with $\gamma - \alpha < 0$ and $\delta - \beta > 0$.

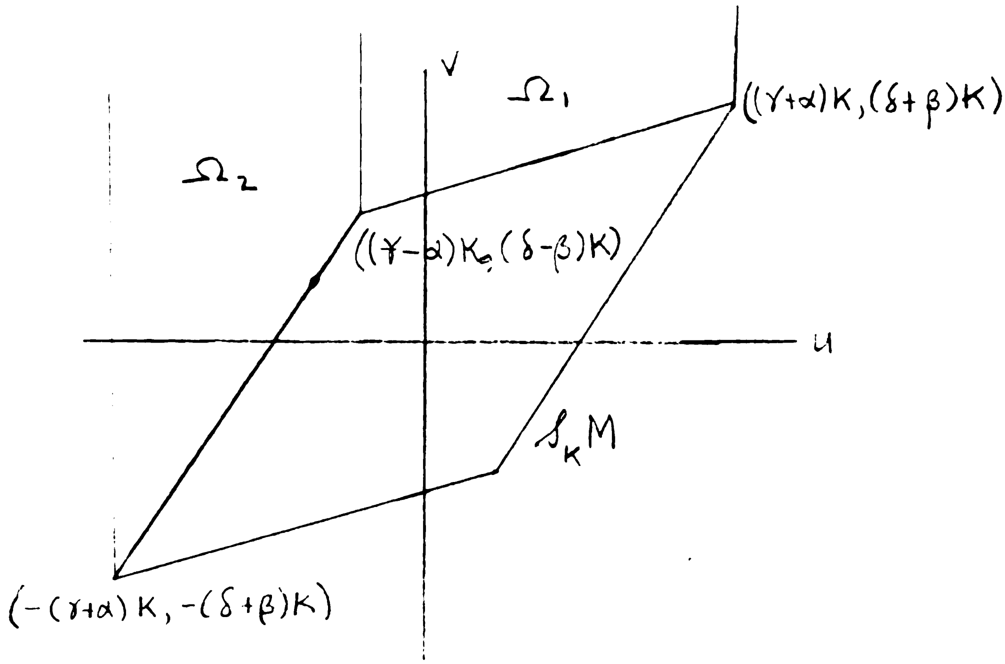

Fig. 4.7

6) $M = (\alpha\beta \mid \gamma\delta) \in \mathcal{M}$ with $\gamma - \alpha < 0$ and $\delta - \beta < 0$.

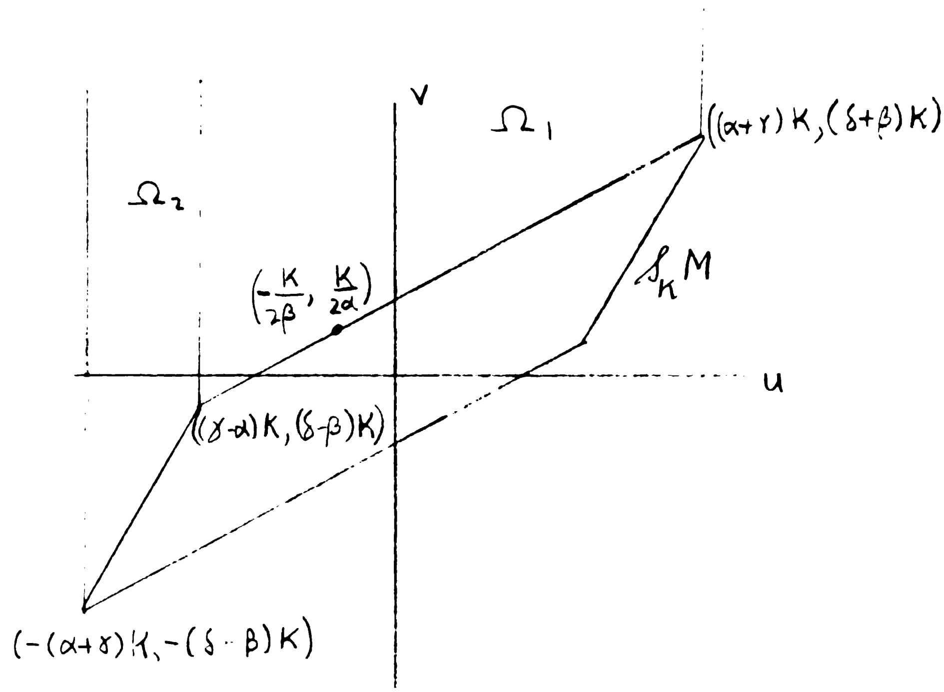

Fig. 4.8

A useful consequence of lemma 2 is the following.

Lemma 3: Let $M \in \mathcal{M}$ then

$$(4.49)\quad G(z,\mathcal{V},A_j,\Gamma,\mu) = \lim_{K\to\infty} \sum_{(c,d)\in \mathcal{O}_K(A_j,\Gamma)M} \frac{e((\mu+\kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2} .$$

Proof: In lemma 1 we proved that

$$G(z,\mathcal{V},A_j,\Gamma,\mu) = \delta_{A_j,I}\, e((\mu+\kappa)z/\lambda) + \sum_{c=-\infty}^{\infty}{}'_j H_1(c,z)$$

$$+ \sum_{c=-\infty}^{\infty}{}''_j H_2(c,z).$$

Now, lemma 2 gives for $M \in \mathcal{M}$

$$(4.50)\quad H_2(c,z) = \lim_{K\to\infty} \sum_{(c,d)\in \mathcal{O}_K(A_j,\Gamma)M}{}' \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

Furthermore, we proved that the double series $H_1(z)$ of (3.17) which sums to $\sum_c{}'_j H_1(c,z)$ is absolutely convergent. Thus, we may write

$$(4.51)\quad H_1(z) = \lim_{K\to\infty} \sum_{(c,d)\in \mathcal{O}_K(A_j,\Gamma)M}{}' \frac{e((\mu+\kappa_j)a/c\,\lambda_j)\left\{e\left[\frac{-(\mu+\kappa_j)}{c\,\lambda_j(cz+d)}\right]-1\right\}}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

Both the limits in (4.50) and (4.51) exist, therefore, we may add them. Using the fact that for $c \neq 0$,

$Vz = (az+b)/(cz+d) = a/c - 1/c(cz + d)$, we see that

$$G(z, \mathcal{V}, A_j, \Gamma, \mu) = \lim_{K \to \infty} \sum_{(c,d) \in \Theta_K(A_j, \ )M} \frac{e((\mu+\kappa_j)V_{c,d}\, z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz + d)^2}$$

where we have incorporated the term corresponding to $c = 0$ in with the limit.

5. The Functional Equation

In this section we prove that the functions $G(z,\mathcal{V},A_j,\Gamma,\mu)$ satisfy the functional equation (2.11). That is, for every $V = (a\ b \mid c\ d)$

$$(5.1)\quad G(Vz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(V)\ (cz + d)^2\ G(z,\mathcal{V},A_j,\Gamma,\mu).$$

We use the results of the preceeding section; namely, lemma 3 and the representations (4.6) and (4.7) for $V \in \Gamma$.

The result (5.1) follows from the special cases:

$$(5.2)\quad G(U^{\lambda} z,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(U^{\lambda})\ G(z,\mathcal{V},A_j,\Gamma,\mu);$$

and for $M = (\alpha\,\beta \mid \gamma\,\delta) \in \mathcal{M}$,

$$(5.3)\quad G(Mz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(M)\ (\gamma z + \delta)^2\ G(z,\mathcal{V},A_j,\Gamma,\mu);$$

finally if, in particular, $S = (0\ -1 \mid 1\ 0) \in \Gamma$, then

$$(5.4)\quad G(Sz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(S)\ z^2\ G(z,\mathcal{V},A_j,\Gamma,\mu).$$

Indeed, suppose first $V = U^{m\lambda} M U^{n\lambda}$ with $M = (\alpha\,\beta \mid \gamma\ \delta) \in \mathcal{M}$. If $V = (a\ b \mid c\ d)$, then $c = \gamma$ and $d = \delta + nc\lambda$. Applying (5.2) and (5.3),

$$\begin{aligned} G(Vz,\mathcal{V},A_j,\Gamma,\mu) &= \mathcal{V}(U^{m\lambda})\ G(MU^{n\lambda} z,\mathcal{V},A_j,\Gamma,\mu) \\ &= \mathcal{V}(U^{m\lambda}M)\ (\gamma U^{n\lambda} z + \delta)^2\ G(U^{n\lambda} z,\mathcal{V},A_j,\Gamma,\mu) \\ &= \mathcal{V}(U^{m\lambda} M)\ \mathcal{V}(U^{n})\ (\gamma z + \gamma n\lambda + \delta)^2\ G(z,\mathcal{V},A_j,\Gamma,\mu) \\ &= \mathcal{V}(V)\ (cz + d)^2\ G(z,\mathcal{V},A_j,\Gamma,\mu). \end{aligned}$$

If, instead, $V = U^{m\lambda} S U^{n\lambda} = (a\ b \mid c\ d)$ then $c = 1$ and $d = n\lambda$. As above,

$$G(Vz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(U^{m\lambda})\ G(SU^{n\lambda}z,\mathcal{V},A_j,\Gamma,\mu)$$

$$= \mathcal{V}(U^{m\lambda}S)\ (U^{n\lambda}z)^2\ G(U^{n\lambda}z,\mathcal{V},A_j,\Gamma,\mu)$$

$$= \mathcal{V}(V)\ (z + n\lambda)^2\ G(z,\mathcal{V},A_j,\Gamma,\mu).$$

There was no loss in generality in assuming $c \geq 0$ since we were dealing with linear fractional transformations rather than matrices.

We defer the proof of (5.2) until the next section where we obtain the Fourier expansion of $G(z,\mathcal{V},A_j,\Gamma,\mu)$. To deal with (5.3), we have from lemma 3,

$$G(z,\mathcal{V},A_j,\Gamma,\mu) = \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)} \frac{e((\mu+\kappa_j)V_{c,d}\ z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

where in that lemma we have taken $M = I$. Now for any $M \in \mathcal{M}$ we see upon substitution

$$G(Mz,\mathcal{V},A_j,\Gamma,\mu) = \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)} \frac{e((\mu+\kappa_j)V_{c,d}\ Mz/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cMz+d)^2}$$

$$= \lim_{K\to\infty} \mathcal{V}(M)(\gamma z+\delta)^2 \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)} \frac{e((\mu+\kappa_j)V_{c,d}Mz/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d}M)(c'z+d')^2}$$

where $V_{c,d}M = (* * \mid c' \; d') = (* * \mid \alpha c + \gamma d, \; \beta c + \delta d)$.

If (c,d) runs over $\mathcal{O}_K(A_j, \Gamma)$ then $(c', d') = (c,d)\, M$ runs over $\mathcal{O}_K(A_j, \Gamma)M$. The terms of the series depend only on the lower row of $V_{c,d}M$; thus,

$$G(Mz, \mathcal{V}, A_j, \Gamma, \mu) = \mathcal{V}(M)(\gamma z + \delta)^2 \lim_{K \to \infty} \sum_{(c,d) \in \mathcal{O}_K(A_j, \Gamma)M} \frac{d((\mu + \kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2},$$

but by lemma 3

$$G(z, \mathcal{V}, A_j, \Gamma, \mu) = \lim_{K \to \infty} \sum_{(c,d) \in \mathcal{O}_K(A_j, \Gamma)M} \frac{e((\mu + \kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2},$$

therefore,

$$G(Mz, \mathcal{V}, A_j, \Gamma, \mu) = \mathcal{V}(M)(\gamma z + \delta)^2 \, G(z, \mathcal{V}, A_j, \Gamma, \mu).$$

Suppose, finally, that $S = (0 \; -1 \mid 1 \; 0) \in \Gamma$. Then from lemma 3 with $M = I$,

$$G(Sz, \mathcal{V}, A_j, \Gamma, \mu) = \lim_{K \to \infty} \sum_{(c,d) \in \mathcal{O}_K(A_j, \Gamma)} \frac{e((\mu + \kappa_j)V_{c,d}\, Sz/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cSz+d)^2}$$

$$= \lim_{K \to \infty} \mathcal{V}(S)\, z^2 \sum_{(c,d) \in \mathcal{O}_K(A_j, \Gamma)} \frac{e((\mu + \kappa_j)V_{c,d}Sz/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d}S)(dz-c)^2}$$

As (c,d) runs over $\mathcal{O}_K(A_j,\Gamma)$, (-d,c) = (c,d) S runs over $\mathcal{O}_K(A_j,\Gamma)S$. But $\mathcal{O}_K(A_j,\Gamma)S = \mathcal{O}_K(A_j,\Gamma)$. Thus,

$$G(Sz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(S)\ z^2 \lim_{K \to \infty} \sum_{(c,d)\in \mathcal{O}_K(A_j,\Gamma)} \frac{e((\mu+\kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

$$= \mathcal{V}(S)\ z^2\ G(z,\mathcal{V},A_j,\Gamma,\mu).$$

6. The Fourier Expansion

As mentioned in sectinn 5 we must yet prove that

$$(6.1) \quad G(U^{\lambda}z, \mathcal{V}, A_j, \Gamma, \mu) = \mathcal{V}(U^{\lambda}) G(z, \mathcal{V}, A_j, \Gamma, \mu).$$

This is accomplished by expanding $G(z, \mathcal{V}, A_j, \Gamma, \mu)$ in a Fourier series, which, furthermore, will give the behavior of $G(z, \mathcal{V}, A_j, \Gamma, \mu)$ at the infinite cusp.

We begin with the series

$$(6.2) \qquad H_2(z) = {\sum_{c=-\infty}^{\infty}}'_{j} H_2(c,z)$$

$$= {\sum_{c=-\infty}^{\infty}}'_{j} (2\pi i)^2/c^2\lambda^2 \sum_{n=0}^{\infty} (n+\kappa) W_c(n,\mu) e((n+\kappa)z/\lambda)$$

where we have used (3.13). The above series was shown to be absolutely uniformly convergent for $y \geq y_0 > 0$. Each of the functions $H_2(c,z)$ is regular in $y \geq y_0 > 0$. Thus, by the Weierstrass double series theorem

$$(6.3) \quad H_2(c,z) = \sum_{n=0}^{\infty} e((n+\kappa)z/\lambda) {\sum_{c=-\infty}^{\infty}}'_{j} (-2\pi i/c\lambda)^2 W_c(n,\mu)(n+\kappa)$$

Following lemma 1 we proved the double series $H_1(z)$ (3.17), whose sum is $\sum_{c=-\infty}^{\infty}{}_j H_1(c,z)$, converges absolutely uniformly as a double series for $y \geq y_0 > 0$. Thus we may rearrange the order of summation in the manner of $H_2(z)$. Proceeding from (3.17a)

$$H_1(z) = \sum_{m=1}^{\infty} \sum_{c=-\infty}^{\infty}{}_j \frac{(-2\pi i(\mu+\kappa_j))^m}{c^m \lambda_j^m \, m!} \sum_{d=-\infty}^{\infty}{}_j \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(cz+d)^{m+2}} \tag{6.4}$$

Now, as in $H_2(z)$, we can write the inner most sum (on d; see section 3, the development following (3.12))

$$\sum_{d=-\infty}^{\infty}{}_j \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})(cz+d)^{m+2}} = \left\{ \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})} \right\}$$

$$\left\{ \frac{(-i)^{m+2}}{(c\lambda)^{m+2}} \sum_{q=-\infty}^{\infty} \frac{e(q\kappa)}{(-i(z/\lambda + d/c\lambda) + qi)^{m+2}} \right\}$$

$$= \left\{ \sum_{d \in \mathcal{D}_c(A_j,\Gamma)} \frac{e((\mu+\kappa_j)a/c\,\lambda_j)}{v(A_j^{-1}V_{c,d})} \cdot \frac{(-2\pi i)^{m+2}}{(c\,\lambda)^{m+2}\,\Gamma(m+2)} \right\} \cdot$$

$$\left\{ \sum_{n=0}^{\infty} (n+\kappa)^{m+1} \quad e((n+\kappa)z/\lambda + d/c\lambda)) \right\}$$

where we have used the Lipschitz formula (2.20). Interchanging orders of summation, valid because of absolute convergence, and introducing the Kloosterman sums, we obtain finally

$$\sum_{d=-\infty}^{\infty}{}_{j} \cdots = \sum_{n=0}^{\infty} e((n+\kappa)z/\lambda) \frac{(-2\pi i)^{m+2} (n+\kappa)^{m+1}}{(c\lambda)^{m+2} (m+1)!} W_c(n,\mu).$$

Introduce the above expression for the inner sum in (6.4):

$$(6.5) \quad H_1(z) = \sum_{n=0}^{\infty} e((n+\kappa)z/\lambda)$$

$$\sum_{m=1}^{\infty} \sum_{c=-\infty}^{\infty}{}'_{j} \frac{(-2\pi i)^{2m+2}(n+\kappa)^{m+1}(\mu+\kappa_j)^m}{c^{2m+2} \lambda_j^m \lambda^{m+2} m!(m+1)!} W_c(n,\mu)$$

On comparing (6.3) and (6.5) one sees that (6.3) corresponds to the missing term m = 0 in (6.5). Thus, on adding (6.3) and (6.5), we obtain

$$G(z,\tau,A_j,\Gamma,\mu) = \delta_{A_j,I} e((\mu+\kappa)z/\lambda)$$

(6.6)

$$+ \sum_{\substack{n=0 \\ n+\kappa>0}}^{\infty} c_n \, e(n+\kappa)z/\lambda)$$

where

$$c_n = \sum_{\substack{c=-\infty \\ c\in \mathcal{C}(A_j,\Gamma)}}^{\infty}{}' W_c(n,\mu) \sum_{m=0}^{\infty} \frac{(-2\pi i)^{2m+2}(n+\kappa)^{m+1}(\mu+\kappa_j)^m}{c^{2m+2} \lambda^{m+2} \lambda_j^m (m+1)! m!}$$

(6.7)

$$= c_n(\tau, A_j, \Gamma, \mu)$$

We note in (6.3) and (6.5) that if $\kappa = 0$ then the coefficient $c_0 = 0$. This is the reason we restrict our sum in (6.6) so that $n + \kappa > 0$.

A simplification of the coefficients c_n can be effected if one used the Bessel function [18, p.358]

$$(6.8) \qquad J_1(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (z/2)^{2m+1}}{m!\ (m+1)!} \ .$$

The sum on m in (6.7) can be written in the form

$$-\frac{2\pi}{|c|\lambda}\ \frac{\sqrt{n+\kappa}}{\sqrt{\mu+\kappa_j}}\ \frac{\sqrt{\lambda_j}}{\sqrt{\lambda}} \sum_{m=0}^{\infty} \frac{(-1)^m (2\pi)^{2m+1} (\sqrt{\mu+\kappa_j}\ \sqrt{n+\kappa}\,)^{2m+1}}{m!(m+1)!(\sqrt{\lambda_j}\ \sqrt{\lambda}\,)^{2m+1}\ |c|^{2m+1}}$$

$$(6.9) = -\frac{2\pi}{|c|\lambda}\left(\frac{\mu+\kappa_j}{\lambda_j}\right)^{-\frac{1}{2}}\left(\frac{n+\kappa}{\lambda}\right)^{\frac{1}{2}} J_1\left(\frac{4\pi}{|c|}\sqrt{\frac{(\mu+\kappa_j)(n+\kappa)}{\lambda_j\ \lambda}}\right)$$

We assumed that $\mu \neq 0$, therefore, $\mu + \kappa_j \neq 0$. Using this expression, we may write the coefficients c_n in the following form

$$c_n = \frac{-2\pi}{\lambda}\left(\frac{\mu+\kappa_j}{\lambda_j}\right)^{1/2}\left(\frac{n+\kappa}{\lambda}\right)^{1/2} \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma)}}^{\infty}{}' |c|^{-1} W_c(n,\mu) J_1\left(\frac{4\pi}{c}\sqrt{\frac{(\mu+\kappa_j)(n+\kappa)}{\lambda_j\ \lambda}}\right)$$

(6.10)

When $\mu + \kappa_j < 0$ we can replace $\mu + \kappa_j$ by $|\mu + \kappa_j|$ and then we must also replace $J_1(z)$ by $I_1(z) = i^{-1}J_1(iz)$ [18, 372], the Bessel function with purely imaginary argument.

We now derive (6.1). Replace a by $U \;\; z = z + \lambda$ in

$$(6.6) \quad G(U^{\lambda} z, \upsilon, A_j, \Gamma, \mu) = \delta_{A_j, I}\, e((\mu+\kappa)(z+\lambda)/\lambda) + \sum_{\substack{n=0 \\ n+\kappa>0}}^{\infty} c_n\, e((n+\kappa)(z+\lambda)/\lambda)$$

$$= e(\kappa) G(z, \upsilon, A_j, \Gamma, \mu) = \upsilon(U^{\lambda}) G(z, \upsilon, A_j, \Gamma, \mu)$$

As $z \to i\infty$ within the fundamental region $R(\Gamma)$, $G(z, \upsilon, A_j, \Gamma, \mu)$ tends to a definite limit, finite or infinite. We see that if $\mu + \kappa > 0$ or $A_j \neq I$ this limit is 0. If $A_j = I$ and $\mu + \kappa < 0$ then $G(z, \upsilon, A_j, \Gamma, \mu)$ has a pole at $i\,\infty$.

7. The Behavior at the Cusps

In order to show that $G(z,v,A_j,\Gamma,\mu)$ has the correct behavior at each of the parabolic fixed points of Γ, we establish the formula

$$(7.1) \qquad G(z,v,A_j,\Gamma,\mu)\mid B = G(z,v',A_jB,B^{-1}\Gamma B,\mu)$$

for each $B \in \Gamma(1)$(see (2.13) for the definition of $\mid B$). $v' = v(B^{-1}\Gamma B,-2)$ is the multiplier induced on $B^{-1}\Gamma B$ by $v = v(\Gamma,-2)$. This formula is of interest in its own right.

Since Γ is a subgroup of the modular group and $B \in \Gamma(1)$, then $B^{-1}\Gamma B$ is a subgroup of the modular group. If $R(\Gamma)$ is a fundamental region for Γ then $B^{-1}R(\Gamma)$ is a fundamental region for $B^{-1}\Gamma B$. Write $R(B^{-1}\Gamma B) = B^{-1}R(\Gamma)$. If $R(\Gamma)$ is bounded by a finite number of sides consisting of straight lines and circular arcs then so is $R(B^{-1}\Gamma B)$. Furthermore, if the parabolic cusps of $R(\Gamma)$ are inequivalent, so are the parabolic cusps of $R(B^{-1}\Gamma B)$. If $A_j^{-1}\infty$ is a cusp of $R(\Gamma)$ then $(A_jB)^{-1}\infty = B^{-1}A_j^{-1}\infty$ is a cusp of $R(B^{-1}\Gamma B)$. Thus, all the developments of sections 2,3,4,5 and 6 hold for the Poincaré series in the right member of (7.1).

We shall assume for the moment that (7.1) holds and show that $G(z,v,A_j,\Gamma,\mu)$ has the proper behavior at the cusps. In section 6 we showed that $G(z,\mathcal{V},A_j,\Gamma,\mu)$ has the proper behavior at $z = i\infty$. We consider first the remaining cusps of $R(\Gamma)$. Let $A_k^{-1}\infty = p_k$ for $2 \leq k \leq \sigma(\Gamma)$. Then from (7.1) applied to $G(z,\mathcal{V}_k,A_jA_k^{-1},\Gamma_k,\mu)$ with $B = A_k$ and $\Gamma_k = A_k\Gamma A_k^{-1}$ we get

$$(7.2)\quad G(z,\mathcal{V}_k,A_jA_k^{-1},\Gamma_k,\mu) \mid A_k = G(z,\mathcal{V},A_j,\Gamma,\mu).$$

$\mathcal{V}_k$ is the multiplier system on Γ_k induced by $\mathcal{V}$ on Γ. $\mathcal{V}_k$ induces $\mathcal{V}$ on Γ. Rewriting (7.2)

$$(c_kz + d_k)^{-2}G(A_kz,\mathcal{V}_k,A_jA_k^{-1},\Gamma_k,\mu) = G(z,\mathcal{V},A_j,\Gamma,\mu)$$

where $A_k = (a_k\ b_k \mid c_k\ d_k)$. As $z \to p_k = -d_k/c_k$, $A_kz \to i\infty$. Since $G(A_kz,\mathcal{V}_k,A_jA_k^{-1},\Gamma_k,\mu) = O(|e(\omega_kA_kz/\lambda_k)|)$ with $\omega_k = \kappa_k$ if $\kappa_k > 0$ and $k \neq j$, $\omega_k = 1$ if $\kappa_k = 0$ and $j \neq k$, and if $k = j$ then $\omega_k = \mu + \kappa_j$, it follows that $G(z,\mathcal{V},A_j,\Gamma,\mu)$ tends exponentially to a definite limit as $z \to p_k$. Indeed, if $k \neq j$ each of the terms of the Fourier expansion of $G(A_kz,\mathcal{V}_k,A_jA_k^{-1},\Gamma_k,\mu)$ tends exponentially to 0 as $z \to p_k$.

Now let p be a parabolic fixed point of Γ. Then there is a $V = (a\ b \mid c\ d)$ such that $V^{-1}p = p_m$ is one of the cusps of $R(\Gamma)$. The functional equation (5.1) gives $G(Vz,\mathcal{V},A_j,\Gamma,\mu) = \mathcal{V}(V)(cz + d)^2\, G(z,\mathcal{V},A_j,\Gamma,\mu)$. As $a \to p_m$ in a parabolic sector in $R(\Gamma)$, $Vz \to p$ in a parabolic sector at p. $G(z,\mathcal{V},A_j,\Gamma,\mu)$ tends exponentially to a definite limit, therefore, $G(Vz, \mathcal{V}, A_j, \Gamma, \mu)$ does also. This completes the proof that $G(z, \mathcal{V}, A_j, \Gamma, \mu)$ has the proper behavior.

It remains to establish the formula (7.1). We do this by induction after we have established

$$G(z, \mathcal{V}, A_j, \Gamma, \mu) \mid U = G(z, \mathcal{V}', A_j U, U^{-1} \Gamma U, \mu), \tag{7.3}$$

and

$$G(z, \mathcal{V}, A_j, \Gamma, \mu) \mid S = G(z, \mathcal{V}'', A_j S, S^{-1} \Gamma S, \mu) \tag{7.4}$$

for the two generators $U = (1\ 1 \mid 0\ 1)$ and $S = (0\ -1 \mid 1\ 0)$ of $\Gamma(1)$. $\mathcal{V}'$ is the character on $U^{-1}\Gamma U$ induced by $\mathcal{V}$ on Γ. $\mathcal{V}''$ is similarly defined.

Proof of (7.3): In (3.2) we defined $H(c,z)$; and

$$G(U \ \mathcal{V}, A_j, \Gamma, \mu) = \sum_{c=-\infty}^{\infty}{}_j H(c, Uz)$$

where

$$H(c,Uz) = \sum_{d=-\infty}^{\infty}{}_j \bar{\mathcal{V}}(A_j^{-1} V_{c,d})(cUz+d)^{-2} e((\mu+\kappa_j) V_{c,d} Uz/\lambda_j).$$

Now

$$\mathcal{C}(A_j, \Gamma) = \mathcal{C}(A_j U, U^{-1} \Gamma U)$$

and

$$\mathcal{D}(c, A_j U, U^{-1} \Gamma U) = \{c+d : d \in \mathcal{D}(c, A_j, \Gamma)\}.$$

The proof of these facts is almost immediate. Furthermore, if ϑ' is the multiplier system induced on $U^{-1}\Gamma U$ and $V'_{c,d+c} = V_{c,d}U^{\lambda} \in A_j \Gamma U$, then

$$\vartheta(A_j^{-1}V_{c,d}) = \vartheta'(U^{-1}A_j^{-1}V_{c,d}U) = \vartheta'((A_jU)^{-1}V'_{c,c+d}).$$

Also,

$$\lambda' = \lambda(A_jU, U^{-1}\Gamma U) = \lambda(A_j, \Gamma) = \lambda_j,$$

and

$$\kappa = \kappa(A_jU, U^{-1}\Gamma U) = \kappa(A_j, \Gamma) = \kappa_j.$$

Substituting these results into $H(c,Uz)$,

$$H(c,Uz) = \sum_{d \in \mathcal{D}(c,A_j,\Gamma)} \bar{\vartheta}'((A_jU)^{-1}V'_{c,d+c})(cz+d+c)^{-2}e((\mu+\kappa_j)V'_{c,d+c}z/\lambda_j)$$

$$= \sum_{d \in \mathcal{D}(c,A_jU,U^{-1}\Gamma U)} \bar{\vartheta}'((A_jU)^{-1}V_{c,d})(cz+d)^{-2}e((\mu+\kappa_j)V_{c,d}z/\lambda_j)$$

$$= \tilde{H}(c,z)$$

since as d runs over $\mathcal{D}(c,A_j,\Gamma)$ $d' = d+c$ runs over $\mathcal{D}(c,A_jU,U^{-1}\Gamma U)$. Thus,

$$\sum_{c=-\infty}^{\infty}{}_j H(c,Uz) = \sum_{c \in \mathcal{C}(A_jU,\, U^{-1}\Gamma U)} \tilde{H}(c,z) = G(z,\vartheta',A_jU,U^{-1}\Gamma U,\mu).$$

Proof of (7.4). From the definition of the operator

$|S$ and from lemma 3 with M = I, we see

$G(z,\mathcal{V},A_j,\Gamma,\mu)|S$

$$= z^{-2} \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}_K(A_j,\Gamma)} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})(cSz+d)^{-2} e((\mu+\kappa_j)V_{c,d}Sz/\lambda_j)$$

Let $V'_{d,-c} = V_{c,d}S = (b, -a | d, -c)$. $V'_{d,-c} \in A_j\Gamma S =$ $A_jS\, S^{-1}\Gamma\, S$. Thus, $(d,-c)\in\mathcal{O}(A_j,\Gamma)\, S$ and

$$\mathcal{O}_K(A_j,\Gamma)\, S = \mathcal{O}_K(A_jS,S^{-1}\Gamma\, S).$$

Moreover,

$$\mathcal{V}(A_j^{-1}V_{c,d}) = \mathcal{V}''(S^{-1}A_j^{-1}V_{c,} S) = \mathcal{V}''((A_jS)^{-1}V'_{c,-})$$

and

$$\lambda(A_jS,S^{-1}\Gamma\, S) = \lambda_j, \qquad \kappa(AS,S^{-1}\Gamma\, S) = \kappa_j.$$

Therefore,

$$z^{-2} \operatorname{Lim}_{K\to\infty} \sum_{(c,d)\in\mathcal{O}(A_j,\Gamma)} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})(cSz+d)^{-2}\, e((\mu+\kappa_j)V_{c,d}Sz/\lambda_j)$$

$$= \lim_{K\to\infty} \sum_{(c,d)\in\mathcal{O}(A_jS,S^{-1}\Gamma S)} \mathcal{V}''((A_jS)^{-1}V_{c,d})(cz+d)^{-2}\, e((\mu+\kappa_j)V_{c,d}z/\lambda_j)$$

$$= G(z,\mathcal{V}'',A_jS,S^{-1}\Gamma\, S,\mu).$$

This last equality follows from lemma 3 with M = I.

Since S and U generate $\Gamma(1)$, we can write any B $\in\Gamma(1)$ in the form $B = S^{a_1}U^{b_1} \dots S^{a_n}U^{b_n}$, $a_i \geq 1, a_0 \geq 0$,

$b_1 \geq 1$ and $b_n \geq 0$. Now we need to know that

$$F(z) \mid V_1V_2 = (F(z) \mid V_1) \mid V_2.$$

Let v' be the multiplier system on $V_1^{-1}\Gamma V_1$ induced by v on Γ, v'' be the multiplier system on $(V_1V_2)^{-1}\Gamma V_1V_2$ induced by v' on $V_1^{-1}\Gamma V_1$, and let v''' be the multiplier system on $(V_1V_2)^{-1}\Gamma V_1V_2$ induced by v on Γ. Then $v''' = v'$. (7.1) is now proved by induction from (7.3), (7.4) and the above mentioned facts.

8. Kloosterman Sums, Main Theorem, and Examples

We introduced Kloosterman sums associated with our modular forms of dimension -2 belonging to Γ and the multiplier system $\mathcal{V}=\mathcal{V}(\Gamma,-2)$ in (2.18). At the same time we assumed that for fixed $\mu \neq 0$ these sums could be estimated as $O(|c|^{1/2+\varepsilon})$. With this assumption we have proved the following theorem.

THEOREM 1: Let Γ be a subgroup of finite index in the modular group $\Gamma(1)$ and let $\mathcal{V}=\mathcal{V}(\Gamma,-2)$ be a multiplier system for Γ and the dimension -2. Let $A_j^{-1}\infty = p_j$ be a cusp of the fundamental region of Γ ($A_j^{-1}\infty = \infty$ if and only if $A_j = I$). Then

$$(8.1)\quad G(z,\mathcal{V},A_j,\Gamma,\mu) = \sum_{\substack{c=-\infty \\ c\in\mathcal{C}(A_j,\Gamma)}}^{\infty} \ \sum_{\substack{d=-\infty \\ d\in\mathcal{D}(c,A_j,\Gamma)}}^{\infty} \frac{e((\mu+\kappa_j)V_{c,d}z/\lambda_j)}{\mathcal{V}(A_j^{-1}V_{c,d})(cz+d)^2}$$

is a modular form of dimension -2 for Γ and the multiplier system $\mathcal{V}=\mathcal{V}(\Gamma,-2)$, provided the Kloosterman sum

$$(8.2)\quad W_c(n+\kappa,A_j,\mu+\kappa_j) = \sum_{d\in\mathcal{D}_c(A_j,\Gamma)} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})e((n+\kappa)/c\lambda+(\mu+\kappa_j)/c\lambda_j)$$

can be estimated as $O(|c|^{1/2+\varepsilon})$ for fixed $\mu \neq 0$. The series in (8.1) is not aboslutely convergent; we understand that the summation is to be performed in the order

$$\lim_{K\to\infty} \sum_{\substack{c=0 \\ c\in\mathcal{C}(A_j,\Gamma)}}^{K} \sum_{\substack{d=-\infty \\ d\in\mathcal{D}(c,A_j,\Gamma)}}^{\infty} \cdots + \lim_{K\to\infty} \sum_{\substack{c=-K \\ c\in\mathcal{C}(A_j,\Gamma)}}^{-1} \sum_{\substack{d=-\infty \\ d\in\mathcal{D}(c,A_j,\Gamma)}}^{\infty} \cdots$$

The sets $\mathcal{C}(A_j,\Gamma)$, $\mathcal{D}(c,A_j,\Gamma)$ and $\mathcal{D}_c(A_j,\Gamma)$ are defined in (2.16). Furthermore, $G(z,\mathcal{V},A_j,\Gamma,\mu)$ has the Fourier expansion

(8.3) $$\delta_{A_j,I}\, e((\mu+\kappa)z/\lambda) + \sum_{n+\kappa>0} c_n(\mathcal{V},A_j,\Gamma,\mu)e((n+\kappa)z/\lambda)$$

where

(8.4) $$c_n = -\pi/\lambda \left[\frac{\lambda_j}{\mu+\kappa_j}\,\frac{n+\kappa}{\lambda}\right]^{\frac{1}{2}} \sum_{n+\kappa>0} \frac{W_c(n,\mu)}{|c|} J_1\left(\frac{4\pi}{|c|}\sqrt{\frac{\mu+\kappa_j}{\lambda_j}\,\frac{n+\kappa}{\lambda}}\right)$$

where we use the positive square root. $J_1(z)$ is the Bessel function defined in (6.8).

In regard to the estimates of the Kloosterman sums Petersson [11] has proved the following theorem.

<u>Theorem (Peterson)</u>: Let Γ be a congruence subgroup of the modular group, $\mathcal{V}$ an abelian character for Γ, and suppose there is a principal congruence subgroup $\Gamma(N)$ such that $\Gamma(N)\subset\Gamma$ and $\mathcal{V}=\mathcal{V}(\Gamma,-2)$ is identically 1 on $\Gamma(N)$. Then the Kloosterman sum (8.2) has the estimate $\mathcal{O}(|c|^{1/2+\varepsilon})$ for fixed $\mu\neq 0$. The constant in the $\mathcal{O}$-symbol depends upon $\mathcal{V},\Gamma,\mu$ and ε but is independent of n.

We shall give Petersson's proof of this result. The material given in the next few pages is an elaboration of pages 16, 17 and 18 of [11]. The proof proceeds by showing how to reduce the sum (8.2) to a sum of original Kloosterman sums. Then the results of Salié [15] and Weil [17] for these sums are applied to give the final result.

$\Gamma(N)$ is normal in Γ. Suppose Γ has the coset decomposition

$$(8.5) \qquad \Gamma = \bigcup_{s=1}^{\nu} K_s \, \Gamma(N), \qquad K_s \in \Gamma$$

where $\nu = [\Gamma : \Gamma(N)]$. Then

$$(8.6) \qquad A_j \Gamma = \bigcup_{s=1}^{\nu} A_j K_s \Gamma(N).$$

We state a lemma.

<u>Lemma 4</u>: $\mathcal{O}(A_j K_m, \Gamma(N))$ and $\mathcal{O}(A_j K_s, \Gamma(N))$ are either disjoint or identical; they are identical if and only if $K_m = P_j^n K_s M$ where n is an integer, $P_j = A_j^{-1} U^{\lambda_j} A_j$ and $M \in \Gamma(N)$. Finally, for $c \in \mathcal{C}(A_j, \Gamma)$

$$(8.7) \qquad \mathcal{D}(c, A_j, \Gamma) = \bigcup_{s=1}^{\nu} \mathcal{D}(c, A_j K_s, \Gamma(N)).$$

Proof: Suppose first, $K_m = P_j^n K_s M$ then $A_j K_m = U^{n\lambda_j} A_j K_s M$. If $(c,d) \in \mathcal{O}(A_j K_s M, \Gamma(N)) = \mathcal{O}(A_j K_s, \Gamma(N))$ then there is a $V \in A_j K_s \Gamma(N)$ of the form $V = (a \; b \mid c \; d)$. Furthermore,

$U^{n\lambda_j} V = (* * | c\ d) \in A_j K_m \Gamma(N)$. Thus, $\mathcal{O}(A_j K_s, \Gamma(N)) \subseteq \mathcal{O}(A_j K_m, \Gamma(N))$. If, on the other hand, $(c,d) \in \mathcal{O}(A_j K_m, \Gamma(N))$ then there is a $V = (a\ b | c\ d) \in A_j K_m \Gamma(N) = U^{n\lambda_j} A_j K_s \Gamma(N)$. Thus, $V = U^{n\lambda_j} V'$ with $V' \in A_j K_s \Gamma(N)$, but V' has the lower row $\{c,d\}$. Therefore, $(c,d) \in \mathcal{O}(A_j K_s, \Gamma(N))$.

Now suppose $(c,d) \in \mathcal{O}(A_j K_m, \Gamma(N)) \cap \mathcal{O}(A_j K_s, \Gamma(N))$. We shall show that, indeed, the above two sets are equal and that $K_m = P_j^n K_s M$ where $M \in \Gamma(N)$. By hypothesis, there are two matrices V and V' such that $V = (a\ b | c\ d) \in A_j K_m \Gamma(N)$ and $V' = (a'\ b' | c'\ d') \in A_j K_s \Gamma(N)$. Write $V = A_j K_m M$ and $V' = A_j K_s M'$ where $M, M' \in \Gamma(N)$. Since V and V' have the same second line, $V = U^r V'$. Thus, $A_j K_m M = U^r A_j K_s M'$ or $K_s = A_j^{-1} U^r A_j M' M^{-1}$. We see that $A_j^{-1} U^r A_j \in \Gamma$ and fixes $p_j = A_j^{-1} \infty$. Hence, $A_j^{-1} U^r A_j \in \Gamma_{p_j}$, the cyclic group which fixes p_j and is generated by $P_j = A_j^{-1} U^{\lambda_j} A_j$. Therefore, $A_j^{-1} U^r A_j = P_j^n$ for some n. By part one of our proof the sets are equal.

Let $K_1, K_2, \ldots, K_{\nu^*}$ be a complete set of representatives of $\Gamma / \Gamma(N)$ no two of which differ by a power of P_j. We shall prove that for $c \in \mathcal{C}(A_j, \Gamma)$

$$(8.8) \qquad \mathcal{D}(c, A_j, \Gamma) = \bigcup_{s=1}^{\nu^*} \mathcal{D}(c, A_j K_s, \Gamma(N))$$

and the sets $\mathscr{D}(c, A_j K_s, \Gamma(N))$, $s = 1,2,\ldots,\nu^*$ are pairwise disjoint. This is a restatement of equation (8.7) of lemma 4 with only the distinct $\mathscr{D}(c, A_j K_s, \Gamma(N))$ chosen. The proof is immediate from the decomposition (8.6), the way in which the first K_m's were chosen and the first statement of lemma 4.

Before turning to the Kloosterman sums we need some auxiliary results concerning Γ and $\mathscr{V}$. U^λ is the least translation in Γ and $U^N \in \Gamma(N) \subset \Gamma$ fixes ∞; therefore, $g\lambda = N$ for some integer g. Similarly $g_j \lambda_j = N$. Now consider $\mathscr{V}(U^N) = e(g\kappa) = 1$, and $\mathscr{V}(A_j^{-1} U^N A_j) = e(g_j \kappa_j) = 1$. Hence, $g\kappa$ and $g_j \kappa_j$ are integers.

Consider the sum (8.2)

$$W_c(n+\kappa, A_j, \mu+\kappa_j) = \sum_{d \in \mathscr{D}_c(A_j, \Gamma)} \bar{\mathscr{V}}(A_j^{-1} V_{c,d}) e((n+\kappa)d/c\lambda + (\mu+\kappa_j)a/c\,\lambda_j).$$

The summation conditions mean that the sum is extended over those matrices V in $A_j\Gamma$ with $V = (a\ b \mid c\ d)$ and $d \in [0, c\lambda]$. The terms of this sum are periodic in a with period $c\lambda_j$ and periodic in d with period $c\lambda$ (see material) following (4.17a). One could make a unique by requiring $0 < a \leq |c|\lambda_j$. Therefore, we can write

$$gW_c(n,\mu) = g \sum_{\substack{d=1 \\ d\in \mathcal{D}(c,A_j,\Gamma)}}^{|c|\lambda} \cdots$$

$$(8.9) \quad = \sum_{\substack{d=1 \\ d\in \mathcal{D}(c,A_j,\Gamma)}}^{g|c|\lambda} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})\ e((n+\kappa)d/c\lambda + (\mu+\kappa_j)/c\lambda_j)$$

$$= \sum_{\substack{d=1 \\ d\in \mathcal{D}(c,A_j,\Gamma)}}^{N|c|} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})\ e([g(n+\kappa)d+g_j(\mu+\kappa_j)]/cN)$$

where we have used the periodicity in d and that $g\lambda = N$ and $g_j\lambda_j = N$. Now using the decomposition $\mathcal{D}(c,A_j,\Gamma) = \bigcup_{s=1}^{\nu^*} \mathcal{D}(c,A_jK_s,\Gamma(N))$ and the fact that these latter sets are pairwise disjoint,

$$(8.10) \quad gW_c(n,\mu)= \sum_{s=1}^{\nu^*} \sum_{\substack{d=1 \\ d\in \mathcal{D}(c,A_jK_s,\Gamma(N))}}^{N|c|} \bar{\mathcal{V}}(A_j^{-1}V_{c,d})\ e([g(n+\kappa)d+g_j(\mu+\kappa_j)a]/cN).$$

Now $V_{c,d} = (a\ b\,|\ c\ d) \in A_jK_s\Gamma(N)$. Let $\rho(c)$ denote the non-empty set of integers s, $1 \leq s \leq \nu^*$, for which $\mathcal{D}(c,A_jK_s,\Gamma(N)) \neq \emptyset$. Write $V_{c,d} = A_jK_sM(a,c,d)$ with $M(a,c,d) \in \Gamma(N)$. Then $\bar{\mathcal{V}}(A_j^{-1}V_{c,d}) = \bar{\mathcal{V}}(K_s)\bar{\mathcal{V}}(M(a,c,d)) = \bar{\mathcal{V}}(K_s)$. Introducing this into (8.10) we obtain

$$(8.11) \quad gW_c(n,\mu) = \sum_{s\in\rho(c)} \bar{\mathcal{V}}(K_s) \sum_{\substack{d=1 \\ d\in \mathcal{D}(c,A_jK_s,\Gamma(N))}}^{N|c|} e([g(n+\kappa)d + g_j(\mu+\kappa_j)a]/cN).$$

Define the integers $m = g(n+\kappa)$ and $\omega_j = g_j(\mu+\kappa_j)$. Define

$$(8.12)\quad S_{cN}(m, A_jK_s, \Gamma(N), \omega_j) = \sum_{\substack{d=1 \\ d\in \mathscr{D}(c, A_jK_s, \Gamma(N))}} e((md + \omega_j a)/cN)$$

where the summation conditions mean that d runs over a set of integers with $1 \leq d \leq N\,|c|$ and with each d there must be associated an element $V = (a\ b\,|\ c\ d) \in A_jK_s\Gamma(N)$. We may suppose $1 \leq a \leq N|c|$, for $U^{rN}V = (a+rcN\ *\,|\,c\ d) \in A_jK_s\Gamma(N)$. We see that

$$(8.13)\quad gW_c(n,\mu) = \sum_{s\in\rho(c)} \bar{\upsilon}(K_s)\, S_{cN}(m, A_jK_s, \Gamma(N), \omega_j).$$

Let us consider the sums S_{cN} more carefully. Suppose $M = (\alpha\ \beta\,|\ \gamma\delta) \in \Gamma(1)$ and $c \in \mathscr{C}(M, \Gamma(N))$ then

$$(8.14)\quad S_{cN}(m, M, \Gamma(N), \omega) = \sum_{d\in \mathscr{D}_c(M, \Gamma(N))} e((md+\omega a)/cN).$$

We can state the summation conditions in the following form:

$$(8.15)\quad \begin{array}{ll} d \bmod cN; & (d,c) = 1; \\ d \equiv \delta,\ a \equiv \alpha \bmod N, & ad \equiv 1 + c\beta \bmod cN. \end{array}$$

Now let $M' = U^{k*}MU^k = (\alpha'\ \beta'\,|\,\gamma'\ \delta')$. Then

$$(8.16)\quad \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha+k^*\gamma & \beta+k\alpha+k^*(\delta+k\gamma) \\ \gamma & \delta+k\gamma \end{pmatrix}.$$

The object of introducing M' is to replace the last congruence of (8.15) by $ad \equiv 1 \bmod cN$. This will require an appropriate choice of k and k*. We have

$$(8.17) \quad S_{cN}(m,M;\Gamma(N),\omega) = \sum_{d' \in \mathcal{D}_c(M',\Gamma(N))} e((md' + \omega a')/cN).$$

If $d \in \mathcal{D}_c(M,\Gamma(N))$ and $V = (a\ b \mid c\ d) \in M\Gamma(N)$ is the matrix determined by this choice of d, then by the normality of $\Gamma(N)$ it follows that $U^{k^*}VU^k = V' \in M'\Gamma(N)$ and $d' = d+ck \in \mathcal{D}(c,M',\Gamma(N))$. Furthermore, as d runs over $\mathcal{D}_c(M,\Gamma(N))$, $d' = d + ck$ runs over a complete set of integers congruent modulo cN to $\mathcal{D}_c(M',\Gamma(N))$. Thus,

$$(8.18) \quad S_{cN}(m,M',\Gamma(N),\omega) = \sum_{d \in \mathcal{D}_c(M,\Gamma(N)} e([(d+kc)m+\omega(a+k^*)c)]/cN)$$

$$= e((km+k^*\omega)/N)\, S_{cN}(m,M,\Gamma(N),\omega).$$

We return to the choice of k and k^*. Since $(\delta,\gamma) = 1$ there are infinitely many primes in the sequence $\delta + k\gamma$; choose one which is not a prime factor of N. This fixes k. Then we can choose k^* so that $\beta' \equiv 0 \bmod N$ (see (8.16)). Now the summation conditions $d \in \mathcal{D}_c(M',\Gamma(N))$ for $S_{cN}(m,M',\Gamma(N),\omega)$ become (see (8.15)):

$$d \bmod cN; \qquad (d,c) = 1;$$
$$d \equiv \delta' \bmod N, \qquad ad \equiv 1 \bmod cN.$$

However, $ad \equiv 1 \bmod cN$ implies $(a,cN) = (d,cN) = 1$, and

$a \equiv \alpha'$ mod N and $d \equiv \delta'$ mod N are equivalent. To see this last remark, note that $ad - bc = 1 = \alpha\delta - \beta\gamma$, $\beta' \equiv b \equiv 0$ mod N; thus, $ad \equiv \alpha'\delta'$ mod N. Now use $(a,N) = (d,N) = 1$. Therefore, we are permitted to write

$$(8.19) \quad S_{cN}(m,M',\Gamma(N),\omega) = \sum_{\substack{d*(cN) \\ ad \equiv 1(cN) \\ a \equiv \alpha'(N)}} d((md + \omega a)/cN)$$

where the asterisk indicates that d runs over a reduced residue system modulo cN.

We are now in a position to reduce the above to a sum of Kloosterman sums. We use the fact that

$$1/N \sum_{t=1}^{N} e((r-\alpha')t/N) = \begin{cases} 1 & \text{when } r \equiv \alpha' \text{ mod } N, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$S_{cN}(m,M',\Gamma(N),\omega) = \sum_{\substack{d*(cN) \\ ad\equiv 1(cN)}} e((md+\omega a)/cN)\ 1/N \sum_{t=1}^{N} e((a-\alpha')t/N)$$

$$= 1/N \sum_{t=1}^{N} e(-\alpha' t/N) \sum_{\substack{d*(cN) \\ ad\equiv 1(cN)}} e([md+(\omega+tc)a]/cN)$$

$$(8.20) \qquad = 1/N \sum_{t=1}^{N} e(-\alpha' t/N)\, S(m,\omega+tc;cN)$$

where

$$(8.21)\qquad S(u,v;q) = \sum_{\substack{d^*(q)\\ ad\equiv 1(q)}} e((ud + va)/q)$$

is the classical Kloosterman sum. We remark that if either u or v $\equiv$ 0 mod q, this sum reduces to the Ramanujan sum. We shall not be required, however, to make a distinction.

Thus, we see from (8.13), (8.18) and (8.20) that

$$(8.22)\quad W_c(n+\kappa,\mu+\kappa_j) = 1/gN\left\{\sum_{s\in\rho(c)} \bar{\nu}(K_s)\, e(-[k_s m + k_s^* \omega_j]/N)\right\} \left\{\sum_{t=1}^{N} e(-\alpha_s' t/N)\, S(m, \omega_j + ct;\, cN)\right\}$$

where k_s, k_s^*, α_s' are integers depending upon $A_j K_s$; $m = g(n+\kappa)$ and $\omega_j = g_j(\mu + \kappa j)$ are integers.

We now concern ourselves with the estimation of the sum appearing in (8.21). It is known [1.1, p.91] that S(u,v;q) is multiplicative; if q has the prime power decomposition $p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$, then to each integer u and to each integer v there are integers $v_1, v_2, \ldots, v_r$ such that

$$(8.23)\qquad S(u,v;q) = \prod_{t=1}^{r} S(u, v_t; p_t^{m_t}).$$

Salié [15] proves that if $q = p^m$, p a prime number, and $(u,v,q) \neq p^{m-1}, p^m$, then

$$(8.24) \qquad |S(u,v;q)| < 2\sqrt{2}\, |q|^{1/2} d^{1/2}$$

where

$$(8.24a) \qquad d = \min\{(u,q), (v,q)\} .$$

Weil [17] proves for p a prime, $uv \not\equiv 0 \bmod p$ and $(u,v,q) = 1$, that

$$(8.25) \qquad |S(u,v;p)| < 2\, |p|^{1/2}.$$

In the case in which $(u,p) = 1$ and $v \equiv 0 \bmod p$ we find $S(u,v;p) = -1$. The case $q = p^m$, $m \geq 2$, p a prime and $(u,v,q) = p^{m-1}$ is handled by using (8.25) and the known result [1.1, p.90] that $S(u,v;q) = p^{m-1}S(u/p^{m-1},v/p^{m-1};p)$:

$$(8.26) \qquad |S(u,v;q)| < 2\, p^{m-1}\sqrt{p} .$$

The one final case to consider is $q = p^m$ and $(u,v,q) = q$. Then,

$$(8.27) \qquad S(u,v;q) = \varphi(q) = p^m - p^{m-1} < q.$$

We can write the inequalities (8.24), (8.25), (8.26) and (8.27) uniformly as

$$(8.28) \qquad |S(u,v;q)| < 2\sqrt{2}\, |q|^{1/2} d^{1/2} .$$

Now from (8.23) and (8.28) we obtain

$$|S(u,v;q)| < (2\sqrt{2})^r |q|^{1/2} \prod_{t=1}^{r} d_t^{1/2}$$

where $d_t = \min\left\{(u, p_t^{m_t}),\ (v_t, p_t^{m_t})\right\}$. Thus,

(8.29) $\quad |S(u,v;q)| < (2\sqrt{2})^r\ |q|^{1/2}\ (u,q)^{1/2}.$

Now using the symmetry of S(u,v;q) in u and v,

(8.30) $\quad |S(u,v;q)| < (2\sqrt{2})^r\ |q|^{1/2} (v,q)^{1/2}.$

Therefore, combining (8.29) and (8.30)

(8.31) $\quad |S(u,v;q)| < (2\sqrt{2})^r\ |q|^{1/2}\ d^{1/2}.$

Let $r = r(q)$ be the number of prime divisors of q. Then it is known [3.1, p. 46] that

$$(2\sqrt{2})^{r(q)} < C_\varepsilon\ q^\varepsilon, \qquad \varepsilon > 0,$$

therefore,

(8.32) $\quad |S(u,v;q)| < C_\varepsilon\ |q|^{1/2+\varepsilon}\ d^{1/2}.$

Now using (8.22) and (8.32)

$$|W_c(n+\kappa, \mu+\kappa_j)| < (\nu^*/gN) \sum_{t=1}^{N} C_\varepsilon\, cN^{\ 1/2+\varepsilon} \min\left\{(m,cN)^{\frac{1}{2}}(\omega_j+ct,cN)^{\frac{1}{2}}\right\}$$

(8.33)

$$< (\nu/gN)\ C_\varepsilon\ |cN|^{1/2+\varepsilon} \sum_{t=1}^{N} (\omega_j+ct,cN)^{1/2}.$$

Now $\omega_j = g_j(\mu+\kappa_j) \neq 0$ since $\mu \neq 0$. Then the above sum does not exceed

$$N^{1/2} \sum_{t=1}^{N} (\omega_j+ct,c)^{1/2} \leq N^{3/2}\ (\omega_j,c)^{1/2} \leq N^{3/2}\,\omega_j.$$

Therefore,

$$(8.34) \qquad |W_c(n+\kappa,\mu+\kappa_j)| < C(\mathcal{V},\Gamma,\varepsilon,\mu)\ |c|^{1/2+\varepsilon}$$

where $C(\mathcal{V},\Gamma,\varepsilon,\mu)$ is a positive constant depending on the parameters indicated.

We now give a series of examples in which our method applies. From theorem 1 we must only show in each case that the Kloosterman sums have the proper estimate.

EXAMPLE 1: van Lint [7] has shown that the commutator subgroup, $\Gamma'(1)$, of the full (inhomogeneous) modular group contains the principal congruence subgroup of level 12, $\Gamma(12)$. That is,

$$\Gamma(12) \subset \Gamma'(1) \subset \Gamma(1).$$

Recall that $\mathcal{V}(-I) = 1$. Every character (there are only six) on $\Gamma(1)$ will be identically 1 on $\Gamma'(1)$, hence on $\Gamma(12)$. By Petersson's theorem the Kloosterman sums associated with $\Gamma = \Gamma(1)$ and any character $\mathcal{V}$ on $\Gamma(1)$ will have the proper estimate. These constitute the cases considered by Lehner [6]. We can make the following easy extension. Suppose Γ is a congruence subgroup of level N and $\mathcal{V} = \mathcal{V}(\Gamma,-2)$ is a character on $\Gamma(1)$ restricted to Γ. Since $\Gamma(12N) \subset \Gamma(12)$, we see that $\mathcal{V} = 1$ on $\Gamma(12N)$. Again Petersson's theorem applies.

EXAMPLE 2: We consider a case similar to example 1, in fact, example 1 suggested this consideration. Let $\Gamma = \Gamma_2$, the subgroup of the modular group generated by

$US = (1, -1 \mid 1\ 0)$ and $SU = (0\ -1 \mid 1\ 1)$. Γ_2 is a normal subgroup of index 2 in $\Gamma(1)$. Its fundamental region consists of two replicas of the fundamental region of $R_0 = R(\Gamma(1))$, namely, $R_0 \cup UR_0 = R(\Gamma_2)$. We shall prove

<u>THEOREM 2</u>: The commutator subgroup of Γ_2 contains the principal congruence subgroup of level 6:

$$\Gamma(6) \subset \Gamma_2' .$$

The proof of this theorem rests on the following well known result: If Γ^* is a subgroup of Γ, Γ/Γ' is a finite group, and every character of Γ is identically 1 on Γ^* then $\Gamma^* \subset \Gamma'$.

Γ_2 is generated by US and SU, and $(US)^3 = (SU)^3 = -I$. If $v = v(\Gamma_2, -2)$ is a character on Γ_2, we find

$$v((US)^3) = v((SU)^3) = v(-I) = 1.$$

Therefore,

$$v(US) = e(k/3), \quad v(SU) = e(m/3), \quad k,m = 0,1,2.$$

Let $U_1 = U^2 = (1\ 2 \mid 0\ 1)$ and $U_2 = S^{-1}U^2S = (1\ 0 \mid -2\ 1)$. We use the Reidemeister-Schreier method for calculating generators for $\Gamma(6)$ as a subgroup of $\Gamma(2)$, which is generated by U_1 and U_2. We omit the details of this calculation; the method is given in [3] and [9]. If $V \in \Gamma(2)$ and $V = U_1^{a_1}U_2^{b_1} \cdots U_1^{a_n}U_2^{b_n}$ then $\sum a_i$, $\sum b_i$ is called the exponent sum of V in the generator U_1, U_2, respectively. Using the relations $U_1 = -US \cdot SU$ and $U_2 = -SU \cdot US$ we can calculate the exponent sums of V in the generators US and SU of Γ_2.

$\Gamma(6)$ is a free group. Let V_i, $i = 1, 2, \ldots, 13$ be the free generators of $\Gamma(6)$, τ_i and ω_i the exponent sum of V_i in U_1 and U_2 respectively, and, finally, ξ_i η_i the exponent sum of V_i in US and SU, respectively. The results of our calculation are contained in Table 8.1. Now from Table 8.1 we see that ξ_i and η_i are multiples of 3. Therefore, if $\upsilon = \upsilon(\Gamma_2, -2)$ is any character on Γ_2, then for any generator V_i of $\Gamma(6)$, we see that

$$\upsilon(V_i) = \upsilon(US)^{3n} \upsilon(SU)^{3n} = e(3kn/3) \cdot e(3mn/3) = 1$$

n an integer. Hence, υ is identically 1 on $\Gamma(6)$. This completes the proof of Theorem 2.

In Table 8.1 we have let ρ_i and σ_i be the exponent sum of V_i in terms of the generators US and S, respectively. Using the relations $U_1 = (US \cdot S)^2$ and $U_2 = S \cdot US \cdot S \cdot US$ we can calculate the exponent sums of V in the generators of S and US. Then from Table 8.1 we can see that van Lint's result can be improved to $\Gamma(6) \subset \Gamma'(1)$.

Now we apply Petersson's theorem to obtain the estimate on the Kloosterman sums for Γ_2 and υ.

Table 8.1 Generators and exponent sums for $\Gamma(6)$.

Generators of $\Gamma(6)$	Representation in terms of U_1 and U_2	ρ_i US	σ_i S	τ_i U_1	ω_i U_2	ξ_i US	η_i SU
V_1	U_1^3	6	6	3	0	3	3
V_2	$U_2^2U_1U_2^{-1}U_1^{-2}$	0	0	-1	1	0	0
V_3	$U_2U_1U_2^{-2}U_1^{-1}$	0	0	1	-1	0	0
V_4	$U_1U_2^2U_1U_2^{-1}$	6	6	2	1	3	3
V_5	$U_1U_2U_1^2U_2^{-2}U_1^{-2}$	0	0	1	-1	0	0
V_6	$U_1^2U_2^2U_1U_2^{-1}U_1^{-1}$	6	6	2	1	3	3
V_7	$U_1^2U_2U_1^2U_2^{-2}$	6	6	4	-1	3	3
V_8	U_2^3	6	6	3	0	3	3
V_9	$U_2U_1U_2U_1^{-1}U_2^{-1}U_1^{-1}$	0	0	-1	1	0	0
V_{10}	$U_1U_2^3U_1^{-1}$	6	6	0	3	3	3
V_{11}	$(U_1U_2)^2U_1^{-1}U_2^{-1}U_1^{-2}$	0	0	-1	1	0	0
V_{12}	$U_1^2U_2^3U_1^{-2}$	6	6	0	3	3	3
V_{13}	$U_1^2U_2U_1U_2U_1^{-1}U_2^{-1}$	6	6	2	1	3	3

EXAMPLE 3. Let $\Gamma = \Gamma_0(q)$ where q is a prime of the form $4m+1$. $\Gamma_0(q)$ is defined to be the group consisting of all those elements $V = (a\ b \mid c\ d) \in \Gamma(1)$ for which $c \equiv 0 \bmod q$. Let the multiplier system for $\Gamma_0(q)$ and the dimension -2 be defined by

$$\mathcal{V}(V) = \mathcal{V}((a\ b \mid c\ d)) = \left(\frac{d}{q}\right)$$

where $\left(\frac{d}{q}\right)$ is the Legendre symbol. Hecke [4, p. 809] calls a modular form for this group and multiplier system, a form of type $(-2, q, \left(\frac{d}{q}\right))$ or of real type. We see $\Gamma(q) \subset \Gamma_0(q) \subset \Gamma(1)$; since -1 is a quadratic residue of primes of the form $4m+1$, we have

$$\mathcal{V}(V) = \left(\frac{d}{q}\right) = \left(\frac{\pm 1}{q}\right) = 1$$

for $V \in \Gamma(q)$. Petersson's theorem applies.

EXAMPLE 4. The principal congruence subgroup $\Gamma(2)$ of level 2 is a free group with two generators U_1 and U_2. One can obtain a character on $\Gamma(2)$ by defining its values on U_1 and U_2. Say,

$$\mathcal{V}(U_1) = e(\alpha/8), \quad \mathcal{V}(U_2) = e(\beta/8), \qquad 0 \leq \alpha, \beta < 8.$$

We choose this notation to agree with Maak [8]. Any element of $\Gamma(2)$ has a representation of the form $V = U_1^{a_1} U_2^{b_1} \cdots U_1^{a_n} U_2^{b_n}$. Then,

$$\mathcal{V}(V) = \mathcal{V}(U_1)^{\sum a_i}\,\mathcal{V}(U_2)^{\sum b_i} = e(\textstyle\sum a_i\alpha/8)e(\sum b_i\beta/8).$$

Maak, in the paper cited above, solved the more difficult problem of determining $\mathcal{V}(V)$ in terms of the elements a, b, c, d of V. We shall not have occasion to use his results. We use the Reidemeister-Schreier method to calculate the free generators of $\Gamma(4)$ and $\Gamma(8)$. In tables 8.2 and 8.3 are tabulated these generators along with the exponent sums τ_i and ω_i of these generators in terms of U_1 and U_2 respectively. The last column gives the value of the character.

Table 8.2 Generators and exponent sums for $\Gamma(4)$

Generators of $\Gamma(4)$	Representation in terms of U_1 and U_2	τ_i U_1	ω_i U_2	$\mathcal{V}(T_i)$
T_1	U_1^2	2	0	$e(\alpha/4)$
T_2	U_2^2	0	2	$e(\beta/4)$
T_3	$U_2U_1^2U_2^{-1}$	2	0	$e(\alpha/4)$
T_4	$U_1U_2U_1^{-1}U_2^{-1}$	0	0	1
T_5	$U_2U_1U_2U_1^{-1}$	2	0	$e(\beta/4)$

We see that the $v = v(\Gamma(2), -2)$ determined by

$$
(8.35)\qquad
\begin{aligned}
&1)\ \alpha = 0, \quad \beta = 0\\
&2)\ \alpha = 0, \quad \beta = 4\\
&3)\ \alpha = 4, \quad \beta = 0\\
&4)\ \alpha = 4, \quad \beta = 4
\end{aligned}
$$

will be identically 1 on $\Gamma(4)$.

From the data given in Table 8.3 we see that, in addition to the above multiplier systems, the multiplier systems determined by

$$
(8.36)\qquad
\begin{aligned}
&5)\ \alpha = 0, \quad \beta = 2\\
&6)\ \alpha = 4, \quad \beta = 2\\
&7)\ \alpha = 0, \quad \beta = 6\\
&8)\ \alpha = 4, \quad \beta = 6
\end{aligned}
$$

are identically 1 on $\Gamma(8)$.

Petersson's theorem applies to these cases. We shall return to these examples at the end of the next section.

Table 8.3 Generators and exponent sums for $\Gamma(8)$

Generators for $\Gamma(8)$	Representation in terms of T_i	τ_i (U_1)	ω_i (U_2)	$v(X_i)$
X_1	T_1^2	4	0	$e(\frac{\alpha}{2})$
X_2	$T_2T_1T_2^{-1}T_1^{-1}$	0	0	1
X_3	$T_4T_1T_4^{-1}T_1^{-1}$	0	0	1
X_4	$T_1T_2T_1T_2^{-1}$	4	0	$e(\frac{\alpha}{2})$
X_5	$T_1T_4T_1T_4^{-1}$	4	0	$e(\frac{\alpha}{2})$
X_6	$T_5T_1T_5^{-1}T_1^{-1}$	0	0	1
X_7	$T_1T_5T_1T_5^{-1}$	4	0	$e(\frac{\alpha}{2})$
X_8	T_2^2	0	4	$e(\frac{\beta}{2})$
X_9	$T_4T_2T_5^{-1}T_1^{-1}$	-2	0	$e(-\frac{\alpha}{4})$
X_{10}	$T_5T_2T_4^{-1}T_1^{-1}$	-2	4	$e(-\frac{\alpha}{4}+\frac{\beta}{2})$
X_{11}	$T_1T_2^2T_1^{-1}$	0	4	$e(\frac{\beta}{2})$
X_{12}	$T_1T_4T_2T_5^{-1}$	2	0	$e(\frac{\alpha}{2})$
X_{13}	$T_1T_5T_2T_4^{-1}$	2	4	$e(\frac{\alpha}{4}+\frac{\beta}{2})$
X_{14}	$T_3T_1^{-1}$	0	0	1
X_{15}	T_1T_3	4	0	$e(\frac{\alpha}{2})$
X_{16}	$T_4T_3T_4^{-1}T_1^{-1}$	0	0	1
X_{17}	$T_5T_3T_5^{-1}T_1^{-1}$	0	0	1

Table 8.3 continued

Generators for $\Gamma(8)$	Representation in terms of T_i	τ_i (U_1)	ω_i (U_2)	$\nu(X_i)$
X_{18}	$T_1T_2T_3T_2^{-1}$	4	0	$e(\frac{\alpha}{2})$
X_{19}	$T_1T_4T_3T_4^{-1}$	4	0	$e(\frac{\alpha}{2})$
X_{20}	$T_1T_5T_3^{-1}T_5^{-1}$	0	0	1
X_{21}	$T_2T_4T_5^{-1}T_1^{-1}$	-2	0	$e(-\frac{\alpha}{4})$
X_{22}	T_4^2	0	0	1
X_{23}	$T_5T_4T_2^{-1}T_1^{-1}$	-2	0	$e(-\frac{\alpha}{4})$
X_{24}	$T_1T_2T_4T_5^{-1}$	2	0	$e(\frac{\alpha}{4})$
X_{25}	$T_1T_4^2T_1^{-1}$	0	0	1
X_{26}	$T_1T_5T_4T_2^{-1}$	2	0	$e(\frac{\alpha}{4})$
X_{27}	$T_2T_5T_4^{-1}T_1^{-1}$	-2	4	$e(-\frac{\alpha}{4} + \frac{\beta}{2})$
X_{28}	$T_4T_5T_2^{-1}T_1^{-1}$	-2	0	$e(\frac{\alpha}{4})$
X_{29}	T_5^2	0	4	$e(\frac{\beta}{2})$
X_{30}	$T_1T_5^2T_1^{-1}$	0	4	$e(\frac{\beta}{2})$
X_{31}	$T_1T_2T_5T_4^{-1}$	2	4	$e(\frac{\alpha}{4} + \frac{\beta}{2})$
X_{32}	$T_1T_4T_5T_2^{-1}$	2	0	$e(\frac{\alpha}{4})$
X_{33}	$T_2T_3T_2^{-1}T_1^{-1}$	0	0	1

9. The Inner Product Formula

The set of all cusp forms $\mathcal{C}^{+}(\Gamma,-2,\vartheta)$ is a finite dimensional vector space. Petersson [10] introduced an inner product

$$(9.1) \qquad (F(z), G(z); R(\Gamma)) = \int_{R(\Gamma)} F(z)\,\overline{G(z)}\; dx\, dy$$

on this space. The integral is a Lebesgue integral, and $R(\Gamma)$ is a fundamental region for Γ. The integral is known to converge and be independent of $R(\Gamma)$ [10, pp. 494-496]. The object of this section is to establish the <u>inner</u> <u>product</u> <u>formula</u>:

<u>THEOREM 3</u>: For $\mu \geq 1$, $F(z)$ $\mathcal{C}^{+}(\Gamma,-2,\vartheta)$, we have

$$(9.2) \qquad (F(z), G(z,\vartheta,A_j,\Gamma,\mu); R(\Gamma)) = \frac{a_\mu(F,A_j,\Gamma)\,\lambda_j^2}{4\pi(\mu+\kappa_j)}$$

where $a_\mu(F,A_j,\Gamma)$ is the μ-th coefficient in expansion of $F(z)$ at the cusp $A_j^{-1}\infty = p_j$ (see(2.12)).

We start with

Lemma 5: If $F(z)\in\mathcal{C}^{+}(\Gamma,-2,\vartheta)$ then for $y \geq y_0 > 0$

$$(9.3) \qquad F(z) = \mathcal{O}(\exp[-2\omega y/\lambda])$$

where $\omega = \kappa$ if $\kappa > 0$, otherwise $\omega = 1$.

Proof: $F(z)$ has the expansion in (2.12) with $A_j = I$ where $s + \kappa > 0$ since $F(z)$ is a cusp form. This Fourier series converges absolutely uniformly in $y \geq y_0 > 0$. The result follows.

Let $R(\Gamma)$ be a fundamental region for Γ which is connected and lies in the strip $\xi < x < \xi + \lambda$ where ξ is a cusp $A_k^{-1}\infty$. This fundamental region is to be bounded by a finite number of straight lines and circular arcs. Each parabolic cycle is to consist of a single element. We begin the proof of (9.2) with $A_j = I$; let

$$J = (G(z,\mathcal{V},I,\Gamma,\mu),\ F(z);\ R). \tag{9.4}$$

For each $p_j = A_j^{-1}\infty$, $j = 2, 3, \ldots, \sigma(\Gamma)$ let $R_j = R_{p_j}$ be a parabolic sector of R at p_j. We suppose the sectors are chosen small enough so that, for $j \neq k$, $R_j \cap R_k = \emptyset$. Let $R_1 = R - \bigcup_{j=2}^{\sigma} R_j$. We are now able to write

$$J = \sum_{j=1}^{\sigma} (G(z,\mathcal{V},I,\Gamma,\mu),\ F(z);\ R_j) = \sum_{j=1}^{\sigma} J_j \tag{9.5}$$

where R_j indicates the set over which the functions are integrated.

We now introduce two results of Petersson [10]. If $\mathcal{B}$ is an "admissible" region, then for $V = (a\ b \mid c\ d)$ a real unimodular matrix

$$\int_{V^{-1}\mathcal{B}} (F(z)|V)\ (\bar{G}(z)|V)dxdy = \int_{\mathcal{B}} F(z)\ \bar{G}(z)\ dx\ dy; \tag{9.6}$$

and for $L \in \Gamma$

(9.7) $$\int_{L\mathcal{B}} F(z)\,\bar{G}(z)\,dx\,dy = \int_{\mathcal{B}} F(z)\,\bar{G}(z)\,dx\,dy.$$

Suffice it to say that R, $R_1, \ldots R_\sigma$ are "admissible" regions.

We now apply (9.6) to J_k, $2 \leq k \leq \sigma$, with $V = A_k^{-1}$.

(9.8) $$J_k = \int_{A_kR_k} \left\{G(z,\mathcal{V},I,\Gamma,\mu) \mid A_k^{-1}\right\} \left\{ \bar{F}(z) \mid A_k^{-1} \right\} dx\,dy$$

$$= \int_{A_kR_k} G(z,\mathcal{V}_k,A_k^{-1},\Gamma_k,\mu) \quad \bar{F}(z) \mid A_k^{-1} \quad dx\,dy$$

The last equality comes from (7.1); $\Gamma_k = A_k \Gamma A_k^{-1}$.

We now use the representation

(9.9) $$G(z,\mathcal{V}_k,A_k^{-1}, \Gamma_k,\mu) = \sum_{c \in \mathcal{C}(A_k^{-1},\, \Gamma_k)} H(c,z).$$

This converges absolutely uniformly for $y \geq y_0 > 0$. The parabolic sector R_k is mapped by A_k onto a strip $\mathcal{S}_k = \left\{z = x + iy\text{:}\ \xi_k < x < \xi_k + \lambda_k,\ \ y > \eta_k > 0\right\}$. Introduce (9.9) into (9.8) and interchange summation and integration; thus,

(9.10) $$J_k = \sum_{c \in \mathcal{C}(A_j^{-1},\, \Gamma_k)} \int_{A_kR_k} H(c,z)\,\bar{F}(z) \mid A_k^{-1}\,dx\,dy\ .$$

We now justify this interchange of integration and summation. We see from (3.10), (3.12), (3.14) and lemma 5 if $c \neq 0$

$$\left|H(c,z)F(z)\,|A_k^{-1}\right| < C(\eta_k)\exp[-2\pi\omega y/\lambda]\Big\{(1/c^2y^2)\exp[2\pi(\mu+1)/y$$

$$+(C/|c|^{5/2}\, y\,\sin\delta\,)\exp[2\pi(\mu+1)/\sqrt{y\sin\delta}\,]$$

$$+(C_\varepsilon/\,|c|^{3/2-\varepsilon})(1-\exp[-\,2\pi y/\lambda])^{-2}$$

Recall that δ = arg. z; since $\xi_k < x < \xi_k + \lambda_k$, $y > \eta_k > 0$, we see that $\sin\delta$ is bounded away from 0. If $c = 0$

$$\left|\,H(0,z)\;F(z)\,|\;A_k^{-1}\,\right| < C(\eta_k)\;\exp[-2\;(\mu+\kappa+\omega)/\lambda\;].$$

Therefore,

$$\sum_c \int_{\eta_k}^{\infty} \left|H(c,z)\;F(z)|\;A_k^{-1}\right|\,dy < +\infty,$$

and this completes the justification of interchange of summation and integration.

H(c,z) is the series

$$\sum_{d\in\mathcal{D}(c,A_k^{-1},\Gamma_k)} \bar{\nu}_k(A_kV_{c,d})(cz+d)^{-2}e((\mu+\kappa)V_{c,d}z/\lambda).$$

Recall that $\kappa = \kappa(I,\Gamma) = \kappa(A_k^{-1},\Gamma_k)$ and $\lambda = \lambda(A_k^{-1},\Gamma_k)$. This series is absolutely uniformly convergent for $y \geq y_0 > 0$. Introduce this expression into the terms of

(9.10) and interchange orders of summation and integration. We obtain

(9.11)

$$J_k = \sum_{c \in \mathcal{C}(A_k^{-1}, \Gamma_k)} \sum_{d \in \mathcal{D}(c, A_k^{-1}, \Gamma_k)} \iint_{A_k R_k} \frac{e((\mu+\kappa)V_{c,d} z/\lambda)}{\mathcal{V}_k(A_k V_{c,d})(cz+d)^2} \bar{F}(z) \mid A_k^{-1} dxdy.$$

We understand that the summation in (9.11) is to be carried out in the same manner as indicated in Theorem 1.

We now justify the interchange of summation and integration on d. We see from lemma 5 and (3.5) that if $d \neq 0$

$$\left| \frac{e((\mu+\kappa)V_{c,d} z/\lambda)}{\mathcal{V}_k(A_k V_{c,d})(cz+d)^2} F(z) \mid A_k^{-1} \right| < C(\eta_k) \frac{\exp[-2\pi\omega y/\lambda]}{|cz+d|^2}$$

$$\leq C(\eta_k)(1/d^2) \exp[-2\pi\omega y/\lambda].$$

Therefore,

$$\sum_d \int_{\eta_k}^{\infty} \left| \frac{e((\mu+\kappa)V_{c,d} z/\lambda)}{\mathcal{V}_k(A_k^{-1} V_{c,d})(cz+d)^2} \bar{F}(z) \mid A_k^{-1} \right| dy$$

$$< C(\eta_k) \left\{ 1 + \sum_d{}' 1/d^2 \right\} \int_{\eta_k}^{\infty} \exp[-2\pi\omega y/\lambda] dy$$

$$< +\infty.$$

In the terms of the series (9.11) we make the change of variable $w = A_k V_{c,d}\, z$. The Jacobian of the transformation is

$$\left| -(cd_k+bc_k)w + (cb_k+aa_k) \right|^{-4}.$$

Since $F(z) \mid A_k^{-1} \in \mathcal{C}(\Gamma_k,-2,\upsilon_k)$

$$F(V_{c,d}^{-1}A_k^{-1}w) \,|\, A_k^{-1} = \upsilon_k(V_{c,d}^{-1}A_k^{-1})(-(cd_k+ac_k)w+(cb_k+aa_k))^2 F(w) \,|\, A_k^{-1}$$

On computing one finds that

$$(cV_{c,d}^{-1}\; A_k^{-1}w+ d)^{-2} = (-c_k w + a_k)^2(-(cd_k+ac_k)w+(cb_k+aa_k))^{-2}.$$

Therefore, substituting these results into the terms of (9.11), we find they become

$$\int_{A_k V_{c,d} A_k R_k} e((\mu+\kappa)A_k^{-1}w/\lambda)(-c_k w + a_k)^{-2}\, \overline{F}(z) \mid A_k^{-1}\, du dv.$$

Once again we make a change of variables. Let $z = A_k^{-1}w$ then the above integral becomes

$$\int_{V_{c,d} A_k R_k} e((\mu+\kappa)z/\lambda)\, \overline{F}(z)\, dx\, dy\, .$$

We have used the fact that the Jacobian is $|c_k z + d_k|^{-4}$, and the identity

$$(c_k z + d_k)^{-2}F(A_k z) \mid A_k^{-1} = (c_k z+d_k)^{-2}(c_k A_k z + d_k)^{-2}\, F(z)$$

$$= F(z)$$

Thus,

$$(9.12) \qquad J_k = \sum_c \sum_d \int_{V_{c,d}A_kR_k} e((\mu+\kappa)z/\lambda)\,\bar{F}(z)\,dx\,dy.$$

c and d are summed over the same sets as in (9.11) and summation is carried out in the manner of Theorem 1.

We mentioned that $V_{c,d} \in \Gamma A_k^{-1}$ so that $V_{c,d}A_k \in \Gamma$.

As (c,d) runs over $\mathcal{O}(A_k^{-1}, \Gamma_k)$, $(c,d)A_k$ runs over $\mathcal{O}(I, \Gamma)$. Let $\mathcal{R}$ be a complete set of matrices in Γ with different lower row. We shall make an appropriate choice of the upper row. Let $[U^\lambda]$ denote the cyclic group generated by U^λ. Then

$$(9.13) \qquad \Gamma = \bigcup_{\in \mathcal{R}} [U^\lambda]\, V .$$

Let

$$(9.14) \qquad \mathcal{D} = \bigcup_{V \in \mathcal{R}} VR(\Gamma) .$$

$\mathcal{D}$ is a fundamental region for $[U^\lambda]$. We suppose that $V_{c,d} \in \Gamma A_k^{-1}$ is chosen so that $V_{c,d}A_k \in \mathcal{R}$. Define

$\mathcal{B}_k = \bigcup_{V_{c,d}A_k \in \mathcal{R}} V_{c,d}\, A_k\, R_k$ for $k = 1, 2, \ldots, \sigma$. Then,

by the completed additivity of the integral, we see from (9.12)

$$(9.15) \qquad J_k = \int_{\mathcal{B}_k} e((\mu+\kappa)z/\lambda)\,\bar{F}(z)\,dxdy .$$

Furthermore, $\bigcup_{k=1}^{\sigma} \mathcal{B}_k = \mathcal{D}$. Thus, from (9.5) we se

that

(9.16) $$J = \int_{\mathcal{D}} e((\mu+\kappa)z/\lambda)\, \overline{F}(z) \; dxdy \; .$$

The strip $\mathcal{S} = \{z = x + iy: \xi < x < \xi + \lambda \;, \; y > 0\}$ is a fundamental region for $[U^{\lambda}]$. Due to the way we chose $R(\Gamma)$, there is a determination of the V's $\in \mathcal{Q}$ so that $\mathcal{D} = \mathcal{S}$. We shall use this determination.

Define $\mathcal{D}(y_0,y_1) = \{ z \in \mathcal{D} : y_0 < y < y_1 \}$ then

$$J = \lim_{\substack{y_1 \to \infty \\ y_0 \to 0}} J(y_0,y_1) = \lim_{\substack{y_1 \to \infty \\ y_0 \to 0}} \int_{\mathcal{D}(y_0,y_1)} e((\mu+\kappa)a/\lambda)\overline{F}(z)dxdy.$$

By the lemma of this section

$$\int_{\mathcal{D}(y_0,y_1)} \left| F(z)e((\mu+\kappa)z/\lambda) \right| dxdy < \frac{C(y_0)\lambda^2}{2\pi(\mu+\kappa+\omega)} \exp[-2\pi(\mu+\kappa+\omega)y_0/\lambda]$$

If now $y_1 \to \infty$, the integral tends to a limit $J(y_0,\infty)$. Introduce the Fourier expansion for $F(z)$ and interchange orders of summation and integration:

$$J(y_0,\infty) = \sum_{n+\kappa>0} \overline{a}_n(F,I,\Gamma) \int_{\mathcal{D}(y_0,\infty)} \overline{e((n+\kappa)z/\lambda)}\, e((\mu+\kappa)z/\lambda)dxdy$$

The interchange of order is justified by the Fourier series converging absolutely uniformly in $y \geq y_0 > 0$. Consider

$$\int_{\mathcal{D}(y_0,\infty)} \overline{e((n+\kappa)z/\lambda)}\, e((\mu+\kappa)z/\lambda)\, dx\, dy$$

$$= \int_{y_0}^{\infty} \exp[-2\pi(n+\kappa+\mu)y/\lambda]dy \int_{\xi}^{\xi+\lambda} \exp[2\pi i(\mu-n)x/\lambda]dx$$

$$= \frac{\lambda^2}{4\pi(\mu+\kappa)} \exp[-4\pi(\mu+\kappa)y_0/\lambda]\ \delta_{\mu,n}\ .$$

$\delta_{\mu,n}$ is the Kronecker symbol. Thus,

$$J(y_0,\infty) = \frac{\overline{a}_\mu(F,I,\Gamma)\lambda^2}{4\pi(\mu+\kappa)} \exp[-4\pi(\mu+\kappa)y_0/\lambda].$$

Now let $y_0 \longrightarrow 0$. Since $(F, G; R) = \overline{(G, F; R)}$ the result (9.2) follows for $A_j = I$.

Consider

$$J(A_j) = (F, G(z,\nu,A_j,\Gamma,\mu); R(\Gamma)).$$

Then by (9.6)

$$J(A_j) = (F|A_j^{-1}, G(z,\nu,A_j,\Gamma,\mu)|A_j^{-1}; R(\Gamma_j))$$

$$= (F|A_j^{-1}, G(z,\nu_j,I,\Gamma_j,\mu); R(\Gamma_j)).$$

Now $\lambda_j = \lambda(A_j,\Gamma) = \lambda(I,\Gamma_j)$ and $\kappa_j = \kappa(A_j,\Gamma) = \kappa(I,\Gamma_j)$.

Thus, from (2.15) and the case already proved

$$(F|A_j^{-1},\ G(z,\mathcal{V}_j,I,\Gamma_j,\mu);\ R(\Gamma_j)) = \frac{a_\mu(F,A_j,\Gamma)\,\lambda_j^2}{4\pi(\mu+\kappa_j)}$$

This completes the proof of the inner product formula.

The following well known result is an immediate consequence of the inner product formula.

Theorem: The vector space spanned by the Poincare series $G(z,\mathcal{V},A_j,\Gamma,\mu)$, $\mu = 1, 2, \ldots$ equals $\mathcal{C}^+(\Gamma, -2, \mathcal{V})$.

We return to the examples we considered at the end of section 8.

EXAMPLE 1. $\Gamma = \Gamma(1)$; there are six characters on $\Gamma(1)$. In only one instance is the dimension of the space $\mathcal{C}^+(\Gamma(1),-2,\mathcal{V})$ positive. Petersson [12, p. 189] gives a formula for calculating the dimension of $\mathcal{C}^+$. Using his formula, we find that when $\mathcal{V}(S) = -1$ and $\mathcal{V}(US) = e(-1/3)$ is the character defined on the generators S and US of $\Gamma(1)$, the dimension of $\mathcal{C}^+(1),-2,\Gamma)$ is 1. This proves that in this instance not all the functions $G(z,\mathcal{V},I,\Gamma,\mu)$ with $\mu > 0$ are identically 0. However, for the remaining five characters on $\Gamma(1)$ and for $\mu > 0$, we have $G(z,\mathcal{V},I,\Gamma,\mu) = 0$.

EXAMPLE 2. $\Gamma = \Gamma_2 = [SU, US]$. There are nine characters on Γ_2. As in example 1, there is only one character for which dim $\mathcal{C}^+(\Gamma_2,-2,\mathcal{V})$ is positive. If $\mathcal{V}(SU) = e(-1/3) = \mathcal{V}(US)$ then the dimension is 1.

EXAMPLE 3. In the case $\Gamma = \Gamma_0(q)$, $\mathcal{V}(V) = (\frac{d}{q})$, $V = (a\ b \mid c\ d)$, q a prime of the form 4 m + 1, Hecke [4, p. 815] gives the dimension of $\mathcal{C}^+(\Gamma_0(q),-2,\mathcal{V})$. The first prime for which this dimension is positive is q = 29, in which case the dimension is 2.

EXAMPLE 4. If $\Gamma = \Gamma(2)$ and $\mathcal{V}$ is the multiplier system determined by $\mathcal{V}(U_1) = e(1/2)$, $\mathcal{V}(U_2) = e(1/4)$, then the dimension of the space $\mathcal{C}^+(\Gamma(2),-2,\mathcal{V})$ is 1. In the other seven cases considered the dimension is 0.

BIBLIOGRAPHY

1. L. E. Dickson, Studies in the Theory of Numbers, Univ. of Chicago Press, 1930.

1.1. T. Estermann, Beweis eines Sates von Kloosterman, Hamb. Abh. 7(1930), 82-98.

2. L. R. Ford, Automorphic Functions, Second edition, New York, 1951.

2.1. R. C. Gunning, Lectures on Modular Forms, Lecture Notes, Math. Dept. Princeton Univ., 1958.

3. M. Hall, Jr., The Theory of Groups, New York, 1959.

3.1. G. Hardy, Ramanujan, Twelve Lectures on Subjects Suggested by His Life and Work, Cambridge, 1940.

4. E. Hecke, Mathematische Werke, Göttingen, 1959.

5. M. I. Knopp, Fourier series of automorphic forms of non-negative dimension, Illinois J. Math. 5(1961), 18-42.

5.1. ________, Automorphic forms of non-negative dimension and exponential sums, Michigan Math. J. 7(1960) 257-287.

6. J. Lehner, On Modular forms of negative dimension, Michigan Math. J. 6(1959), 71-88.

7. J. H. van Lint, On the multiplier system of the Riemann-Dedekind function η, Neder. Akad. Wetensch. Indag. Math. 20(1958), 522-527.

8. W. Maak, Fastperiodische Funktionen auf der Modulgruppe, Math. Scand. 3 (1955), 44 - 48.

9. W. Magnus, Discrete Groups, N. Y. U. Inst. of Math. Sci., 1952.

10. H. Petersson, Metrisung der Automorphen Formen und der Theorie der Poincaréschen Reihen, Math. Ann. 117(1940), 453 - 537.

11. ____________, Über Modulfunktionen und Partitionenprobleme, Abh. D. Akad. Wiss. Berlin, 2, 1954, 1 - 59.

12. ____________, Automorphe Formen als metrische Invarianten I, Math Nachr. 1(1948), 158 - 212.

13. H. Rademacher, The Fourier series and the functional equation of the absolute modular invariant $J(\tau)$, Amer. J. Math. 61 (1939), 237 - 248.

14. R. A. Rankin, On horocyclic groups, Proc. London Math. Soc. (3) 4 (1954), 219 - 234.

15. H. Salié, Über die Kloostermanschen Summen $S(u,v;q)$, Math. Zeit. 34 (1932), 91 - 109.

16. E. C. Titchmarsh, The Theory of Functions, Second Edition, Oxford Univ. Press, 1952.

17. A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. U. S. A. 34 (1948), 204 - 207.

18. E. T. Whittaker, and G. N. Watson, Course of Modern Analysis, Fourth Edition, Cambridge, 1958.