DIRECT ANALYSIS OF IMPLIED VOLATILITY FOR EUROPEAN OPTIONS

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ABSTRACT

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We show existence and uniqueness of a strong solution to a linear non-uniformly parabolic equation, which gives the fair price of a normalized European call option. We then provide a direct link between local and implied volatilities in the form of a quasilinear degenerate parabolic partial differential equation. We also establish closed-form asymptotic formulae for the implied volatility near expiry as well as for deep in and out of the money options, using a generalized comparison principle on bounded domains.

To my parents, for their faith and love

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For any errors or inadequacies that may remain in this work, of course, the responsibility is entirely my own.

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Introduction

A *European call option* on an *underlying security* (or *underlying*) X, with *strike price* K and *exercise date (or expiry)* T is a financial contract between two parties, the buyer (or " holder") and the seller (or "writer"), written at time t with the following properties: i) The holder of the contract has, exactly at the time $t = T$, has the right to buy X at the price K, ii) The holder of the option has no obligation to buy the security. However, The seller is obligated to sell the underlying should the buyer so decide. The buyer pays a fee (called a *premium*) for this right. The *underlying security* (or *underlying*) is the commodity or financial instrument that can be sold or bought when an option holder decides to exercise his contract.

The Black-Scholes model [10], [35] of a call option on a *stock* has gained wide recognition in both academia and industry. It makes the following explicit assumptions:

- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- It is possible to buy and sell any amount, even fractional, of stock (this includes short

selling).

- The above transactions do not incur any fees or costs (i.e., frictionless market).

- The stock price S_t follows a geometric Brownian motion with constant drift and volatility:

$$dS_t = S_t(\mu dt + \Sigma dW_t),$$

where t is time, μ and Σ are constants and W_t is a standard Brownian motion. The parameter σ is called the *volatility* of the stock S_t. It is the relative rate at which the price of a security moves up and down.

- The underlying security does not pay a dividend. [1]

The price $C(S_t, t; K, T)$ of a European call option written on S_t with strike K and maturity T satisfies the linear backward parabolic partial differential equation

$$C_t + \frac{\Sigma^2}{2} S^2 C_{SS} + rSC_S - rC = 0 \quad \text{in } (0, +\infty) \times (0, T) \tag{1.1a}$$

$$C(S, T) = (S - K)_+, \tag{1.1b}$$

where r is the risk-free short-term interest rate. It is well known that, the solution to equation (1.1) is:

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)} N(d_2), \tag{1.2}$$

[1] Although the original model assumed no dividends, trivial extensions to the model can accommodate a continuous dividend yield factor.

where

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\Sigma^2}{2})(T-t)}{\Sigma\sqrt{T-t}}, \tag{1.3}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\Sigma^2}{2})(T-t)}{\Sigma\sqrt{T-t}} = d_1 - \Sigma\sqrt{T-t}, \tag{1.4}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy. \tag{1.5}$$

The expression (1.2) is known as the *Black-Scholes formula*

Once we have a measure of the (statistical) volatility for any underlying, we can plug the value into a standard options pricing model[2] and calculate the fair market value of an option. A model's fair market value[3], however, is often out of line with the actual market value for that same option. This is known as option mispricing. To understand the reason, we need to look closer at the role *implied volatility* plays in the equation. The *implied volatility* of the option is the volatility that, when used in a particular pricing model, yields a theoretical value for the option equal to the current market price of that option. It is the expected volatility the market is pricing into the option.

Often, the implied volatility of an option is a more useful measure of the option's relative value to other options than is its price. The reason is that the price of an option depends most directly on the price of its underlying asset. If an option is held as part of a *delta neutral*[4] portfolio (that is, a portfolio that is hedged against small moves in the underlying's

[2]for example, the Black-Scholes Formula

[3]for example, (1.2)

[4]In finance, delta neutral describes a portfolio of related financial securities, in which the portfolio value remains unchanged due to small changes in the value of the underlying security, *i.e.* $\Delta = \frac{\partial V}{\partial S}$. Such a portfolio typically contains options and their corresponding underlying securities such that positive and negative delta components offset, resulting in the portfolio's value being relatively insensitive to changes in the value of the underlying

price), then the next most important factor in determining the value of the option will be its implied volatility.

Another way to look at implied volatility is to think of it as a price, not as a measure of future stock moves. In this view it simply is a more convenient way to communicate option prices than currency. Prices are different in nature from statistical quantities: one can estimate volatility of future underlying returns using any of a large number of estimation methods, however the number one gets is not a price. A price requires two counterparties, a buyer and a seller. Prices are determined by supply and demand. Statistical estimates depend on the time-series and the mathematical structure of the model used. It is a mistake to confuse a price, which implies a transaction, with the result of a statistical estimation, which is merely what comes out of a calculation. Implied volatilities are prices: they have been derived from actual transactions. Seen in this light, it should not be surprising that implied volatilities might not conform to what a particular statistical model would predict. [Wikipedia]

In general, the value of an option depends on an estimate of the future realized price volatility, Σ, of the underlying. Or, mathematically: $C = f(\cdot, \Sigma)$[5]. The function f is monotonically increasing in Σ, [6] meaning that a higher value for volatility results in a higher theoretical value of the option. Conversely, by the inverse function theorem [1], there can be at most one value for φ that, when applied as an input to $f(\cdot, \varphi)$, will result in a particular value for C.

I.) In principle, by regarding σ as constant, the implied volatility can be inferred by

security.

[5] where $\cdot$ represents S, K, T, t, and r.

[6] We give a proof for such property based on the Black-Scholes model, using the generalized Maximum Principle we derive.

inverting the closed form of the solution to (1.1), the option price equation. However, this is known to be computationally difficult, especially near expiry or far from the money.

II.) Another shortcoming is that the Black-Scholes model assumes the volatility of the underlying asset S price as constant. However, options based on the same underlying asset but with different strike value and expiration times will yield different implied volatilities. The *volatility smile*[7] or *smirk*[8] is a well-known manifestation of this phenomenon. This is generally viewed as evidence that the security's volatility is not constant.

There have been various attempts to extend the Black-Scholes theory to account for the volatility smile and the term structure. One class of models introduces a none-traded source of risk such as jumps [36] or *stochastic volatility*, including those given by Hull and White [29], and Heston [26]. Rubinstein [39], Derman and Kani [20] have independently constructed a discrete approximation to the risk-neutral process for the underlying asset in the form of a bi/trinomial tree, which are extensions of the original Cox et al. [18] binomial trees. Bouchouev and Isakov [12, 31] reduce the identification of volatility to an inverse parabolic problem with the final observation and establish uniqueness and stability results under certain assumptions. Then, they obtain a non-linear Fredholm integral equation for unknown volatility after dropping terms of higher orders in time to maturity and solve the equation iteratively. Deng, Yu, and Yang [19] use an optimal control framework to discuss an inverse problem of determining the implied volatility when the average option premium,

[7] In markets such as FX options or equity index options

[8] such as for longer term equity options

namely the average value of option premium corresponding with a fixed strike price and all possible maturities from the current time to a chosen future time, is known.

Furthermore, the difficulty regarding inverting the extended BS formula is asserted by *the impossibility of obtaining volatility as a closed form function of the option value and the remaining variables* (see [28, 43, 9]; a similar assertion can also be found in the introduction of the forthcoming paper by Teichmann and Schachermayer [41]). Most works on this problem assume such impossibility and proceed in two broad directions: one theoretical, that attempts to obtain abstract mathematical properties of the implied volatility, such as partial differential equations governing it or similar approaches (see [22, 11, 5]), and the other research direction somewhat more practical, centering on obtaining approximate formulas and testing them against market data [14, 17, 27]. We list a few approximations of the inversion of the Black-Scholes formula. In the following, $\tau = T - t$ is the *time to maturity.*

- Li [37] developed a closed-form method for the implied volatility based on *rational approximation.* Rational approximation has been used extensively in both physical and social sciences but his paper is among the first to apply this approximation to the problem of inverting the Black-Scholes formula. His approximation scheme is much faster than typical solver methods and very accurate for both at-the-money and away-from-the-money options.

- Chargoy-Coronaa and Ibarra-Valdez [32] use *elementary arithmetic operations and functions*, together with the normal distribution function and its inverse, to obtain the asymptotic and approximate formulas for the option value, and an approximate formula for the implied volatility. Define the log-moneyness as $\alpha = \log(X/Ke^{-r\tau}) =$

$\log(X/K) + r\tau$, $\theta = \Sigma\sqrt{\tau}$. The authors have the approximating option value:

$$u_a(\theta, \alpha, X) = Xe^{-\alpha}[2N(\theta/2) - 1],$$

and an approximation formula for the volatility:

$$\Sigma_a = \frac{2}{\sqrt{\tau}}\varphi\left(\frac{u_a e^{r\tau} + K}{2K}\right),$$

where φ is the inverse of N, the standard normal cumulative distribution function. They also showed an error estimate in Theorem 3: There are $a, b > 0$, with b small enough, such that for all $\theta > 0$, $|\alpha| < a$, and $s > (1+b)u_0$ it holds

$$|\Sigma - \Sigma_a| \leq \frac{2\sqrt{2\pi}}{\sqrt{\tau}}|\alpha|e^{|\alpha|}\exp\left\{\frac{1}{2}\varphi\left(\frac{(1 + \frac{e^{|\alpha|}}{1+a} + |\alpha|e^{|\alpha|})^2}{2}\right)\right\}.$$

- Brenner and Subrahmanyam [14] provided an elegant formula to compute an implied volatility that is accurate when a stock price is exactly equal to a discounted strike price:

$$\Sigma \approx \sqrt{\frac{2\pi}{\tau}}\frac{C}{S}.$$

Feinstein [23] independently derived an essentially identical formula.

- Corrado and Miller [17] provided an improved quadratic formula which is valid when stock prices deviate from discounted strike prices:

$$\Sigma \approx \sqrt{\frac{2\pi}{\tau}}\frac{1}{S+K}\left[C - \frac{S-K}{2} + \sqrt{(C - \frac{S-K}{2})^2 - \frac{(S-K)^2}{\pi}}\right].$$

- Bharadia et al. [7] derive a highly simplified but less accurate volatility approximation:

$$\Sigma \approx \sqrt{\frac{2\pi}{\tau}} \frac{C - (S-K)/2}{S - (S-K)/2}.$$

- Chance [16] provides a direct method of obtaining an accurate estimate of the implied volatility of a call option. His estimate is based on the formula for at-the-money options developed by Brenner and Subrahmanyam [14]. The adjusted formula by Chance [16] is quite accurate for options no more than 20% in- or out-of-the-money and is simple to program and compute. Later, Chambers and Nawalkha [15] developed a simplified extension of the Chance [16] model. The approach taken in these two papers uses the first and second derivatives of the call price with respect to volatility. In addition, they need a reasonable estimate of volatility to serve as a starting point to the approximation. More recently, S. Li [40] used the *Taylor series expansion* to the third order for the standard normal cumulative distribution function $N(x)$ and obtained new approximations that are valid for a wide band of option moneyness and time to expiration:

  - At-the-money calls: $(S = K)$

$$\Sigma \approx \frac{2\sqrt{2}}{\sqrt{T}} Z - \frac{1}{\sqrt{\tau}} \sqrt{8Z^2 - \frac{6\alpha}{\sqrt{2}Z}},$$

    where $\alpha = \dfrac{\sqrt{2\pi}C}{S}$, and $Z = \cos\left[\dfrac{1}{3}\arccos\left(\dfrac{3\alpha}{\sqrt{32}}\right)\right]$. In this case, Li's formula is significantly more accurate than Brenner-Subrahmanyam's [14].

  - In- or out-of-the-money calls: (Define $\eta = \dfrac{K}{S}$ that measures the moneyness of an option: $\eta = 1$ represents at-the-money, $\eta > 1$ represents out-of-the-money, and $\eta < 1$ represents in-the-money.)

$$\Sigma \approx \begin{cases} \frac{2\sqrt{2}}{\sqrt{\tau}}\tilde{Z} - \frac{1}{\sqrt{\tau}}\sqrt{8\tilde{Z}^2 - \frac{6\tilde{\alpha}}{\sqrt{2}\tilde{Z}}} & \text{if } \rho \le 1.4 \\ \frac{\tilde{\alpha}+\sqrt{\tilde{\alpha}^2 - \frac{4(\eta-1)^2}{1+\eta}}}{2\sqrt{\tau}} & \text{if } \rho > 1.4, \end{cases}$$

where $\tilde{\alpha} = \frac{\sqrt{2\pi}}{1+\eta}\left[\frac{2C}{S} + \eta - 1\right]$, $\tilde{Z} = \cos\left[\frac{1}{3}\arccos(\frac{3\tilde{\alpha}}{\sqrt{32}})\right]$,
and $\rho = \frac{|\eta - 1|}{(C/S)^2} = \frac{|X - S|S}{C^2}$

Our goal is to overcome the challenges I.) and II.), and hence, to directly analyze the implied volatility. Indeed, this approach allows us to shed light on qualitative properties that would otherwise be more difficult to establish[6]. To accomplish these goals, we take the following steps:

1. we assume that the volatility Σ depends on the variables t (time) and S_t (stock price at time t), giving rise to the so-called *local volatility* model. The dynamics of the underlying asset is then governed by the stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \Sigma(S_t, t)dW_t, \tag{1.6}$$

where μ, the expected rate of return of the stock, is constant; the local volatility $\Sigma(x,t)$ satisfies certain smoothness and growth/decay conditions which we will state in the following section; and W_t is a standard Brownian motion. Since in this model Σ is a function of time and stock price, the stock price is no longer a geometric Brownian motion and the Black-Scholes-Merton formula would no longer apply. However, one can still use a *Feynman-Kač*-like representation formula to derive a parabolic partial differential equation for the price $C(S_t = S, t; K, T)$ of a call option:

$$C_t(S,t) + \frac{\Sigma^2(S,t)}{2} S^2 C_{SS}(S,t) + rSC_S - rC(S,t) = 0$$

$$\text{in } (0, +\infty) \times (0, T) \tag{1.7a}$$

$$C(S,T) = (S-K)_+. \tag{1.7b}$$

We will derive equation (1.7) in the appendix.

To simplify the analysis, we transform the equation through the change of variables throughout the paper:

$$\tau = T - t, \qquad x = \ln(S/K) + r\tau, .$$

From now on, T and K are fixed.

Even though in equation (1.7) we consider only one underlying, many of our results can be extended into higher dimensions, say $d \geq 1$. We denote, unless specify, $\Omega_T := \mathbb{R}^d \times (0,T)$, $\Omega := \mathbb{R}^d \times [0,T]$, and $\Omega_0 := \mathbb{R}^d \times [0,T)$, where T can be any positive number, and d can be one or larger, depending on the contents. Then, the normalized price

$$v(x,\tau) = e^{r\tau} C(S, T-\tau; K, T)/K$$

satisfies

$$v_\tau = \frac{1}{2}\sigma^2(x,\tau)(v_{xx} - v_x) \quad \text{in } \Omega_T, \tag{1.8a}$$

$$v(x,0) = (e^x - 1)_+. \tag{1.8b}$$

Some fundamental questions such as the existence, uniqueness, and positivity of the solution to equation (1.8) under various assumptions are discussed in detail in Chapter 2. The main result of that chapter involves removing the assumption of the uniform bounds on the local volatility previous made in [5].

We do so by first freeze σ whenever it is greater than m or less than $1/n$. We denote such functions σ_n^m. Then we take a sequence of mollified version of each σ_n^m, *i.e.* $\sigma_n^{m\varepsilon}$. Replacing σ by $\sigma_n^{m\varepsilon}$ in equation (1.8) yields a sequence of uniform parabolic differential equations with Hölder continuous coefficients. Hence, the existence of the solution to the corresponding uniform parabolic equations and their properties are known from classical theorems pertaining to parabolic differential equations. We then show convergence of the solutions to these equations of the form (1.8), using a generalized Maximum Principle, uniform Schauder interior estimates, Sobolev embedding theorem and the Arzelà-Ascoli theorem. Next, we prove the limit of the solutions lies in $C(\Omega) \cap W^{2,1,p}_{Loc}(\Omega_T)$, and is the solution of the limiting equation, (1.8) almost everywhere. We conclude this section by showing the positivity, monotonicity, and uniqueness of the solution to (1.8).

2. In Chapter 3, we view the implied volatility, φ, as a suitable such that $v := u(x, \tau\varphi^2)$ satisfies (1.8). The function u has explicitly definition, and is monotonically increasing

in the time variable. Consequently, we give the implied volatility, φ, as the unique solution to a well-posed degenerate[9] initial value quasilinear parabolic problem:

$$(\tau\varphi^2)_\tau - \sigma^2(x,\tau)(1 - x\frac{\varphi_x}{\varphi})^2 + \tau\varphi\varphi_{xx} - \frac{1}{4}\tau^2\varphi^2\varphi_x^2 = 0$$

in Ω.

We also introduce the "associated local volatility", $\sigma[\psi]$, of any suitable given function ψ. This concept, together with the generalized comparison and maximum principle, as well as the monotonicity of u in time variable allow us to compare the implied volatility with any suitable function ψ – a key property for establishing the asymptotics in the following chapters.

3. The next step is to find the asymptotics of the solution for options near expiry($\tau = 0$) and for deep in the money[10]/far out of the money[11] options. Deriving and proving those asymptotics are main results for Chapter 4 and 5, respectively.

   In Chapter 4, we solve the formal limiting equation of (3.7) at the expire and use its solution as benchmark for sequences of sub and super solutions to (3.7) in Ω_T. Then we prove that actual convergence takes place through a generalized comparison principle. This theorem allows us to compare the solutions to parabolic equations on a bounded domain of Ω_T. The proof of this Comparison Principle is a main mathematical contribution in this section. It is done through exhaustive applications of fundamental solutions for parabolic equations, with bounded and unbounded coefficients.

[9] at $\tau = 0$
[10] $S_T/K >> 1$
[11] $S_T/K << 1$

4. The methodology for proving the asymptotics in Chapter 5 is similar to the one in section 4. However, we are more focused on finding the right auxiliary functions through which the comparisons are carried over.

5. Chapter 6 is dedicated to the numerical simulation of the implied volatility. The obstacle remains that this equation is degenerate at its initial time. To circumvent this difficulty, we numerically solve for another variable that is one-to-one to φ, using finite difference method and the asymptotics we derived in the previous two chapters. In addition, we give one-term and two-term approximation formula for the implied volatility.

   Our numerical results illustrate the "volatility smile", a long-observed pattern in which at-the-money options tend to have lower implied volatilities than in- or out-of-the-money options. Additionally, the comparison between the asymptotic formula and the numerically computed smile shows a satisfactory agreement.

Chapter 2

Existence of the Solution to Equation (1.8)

We want to understand the behavior of the implied volatility near expiry in a more general setting than has previously been done. Instead of assuming the local volatility $\sigma(x, \tau)$ satisfies $0 < \underline{\sigma} \leq \sigma \leq \bar{\sigma} < \infty$ as in [5], we allow the more realistic situation:

Condition H0:

1. *σ^2 is positive and uniformly continuous in Ω.*

2. *It follows from 1. that for any compact set $C \subset \Omega$, there exists $n(C) \in \mathbb{N}$, such that $1/n(C) \leq \sigma \leq n(C)$. However, as $|x| \to \infty$, σ may grow to infinity, or decay to zero at a rate that is less than linear, uniformly in τ. Furthermore, there are step function $-1 < p(x) < 1$, which may only have discontinuity at zero, and constant $\kappa_1 > 1$ such that*

$$\frac{1}{\kappa_1}(1 + x^2)^{p(x)/2} \leq \frac{1}{2}\sigma^2(x, \tau) \leq \kappa_1(1 + x^2)^{p(x)/2}$$

in Ω.

From now on, we denote $p_+ = \max_{x\in\mathbb{R}}\{p(x), 0\}$, and $p_- = \max_{x\in\mathbb{R}}\{-p(x), 0\}$.

2.1 Auxiliary Notions, and Function Spaces

In order to make precise statements, let us specify the technical conditions that we impose, and state some notations, terminologies, and functional spaces that we shall need.

2.1.1 Auxiliary Notation

The *parabolic distance* between $P = (x_P, t_P)$ to $Q = (x_Q, t_Q) \in \Omega$ is defined by

Definition 2.1.

$$d(P, Q) = (\,\|x_P - x_Q\|^2 + |t_P - t_Q|\,)^{1/2}, \tag{2.1}$$

where $\|x\|$ *is the Euclidean norm* $\sum_{i=1}^{d}(x_i^2)^{1/2}$. *The concept of Hölder continuity in this paper will always be defined with respect to the metric (2.1).*

We define the *round cubes* in $\mathbb{R}^{d+1}$:

$$Q_r = (-r, r) \times (-r^2, 0], \tag{2.2a}$$

$$Q_r(x, t) = Q_r + (x, t). \tag{2.2b}$$

2.1.2 Function Spaces

As is classical when studying parabolic problems, we make use of the following anisotropic Sobolev spaces:

Definition 2.2.

$$W^{2,1,p}(\Omega_T) = \left\{ w \middle| \int_{\Omega_T} |w_{xx}|^p + |w_\tau|^p + |w|^p < \infty \right\}, \tag{2.3a}$$

$$W^{2,0,\infty}(\Omega_T) = \{w|\ |w_{xx}|_\infty + |w|_\infty < \infty\}, \tag{2.3b}$$

endowed with their natural norms. Similarly, we define $W^{2,1,p}_{loc}(\Omega_T)$ *and* $W^{2,0,\infty}_{loc}(\Omega_T)$ *as*

$$W^{2,1,p}_{loc}(\Omega_T) = \left\{ w \middle| \int_{\mathscr{D}} |w_{xx}|^p + |w_\tau|^p + |w|^p < \infty,\ \forall \bar{\mathscr{D}} \subset \overset{\circ}{\Omega}_T \right\}, \tag{2.4a}$$

$$W^{2,0,\infty}_{loc}(\Omega_T) = \{w|\ |w_{xx}|_{\infty(\mathscr{D})} + |w|_{\infty(\mathscr{D})} < \infty,\ \forall \bar{\mathscr{D}} \subset \overset{\circ}{\Omega}_T\}. \tag{2.4b}$$

Definition 2.3. *Given* $\mathcal{D} \subseteq \Omega$.

1. $C(\mathcal{D})$ *is the set of all real-valued, continuous functions* $v(x,t)$ *defined on* $\mathcal{D}$ *such that the norm*

$$\|v\|_{C(\mathcal{D})} = \sup_{(x,t)\in\mathcal{D}} |v(x,t)| \tag{2.5}$$

is finite.

2. *Given* $\alpha \in (0,1)$.

   (a) *Define* $C^\alpha(\mathcal{D})$ *as the subspace of* $C(\mathcal{D})$ *consisting of all functions* v *such that the norms*

$$\|v\|_{C^\alpha(\mathcal{D})} = \|v\|_{C(\mathcal{D})} + \sup_{\substack{P,Q\in\mathcal{D}\\ P\neq Q}} \frac{|v(P)-v(Q)|}{d(P,Q)^\alpha} \tag{2.6}$$

are finite.

(b) *Moreover, we say* v *is in* $C^{2+\alpha}(\mathcal{D})$ *whenever the norm*

$$\|v\|_{C^{2+\alpha}(\mathcal{D})} = \|v\|_{C^{\alpha}(\mathcal{D})} + \|v_x\|_{C^{\alpha}(\mathcal{D})} + \|v_{xx}\|_{C^{\alpha}(\mathcal{D})} + \|v_t\|_{C^{\alpha}(\mathcal{D})} \tag{2.7}$$

is finite.

3. *As convention,* $C^{2,1}(\mathcal{D})$ *is the set of all functions in* $\mathcal{D}$ *having continuous second space-derivative and first time-derivative.*

Some results concern Hölder continuity in x *but not in* t*. We therefore define* $\| \ \|_{C^{*,\alpha}(\mathcal{D})}$, $C^{*,\alpha}(\mathcal{D})$, $\| \ \|_{C^{*,2+\alpha}(\mathcal{D})}$, *and* $C^{*,2+\alpha}(\mathcal{D})$ *as above.*

Now, we give notations and spaces relating interior Schaulder estimates. These estimates were first introduced by Brandt [13]. Knerr [33] then extended Brandt's result to give an estimate on the time derivative of the solutions. Their results show that if the coefficients of L is locally Hölder, and bounded below, then the Hölder norms of derivatives of the solution to $Lu = f$ can be bounded above by the norms of f and the initial condition of u. Consequently, as we will show in the proof of our existence theorem, if σ were locally Hölder, then the solution to (1.8) would be in the Hölder space, which we define next.

Definition 2.4. *Let* $\tilde{\mathcal{D}}$ *be a bounded domain in* $\Omega := \mathbb{R} \times \mathbb{R}^+$*. For any point* $P = (x, t)$ *in* $\tilde{\mathcal{D}}$ *we denote by* $T(P)$ *the set of points* Q *on the boundary of* $\tilde{\mathcal{D}}$ *which can be connected to* P *by a simple continuous arc along which the* t *coordinate is nondecreasing from* Q *to* P*. For all* P*,* Q *in* $\tilde{\mathcal{D}}$*, we introduce the metric*

$$|P - Q| = max\{|x^P - x^Q|,\ 4\sqrt{2}\,|t^P - t^Q|^{1/2}\}.$$

Consequently, let

$$d_P = \inf\{|P - Q| : \ Q \in T(P)\},$$

and

$$d_{PQ} = min\{d_P, \ d_Q\}.$$

1. *Given* $\mathcal{D} \subseteq \tilde{\mathcal{D}}$, $C(\mathcal{D})$ *is the set of all real-valued, continuous functions* $v(x,t)$ *defined on* $\mathcal{D}$ *such that the norm*

$$\|v\|_{C(\mathcal{D})} = \sup_{(x,t)\in\mathcal{D}} |v(x,t)|$$

   is finite.

2. *Given* $\alpha \in (0,1)$.

   (a) *Let* $C^{\alpha}(\mathcal{D})$ *be the space of functions in* $C(\mathcal{D})$ *consisting of all functions* v *such that the norms*

$$\|v\|_{C^{\alpha}(\mathcal{D})} = \|v\|_{C(\mathcal{D})} + \sup_{\substack{P,Q\in\mathcal{D}\\ P\neq Q}} \frac{|v(P) - v(Q)|}{|P - Q|^{\alpha}}$$

   are finite.

   (b) *For* $m = 0, 1, 2$ *and* $0 < \alpha < 1$ *and for any sufficiently smooth function* $v : \ \mathcal{D} \to \mathbb{R}$ *we define*

$$\|v\|_{\mathcal{D}}^{0,m} = \sup_{P\in\mathcal{D}} |d_P^m v(P)|, \tag{2.8}$$

$$H_{\mathcal{D}}^{\alpha,m}(v) = \sup_{P,Q\in\mathcal{D}} d_{P,Q}^{m+\alpha} \frac{|v(P) - v(Q)|}{|P - Q|^{\alpha}}, \tag{2.9}$$

$$\|v\|_{\mathcal{D}}^{\alpha,m} = \|v\|_{\mathcal{D}}^{0,m} + H_{\mathcal{D}}^{\alpha,m}(v). \tag{2.10}$$

3. The the space $H_{\mathcal{D}}^{2+\alpha}$ is the collection of functions such that

$$\|v\|_{\mathcal{D}}^{2+\alpha} = \|v\|_{\mathcal{D}}^{\alpha} + \|v_x\|_{\mathcal{D}}^{\alpha,1} + \|v_{xx}\|_{\mathcal{D}}^{\alpha,2} + \|v_t\|_{\mathcal{D}}^{\alpha,2} \tag{2.11}$$

are finite.

Some results concern Hölder continuity in x but not in t. We therefore define $H_{\mathcal{D}}^{*,\alpha,m}$, $\|\cdot\|_{\mathcal{D}}^{*,\alpha,m}$, and $\|\cdot\|_{\mathcal{D}}^{*,2+\alpha}$ as above except in (2.9) we further restrict P and Q so that $P = Q + \eta\mathbf{e}$ for some scalar η, where $\mathbf{e} = (1,0)$.

Furthermore, we say v is in $H^{2+\alpha}(\Omega)$ if $v \in H_{\mathcal{D}}^{2+\alpha}$ for any bounded domain $\mathcal{D} \in \Omega$.

Note: $\mathcal{D} = \tilde{\mathcal{D}}$ is allowed here but later we will require $\bar{\mathcal{D}} \subset \overset{\circ}{\tilde{\mathcal{D}}}$ so that d_P is bounded below for $P \in \mathcal{D}$.

Remark 2.5. *The estimates using norms of type $C^{2+\alpha}(\mathcal{D})$ are called* **boundary** *estimates, and the ones that are using norms of $H_{\mathcal{D}}^{2+\alpha}$ type are called* **interior** *estimates [24].*

Definition 2.6. *Let*

$$[u]_{p,\alpha}(x,t) = \sup_r \frac{1}{r^\alpha}\left(\int_{Q_r(x,t)} |u - u(x,t)|^p\right)^{\frac{1}{p}}. \tag{2.12}$$

As in [42], we say a function is $C_{p,\alpha}$ at (x,t) if $[u]_{p,\alpha}(x,t) < \infty$.

We shall show, in both cases, our solution to (1.8) has the following bounds in Ω_T, and it is the only solution in this class.

Definition 2.7. *We define the set of functions defined in Ω_T that has no more than expo-*

nential growth in the pace variable as

$$\mathscr{E}_T := \{w|, \exists\, C, \beta > 0, \forall (x,\tau) \in \Omega_T,\ |w(x,\tau)| \leq Ce^{\beta|x|}\}. \tag{2.13}$$

2.1.3 Parabolic Terminology

Consider the operator

$$Lu \equiv \frac{\partial u}{\partial t} - \left\{ \sum_{i,j=1}^{d} a_{i,j}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u \right\} \tag{2.14}$$

$$:= \frac{\partial u}{\partial t} - F\left(\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial x_i}, u, x, \tau\right) \tag{2.15}$$

in a (d+1)-dimensional domain $\Omega := \mathbb{R}^d \times [0,T]$, where T can be any positive number. The coefficients a_{ij}, b_i, c are defined in Ω. We always take $(a_{ij}(x,t))$ to be a symmetric matrix, *i.e.* $a_{ij} = a_{ji}$. If the matrix $(a_{ij}(x,t))$ is positive definite, *i.e*, if for every real vector $\xi = (\xi_1, \cdots, \xi_n) \neq 0$, $\sum a_{ij}(x,t)\xi_i\xi_j > 0$, then we say that the operator L is of *parabolic type* (or that L is *parabolic*) at point (x,t). If L is parabolic at all the points of Ω then we say that L is parabolic in Ω. If there exist positive constants $\bar{\lambda}_0$, $\bar{\lambda}_1$ such that, for any real vector ξ,

$$\bar{\lambda}_0|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x,t)\xi_i\xi_j \leq \bar{\lambda}_1|\xi|^2 \tag{2.16}$$

for all $(x,t) \in \Omega$ then we say that L is *uniformly parabolic* in Ω. We refer $\bar{\lambda}_0$ and $\bar{\lambda}_1$ as *parabolic constants.*

Besides the parabolic constants, we also make use of the following bounds on $(a^{i,j}(x,t))$, the inverse matrix to $a_{i,j}$. From the above inequalities, it follows that

$$\lambda_0|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x,t)\xi_i\xi_j \leq \lambda_1|\xi|^2, \tag{2.17}$$

for some $0 < \lambda_0,\ \lambda_1 < \infty$.

2.2 Primary Results on the Fundamental Solution, Green's Function, and Properties of Solutions to Parabolic Equations

Let us review some concepts concerning parabolic operators and associated Cauchy problems.

Given a function $f(x,t)$ in Ω and a function $\varphi(x)$ in $\mathbb{R}^d$, the problem of finding a function $u(x,t)$ satisfying the following parabolic equation, and the *initial condition*

$$Lu(x,t) = f(x,t) \quad \text{in } \Omega_0 \equiv \mathbb{R}^d \times (0,T] \tag{2.18a}$$

$$u(x,0) = \varphi(x) \quad \text{on } \mathbb{R}^d \tag{2.18b}$$

is called a *Cauchy problem* (in the strip $0 \leq t \leq T$). The solution is always required to be continuous in Ω.

The functions $f(x,t)$ and $\varphi(x)$ will be assumed to satisfy the boundedness conditions

$$|f(x,t)| \leq \text{const.} \exp[h|x|^2], \tag{2.19}$$

$$|\varphi(x)| \leq \text{const.} \exp[h|x|^2] \tag{2.20}$$

where h is any positive constant satisfying

$$h < \frac{\lambda_0}{4T}. \tag{2.21}$$

Let us now mention the following assumptions, as they are critical conditions in classical existence theorems for equations of parabolic type.

($\mathscr{A}1$) L is uniformly parabolic in Ω;

($\mathscr{A}2$) $a_{ij} \in C^{\alpha}(\Omega)$, b_i, $c \in C^{*,\alpha}(\Omega)$ for some $0 < \alpha < 1$.

In the following, we recall

i.) the definition of the fundamental solution to parabolic operator of form (2.14);

ii.) sufficient conditions for the existence of a fundamental solution;

iii.) the expression of solution to the Cauchy problem (2.18), which is closely related to the fundamental solution of the corresponding parabolic equation.

Definition 2.8. *[24] A* ***fundamental solution*** *of $Lu = 0$ is a function $\Gamma(x, \tau; \xi, s)$ defined for all $(x, \tau) \in \Omega$, $(\xi, s) \in \Omega$, $\tau > s$, which satisfies the following conditions:*

i) for each fixed $(\xi, s) \in \Omega$, it satisfies, as a function of (x, τ) ($x \in \mathbb{R}^d$, $s < \tau < T$), the equation $L\Gamma = 0$.

ii) Given any continuous function $f = f(x)$ in Ω, such that

$$|f| \leq const \cdot \exp[h|x|^2]$$

for some positive constant $h < \frac{\lambda_0}{4T}$, *one has, for all* $x_0 \in \mathbb{R}^d$,

$$\lim_{\tau \to s+} \int_{\mathbb{R}^d} \Gamma(x_0, \tau; \xi, s) f(\xi) d\xi = f(x_0). \tag{2.22}$$

Theorem 2.9. *(Existence of the Fundamental Solution to operator (2.14))[24], [30] Consider operator* L *defined in (2.14), and assume* $(\mathscr{A}1)$, *and* $(\mathscr{A}2)$ *hold. Then the fundamental solution* $\Gamma(x, t; \xi, \tau)$ *of* $Lu = 0$ *exists.*

Theorem 2.10. *(Existence of the solution to the Cauchy problem (2.18))[24] Suppose that* L *satisfies* $(\mathscr{A}1)$, *and* $(\mathscr{A}2)$ *and let* $f(x,t)$, $\varphi(x)$ *be continuous functions in* Ω *and* $\mathbb{R}^d$ *respectively, satisfying (2.19). Assume also that* $f(x,t)$ *is locally Hölder continuous (exponent* α*) in* $x \in \mathbb{R}^d$, *uniformly with respect to* t. *Then the function*

$$u(x,t) = \int_{\mathbb{R}^d} \Gamma(x, t; \xi, 0) \varphi(\xi) d\xi - \int_0^t \int_{\mathbb{R}^d} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau \tag{2.23}$$

is a solution of the Cauchy problem (2.18) and

$$|u(x,t)| \leq const. \exp[\kappa |x|^2] \quad \text{for } (x,t) \in \Omega, \tag{2.24}$$

where κ *is a constant depending only on* h, λ_0, *and* T.

Next, we give an extension of Friedman's [24] Maximum Principle of solutions to parabolic inequalities on unbounded domains:

Theorem 2.11. *Let* L *be a parabolic operator with continuous coefficients in* Ω_T, *and let*

$$a_{i,j}(x,y), |b_i(x,y)| \leq M(|x|^\varepsilon + 1), \quad |c(x,y)| \leq M|x|^{2-\varepsilon}, \tag{2.25}$$

be satisfied for some $M > 0$*, and* $0 \leq \varepsilon \leq 1$*. Assume* $Lu \geq 0$ *in* Ω_T *and that*

$$u(x,t) \geq -B\ exp[\beta|x|^{2-\varepsilon}] \quad in\ \Omega \tag{2.26}$$

for some positive constants B *and* β*. If* $u(x,0) \geq 0$ *in* $\mathbb{R}^d$*, then* $u(x,t) \geq 0$ *in* Ω*. Similarly, if* $u(x,0) \leq 0$ *in* $\mathbb{R}^d$ *then* $u(x,t) \leq 0$ *in* Ω*.* [1]

Proof. For the moment, fix μ, $\nu > 0$, and $\kappa > \beta$. Consider the auxiliary function

$$H(x,t) = exp[\frac{\kappa}{1-\mu t}\gamma(|x|) + \nu t] \quad \left(0 \leq t \leq \frac{1}{2\mu}\right) \tag{2.27}$$

where $\gamma(|z|)$ is a C^2 function defined on $\mathbb{R}^d$ such that

$$\gamma(r) = \begin{cases} r^{2-\varepsilon} & \text{for } r > 1 \\ r^2 & \text{for } r \leq 1/2. \end{cases}$$

[2] We shall see, fix κ, by properly choosing μ and ν, LH/H is non-negative on $\mathbb{R} \times [0, 1/(2\mu)]$.

[1] Recall $\Omega_T := \mathbb{R}^d \times (0,T)$, $\Omega := \mathbb{R}^d \times [0,T]$, and $\Omega_0 := \mathbb{R}^d \times (0,T]$.

[2] We make C^2 connection at for $r = 1$ and $r = 1/2$.

- First, let us consider $|x| > 1$,

$$
\begin{aligned}
&\frac{LH}{H} \\
&= \left[\mu\frac{\kappa\gamma(|x|)}{(1-\mu t)^2} + \nu\right] - \frac{\kappa}{1-\mu t}(2-\varepsilon)|x|^{-\varepsilon}(\sum_{i=1}^{d} a_{i,i}) \\
&\quad - \left[\left(\frac{\kappa}{1-\mu t}(2-\varepsilon)\right)^2 |x|^{-2\varepsilon} - \frac{\kappa}{1-\mu t}(2-\varepsilon)\,\varepsilon|x|^{-2-\varepsilon}\right](\sum_{i,j=1}^{d} a_{i,j}x_i x_j) \\
&\quad - \frac{\kappa}{1-\mu t}(2-\varepsilon)|x|^{-\varepsilon}(\sum_{i=1}^{d} b_i x_i) - c \\
&\geq \left[\mu\frac{\kappa}{(1-\mu t)^2}|x|^{2-\varepsilon} + \nu\right] - \frac{\kappa}{1-\mu t}(2-\varepsilon)|x|^{-\varepsilon}\left[dM(|x|^{\varepsilon}+1)\right] \\
&\quad - \left[\left(\frac{\kappa}{1-\mu t}(2-\varepsilon)\right)^2 |x|^{-2\varepsilon} + \frac{\kappa}{1-\mu t}\,(2-\varepsilon)\,\varepsilon|x|^{-2-\varepsilon}\right][d^2M(|x|^{\varepsilon}+1)|x|^2] \\
&\quad - \frac{\kappa}{1-\mu t}(2-\varepsilon)|x|^{-\varepsilon}\left[dM(|x|^{\varepsilon}+1)|x|\right] - c
\end{aligned}
$$

- Next, consider $|x| \leq 1$,

$$
\begin{aligned}
&\frac{LH}{H} \\
&= \left[\mu\frac{\kappa\gamma(|x|)}{(1-\mu t)^2} + \nu\right] - \frac{2\kappa}{1-\mu t}(\sum_{i=1}^{d} a_{i,i}) - \left(\frac{2\kappa}{1-\mu t}\right)^2(\sum_{i,j=1}^{d} a_{i,j}x_i x_j) \\
&\quad - \frac{2\kappa}{1-\mu t}(\sum_{i=1}^{d} b_i x_i) - c \\
&= \left[\mu\frac{\kappa|x|^2}{(1-\mu t)^2} + \nu\right] - \frac{2\kappa}{1-\mu t}(\sum_{i=1}^{d} a_{i,i}) - \left(\frac{2\kappa}{1-\mu t}\right)^2(\sum_{i,j=1}^{d} a_{i,j}x_i x_j) \\
&\quad - \frac{2\kappa}{1-\mu t}(\sum_{i=1}^{d} b_i x_i) - c.
\end{aligned}
$$

Since $a_{i,j}$, b_i, and c are continuous, and since the set $0 \le |x| \le 1$ is compact and $1 \le \dfrac{1}{1-\mu t} \le 2$ for $0 \le t \le \dfrac{1}{2\mu}$, one can find both upper and lower bounds for $\dfrac{LH - \frac{\partial H}{\partial t}}{H}$ on $|x| \le 1$, *i.e.*, the terms after the first one.

Thus, given any $\kappa > 0$ we can choose sufficiently large positive numbers μ, ν such that

$$\frac{LH}{H} \ge 0 \quad \text{for all } x \in \mathbb{R}^d,\ t \in \left[0, \frac{1}{2\mu}\right]. \tag{2.28}$$

Let $v = u/H$, where H is defined by (2.27) with a fixed $\kappa > \beta$ and with μ, ν such that (2.28) holds. From (2.26) it follows that

$$\liminf_{|x|\to\infty} v(x,t) \ge 0,$$

uniformly with respect to $t \in [0, 1/(2\mu)]$.

Since $Lu = L(Hv)$, v satisfies the equation

$$\bar{L}v \equiv \frac{\partial v}{\partial t} - \left\{ \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \bar{b}_i \frac{\partial v}{\partial x_i} + \bar{c}v \right\} = \bar{f},$$

where $\bar{f} = (Lu)/H \ge 0$ and

$$\bar{b}_i = b_i + 2\sum_{j=1}^{d} a_{i,j} \frac{\partial H/\partial x_j}{H}, \quad \bar{c} = -\frac{LH}{H}.$$

Now, for any $\delta > 0$, $v(x,t) + \delta > 0$ on $t = 0$ and on $|x| = R$, $0 \le t \le 1/2\mu$ provided R is sufficiently large. Since $\bar{L}(v + \delta) = \bar{f} - \bar{c}\delta \ge 0$, by the classical maximum principle [24], $v(x,t) + \delta > 0$ if $|x| < R$, $0 \le t \le 1/2\mu$. Taking (x,t) to be fixed and $\delta \to 0$, it follows that $v(x,t) \geq 0$ on $\mathbb{R}^d \times [0, 1/(2\mu)]$. The same is true for $u(x,t) = H(x,t)v(x,t)$. We can now proceed step by step in time to prove the positivity of u in Ω_0, since the choice of μ is uniform in Ω_0, *i.e.*, only depends on M, ε, κ. $\square$

Remark 2.12. *Theorem 2.11 can be extended to the case where* $u \in C(\Omega) \cap W^{2,1,p}_{loc}(\Omega_T)$ *satisfies* $Lu \geq 0$ a.e. *in* Ω_T. *[5]*

2.3 The Existence of the Solution to the Non-uniform Parabolic Equation (1.8)

The difficulty in proving the existence and uniqueness of the solution to (1.8) are (i) this is a non-uniformly parabolic equation on an unbounded domain; (ii) the coefficient σ is only assumed to be uniformly continuous. Our strategy is to first replace σ by its mollified cut-off versions $\sigma_n^{m\varepsilon}$, as shown in (2.29)–(2.31). Then show the convergence of the corresponding solutions as $m, n \to \infty$ and $\varepsilon \to 0$. Last, but not least, we show the limit satisfies the limiting equation. We would also like to point out, throughout our proof, we work with $w = v - (e^x - 1)$, if v exists. Notice that w satisfies the same differential equation, and it has the same smoothness as v, but it has bounded initial value $(e^x - 1)_-$. This additional boundedness allows us to show convergence in several places that might otherwise be difficult. Once we established the properties of w, we will add $e^x - 1$ to w and get back to v.

Now, we introduce the following "cut-off" volatility functions, which will be used below:

For all $n, m \in \mathbb{N}$, define

$$\sigma_n^m(x,\tau) = \begin{cases} m & \sigma(x,\tau) \geq m, \\ \sigma(x,\tau) & 1/n < \sigma(x,\tau) < m, \\ 1/n & \sigma(x,\tau) \leq 1/n. \end{cases} \tag{2.29}$$

Similarly, for each $m, n \in \mathbb{N}$, we define $\sigma^m(x,\tau)$ and $\sigma_n(x,\tau)$ as the cut-off version of σ from above and below, respectively.

Next, we take a standard mollifier $\rho^\varepsilon(x,\tau)$, which satisfies

$$\rho^\varepsilon \geq 0, \quad \rho^\varepsilon \in C_0^\infty(\mathbb{R}^2), \quad supp(\rho^\varepsilon) \subset B_\varepsilon(0), \tag{2.30a}$$

$$\int \rho^\varepsilon = 1. \tag{2.30b}$$

$$\text{Consequently,} \quad \rho^\varepsilon(x,\tau) \to \delta(x,\tau) \text{ as } \varepsilon \to 0 \text{ in distribution sense.} \tag{2.30c}$$

Fix m,n, and define $\tilde{\sigma}_n^m(x,-\tau) = \sigma_n^m(x,\tau)$ for all $\tau \geq 0$, and set for all $\varepsilon > 0$

$$\sigma_n^{m\varepsilon}(x,t) = \rho^\varepsilon * \tilde{\sigma}_n^m(x,t), \tag{2.31}$$

where $*$ denotes the convolution in $\mathbb{R}^2$.

Remark 2.13. *Given continuous functions h_i, $i = 1, 2$, in Ω, and ρ^ε. If both $\rho^\varepsilon * h_i$ are locally integrable, and $h_1 \geq h_2$ in Ω, then $\rho^\varepsilon * h_1 \geq \rho^\varepsilon * h_2$ in Ω_T. To see why, we take*

$(x,t) \in \Omega_T$, *and the difference*

$$\begin{aligned}
&\rho^\varepsilon * h_1(x,t) - \rho^\varepsilon(x,t) * h_2 \\
=&\int_0^T \int_{\mathbb{R}} \rho^\varepsilon(x-\xi, t-\tau) \cdot (h_1 - h_2)(\xi,\tau) d\xi d\tau \\
\geq&0.
\end{aligned}$$

In particular, given (m,n)*,* $1/n \leq \sigma_n^{m\varepsilon} \leq m$ *in* Ω*, uniformly in* ε*. Moreover, given* ε*,* $\sigma_n^{m\varepsilon}$ *is non-increasing in* n *and non-decreasing in* m *in* Ω*.*

Note that $\sigma_n^{m\varepsilon} \in C^\infty(\Omega)$, but this, a priori, does not imply that it is uniform Hölder continuous. To that end, we need to show

Lemma 2.14. *Fix* m*,* n*, and* ε*,* $(\sigma_n^{m\varepsilon})^2 \in C^\alpha(\Omega)$ *for all* $\alpha \in (0,1)$*.*

Proof. We only need to show, for fixed m, n, and ε,

$$\sup_{(x_0,\tau_0),(x,\tau)\in\Omega} \frac{|(\sigma_n^{m\varepsilon})^2(x,\tau) - (\sigma_n^{m\varepsilon})^2(x_0,\tau_0)|}{(|x-x_0|^2 + |\tau-\tau_0|)^{\alpha/2}} < \infty \qquad \text{for all } \alpha \in (0,1).$$

Take (x_0,τ_0), $(x,\tau) \in \Omega$

i) $|x-x_0| \geq 1$ or $|\tau-\tau_0| \geq 1$:

$$\begin{aligned}
&\frac{|(\sigma_n^{m\varepsilon})^2(x,\tau) - (\sigma_n^{m\varepsilon})^2(x_0,\tau_0)|}{(|x-x_0|^2 + |\tau-\tau_0|)^{\alpha/2}} \\
\leq&|(\sigma_n^{m\varepsilon})^2(x,\tau) - (\sigma_n^{m\varepsilon})^2(x_0,\tau_0)| \\
\leq&\left(m - \frac{1}{n}\right)\left(m + \frac{1}{n}\right)
\end{aligned}$$

for all $\alpha \in (0,1)$.

ii) $|x-x_0|<1$ and $|\tau-\tau_0|<1$:

$$\begin{aligned}&\frac{|(\sigma_n^{m\varepsilon})^2(x,\tau)-(\sigma_n^{m\varepsilon})^2(x_0,\tau_0)|}{(|x-x_0|^2+|\tau-\tau_0|)^{\alpha/2}}\\ &\leq \max\left\{\frac{|(\sigma_n^{m\varepsilon})^2(x,\tau)-(\sigma_n^{m\varepsilon})^2(x_0,\tau_0)|}{|x-x_0|},\frac{|(\sigma_n^{m\varepsilon})^2(x,\tau)-(\sigma_n^{m\varepsilon})^2(x_0,\tau_0)|}{|\tau-\tau_0|}\right\}\end{aligned}$$

for all $\alpha\in(0,1)$.

By the mean value theorem, it is sufficient to show $\sup\limits_{\Omega}(\sigma_n^{m\varepsilon})^2_x$, and $\sup\limits_{\Omega}(\sigma_n^{m\varepsilon})^2_\tau$ are finite. In fact, since $\sigma_n^{m\varepsilon}$ is bounded above in Ω, it boils down to show $\sup\limits_{\Omega}|(\sigma_n^{m\varepsilon})_x|$, and $\sup\limits_{\Omega}|(\sigma_n^{m\varepsilon})_\tau|$ are finite. It is straight forward that

$$\begin{aligned}|(\sigma_n^{m\varepsilon})_x|(x,\tau)&=|\int_{\mathbb{R}}\int_{\mathbb{R}}\rho_x^{\varepsilon}(\xi-x,\eta-\tau)\sigma_n^m(\xi,\eta)d\xi d\eta|\\ &=|\int_{\tau-\varepsilon}^{\tau+\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}\rho_x^{\varepsilon}(\xi-x,\eta-\tau)\sigma_n^m(\xi,\eta)d\xi d\eta|\\ &\leq m\int_{\tau-\varepsilon}^{\tau+\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}|\rho_x^{\varepsilon}(\xi-x,\eta-\tau)|\,d\xi d\eta,\qquad \text{for all } (x,\tau)\in\Omega_T.\end{aligned}$$

Therefore, $\sup\limits_{\Omega}|(\sigma_n^{m\varepsilon})_x|<\infty$. Similarly, $\sup\limits_{\Omega}|(\sigma_n^{m\varepsilon})_\tau|<\infty$.

□

Given m, n, and ε, define

$$L_n^{m\varepsilon}:=\frac{\partial}{\partial t}-\frac{1}{2}(\sigma_n^{m\varepsilon})^2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right).$$

We now show the existence, uniqueness, and monotonicity of solution to

$$L_n^{m\varepsilon} w_n^{m\varepsilon} = 0 \quad \text{in } \Omega_T \tag{2.32a}$$

$$w_n^{m\varepsilon}(x,0) = (e^x - 1)_-, \tag{2.32b}$$

where

$$(e^x - 1)_- = \begin{cases} 1 - e^x & \text{if } x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$$

Lemma 2.15. *The solution, $w_n^{m\varepsilon}(x,\tau)$ to (2.32) exists and has the following properties*

i) Smoothness*: $w_n^{m\varepsilon} \in C(\Omega) \cap C^{2,1}(\Omega_T)$.*

ii) Positivity*: $w_n^{m\varepsilon}(x,\tau) \geq (e^x - 1)_-$ in Ω, and $0 < w_n^{m\varepsilon}(x,\tau) < 1$ in Ω_T.*

iii) Monotonicity*: $(w_n^{m\varepsilon})_\tau(x,\tau) > 0$ in Ω_T. Hence, $w_n^{m\varepsilon}(x,\tau) > (e^x - 1)_-$ in Ω_T.*

iv) Uniqueness*: w_n^m is the only solution to (1.8) in the class $\mathscr{E}_T$.*

Proof. i) Smoothness.

By [24], the solution to equation (2.32) exists, and can be written as:

$$w_n^{m\varepsilon}(x,\tau) = \int_{\mathbb{R}} \Gamma(x,\tau;\xi,0)(e^\xi - 1)_- \, d\xi, \tag{2.33}$$

where Γ is the fundamental solution to equation (2.32a), defined in Definition 2.8. Consequently, we have $w_n^{m\varepsilon} \in C^{2,1}(\Omega_T)$ from the smoothness of Γ and the boundedness of the initial value. By Definition 2.8, we know that $w_n^{m\varepsilon}$ is continuous at $\tau = 0$.

ii) $0 < w_n^{m\varepsilon} < 1$ in Ω_T.

Equation (2.33) implies $|w_n^{m\varepsilon}| \leq 1$. This estimate combined with Condition H0 allows us to use Theorem 2.11. Notice $w_n^{m\varepsilon}$ has nonnegative initial value, and $w_n^{m\varepsilon} - 1$ has non-positive initial value, we conclude $0 \leq w_n^{m\varepsilon} \leq 1$ in Ω. One then applies the strong maximum principle [24] and gets $0 < w_n^{m\varepsilon} < 1$ in Ω_T.

iii) We now show $(w_n^{m\varepsilon})_\tau > 0$, for $\tau > 0$.

Denote $(w_n^{m\varepsilon})_\tau$ by $z_n^{m\varepsilon}$, equation (2.32a) can be rewritten as

$$\frac{1}{2}(\sigma_n^{m\varepsilon})^2[\,(w_n^{m\varepsilon})_{xx} - (w_n^{m\varepsilon})_x\,] = z_n^{m\varepsilon} \quad \text{in } \Omega_T. \tag{2.34}$$

Furthermore, differentiating (2.32a) w.r.t. τ, one gets

$$\begin{aligned}&(w_n^{m\varepsilon})_{\tau\tau} - \frac{1}{2}2\sigma_n^{m\varepsilon}(\sigma_n^{m\varepsilon})_\tau((w_n^{m\varepsilon})_{xx} - (w_n^{m\varepsilon})_x)\\ &-\frac{1}{2}(\sigma_n^{m\varepsilon})^2[(w_n^{m\varepsilon})_{\tau xx} - (w_n^{m\varepsilon})_{\tau x}] = 0 \quad \text{in } \Omega_T,\end{aligned}$$

i.e.,

$$(z_n^{m\varepsilon})_\tau - \frac{1}{2}(\sigma_n^{m\varepsilon})^2[(z_n^{m\varepsilon})_{xx} - (z_n^{m\varepsilon})_x] - 2\frac{(\sigma_n^{m\varepsilon})_\tau}{\sigma_n^{m\varepsilon}} z_n^{m\varepsilon} = 0 \quad \text{in } \Omega_T. \tag{2.35}$$

From (2.34), the facts $D_x(e^x - 1)_- = D_{xx}(e^x - 1)_-$ in $\mathbb{R}\backslash\{0\}$, $D_x(e^x - 1)_-|_{0^+} = 1$, $D_x(e^x - 1)_-|_{0^-} = 0$, one sees $D_{xx}(e^x - 1)_-|_{x=0} = \delta_0(x)$, the Dirac mass at $x = 0$, in the distribution sense. One then defines

$$z_n^{m\varepsilon}(x, 0) = \frac{1}{2}(\sigma_n^{m\varepsilon}(0, 0))^2\delta_0(x), \tag{2.36}$$

in the distributional sense. Plugging (2.36) into (2.23), one realizes $z_n^{m\varepsilon}(x,\tau)$ is the fundamental solution to the uniform parabolic equation (2.35) multiplied by $\frac{1}{2}(\sigma_n^{m\varepsilon}(0,0))^2$. Therefore, by [25], $0 \le z_n^{m\varepsilon}$ in Ω, and $z_n^{m\varepsilon} > 0$ in Ω_T.

iv) Uniqueness. See [24] and (2.33).

□

For the rest of this section, we send $m, n \to \infty$, and then $\varepsilon \to 0$. We aim to show convergence of $w_n^{m\varepsilon}$ to a solution to the limiting equation. *The first stage is to send $m, n \to \infty$, and show convergence of $w_n^{m\varepsilon}$ to a classical solution to the limiting equation. In this stage, we use monotonicity of $(w_n^{m\varepsilon})$ in m and n, and a interior Schauder estimates by [33].* Let us start with the following lemma.

Lemma 2.16. *$w_n^{m\varepsilon}$, the solution to (2.32) increases as $\sigma_n^{m\varepsilon}$ increases. In particular, given $(n_i, m_i, \varepsilon_i)$, $i = 1, 2$. If $\sigma_{n_1}^{m_1\varepsilon_1} \ge \sigma_{n_2}^{m_2\varepsilon_2}$ in Ω, then for the corresponding solutions, $w_{n_1}^{m_1\varepsilon_1} \ge w_{n_2}^{m_2\varepsilon_2}$ in Ω_0.*

Proof. Suppose $w_i(x,t) := w_{n_i}^{m_i\varepsilon_i}$, $i = 1, 2$ are solutions to the equation (2.32) with corresponding parameters $\sigma_i := \sigma_{n_i}^{m_i\varepsilon_i}$ in Ω.

Let $\Delta := w_1(x,\tau) - w_2(x,\tau)$, then Δ satisfies

$$\Delta_\tau - \frac{1}{2}(\sigma_1)^2(\Delta_{xx} - \Delta_x) = \left[\left(\frac{\sigma_1}{\sigma_2}\right)^2 - 1\right](w_2)_\tau \quad \text{in } \Omega_T$$

$$\Delta(x,0) = 0 \quad \text{on } \mathbb{R}.$$

By Lemma 2.15, $(w_2)_\tau > 0$ in Ω_T. From our assumption, $\left[\left(\frac{\sigma_1}{\sigma_2}\right)^2 - 1\right] \ge 0$ in Ω, the right hand side of the above equation is non-negative in Ω. Now, one can employ Theorem 2.11, to the above equation and conclude $\Delta \ge 0$ in Ω_0. That is, $w_1 \ge w_2$ in Ω_0. □

Remark 2.17. *By Lemma 2.16, given* m, ε*, the sequence* $w_n^{m\varepsilon}$ *is non-increasing with respect to* n*. Similarly, for fixed* n, ε*,* $w_n^{m\varepsilon}$ *is non-decreasing with respect to* m*. Moreover,* $w_n^{m\varepsilon}$ *is bounded above by* 1*, and bounded below by* 0*, uniformly for all* m, n, ε*. Therefore, the sequence* $w_n^{m\varepsilon}$ *converges to* $w^{m\varepsilon}$*, say, as* $n \to \infty$*. Similarly, if one lets* $m \to \infty$*, then* $w_n^{m\varepsilon}$ *converges on* Ω_0 *to* w_n^{ε}*, say.*

Next, we show convergence of various derivatives of $w_n^{m\varepsilon}(x,\tau)$ as $m \to \infty$, or $n \to \infty$. Let us first mention the following interior estimates ([13] and [33]) on solutions.

Proposition 2.18. *Suppose* $v: \mathcal{D} \to \mathbb{R}$ *is a classical solution to (1.8) on a bounded domain* $\mathcal{D} \subset \mathbb{R}^d \times \mathbb{R}^+$ *and there exist constants* $\kappa,\ \nu > 0$*, and* $\alpha,\ 0 < \alpha < 1$*, such that*

$$\|\sigma^2\|_{C^{*,\alpha}(\mathcal{D})} \leq \kappa \tag{2.37}$$

and

$$\sigma^2(x,t) \geq \nu \tag{2.38}$$

hold. Then there exists a constant $C > 0$*, depending only on* κ*,* ν*,* α*, and* d*, such that:*

$$\|v\|_{\mathcal{D}}^{2+\alpha} \leq C\|v\|_{\mathcal{D}}^{0}. \tag{2.39}$$

We shall provide local estimates on the derivatives of $w_n^{m\varepsilon}$, which are needed in several places.

Lemma 2.19. *Let* $\mathscr{D} \subset \bar{\mathscr{D}} \subset \mathring{\Omega}_T$*. For each fixed* ε *and* n *the sequence* $\{w_n^{m\varepsilon}\}$ *is uniformly bounded in* $C^{\alpha}(\mathscr{D})$ *and converges to a function* w_n^{ε}*. Furthermore, the derivatives* $\{(w_n^{m\varepsilon})_x\}$*,* $\{(w_n^{m\varepsilon})_{xx}\}$*, and* $\{(w_n^{m\varepsilon})_\tau\}$ *are also uniformly bounded in* $C^{\alpha}(\mathscr{D})$ *and converges* *to the respective derivatives of* w_n^ε.

Proof. Fix any domain $\mathscr{D} \subset \bar{\mathscr{D}} \subset \mathring{\Omega}_T$, let $\hat{\mathscr{D}} \subset \Omega_T$ be a bounded domain such that $\bar{\mathscr{D}} \subset \mathring{\hat{\mathscr{D}}}$. Define the norms and distance functions given at the start of Section 2.1 in terms of $\hat{\mathscr{D}}$. For instance, for $P \in \mathscr{D}$, $T(P)$ is the set of points Q in $\partial\hat{\mathscr{D}}$ which can be connected to P by a simple continuous arc along which the time coordinate is nondecreasing from Q to P. Now, define $d_P = \inf\{|P-Q| : Q \in T(P)\}$. Since $\inf\limits_{P\in\mathscr{D}}\{d_P\} > 0$, $\|\cdot\|_{\mathscr{D}}^{2+\alpha}$ is equivalent to the usual norm on $C^{2+\alpha}(\bar{\mathscr{D}})$ which is compactly embedded in $C^2(\bar{\mathscr{D}})$. This, with Proposition 2.18 and the monotonicity of $w_n^{m\varepsilon}$ as $m \to \infty$ implies $w_n^{m\varepsilon}$ converge to $w_n^\varepsilon \in C^{2+\alpha}$ uniformly as $m \to \infty$. This establishes the lemma. □

The second stage is to let $\varepsilon \to 0$. *Note that by doing so, we will lose local uniform Hölder constant on* σ^ε, *a key assumption in Proposition 2.18. As in [5], here we make use of the following* $W_{loc}^{2,1,p}$ *estimates for the solutions to* uniform parabolic *equations [42].*

Proposition 2.20. *([42]* THEOREM *5.9) Let u be a continuous solution of*

$$u_t - F\left(\frac{\partial^2}{\partial x_i \partial x_j}u, x, t\right) = g(x,t)$$

for some continuous function g, with $\|u\|_{L^\infty(Q_1)} < \infty$, *and let*

$$\theta(x,t;y,\tau) = \sup_M \frac{|F(M,x,t) - F(M,y,\tau)|}{|M|+1},$$

where the supremum is taken over the set of symmetric matrices. If

$$v_t - F(\frac{\partial^2}{\partial x_i \partial x_j}v, x, t) = 0$$

has interior $C_{1,1}$ estimates and

$$\lim_{r\to 0}\|\theta(\cdot,\cdot;\, y,\tau)\|_{L^\infty(Q_r(y,\tau))} \le \delta_0(p,\bar{\lambda}_0,\bar{\lambda}_1)$$

for all $(y,\tau)\in Q_1$, then for $p>d+1$,

$$\int_{Q_{1/2}} |u_\tau|^p + |u_{xx}|^p \le C\left(\int_{Q_1} |g|^p + \|u\|_{L^\infty(Q_1)}\right), \tag{2.40}$$

where C only defends on the dimension d and the parabolic constants.

We now give the main theorem of this section.

Theorem 2.21. *Given σ satisfies Condition H0, a strong solution, $v(x,\tau)$ to (1.8) exists almost everywhere and*

1. Smoothness*: $v\in C(\Omega)\cap W^{2,1,p}_{loc}(\Omega_T)$ for any $1<p<\infty$.*
2. Positivity*: $v(x,\tau)\ge (e^x-1)_+$ in Ω,*
3. Monotonicity*: $v_\tau(x,\tau)\ge 0$ in Ω_T, and $v(x,\tau)$ is strictly greater than $v(x,0)$ for $\tau>0$, i.e., $v(x,\tau)>(e^x-1)_+$ in Ω_T.*
4. Growth rate and uniqueness*: $v(x,\tau)<e^x$ in Ω, and it is the only solution to (1.8) in the class $\mathscr{E}_T$.*

Proof. 1) Our strategy to show the ***existence*** and ***smoothness*** of the solution to (1.8) is to use a two-stage approximation then show the convergence as $\sigma_n^{m\varepsilon}\to\sigma$.

STAGE I.

STEP 1. Fix $n>0$, $\varepsilon>0$, and let $m\to\infty$.

For given domain $\mathscr{D}$ in Ω, by Lemma 2.19, $w_n^{m\varepsilon}$ converges to w_n^ε uniformly on $\bar{\mathscr{D}}$ as $m \to \infty$. Moreover, $L_n^\varepsilon w_n^\varepsilon = 0$ on $\bar{\mathscr{D}}$. Because $\mathscr{D}$ is arbitrary, $w_n^\varepsilon \in C^{2,1}(\Omega_T)$ and $L_n^\varepsilon w_n^\varepsilon = 0$ in Ω_T everywhere. Furthermore, w_n^ε has all the properties in Lemma 2.15, except $(w_n^\varepsilon)_\tau$ is only non-negative.

STEP 2. Fix ε, let $n \to \infty$.

Because of monotonicity in n, the same analysis as in Lemma 2.16 and Remark 2.17 shows that $\sigma_n^\varepsilon \to \sigma^\varepsilon$ as $n \to \infty$. Note that w_n^ε is a classical solution to $L_n^\varepsilon v = 0$ in Ω_T, we apply the same procedures as in STEP 1. and conclude that w^ε is a classical solution to $L^\varepsilon v = 0$ in Ω_T everywhere. In addition, it has all properties as $w_n^{m\varepsilon}$ listed in Lemma 2.15, expect it is only non-decreasing in τ.

Remark 2.22. *If σ were locally Hölder continuous, then we would have no need to use mollifiers and we would have the existence of the unique classical solution to (1.8) in $H^{2+\alpha}(\Omega_T)$.*

STAGE II. Now, letting $\varepsilon \to 0$, we shall derive the existence of the solution v to (1.8).

On any bounded domain $\mathscr{D} = (-R, R) \times (\tau_1, \tau_2)$, the σ^ε are uniformly continuous, uniformly bounded, and bounded away from 0, uniformly in ε. Hence we have a family of uniformly parabolic operators with common upper and positive lower bounds on the ellipticity constants. Instead of solving a Cauchy problem, we treat (1.8) as a boundary problem, using w^ε restricted to the parabolic boundary to obtain interior estimates. Hence, we are in the same situation as [5]. By Proposition 2.20 and the uniform bounds on w^ε in Ω, we have uniform $W^{2,1,p}(\mathscr{D})$ estimates on w^ε for all $p > 1$. By the Sobolev embedding theorem and the Arzelà-Ascoli theorem, there is a subsequence w^{ε_j} which converges locally uniformly to w in $\mathscr{D}$, a viscosity solution to the limiting equation. By [42] (THEOREM 4.21) again, $w \in W^{2,1,p}_{loc}$ and has second derivatives in x almost everywhere. Therefore, w satisfies the limiting equation point wise a.e. in $\mathscr{D}$.

Additionally, the whole family w^{ε} converges to the same limit, since uniqueness follows from the Maximum Principle.

Adding $e^x - 1$ to w, we get the existence and smoothness of v in the theorem.

2) The ***positivity*** of the solution follows Lemma 2.15 that the sequence $\{v_n^{m\varepsilon}\}$ is non negative in Ω.

3) ***monotonicity***

Since each $v_{n\ \tau}^{m\varepsilon} = w_{n\ \tau}^{m\varepsilon} > 0$ in Ω_T, their limit, v_τ must be non-negative in Ω_T.

We are left to show $v(x,\tau) > v(x,0)$ in Ω_T. Suppose there is $(x_0, \tau_0) \in \Omega$ such that $v(x_0, \tau_0) = v(x_0, 0)$. Since $v_\tau \geq 0$, we conclude that $v(x_0, \tau) \equiv v(x_0, 0)$ for $0 \leq \tau \leq \tau_0$, and $v_\tau(x_0, \tau) \equiv 0$ on $[0, \tau_0]$. Recall the initial value of v:

$$v(x, 0) = (e^x - 1)_+ \geq 0 \quad \text{ on } \Omega.$$

If $x_0 \leq 0$ then $v(x_0, \tau_0) \equiv v(x_0, 0) = 0$, so v has a minimum in $\overset{\circ}{\Omega}$, a contradiction to the maximum principle, which holds for all parabolic equations. So, $x_0 > 0$. As before, with $w(x,\tau) = v(x,\tau) - (e^x - 1)$ then $w(x_0, \tau) \equiv w(x_0, 0) = 0$ for $\tau \in [0, \tau_0]$. So w has a minimum in $\overset{\circ}{\Omega}$, which contradicts to the maximum principle. So, $x_0 < 0$.

In conclusion, there is no $(x_0, \tau_0) \in \Omega$ such that $v(x_0, \tau_0) = v(x_o, 0)$, *i.e.*, $v(x,\tau) > v(x_0, 0)$ in Ω_T.

4) The ***growth rate*** of v follows from the fact that each $w_n^{m\varepsilon}$ values in $[0, 1)$ in Ω.

5) The ***uniqueness*** of $v(x, \tau)$ as discussed in the first part of this proof, follows the Strong Maximum Principle (Theorem 2.11).

□

Chapter 3

Directly Computing the Implied Volatility

3.1 The Implied Volatility φ

Let u be the solution to

$$u_t = \frac{1}{2}(u_{xx} - u_x) \text{ in } \Omega_T, \qquad \text{and } u(x,0) = (e^x - 1)_+. \tag{3.1}$$

The explicit solution to (3.1) is readily seen to be

$$u(x,t) = e^x N\left(\frac{x}{\sqrt{t}} + \frac{1}{2}\sqrt{t}\right) - N\left(\frac{x}{\sqrt{t}} - \frac{1}{2}\sqrt{t}\right), \tag{3.2}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy.$$

Then the derivatives of u are,

$$u_x(x,t) = e^x N\left(\frac{x}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) > 0, \tag{3.3a}$$

$$u_{xx}(x,t) = e^x N\left(\frac{x}{\sqrt{t}} + \frac{\sqrt{t}}{2}\right) - \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{t}}e^{-(\frac{x}{\sqrt{t}} - \frac{\sqrt{t}}{2})^2}, \tag{3.3b}$$

$$u_t(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{t}}e^{-\left(\frac{x}{\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} > 0 \quad \text{in } \Omega_T. \tag{3.3c}$$

Additionally, we will make use of the following identities:

$$\frac{u_{xt}}{u_t}(x,t) = \frac{1}{2} - \frac{x}{t}, \tag{3.4a}$$

$$\frac{u_{tt}}{u_t}(x,t) = -\frac{1}{2t} + \frac{x^2}{2t^2} - \frac{1}{8}. \tag{3.4b}$$

Given constant $\varphi > 0$, one sees that $v(x,\tau) \equiv u(x, \tau\varphi^2)$ satisfies

$$v_\tau = \frac{\varphi^2}{2}(v_{xx} - v_x).$$

So, v is a solution to equation (1.8) if and only if $\sigma \equiv \varphi$. Thus, for constant volatility, φ may be determined by inverting u with respect to the time variable. More generally, if one can find $\varphi(x,\tau) \geq 0$, so that

$$v(x,\tau) = u(x, \tau\varphi^2(x,\tau)) \tag{3.5}$$

satisfies equation (1.8) for all $(x,\tau) \in \Omega$, then $\varphi(x,\tau)$ is called the implied volatility. One main result in this chapter is to find an equation which $\varphi(x,\tau)$ satisfies. In the next chapter, we will study its asymptotics as $\tau \to 0$ (near expiry) and $x \to \pm\infty$ (far out-of-the-money/deep in-the-money option).

We now state an existence, uniqueness lemma for the implied volatility.

Lemma 3.1. *Given $v(x,\tau)$, the solution to (1.8), the choice of φ as described in (3.5) exists, is unique, and positive in Ω_T.*

Proof. To find $\varphi(x,\tau)$, one just inverts $u(x,t)$ with respect to the time variable. Since v exists, $(e^{+}-1)_{+} \le v(x,\tau) < e^x$ and is unique (Theorem 2.21). On the other hand, $u(x,0) = (e^x-1)_{+}$, $\lim_{\tau\to\infty} u(x,\tau) = e^x$, and $u_t > 0$ (equation (3.3)) in Ω, the inverse function theorem [1] assures us the *existence* and *uniqueness* of such inverse.

The *positivity* of φ in $\Omega_T = \mathbb{R}\times(0,T)$ follows from the monotonicity of v in time. If there is $(x_0,\tau_0) \in \Omega_T$ such that $\varphi(x_0,\tau_0) = 0$, then, $v(x_0,\tau_0) = v(x_0,\tau_0\varphi^2(x_0,\tau_0)) = v(x_0,0)$, a contradiction to Theorem 2.21. □

However, as stated in the introduction section, our strategy is *not* to invert $u(x,\tau\varphi^2)$. Instead, we see the implied volatility as the unique solution to a well-posed degenerate quasilinear parabolic problem.

For any $\psi \in W^{2,1,p}_{loc}(\Omega_T)$, denote by H the quasilinear operator [5]

$$H[\psi] \equiv H(x,\tau,\psi,D\psi,D^2\psi) = (1 - x\frac{\psi_x}{\psi})^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2. \tag{3.6}$$

Theorem 3.2. *Under Condition H0, suppose that the implied volatility φ is defined by (3.5) where v and u are solutions to (1.8) and (3.1), respectively. Then, $\varphi \in W^{2,1,p}_{loc}(\Omega_T)$ for all $p > 1$, and it satisfies*

$$(\tau\varphi^2)_\tau - \sigma^2(x,\tau)H[\varphi] = 0 \quad a.e.\ in\ \Omega_T. \tag{3.7}$$

Proof. We first show that $\varphi \in W^{2,1,p}_{loc}(\Omega_T)$. Therefore, we need estimates on φ and its various (weak) derivatives. We do so by the following two steps.

1. We express φ, and its derivatives in terms of u, v and their derivatives.

   Denote $u^{-1}(y;x) : \mathbb{R}^+ \to \mathbb{R}^+$ the inverse of $u(x,\tau)$ with respect to the time variable, *i.e.*, $u(x, u^{-1}(y;x)) = y$ for any $y \in \mathbb{R}^+$. Then given $(x,\tau) \in \Omega_T$,

$$\varphi^2(x,\tau) = u^{-1}(v(x,\tau);x)/\tau. \tag{3.8}$$

   By formally taking the derivative of $v(x,\tau) = u(x, \tau\varphi^2(x,\tau))$ with respect to x, and using (3.4), one obtains

$$v_x = u_x + 2\tau\varphi\varphi_x u_\tau. \tag{3.9}$$

   Hence,

$$\varphi_x(x,\tau) = \frac{v_x - u_x}{2\tau\varphi u_\tau}(x,\tau). \tag{3.10}$$

   Taking a second derivative of v with respect to x yields

$$v_{xx} = u_{xx} + 4\tau\varphi\varphi_x u_{x\tau} + 2\tau(\varphi_x^2 + \varphi_{xx}\varphi)u_\tau + (2\tau\varphi\varphi_x)^2 u_{\tau\tau}.$$

   The above equality and identities (3.4) imply

$$\begin{aligned}\varphi_{xx}(x,\tau) &= \frac{v_{xx} - u_{xx}}{2\tau u_\tau \varphi} - \left[\frac{2\varphi\varphi_x u_{x\tau}}{u_{x\tau}} + \frac{2\tau(\varphi\varphi_x)^2 u_{\tau\tau}}{u_\tau\varphi} + \frac{\varphi_x^2}{\varphi}\right], \\ &= \frac{v_{xx} - u_{xx}}{2t\varphi u_\tau}(x,\tau) - \left[\varphi_x + \frac{x\varphi_x(x\varphi_x - 2\varphi)}{t\varphi^3} - \frac{t\varphi\varphi_x^2}{4}\right](x,\tau).\end{aligned} \tag{3.11}$$

Similarly, differentiating v with respect to the time variable gives

$$v_\tau = u_\tau(\varphi^2 + 2\tau\varphi\varphi_\tau).$$

Solving for φ_τ:

$$\varphi_\tau(x,\tau) = \frac{v_\tau}{2\tau u_\tau \varphi}(x,\tau) - \frac{\varphi}{2\tau}(x,\tau). \tag{3.12}$$

2. Since $v \in W^{2,1,p}_{loc}(\Omega_T)$ (Theorem 2.21), $u \in C^\infty(\Omega_T)$, $u_\tau > 0$ and $\varphi > 0$ in Ω_T, it follows that φ and all its (weak) derivatives stated above are locally $p-$integrable for any $p > 1$. This implies a finite $W^{2,1,p}_{loc}$ bound on φ.

Next, we show how to derive equation (3.7). Define $w(x,\tau) = u(x, \tau\varphi^2(x,\tau))$.

Since $\varphi \in W^{2,1,p}_{loc}$, differentiating w with respect to x and τ yields:

$$w_x(x,\tau) = u_x + u_\tau 2\tau\varphi\varphi_x, \tag{3.13a}$$

$$w_{xx}(x,\tau) = u_{xx} + u_{x\tau}4\tau\varphi\varphi_x + u_{\tau\tau}(2\tau\varphi\varphi_x)^2 + u_\tau 2\tau(\varphi_x^2 + \varphi\varphi_{xx}), \tag{3.13b}$$

$$w_\tau(x,\tau) = u_\tau(\varphi^2 + 2\tau\varphi\varphi_\tau). \tag{3.13c}$$

From (3.13), (3.3), and (3.4),

$$
\begin{aligned}
& w_\tau - \frac{1}{2}\sigma^2(w_{xx} - w_x) \\
\overset{(3.13)}{=} & u_\tau(\varphi^2 + 2\tau\varphi\varphi_\tau) - \frac{1}{2}\sigma^2 \\
& [u_{xx} + u_{x\tau}4\tau\varphi\varphi_x + u_{\tau\tau}(2\tau\varphi_x\varphi_{xx})^2 + u_\tau 2\varphi(\varphi_x^2 + \varphi\varphi_{xx} - \varphi\varphi_x)] \\
\overset{(3.3)}{=} & u_\tau(x, \tau\varphi^2)\{2\tau\varphi\varphi_\tau + \varphi^2 - \frac{\sigma^2}{2}[2 - 2\tau\varphi\varphi_x \\
& + 2\tau(\varphi_x^2 + \varphi\varphi_{xx}) + 4\tau\varphi\varphi_x\frac{u_{\tau\tau}}{u_\tau}(x, \tau\varphi^2) + 4\tau^2\varphi^2\varphi_x^2\frac{u_{\tau\tau}}{u_\tau}(x, \tau\varphi^2)]\} \\
\overset{(3.4)}{=} & u_\tau(x, \tau\varphi^2)\{2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2\left[\left(1 - \frac{x\varphi_x}{\varphi}\right)^2 + \tau\left(\varphi\varphi_{xx} - \frac{\varphi^2\varphi_x^2}{4}\right)\right]\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
F(x, \tau, \psi, D\psi, D^2\psi) &= 2\tau\psi\psi_\tau + \psi^2 - \sigma^2\left[\left(1 - \frac{x\psi_x}{\psi}\right)^2 + \tau\left(\psi\psi_{xx} - \frac{\psi^2\psi_x^2}{4}\right)\right] \\
&= (\tau\psi^2)_\tau - \sigma^2 H(x, \tau, \psi, D\psi, D^2\psi).
\end{aligned}
$$

Then, $w(x, \tau)$ satisfies (1.8) if

$$
w_\tau - \frac{1}{2}\sigma^2(w_{xx} - w_x) = u_\tau(x, \tau\varphi^2)F(x, \tau, \varphi, D\varphi, D^2\varphi) = 0.
$$

Because $u_\tau(x, \tau\varphi^2) > 0$ on $\mathbb{R} \times (0, \infty)$, this is equivalent to

$$
F(x, \tau, \varphi, D\varphi, D^2\varphi) = (\tau\varphi^2)_\tau - \sigma^2(x, \tau)H[\varphi] = 0.
$$

□

A very closely related concept is the so called

3.2 Associated Local Volatility $\sigma[\varphi]$

Definition 3.3. *Let $\mathscr{I}(0,\xi)$ be the class of nonnegative functions $\psi \in W^{2,1,p}_{loc}(\Omega_T)$ for which the "associated local volatility"*

$$\sigma[\psi](x,\tau) = \left(\frac{(\tau\psi^2)_\tau}{H(x,\tau,\psi,D\psi,D^2\psi)}\right)^{1/2} \tag{3.14}$$

is well defined, continuous in Ω_ξ, and satisfies Condition H0. Furthermore, we require the growth condition at zero:

$$\tau\psi^2(x,\tau) \to 0 \quad \textit{as } \tau \to 0, \tag{3.15}$$

since $u(x,\tau\psi^2)$ has to replicate the behavior of $u(x,\tau)$ as $\tau \to 0$.

Remark 3.4. *We call $\sigma[\psi]$ the "associated local volatility" of $\psi \in \mathscr{I}(0,\xi)$ because, for $\omega(x,\tau) = u(x,\tau\psi^2)$,*

$$\omega_\tau - \frac{1}{2}\sigma[\psi]^2(\omega_{xx} - \omega_x) = 0 \quad \textit{on } \Omega_\xi, \tag{3.16a}$$

$$\omega(x,0) = (e^x - 1)_+. \tag{3.16b}$$

Note that $v(x,\tau) = u(x,\tau\varphi^2)$ satisfies both

$$v_\tau - \frac{1}{2}\sigma[\varphi]^2(v_{xx} - v_x) = 0 \quad \textit{on } \Omega_T,$$

$$v(x,0) = (e^x - 1)_+,$$

and equation (1.8). Therefore,

$$\sigma^2[\varphi] = \frac{2v_\tau}{v_{xx} - v_x} = \sigma^2 \qquad on\ \Omega_T.$$

As we will see later, the "associated local volatility" provides a way to back out the local volatility from the solution to equation (3.7). In the next chapter, where we are looking for the asymptotics of the implied volatility, it becomes an essential tool to form super and sub solutions, on which variance comparison theorems are performed.

Chapter 4

The Asymptotic of φ as $\tau \to 0$

Condition H1

In this chapter, we assume, in addition to Condition H0, σ is *uniformly Lipschitz* in Ω.

4.1 The Fundamental Solutions, Green's Functions, etc.

4.1.1 Uniform Parabolic Equations

We dedicate this subsection to giving an estimate on the fundamental solution (Proposition 4.5), starting with the following notations and conditions.

Definition 4.1. *Let Σ denote an open domain in $\mathbb{R}^d$. It is not necessary that Σ be bounded and $\Sigma = \mathbb{R}^d$ is not excluded. Let I denote an interval included in $[0, T]$. A function $w = w(x,t)$ defined and measurable on $\mathcal{D} = \Sigma \times I$ is said to belong to the class $L^{p,q}(\mathcal{D})$ with $1 \le p, q < \infty$ if*

$$\|w\|_{p,q} = \left[\int_I \left(\int_{\Sigma} |w|^p dx \right)^{q/p} \right]^{1/q} < \infty.$$

In case either p or q is infinite, $\|w\|_{p,q}$ is defined in a similar fashion using L^∞ norms rather than integrals.

Recall the operator L (2.14):

$$L \equiv \frac{\partial}{\partial t} - \left\{ \sum_{i,j=1}^{d} a_{i,j}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,t) \frac{\partial}{\partial x_i} + c(x,t) \right\}$$

in a (d+1)-dimensional domain Ω.

The following conditions are collectively referred to as **Condition H2:** *(To show the bounds of the fundamental solutions and the Green's Functions to uniform parabolic equations.)*

There exist ν, M, M_0 and R_0 such that $\nu > 0$, $M < \infty$, $0 \le M_0 < \infty$ and $0 \le R_0 \le \infty$, and such that the coefficients of L satisfy the following conditions:

1. *For all $\xi \in \mathbb{R}^d$ and for almost all (x,t)*

$$a_{i,j}(x,t)\xi_i\xi_j \ge \nu|\xi|^2 \quad \text{and } |a_{i,j}(x,t)| \le M.$$

2. (a) *Let $Q_0 = \{|x| < R_0\} \times (0,T]$. Each of the coefficients $b_i(x,t) - \sum_{j=1}^{d}(a_{i,j})_{x_j}(x,t)$ is contained in some space $L^{p,q}(Q_0)$, where p, q are such that*

$$(*) \qquad 2 < p,\ q \le \infty \quad \text{and } \frac{d}{2p} + \frac{1}{q} < \frac{1}{2};$$

and

(b) $\left| b_i(x,t) - \sum_{j=1}^{d}(a_{i,j})_{x_j}(x,t) \right| \le M_0$ *for all $|x| \ge R_0$ and $t \in (0,T]$.*

3. (a) $\|c(x,t)\|_{p,q} < \infty$, *where the norms are taken over for all cylinders of the form* $R(\varrho) \times (0,T]$, *contained in* $\mathbb{R}^d \times I$. $R(\varrho)$ *denotes an open cube in* $\mathbb{R}^d$ *of edge length* ϱ *and* $\varrho = \min\{1, \sqrt{T}\}$. p *and* q *are such that*

$$(**) \qquad 1 < p,\ q \leq \infty \quad \text{and } \frac{d}{2p} + \frac{1}{q} < 1.$$

and

(b) $c(x,t) \leq M_0$ *for almost all* $|x| \geq R_0$ *and* $t \in (0,T]$.

All (strong) derivatives are defined and measurable in Ω_T.

Remark 4.2. *Consider the operator* L *defined as*

$$L \equiv \frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2(x,\tau)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right). \tag{4.1}$$

Since $d = 1$, *its coefficients satisfies Condition H2 if there exists* ν, M, M_0 *and* R_0 *such that* $0 < \nu$, $M < \infty$, $0 \leq M_0 < \infty$ *and* $0 \leq R_0 \leq \infty$, *and such that*

1. *For almost all* (x,t)

$$\frac{1}{2}\sigma^2(x,t) \geq \nu \quad \text{and} \quad \left|\frac{1}{2}\sigma^2(x,t)\right| \leq M.$$

2. *Let* $Q_0 = (-R_0, R_0) \times (0,T]$, *then* $(\frac{1}{2}\sigma^2(x,t))_x$ *is contained in some space* $L^{p,q}(Q_0)$, *where* p, q *are such that*

$$(*) \qquad 2 < p,\ q \leq \infty \quad \text{and } \frac{1}{2p} + \frac{1}{q} < \frac{1}{2};$$

and $|(\frac{1}{2}\sigma^2)_x| \leq M_0$ *for all* $|x| \geq R_0$ *and* $t \in (0, T]$.

Remark 4.3. *The statement "*$\mathscr{C}$ *depends on the structure of* L*" means that* $\mathscr{C}$ *is determined by the quantities* ν, M, d, $\|(\frac{1}{2}\sigma^2)_x\|_{p,q(Q_0)}$ *and* p, q, d *occurring in Condition H2.*

The following lemma says if σ satisfies Condition H1, and it has upper and lower bounds, then if one defines σ^ε as in (2.31), then $(\sigma^\varepsilon)^2$ satisfies Condition H2, with p, q any integers satisfying $(*)$.

Lemma 4.4. *Suppose* $f \in \mathbb{R}$ *is Lipschitz continuous with Lipschitz constant* K. *Let* $\rho^\varepsilon(x)$ *be a standard mollifier, that is, a function satisfying (2.30). Denote*

$$f^\varepsilon(x) = (f * \rho^\varepsilon)(x) = \int_{\mathbb{R}} f(x-\xi)\rho^\varepsilon(\xi)d\xi = \int_{\mathbb{R}} f(\xi)\rho^\varepsilon(x-\xi)d\xi.$$

Then $|(f^\varepsilon)'(x)| \leq K$ *on* $\mathbb{R}$*, uniformly in* ε.

Proof. Note that

$$\begin{aligned}
&\left|\frac{f^\varepsilon(x+h) - f^\varepsilon(x)}{h}\right| \\
=&\left|\int_{\mathbb{R}} \frac{f(x+h-\xi) - f(x-\xi)}{h}\rho^\varepsilon(\xi)\, d\xi\right| \\
\leq&\int_{\mathbb{R}} \frac{|f(x+h-\xi) - f(x-\xi)|}{h}\ \rho^\varepsilon(\xi)\, d\xi \\
\leq&\int_{\mathbb{R}} K \cdot \rho^\varepsilon(\xi)d\xi \\
=&K
\end{aligned}$$

uniformly in $x \in \mathbb{R}$, $\varepsilon > 0$, and $h > 0$. Therefore,

$$|(f^\varepsilon)'(x)| = \lim_{h\to 0}\left|\frac{f^\varepsilon(x+h) - f^\varepsilon(x)}{h}\right| \leq L,$$

for all $x \in \mathbb{R}$, $\varepsilon > 0$. □

Proposition 4.5. *(**Bounds on Fundamental Solutions**)([3] Theorem 7[1]) Consider operator L defined in (4.1). If σ^2 satisfies Condition H2, then the fundamental solution to L exists. Moreover, there exists positive α_1, $\mathscr{C}_1$, $\mathscr{C}_2$ depending only on T and the structure of L, and positive α_2 depending only on the structure of L such that*

$$\mathscr{C}_1(t-\tau)^{-1/2}e^{-\frac{\alpha_1|x-\xi|^2}{t-\tau}} \leq \Gamma(x,t;\xi,\tau) \leq \mathscr{C}_2(t-\tau)^{-1/2}e^{-\frac{\alpha_2|x-\xi|^2}{t-\tau}}, \tag{4.2}$$

for all x, $\xi \in \mathbb{R}$, $0 \leq \xi < t \leq T$.

Remark 4.6. *The purpose of this remark is to provide exact formulas for $\mathscr{C}_1$, $\mathscr{C}_2$, α_1 and α_2. For the reader's convince, we keep the notation consistent with original papers [3], [4] whenever we can. To this end, let us first define a few constants that will appear in different parts of this remark. As a reminder, p, q, M, M_0, and ν are defined in Condition H2. For $d \in \mathbb{N}$, define*

$$\omega_d = \begin{cases} \frac{\pi^{d/2}}{(d/2)!} & d \text{ even} \\ \\ \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{d!!} & d \text{ odd} \end{cases},$$

$\iota = 16/T$, $\tilde{\theta} = 1 - \frac{1}{q} - \frac{d}{2p}$ [2], $\tilde{\sigma} = 1 + 2\tilde{\theta}/d > 1$, *and* $r_0 = 1/(1+\tilde{\sigma})$. *Let μ, λ, and K be constants depend only on the dimension d.*

[1]It is a generation of Harnack Inequality for Parabolic Equations.[38]

[2]In our case, p, q can be any numbers satisfying $(*)$, $d = 1$, and we have $\tilde{\theta} \in [1/2, 1]$.

Define the function g as $g(x) = \dfrac{x}{(x^2-1)^2}$. Furthermore, denote

$$\alpha = \frac{1}{8} \cdot \left[\left(1 + \frac{4d^2M^2}{\nu}\right) + \min(1,\nu) \right]^{-1}, \quad \textit{and}$$
$$\beta = M_0 \left(1 + 2dM_0 + \frac{4dM_0}{\nu}\right).$$

1. *We introduce the following constants:*

$$c_{11} = \nu^{-1}M_0^2 + 16\nu^{-1}M^2 + 12,$$
$$c_{12} = [144K(2 + 4c_{11} + \nu^{-1}c_{11})]^{\frac{\tilde{\sigma}}{2(\tilde{\sigma}-1)}} \cdot \tilde{\sigma}^{g(\tilde{\sigma})},$$
$$B = 2^{-d}\left\{ (\frac{\nu}{\omega_d} d^{1+\frac{d}{2}})^{-1} + 2^{-d} + \sqrt{2^{-d}[c_{11} + 64(1 + 2\nu^{-1}M^2)]} \right\},$$
$$\gamma = \mu^{B^2/\lambda} \cdot c_{12}^{1/r_0},$$

$$\tilde{C}_1 = \ln\gamma \cdot \max\left\{32, \frac{9}{8} + \frac{T}{8} + \frac{9d}{32}\frac{T}{\delta^2}\right\}, \tag{4.3a}$$

$$\textit{and } C_1 = \ln\gamma \cdot \max\{32, d\}. \tag{4.3b}$$

[3] *The value of $\mathscr{C}_1$ is*

$$\mathscr{C}_1 = \frac{1}{\omega_d\, \iota^{d/2}} \exp\{-2\tilde{C}_1(\iota + 1)\} \cdot \exp\{-C_1(T+1)\}. \tag{4.4}$$

2. (a) *Let θ denote the minimum value of $1 - d/(2p) - 1/q$ for all pairs of (p,q) involved.*

---

[3]Note that for each fixed δ, when $d < 32$ and T is small enough, $\tilde{C}_1 = C_1 = 32\ln\gamma$.

[4] *Now, let*

$$
\begin{aligned}
\hat{\sigma} &= \min\left[1, \left\{\frac{\min\{1,\nu\}}{8K\left[2\left(1+\frac{2}{\nu}\right)(M_0^2+M^2)+\frac{1}{2}\right]}\right\}^{1/\theta}\right],\\
c_{211} &= 2^{1+T/\hat{\sigma}},\\
c_{212} &= \left(c_{211}\int_{\mathbb{R}^d}\exp\{-\|z\|^2\}dz\,\alpha^{-d/2}\right)^{1/2},\\
\text{and } c_{213} &= c_{212}\cdot\exp\left(\frac{9\alpha}{4}+\beta T\right).
\end{aligned}
$$

Define, [5]

$$
C_{21} = \sqrt{27+(c_{213})^2}. \tag{4.5}
$$

(b) Define

$$
\begin{aligned}
c_{221} &= \nu^{-1}M_0^2+1,\\
\tilde{\mu} &= \left\{\frac{\min(\nu,1)}{8}Kc_{221}\right\}^{1/\tilde{\theta}},\\
c_{222} &= 9+16\nu^{-1}M^2,\\
c_{223} &= c_{221}+2c_{222},\\
c_{224} &= 2K\left[5+(\nu^{-1}+4)c_{223}\right],\\
c_{225} &= (36c_{224})^{\tilde{\sigma}/(\tilde{\sigma}-1)}(2\tilde{\sigma})^{2g(\tilde{\sigma})},\\
\text{and}\quad c_{226} &= 2^{10+1/\tilde{\mu}}K(1+\nu^{-1})(1+c_{222}).
\end{aligned}
$$

[4] In our case, p, q can be any number satisfying $(*)$, and $d = 1$, we have $\theta > 1/2$.

[5] α, β are defined earlier in the remark.

We then let [6]

$$C_{22} = \sqrt{c_{225}\, c_{226}}. \tag{4.6}$$

(c) Define

$$c_{232} = 4K(2 + 4c_{11} + \nu^{-1}c_{11}),$$

$$c_{233} = 4c_{232}(1 + \tilde{\sigma}^2)(\tilde{\sigma} - 1)^{-2},$$

$$\text{and } c_{234} = (9c_{233})^{\frac{1+2\tilde{\theta}}{4\tilde{\theta}}} \cdot \tilde{\sigma}^{g(\tilde{\sigma})}.$$

Denote [7]

$$C_{23} = \tilde{\mu}^{B^2/\lambda}(c_{234})^{2/r_0}. \tag{4.7}$$

We finally arrive

$$\mathscr{C}_2 = C_{21}C_{22}C_{23}e^{\alpha/2}. \tag{4.8}$$

3.

$$\alpha_1 = 2\tilde{C}_1 \tag{4.9}$$

4.

$$\alpha_2 = \alpha/8, \tag{4.10}$$

where α is defined at the beginning of this remark. $\tilde{C}_1$ is defined in (4.3).

It is worth noting

[6] K, $\tilde{\theta}$, $\tilde{\sigma}$, and $g(x)$ are defined at the beginning of this remark.
[7] c_{11}, and B were defined before in case 1.

- *Keeping the other parameters fixed,* $\mathscr{C}_1 \sim O(T^{d/2})$*, as* $T \to 0$.

- *Keeping the other parameters fixed and fixing* $\bar{T} < \infty$*,* C_{21} *has uniform upper and lower bounds for* $0 \leq T \leq \bar{T}$*;* C_{22}*,* C_{23}*, and* α *depend only on the structure of* L*. Therefore,* $\mathscr{C}_2$ *has uniform upper and lower bounds for all* $0 \leq T \leq \bar{T}$.

- *Since* γ *depends only on the structure of* L*,* $\alpha_1 = 2\ln\gamma \cdot \max\{32, \frac{9}{8} + \frac{T}{8} + \frac{9dT}{32\delta^2}\}$ *depends only on the structure of* L *for* T *small enough. More precisely,*

$$\alpha_1 \sim O\left(\max\left\{\frac{1}{\nu^2}, \frac{M_0^2 + M^2}{\nu}\right\}\right), \quad \text{and } \alpha_2 \sim O\left(\frac{\nu}{M^2}\right),$$

when T *is small enough.*

4.1.2 Parabolic Equations with unbounded Coefficients

The goal of this part is to show $w^\varepsilon_\tau > 0$ *in* Ω_T*, where* w^ε *satisfies*

$$\begin{aligned} w^\varepsilon_\tau &= \frac{1}{2}(\sigma^\varepsilon)^2(w^\varepsilon_{xx} - w^\varepsilon_x) \\ w^\varepsilon(x,0) &= (e^x - 1)_-. \end{aligned}$$

Let us recall some properties relating to the Green's Functions.

Proposition 4.7. *(Ch3, Sec 7 [24], Theorem 16) Given* M*,* $T_\alpha > 0$*, denote* $\sum_\alpha^M = (-M, M) \times (0, T_\alpha]$*. Let* L_α^M *in the form of (2.14) be a parabolic operator in* $\sum_\alpha^M$*. If* L_α^M *satisfies*

(A) the coefficients of L_α^M *are uniformly Hölder continuous (exponent* α*)* [8] *in* $\overline{\sum_\alpha^M}$ *and*

$$\|a_{11}\|_{C^\alpha(\sum_\alpha^M)},\ \|b_1\|_{C^\alpha(\sum_\alpha^M)},\ \|c\|_{C^\alpha(\sum_\alpha^M)} \leq K_1,$$

(B) for any $(x,t) \in \sum_\alpha^M$,

$$a_{1,1}(x,t) \geq K_2 > 0,$$

then the Green's function $\gamma_\alpha^M(x,t;\xi,\tau)$ *for* L_α^M *with zero Dirichlet initial data exists, and has the following properties:*

a) For any $0 \leq \tau < T_\alpha$*, and for any continuous, compactly supported function* f *on* $B_\tau := \{|x| < M\} \times \{t = \tau\}$*, the function*

$$u(x,t) = \int_{B_\tau} \gamma_\alpha^M(x,t;\xi,\tau) f(\xi) d\xi$$

is a solution to $L_\alpha^M v = 0$ *in* $\{|x| < M\} \times \{\tau < t \leq T_\alpha\}$*, and it satisfies the initial and boundary conditions*

$$\lim_{t \to \tau^+} u(x,t) = f(x) \quad \text{for } x \in \bar{B}_\tau,$$

$$u(x,t) = 0 \quad \text{on } \{|x| = M\} \times \{\tau < t \leq T_\alpha\}.$$

b) (Ch3, Sec 7 [24], Corollary 1) As a function of $(x,t) \in \sum_\alpha^M$, $L_\alpha^M \gamma_\alpha^m = 0$*. Furthermore, for* $(\xi,\tau) \in \sum_\alpha^M \cup [-M,M] \times \{0\}$, $\gamma_\alpha^M(x,t;\xi,\tau) = 0$ *on* $\{|x| = M\} \times \{\tau < t \leq T_\alpha\}$ *and*

[8] Recall a function $f(x,\tau)$ defined on a bonded closed set S of $\mathbb{R}^2$ is said to be Hölder continuous of exponent α in S [24] if there exists a constant A such that for all (x,τ), $(x^0,\tau^0) \in S$,

$$|f(x,\tau) - f(x^0,\tau^0)| \leq A(|x - x^0|^2 + |\tau - \tau^0|)^{\alpha/2}.$$

$\gamma_\alpha^M(x,t;\xi,\tau) > 0$ on $\{|x| < M\} \times \{\tau < t \leq T_\alpha\}$.

c) ***(Bounds on Green's Functions)****([3] Theorem 8)*

Suppose furthermore that, if we restrict σ^2 on a bounded domain, then it satisfies Condition H2. Let $(\Omega^M)'$ be a subinterval of $(-M, M)$, and let $\delta > 0$ be the distance from $(\Omega^M)'$ to $(-M, M)$. Then,

(a) There exist positive constants $\mathscr{C}_2$ depending on T_α and the structure of L, and α_2 depending on the structure of L, such that for all (x,t) and (ξ,τ) in $\sum_\alpha^M$ with $t > \tau$,

$$\gamma_\alpha^M(x,t;\xi,\tau) \leq \mathscr{C}_2(t-\tau)^{-1/2} \exp\left\{-\alpha_2 \frac{|x-\xi|^2}{8(t-\tau)}\right\}. \tag{4.11}$$

(b) There exist positive constants $\mathscr{C}_1$ and α_1, depending on δ, T_α and the structure of L, such that

$$\gamma_\alpha^M(x,t;\xi,\tau) \geq \mathscr{C}_1(t-\tau)^{-1/2} \exp\left\{-\alpha_1 \frac{|x-\xi|^2}{t-\tau}\right\}. \tag{4.12}$$

holds for all x, $\xi \in (\Omega^M)'$ and either

$$\tau < t \leq \min\left\{T_\alpha, \tau + \frac{T_\alpha}{8} d^2(\xi, \partial(\Omega^M)')\right\}$$

for arbitrary $\tau \in [0, T_\alpha)$ or

$$\max\left\{0, t - \frac{T_\alpha}{8} d^2(x, \partial(\Omega^M)')\right\} \leq \tau < t$$

for arbitrary $t \in (0, T_\alpha]$.

Remark 4.8. *The structures and values of* $\mathscr{C}_1$, $\mathscr{C}_2$, α_1 *and* α_2 *are similar to those bounds on Fundamental Solution,* i.e., *in Remark 4.6.*

We are interested in the properties of the operator L_α^M on $\sum_\alpha^M$ as $M \to \infty$, *i.e.*, the properties of the operator L_α in $\bar{\Omega}_{T_\alpha}$.

Let $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $S = \mathbb{R}^d \times (0, t]$ and $\overline{S} = \mathbb{R}^d \times [0, T]$ for some fixed $T > 0$. Consider the differential operator L given in the form (2.14). We collectively refer the following conditions as **Condition H3:** *(To show the bounds of* v_τ^ε *.)*

Assume that $a_{i,j}$, $(a_{i,j})_{x_i}$, $b_i - \sum_{j=1}^d a_{i,j}$, $[b_i - \sum_{j=1}^d a_{i,j}]_{x_i}$ *are Hölder continuous, and* $a_{i,j}$ *is twice differentiable in* x *on every compact subset of* $\overline{S}$. *We further assume there exist constants* ν, $\tilde{\kappa}_2 > 0$ *such that*

1.

$$\nu|\xi|^2 \le a_{i,j}(x,t)\xi_i\xi_j \le \tilde{\kappa}_2\sqrt{|x|^2+1}|\xi|^2;$$

2.

$$|(a_{i,j})_x|, \left|b_i(x,t) - \sum_{j=1}^d (a_{i,j})_{x_j}(x,t)\right| \le \tilde{\kappa}_2\sqrt{|x|^2+1}; \quad \textit{and}$$

3.

$$c(x,t),\ c(x,t) - \left[b_i(x,t) - \sum_{j=1}^d (a_{i,j})_{x_j}(x,t)\right]_{x_i} \le \tilde{\kappa}_2\sqrt{|x|^2+1}$$

for all $(x,t) \in \overline{S}$, *and* $\xi \in \mathbb{R}^d$.

Remark 4.9. *Consider operators of the form*

$$L \equiv \frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2\left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right] - 2\frac{\sigma_\tau}{\sigma}. \tag{4.13}$$

They satisfy Condition H3 if σ^2, $(\sigma^2)_x$, and $(\sigma^2)_{xx}$ are Hölder continuous on every compact subset of Ω, and if there exist constants ν, κ_2, $\kappa_3 > 0$ such that

1. $$\nu \leq \frac{1}{2}\sigma^2(x,t) \leq \kappa_2\sqrt{x^2+1},$$

2. $$\left|(\frac{1}{2}\sigma^2)_x\right|,\ 2\frac{\sigma_t}{\sigma} \leq \kappa_2\sqrt{x^2+1}, \quad \textit{and}$$

3. $$\sigma\sigma_{xx} \leq \kappa_3\sqrt{x^2+1}$$

for all $(x,t) \in \Omega$.

Similar as discussed before, suppose σ satisfies Condition H1, and it is bounded below, *then σ^ε satisfies Condition H3. By Lemma 4.4, The constants ν and κ_2 are uniform in ε. But $\kappa_3(\varepsilon)$ may approach to infinity as $\varepsilon \to 0$. This is because $\rho^\varepsilon \in C_0^\infty$, when differentiating σ^ε, the derivatives are passed onto ρ^ε, the upper bounds of which depend on ε. See the proof for Lemma 2.14 for a detailed calculation. By Condition H0, σ^2 has no more than linear growth. Therefore, $\kappa_3(\varepsilon)$ really is the upper bound for $\frac{\partial^2}{\partial x^2}\rho^\varepsilon$.*

For operators of form 4.13, we start our analysis with a sufficiently smooth, bounded below σ^2. To this end, let us take $\tilde{\sigma}(x,-\tau) = \sigma(x,\tau)$ for all $\tau \leq 0$ as an extension of σ, and define for any $\varepsilon > 0$, ρ^ε a standard mollifier. Define σ^ε by

$$\sigma^\varepsilon = \rho^\varepsilon * \tilde{\sigma}.$$

Given $n \in \mathbb{N}$, $\sigma_n^\varepsilon(x,\tau;\eta)$ is a smooth function in Ω_T such that

$$\sigma_n^\varepsilon(x,\tau) = \begin{cases} \sigma^\varepsilon(x,\tau) & \sigma^\varepsilon(x,\tau) \geq \frac{2}{n} \\ \frac{1}{n} & \sigma^\varepsilon(x,\tau) \leq \frac{1}{n}. \end{cases}$$

Clearly, if $\sigma^2 \geq \nu^2 > 0$ in Ω_T, then $\sigma_n^\varepsilon(x,\tau) \equiv \sigma^\varepsilon(x,\tau)$ in Ω_T for all $n > 1/(2\nu)$.

Lemma 4.10. *([2]) Given T, ε, n, and $\alpha > 0$, there exists $0 < T_\alpha(n,\varepsilon) < T$ such that the Cauchy problem*

$$(z_{n,\alpha}^\varepsilon)_t = \frac{1}{2}(\sigma_n^\varepsilon)^2[(z_{n,\alpha}^\varepsilon)_{xx} - (z_{n,\alpha}^\varepsilon)_x] + 2\frac{(\sigma_n^\varepsilon)_t}{\sigma_n^\varepsilon} z_{n,\alpha}^\varepsilon \quad \text{in } \Omega_{T_\alpha} := \mathbb{R} \times (0, T_\alpha], \tag{4.14a}$$

$$z_{n,\alpha}^\varepsilon(x,0) = \frac{1}{2}(\sigma_n^\varepsilon(0,0))^2 \delta_0(x) \tag{4.14b}$$

has a solution for $0 < t \leq T_{\alpha(n,\varepsilon)}$. It is strictly positive in Ω_{T_α}.

Proof. As argued in Lemma 2.15, the solution to (4.14) is the fundamental solution to the corresponding operator multiplied by $\frac{1}{2}(\sigma_n^\varepsilon(0,0))^2 > 0$. Hence, it is equivalent to show the existence and strict positivity of the fundamental solution to the corresponding operator. We further provide the expression of the corresponding fundamental solution through our proof. Further estimates on the fundamental solution will be discussed in the following remark.

- **Notations and definitions**

  Throughout *the proof for this lemma*, we define the operator L_n^ε associated with equation (4.14) on T as:

$$L_n^\varepsilon \equiv \frac{\partial}{\partial t} - \frac{1}{2}(\sigma_n^\varepsilon)^2 \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) - 2\frac{(\sigma_n^\varepsilon)_\tau}{\sigma_n^\varepsilon}, \quad (x,t) \in \Omega_T. \tag{4.15}$$

Given parameter $\alpha > 0$, we introduce the following definitions and notations ([2]):

i) $\beta(\varepsilon, \alpha) = \max\{\kappa_2, \kappa_3(\varepsilon)\}(\alpha + 2)^2 > 0$, where κ_2, κ_3 are defined in Remark 4.9.

ii) $T_\alpha(\varepsilon) = \min\left\{T, \dfrac{1}{2\beta}\right\} > 0$ is a non-increasing function of α such that

$$\lim_{\alpha\to\infty} T_\alpha(\varepsilon) = 0 \quad \text{and} \quad \lim_{\alpha\to 0} T_\alpha(\varepsilon) = T_0(\varepsilon) \le T.$$

Thus, in particular,

$$0 < T_\alpha(\varepsilon) < T_0(\varepsilon) \quad \text{for } \alpha \in (0, \infty).$$

To simplify notation, for the rest of the proof, we use β, κ_2, and T_α.

iii) Define

$$g_\alpha(x,t) = (\alpha + \beta t)\sqrt{x^2+1}. \tag{4.16}$$

iv) For any $u(x,t)$ defined on $\overline{\Omega}_{T_\alpha}$, we define

$$u_\alpha(x,t) = e^{-g_\alpha(x,t)}u(x,t); \tag{4.17}$$

$$L^\varepsilon_{n,\alpha} = \frac{\partial}{\partial t} - \frac{1}{2}(\sigma^\varepsilon_n)^2\frac{\partial^2}{\partial x^2} - a^\varepsilon_{n,\alpha}\frac{\partial}{\partial x} - c^\varepsilon_{n,\alpha}, \tag{4.18}$$

where

$$a^\varepsilon_{n,\alpha} = -\frac{1}{2}(\sigma^\varepsilon_n)^2 + (\sigma^\varepsilon_n)^2(g_\alpha)_x,$$

and

$$
\begin{aligned}
c^{\varepsilon}_{n,\alpha} &= 2\frac{(\sigma^{\varepsilon}_n)_t}{\sigma^{\varepsilon}_n} + \frac{1}{2}(\sigma^{\varepsilon}_n)^2(g_\alpha)^2_x - \frac{1}{2}(\sigma^{\varepsilon}_n)^2(g_\alpha)_x + \frac{1}{2}(\sigma^{\varepsilon}_n)^2(g_\alpha)_{xx} - (g_\alpha)_t \\
&= 2\frac{(\sigma^{\varepsilon}_n)_t}{\sigma^{\varepsilon}_n} + \frac{1}{2}(\sigma^{\varepsilon}_n)^2(\alpha+\beta t)\left[(\alpha+\beta t)\frac{x^2}{x^2+1} - \frac{x}{\sqrt{x^2+1}} + \frac{1}{(x^2+1)^{3/2}}\right] \\
&- \beta\sqrt{x^2+1}. \\
&\le (|x|^2+1)^{1/2}\{\kappa_3 + \kappa_2(\alpha+\beta t)^2 + 2\kappa_2(\alpha+\beta t) - \beta\}.
\end{aligned}
$$

The last inequality holds because $(\sigma^{\varepsilon}_n)^2$ satisfies Condition H1. Hence, $c^{\alpha} < 0$ on $\overline{\Omega}_{T_\alpha}$ for $\beta = \max\{\kappa_2, \kappa_3\}(\alpha+2)^2$, and $T_\alpha = \min\left\{T, \dfrac{1}{2\beta}\right\}$.

Clearly,

$$
L^{\varepsilon}_{n,\alpha} u = e^{g_\alpha} L^{\varepsilon}_{n,\alpha} u_\alpha,
$$

and $L^{\varepsilon}_{n,\alpha} u = 0$ in Ω_{T_α} if and only if $L^{\varepsilon}_{n,\alpha} u_\alpha = 0$ in Ω_{T_α}. Moreover, if $\gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau)$ is a fundamental solution of $L^{\varepsilon}_{n,\alpha} v = 0$ then

$$
\Gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau) = e^{g_\alpha(x,t) - g_\alpha(\xi,\tau)} \gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau) \tag{4.19}
$$

is a fundamental solution of $L^{\varepsilon}_n u = 0$.

- **The increasing sequence of Green's Functions**

Let $\sum^M_{T_\alpha} = (-M, M) \times (0, T_\alpha]$ for $M = 1, 2, \cdots$. We use the superscript "M" to indicate the operator, and the corresponding Green's Function are restricted on $(-M, M)$ for the space variable. In each $\overline{\sum}^M_{T_\alpha}$, the leading coefficient of L^M_α is bounded away from zero, and uniformly Hölder continuous. Hence, the Green's function $\gamma_{n,\alpha}^{\varepsilon,M}(x,t;\xi,\tau)$ for $L_{n,\alpha}^{\varepsilon,M}$ with zero Dirichlet data in each $\sum_{T_\alpha}^{M}$ exists, and has the properties in Proposition 4.7. Let $0 < \eta << 1$, and let $\varphi^{M\eta}$ be a smooth function such that $\varphi^{M\eta} \equiv 1$ for $|x| \leq M - 2\eta$, $\varphi^{M\eta} \equiv 0$ for $|x| \geq M - \eta$ and $0 \leq \varphi^{M\eta} \leq 1$. Consider the boundary value problem

$$
\begin{aligned}
L_{n,\alpha}^{\varepsilon,M} v = 0 \qquad & \text{in } (-M, M) \times (\tau, T_\alpha], \\
v(x,\tau) = \varphi^{M\eta} \quad & \text{for } |x| \leq M, \\
v = 0 \qquad & \text{on } \{|x| = M\} \times [\tau, T_\alpha].
\end{aligned}
$$

The solution is given by

$$
v^{M\eta}(x,t) = \int_{|\xi|<M} \gamma_{n,\alpha}^{\varepsilon,M}(x,t;xi,\tau)\varphi^{M\eta}(\xi)d\xi.
$$

Since $c_{n,\alpha}^{\varepsilon} < 0$ it follows from the maximum principle for parabolic equations [24] that $0 \leq v^{M\eta}(x,t) \leq 1$. Thus, letting $\eta \to 0$, we obtain

$$
\int_{|\xi|<M} \gamma_{n,\alpha}^{\varepsilon,M}(x,t;\xi,\tau)d\xi \leq 1 \quad \text{for } |x| \leq M \text{ and } 0 \leq \tau < t \leq T_\alpha. \tag{4.20}
$$

The operator adjoint to $L_{n,\alpha}^{\varepsilon}$ is

$$
\tilde{L}_{n,\alpha}^{\varepsilon} = -\frac{\partial}{\partial t} - \frac{1}{2}(\sigma_n^{\varepsilon})^2 \frac{\partial^2}{\partial x^2} - \tilde{a}_{n,\alpha}^{\varepsilon}\frac{\partial}{\partial x} - \tilde{c}_{n,\alpha}^{\varepsilon},
$$

where

$$\tilde{a}^{\varepsilon}_{n,\alpha} = \frac{1}{2}(\sigma^{\varepsilon}_{n})^2 + (\sigma^{\varepsilon}_{n})^2 \frac{x}{\sqrt{x^2+1}}(\alpha+\beta t) + 2\sigma^{\varepsilon}(\sigma^{\varepsilon}_{n})_x,$$

and

$$\begin{aligned}\tilde{c}^{\varepsilon}_{n,\alpha} =& 2\frac{(\sigma^{\varepsilon}_{n})_t}{\sigma^{\varepsilon}_{n}} + \frac{1}{2}(\sigma^{\varepsilon}_{n})^2 \left[\frac{x^2}{x^2+1}(\alpha+\beta t)^2 - \frac{x}{\sqrt{x^2+1}}(\alpha+\beta t) - (x^2+1)^{-3/2}(\alpha+\beta t)\right] \\ & - (g_\alpha)_t + \sigma^{\varepsilon}_{n}(\sigma^{\varepsilon}_{n})_x[1 - \frac{2x}{\sqrt{x^2+1}}(\alpha+\beta t)] + (\sigma^{\varepsilon}_{n})^2_x + \sigma^{\varepsilon}_{n}(\sigma^{\varepsilon}_{n})_{xx}.\end{aligned}$$

Clearly we have $\tilde{c}^{\varepsilon}_{n,\alpha} < 0$ in $\overline{\Omega}_{T_\alpha}$. As a function of (ξ, τ), $\gamma^{\varepsilon,M}_{n,\alpha}$ is the Green's Function for $\tilde{L}^{\varepsilon}_{n,\alpha}$ in $\Sigma^{M}_{T_\alpha}$, [24]. By an argument similar to the one employed above we have

$$\int_{|x|<M} \gamma^{\varepsilon,M}_{n,\alpha}(x,t;\xi,\tau)dx \leq 1 \quad \text{for } |\xi| \leq M \text{ and } 0 \leq \tau < t \leq T_\alpha. \tag{4.21}$$

Let $\xi \in \mathbb{R}$ be fixed and let $\psi(x)$ be a smooth, non-negative, compactly supported function such that ξ is in the interior of the support of ψ. Consider the functions

$$w_j(x,t) = \int_{|\zeta|<j} \gamma^{\varepsilon,j}_{n,\alpha}(x,t;\zeta,0)\psi(\zeta)d\zeta$$

for $j = M$ and $M+1$, where M is chosen so large that the support of ψ is contained in $|x| < M$. Clearly $L^{\varepsilon,j}_{n,\alpha}w_j = 0$, $j = M, M+1$ in $\sum^{M}_{T_\alpha}$; $w_j = \psi$, $j = M, M+1$ for $t = \tau$; and $w_{M+1} \geq 0 = w_M$, on $|x| = M$, $t \geq 0$. Since $L^{\varepsilon,j}_{n,\alpha}$ is parabolic and $c^{\alpha} \leq 0$, by the maximum principle, $w_{M+1} \geq w_M$, in $\overline{\sum}^{M}_{T_\alpha}$. We now replace $\psi(\zeta)$ by a sequence of distribution functions which have compact support and approximate δ_ξ, the Dirac measure concentrated at ξ. Since $\gamma^{\varepsilon,M}_{n,\alpha}(x,t;\zeta,0)$ is bounded and continuous on $\overline{\sum}^{M}_{T_\alpha}$, $w_M \overset{p.w.}{\to} \gamma^{\varepsilon,M}_{n,\alpha}$ in $\sum^{M}_{T_\alpha}$, and it follows that $\gamma^{\varepsilon,M+1}_{n,\alpha} \geq \gamma^{\varepsilon,M}_{n,\alpha}$ in $\overline{\sum}^{M}_{T_\alpha}$. Thus, if we extend the definition of $\gamma_{n,\alpha}^{\varepsilon,M}(x,t;\xi,\tau)$ by setting $\gamma_{n,\alpha}^{\varepsilon,M} = 0$ for $|x|$ or $|\xi| \geq m$, we have

$$0 \leq \gamma_{n,\alpha}^{\varepsilon,1}(x,t;\xi,\tau) \leq \cdots \leq \gamma_{n,\alpha}^{\varepsilon,M}(x,t;\xi,\tau) \leq \gamma_{n,\alpha}^{\varepsilon,M+1}(x,t;\xi,\tau) \leq \cdots \tag{4.22}$$

for all $x, \xi \in \mathbb{R}$ and $0 \leq \tau < t \leq T_\alpha$. From the properties (4.20), (4.21), (4.22) it follows that the sequence $\{\gamma_{n,\alpha}^{\varepsilon,M}\}$ has a finite limit almost everywhere.

- **The properties of** $\gamma_{n,\alpha}^{\varepsilon} = \lim_{M\to\infty} \gamma_{n,\alpha}^{\varepsilon,M}$**.** In fact, from inequality (2.13) in [2]:

$$\gamma_{n,\alpha}^{\varepsilon,M}(x,t;\xi,\tau) \leq \kappa(t-\tau)^{-\frac{1}{2}},$$

where κ is a constant depending on the dimension of the domain, and $\nu(n)$. But independent of α. Given $x \in \mathbb{R}$, $t > 0$, $\{\gamma_{n,\alpha}^{\varepsilon,M}\}_M$ is monotone nondecreasing in M and has a uniform upper bound, therefore, it has a limit $\gamma_{n,\alpha}^{\varepsilon}$ as $M \to \infty$. It is shown in [2] that $\gamma_{n,\alpha}^{\varepsilon}$ is the fundamental solution to $L_{n,\alpha}^{\varepsilon}$ in Ω_{T_α}. Furthermore,

Proposition 4.11. *i) [2]* THEOREM II*: Given $\alpha > 0$, a fundamental solution of $Lu = 0$ is given by*

$$\Gamma_{n,\alpha}^{\varepsilon}(x,t;\xi,\tau) = \exp\{g_\alpha(x,t) - g_\alpha(\xi,\tau)\}\gamma_{n,\alpha}^{\varepsilon}(x,t;\xi,\tau) \tag{4.23}$$

for $x, \xi \in \mathbb{R}$ and $0 \leq \tau < t \leq T_\alpha$.

ii) [2] $0 \leq \gamma_{n,\alpha}^{\varepsilon}(x,t;\xi,\tau) \leq \kappa(n)(t-\tau)^{-1/2}$ for $x \in \mathbb{R}$ and $\tau \leq t \leq T_\alpha$.

iii) [24] $\gamma_{n,\alpha}^{\varepsilon}(x,t;\xi,\tau) > 0$ for $x \in \mathbb{R}$ and $\tau < t \leq T_\alpha$.

- **The existence and positivity of the solution, $z^{\varepsilon}_{n,\alpha} \in \Omega_{T_\alpha}$, to (4.14).**

$$\begin{aligned} z^{\varepsilon}_{n,\alpha}(x,t) &= \frac{1}{2}\,(\sigma^{\varepsilon}_{n}(0,0))^2 \Gamma^{\varepsilon}_{n,\alpha}(x,t;0,0^+) \\ &= \frac{1}{2}\,(\sigma^{\varepsilon}_{n}(0,0))^2 \exp\{(\alpha+\beta)\sqrt{x^2+1}-\alpha\} \cdot \gamma^{\varepsilon}_{n,\alpha}(x,t;0,0^+), \end{aligned} \tag{4.24}$$

□

Remark 4.12. *We further derive upper and lower bounds of $z^{\varepsilon}_{n,\alpha}(x,t)$*

Recall that σ satisfies Condition H1. Lemma 4.4 and remark 4.9 imply σ^{ε}_{n} satisfies Condition H3. Therefore, given $M \in \mathbb{N}$, n and α, we have the following estimates for $\gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau)$.

1. *By Proposition 4.11, there exists positive constant κ such that*

$$\gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau) \le \kappa(n)(t-\tau)^{-1/2} \quad \text{in } \Omega_{T_\alpha}. \tag{4.25}$$

2. *There exist positive constants $\mathscr{C}_1(M)$, and $\alpha_1(M)$, depending on the structure of $L^{\varepsilon,M+1}_{n,\alpha}$,*

and T_α*, such that* [9]

$$
\begin{aligned}
&\gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau)\\
&= \lim_{K\to\infty} \gamma^{\varepsilon,K}_{n,\alpha}(x,t;\xi,\tau)\\
&\geq \gamma^{\varepsilon,M+1}_{n,\alpha}(x,t;\xi,\tau) \qquad (4.26)\\
&\geq \mathscr{C}_1 (t-\tau)^{-1/2} \exp\left\{-\alpha_1 \frac{(x-\xi)^2}{t-\tau}\right\} \qquad (4.27)\\
&\quad \text{in } \sum_{T_\alpha}^{M}
\end{aligned}
$$

where the first inequality is from equation (4.22), and the second is an application of Proposition 4.7.

By equations (4.19) and (4.24), given $\alpha > 0$*,*

$$z^{\varepsilon}_{n,\alpha}(x,t) = \frac{1}{2}\,(\sigma^{\varepsilon}_{n}(0,0))^2 \Gamma^{\varepsilon}_{n,\alpha}(x,t;0,0^+),$$

where

$$\Gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau) = e^{g_\alpha(x,t)-g_\alpha(\xi,\tau)} \gamma^{\varepsilon}_{n,\alpha}(x,t;\xi,\tau),$$

and

$$g_\alpha(x,t) = (\alpha+\beta t)\sqrt{x^2+1}.$$

[9] Note that for fixed M, the structure of $L^{\varepsilon,M}_{n,\alpha}$ is uniform for all ε, n, and $T_\alpha(\varepsilon)$. Moreover, as discussed in Remark 4.6, α_1 depends only on the structure of $L^{\varepsilon,M}_{n,\alpha}$ when T_α is small enough, which happens when ε is small enough. Therefore, $\mathscr{C}_1$ is a function of M, and α_1 is a function of M and α when ε is small enough.

Therefore, given $M \in \mathbb{N}$, when ε is sufficiently small,

$$z^{\varepsilon}_{n,\alpha}(x,t) \leq \frac{1}{2}\,(\sigma^{\varepsilon}_{n}(0,0))^2 \cdot e^{g_\alpha(x,t)-\alpha} \cdot \kappa(n) t^{-1/2} \quad \text{in } \Omega_{T_\alpha}, \tag{4.28}$$

$$z^{\varepsilon}_{n,\alpha}(x,t) \geq \frac{1}{2}\,(\sigma^{\varepsilon}_{n}(0,0))^2 \cdot e^{g_\alpha(x,t)-\alpha} \cdot \mathscr{C}_1(M) t^{-1/2} \exp\left\{-\alpha_1(M,\alpha)\frac{x^2}{t}\right\} \quad \text{in } \sum_{T_\alpha}^{M}. \tag{4.29}$$

Remark 4.13. *1. If further we have $\sigma^2 \geq \nu > 0$ in Ω_T, then $z^{\varepsilon}_{\alpha} = z^{\varepsilon}_{n,\alpha}$ for all $n > 1/\nu$.*

2. One can push the final time T_α up to $T_0 := \min\left\{T, \frac{1}{4k_1}\right\}$ by sending $\alpha \to 0$.

4.2 The Asymptotics of φ as $\tau \to 0$

As in [5], the essential idea for finding the asymptotics of φ as $\tau \to 0$ is to define suitable sub and super solutions of (1.8) from the formal limiting solution of (3.7) and prove that actual convergence takes place through the comparison principle. Hence, let us first find the formal limiting solution to (3.7).

Claim 4.14. *The unique* ***positive*** *solution of*

$$F(x, 0, \varphi^0, D\varphi^0, D^2\varphi^0) = 0,$$

i.e.,

$$(\varphi^0)^2 - \sigma^2(x,0)(1 - x\frac{\varphi^0_x}{\varphi^0})^2 = 0 \tag{4.30}$$

is

$$\varphi^0(x) = \frac{x}{\int_0^x \frac{ds}{\sigma(s,0)}}. \tag{4.31}$$

Solution 4.15. *Our first observation is*

$$\begin{aligned}\left(\frac{x}{\varphi^0}\right)' &= \frac{\varphi^0 - x\varphi^0_x}{\varphi^{0\,2}} \\ &= \frac{1}{\varphi^0}\left(1 - x\frac{\varphi^0_x}{\varphi^0}\right).\end{aligned}$$

This implies $\varphi^0\left(\frac{x}{\varphi^0}\right)' = 1 - x\frac{\varphi^0_x}{\varphi^0}$. *Therefore, (4.30) is equivalent to* $\varphi^0 = \pm\sigma(x,0)\varphi^0\left(\frac{x}{\varphi^0}\right)'$. *Since* $\varphi^0(x)$ *is non-zero, one can further rewrite equation (4.30) as*

$$\frac{1}{\sigma(x,0)} = \pm(\frac{x}{\varphi^0})'.$$

Integrate both sides from on $[0,x]$, *we get*

$$\begin{aligned}\int_0^x \frac{ds}{\sigma(s,0)} &= \pm\int_0^x \frac{s}{\varphi^0(s)}ds \\ &= \pm(\frac{x}{\varphi^0(x)} - \frac{0}{\varphi^(0)}) \\ &= \frac{x}{\varphi^0(x)}.\end{aligned}$$

In the last equation, we choose the "+" sign. It is because both $\sigma(x,0)$ *and* $\varphi^0(x)$ *are strictly positive, so* $\int_0^x \frac{ds}{\sigma(s,0)}$ *and* $\frac{x}{\varphi^0(x)}$ *should have the same sign. Finally, we rearrange terms and get (4.31), the solution to (4.30).*

The upper and lower solutions that we are comparing φ with have the form

$$\bar{\varphi}(x,\tau) = \varphi^0(x)(1+\kappa\tau), \quad \text{and} \tag{4.32}$$

$$\underline{\varphi}(x,\tau) = \varphi^0(x)(1-\kappa\tau) \tag{4.33}$$

in $\Omega_{T\kappa}$, for some given $\kappa > 0$, and $T(\kappa) << 1/\kappa$.

Clearly, $\bar{\varphi} \geq \varphi^0 \geq \underline{\varphi}$ in Ω_T. We shall further show, under some additional regularity conditions, $\sigma[\bar{\varphi}] \geq \sigma[\varphi] \geq \sigma[\underline{\varphi}]$ in some domain of Ω_T. To motivate our generalized comparison principle, let us start with the following estimates.

Claim 4.16. *Assume additional regularity conditions on σ:*

$$\sigma \in C^{2,1}(\Omega) \quad \textit{and } \sigma_{xx} \in L^{\infty}(\Omega_T); \tag{4.34}$$

and the ***Additional Decay Rate*** *of $\sigma(x,0)$ as $x \to 0$:*

$$\sigma_x(x,0) \to 0, \tag{4.35a}$$

$$\sigma_{xx}(x,0) \to 0. \tag{4.35b}$$

Given $R > 0$, there are $\underline{\sigma}(R)$, $\bar{\sigma}(R)$, such that $0 < \underline{\sigma}(R) \leq \sigma \leq \bar{\sigma}(R) < \infty$ on $[-R,R]\times[0,T]$, and

i)

$$\sigma[\bar{\varphi}](x,\tau) = \sigma(x,\tau) \times [1 + \tau(2\kappa - \sigma^2\frac{\varphi^0_{xx}}{2\varphi^0} - \frac{\sigma_\tau}{\sigma}(x,0))\,] + O(\tau^2), \ \ \textit{on } \Omega^R_T; \tag{4.36}$$

ii)

$$\sigma[\underline{\varphi}](x,\tau) = \sigma(x,\tau) \times [1 - \tau(2\kappa - \sigma^2\frac{\varphi^0_{xx}}{2\varphi^0} - \frac{\sigma_\tau}{\sigma}(x,0))\,] + O(\tau^2), \ \ \textit{on } \Omega^R_T. \tag{4.37}$$

Where $O(\tau^2) = O(\underline{\sigma}(R), \bar{\sigma}(R), \|\sigma\|_{W^{2,0,\infty}([-R,R]\times[0,T])}) \cdot \tau^2$.

Proof. We only provide the proof for equation (4.36). The proof for equation (4.37) is similar.

Besides equations (3.14), (4.31), (4.32), and (4.41), we will also use the following equations:

$$\bar{\varphi}_x(x,\tau) = \varphi^0_x(x)(1+\kappa\tau); \tag{4.38a}$$

$$\bar{\varphi}_{xx}(x,\tau) = \varphi^0_{xx}(x)(1+\kappa\tau); \quad \text{and} \tag{4.38b}$$

$$\bar{\varphi}_\tau(x,\tau) = \kappa\varphi^0(x). \tag{4.38c}$$

Denote

$$d(x) = \int_0^x \frac{dy}{\sigma(y,0)}. \tag{4.39}$$

Note that

$$\begin{aligned}
\varphi^0_x(x) &= \frac{\varphi^0(x)}{x}\left[1 - \frac{\varphi^0(x)}{\sigma(x,0)}\right] \\
&= \frac{d(x) - \frac{x}{\sigma(x,0)}}{d^2(x)}, \qquad \text{and} \qquad (4.40\text{a}) \\
\varphi^0_{xx}(x) &= \frac{\varphi^0_x x - \varphi^0}{x}\left[1 - \frac{\varphi^0}{\sigma(x,0)}\right] + \frac{\varphi^0}{x}\frac{-\varphi^0_x\sigma(x,0) + \sigma_x(x,0)\varphi^0}{\sigma^2(x,0)} \\
&= \frac{x\sigma_x(x,0) + 2[\varphi^0(x) - \sigma(x,0)]}{\sigma^2(x,0)d^2(x)}. \qquad (4.40\text{b})
\end{aligned}$$

We also need the following two higher order derivatives:

$$\begin{aligned}
\frac{d^3}{dx^3}\varphi^0(x) &= \frac{\{\sigma_x(x,0) + x\sigma_{xx}(x,0) + 2[\varphi^0(x) - \sigma_x(x,0)]\}\sigma^2(x,0)d^2(x)}{\sigma^4(x,0)d^4(x)} \\
&\quad - \frac{\{x\sigma_x(x,0) + 2[\varphi^0(x) - \sigma(x,0)]\}2\sigma(x,0)d(x)[\sigma_x(x,0)d(x)+1]}{\sigma^4(x,0)d^4(x)}. \qquad (4.40\text{c})
\end{aligned}$$

It is then straight forward from (3.6):

$$
\begin{aligned}
H[\bar{\varphi}] &= \left(1 - x\frac{\varphi_x^0}{\varphi^0}\right)^2 + \tau\varphi^0\varphi_{xx}^0(1+\kappa\tau)^2 - \frac{1}{4}\tau^2{\varphi^0}^2{\varphi_x^0}^2(1+\kappa\tau)^4 \\
&= \left(1 - x\frac{\varphi_x^0}{\varphi^0}\right)^2 + \tau\varphi^0\varphi_{xx}^0 + O(\tau^2), \qquad (4.41)
\end{aligned}
$$

$$
\text{where } O(\tau^2) = \tau^2(2\kappa + \kappa^2\tau)\varphi^0\varphi_{xx}^0 - \frac{1}{4}\tau^2(1+\kappa\tau)^4(\varphi^0\varphi_x^0)^2. \qquad (4.42)
$$

In order to derive equation (4.36), we fix x then find a polynomial approximation of $\sigma[\bar{\varphi}](\tau)$ for $\tau > 0$. Suppose

$$
\sigma[\bar{\varphi}](\tau) = a_0 + a_1\tau + a_2\tau^2 + O(\tau^3). \qquad (4.43)
$$

So,

$$
\sigma^2[\bar{\varphi}](\tau) = a_0^2 + 2a_0a_1\tau + (a_1^2 + 2a_0a_2)\tau^2 + O(\tau^3). \qquad (4.44)
$$

Meanwhile, we Taylor expand $\sigma(x,\tau)$ and $\sigma_\tau(x,\tau)$ at $(x,0)$. Then rearrange terms so we have expressions for $\sigma(x,0)$ and $\sigma_\tau(x,0)$:

$$
\sigma(x,0) = \sigma(x,\tau) - \tau\sigma_\tau(x,0) - \frac{\tau^2}{2}\sigma_{\tau\tau}(x,0) + O(\tau^3), \qquad (4.45)
$$

$$
\sigma_\tau(x,0) = \sigma_\tau(x,\tau) - \tau\sigma_{\tau\tau}(x,0) - \frac{\tau^2}{2}\frac{\partial^3\sigma}{\partial\tau^3}(x,0) + O(\tau^3), \quad \text{and} \qquad (4.46)
$$

$$
\sigma^2(x,0) = \sigma^2(x,\tau) - 2\tau\sigma(x,\tau)\sigma_\tau(x,0) + O(\tau^2). \qquad (4.47)
$$

Collecting the above results:

$$\sigma^2[\bar{\varphi}]H[\bar{\varphi}] = (\tau\bar{\varphi}^2)_\tau,$$

$$\Rightarrow \sigma^2[\bar{\varphi}]H[\bar{\varphi}] = \varphi^0(x)^2(1 + 4\kappa\tau + 3\kappa^2\tau^2),$$

$$\Rightarrow [a_0^2 + 2a_0a_1\tau + O(\tau^2)]\left[\left(\frac{\varphi^0(x)}{\sigma(x,0)}\right)^2 + \varphi^0\varphi^0_{xx}\tau + O(\tau^2)\right]$$

$$= \varphi^0(x)^2(1 + 4\kappa\tau + 3\kappa^2\tau^2),$$

$$\Rightarrow [\underbrace{a_0^2 + 2a_0a_1\tau}_{1} + O(\tau^2)][\underbrace{\varphi^0(x)^2 + \sigma^2(x,0)\varphi^0\varphi^0_{xx}\tau}_{2} + O(\tau^2)]$$

$$= \underbrace{\varphi^0(x)^2\sigma^2(x,0)(1 + 4\kappa\tau + 3\kappa^2\tau^2)}_{3}.$$

We further expand $O(\tau^2)$ terms and get

$$[(1) + (a_1^2 + 2a_0a_2)\tau^2 + O(\tau^3)][(2) + \sigma^2(x,0)(2\kappa\varphi^0\varphi^0_{xx} - \frac{1}{4}{\varphi^0}^2(\varphi^0_{xx})^2)\tau^2 + O(\tau^3)]$$

$$= (3).$$

Plug (4.45), (4.47) to the last equality, then compare coefficients, we have

$$\tau^0 : a_0^2\varphi^0(x)^2 = \sigma^2(x,0)\varphi^0(x)^2,$$

$$\tau^1 : a_0^2\varphi^0\varphi^0_{xx}\sigma^2(x,0) + 2a_0a_1\varphi^0(x)^2 = 4\kappa\sigma^2(x,0)\varphi^0(x)^2, \quad \text{and}$$

$$\tau^2 : a_0^2\left[2\kappa\varphi^0\varphi^0_{xx} - \frac{1}{4}(\varphi^0\varphi^0_x)^2\right] + 2a_0a_1\varphi^0\varphi^0_{xx} + (a_1^2 + 2a_0a_2)\left(\frac{\varphi^0}{\sigma(x,0)}\right)^2 = 3\kappa^2(\varphi^0)^2.$$

This means

$$a_0 = \sigma(x,0), \tag{4.48}$$

$$a_1 = \frac{4\kappa\varphi^0(x)^2\sigma^2(x,0) - \sigma^4(x,0)\varphi^0\varphi^0_{xx}}{2\sigma(x,0)\varphi^0(x)^2}$$

$$= 2\kappa\sigma(x,0) - \frac{\sigma^3(x,0)\varphi^0_{xx}}{2\varphi^0(x)}, \quad \text{and} \tag{4.49}$$

$$a_2 = \frac{3}{2}\kappa^2\sigma(x,0) - \frac{\sigma^3(x,0)}{2}\left[2\kappa\frac{\varphi^0_{xx}}{\varphi^0} - \frac{1}{4}\varphi^{0\,2}_x\right]. \tag{4.50}$$

Hence,

$$\sigma[\bar{\varphi}](x,\tau) = \sigma(x,0) + \left[2\kappa\sigma(x,0) - \frac{\sigma^3(x,0)\varphi^0_{xx}}{2\varphi^0(x)}\right]\tau + a_2\tau^2 + O(\tau^3).$$

Replace $\sigma(x,0)$ by (4.45),

$$\sigma[\bar{\varphi}](x,\tau) = \sigma(x,\tau) + \left[2\kappa\sigma(x,\tau) - \sigma_\tau(x,0) - \frac{\sigma(x,\tau)\sigma^2(x,0)\varphi^0_{xx}}{2\varphi^0(x)}\right]\tau + O(\tau^2), \tag{4.51}$$

$$= \sigma(x,\tau)\left\{1 + \tau\left[2\kappa - \sigma^2(x,0)\frac{\varphi^0_{xx}}{2\varphi^0(x)} - \frac{\sigma_\tau(x,0)}{\sigma(x,\tau)}\right]\right\} + O(\tau^2) \tag{4.52}$$

Replace $\dfrac{1}{\sigma(x,\tau)}$ by $\dfrac{1}{\sigma(x,0)}\left[1 - \dfrac{\sigma_t}{\sigma}(x,0)\tau + \dfrac{1}{2}\left(-\dfrac{\sigma_{\tau\tau}}{\sigma} + 2\dfrac{\sigma^2_\tau}{\sigma^3}\right)(x,0)\tau^2\right] + O(\tau^2)$,

$$= \underbrace{\sigma(x,\tau)\left\{1 + \tau\left[2\kappa - \sigma^2(x,0)\frac{\varphi^0_{xx}}{2\varphi^0(x)} - \frac{\sigma_\tau}{\sigma}(x,0)\right]\right\}}_{4} + O(\tau^2) \tag{4.53}$$

$$= (4) + \tau^2[\underbrace{-\frac{1}{2}\sigma_{\tau\tau}(x,0) - 2\kappa\sigma_\tau(x,0) + \frac{\sigma_\tau(x,0)\sigma^2(x,0)\varphi^0_{xx}}{2\varphi^0}}_{5}$$

$$+ \underbrace{\frac{\sigma^2_\tau(x,0)}{\sigma(x,0)}}_{6} + \underbrace{a_2(\sigma(x,0),\varphi^0,\varphi^0_{xx})}_{7}] + O(\tau^3), \tag{4.54}$$

[10] where $O(\tau^2) = 5 + 6 + 7 + O(\tau^3) = O(\underline{\sigma}(R), \bar{\sigma}(R), \|\sigma\|_{W^{1,2,\infty}([-R,R]\times[0,T])})\tau^2$. Under

[10] Here (5) comes from the Taylor expansion for $\sigma(x,t)$, (6) is the Taylor expansion for

additional regularity condition (4.34), we will show all terms in (5), (6), and (7) are bounded when $x \in [-R, R]$. So, we can write

$$\sigma[\bar{\varphi}](x,t) = a_0 + a_1\tau + O(\tau^2), \text{ where } O = O(\underline{\sigma}, \bar{\sigma}, \|\sigma\|_{W^{2,0,\infty}})\tau^2,$$

where a_0 and a_1 are as in (4.48) and (4.49).

- **Some estimates and bounds on φ^0, φ^0_x , φ^0_{xx}, $D^3\varphi^0(x)$ as $x \to 0$.**

1. From (4.31)

$$\lim_{x\to 0} \varphi^0(x) = \sigma(0,0).$$

This implies $d(x) \sim \dfrac{x}{\sigma(x,0)}$ as $x \to 0$.

2. From (4.40a)

$$\varphi^0_x(x) \to \frac{\frac{x}{\sigma(x,0)} - \frac{x}{\sigma(x,0)}}{(\frac{x}{\sigma(x,0)})^2} = \frac{0}{0} \quad \text{as } x \to 0.$$

So,

$$\begin{aligned}\lim_{x\to 0}\varphi^0_x(x) &\overset{L.P.}{=} \lim_{x\to 0}\frac{d'(x) - d'(x) + \frac{x\sigma_x}{\sigma^2}(x,0)}{2d(x)\sigma^{-1}(x,0)} \\ &= \frac{\sigma_x}{2\sigma}(0,0)\cdot \lim_{x\to 0}\frac{x}{d(x)} \\ &\overset{L.P.}{=} \frac{\sigma_x}{2\sigma}(0,0)\cdot\sigma(0,0) \\ &= \frac{\sigma_x(0,0)}{2}\end{aligned}$$

Since $\sigma_x(x,\tau)$ is continuous in Ω_T, $\sigma_x(x,0)$ is bounded for all $x \in [-R, R]$.

$\sigma^{-1}(x,t)$, and (7) is a function of $(\sigma(x,0), \varphi^0, \varphi^0_{xx})$.

3. Similarly, by (4.40b)

$$\varphi^0_{xx}(x) \to \frac{x\sigma_x(x,0) + (2\sigma(x,0) - 2\sigma(x,0))}{x^2} = \frac{0}{0} \quad \text{as } x \to 0,$$

we need to apply ĹHôpital′s rule. First, let us evaluate $\lim_{x\to 0}\varphi^0_{xx}(x)$:

$$\begin{aligned}
&\lim_{x\to 0}\varphi^0_{xx}(x) \\
=&\lim_{x\to 0}\frac{x\sigma_x(x,0) + 2[\varphi^0(x) - \sigma(x,0)]}{\sigma^2(x,0)\cdot d^2(x)} \\
\overset{(1)}{=}&\frac{1}{\sigma^2(0,0)}\lim_{x\to 0}\left[\frac{x\sigma_x(x,0)}{d^2(x)} + 2\frac{x - \sigma(x,0)d(x)}{d^3(x)}\right] \\
\overset{(2)}{=}&\frac{1}{\sigma^2(0,0)} \\
&\cdot\lim_{x\to 0}\left\{\frac{[\sigma_x(x,0) + x\sigma_{xx}(x,0)]\sigma(x,0)}{2d(x)} - 2\frac{[1 - d'(x)\sigma(x,0) + d(x)\sigma_x(x,0)]\sigma(x,0)}{3d^2(x)}\right\} \\
\overset{(3)}{=}&\frac{1}{\sigma^2(0,0)} \\
&\cdot\left\{\frac{1}{2}\cdot\lim_{x\to 0}\sigma(x,0)\left[\frac{\sigma_x(x,0)}{d(x)} + \varphi^0(x)\sigma_{xx}(x,0)\right] - \frac{2}{3}\lim_{x\to 0}\frac{\sigma_x\sigma(x,0)}{d(x)}\right\} \\
\overset{(4)}{=}&\frac{1}{\sigma^2(0,0)}\cdot\lim_{x\to 0}\left[-\frac{1}{6}\frac{\sigma(x,0)\sigma_x(x,0)}{d(x)} + \frac{\sigma(x,0)\sigma_{xx}(x,0)\varphi^0(x)}{2}\right] \\
\overset{(5)}{=}&\frac{1}{\sigma^2(0,0)}\cdot\left[-\frac{\sigma(0,0)}{6}\lim_{x\to 0}\frac{\sigma_x(x,0)}{d(x)} + \frac{\sigma^2(0,0)}{2}\sigma_{xx}(0,0)\right]
\end{aligned}$$

Equality (1) holds because $\varphi^0(x) = x/d(x)$. Equation (2) is valid because $\sigma_x(x) \to 0$ as $x \to 0$. So, we can apply ĹHôpital′s rule and continue our estimate. (3) comes from $d'(x)\sigma(x,0) = \frac{1}{\sigma(x,0)}\sigma(x,0) = 1$, and $x/d(x) = \varphi^0(x)$. Simplifying and rearranging terms of (3), we get (4). Equation (5) follows $\lim_{x\to 0}\varphi^0(x) = \sigma(0,0)$.

Note that $\sigma_x(x,0)$ and $d(x)$ both approaches to zero as $x \to 0$, apply ĹHôpital′s rule one more time and we get

$$\begin{aligned}\lim_{x\to 0}\varphi^0_{xx}(x) &= -\frac{1}{6\sigma(0,0)}\cdot\lim_{x\to 0}\frac{\sigma_{xx}(x,0)}{\sigma^{-1}(x,0)}+\frac{\sigma_{xx}(0,0)}{2}\\ &= \frac{1}{3}\sigma_{xx}(0,0).\end{aligned}$$

4. By (4.40c)

$$\begin{aligned}&\lim_{x\to 0}D^3\varphi^0(x)\\ =&\lim_{x\to 0}\frac{(3\sigma_x+x\sigma_{xx})\sigma^2d^2+6\sigma^2d-6x\sigma-2x\sigma\sigma^2d^2-6x\sigma\sigma_xd}{\sigma^4(x,0)d^4(x)}\\ =&\lim_{x\to 0}\left[\frac{6\sigma}{\sigma^4}\left(\frac{\sigma d-x}{d^4}\right)+\frac{3\sigma_x+x\sigma_{xx}}{\sigma^2d^2}-\frac{2x\sigma_x^2}{\sigma^3d^2}-6\frac{x\sigma_x}{\sigma^3d^3}\right].\end{aligned}$$

Next, we calculate each of the three limits. Under additional assumption σ is smooth, σ_x, σ_{xx} approach to zero as $x\to 0$, we repeatedly applying ĹHôpital′s rule, and arrive at the following conclusions.

(a)

$$\begin{aligned}&\lim_{x\to 0}\frac{\sigma(x,0)d(x)-x}{d^4(x)}\\ =&\lim_{x\to 0}\frac{\sigma(x,0)(\sigma_x(x,0)d(x)+1-1)}{4d^3(x)}\\ =&\lim_{x\to 0}\sigma(x,0)\frac{\sigma_x^2(x,0)+\sigma\sigma_{xx}(x,0)+\sigma_x(x,0)/d(x)}{12d(x)}\\ =&\frac{\sigma(0,0)}{12}\lim_{x\to 0}\sigma(x,0)[2\sigma_x\sigma_{xx}(x,0)+\sigma_x\sigma_{xx}(x,0)+\sigma\sigma_{xxx}(x,0)]\\ +&\frac{\sigma(0,0)}{12}\lim_{x\to 0}\frac{\sigma(x,0)\sigma_{xx}(x,0)}{2d(x)}\\ =&\frac{\sigma^2(0,0)}{12}\left[\frac{7}{2}\sigma_x\sigma_{xx}(0,0)+\frac{3}{2}\sigma\sigma_{xxx}(0,0)\right].\end{aligned}$$

(b)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{3\sigma_x + x\sigma_{xx}}{d^2(x)}\\
=&\lim_{x\to 0}\frac{\sigma(x,0)(3\sigma_{xx}+\sigma_{xx}+x\frac{\partial^3}{\partial x^3}\sigma)}{2d(x)}\\
=&\frac{\sigma(0,0)}{2}\lim_{x\to 0}\frac{\sigma(4\frac{\partial^3}{\partial x^3}\sigma+\frac{\partial^3}{\partial x^3}\sigma+x\frac{\partial^4}{\partial x^4}\sigma)}{2}\\
=&\frac{\sigma^2(0,0)\cdot 5\frac{\partial^3}{\partial x^3}\sigma(0,0)}{4}.
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\lim_{x\to 0}\frac{x\sigma_x}{d^3}\\
=&\lim_{x\to 0}\frac{\sigma(\sigma_x + x\sigma_{xx})}{3d^2}\\
=&\frac{\sigma(0,0)}{3}\lim_{x\to 0}\frac{\sigma(\sigma_{xx}+\sigma_{xx}+x\frac{\partial^3}{\partial x^3}\sigma)}{2d}\\
=&\frac{\sigma^2(0,0)}{3\cdot 2}\lim_{x\to 0}2\frac{\partial^3}{\partial x^3}\sigma+\frac{\partial^3}{\partial x^3}\sigma+x\frac{\partial^4}{\partial x^4}\sigma\\
=&\frac{\sigma^2(0,0)\frac{\partial^3}{\partial x^3}\sigma(0,0)}{2}.
\end{aligned}
$$

Some estimates and bounds on φ^0, φ^0_x, φ^0_{xx} , and $D^3\varphi^0(x)$ as $x \to \pm R$.

Recall equations (4.31), (4.40a), and (4.40b), $\lim_{x\to\pm R}\varphi^0(x)$, $\lim_{x\to\pm R}\varphi^0_x(x)$, and $\lim_{x\to\pm R}\varphi^0_{xx}(x)$, they are obviously bounded.

Some estimates and bounds relating φ^0, φ^0_x, and φ^0_{xx} as $x \to \pm\infty$, and $x \to 0$.

1. $\sigma(x,\tau)d(x) = O(|x|)$ as $|x| \to \infty$, uniformly in τ. Therefore, $\varphi^0_{xx} = O(\frac{1}{|x|})$ as $x \to \pm\infty$.

2. $\frac{(\varphi^0)_x}{\varphi^0} \leq O(\frac{1}{|x|})$ as $|x| \to 0$. Therefore, $\frac{(\varphi^0)_x}{\varphi^0} \cdot x \leq O(1)$ as $|x| \to 0$.

In summary, φ^0, φ^0_x and φ^0_{xx} are all of $O(\underline{\sigma}, \bar{\sigma}, \|\sigma\|_{W^{2,0,\infty}([-R,R]\times[0,T])})$ for $|x| \leq R$. φ^0_x and φ^0_{xx} are bounded in $\mathbb{R}$. □

Next, we remove the additional regularity assumption in Claim 4.16 by an approximation procedure.

Definition 4.17. *Take $\tilde{\sigma}(x, -\tau) = \sigma(x, \tau)$ for all $\tau \leq 0$ as an extension of σ and define for any $\varepsilon > 0$, ρ^ε a standard mollifier. Define σ^ε by*

$$\sigma^\varepsilon = \rho^\varepsilon * \tilde{\sigma}.$$

Given $R >> 1$, and $0 < \eta << 1$, $\sigma^\varepsilon_R(x, \tau; \eta)$ is a smooth function in Ω_T such that

$$\sigma^\varepsilon_R(x,\tau) = \begin{cases} \sigma^\varepsilon(R+\eta,\tau) & [R+\eta,\infty)\times[0,T] \\ \sigma^\varepsilon(x,\tau) & [-R,R]\times[0,T] \\ \sigma^\varepsilon(-R-\eta,\tau) & (-\infty,-R-\eta]\times[0,T]. \end{cases} \tag{4.55}$$

In the following, ${\varphi^\varepsilon_R}^0$ is defined by equation (4.31), with σ replaced by σ^ε_R, and $\bar{\varphi}^\varepsilon_R$ is given by equation (4.32), with φ^0 replaced by ${\varphi^\varepsilon_R}^0$. We also denote $d^\varepsilon_R := \int_0^x \sigma^\varepsilon_R(s,0)ds$. The associated volatility, $\sigma[\bar{\varphi}^\varepsilon_R]$ of $\bar{\varphi}^\varepsilon_R$ is given by (3.14), with ψ been replaced by $\bar{\varphi}^\varepsilon_R$. As mentioned before, if σ satisfies Condition H1, then $(\sigma^\varepsilon_R)^2$ satisfies both Condition H2, and Condition H3. Moreover, we have estimates of the bounds appear in these conditions. [11]

[11] M_0, and κ_2 are constants for all (R, ε). $\nu(R)$, $M(R)$ depend on R, uniformly in ε, and $\kappa_3(R, \varepsilon)$ depends on both R and ε.

The following claim, gives bounds relating $\sigma[\bar{\varphi}_R^\varepsilon]$.

Claim 4.18. *Given* (R, ε),

$$L_R^\varepsilon := \frac{\partial}{\partial \tau} - \frac{1}{2}\sigma^2[\bar{\varphi}_R^\varepsilon](\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})$$

satisfies ***Condition H2****.*

Proof. Let us first estimate the components of $\sigma[\bar{\varphi}_R^\varepsilon]$. Recall

$$\sigma[\bar{\varphi}_R^\varepsilon]^2(x,\tau) = \frac{\frac{\partial}{\partial \tau}(\tau \bar{\varphi}_R^\varepsilon)^2}{H[\bar{\varphi}_R^\varepsilon]}.$$

where,

$$\begin{aligned} \frac{\partial}{\partial \tau}\tau(\bar{\varphi}_R^\varepsilon)^2 &= (\bar{\varphi}_R^\varepsilon)^2 + 2\bar{\varphi}_R^\varepsilon(\bar{\varphi}_R^\varepsilon)_\tau \tau \\ &= ({\varphi_R^\varepsilon}^0)^2(1+\kappa\tau)(1+3\kappa\tau), \end{aligned}$$

$$H[\bar{\varphi}_R^\varepsilon] = \left(1 - x\frac{({\varphi_R^\varepsilon}^0)_x}{{\varphi_R^\varepsilon}^0}\right)^2 + \tau(1+2\kappa\tau+\kappa^2\tau^2){\varphi_R^\varepsilon}^0({\varphi_R^\varepsilon}^0)_{xx} - \frac{1}{4}\tau^2(1+\kappa\tau)^4[\,{\varphi_R^\varepsilon}^0({\varphi_R^\varepsilon}^0)_x\,]^2,$$

and

$$1 - x\frac{({\varphi_R^\varepsilon}^0)_x}{{\varphi_R^\varepsilon}^0} = \frac{{\varphi_R^\varepsilon}^0}{\sigma_R^\varepsilon}.$$

We hence have

$$\begin{aligned} &\sigma[\bar{\varphi}_R^\varepsilon]^2(x,\tau) \\ =&\frac{(1+\kappa\tau)^3\,(1+3\kappa\tau)}{[\sigma_R^\varepsilon(x,0)]^{-2} + \tau(1+2\kappa\tau+\kappa^2\tau^2)\frac{({\varphi_R^\varepsilon}^0)xx}{{\varphi_R^\varepsilon}^0} - \frac{1}{4}\tau^2(1+\kappa\tau)^4[({\varphi_R^\varepsilon}^0)_x]^2} \end{aligned}$$

Note that fix R, σ_R^ε and $(\varphi_R^{\varepsilon\,0})$ are bounded above and below by positive constants in Ω. The bounds are of order $O(|R|^{p(\pm R)})$. In addition, by Condition H0, $\varphi_R^{\varepsilon\,0}/\sigma_R^\varepsilon$ is bounded below and above in $\mathbb{R}$ by positive constants, uniformly in all (R,ε). Therefore, by choosing $\kappa(R)$ big enough, and then choose $T_\delta(R)$ small enough, we have $H[\bar{\varphi}_R^\varepsilon]$ bounded below and above on $\Omega_{T_\delta(R)}$. In conclusion, we have $\sigma[\bar{\varphi}_R^\varepsilon]$ be smooth, strictly positive(away from zero), and bounded[12] above in Ω, *uniformly in* ε.

Next, we estimate the derivatives of $\sigma[\bar{\varphi}_R^\varepsilon]^2$.

$$
\begin{aligned}
&\frac{\partial}{\partial x}\,\sigma[\bar{\varphi}_R^\varepsilon]^2\\
=&\frac{(1+\kappa\tau)(1+3\kappa\tau)}{H[\bar{\varphi}_R^\varepsilon]^2}\left\{2\varphi_R^{\varepsilon\,0}\varphi_{Rx}^{\varepsilon\,0}H[\bar{\varphi}_R^\varepsilon]-\frac{\partial}{\partial x}H[\bar{\varphi}_R^\varepsilon](\varphi_R^{\varepsilon\,0})^2\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial \tau}\,\sigma[\bar{\varphi}_R^\varepsilon]^2\\
=&\frac{1}{2}\frac{(\varphi_R^{\varepsilon\,0})^2}{H[\bar{\varphi}_R^\varepsilon]^2}\\
&\cdot\left\{\left[\frac{\partial}{\partial\tau}(1+\kappa\tau)(1+3\kappa\tau)\right]H[\bar{\varphi}_R^\varepsilon]-\frac{\partial}{\partial\tau}H[\bar{\varphi}_R^\varepsilon](1+\kappa\tau)(1+3\kappa\tau)\right\}.
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{\partial}{\partial x}H[\bar{\varphi}_R^\varepsilon]\\
=&2\left(1-x\frac{\varphi_{Rx}^{\varepsilon\,0}}{\varphi_R^{\varepsilon\,0}}\right)\left(-\frac{\varphi_{Rx}^{\varepsilon\,0}}{\varphi_R^{\varepsilon\,0}}-x\frac{\varphi_{Rxx}^{\varepsilon\,0}\varphi_R^{\varepsilon\,0}-\varphi_{Rx}^{\varepsilon\,0\,2}}{\varphi_R^{\varepsilon\,0\,2}}\right)+(\tau+2\tau^2\kappa+\kappa^2\tau^3)\\
&\cdot(\varphi_{Rx}^{\varepsilon\,0}\varphi_{Rxx}^{\varepsilon\,0}+\varphi_{Rx}^{\varepsilon\,0}(D^3\varphi_R^{\varepsilon\,0}))-\frac{1}{2}\tau^2(1+\kappa\tau)^4\varphi_R^{\varepsilon\,0}\varphi_{Rx}^{\varepsilon\,0}(\varphi_{Rx}^{\varepsilon\,0\,2}+\varphi_R^{\varepsilon\,0}\varphi_{Rxx}^{\varepsilon\,0}),
\end{aligned}
$$

[12]The bounds are of order $R^{p(\pm R)}$.

and

$$\frac{\partial}{\partial \tau} H[\bar{\varphi}_R^\varepsilon]$$
$$=({\varphi_R^\varepsilon}^0 {\varphi_{Rxx}^\varepsilon}^0)(1+4\kappa\tau+3\kappa^2\tau^2) - ({\varphi_R^\varepsilon}^0 {\varphi_{Rxx}^\varepsilon}^0)^2 \left[\frac{1}{2}\tau(1+\kappa\tau)^4 + \tau^2\kappa(1+\kappa\tau)^3\right].$$

From Claim 4.16, the first factor of the first term of $\frac{\partial}{\partial x} H[\bar{\varphi}_R^\varepsilon]$ is bounded below and above by constants for all (R, ε). The second factor of the first term of $\frac{\partial}{\partial x} H[\bar{\varphi}_R^\varepsilon]$ is $O(1/|x|)$ as $|x| \to \infty$, uniformly for all (R, ε). Moreover, for given $R > 0$, on can choose $T_\delta(R, \varepsilon)$ small enough so that $\tau D^3 {\varphi_R^\varepsilon}^0$ is also bounded on $\Omega_{T_\delta(R,\varepsilon)}$, and the bounds are uniform for all ε if one chooses the appropriate $T_\delta(R, \varepsilon)$. In addition, we notice that ${\varphi_R^\varepsilon}^0 {\varphi_{Rx}^\varepsilon}^0$ is uniformly[13] bounded by constants on $\mathbb{R}$. This means $\frac{\partial}{\partial x} H[\bar{\varphi}_R^\varepsilon]$ has uniform[14] bounds on $\Omega_{T_\delta(R,\varepsilon)}$. For the same reason, we have the same conclusion for $\frac{\partial}{\partial \tau} H[\bar{\varphi}_R^\varepsilon]$. Therefore, *both* $\dfrac{\partial}{\partial x}\sigma[\bar{\varphi}_R^\varepsilon]^2$ *and* $\dfrac{\partial}{\partial \tau}\sigma[\bar{\varphi}_R^\varepsilon]^2$ *are bounded on* $\Omega_{T_\delta(R,\varepsilon)}$*, and the bounds are uniform for all* (R, ε).

To summarize, fixing (R, ε), $\sigma[\bar{\varphi}_R^\varepsilon]^2$ has the following properties:

- $\sigma[\bar{\varphi}_R^\varepsilon]^2$ is bounded below and above by positive numbers in Ω. Moreover, *the bounds in* $\Omega_{T_\delta(R)}$ *are of order* $R^{p(R)}$*, and are uniform in* ε.

- $\sigma[\bar{\varphi}_R^\varepsilon]^2$ is uniformly Lipschitz on $\Omega_{T_\delta(R,\varepsilon)}$, with *uniform* Lipschitz constants for all (R, ε).

□

Now, let us further motivate our generalized Comparison Principle by estimating $\left(\dfrac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 - 1$.

[13] for all (R, ε)
[14] for all (R, ε)

Corollary 4.19. *Given R, by choosing $\kappa(R) > 1$ big enough and $T_\delta(R)$ small enough, one can construct $\sigma[\bar{\varphi}^\varepsilon_R]$ such that*

$$\left(\frac{\sigma[\bar{\varphi}^\varepsilon_R]}{\sigma^\varepsilon}\right)^2 - 1 = 2\tau\left(2\kappa - {\sigma^\varepsilon_R}^2 \frac{({\varphi^\varepsilon_R}^0)_{xx}}{2{\varphi^\varepsilon_R}^0} - \frac{(\sigma^\varepsilon_R)_\tau}{\sigma^\varepsilon}\right) + O(\tau^2) > 0$$

for $(x,\tau) \in [-R, R] \times (0, T_\delta]$; and (4.56a)

$$\left(\frac{\sigma[\bar{\varphi}^\varepsilon_R]}{\sigma^\varepsilon}\right)^2 - 1 = \left(\frac{\sigma^\varepsilon_R}{\sigma^\varepsilon}\right)^2 \cdot 2\tau\left(2\kappa - {\sigma^\varepsilon_R}^2 \frac{({\varphi^\varepsilon_R}^0)_{xx}}{2{\varphi^\varepsilon_R}^0} - \frac{(\sigma^\varepsilon_R)_\tau}{\sigma^\varepsilon}\right) + \left[\left(\frac{\sigma^\varepsilon_R}{\sigma^\varepsilon}\right)^2 - 1\right] + O(\tau^2)$$

for $(x,\tau) \in [-R-\eta, R+\eta]^C \times (0, T_\delta]$. (4.56b)

Moreover, if one choses $T_\delta \leq \dfrac{1}{6\kappa}$, then

$$2\tau\kappa \leq 2\tau\left(2\kappa - {\sigma^\varepsilon_R}^2 \frac{({\varphi^\varepsilon_R}^0)_{xx}}{2{\varphi^\varepsilon_R}^0} - \frac{(\sigma^\varepsilon_R)_\tau}{\sigma^\varepsilon}\right) \leq 1 \text{ in } |x| \leq R, \tag{4.56c}$$

for all $\varepsilon > 0$. We have similar estimates for $\left(\dfrac{\sigma[\underline{\varphi}^\varepsilon_R]}{\sigma^\varepsilon}\right)^2 - 1$.

Proof. *Since σ satisfies Condition H1, for given R, the bounds of σ^ε_R and $(\sigma^\varepsilon_R)_x$ in Ω_T are uniform for all ε. Consequently, fix R, x, we have uniform bounds for ${\varphi^\varepsilon_R}^0$, $({\varphi^\varepsilon_R}^0)_x$ and $({\varphi^\varepsilon_R}^0)_{xx}$ (for all ε). However, for any pair of R, and x, $D^3({\varphi^\varepsilon_R}^0)$ may approach to $\pm\infty$ as $\varepsilon \to 0$.*

Claim 4.16 says

$$\begin{aligned}
&\frac{\sigma[\bar{\varphi}^\varepsilon_R]}{\sigma^\varepsilon}(x,\tau) \\
=&\frac{\sigma^\varepsilon_R}{\sigma^\varepsilon} \cdot \frac{\sigma[\bar{\varphi}^\varepsilon_R]}{\sigma^\varepsilon_R}(x,\tau) \\
=&\frac{\sigma^\varepsilon_R}{\sigma^\varepsilon} \cdot \left\{1 + \tau\left(2\kappa - {\sigma^\varepsilon_R}^2 \frac{({\varphi^\varepsilon_R}^0)_{xx}}{2{\varphi^\varepsilon_R}^0} - \frac{(\sigma^\varepsilon_R)_\tau}{\sigma^\varepsilon_R}\right)(x,0) + O(\tau^2)\right\}.
\end{aligned}$$

We now give estimates on $\dfrac{\sigma_R^\varepsilon}{\sigma^\varepsilon}$ and $2\kappa - {\sigma_R^\varepsilon}^2 \dfrac{({\varphi_R^\varepsilon}^0)_{xx}}{2{\varphi_R^\varepsilon}^0} - \dfrac{(\sigma_R^\varepsilon)_\tau}{\sigma^\varepsilon}$.

1. $\dfrac{\sigma_R^\varepsilon}{\sigma^\varepsilon} = 1$ for $|x| \le R$, and

$$\frac{1}{\kappa_1^2}\left(\frac{1+R^2}{1+x^2}\right)^{-p(x)/4} \le \frac{\sigma_R^\varepsilon}{\sigma^\varepsilon} \le \kappa_1^2\left(\frac{1+R^2}{1+x^2}\right)^{-p(x)/4}$$

for $|x| \ge R + \eta$ [15]. We have

$$\frac{\sigma_R^\varepsilon}{\sigma^\varepsilon} \le \begin{cases} \kappa_1^2(1+x^2)^{-p(x)/4} & \text{if} \quad p(x) < 0 \\ \kappa_1^2 & \text{if} \quad p(x) > 0. \end{cases} \tag{4.57}$$

2. (a) The definition of σ_R^ε shows that $(\sigma_R^\varepsilon)_x$, $(\sigma_R^\varepsilon)_{xx}$, and $(\sigma_R^\varepsilon)_\tau$ are bounded on $\sum_T^R$, and they all vanish on $(\sum_T^{R+\eta})^C$. Hence, for $|x| \ge R+1$,

$$\begin{aligned} {\varphi_R^\varepsilon}^0(x) &= \frac{x}{d_R^\varepsilon(x)}, \\ ({\varphi_R^\varepsilon}^0)_x(x) &= \frac{d_R^\varepsilon(x) - \frac{x}{\sigma_R^\varepsilon(x,0)}}{(d_R^\varepsilon(x))^2}, \\ ({\varphi_R^\varepsilon}^0)_{xx}(x) &= \frac{2[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(x,0)]}{(\sigma_R^\varepsilon(x,0)d_R^\varepsilon(x))^2}. \end{aligned}$$

[15] The constant κ_1 and the function $-1 < p(x) < 1$ are defined in Condition H0.

Similar method as in Claim 4.16 shows

$$
\begin{aligned}
\lim_{x\to\pm\infty} {\varphi_R^\varepsilon}^0(x) &= \lim_{x\to\pm\infty} \sigma_R^\varepsilon(x,0) = \sigma_R^\varepsilon(\pm R,0);\\
\lim_{x\to\pm\infty} ({\varphi_R^\varepsilon}^0)_x(x) &= \frac{1}{2}\lim_{x\to\pm\infty} (\sigma_R^\varepsilon)_x(x,0)\\
&= 0;\\
\lim_{x\to\pm\infty} ({\varphi_R^\varepsilon}^0)_{xx}(x) &= \lim_{x\to\pm\infty} \frac{x(\sigma_R^\varepsilon(x,0))_x + 2[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(x,0)]}{(\sigma_R^\varepsilon(x,0)d_R^\varepsilon(x))^2}\\
&= \lim_{x\to\pm\infty} \frac{2[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(\pm R,0)]}{(\sigma_R^\varepsilon(\pm R,0)d_R^\varepsilon(x))^2}\\
&= 0;\\
\lim_{x\to\pm\infty} x({\varphi_R^\varepsilon}^0)_x(x) &= \lim_{x\to\pm\infty} x(\sigma_R^\varepsilon)_x(x,0)\\
&= \lim_{x\to\pm\infty} x\cdot 0\\
&= 0;\\
\lim_{x\to\pm\infty} x({\varphi_R^\varepsilon}^0)_{xx}(x) &= \lim_{x\to\pm\infty} x\frac{x(\sigma_R^\varepsilon(x,0))_x + 2[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(x,0)]}{(\sigma_R^\varepsilon(x,0)d_R^\varepsilon(x))^2}\\
&= \lim_{x\to\pm\infty} \frac{x^2\cdot 0}{(\sigma_R^\varepsilon(x,0)d_R^\varepsilon(x))^2} + \lim_{x\to\pm\infty} \frac{2x[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(x,0)]}{(\sigma_R^\varepsilon(x,0)d_R^\varepsilon(x))^2}\\
&= 0 + \lim_{x\to\pm\infty} \frac{2x}{(d_R^\varepsilon(x))^2}\cdot \lim_{x\to\pm\infty} \frac{[{\varphi_R^\varepsilon}^0(x) - \sigma_R^\varepsilon(x,0)]}{(\sigma_R^\varepsilon(x,0))^2}\\
&= 0.
\end{aligned}
$$

The definitions of σ_R^ε, and d_R^ε imply that $|xd_R^\varepsilon|$ is bounded away from zero when $|x| \geq R+\eta$. Therefore,

i. the magnitude of $(\varphi_R^\varepsilon)^2\dfrac{({\sigma_R^\varepsilon}^0)_{xx}}{2{\varphi_R^\varepsilon}^0} = \dfrac{{\varphi_R^\varepsilon}^0 - \sigma_R^\varepsilon}{xd_R^\varepsilon}$ is bounded above for $|x| \geq R+\eta$.

ii. The magnitude of $\dfrac{(\sigma_R^\varepsilon)_\tau}{\sigma_R^\varepsilon}$ is bounded for $|x| \geq R + \eta$.

iii. The magnitude of $x({\varphi_R^\varepsilon}^0)_x$ and $x({\varphi_R^\varepsilon}^0)_{xx}$ are also bounded for $|x| \geq R + \eta$.

(b) The estimates in Claim 4.16 show that $|({\varphi_R^\varepsilon}^0)_{xx}|$ has uniform bounds on $\sum_T^R$; The definitions of σ_R^ε, ${\varphi_R^\varepsilon}^0$ show that the magnitudes of both $(\sigma_R^\varepsilon)^2 \dfrac{({\varphi_R^\varepsilon}^0)_{xx}}{2{\varphi_R^\varepsilon}^0}$ and $\dfrac{(\sigma_R^\varepsilon)_\tau}{\sigma_R^\varepsilon}$ are bounded above for $|x| \leq R$.

Hence, given (R, ε), there exists $\kappa(R) > 1$ such that

$$\left|(\sigma_R^\varepsilon)^2 \frac{({\varphi_R^\varepsilon}^0)_{xx}}{2{\varphi_R^\varepsilon}^0}\right| + \left|\frac{(\sigma_R^\varepsilon)_\tau}{\sigma_R^\varepsilon}\right| \leq \kappa \quad \text{on } \mathbb{R}.$$

Consequently,

$$\kappa \leq 2\kappa - {\sigma_R^\varepsilon}^2 \frac{({\varphi_R^\varepsilon}^0)_{xx}}{2{\varphi_R^\varepsilon}^0} - \frac{(\sigma_R^\varepsilon)_\tau}{\sigma^\varepsilon} \leq 3\kappa$$

for given (R, ε) and proper $\kappa(R)$ and $T_\delta(R)$.

Summing up, by choosing $\kappa(R) > 1$ big enough and $T_\delta(R)$ small enough[16], one can construct $\sigma[\bar{\varphi}_R^\varepsilon]$ such that (4.56) are satisfied.

□

When finding the asymptotics of a function, we find two sequences of functions with known asymptotics that converge to the target function from below and above. The following generalized comparison principle [5] allows us to transfer the comparison between implied volatilities, which are solutions to two different PDEs, into the comparison between "associated local volatilities", which are straightforward functions in class $\mathscr{I}(0, \xi)$ of our choice. Let us now state and prove the following critical result.

[16] for example, $T_\delta \leq 1/(6\kappa)$

Lemma 4.20. *(Comparison Principle) Assume σ satisfies Condition H1 and is bounded below in Ω. Given $R >> 1$, $\varepsilon > 0$, define σ^{ε}, $\bar{\varphi}^{\varepsilon}_R$, and $\sigma[\bar{\varphi}^{\varepsilon}_R]$ as in Corollary 4.19. We also define σ_{ε}, $\underline{\varphi}^{\varepsilon}_R$, and $\sigma[\underline{\varphi}^{\varepsilon}_R]$ in a similar way.*

From Theorem 2.21, there are $\bar{v}^{\varepsilon}_R$, $\underline{v}^{\varepsilon}_R$, and $v^{\varepsilon} \in C(\Omega) \cap W^{2,1,p}(\Omega_T)$ such that

$$
\begin{aligned}
(\bar{v}^{\varepsilon}_R)_\tau &= \frac{1}{2}(\sigma[\bar{\varphi}^{\varepsilon}_R])^2((\bar{v}^{\varepsilon}_R)_{xx} - (\bar{v}^{\varepsilon}_R)_x), \quad (\bar{v}^{\varepsilon}_R)(x,0) = (e^x - 1)_+;\\
(\underline{v}^{\varepsilon}_R)_\tau &= \frac{1}{2}(\sigma[\underline{\varphi}^{\varepsilon}_R])^2((\underline{v}^{\varepsilon}_R)_{xx} - (\underline{v}^{\varepsilon}_R)_x), \quad (\underline{v}^{\varepsilon}_R)(x,0) = (e^x - 1)_+;\\
(v^{\varepsilon})_\tau &= \frac{1}{2}(\sigma^{\varepsilon})^2(v^{\varepsilon}_{xx} - v^{\varepsilon}_x), \qquad v^{\varepsilon}(x,0) = (e^x - 1)_+.
\end{aligned}
$$

Moreover, we have the following comparison between the corresponding implied volatilities [17] *, $\bar{\varphi}^{\varepsilon}_R$, $\underline{\varphi}^{\varepsilon}_R$, and φ^{ε}_R on a bounded domain.*

i) There exist $N(R) > 1$, $\kappa(R,\varepsilon)$ sufficiently large, and $T_\delta(R,\varepsilon)$ sufficiently small such that

$$
\bar{\varphi}^{\varepsilon}_R \geq \varphi^{\varepsilon} \quad \textit{in} \sum_{T_\delta}^{R/N}. \tag{4.58}
$$

ii) Similarly, for the same $N(R) > 1$, $\kappa(R,\varepsilon)$ and $T_\delta(R,\varepsilon)$,

$$
\underline{\varphi}^{\varepsilon}_R \leq \varphi^{\varepsilon} \quad \textit{in} \sum_{T_\delta}^{R/N}, \tag{4.59}
$$

Proof. We only give the proof for inequality (4.58). Inequality (4.59) can be proved similarly.

Given (R,ε). Since $\bar{v}^{\varepsilon}_R = u(x, (\bar{\varphi}^{\varepsilon}_R)^2\tau)$, $v^{\varepsilon} = u(x, \tau(\varphi^{\varepsilon})^2)$, and $u_\tau(\cdot,\tau) > 1$ in Ω_T, instead

[17] Recall from previous sections: Given v, the solution to (1.8), the function φ such that $u(x, \tau\varphi^2) = v(x,\tau)$ is the implied volatility associated with σ^2. Here u is the solution to (3.1), and is explicitly given in (3.2).

of (4.58), it is equivalent to show the difference function

$$\Delta(x,\tau) := \bar{v}_R^{\varepsilon}(x,\tau) - v^{\varepsilon}(x,\tau) \geq 0 \quad \text{in } \Sigma_{T_\delta(R,\varepsilon)}^{R/N(R)}, \tag{4.60}$$

for some $N(R) > 0$. In fact, Δ satisfies the following equation:

$$\Delta_\tau - \frac{1}{2}(\sigma[\bar{\varphi}_R^{\varepsilon}])^2(\Delta_{xx} - \Delta_x) = \left[\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1\right] v_\tau^{\varepsilon} \quad \text{in } \Omega_{T_\delta} \tag{4.61}$$

$$\Delta(x,0) = 0 \quad \text{in} \quad \mathbb{R}. \tag{4.62}$$

We formally write the solution to equation (4.61) as the following double integral, and show positivity in $\sum_{T_\delta}^{R/N}$, where the value of $N > 1$ will be specified in the proof. The integrability of this double integral will become clear as we proceed.

$$\begin{aligned}
&\Delta(x,\tau) \\
&= \int_0^\tau ds \int_{\mathbb{R}} \Gamma_0(x,\tau;\xi,s) \left[\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1\right] v_\tau^{\varepsilon}(\xi,s)d\xi, &(4.63)\\
&= \frac{1}{2}(\sigma^{\varepsilon}(0,0))^2 \int_0^\tau ds \int_{[-R,R]} \Gamma_0 \left[\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1\right] e^{(\alpha+\beta s)\sqrt{\xi^2+1}-\alpha}\gamma^{\varepsilon}(\xi,s;0,0)d\xi \\
&+ \frac{1}{2}(\sigma^{\varepsilon}(0,0))^2 \int_0^\tau ds \int_{[-R-\eta,-R]\cup[R,R+\eta]} \Gamma_0 \left[\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1\right] e^{(\alpha+\beta s)\sqrt{\xi^2+1}-\alpha}\gamma^{\varepsilon}(\xi,s;0,0)d\xi \\
&+ \frac{1}{2}(\sigma^{\varepsilon}(0,0))^2 \int_0^\tau ds \int_{[-R-\eta,R+\eta]^C} \Gamma_0 \left[\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1\right] e^{(\alpha+\beta s)\sqrt{\xi^2+1}-\alpha}\gamma^{\varepsilon}(\xi,s;0,0)d\xi \\
&:= I_1 + I_2 + I_3. &(4.64)
\end{aligned}$$

The goal is to show $I_1 > |I_3|$ on a bounded domain of Ω_T. But first, let us review a few estimates on terms appear in the last equation.

Define the operator *for this proof*:

$$L_0 := \frac{\partial}{\partial \tau} - \frac{1}{2}(\sigma[\bar{\varphi}^{\varepsilon}_R])^2(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}).$$

And,

$$L_1 := \frac{\partial}{\partial \tau} - \frac{1}{2}(\sigma^{\varepsilon})^2(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}) - 2\frac{\sigma^{\varepsilon}_{\tau}}{\sigma^{\varepsilon}}.$$

1. Recall Claim 4.18, $\frac{1}{2}(\sigma[\bar{\varphi}^{\varepsilon}_R])^2$ is smooth and satisfies Condition H2. Based on our assumption on σ^2, the constants $\nu > 0$, $R_0 = 0$, and $\|\cdot\|_{p,q} = 0$ are uniform in (R, ε). The constants μ, M, and M_0 depend only on R. By Proposition 4.5 and Remark 4.6, the fundamental solution Γ_0 to L_0 exists and there exist positive constants $\mathscr{C}_1$, α_1, α_2, depending only on the structure of L, *i.e.*, R, such that

$$\frac{1}{\mathscr{C}_1}(\tau - s)^{-1/2} e^{-\frac{\alpha_1|x-\xi|^2}{\tau - s}} \leq \Gamma_0(x, \tau; \xi, s) \leq \mathscr{C}_1(\tau - s)^{-1/2} e^{-\frac{\alpha_2|x-\xi|^2}{\tau - s}}. \quad (4.65)$$

2. Recall Lemma 4.10, there exist positive constants $\mathscr{C}_2$, and α_3, depending on the structure of L_1 and T_δ, such that

$$v^{\varepsilon}_{\tau}(\xi, s; 0, 0) = \frac{1}{2}(\sigma^{\varepsilon}(0, 0))^2 \cdot \exp\{(\alpha + \beta s)\sqrt{\xi^2 + 1} - \alpha\} \cdot \gamma^{\varepsilon}(\xi, s; 0, 0),$$

where constants $\alpha > 0$, $\beta(\alpha) = \max\{\kappa_2, \kappa_3\}(\alpha + 2)^2$ are fixed for all (R, ε). [18] By Remark 4.12,

(a)

$$\gamma^{\varepsilon}(\xi, s; 0, 0) \leq \mathscr{C}_2 s^{-1/2} \quad \text{in } \Omega_{T_\delta}, \quad (4.66)$$

[18] κ_2 and κ_3 are defined in **Condition H3**.

and

(b)

$$\gamma^{\varepsilon}(\xi, s; 0, 0) \geq \frac{1}{\mathscr{C}_2} s^{-1/2} \exp\left\{-\alpha_3 \frac{\xi^2}{s}\right\} \quad \text{in} \quad \sum_{T_\delta}^{R}. \tag{4.67}$$

3. Recall equations (4.56), and (4.57) in Remark 4.19

$$\begin{aligned}
&\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1 \\
=&2\tau\left(2\kappa - {\sigma_R^{\varepsilon}}^2 \frac{({\varphi_R^{\varepsilon}}^0)_{xx}}{2{\varphi_R^{\varepsilon}}^0} - \frac{(\sigma_R^{\varepsilon})_\tau}{\sigma^{\varepsilon}}\right) + O(\tau^2) \\
\geq&2\tau\kappa + O(\tau^2) > 0
\end{aligned}$$

for $(x, \tau) \in [-R, R] \times (0, T_\delta]$;

and

$$\begin{aligned}
&\left(\frac{\sigma[\bar{\varphi}_R^{\varepsilon}]}{\sigma^{\varepsilon}}\right)^2 - 1 \\
=&\left(\frac{\sigma_R^{\varepsilon}}{\sigma^{\varepsilon}}\right)^2 \cdot 2\tau\left(2\kappa - {\sigma_R^{\varepsilon}}^2 \frac{({\varphi_R^{\varepsilon}}^0)_{xx}}{2{\varphi_R^{\varepsilon}}^0} - \frac{(\sigma_R^{\varepsilon})_\tau}{\sigma^{\varepsilon}}\right) + \left[\left(\frac{\sigma_R^{\varepsilon}}{\sigma^{\varepsilon}}\right)^2 - 1\right] + O(\tau^2) \\
\leq&\left(\frac{\sigma_R^{\varepsilon}}{\sigma^{\varepsilon}}\right)^2 \cdot 2\tau \cdot 3\kappa + \left[\left(\frac{\sigma_R^{\varepsilon}}{\sigma^{\varepsilon}}\right)^2 - 1\right] + O(\tau^2)
\end{aligned}$$

for $(x, \tau) \in [-R-\eta, R+\eta]^C \times (0, T_\delta]$,

where

$$\frac{\sigma_R^{\varepsilon}}{\sigma^{\varepsilon}} \leq \begin{cases} \kappa_1^2 (1+x^2)^{-p(x)/4} & \text{if} \quad p(x) < 0 \\ \kappa_1^2 & \text{if} \quad p(x) > 0. \end{cases}$$

Now, we are readily estimating I_1 and I_3.

1. Lower bound for I_1:

$$
\begin{aligned}
& I_1 \\
:=& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 \int_0^\tau ds \int_{[-R,R]} \Gamma_0 \left[\left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 - 1\right] e^{(\alpha+\beta s)\sqrt{\xi^2+1}-\alpha} \gamma^\varepsilon(\xi, s; 0,0) d\xi \\
\geq& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 \frac{1}{\mathscr{C}_1\mathscr{C}_2} e^{-\alpha} \int_{[-R,R]} (\tau - s)^{-1/2} \\
& \cdot \exp\left\{-\alpha_1 \frac{(x-\xi)^2}{\tau - s}\right\} \cdot s \cdot \exp\left\{(\alpha+\beta s)\sqrt{\xi^2+1}\right\} \cdot s^{-1/2} \exp\left\{-\alpha_3 \frac{\xi^2}{s}\right\} d\xi \\
=& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 \frac{1}{\mathscr{C}_1\mathscr{C}_2} e^{-\alpha} \int_0^\tau ds \int_{[-R,R]} (\tau - s)^{-1/2} \\
& \cdot \exp\left\{-\alpha_1 \frac{(x-\xi)^2}{\tau - s}\right\} \cdot s \cdot \exp\left\{(\alpha+\beta s)\sqrt{\xi^2+1}\right\} \cdot s^{-1/2} \exp\left\{-\alpha_3 \frac{\xi^2}{s}\right\} d\xi \\
=& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 \frac{1}{\mathscr{C}_1\mathscr{C}_2} e^{-\alpha} \int_0^\tau \left(\frac{s}{\tau - s}\right)^{1/2} ds \\
& \cdot \int_{[-R,R]} \exp\left\{-\alpha_1 \frac{(x-\xi)^2}{\tau - s} + (\alpha+\beta s)\sqrt{\xi^2+1} - \alpha_3 \frac{\xi^2}{s}\right\} d\xi.
\end{aligned}
$$

The first inequality holds because when $|\xi| \leq R$,

$$
\begin{aligned}
& \left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 (\xi, s) - 1 \\
&= 2s\left(2\kappa - {\sigma_R^\varepsilon}^2 \frac{(\varphi_R^{\varepsilon\,0})_{xx}}{2\varphi_R^{\varepsilon\,0}} - \frac{(\sigma_R^\varepsilon)_\tau}{\sigma^\varepsilon}\right)(\xi) + O(s^2) > 0 \\
&\geq 2s \quad \text{on } \sum_{T_\delta}^{R},
\end{aligned}
$$

holds for appropriate κ and T_δ, uniformly in (R, ε).

Now, fix $\tau \leq T_\delta$, we show a lower bound of

$$
\int_{[-R,R]} \exp\left\{-\alpha_1 \frac{(x-\xi)^2}{\tau - s} + (\alpha+\beta s)\sqrt{\xi^2+1} - \alpha_3 \frac{\xi^2}{s}\right\} d\xi
$$

for all $0 \le s < \tau$.

(a) The first step is to complete the following square:

$$
\begin{aligned}
& -\alpha_1 \frac{(x-\xi)^2}{\tau - s} + (\alpha + \beta s)\sqrt{\xi^2+1} - \alpha_3 \frac{\xi^2}{s} \\
= & -\left(\frac{\sqrt{\alpha_1 s + \alpha_3(\tau - s)}\xi + \frac{\alpha_1 s}{\sqrt{\alpha_1 s + \alpha_3(\tau-s)}} x}{\sqrt{(\tau - s)s}} \right)^2 - \frac{\alpha_1\alpha_3}{\alpha_1 s + \alpha_3(\tau - s)} x^2.
\end{aligned}
$$

(b) Now we have

$$
\begin{aligned}
& \int_{[-R,R]} \exp\left\{ -\alpha_1 \frac{(x-\xi)^2}{\tau - s} + (\alpha + \beta s)\sqrt{\xi^2+1} - \alpha_3 \frac{\xi^2}{s} \right\} d\xi \\
= & \sqrt{2\pi}\ \exp\left\{ -\frac{\alpha_1\alpha_3}{\alpha_1 s + \alpha_3(\tau - s)} x^2 \right\} \\
& \cdot \int_{[-R,R]} \frac{1}{\sqrt{2\pi}}\ \exp\left\{ -\frac{1}{2} \cdot \left(\frac{\sqrt{\alpha_1 s + \alpha_3(\tau - s)}\xi + \frac{\alpha_1 s}{\sqrt{\alpha_1 s + \alpha_3(\tau-s)}} x}{\sqrt{(\tau - s)s/2}} \right)^2 \right\} d\xi.
\end{aligned}
$$

Again, denote

$$
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi.
$$

Changing variables, one have the last integral equals to

$$
\begin{aligned}
& \sqrt{\frac{(\tau - s)s}{2[\alpha_1 s + \alpha_3(\tau - s)]}} \\
& \cdot [N\left(\frac{\sqrt{\alpha_1 s + \alpha_3(\tau - s)}R + \frac{\alpha_1 s}{\sqrt{\alpha_1 s + \alpha_3(\tau-s)}} x}{\sqrt{(\tau - s)s/2}} \right) \\
& - N\left(\frac{-\sqrt{\alpha_1 s + \alpha_3(\tau - s)}R + \frac{\alpha_1 s}{\sqrt{\alpha_1 s + \alpha_3(\tau-s)}} x}{\sqrt{(\tau - s)s/2}} \right)]
\end{aligned}
$$

for $|x|, |\xi| \le R$.

(c) Next, we show the interval

$$\left[\frac{\sqrt{\alpha_1 s+\alpha_3(\tau-s)}(\pm R)+\frac{\alpha_1 s}{\sqrt{\alpha_1 s+\alpha_3(\tau-s)}}x}{\sqrt{(\tau-s)s/2}}\right]$$
$$=\left[\pm\sqrt{2\frac{\alpha_1 s+\alpha_3(\tau-s)}{(\tau-s)s}}R+\frac{\sqrt{2}\alpha x}{\sqrt{\alpha_1 s+\alpha_3(\tau-s)}}\cdot\sqrt{\frac{s}{\tau-s}}\right]$$

- Covers the origin. We simply take the ratio of between half of the length of the interval and the magnitude of the center of that interval.

$$\begin{aligned}
&\sqrt{2\frac{\alpha_1 s+\alpha_3(\tau-s)}{(\tau-s)s}}R/\frac{\sqrt{2}\alpha|x|}{\sqrt{\alpha_1 s+\alpha_3(\tau-s)}}\cdot\sqrt{\frac{s}{\tau-s}}\\
=&\frac{R}{\alpha|x|}\frac{\alpha_1 s+\alpha_3(\tau-s)}{s}\\
=&\frac{R}{\alpha|x|}\left[\alpha_1+\alpha_3\left(\frac{\tau}{s}-1\right)\right]\\
>&\frac{R}{\alpha|x|}\alpha_1, \quad \text{since } 0\le s<\tau\\
>&1.
\end{aligned}$$

The last inequality holds uniformly for all (R,ε), if we choose $\alpha \geq 1$ [19] . This is because $\alpha_1 > 1$ for any$R$ sufficiently large and any $\varepsilon > 0$.

- Furthermore, the length of the above interval has a uniform lower bound for

[19]Recall the summery of Remark 4.6,

$$\alpha_1 \sim O\left(\max\left\{\frac{1}{\nu^2},\frac{M_0^2+M^2}{\nu}\right\}\right),$$

where ν, M are lower and upper bounds of $\sigma[\varphi_R^{\bar{\varepsilon}}]$, respectively, M_0 is the upper bound for $|\frac{\partial}{\partial x}\sigma[\varphi_R^{\bar{\varepsilon}}]|$.

all $0 \le s < \tau$. The minimum length occurs when

$$\frac{\partial}{\partial s} 2 \cdot \sqrt{2 \frac{\alpha_1 s + \alpha_3(\tau - s)}{(\tau - s)s}} = \sqrt{2}[(\tau - s)s]^{-3/2} \frac{\alpha_1 s^2 - \alpha_3(\tau - s)^2}{\sqrt{\alpha_1 s + \alpha_3(\tau - s)}} = 0.$$

Therefore, the minimal length of the interval is at $s = \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_1} + \sqrt{\alpha_3}} t$, and that length is $2R\sqrt{2(\sqrt{\alpha_1} + \sqrt{\alpha_3})^2 / \tau}$, which increases as R increases, or as τ decreases. That means, the length of the above interval is bounded below.

- In conclusion, there exists constant ϑ, such that $N(+) - N(-) \ge \vartheta$ for all R, whenever $\kappa(R, \varepsilon)$ big enough, and ε, $T_\delta(R, \varepsilon)$ small enough.

(d) Plugging the result to part (b), we have

$$\begin{aligned} &\int_{[-R,R]} \exp\left\{ -\alpha_1 \frac{(x - \xi)^2}{\tau - s} + (\alpha + \beta s)\sqrt{\xi^2 + 1} - \alpha_3 \frac{\xi^2}{s} \right\} d\xi \\ \ge &\sqrt{2\pi} \exp\left\{ -\frac{\alpha_1 \alpha_3}{\alpha_1 s + \alpha_3(\tau - s)} x^2 \right\} \cdot \sqrt{\frac{(\tau - s)s}{2[\alpha_1 s + \alpha_3(\tau - s)]}} \cdot \vartheta. \end{aligned}$$

Hence

$$
\begin{aligned}
&I_1 \\
\geq&\frac{1}{2}(\sigma^{\varepsilon}(0,0))^2\frac{1}{\mathscr{C}_1\mathscr{C}_2}e^{-\alpha}\int_0^{\tau}2\left(\frac{s}{\tau-s}\right)^{1/2}ds \\
&\cdot\int_{[-R,R]}\exp\left\{-\alpha_1\frac{(x-\xi)^2}{\tau-s}+(\alpha+\beta s)\sqrt{\xi^2+1}-\alpha_3\frac{\xi^2}{s}\right\}d\xi \\
\geq&\sqrt{2\pi}(\sigma^{\varepsilon}(0,0))^2\frac{1}{\mathscr{C}_1\mathscr{C}_2}e^{-\alpha}\cdot\vartheta \\
&\cdot\int_0^{\tau}\left(\frac{s}{\tau-s}\right)^{1/2}\exp\left\{-\frac{\alpha_1\alpha_3}{\alpha_1 s+\alpha_3(\tau-s)}x^2\right\}\cdot\sqrt{\frac{(\tau-s)s}{2[\alpha_1 s+\alpha_3(\tau-s)]}}ds \\
=&\sqrt{2\pi}(\sigma^{\varepsilon}(0,0))^2\frac{1}{\mathscr{C}_1\mathscr{C}_2}e^{-\alpha}\cdot\vartheta \\
&\cdot\int_0^{\tau}\exp\left\{-\frac{\alpha_1\alpha_3}{\alpha_1 s+\alpha_3(\tau-s)}x^2\right\}\cdot\frac{s}{\sqrt{2[\alpha_1 s+\alpha_3(\tau-s)]}}ds \\
\geq&\sqrt{2\pi}(\sigma^{\varepsilon}(0,0))^2\frac{1}{\mathscr{C}_1\mathscr{C}_2}e^{-\alpha}\cdot\vartheta \\
&\cdot\int_0^{\tau}\exp\left\{-\frac{\alpha_1\alpha_3}{\min\{\alpha_1,\alpha_3\}\tau}x^2\right\}\cdot\frac{s}{\sqrt{2\max\{\alpha_1,\alpha_3\}\tau}}ds \\
=&\sqrt{\frac{\pi}{2}}(\sigma^{\varepsilon}(0,0))^2\frac{1}{\mathscr{C}_1\mathscr{C}_2}e^{-\alpha}\cdot\vartheta \\
&\cdot\tau^{3/2}\exp\left\{-\frac{\alpha_1\alpha_3}{\min\{\alpha_1,\alpha_3\}\tau}x^2\right\}(\max\{\alpha_1,\alpha_3\})^{-1/2}.
\end{aligned}
$$

2. Upper bound for $|I_3|$:

$$
\begin{aligned}
&|I_3| \\
\leq& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 e^{-\alpha} \int_0^\tau ds \int_{[-R-\eta,R+\eta]^C} \Gamma_0 \, e^{(\alpha+\beta s)\sqrt{\xi^2+1}} \gamma^\varepsilon(\xi,s;0,0) \left| \left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 - 1 \right| d\xi \\
\leq& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 e^{-\alpha} \mathscr{C}_1 \mathscr{C}_2 \int_0^\tau ds \int_{[-R-\eta,R+\eta]^C} (\tau-s)^{-1/2} e^{-\alpha_2 \frac{(x-\xi)^2}{\tau-s}} \, e^{(\alpha+\beta s)\sqrt{\xi^2+1}} \, s^{-1/2} \\
&\cdot \left| \left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 - 1 \right| d\xi \\
=& \frac{1}{2}(\sigma^\varepsilon(0,0))^2 e^{-\alpha} \mathscr{C}_1 \mathscr{C}_2 \int_0^\tau ds \int_{[-R-\eta,R+\eta]^C} [(\tau-s)s]^{-1/2} \\
&\cdot \exp\left\{ -\alpha_2 \frac{(x-\xi)^2}{\tau-s} + (\alpha+\beta s)\sqrt{\xi^2+1} \right\} \cdot \left| \left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2 - 1 \right| d\xi.
\end{aligned}
$$

Next, we estimate each term of the integrand.

-
$$
\begin{aligned}
&-\alpha_2 \frac{(x-\xi)^2}{\tau-s} + (\alpha+\beta s)\sqrt{\xi^2+1} \\
=& \frac{-\alpha_2\xi^2 + 2\alpha_2 x\xi + (\alpha+\beta s)(\tau-s)\sqrt{\xi^2+1} - \alpha_2 x^2}{\tau-s} \\
\overset{(1)}{\leq}& \frac{-\alpha_2\xi^2 + 2\alpha_2 x\xi + \alpha\tau\sqrt{\xi^2+1} - \alpha_2 x^2}{\tau-s} \\
\overset{(2)}{\leq}& \frac{-\alpha_2\xi^2 + \frac{2}{N}\alpha_2\xi^2 + \alpha\tau\sqrt{\xi^2+1} - \alpha_2 x^2}{\tau-s} \\
\overset{(3)}{\leq}& \frac{-\left(\frac{3}{4} - \frac{2}{N}\right)\alpha_2\xi^2 - \alpha_2 x^2}{\tau-s}.
\end{aligned}
$$

Here, inequalities (1), (2), (3) hold for the following reasons:

(1) $(\alpha+\beta s)(\tau-s)$, $0 \leq s < \tau \leq T_\delta$ takes its maximum value when $s = \frac{\beta\tau-\alpha}{2\beta} =$

$\frac{\tau}{2} - \frac{\alpha}{2\max\{\kappa_2,\kappa_3\}(\alpha+1)^2}$, which is negative when T_δ is very small. So, if we make T_δ small enough, then $(\alpha+\beta s)(\tau-s)$ takes its maximum value at $s=0$. Therefore, $(\alpha+\beta s)(\tau-s) \le \alpha\tau$ for $0 \le s < \tau \le T_\delta << 1$.

(2) For $|x| \le R/N$, and $|\xi| \ge R+\eta > R$, $|x\xi| < \frac{R}{N}|\xi| < \frac{1}{N}\xi^2$.

(3) When $T_\delta < \frac{\alpha_2}{4\alpha}$ and R is sufficiently large,

$$\alpha\tau\sqrt{\xi^2+1} \le \alpha T_\delta\sqrt{\xi^2+1} \le \alpha\frac{\alpha_2}{4\alpha}\xi^2 = \frac{1}{4}\alpha_2\xi^2.$$

So, $-\alpha_2\xi^2 + \alpha\tau\sqrt{\xi^2+1} \le -\frac{3}{4}\alpha_2\xi^2$.

•

$$\begin{aligned}
&\left|\left(\frac{\sigma[\bar\varphi_R^\varepsilon]}{\sigma^\varepsilon}\right)^2(\xi,s)-1\right| \\
\le&\left(\frac{\sigma_R^\varepsilon}{\sigma^\varepsilon}\right)^2\cdot 2\tau\left(2\kappa - \sigma_R^{\varepsilon\,2}\frac{(\varphi_R^{\varepsilon\,0})_{xx}}{2\varphi_R^{\varepsilon\,0}} - \frac{(\sigma_R^\varepsilon)_\tau}{\sigma^\varepsilon}\right) + \left|\left(\frac{\sigma_R^\varepsilon}{\sigma^\varepsilon}\right)^2 - 1\right| + O(\tau^2) \\
\le&\kappa_1^4(1+\xi^2)^2\cdot 2\frac{1}{3\kappa}3\kappa + \kappa_1^4(1+\xi^2)^2 + 1 + 1 \\
=&5\kappa_1^4(1+\xi^2)^2 + 2, \quad \text{if one chooses } \tau \le T_\delta \le \frac{1}{6\kappa}.
\end{aligned}$$

The validation of the first inequality is discussed in Remark 4.19. The second inequality holds uniformly for all (R,ε) when we choose $\kappa(R,\varepsilon) > 1$ large enough and $T_\delta(R,\varepsilon) \le 1/\kappa(R,\varepsilon)$.

Therefore, for R big enough, one can choose T_δ so small that

$$5\kappa_1^4(1+\xi^2)^2 + 2 \le \exp\left\{\frac{\frac{1}{4}\alpha_2\xi^2}{\tau-s}\right\} \qquad \text{for all } 0 \le s < \tau \le T_\delta,\ |\xi| \le R+\eta,$$

uniformly for all $\varepsilon > 0$.

Therefore

$$\exp\left\{-\alpha_2\frac{(x-\xi)^2}{\tau-s}+(\alpha+\beta s)\sqrt{\xi^2+1}\right\}\cdot\left|\left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2-1\right|$$
$$\leq\exp\left\{\frac{-\left(\frac{1}{2}-\frac{2}{N}\right)\alpha_2\xi^2-\alpha_2 x^2}{\tau-s}\right\}$$
$$:=\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}\exp\left\{-\alpha_2\frac{x^2}{\tau-s}\right\}\qquad\text{for all } 0\leq s<\tau\leq T_\delta,\ |\xi|\leq R+\eta,$$

uniformly for all $\varepsilon > 0$,

where $\quad \tilde{\alpha}_2(N,\alpha_2)=\left(\frac{1}{2}-\frac{2}{N}\right)\alpha_2(R)$.

So

$$|I_3|$$
$$\leq\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\int_0^\tau ds\int_{[-R-\eta,R+\eta]^C}(\tau-s)^{-1/2}\ s^{-1/2}$$
$$\cdot\exp\left\{-\alpha_2\frac{(x-\xi)^2}{\tau-s}+(\alpha+\beta s)\sqrt{\xi^2+1}\right\}\cdot\left|\left(\frac{\sigma[\bar{\varphi}_R^\varepsilon]}{\sigma^\varepsilon}\right)^2-1\right|d\xi$$
$$\leq\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\int_0^\tau(\tau-s)^{-1/2}\ s^{-1/2}\exp\left\{-\alpha_2\frac{x^2}{\tau-s}\right\}ds$$
$$\cdot\int_{[-R-\eta,R+\eta]^C}\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}d\xi$$
$$\leq\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\int_0^\tau(\tau-s)^{-1/2}\ s^{-1/2}$$
$$\cdot\exp\left\{-\alpha_2\frac{x^2}{\tau-s}\right\}\cdot\frac{\tau-s}{2\tilde{\alpha}_2}\cdot 2\exp\left\{-\tilde{\alpha}_2\frac{(R+\eta)^2}{\tau-s}\right\}ds.$$

The last inequality holds because

$$\frac{d}{d\xi}\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}=\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}\cdot\left(-\frac{2\tilde{\alpha}_2}{\tau-s}\xi\right)d\xi,$$

and consequently,

$$\begin{aligned}
&\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}d\xi\\
=&\frac{\tau-s}{2\tilde{\alpha}_2\xi}\cdot\left|\frac{d}{d\xi}\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}\right|\\
\leq&\frac{\tau-s}{2\tilde{\alpha}_2}\cdot\left|\frac{d}{d\xi}\exp\left\{-\tilde{\alpha}_2\frac{\xi^2}{\tau-s}\right\}\right|\quad\text{since } |\xi|>R+\eta>>1.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&|I_3|\\
\leq&\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\tilde{\alpha}_2{}^{-1}\\
&\cdot\int_0^\tau\left(\frac{\tau-s}{s}\right)^{1/2}\cdot\exp\left\{-\alpha_2\frac{x^2}{\tau-s}\right\}\cdot\exp\left\{-\tilde{\alpha}_2\frac{(R+\eta)^2}{\tau-s}\right\}ds\\
\leq&\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\tilde{\alpha}_2{}^{-1}\\
&\cdot\int_0^\tau\left(\frac{\tau}{s}\right)^{1/2}\cdot\exp\left\{-\alpha_2\frac{x^2}{\tau}\right\}\cdot\exp\left\{-\tilde{\alpha}_2\frac{(R+\eta)^2}{\tau}\right\}ds\\
=&\frac{1}{2}(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\tilde{\alpha}_2{}^{-1}\tau\cdot2\cdot\exp\left\{-\alpha_2\frac{x^2}{\tau}\right\}\cdot\exp\left\{-\tilde{\alpha}_2\frac{(R+\eta)^2}{\tau}\right\}\\
=&(\sigma^\varepsilon(0,0))^2e^{-\alpha}\mathscr{C}_1\mathscr{C}_2\tilde{\alpha}_2{}^{-1}\tau\cdot\exp\left\{-\alpha_2\frac{x^2}{\tau}\right\}\cdot\exp\left\{-\tilde{\alpha}_2\frac{(R+\eta)^2}{\tau}\right\},\\
&\text{where } \tilde{\alpha}_2(N,\alpha_2)=\left(\frac{1}{2}-\frac{2}{N}\right)\alpha_2(R).
\end{aligned}$$

3. We shall show, $I_1 > |I_3|$ in $\sum_{T_\delta}^{R/N(R)}$, and the difference is independent of η. Recall

$$\begin{aligned} & I_1 \\ \geq & \sqrt{\frac{\pi}{2}}(\sigma^\varepsilon(0,0))^2 \frac{1}{\mathscr{C}_1\mathscr{C}_2} e^{-\alpha} \cdot \vartheta \\ & \cdot \tau^{3/2} \exp\left\{ -\frac{\alpha_1\alpha_3}{\min\{\alpha_1,\alpha_3\}\tau} x^2 \right\} (\max\{\alpha_1,\alpha_3\})^{-1/2}. \end{aligned}$$

And

$$\begin{aligned} & |I_3| \\ \leq & (\sigma^\varepsilon(0,0))^2 e^{-\alpha} \mathscr{C}_1\mathscr{C}_2 \tilde{\alpha}_2{}^{-1} \tau \cdot \exp\left\{ -\alpha_2 \frac{x^2}{\tau} \right\} \cdot \exp\left\{ -\tilde{\alpha}_2 \frac{(R+\eta)^2}{\tau} \right\}, \\ & \text{where } \tilde{\alpha}_2(N,\alpha_2) = \left(\frac{1}{2} - \frac{2}{N} \right) \alpha_2(R). \end{aligned}$$

For fixed R, ε, and $T_\delta(R,\varepsilon)$, α_1, α_2, $\mathscr{C}_1$ and $\mathscr{C}_2$ are constants. We are interested in the comparison between I_1 and $|I_3|$ as $\tau \to 0$, in which case both bounds will be dominated by the exponential factor. Therefore, it is sufficient to show

$$\begin{aligned} \alpha_2 x^2 + \tilde{\alpha}_2(R+\eta)^2 > \frac{\alpha_1\alpha_3}{\min\{\alpha_1,\alpha_3\}} x^2, \quad \textit{i.e.}, \\ \left[\left(\frac{1}{2} - \frac{2}{N} \right) \alpha_2 \right] (R+\eta)^2 > (\max\{\alpha_1,\alpha_3\} - \alpha_2)\, x^2. \end{aligned}$$

Note $x^2 \le R^2/N^2$, and $(R+\eta)^2 > R^2$, so the last inequality holds if

$$\left[\left(\frac{1}{2}-\frac{2}{N}\right)\alpha_2\right] > \left(\max\{\alpha_1,\alpha_3\}-\alpha_2\right)/N^2, \quad i.e.$$

$$\frac{1}{2}N^2 - 2N > \frac{\max\{\alpha_1,\alpha_2\}-\alpha_2}{\alpha_2},$$

$$\text{therefore, we need} \quad N > 2+\sqrt{2}\sqrt{1+\frac{\max\{\alpha_1,\alpha_3\}}{\alpha_2}}.$$

Recall Remark 4.6, and the summary in Claim 4.18,
$\alpha_1,\ \alpha_3 \sim \max\{\frac{1}{\nu^2(R)}, \frac{M^2(R)+M_0^2}{\nu(R)}\}$, and $\alpha_2 \sim \frac{\nu(R)}{M^2(R)}$, *uniformly in* ε. Here $M(R)$, and $\nu(R)$ are the upper and lower bound for $\sigma^2[\bar{\varphi}_R^\varepsilon]$ on Ω_{T_δ}, respectively. M_0 is the upper bound [20] for $\frac{\partial}{\partial x}\sigma^2[\bar{\varphi}_R^\varepsilon]$ on $\Omega_{T_\delta(R,\varepsilon)}$. Thus,

$$N \sim \left(\max\left\{\frac{1}{\nu^2(R)}, \frac{M^2(R)+M_0^2}{\nu(R)}\right\}\cdot\frac{M^2(R)}{\nu(R)}\right)^{1/2}$$

only depends on R.

□

Our next main result in this section is:

Theorem 4.21. *Suppose* $p_+ < 1/2$, *and* $p_- = 0$.

i) In the limit $\tau \to 0$, *the implied volatility* φ *is the harmonic-mean of the local volatility, namely, given* $x \in \mathbb{R}$,

$$\lim_{\tau\to 0}\frac{1}{\varphi(x,\tau)} = \int_0^1 \frac{ds}{\sigma(sx,0)}. \tag{4.68}$$

ii) Conversely, if $\tilde{\varphi} \in W^{2,1,p}_{loc}(\Omega)$ *(for any* $p>1$*), satisfies (3.7) and (4.68), then* $\tilde{\varphi} \equiv \varphi$.

[20] uniformly for all (R,ε)

Proof. i) Given $(R,\varepsilon) > 0$. Applying the comparison Lemma 4.20, we get

$$\bar{\varphi}_R^{\varepsilon} \geq \varphi^{\varepsilon} \quad \text{in} \quad \sum_{T_\delta}^{R/N(R)} \tag{4.69}$$

for some $N(R) > 1$, and $0 < T_\delta(R,\varepsilon) << \min\{T,1\}$.

Similarly, from Lemma 4.20

$$\underline{\varphi}_{\varepsilon,R} \leq \varphi^{\varepsilon} \quad \text{in} \quad \sum_{T_\delta}^{R/N}, \tag{4.70}$$

without lost of generality, for the same $N(R) > 1$, and $0 < T_\delta(R,\varepsilon) << \min\{T,1\}$.

Therefore, for all (R,ε), there exists $(\kappa, T_\delta)(R,\varepsilon)$, and $N(R)$ such that

$$\begin{aligned} & x\left(\int_0^x \frac{ds}{\sigma_R^{\varepsilon}(s,0)}\right)^{-1}(1-\kappa\tau) \\ \leq & \varphi^{\varepsilon}(x,\tau) \\ \leq & x\left(\int_0^x \frac{ds}{\sigma_R^{\varepsilon}(s,0)}\right)^{-1}(1+\kappa\tau) \end{aligned} \tag{4.71}$$

for all $(x,\tau) \in \sum_{T_\delta(R,\varepsilon)}^{R/N(R)}$. This yields

$$\begin{aligned} & x\left(\int_0^x \frac{ds}{\sigma_R^{\varepsilon}(s,0)}\right)^{-1} \\ \leq & \liminf_{\tau\to 0} \varphi^{\varepsilon}(x,\tau) \\ \leq & \limsup_{\tau\to 0} \varphi^{\varepsilon}(x,\tau) \\ \leq & x\left(\int_0^x \frac{ds}{\sigma_R^{\varepsilon}(s,0)}\right)^{-1} \end{aligned} \tag{4.72}$$

for all $\varepsilon > 0$.

Now, let $\varepsilon \to 0$, then $\sigma_R^\varepsilon(x,0) \to \sigma_R(x,0)$, which equals to σ in $\sum_{T_\delta(R,\varepsilon)}^{R/N(R)}$. On the other hand, by theorem 2.21, $\varphi^\varepsilon \to \varphi$ as $\varepsilon \to 0$. Equation (4.72) hence implies

$$\lim_{\tau\to 0}\varphi(x,\tau) = x\left(\int_0^x \frac{ds}{\sigma_R(s,0)}\right)^{-1} = x\left(\int_0^x \frac{ds}{\sigma(s,0)}\right)^{-1} \quad \text{in} \sum_{T_\delta(R,\varepsilon)}^{R/N(R)}. \tag{4.73}$$

To show the above equality holds on $\Omega_{T_\delta(R,\varepsilon)}$*, it is sufficient to show* $N(R) \sim \dfrac{M^{3/2}(R)}{\nu(R)} = o(R)$ *as* $R \to \infty$, uniformly in ε. *Recall* $p_+ = \max_{x\in\mathbb{R}}\{p(x),0\}$*, and* $p_- = \max_{x\in\mathbb{R}}\{-p(x),0\}$*, then* $\alpha_1 \sim O(R^{2p_+ + p_-})$*, and* $\alpha_2 \sim O(R^{-p_- - 2p_+})$, uniformly in ε[21]. *Therefore,*

$$\sqrt{\frac{\max\{\alpha_1,\alpha_3\}}{\alpha_2}} = O(R^{2p_+ + p_-}) = o(R) \text{ if } 2p_+ + p_- < 1.$$

Since R is an arbitrary large number, $N(R) = o(R)$ as $R \to \infty$ for $p_+ < 1/2$, and $p_- = 0$. Equation (4.73) gives the point-wise convergence of φ as $\tau \to 0$.

ii) We now show the uniqueness part of Theorem 4.21.

Suppose $\tilde\varphi \in W^{2,1,p}_{Loc}(\Omega_T)$, also satisfies (3.7) and (4.68). Let

$$\Delta(x,\tau) = u(x,\tau\varphi^2(x,\tau)) - u(x,\tau\tilde\varphi^2(x,\tau)).$$

From part i),

$$\Delta_\tau = \frac{1}{2}\sigma^2(\Delta_{xx} - \Delta_x) \quad \text{in } \Omega_T.$$

[21] for $\varepsilon sufficientlysmall$

Furthermore, one can extend $\Delta(x,\tau)$ to a continuous function in Ω, which gives

$$\Delta(x,0) = 0 \quad \forall x \in \mathbb{R}.$$

To see how, we take any $x \in \mathbb{R}$, then compute the limit of Δ as τ goes to zero:

$$\begin{aligned}
\lim_{\tau\to 0}\Delta(x,\tau) &= \lim_{\tau\to 0} u(x,\tau\varphi^2(x,\tau)) - \lim_{\tau\to 0} u(x,\tau\tilde{\varphi}^2(x,\tau)) \\
&\overset{u\in C(\Omega)}{=} u(x,\lim_{\tau\to 0}\tau\varphi^2(x,\tau)) - u(x,\lim_{\tau\to 0}\tau\tilde{\varphi}^2(x,\tau)) \\
&\overset{(1)}{=} u(x,0) - u(x,0) \\
&= 0.
\end{aligned}$$

Equality (1) holds since $0 < \lim_{\tau\to 0}\frac{1}{\varphi(x,\tau)} = \lim_{\tau\to 0}\frac{1}{\tilde{\varphi}(x,\tau)} = \int_0^1 \frac{ds}{\sigma(sx,0)} < \infty$. Notice that $|\Delta(x,\tau)| \leq |u(x,\tau\varphi^2(x,\tau))| + |u(x,\tau\tilde{\varphi}^2(x,\tau))| < 2e^x$, one may apply the generalized Maximum Principle, Theorem 2.11 and conclude $\Delta \geq 0$ and $\Delta \leq 0$ in Ω, *i.e.*, $\tilde{\varphi} \equiv \varphi$.

□

Chapter 5

The Asymptotic of φ as $x \to \pm\infty$

Throughout this section, assume σ satisfies Condition H0. Furthermore, we assume $\lim_{x\to+\infty} \sigma(x,\tau) = \sigma_+(\tau)$ (respectively $x \to -\infty$, $\sigma_-(\tau)$), locally uniformly in τ , with $\sigma_\pm$ continuous. The main theorem in this section says:

Theorem 5.1. *I. If $\sigma_{+/-} = \infty$ and $p_- \in [0, 1/2)$, then $\lim_{x\to\pm\infty} \varphi(x,\tau) = \infty$, uniformly in $t \in (0,T]$.*

II. If $\sigma_{+/-} = 0$ and $p_+ = 0$, then $\lim_{x\to\pm\infty} \varphi(x,\tau) = 0$, uniformly in $t \in (0,T]$.

III. If $p_+ = p_- = 0$, then

$$\lim_{x\to\pm\infty} \varphi(x,\tau) = \left(\frac{1}{\tau}\int_0^\tau \sigma_\pm^2(s)ds\right)^2 . \tag{5.1}$$

Remark 5.2. *The cases where $\sigma_\pm$ is positive and finite are proved in [5]. We provide the prove for the case involving $\sigma_{+/-} = \infty$, or $\sigma_{+/-} = 0$.*

Recall the following "cut-off volatilities" functions, which will be used in the proof of

Theorem 5.1. For each $m, n \in \mathbb{N}$, define

$$\sigma_n^m(x,\tau) = \begin{cases} m & \sigma(x,\tau) \geq m \\ \sigma(x,\tau) & 1/n < \sigma(x,\tau) < m \\ 1/n & \sigma(x,\tau) \leq 1/n. \end{cases}$$

Similarly, for each $m, n \in \mathbb{N}$, we define $\sigma^m(x,\tau)$ and $\sigma_n(x,\tau)$ as the cut-off version of σ from the above and below, respectively. By Theorem 2.21, σ, σ^m, σ_n, and σ_n^m each has a corresponding implied volatility: φ, φ^m, φ_n, and φ_n^m respectively.

The **proof of Theorem 5.1** takes the following steps:

I. The asymptotic of φ if $\sigma_{+/-} = \infty$

The case where $\sigma_+ = \infty$ is equivalent to: $\liminf_{x\to+\infty} \varphi(x,\tau) \geq m$ uniformly in τ, for all $m \in \mathbb{N}$. To this end, we need the following auxiliary function.

Lemma 5.3. *Given $p \in (0, 1/2)$, define: $M = \dfrac{2}{(1-p)^2}$, $z_0 = -\left(\dfrac{13p+3}{3+3p}\right)^{1/p}$, and $z_1 = \left(1 - \dfrac{4}{1+p}\right) z_0$. Take any $\eta \in (0, 0.1)$, and $\kappa \geq 1$.*[1] *For $0 < \varepsilon^p < \min\left\{\dfrac{2}{\sqrt{2\pi p}}, 1\right\}$, let $Y = \dfrac{p}{2}\dfrac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p} < 1$. $A > z_1 > 2$,*[2] *then there exists $\underline{\psi} \in C^2(\mathbb{R})$ satisfying the following properties:*

(i) $\underline{\psi} \in W^{2,0,\infty}(\mathbb{R})$. *In addition,* $\|\underline{\psi}\|_{W^{2,0,\infty}(\mathbb{R})} = |\underline{\psi}|_\infty + |\underline{\psi}_{xx}|_\infty$ *is independent of* A, *and* ε

(ii) $\underline{\psi}(z) \leq \frac{1}{1+\eta} = \lim_{z\to+\infty} \underline{\psi}(z) \quad \forall z \in \mathbb{R}$

[1] Later, we will use the inequalities $\dfrac{1+\eta}{\sqrt{M}} < \dfrac{1+0.1}{2} < 1$ and $\kappa > \dfrac{1+\eta}{\sqrt{M}}$.

[2] for $p \in (0, 1/2)$, $z_1(p)$ has the smallest value when $p = 9/31$, and this value is bigger than 2.848125

(iii) $\underline{\psi}(z) \leq \min\left\{\frac{\varepsilon^p}{\sqrt{M}\kappa}, \frac{\varepsilon^p}{\sqrt{M}\kappa|z|^p}\right\} \quad \forall z \in (-\infty, 0)$, *and*

$$\underline{\psi}(z) \leq \frac{\varepsilon^p}{\sqrt{M}\kappa} \quad \forall z \in [0, z_1 + A)$$

(iv) $\left|\frac{z\underline{\psi}'(z)}{\underline{\psi}(z)}\right| \leq p \quad \forall z \in \mathbb{R}$

(v) $\frac{z\underline{\psi}'(z)}{\underline{\psi}(z)} \to 0,\ \underline{\psi}'(z) \to 0,\ \underline{\psi}''(z) \to 0$ *as* $z \to +\infty$.

Proof. We prove this lemma in the Appendix. □

We also make use of the following Lemma when prove item II, the case $\sigma_- = \infty$ in Theorem 5.1.

Lemma 5.4. *Given* $p \in (0, 1/2)$*, define:* $M = \frac{2}{(1-p)^2}$, $z_0 = -\left(\frac{13p+3}{3+3p}\right)^{1/p}$, *and* $z_1 = \left(1 - \frac{4}{1+p}\right) z_0$. *Now take any* $\eta \in (0, 0.1)$*, and* $\kappa \geq 1$. *For* $0 < \varepsilon^p < \min\left\{\frac{2}{\sqrt{2\pi}p}, 1\right\}$, *let* $Y = \frac{p}{2}\frac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p} < 1$.

Take any $A > z_1 > 1$*, then there exists* $\underline{\tilde{\psi}} \in C^2(\mathbb{R})$ *satisfying the following properties:*

(i) $\underline{\tilde{\psi}} \in W^{2,0,\infty}(\mathbb{R})$, *with* $\|\underline{\tilde{\psi}}\|_{W^{2,0,\infty}}$, *independent of* A, *and* ε

(ii) $\underline{\tilde{\psi}}(z) \leq \frac{1}{1+\eta} = \lim_{z\to-\infty} \underline{\tilde{\psi}}(z) \quad \forall z \in \mathbb{R}$

(iii) $\underline{\tilde{\psi}}(z) \leq \min\left\{\frac{\varepsilon^p}{\sqrt{M}\kappa}, \frac{\varepsilon^p}{\sqrt{M}\kappa|z|^p}\right\} \quad \forall z \in (0, \infty)$, *and*

$$\underline{\tilde{\psi}}(z) \leq \frac{\varepsilon^p}{\sqrt{M}\kappa} \quad \forall z \in (-z_1 - A, 0]$$

(iv) $\left|\frac{z\underline{\tilde{\psi}}'(z)}{\underline{\tilde{\psi}}(z)}\right| \leq p \quad \forall z \in \mathbb{R}$

(v) $\frac{z\underline{\tilde{\psi}}'(z)}{\underline{\tilde{\psi}}(z)} \to 0,\ \underline{\tilde{\psi}}'(z) \to 0,\ \underline{\tilde{\psi}}''(z) \to 0$ *as* $z \to -\infty$.

Proof. If $\underline{\psi}(x)$ satisfies Lemma 5.3, then $\underline{\tilde{\psi}}(x) = \underline{\psi}(-x)$ satisfies Lemma 5.4. □

Proof for item I, $\sigma_+ = \infty$**.**

step 1: Comparison between local volatilities.

We fix an $m \in \mathbb{N}$, and show that $\lim_{x\to+\infty} \varphi(x,\tau) \geq m$, uniformly in τ.

Note $\sigma^m(x,\tau) \to m$ as $x \to +\infty$, uniformly in $\tau \in [0,T]$. Hence, given $\eta \in (0, 0.1)$, there exists $\tilde{A}$ such that

$$\frac{m}{\sigma^m(x,\tau)} \leq \sqrt{1+\eta} \quad \forall (x,\tau) \in [\tilde{A}, +\infty) \times [0,T]. \tag{5.2}$$

We denote the *decay* rate of σ as $p = -\min\{0, \lim_{x\to-\infty} p(x)/2\}$, where $p(x)$ is defined in Condition H0. We now denote $M := 2/(1-p)^2$. Also, by Condition H0,

$$\min\left\{\frac{1}{\kappa_1|x|^p}, \frac{1}{\kappa_1}\right\} \leq \frac{\sigma^m}{m} \leq \kappa_1 \qquad \text{for } (x,\tau) \in (-\infty,0) \times [0,T],\ m,\, n \geq 1, \tag{5.3a}$$

$$\frac{1}{\kappa_1} \leq \frac{\sigma^m}{m} \leq \kappa_1 \qquad \text{for } (x,\tau) \in [0,+\infty) \times [0,T],\ m,\, n \geq 1. \tag{5.3b}$$

Now, we set

$$\underline{\varphi}^m(x,\tau) = m\underline{\psi}(\varepsilon x) \tag{5.4}$$

where $\underline{\psi}$ is given in Lemma 5.3, the values of η, $\kappa = \kappa_1$ being defined as above, and A, ε to be found. A simple computation yields

$$H[\underline{\varphi}^m](x,\tau) = \left(1 - \varepsilon x \frac{\underline{\psi}'}{\underline{\psi}}(\varepsilon x)\right)^2 + \tau m^2 \varepsilon^2 \underline{\psi}\,\underline{\psi}''(\varepsilon x) - \frac{1}{4}\varepsilon^2 \tau^2 m^4 \underline{\psi}^2 \underline{\psi}'^2(\varepsilon x). \tag{5.5}$$

Clearly, by (i) and (iv) in Lemma 5.3, we can choose

$0 < \varepsilon = \varepsilon(\|\underline{\psi}\|_{W^{2,\infty}}, T, m, p) < 1$, independent of A such that

$$0 < \frac{(1-p)^2}{2} := \frac{1}{M} \leq H[\underline{\varphi}^m](x,\tau) \leq \frac{4}{M} < \infty \qquad \text{on } \Omega. \tag{5.6}$$

By (v) in Lemma 5.3, there is $B > 0$, for which $\varepsilon x \geq B$ implies

$$H[\underline{\varphi}^m] \geq \frac{1}{1+\eta} \quad \text{for } \tau \in [0,T]. \tag{5.7}$$

Setting $A = \max\{B, \varepsilon\tilde{A}, 2\}$, we see that (5.6) holds for all $z = \varepsilon x \in \mathbb{R}$, $\tau \in (0,T)$; (5.7) holds for $z = \varepsilon x \geq A$, $\tau \in [0,T]$.

Next, we compute the local volatility (3.14) associated with $\underline{\varphi}$, that is

$$\sigma[\underline{\varphi}^m]^2(x,\tau) = m^2 \frac{\underline{\psi}^2(\varepsilon x)}{H[\underline{\varphi}^m](x,\tau)}. \tag{5.8}$$

We now estimate $\sigma[\underline{\varphi}^m]^2(x,\tau)$ in the following cases:

- $z = \varepsilon x \in (-\infty, 0)$. So $x \in (-\infty, 0)$:

$$\begin{aligned}
&m^2 \frac{\underline{\psi}^2(\varepsilon x)}{H[\underline{\varphi}^m](x,\tau)} \\
\overset{(1)}{\leq}\, & m^2 \cdot \left(\min\left\{ \frac{\varepsilon^p}{\sqrt{M}\kappa_1 |\varepsilon x|^p}, \frac{\varepsilon^p}{\sqrt{M}\kappa_1} \right\} \right)^2 \cdot M \\
\overset{(2)}{\leq}\, & \left(\sigma^m / \min\left\{ \frac{1}{\kappa_1 |x|^p}, \frac{1}{\kappa_1} \right\} \right)^2 \cdot \left(\min\left\{ \frac{\varepsilon^p}{\sqrt{M}\kappa_1 |\varepsilon x|^p}, \frac{\varepsilon^p}{\sqrt{M}\kappa_1} \right\} \right)^2 \cdot M \\
=\, & (\sigma^m)^2(x,\tau) \cdot \frac{1}{\left(\frac{1}{\kappa_1}\right)^2 \left(\min\left\{ \frac{1}{|x|^p}, 1 \right\} \right)^2} \cdot \frac{1}{M\kappa_1^2} \left(\min\left\{ \frac{1}{|x|^p}, \varepsilon^p \right\} \right)^2 \cdot M \\
\overset{(3)}{\leq}\, & (\sigma^m)^2(x,\tau).
\end{aligned}$$

Inequalities (1) holds by (*iii*) in Lemma 5.3, and (5.6); (2) is from (5.3a); (3) holds due to $0 < \varepsilon^p < 1$.

- $z = \varepsilon x \in [0, A + z_1)$. So $x \geq 0$:

$$
\begin{aligned}
& m^2 \frac{\underline{\psi}^2(\varepsilon x)}{H[\underline{\varphi}^m](x,\tau)} \\
\overset{(1)}{\leq} & m^2 \underline{\psi}^2(\varepsilon x) \cdot M \\
\overset{(2)}{\leq} & m^2 \left(\frac{\varepsilon^p}{\sqrt{M}\kappa_1} \right)^2 \cdot M \\
\overset{(3)}{\leq} & (\sigma^m \kappa_1)^2 \, \frac{\varepsilon^{2p}}{M\kappa_1^2} \cdot M \\
= & (\sigma^m)^2 \varepsilon^{2p} \\
\overset{(4)}{\leq} & (\sigma^m)^2.
\end{aligned}
$$

Inequalities (1) golds by (5.6); (2) holds by (*iii*) in Lemma 5.3, and (5.6) ; (3) holds by (5.3b); (4) holds since $0 < \varepsilon^p < 1$.

- $z = \varepsilon x \in [A + z_1, \infty)$. So $x \geq \max\{B/\varepsilon, \tilde{A}\}$:

$$
\begin{aligned}
& m^2(\tau) \frac{\underline{\psi}^2(\varepsilon x)}{H[\underline{\varphi}^m](x,\tau)} \\
\overset{(1)}{\leq} & m^2(\tau) \left(\frac{1}{1+\eta} \right)^2 (1+\eta) \\
= & m^2(\tau) \frac{1}{1+\eta} \\
\overset{(2)}{\leq} & \frac{(\sigma^m)^2}{\kappa_1} \, \frac{1}{1+\eta} \\
\overset{(3)}{\leq} & (\sigma^m)^2(x,\tau).
\end{aligned}
$$

Inequalities (1) holds by (*ii*) in Lemma 5.3, and (5.7); (2) holds by (5.2); (3) holds because both κ and $1+\eta$ are greater than one.

In summary,

$$\sigma[\underline{\varphi}^m](x,\tau) \leq \sigma^m(x,\tau) \leq \sigma(x,\tau) \quad \text{in } \Omega_T \quad \text{for all } m \in \mathbb{N}. \tag{5.9}$$

Step 2: Comparison between implied volatilities.

Let $\underline{v}^m(x,\tau) = u(x,(\underline{\varphi}^m)^2\tau)$, where u is the solution to (3.1). One easily verifies, $\underline{v}^m$ satisfied equation (1.8) with σ^2 replaced by $\sigma^2[\underline{\varphi}^m]$. Moreover, $0 < \underline{v}^m(x,\tau) < e^x$ on Ω_T. On the other hand, Theorem 2.21 says $v = u(x,\tau\varphi^2)$ is the unique solution to equation (1.8), which has no more than exponential growth. Let $\Delta = \underline{v}^m - v$, then Δ satisfies

$$\Delta_\tau - \frac{1}{2}\sigma[\underline{\varphi}^m]^2(x,\tau)(\Delta_{xx} - \Delta_x) = \left[\left(\frac{\sigma[\underline{\varphi}^m]}{\sigma}\right)^2 - 1\right] v_\tau \ \text{ in } \Omega_T \tag{5.10a}$$

$$\Delta(x,0) = 0 \quad \text{in } \mathbb{R}. \tag{5.10b}$$

Since

1. $\sigma[\underline{\varphi}^m]^2(x,\tau)$ is continuous and non-negative on Ω;
2. $\sigma^2[\underline{\varphi}^m](x,\tau) \leq m^2 \cdot \frac{M}{4} \cdot \frac{1}{1+\eta}$ in Ω.[3]
3. $\left[\left(\frac{\sigma[\underline{\varphi}^m]}{\sigma}\right)^2 - 1\right] v_\tau < 0$ in Ω .

[3]Because
(a) equation (5.8),

$$\sigma[\underline{\varphi}^m]^2(x,\tau) = m^2 \frac{\underline{\psi}^2(\varepsilon x)}{H[\underline{\varphi}^m](x,\tau)},$$

We can apply the Maximum Principle ([24] or Theorem 2.11) to Δ, and conclude

$$\Delta = \underline{v}^m - v \leq 0 \quad \text{ in } \Omega_T.$$

Furthermore, note that $\underline{v}^m = u(x, \tau(\underline{\varphi}^m)^2)$, $v = u(x, \tau\varphi^2)$, and $u_\tau(x, \cdot) > 0$ in Ω_T, we have

$$\underline{\varphi}^m \leq \varphi \quad \text{in } \Omega_T.$$

This implies

$$\frac{1}{1+\eta} m = \lim_{x\to+\infty} \underline{\varphi}^m(x,\tau) \leq \liminf_{x\to+\infty} \varphi(x,\tau).$$

Sending $\eta \to 0$, we get

$$\liminf_{x\to+\infty} \varphi(x,\tau) \geq m \quad \text{for all } m \in \mathbb{N}. \tag{5.11}$$

The proof for the case $\sigma_- = \infty$ follows the same argument, using Lemma 5.4 and auxiliary function $\underline{\tilde{\psi}}(z) = \underline{\psi}(-z)$.

□

II. The asymptotic of φ if $\sigma_+ = 0$, $\sigma_- < \infty$, or vise-versa.

(b) equation (5.6)

$$0 < \frac{(1-p)^2}{2} := \frac{1}{M} \leq H[\underline{\varphi}^m](x,\tau) \leq \frac{4}{M} < \infty \qquad \text{in } \Omega,$$

and

(c) item (ii) in Lemma 5.3, $\underline{\psi}(z) \leq \frac{1}{1+\eta} = \lim_{z\to+\infty} \underline{\psi}(z) \quad \forall z \in \mathbb{R}$.

Without loss of generality, we give the proof for the case where $\sigma_+ = 0$. It is sufficient to show $\liminf_{x\to+\infty} \varphi(x,\tau) \le 1/n$ uniformly in τ, for all $n \in \mathbb{N}$. To this end, we need the following auxiliary function[5]. The other case can be proved using the same symmetric argument as we did for item I.

Lemma 5.5. *Given $A > 1$, $\eta \in (0, 0.1)$, and $\kappa > 1/(1-\eta)$, there exists $\overline{\psi} \in C^2(\mathbb{R})$ satisfying the following properties:*

(i) $\overline{\psi} \in W^{2,0,\infty}(\mathbb{R})$ *with* $\|\overline{\psi}\|_{W^{2,0,\infty}}$ *independent of A.*

(ii) $\overline{\psi}(z) \ge \frac{1}{1-\eta} = \lim_{z\to+\infty} \overline{\psi}(z) \quad \forall z \in \mathbb{R}$,

(iii) $\overline{\psi}(z) \ge 2\kappa \quad \forall z \in (-\infty, A)$.

(iv) $\left|\dfrac{z\overline{\psi}'(z)}{\overline{\psi}(z)}\right| \le 1/2 \quad \forall z \in \mathbb{R}$,

(v) $\dfrac{z\overline{\psi}'(z)}{\overline{\psi}(z)} \to 0,\ \overline{\psi}'(z) \to 0,\ \overline{\psi}''(z) \to 0$ *as* $z \to +\infty$.

Proof for item II:

We fix an $n \in \mathbb{N}$, and show that $\lim_{x\to+\infty} \varphi(x,\tau) \le 1/n$, uniformly in τ. By assumption in Condition H0, $\sigma_n(x,\tau) \to 1/n$ as $x \to +\infty$, uniformly in $\tau \in [0,T]$. Therefore, given $\eta \in (0, 0.1)$, there exists $\tilde{A}$ such that

$$\frac{1}{n}\sigma_n(x,\tau) \ge \sqrt{1-\eta} \quad \forall (x,\tau) \in [\tilde{A}, +\infty) \times [0,T]. \tag{5.12}$$

In addition, from Condition H0,

$$\frac{1}{\kappa_1} \le \frac{1/n}{\sigma_n} \le \kappa_1 \qquad \text{for } (x,\tau) \in \Omega_T. \tag{5.13}$$

Now, we set

$$\overline{\varphi}_n(x,\tau) = \frac{1}{n}\overline{\psi}(\varepsilon x) \tag{5.14}$$

where $\overline{\psi}$ is given in Lemma 5.5, the values of η, $\kappa = \kappa_1$ being defined as above, and A, ε to be found. A simple computation yields

$$H[\overline{\varphi}_n](x,\tau) = \left(1 - \varepsilon x \frac{\overline{\psi}'}{\overline{\psi}}(\varepsilon x)\right)^2 + \tau \frac{1}{n^2}\varepsilon^2 \overline{\psi}\,\overline{\psi}''(\varepsilon x) - \frac{1}{4}\varepsilon^2\tau^2\frac{1}{n^4}\overline{\psi}^2\overline{\psi}'^2(\varepsilon x) \tag{5.15}$$

Clearly, by (i) and (iv) in Lemma 5.5, we can choose

$0 < \varepsilon = \varepsilon(\|\overline{\psi}\|_{W^{2,\infty}}, T, 1/n) < 1$, independent of A such that

$$\frac{1}{4} \le H[\overline{\varphi}_n](x,\tau) \le 2 < \infty \qquad \text{in } \Omega. \tag{5.16}$$

By (v) in Lemma 5.5, there is $B > 0$, for which $\varepsilon x \ge B$ implies

$$H[\overline{\varphi}_n] \le \frac{1}{1-\eta} \quad \text{for } \tau \in [0,T]. \tag{5.17}$$

Setting $A = \max\{B, \varepsilon\tilde{A}\}$, we see that (5.16) holds for all $z = \varepsilon x \in \mathbb{R}$, $\tau \in (0,T)$, and (5.17) holds for $z = \varepsilon x \ge A$, $\tau \in [0,T]$.

Next, we compute the local volatility (3.14) associated with $\overline{\varphi}$, *i.e.*,

$$\sigma[\overline{\varphi}_n]^2(x,\tau) = \frac{1}{n^2}\frac{\overline{\psi}^2(\varepsilon x)}{H[\overline{\psi}_n](x,\tau)}. \tag{5.18}$$

Similar, but much simpler than the estimates for item I, we have

$$\sigma[\overline{\varphi}_n](x,\tau) \leq \sigma_n(x,\tau) \leq \sigma(x,\tau) \quad \text{in } \Omega_T \quad \text{for all } n > 0. \tag{5.19}$$

Step 2: Comparison between implied volatilities.

Similar as item I, we have

$$\overline{\varphi}_n \geq \varphi \quad \text{in } \Omega_T, \quad for all n \in \mathbb{N}.$$

This implies

$$\frac{1}{1-\eta}\frac{1}{n} = \lim_{x\to+\infty} \overline{\varphi}_n(x,\tau) \geq \limsup_{x\to+\infty} \varphi(x,\tau).$$

Sending $\eta \to 0$,

$$\limsup_{x\to+\infty} \varphi(x,\tau) \leq \frac{1}{n} \quad \text{for all } n. \tag{5.20}$$

Proof for item III: See [5].

Chapter 6

Numerical Implementation

6.1 The Finite Difference Method

Recall, for any $\psi \in W^{2,1,p}_{Loc}(\Omega_T)$, denote by H the quasilinear operator

$$H[\psi] \equiv H(x, \tau, \psi, D\psi, D^2\psi) = (1 - x\frac{\psi_x}{\psi})^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2.$$

The equation which the implied volatility φ satisfies in Ω is

$$(\tau\varphi^2)_\tau - \sigma^2(x, \tau)H[\varphi] = 0,$$

with its asymptotic

$$\lim_{\tau \to 0} \frac{1}{\varphi(x, \tau)} = \int_0^1 \frac{ds}{\sigma(sx, 0)}.$$

Note that equation (3.7) is singular at $\tau = 0$. To overcome this obstacle, we numerically solve for another variable that is one-to-one w.r.t. $\varphi > 0$, and has no singularity in Ω. This intermediate variable we are considering is $R = \tau\varphi^2$, which satisfies the Cauchy problem

$$R_\tau - \frac{\sigma^2}{2} R_{xx} = \sigma^2 (1 - \frac{x}{2} \frac{R_x}{R})^2 - \frac{\sigma^2}{4} \frac{R_x^2}{R} - \frac{\sigma^2}{16} R_x^2 \tag{6.1a}$$

$$R(x, 0) = 0. \tag{6.1b}$$

We claim it is not singular in Ω, for the following reasons.

i) $R > 0$ on Ω_T.

ii) When $\tau = 0$, $R = 0$. However, (6.1) is non-sigular if $\frac{R_x}{R}$ has a finite limit. To this end, we need to find the limit of $\frac{R_x}{R}$ as $\tau \to 0$.

$$\begin{aligned}\lim_{\tau \to 0} \frac{R_x}{R} &= 2 \lim_{\tau \to 0} \frac{\varphi_x}{\varphi} \\ &= 2 \frac{\frac{d(x) - x/\sigma(x,0)}{d^2(x)}}{x/d(x)} \\ &= 2 \frac{\int_0^x \frac{ds}{\sigma(s,0)} - \frac{x}{\sigma(x,0)}}{x \int_0^x \frac{ds}{\sigma(s,0)}},\end{aligned}$$

where $d(x) = \displaystyle\int_0^x \frac{ds}{\sigma(s,0)}$. Since $\sigma(x, 0)$ is strictly positive and finite on $\mathbb{R}$, we have

(a) $0 < |x \int_0^x \frac{ds}{\sigma(s,0)}| < \infty$, and

(b) $| \int_0^x \frac{ds}{\sigma(s,0)} - \frac{x}{\sigma(x,0)}| < \infty$

for $x \in \mathbb{R}\backslash\{0\}$. Our task now boils down to show $\lim_{x\to 0} \frac{\int_0^x \frac{ds}{\sigma(s,0)} - \frac{x}{\sigma(x,0)}}{x\int_0^x \frac{ds}{\sigma(s,0)}} < \infty$. In fact,

$$\begin{aligned}
&\lim_{x\to 0} \frac{\int_0^x \frac{ds}{\sigma(s,0)} - \frac{x}{\sigma(x,0)}}{x\int_0^x \frac{ds}{\sigma(s,0)}} \\
=&\lim_{x\to 0} \frac{d(x) - \frac{x}{\sigma(x,0)}}{xd(x)} \\
=&\lim_{x\to 0} \frac{\frac{1}{\sigma(x,0)} - \left(\frac{1}{\sigma(x,0)} - \frac{x\sigma'(x,0)}{\sigma^2(x,0)}\right)}{d(x) + xd'(x)} \\
=&\lim_{x\to 0} \frac{\frac{\sigma'(x,0)}{\sigma^2(x,0)}}{\frac{d(x)}{x} + \frac{1}{\sigma(x,0)}} \\
=&\frac{\frac{\sigma'(0,0)}{\sigma^2(0,0)}}{\left(\lim_{x\to 0}\frac{d(x)}{x}\right) + \frac{1}{\sigma(x,0)}} \\
=&\frac{\frac{\sigma'(0,0)}{\sigma^2(0,0)}}{\frac{1}{\sigma(x,0)} + \frac{1}{\sigma(x,0)}} \\
=&\frac{1}{2}\frac{\sigma'(0,0)}{\sigma(0,0)} < \infty.
\end{aligned}$$

Hence, equation (6.1) is well-defined and has no singularity in Ω.

Next, we solve equation (6.1) on $[x_L, x_U] \times [0,T]$ using finite-difference method. Let $R_{i,j}$ be the approximation of $R(x_i,\tau_j)$, where $x_L = x_0 < x_1 < \cdots < x_n = x_U$, and $0 = \tau_0 << \tau_1 \cdots < \tau_m = T$. We approximate the derivatives $R(x_i,\tau_{j+1})_\tau$, and $R(x_i,\tau_{j+1})_{xx}$ at (x_0,τ_{j+1}) by

$$\begin{aligned}
(R_\tau)_{0,j+1} &= \frac{R_{0,j+1} - R_{0,j}}{\Delta\tau}, \\
(R_{xx})_{0,j+1} &= \frac{R_{0,j+1} - 2R_{1,j+1} + R_{2,j+1}}{\Delta x^2}.
\end{aligned}$$

The derivatives at (x_n, τ_{j+1}) are handled in the same way. Now, we are readily writing down our scheme

$$R_\tau - \frac{\sigma^2}{2} R_{xx} = \sigma^2 \left(1 - \frac{x}{2}\frac{R_x}{R}\right)^2 - \frac{\sigma^2}{16} R_x^2 - \frac{\sigma^4}{4}\frac{R_x}{R}, \tag{6.2a}$$

$$i.e., \quad LR_{i,j+1} = G(R_{i,j}). \tag{6.2b}$$

In matrix form, it is

$$\begin{pmatrix} \frac{(\Delta x)^2}{\Delta\tau\sigma_{0,j}^2} - \frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ -\frac{1}{2} & \frac{(\Delta x)^2}{\Delta\tau\sigma_{1,j}^2} - \frac{1}{2} & -\frac{1}{2} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & -\frac{1}{2} & \frac{(\Delta x)^2}{\Delta\tau\sigma_{i,j}^2} - \frac{1}{2} & -\frac{1}{2} \\ & & -\frac{1}{2} & 1 & \frac{(\Delta x)^2}{\Delta\tau\sigma_{n,j}^2} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} R_{0,j+1} \\ \vdots \\ R_{i,j+1} \\ \vdots \\ R_{n,j+1} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta x}{\sigma_{0,j}}\right)^2 \left[\frac{R_{0,j}}{\Delta\tau} + G(R_{0,j})\right] \\ \vdots \\ \left(\frac{\Delta x}{\sigma_{i,j}}\right)^2 \left[\frac{R_{i,j}}{\Delta\tau} + G(R_{i,j})\right] \\ \vdots \\ \left(\frac{\Delta x}{\sigma_{n,j}}\right)^2 \left[\frac{R_{n,j}}{\Delta\tau} + G(R_{n,j})\right] \end{pmatrix}. \tag{6.3}$$

6.2 Approximations to the Implied Volatility φ

Once we found the asymptotic of φ as $\tau \to 0$, we can get further terms of φ by Taylor expanding φ in powers of τ. This gives simple approximations to φ. The goal for this section is to find functions $\varphi^0(x)$ and $\varphi^1(x)$, so that

$$\varphi(x,\tau) = \varphi^0(x,\tau)[1 + \varphi^1(x,\tau)\tau + O(\tau^2)] \tag{6.4}$$

satisfies (3.7), *i.e.*,

$$(\tau\varphi^2)_\tau = \left(1 - x\frac{\varphi_x}{\varphi}\right)^2 + \sigma^2\tau\varphi\varphi_{xx} + O(\tau^2) \quad \text{in } \Omega_T, \quad \text{and}$$
$$\varphi(x,0) = \varphi^0(x).$$

Recall

$$\varphi^0(x) = \frac{x}{d(x)}, \quad \text{where } d(x) = \int_0^x \frac{ds}{\sigma(s,0)}. \tag{6.6}$$

So, we only have φ^1 to solve. Matching terms involving τ, we have

$$O(\tau^1): \quad 6(\varphi^0)^4\varphi^1 = 2\sigma^2[\varphi^0 - x(\varphi^0)'][\varphi^0\varphi^1 - x(\varphi^0)'\varphi^1 - x\varphi^0(\varphi^1)']$$
$$+ (\varphi^0)^3\sigma^2(\varphi^0(x))''. \tag{6.7}$$

Plug

$$d'(x) = \frac{1}{\sigma(x,0)},$$
$$\varphi^0(x) = \frac{x}{d(x)},$$
$$(\varphi^0)'(x) = \frac{1}{d(x)} - \frac{xd'(x)}{d^2(x)}, \quad \text{and}$$
$$(\varphi^0)''(x) = -\frac{2d'(x)}{d^2(x)} + \frac{x\sigma'(x,0)d'^2(x)}{d^2(x)} + \frac{2xd'^2(x)}{d^3(x)}$$

into (6.7), we get an equivalent equation that φ^1 satisfies

$$2\varphi^1(x) + \sigma(x,0)d(x)(\varphi^1)'(x) = -\frac{\sigma(x,0)}{xd(x)} + \frac{\sigma'(x,0)}{2d(x)} + \frac{1}{d^2(x)}. \tag{6.8}$$

Claim 6.1. *The solution to (6.8) is:*

$$\varphi^1(x) = -\frac{1}{d^2(x)}\left[\ln\left(\frac{x}{d(x)\sqrt{\sigma(x,0)}}\right) - \ln\sqrt{\sigma(x,0)}\right].$$

Solution: Multiplying both sides of (6.8) by $\dfrac{1}{\sigma(x,0)d(x)} = \ln(d(x))'$, we get

$$\begin{aligned}
\frac{2d'(x)}{d(x)}\varphi^1(x) + (\varphi^1(x))' &= -\frac{1}{d^2(x)}\left(\frac{1}{x} - \frac{\sigma'(x,0)}{2\sigma(x,0)} - \frac{d'(x)}{d(x)}\right), \quad \textit{i.e.,}\\
\frac{2d'(x)}{d(x)}\varphi^1(x) + (\varphi^1(x))' &= -\frac{1}{d^2(x)}\left[(\ln x)' - \frac{1}{2}(\ln\sigma(x,0))' - (\ln d(x))'\right], \quad \textit{i.e.,}\\
\frac{2d'(x)}{d(x)}\varphi^1 + (\varphi^1)' &= -\frac{1}{d^2}\left[\ln\left(\frac{x}{d\sqrt{\sigma(x,0)}}\right)\right]'.
\end{aligned}$$

Now, multiply both sides of the above equation by $d^2(x) = \exp\left\{2\int(\ln d)'d\xi\right\}$, we can further simplify (6.8) to

$$2d(x)d'(x)\varphi^1(x) + d^2(x)(\varphi^1(x))' = -\left(\ln\frac{x}{d(x)\sqrt{\sigma(x,0)}}\right)'.$$

Hence,

$$(d(x)^2\varphi(x)^1)' = -\left(\ln\frac{x}{d(x)\sqrt{\sigma(x,0)}}\right)'.$$

Therefore

$$\varphi^1(x) = -\frac{1}{d^2(x)}\left[\ln(\frac{x}{d(x)\sqrt{\sigma(x,0)}}) + C\right]$$
$$= -\frac{1}{d^2(x)}\left[\ln(\frac{x}{d(x)\sqrt{\sigma(x,0)}}) - \ln\sqrt{\sigma(x,0)}\right]. \tag{6.9}$$

We chose the constant C to be $-\ln\sqrt{\sigma(x,0)}$, so that $\varphi^1(x)$ does not blow up as x approaches to zero.

□

To summarize this subsection, we have the following approximation to the implied volatility $\varphi(x,\tau)$ when time to expire is small:

$$\varphi(x,\tau) = \frac{x}{d}\left\{1 - \tau\frac{1}{d^2}\left[\ln\left(\frac{x}{d\sqrt{\sigma(x,0)}}\right) - \ln\sqrt{\sigma(x,0)}\right] + O(\tau^2)\right\}, \tag{6.10}$$

where $d(x) = \int_0^x \frac{ds}{\sigma(s,0)}$.

6.3 Numerical Example

We examine the accuracy of the asymptotic formula in (6.6) and (6.10) by comparing it to benchmark prices computed by solving (6.1) on a refined finite difference grid. Moreover, we illustrate the gain in accuracy provided by the two-term expansion (6.10) in Figures 6.1 and 6.2. We observe a satisfactory agreement between the asymptotic formula and the numerically computed smile.

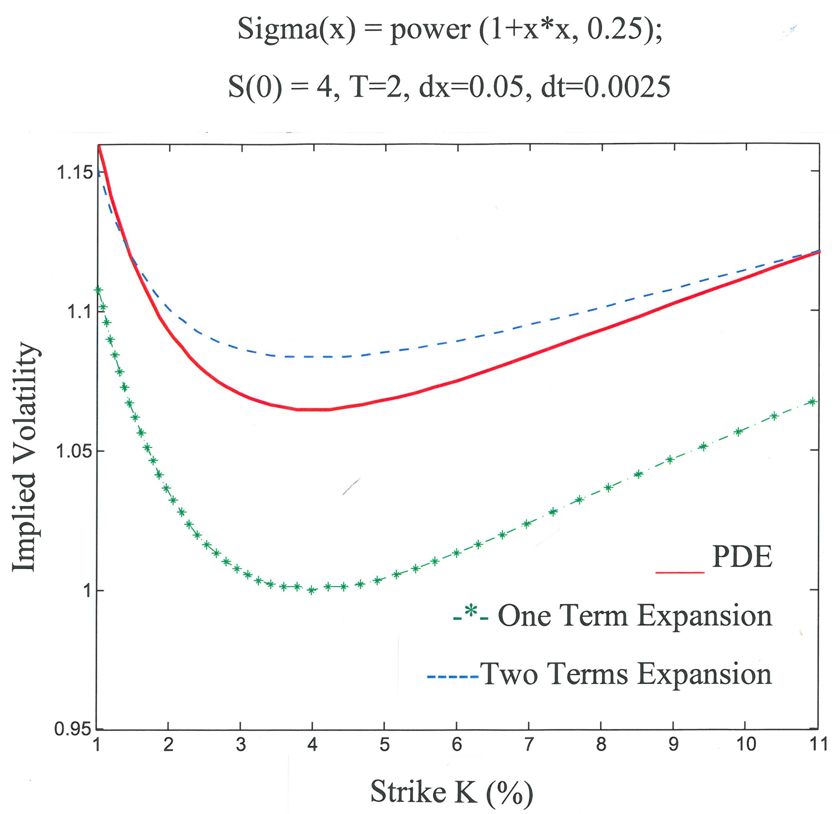

Figure 6.1: Implied Volatility (For interpretation of the references to color in this and other figures, the reader is referred to the electronic version of this dissertation.)

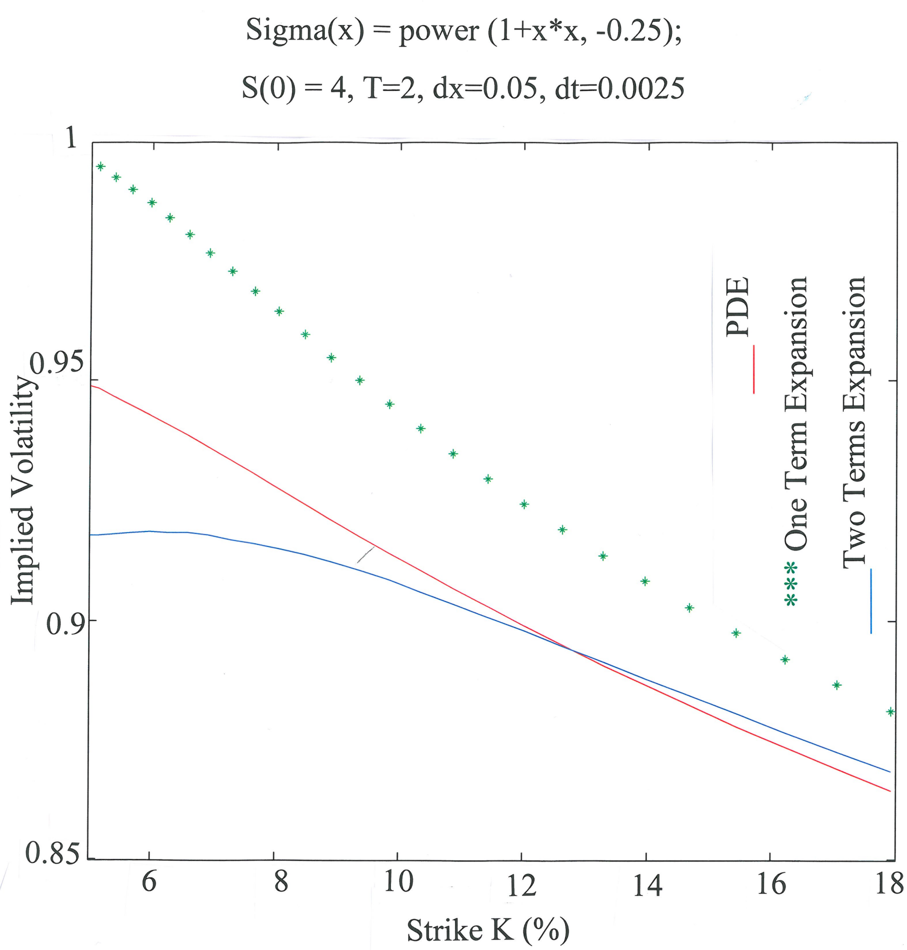

Figure 6.2: Implied Volatility CEV

Chapter 7

The Calibration Problem

In this section, we follow the ideas in [5]. One problem relevant in practice is the calibration problem–one wants to recover the value of the parameters of the model from market data. The asymptotics in Theorem 4.21 *ii)* exhibits a *linear* relation between the *inverses* of the local and implied volatilities. This leads us to propose the following penalized functional for the calibration problem:

$$J^{\varepsilon}(\sigma) = \varepsilon \int \left| \nabla \left(\frac{1}{\sigma} \right) \right|^2 dx + \sum_{i,j} \left[\left(\frac{1}{\varphi} - \frac{1}{\varphi^*} \right) (x_i, \tau_j) \right]^2 . \tag{7.1}$$

where φ^* are market implied volatilities [1], to be minimized over a suitable functional space. We suspect that this minimization problem is well posed, at least for short time-to-maturities τ_j. Indeed in this case J^{ε} is close to a convex functional. As a matter of fact we shall prove this property in the limiting case, that is, $\tau_j \equiv \tau \to 0$. Specifically, we denote by $\xi(x) = \sigma(x, 0)^{-1}$ the inverse of the local volatility and $\zeta(x) = \int_0^x \xi(y)dy/x$ the inverse of

[1] by inverting the BlackScholes formula for σ

the implied volatility[2] in the limit $\tau \to 0$. We can consider the functional in terms of ξ, ζ and write (abusing the notation)

$$J^{\varepsilon}(\xi) = \varepsilon \int \xi'^2 \, dx + \sum_i (\zeta(x_i) - \zeta^*(x_i))^2, \tag{7.2}$$

where $\zeta^* = \dfrac{1}{\varphi^*}$.

We assume that these volatilities are *consistent*, *i.e.*, that there exists $\sigma_0(x, 0)$ for which the solution to (3.6) and (3.7) with $\sigma(x) \equiv \sigma_0(x, 0)$ asymptotically replicates market prices, *i.e.*, such that

$$\lim_{\tau \to 0} \varphi(x_i, \tau) = \varphi^*(x_i) \equiv \frac{1}{\zeta^*(x_i)}.$$

It follows that $J^{\varepsilon}(\xi_0)|_{\varepsilon=0} = 0$, with $\xi_0 = \sigma_0^{-1}$. This means that, by assumption, we have a solution to the exact asymptotic calibration problem. As a consequence, there are in fact infinitely many of them, as can easily be seen from the argument in the proof of Theorem 7.1 below. The whole point is to choose one of these solutions in a stable way. This is the question that the following result addresses.

Theorem 7.1. *i) For any $\varepsilon > 0$, there exists a unique solution of the minimization problem*

$$\inf_{\xi \in H^1(\mathbb{R})} J^{\varepsilon}(\xi), \tag{7.3}$$

denoted by ξ_ε.

ii) When $\varepsilon \to 0$, ξ_ε converges uniformly in $\mathbb{R}$ to a solution $\hat{\xi}$ of the exact asymptotic

[2] Theorem 4.21

calibration problem, i.e.,

$$\hat{\zeta}(x_i, 0) \equiv \int_0^1 \hat{\xi}(sx_i)\, ds = \zeta^*(x_i). \tag{7.4}$$

Proof. i) The existence of the infinium of J^ε follows from the best approximation property [21] of a Hilbert space: "If C is a non-empty closed convex subset of a Hilbert space H and x a point in H, there exists a unique point $y \in C$ which minimizes the distance between x and points in C."

For the uniqueness of the minimizer, let us first compute the Euler equation. Note that ξ_ε being a solution for the minimizing problem (7.3) implies

$$\left.\frac{\partial J^\varepsilon(\xi_\varepsilon + hv)}{\partial h}\right|_{h=0} = 0 \quad \forall v \in H^1(\mathbb{R}).$$

That is

$$\begin{aligned}
&\left.\frac{\partial J^\varepsilon(\xi_\varepsilon + hv)}{\partial h}\right|_{h=0} \\
=&2\varepsilon \int_{\mathbb{R}} (\xi_\varepsilon + hv)'v' + 2\sum_i \left(\frac{1}{x_i}\int_0^{x_i} \xi_\varepsilon + hvdy - \zeta^*(x_i)\right)\left(\frac{1}{x_i}\int_0^{x_i} vdy\right)\Bigg|_{h=0} \\
\overset{(1)}{=}&2\varepsilon\left(\xi_\varepsilon' v|_{-\infty}^{\infty} - \int \xi_\varepsilon'' vdy\right) + 2\sum_i (\zeta_\varepsilon(x_i) - \zeta^*(x_i))\left(\frac{1}{x_i}\int_0^{x_i} vds\right) \\
\overset{(2)}{=}&2\left[-\varepsilon\int \xi_\varepsilon'' vdy + \sum_i (\zeta_\varepsilon(x_i) - \zeta^*(x_i))\left(\frac{1}{x_i}\int_0^{x_i} vdy\right)\right] \\
=&2\int\left[-\varepsilon\xi_\varepsilon'' + \sum_i (\zeta_\varepsilon(x_i) - \zeta^*(x_i))\frac{1}{x_i}\mathbf{1}_{(0,x_i)}\right] vdy \\
=&0 \quad \forall v \in H^1(\mathbb{R}).
\end{aligned}$$

Equalities (1) holds because $\zeta(x) = \frac{1}{x}\int_0^x \xi(y)dy$, (2) holds since $\xi_\varepsilon \in H^1(\mathbb{R})$.

Therefore, the Euler equation turns out to be:

$$-\varepsilon\xi_\varepsilon'' + \sum_i \frac{1}{x_i}\mathbf{1}_{(0,x_i)}(\zeta_\varepsilon(x_i) - \zeta_i^*) = 0. \tag{7.5}$$

Next, we show uniqueness of the solution to (7.5). Take ξ_ε and $\tilde{\xi}_\varepsilon$ as two solutions to (7.5), with corresponding quantities ζ_ε , $\tilde{\zeta}_\varepsilon$,respectively. We first take the difference of two corresponding Euler equations. Then multiply both sides by $\xi_\varepsilon - \tilde{\xi}_\varepsilon$ and integrate over $\mathbb{R}$, we get

$$\begin{aligned}
-\varepsilon(\xi_\varepsilon'' - \tilde{\xi}_\varepsilon'') + \sum_i (\zeta(x_i)_\varepsilon - \tilde{\zeta}_\varepsilon(x_i))\frac{1}{x_i}\mathbf{1}_{(0,x_i)} = 0 \\
\Rightarrow \int_{\mathbb{R}} -\varepsilon(\xi_\varepsilon'' - \tilde{\xi}_\varepsilon'')(\xi_\varepsilon - \tilde{\xi}_\varepsilon)dy + \sum_i \int_{\mathbb{R}} (\xi_\varepsilon - \tilde{\xi}_\varepsilon)(\zeta_{\varepsilon_i} - \tilde{\zeta}_{\varepsilon_i})\frac{1}{x_i}\mathbf{1}_{(0,x_i)}dy = 0 \\
\Rightarrow \varepsilon \int_{\mathbb{R}} (\xi_\varepsilon' - \tilde{\xi}_\varepsilon')^2 dy + \sum_i (\zeta_{\varepsilon_i} - \tilde{\zeta}_{\varepsilon_i})^2 = 0.
\end{aligned}$$

This implies $\xi_\varepsilon' = \tilde{\xi}_\varepsilon'$ *a.e.* and $\zeta_{\varepsilon_i} = \tilde{\zeta}_{\varepsilon_i}$. Therefore, $\xi(x) = \tilde{\xi}(x)\ \forall x \in \mathbb{R}$.

ii) We now show the convergence of ξ_ε as $\varepsilon \to 0$. Multiply both sides of (7.5) by ξ_ε, then integrate over $\mathbb{R}$. This gives us $\varepsilon \int_{\mathbb{R}} {\xi_\varepsilon'}^2 + \sum_i (\zeta_\varepsilon(x_i) - \zeta_i^*)\zeta_\varepsilon(x_i) = 0$, *i.e.*,

$$\varepsilon \int_{\mathbb{R}} {\xi_\varepsilon'}^2 + \sum_i \zeta_\varepsilon^2(x_i) = \sum_i \zeta_\varepsilon(x_i)\zeta_i^*. \tag{7.6}$$

Therefore, $\sum_i \zeta_\varepsilon^2(x_i) \overset{(*)}{\leq} \sum_i \zeta_\varepsilon(x_i)\zeta_i^*$. On the other hand,

$$\begin{aligned}\sum_i (\zeta_\varepsilon(x_i) - \zeta_i^*)^2 &= \sum_i \zeta_\varepsilon^2(x_i) + \sum_i \zeta_i^{*2} - 2\sum_i \zeta_\varepsilon(x_i)\zeta_i^* \\ &\overset{(*)}{\leq} \sum_i \zeta_i^{*2} - \sum_i \zeta_\varepsilon(x_i)\zeta_i^*. \end{aligned} \tag{7.7}$$

Hence,

$$\begin{aligned}\inf_{\xi \in H^1(\mathbb{R})} J^\varepsilon(\xi) &= \varepsilon \int_{\mathbb{R}} \xi_\varepsilon'^2 + \sum_i (\zeta_\varepsilon(x_i) - \zeta^*(x_i))^2 \\ &= \underbrace{\sum_i \zeta_\varepsilon(x_i)\zeta_i^* - \sum_i \zeta_\varepsilon^2(x_i)}_{(7.6)} + \sum_i (\zeta_\varepsilon(x_i) - \zeta^*(x_i))^2 \\ &\leq \sum_i \zeta_\varepsilon(x_i)\zeta_i^* + \underbrace{\sum_i \zeta_i^{*2} - \sum_i \zeta_\varepsilon(x_i)\zeta_i^*}_{(7.7)} \\ &= \sum_i \zeta_i^{*2}. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ in (7.2), one sees the limit of ξ_ε solves the exact asymptotic calibration problem.

□

Chapter 8

Comparing Relative Pricing of Options with Stochastic Volatility

Ledoit, Santa-Clara and Yan molded the implied volatilities of call options of all maturities and strike prices as a joint diffusion with the stock price $S(t)$. They assumed the stock price follows:

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_{S_1} dW_1(t).$$

The implied volatilities V of any fixed time to maturity and moneyness $X \equiv S(t)/K$, where K is the strike price have dynamics given by

$$dV(t,T,X) = \mu_V(t,T,X)dt + \sigma_{V_1}(t,T,X)dW_1(t) + \sigma_{V_2}(t,T,X)dW_2(t),$$

where W_2 is a Brownian motion orthogonal to W_1.

In order for no arbitrage opportunities to exist in trading the stock and its options, the drift of the processes followed by the implied volatilities is constrained in such a way that it is fully characterized by the volatilities of the implied volatilities. The authors equated the drift of the options that we obtain in this manner with the short term interest rate and obtain a constraint on the drift of the implied volatilities. In conclusion, the authors derived the risk-adjusted dynamics of the implied volatilities. They also showed that the Black-Scholes implied volatilities of at-the-money options converge to the underlying asset's instantaneous (stochastic) volatility as the time to maturity goes to zero. This asymptotic agrees with ours if

$$\int_0^x \frac{dy}{\sigma(y,0)} = \frac{x}{\sigma(x,0)} \quad \forall x \in \mathbb{R}.$$

One possibility is that $\sigma(y,0) \equiv \sigma$ for $y \in (0,x)$.

APPENDICIES

Appendix A

Derivation of Equation (1.7)

Proof. We notice under the risk-neutral measure $\tilde{\mathbb{P}}$, the underlying security satisfies

$$\frac{dS_t}{S_t} = rdt + \Sigma(t, S_t)d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$. Furthermore, at time t, the value of a call option is the discounted conditional expectation of the pay-off function $(S_T - K)_+$, *i.e.*, $C(S_t, t) = e^{-r(T-t)}\,\tilde{\mathbb{E}}^{S_t,t}[(S_T - K)_+]$. Differentiating $e^{r(T-t)}C(S_t, t)$, and let $\tau = T - t$, one gets:

$$\begin{aligned}
& d(e^{r(T-t)}C(S_t, t)) \\
=& e^{r\tau}[-rC(S_t, t) + C_t(S_t, t)]dt + e^{r\tau}C_S(S_t, t)dS_t + \frac{1}{2}e^{r\tau}C_{SS}(S_t, t)dS_t dS_t \\
=& e^{r\tau}\left[-rC(S_t, t) + C_t(S_t, t) + rS_t C_S(S_t, t) + \frac{1}{2}\Sigma^2(S_t, t)S_t^2 C_{SS}(S_t, t)\right]dt \\
& + e^{r\tau}C_S(S_t, t)S_t\Sigma(S_t, t)d\tilde{W}_t.
\end{aligned}$$

Since $e^{r(T-t)}C(S_t,t)$ is a martingale, the drift is zero. That is,

$$e^{r\tau}\left[-rC(S_t,t)+C_t(S_t,t)+rS_tC_S(S_t,t)+\frac{1}{2}\Sigma^2(S_t,t)S_t^2C_{SS}(S_t,t)\right]=0.$$

Notice that $e^{r\tau}>0$, and the above equation holds for every possible path of S_t. Therefore, one can replace (S_t,t) by $(S,t)\in\mathbb{R}^+\times(0,T)$, and get equation (1.7). $\square$

Appendix B

Proof for Lemma 5.3.

Given $p \in (0, 1/2)$, $\eta \in (0, 0.1)$, let $M = \dfrac{2}{(1-p)^2}$, $z_0 = -\left(\dfrac{13p+3}{3+3p}\right)^{1/p}$, and $z_1 = \left(1 - \dfrac{4}{1+p}\right) z_0$. For any $\kappa \geq 1 > \dfrac{1+\eta}{\sqrt{M}}$, and $0 < \varepsilon^p < \min\left\{\dfrac{2}{\sqrt{2\pi p}}, 1\right\}$, define $Y = \dfrac{p}{2}\dfrac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p}$, $c = \dfrac{\varepsilon^p}{(1+\eta)\sqrt{M}\kappa}$, and $Z_0 < 0$ such that $N(Z_0) \leq \frac{\varepsilon^p \eta}{\sqrt{M}\kappa}$.

The function $\underline{\psi}(z)$ is defined as:

$$\underline{\psi}(z) = \begin{cases} c \cdot \frac{1}{|z|^p} & z < z_0 \\ c \cdot g(z)/|z_0|^{p+3} & z_0 \leq z \leq z_1 \\ \tilde{\underline{\psi}}(z - z_1) & z_1 < z, \end{cases} \tag{B.1}$$

where

$$g(z) = \begin{cases} g_1(z) = & \frac{p(p+1)}{2}(z-z_0)^2|z_0| - \frac{p(p+1)^2}{6}(z-z_0)^3 + p(z-z_0)|z_0|^2 + |z_0|^3 z_0 \leq z < \frac{z_0+z_1}{2} \\ g_2(z) = & \frac{p(p+1)^2(z-z_1)^3}{12} + |z_0|^{p+3}, \quad \frac{z_0+z_1}{2} \leq z \leq z_1 \end{cases} \tag{B.2}$$

and

$$\underline{\tilde{\psi}}(z) = \frac{1}{1+\eta}\left\{\left(1-\frac{\varepsilon^p}{\sqrt{M\kappa}}\right) N\left[Y\ln(\frac{z}{A}) + Z_0\right] + \frac{\varepsilon^p}{\sqrt{M\kappa}}\right\},$$
$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-y^2/2}dy.$$

Note that $M \geq 2$, $z_0 < 0$, $z_1 > 0$ and $0 < Y < 1$ for all ε defined above.

Before we check all five conditions in lemma 5.3, we need to show:

Proposition B.1. *1.)* $\underline{\psi}(z) \in C^2(\mathbb{R})$*.*

2.) It is non-decreasing on $\mathbb{R}$*, and*

3.) $\underline{\psi}(z) \leq \dfrac{1}{1+\eta} = \lim\limits_{z\to\infty} \underline{\psi}(z)$*.*

Proof. 1.) Notice $\underline{\psi}(z)$ is piece wise defined, and smooth in each piece. To show $\underline{\psi}(z) \in C^2(\mathbb{R})$, we only need to check the smoothness at points where different pieces connect. Simple calculations give:

$$\underline{\psi}'(z) = \quad c\cdot p|z|^{-p-1} \qquad z < z_0$$
$$\underline{\psi}''(z) = \quad c\cdot p(p+1)|z|^{-p-2} \qquad z < z_0;$$

$$g'(z) = \begin{cases} g_1'(z) & = p(p+1)(z-z_0)|z_0| - \frac{p(p+1)^2}{2}(z-z_0)^2 + p|z_0|^2, z_0 \leq z < \frac{z_0+z_1}{2} \\ g_2'(z) & = \frac{p(p+1)^2(z-z_1)^2}{4}, \frac{z_0+z_1}{2} \leq z \leq z_1; \end{cases}$$

$$g''(z) = \begin{cases} g_1''(z) = p(p+1)|z_0| - p(p+1)^2(z-z_0), & z_0 \le z < \frac{z_0+z_1}{2} \\ g_2''(z) = \frac{p(p+1)^2}{2}(z-z_1), & \frac{z_0+z_1}{2} \le z \le z_1; \end{cases}$$

and

$$\begin{aligned} \underline{\psi}'(z) = &\quad \tfrac{1}{1+\eta}\tfrac{1}{\sqrt{2\pi}}\left(1 - \tfrac{\varepsilon^p}{\sqrt{M\kappa}}\right)\exp\{-\Lambda^2/2\}Y\tfrac{1}{z-z_1} \qquad && z > z_1 \\ \underline{\psi}''(z) = &\quad -C_1\exp\{-\Lambda^2/2\}(1+Y\Lambda)Y\tfrac{1}{(z-z_1)^2} && z > z_1, \end{aligned}$$

where

$$C_1 = \frac{1}{1+\eta}\frac{1}{\sqrt{2\pi}}\left(1 - \frac{\varepsilon^p}{\sqrt{M\kappa}}\right) \quad \text{and} \quad \Lambda(z) = Y\ln\frac{z-z_1}{A} + Z_0.$$

Note that the constant C_1 is between 0 and 1, for all A and ε. To show $\underline{\psi}$ is C^2 at z_0 and $(z_0+z_1)/2$, one simply computes left and right limits of $\underline{\psi}$ and its derivatives up to order two, and show they agree at z_0 and $(z_0+z_1)/2$. To see $\underline{\psi}$ is C^2 at z_1, one needs to show

$$\lim_{z\to 0^+}\underline{\tilde{\psi}}'(z) = 0 \quad \text{and} \quad \lim_{z\to 0^+}\underline{\tilde{\psi}}''(z) = 0. \tag{B.3}$$

The key is to show $\exp\left\{-\frac{\Lambda^2}{2}\right\} = o(z^N)$ for any $N \in \mathbb{N}$ as $z \to 0^+$. Indeed,

$$\begin{aligned}
&\exp\{-(Y \ln \frac{z}{A} + Z_0)^2/2\} \\
=&\exp\left\{-Y^2\left(\ln\frac{z}{A}\right)^2/2 - \frac{Z_0^2}{2} - Y\ln\frac{z}{A}Z_0\right\} \\
<&\exp\left\{-Y^2\left(\ln\frac{z}{A}\right)^2/2 - Y\ln\frac{z}{A}Z_0\right\} \\
=&\exp\left\{\ln\frac{z}{A}\cdot\ln\left(\frac{A}{z}\right)^{\frac{Y^2}{2}}\right\}\cdot\exp\left\{\ln\left(\frac{A}{z}\right)^{YZ_0}\right\} \\
=&\left(\frac{z}{A}\right)^{\ln\left(\frac{A}{z}\right)^{\frac{Y^2}{2}}}\cdot\left(\frac{A}{z}\right)^{YZ_0} \\
=&\left(\frac{z}{A}\right)^{\ln\left(\frac{A}{z}\right)^{\frac{Y^2}{2}}-YZ_0},
\end{aligned}$$

and $\lim_{z\to 0^+}\ln\left(\frac{A}{z}\right)^{\frac{Y^2}{2}} = +\infty$. Therefore, $\lim_{z\to 0^+}\exp\{-(Y\ln\frac{z}{A}+Z_0)^2/2\}\cdot\frac{1}{z^N} = 0$ for any given large N.

These complete the proof for the smoothness of $\underline{\psi}$.

2.) The non-decreasing property of $\underline{\psi}$ is a simple evaluation of its first derivative. For $z < z_0$, or $z \geq (z_0 + z_1)/2$, it is clear, $\underline{\psi}'(z) \geq 0$. It is left to show $g_1'(z) \geq 0$ for

$z_0 \leq z < (z_0 + z_1)/2$. Recall

$$
\begin{aligned}
& g_1'(z) \\
=& p(p+1)(z-z_0)|z_0| - \frac{p(p+1)^2}{2}(z-z_0)^2 + p\,|z_0|^2 \\
=& p\left(\frac{(p+1)(z-z_0)}{2} + |z_0|\right)^2 - \frac{3}{4}p(p+1)^2(z-z_0)^2 \\
=& \frac{p}{4}\left[(1+\sqrt{3})(p+1)(z-z_0) + 2\,|z_0|\right]\left[(1-\sqrt{3})(p+1)(z-z_0) + 2\,|z_0|\right]
\end{aligned}
$$

for $z_0 \leq z < \frac{z_0+z_1}{2}$. Moreover, $|z - z_0| \leq \frac{2}{1+p}|z_0|$ implies

$$
(1-\sqrt{3})(p+1)(z-z_0) \geq -|1-\sqrt{3}|(p+1)\frac{2}{p+1}\,|z_0| = -2(\sqrt{3}-1)\,|z_0|.
$$

Therefore, both factors of the last equality are positive and consequently, $g_1'(z) \geq 0$ for $z_0 \leq z < (z_0 + z_1)/2$.

3.) Consequently, $\underline{\psi} \leq \lim_{z\to\infty} \underline{\psi} = \frac{1}{1+\eta}$.

$\square$

Proposition B.2.

$$
\underline{\psi}(z) \in \left(0, \frac{1}{1+\eta}\right) \quad \text{in } \mathbb{R} \tag{B.4}
$$

Proof. It is simply because of the non-decreasing property of $\underline{\psi}$ and the facts $\lim_{z\to-\infty} \underline{\psi}(z) = 0$, $\lim_{z\to+\infty}$ and $\underline{\psi}(z) = \frac{1}{1+\eta}$. $\square$

Proposition B.3. $|\underline{\psi}'(z)|$ *has uniform upper bound on* $\mathbb{R}$ *for all* A, ε[1].

[1]defined in Lemma 5.3

Proof. 1. or $z < z_0 < 0$

$$\begin{aligned}\underline{\psi}'(z) =& c \cdot p \cdot |z|^{-p-1} \\ \leq& c \cdot p \cdot |z_0|^{-p-1},\end{aligned}$$

where $c = \frac{\varepsilon^p}{(1+\eta)\sqrt{M}\kappa} < \frac{1}{(1+\eta)\sqrt{M}\kappa}$, and $z_0 = -\left(\frac{13p+3}{3+3p}\right)^{1/p}$ both have bounds independent of A and ε.

2. For $z_0 \leq z \leq z_1$

$$\underline{\psi}'(z) = c \cdot P_2(z),$$

where $c = \frac{\varepsilon^p}{(1+\eta)\sqrt{M}\kappa} < \frac{1}{(1+\eta)\sqrt{M}\kappa}$ and $P_2(z)$ is a quadratic polynomial of which the coefficients depend only on p and z_0. Hence, the bounds for $\underline{\psi}'(z)$ are finite and independent of A and ε.

3. For $z > z_1$

$$\underline{\psi}'(z) = C_1 \exp\{-\Lambda^2/2\} \frac{Y}{z - z_1},$$

where $C_1 = \frac{1}{1+\eta}\left(1 - \frac{\varepsilon^p}{\sqrt{M}\kappa}\right)\frac{1}{\sqrt{2\pi}}$, and $\Lambda = Y \ln\left(\frac{z - z_1}{A}\right) + Z_0$.

(a) $0 < z - z_1 \leq A$

We first find an upper bound for $\exp[-\Lambda^2/2]$.

$$
\begin{aligned}
&\exp[-\Lambda^2/2] \\
&= \exp\left\{\left[Y \ln\left(\frac{A}{z-z_1}\right) - Z_0\right] \cdot \left[\frac{Y}{2}\ln\left(\frac{z-z_1}{A}\right) + \frac{Z_0}{2}\right]\right\} \\
&= \left[\left(\frac{A}{z-z_1}\right)^{Y} \cdot e^{-Z_0}\right]^{\frac{Y}{2}\ln\left(\frac{z-z_0}{A}\right)+\frac{Z_0}{2}} \\
&= \left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}} \cdot e^{-Z_0\left[\frac{Y}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{Z_0}{2}\right]} \\
&\leq \left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}}.
\end{aligned}
$$

The last inequality holds because of $\frac{Y}{2}\ln\left(\frac{z-z_1}{A}\right) + \frac{Z_0}{2} < 0$ and consequently, $e^{-Z_0\left[\frac{Y}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{Z_0}{2}\right]} \leq 1$ when $0 < Y < 1$, $0 < \frac{z-z_1}{A} \leq 1$, and $Z_0 < 0$.

Therefore, for $0 < z - z_1 \leq A$,

$$
\begin{aligned}
&\underline{\psi}'(z) \leq C_1 \cdot \frac{Y}{A} \cdot \left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}-1}, \\
&\text{where} \quad C_1 = \frac{1}{1+\eta}\frac{1}{\sqrt{2\pi}}\left(1 - \frac{\varepsilon^p}{\sqrt{M}\kappa}\right).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}-1} \\
&= \exp\left\{-\left[\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right) + \frac{Y}{2}Z_0 - 1\right] \cdot \ln\left(\frac{z-z_1}{A}\right)\right\} \downarrow 0
\end{aligned}
$$

as $z - z_1 \to 0$, or $z - z_1 \to \infty$. So it would have an interior maximum on $\mathbb{R}$ when its derivative with respect to z equals to zero. That is,

when $\frac{Y^2}{2}\ln\left(\frac{\tilde{z}-z_1}{A}\right)+\frac{YZ_0}{2}-1=0$. However, this point, $\frac{\tilde{z}-z_1}{A}=\exp\left\{\frac{2}{Y^2}\left(1-\frac{Y}{2}Z_0\right)\right\}>\exp\left\{\frac{2}{Y^2}\right\}>1$ is outside of the interval $\frac{z-z_1}{A}\in(0,1]$, so

$$\left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}-1}\in(0,1]\quad\text{for }\frac{z-z_1}{A}\in(0,1].$$ [2]

Consequently,

$$\begin{aligned}
&|\underline{\psi}'(z)|\\
&\overset{(1)}{\leq} C_1\cdot\frac{Y}{A}\cdot 1\\
&\overset{(2)}{\leq}\frac{1}{1+\eta}\frac{1}{\sqrt{2\pi}}\cdot\frac{p}{2}\frac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M\kappa/2}}\cdot\frac{1}{A}\\
&\overset{(3)}{\leq}\frac{p}{1+\eta}\cdot\frac{\varepsilon^p}{\sqrt{M\kappa}}\cdot\frac{1}{2}\\
&\leq\frac{p}{1+\eta}\cdot\frac{1}{2\sqrt{M\kappa}}.
\end{aligned}$$

Note inequality (1) is from the last estimate on $\left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}-1}$. (2) holds because $C_1<\frac{1}{1+\eta}\frac{1}{\sqrt{2\pi}}$, and $Y<\frac{p}{2}\frac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M\kappa/2}}$ when $\varepsilon^p<1$. (3) holds because $A\geq 2$.

[2] The range of $\left(\frac{A}{z-z_1}\right)^{\frac{Y^2}{2}\ln\left(\frac{z-z_1}{A}\right)+\frac{YZ_0}{2}-1}$ is between the value at $z-z_1=A$ and when $z-z_1\to 0^+$.

(b) $z - z_1 > A$:

$$
\begin{aligned}
|\underline{\psi}'(z)| &= C_1 \cdot \frac{Y}{A} \cdot \frac{A}{z - z_1} \exp\{-\Lambda^2/2\} \\
&\leq C_1 \cdot \frac{Y}{A} \cdot 1 \cdot 1 \\
&= \frac{1}{1+\eta} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{M}\kappa - \varepsilon^p}{\sqrt{M}\kappa} \cdot \frac{p}{2} \frac{\sqrt{2\pi}\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p} \cdot \frac{1}{A} \\
&\leq \frac{1}{1+\eta} \cdot \frac{\varepsilon^p}{\sqrt{M}\kappa} \cdot \frac{p}{2} \cdot \frac{1}{2} \\
&< \frac{p}{1+\eta} \cdot \frac{1}{4\sqrt{M}\kappa}
\end{aligned}
$$

□

Proposition B.4. $|\underline{\psi}''(z)|$ *is finite and the maximum value is independent of* A*.*

Proof. We estimate $|\underline{\psi}''|$ in the following intervals separately.

1. $z < z_0 < 0$:

$$
\underline{\psi}'' = \frac{\varepsilon^p}{(1+\eta)\sqrt{M}\kappa} \cdot p(p+1)|z|^{-p-2}.
$$

Since

(a) $\frac{\varepsilon^p}{(1+\eta)\sqrt{M}\kappa} < \frac{1}{(1+\eta)^2\sqrt{2}}$ for all ε and A[3], and

(b) z_0 depends only on on p, the decay rate of σ^2.

$|\underline{\psi}|''$ is finite and the maximum value independent of A and ε.

2. $z_0 \leq z \leq z_1$:

In this case, $\underline{\psi}''$ is a polynomial on $[z_0, z_1]$. Since all coefficients and the end points z_0, z_1 of this polynomial are finite and depending only on p, the maximum and minimum

[3]satisfies Lemma 5.3

values of $\underline{\psi}''$ are finite and only depending on p.

3. $z > z_1$

$$
\begin{aligned}
|\underline{\psi}''(z)| &= C_1 \left| \frac{1+Y\Lambda}{(z-z_1)^2 exp\left\{\frac{\Lambda^2}{2}\right\}} \right| \cdot Y \\
&= C_1 \cdot \left(\frac{A}{z-z_1}\right)^2 \cdot \exp\left\{-\frac{\Lambda^2}{2}\right\} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y
\end{aligned}
$$

where $\Lambda = Y \ln \dfrac{z-z_1}{A} + Z_0$, and $0 < C_1 = \dfrac{1}{1+\eta}\dfrac{1}{\sqrt{2\pi}}(1 - \dfrac{\varepsilon^p}{\sqrt{M}\kappa}) < 1$. Next, we show

$$
|\underline{\psi}''(z)| \leq 1 + e^{-1/2} \quad \text{for } z > z_1
$$

in the following subintervals.

(a) $0 < z - z_1 \leq A$

$$
\begin{aligned}
&\left(\frac{A}{z-z_1}\right)^2 \cdot \exp\left\{-\frac{\Lambda^2}{2}\right\} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y \\
&= \left(\frac{A}{z-z_1}\right)^2 \cdot \exp\left\{\left[\ln\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}} - \frac{Z_0}{2}\right] \cdot \Lambda\right\} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y \\
&= \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} \cdot e^{-\frac{Z_0}{2}\Lambda} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y \\
&\overset{(1)}{\leq} \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} \cdot e^{-\frac{Z_0}{2}\Lambda} \cdot |1+Y\Lambda| \cdot Y \\
&\leq \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} e^{-\frac{Z_0}{2}\Lambda} Y + \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} e^{-\frac{Z_0}{2}\Lambda} |\Lambda| \, Y^2 \\
&\overset{(2)}{\leq} \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} e^{-\frac{Z_0}{2}\Lambda} + \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda - 2} e^{-\frac{Z_0}{2}\Lambda} |\Lambda|,
\end{aligned}
$$

where inequalities (1) holds because $A > 2$, and (2) holds since $0 < Y < 1$.

Next, we find an upper bound for $\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda}$ and $\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda} \cdot |\Lambda|$.

i.

$$\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda} = \exp\left\{-\frac{\Lambda^2}{2}\right\} \leq 1 \quad \text{in } \mathbb{R}.$$

ii.

$$\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda} \cdot |\Lambda| = -\Lambda \cdot \exp\left\{-\frac{\Lambda^2}{2}\right\}.$$

The last quality has maximum when its derivative,

$$\Lambda' \exp\left\{-\frac{\Lambda^2}{2}\right\}(-1+\Lambda^2),$$

equals to zero, or when $z \to z_1^+$, *i.e.*, when $\Lambda = -1$ [4], or $z \to z_1^+$. Since $1 \cdot \exp\left\{-\frac{(-1)^2}{2}\right\} = e^{-1/2}$, and $-\Lambda \exp\{-\frac{\Lambda^2}{2}\} \to 0$ as $z \to z_1^+$, we get

$$\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}} \cdot |\Lambda| \leq e^{-1/2}.$$

[4] Since $0 < z - z_1 \leq A$, $Y > 0$ and $Z_0 < 0$, the only feasible solution to $-1 + \Lambda^2 = 0$ is $\Lambda = -1$.

In conclusion, when $0 < z - z_1 \leq A$,

$$
\begin{aligned}
&|\underline{\psi}''(z)| \\
=&C_1 \cdot \left(\frac{A}{z-z_1}\right)^2 \cdot \exp\left\{-\frac{\Lambda^2}{2}\right\} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y \\
\leq&\left(\frac{A}{z-z_1}\right)^2 \cdot \exp\left\{-\frac{\Lambda^2}{2}\right\} \cdot \left|\frac{1+Y\Lambda}{A^2}\right| \cdot Y \\
\leq&\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} e^{-\frac{Z_0}{2}\Lambda} \cdot Y + \left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} e^{-\frac{Z_0}{2}\Lambda}|\Lambda| \cdot Y^2 \\
\overset{(1)}{\leq}&1 \cdot Y + e^{-1/2} \cdot Y^2 \\
\overset{(2)}{\leq}&1 + e^{-1/2}.
\end{aligned}
$$

Inequality (1) is from the estimates on $\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda}$ and $\left(\frac{z-z_1}{A}\right)^{-\frac{Y}{2}\Lambda-2} \cdot e^{-\frac{Z_0}{2}\Lambda} \cdot |\Lambda|$. Inequality (2) holds because $0 < Y < 1$ when $\varepsilon^p < \min\{1, 2/(\sqrt{2\pi}p)\}$.

(b) $z - z_1 > A$:

$$
\begin{aligned}
&|\underline{\psi}''(z)| \\
\leq&\left|\frac{1+Y\Lambda}{(z-z_1)^2 \exp\left\{\frac{\Lambda^2}{2}\right\}}\right| \cdot Y \\
\leq&\left[\frac{1}{(z-z_1)^2} + \frac{|Y\Lambda|}{(z-z_1)^2}\right] \frac{Y}{\exp\{\Lambda^2/2\}} \\
\overset{(1)}{\leq}&1 + \frac{Y^2}{(z-z_1)^2} \cdot \frac{|\Lambda|}{\exp\{\Lambda^2/2\}} \\
=&1 + \left(\frac{A}{z-z_1}\right)^2 \cdot \frac{Y^2}{A^2} \cdot \frac{|\Lambda|}{\exp\{\Lambda^2/2\}} \\
\overset{(2)}{\leq}&1 + Y^2 \cdot 1 \cdot 1 \cdot \frac{|\Lambda|}{\exp\{\Lambda^2/2\}} \\
\overset{(2)}{\leq}&1 + 1 \cdot e^{-1/2},
\end{aligned}
$$

where inequalities (1) holds because $|z-z_1| \geq A \geq 2$, and $\dfrac{Y}{\exp \Lambda^2/2} \leq Y < 1$; (2) holds because $z-z_1 > A \geq 2$ and $0 < Y < 1$; (3) holds because $\dfrac{|\Lambda|}{\exp\{\Lambda^2/2\}} < e^{-1/2}$ as discussed in the previous case, and $0 < Y < 1$.

In summary, when $z - z_1 > A$,

$$|\underline{\psi}''(z)| \leq 1 + e^{-1/2}.$$

$\square$

If one exam the proof for Proposition B.1, it is clear that $z\underline{\tilde{\psi}}'(z) \to 0$, $\underline{\tilde{\psi}}'(z) \to 0$, $\underline{\tilde{\psi}}''(z) \to 0$ as $z \to +\infty$, and $\lim_{z\to+\infty} \underline{\tilde{\psi}}(z) = \frac{1}{1+\eta}$. These implies (ii), and (v) in Lemma 5.3.

Now, we left to show (iii) and (iv) in Lemma 5.3.

(iii): It is equivalent to show

$$\underline{\psi} \leq \frac{\varepsilon^p}{\sqrt{M}\kappa|z|^p} \quad z \in (-\infty, -1), \quad \text{and}$$
$$\underline{\psi} \leq \frac{\varepsilon^p}{\sqrt{M}\kappa} \quad z \in [-1, z_1 + A).$$

As usual, we estimate $\underline{\psi}$ in different intervals. A critical property is that $\underline{\psi}$ is non-decreasing in $\mathbb{R}$.

1. $z < z_0 < -1$:

$$
\begin{aligned}
\underline{\psi} &\le \underline{\psi}(z_0) \\
&= \frac{1}{1+\eta}\frac{\varepsilon^p}{\sqrt{M}\kappa}\frac{1}{|z_0|^p} \\
&< \frac{\varepsilon^p}{\sqrt{M}\kappa},
\end{aligned}
$$

since $0 < \dfrac{1}{1+\eta},\ \dfrac{1}{|z_0|^p} < 1$.

2. $z_0 \le z \le z_1$:

$$
\begin{aligned}
\underline{\psi} &\le \underline{\psi}(z_1) \\
&= \frac{1}{1+\eta}\frac{\varepsilon^p}{\sqrt{M}\kappa}\frac{|z_0|^p}{|z_0|^p} \\
&< \frac{\varepsilon^p}{\sqrt{M}\kappa},
\end{aligned}
$$

3. $z_1 < z \le z_1 + A$:

$$
\begin{aligned}
\underline{\psi} &\le \underline{\psi}(z_1 + A) \\
&= \frac{1}{1+\eta}\left\{\left(1 - \frac{\varepsilon^p}{\sqrt{M}\kappa}\right)N(Z_0) + \frac{\varepsilon^p}{\sqrt{M}\kappa}\right\} \\
&= \frac{\varepsilon^p}{\sqrt{M}\kappa}\frac{1}{1+\eta}\left\{\left(\frac{\sqrt{M}\kappa}{\varepsilon^p} - 1\right)N(Z_0) + 1\right\} \\
&\overset{(1)}{\le} \frac{\varepsilon^p}{\sqrt{M}\kappa}\frac{1}{1+\eta}\left\{\left(\frac{\sqrt{M}\kappa}{\varepsilon^p} - 1\right)\cdot\frac{\eta\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p} + 1\right\}, \\
&= \frac{\varepsilon^p}{\sqrt{M}\kappa}\frac{1}{1+\eta}\cdot\left\{\frac{\sqrt{M}\kappa - \varepsilon^p}{\varepsilon^p}\cdot\frac{\eta\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p} + 1\right\} \\
&= \frac{\varepsilon^p}{\sqrt{M}\kappa}
\end{aligned}
$$

Inequality (1) holds because we choose Z_0 so that $N(Z_0) \le \dfrac{\eta\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p}$.

(iv):

1. $z < z_0$:

$$\left|\frac{z\underline{\psi}'}{\underline{\psi}}\right| = \left|\frac{zpz^{p-1}}{z^p}\right| = p.$$

2. $z_0 \le z \le 0$:

$$\left|\frac{z\underline{\psi}'}{\underline{\psi}}\right| \le \left|\frac{z_0\underline{\psi}'(z_0)}{\underline{\psi}(z_0)}\right| = p.$$

The first inequality holds because when $z_0 < z < 0$, $\underline{\psi} > 0$, $\underline{\psi}' > 0$, and $\underline{\psi}'' < 0$. Therefore, $|z|$, $|\overline{\psi}'(z)|$, and $1/|\overline{\psi}(z)|$ are decreasing as $z \uparrow 0$.

3. $0 < z \le z_1$: since z, $\underline{\psi} > 0$, and $\underline{\psi}' > 0$, it is equivalent to show $\dfrac{z\underline{\psi}'}{\underline{\psi}} \le p$, *i.e.*,

$$W(z) := z\underline{\psi}' - p\underline{\psi} \le 0 \qquad \text{for } 0 < z \le z_1.$$

We now exam the inequality over the following two sub-intervals[5].

(a) For $0 < z \le \dfrac{z_0 + z_1}{2}$,

$$W(z) = zg_1'(z) - g_1(z).$$

- Endpoints

$$W\left(\frac{z_0+z_1}{2}\right)$$
$$= \frac{14/3p^2}{1+p} z_0^3,$$
$$<0.$$

[5] We frequently use the equalities $\frac{z_0+z_1}{2} = \frac{p-1}{p+1}z_0$, and $\frac{z_1-z_0}{2} = -\frac{2}{p+1}z_0$.

• Exam the interior extreme points. Since

$$\begin{aligned} W'(z) &= (1-p)g_1'(z) + zg"_1(z) \\ &= \frac{p-3}{2}p(p+1)^2(z-z_0)^2 + p(p+1)(-3)z_0(z-z_0) - 2p^2 z_0^2, \end{aligned}$$

$W'(\tilde{z}) = 0$ when

$$\begin{aligned} \frac{p-3}{2}&(p+1)^2(z-z_0)^2 - 3(p+1)z_0(z-z_0) - 2pz_0^2 = 0 \quad \textit{i.e.,} \\ \tilde{z} - z_0 &= \frac{3z_0 \pm (2p-3)}{(p-3)(p+1)} z_0, \quad \textit{i.e.,} \\ \tilde{z} &= \left(\frac{2p}{(p-3)(p+1)} + 1 \right) z_0 < 0, \quad \text{or} \\ \tilde{z} &= \left(\frac{-2}{p+1} + 1 \right) z_0 = \frac{z_0 + z_1}{2}. \end{aligned}$$

We see that the two extreme points are either unfeasible, or an end point. Therefore, we conclude $W(z) < 0$ for $0 < z \leq \frac{z_0+z_1}{2}$.

(b) When $\frac{z_0+z_1}{2} < z \leq z_1$,

$$\begin{aligned} W(z) &= zg_2'(z) - pg_2(z) \\ &= z\frac{p(p+1)^2(z-z_1)^2}{4} - p\left[\frac{p(p+1)^2(z-z_1)^3}{12} + |z_0|^{p+3} \right]. \end{aligned}$$

• Check endpoints:

$$W\left(\frac{z_0+z_1}{2}\right) = -\frac{14p^2}{3(1+p)}|z_0|^3 < 0$$

$$W(z_1) = -p|z_0|^{p+3} < 0.$$

- Let us now check interior extreme points. Since

$$
\begin{aligned}
&W'(z)\\
=&\frac{p(p+1)^2(z-z_1)^2}{4}+\frac{z}{4}p(p+1)^2\cdot 2(z-z_1)-\frac{p^2(p+1)^2}{12}\cdot 3\cdot (z-z_1)^2\\
=&\frac{p(p+1)^2}{4}(z-z_1)\left[(3-p)(z-z_1)+2z_1\right],
\end{aligned}
$$

$W'(\tilde{z})=0$ if $\tilde{z}=z_1$, or $\tilde{z}=\frac{1-p}{3-p}z_1$. Furthermore,

$$
\begin{aligned}
&W(\frac{1-p}{3-p}z_1)\\
=&\frac{1-p}{3-p}z_1\cdot\frac{p(p+1)^2(\frac{-2}{3-p}z_1)^2}{4}-p\left[\frac{p(p+1)^2(\frac{-2}{3-p}z_1)^3}{12}+|z_0|^{p+3}\right]\\
=&\frac{p(p+1)^2}{(3-p)^3}(1-\frac{1}{3}p)z_1^3-p|z_0|^{p+3}\\
=&-\frac{14p^2}{3(p+1)}|z_0|^3\\
<&0.
\end{aligned}
$$

4. When $z>z_1$,

$$
\left|\frac{z\underline{\psi}'}{\underline{\psi}}\right|=\frac{z\cdot\frac{1}{1+\eta}\cdot(1-\frac{\varepsilon^p}{\sqrt{M\kappa}})\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\Lambda^2/2}\cdot Y\cdot\frac{1}{z-z_1}}{\frac{1}{1+\eta}\left\{(1-\frac{\varepsilon^p}{\sqrt{M\kappa}})N\left[\Lambda\right]+\frac{\varepsilon^p}{\sqrt{M\kappa}}\right\}},
$$

where $\Lambda=Y\ln(\frac{z-z_1}{A})+Z_0$.

- For $0<z-z_1\le A$, we have $z_1<z<2A$ [6]. By Proposition B.3,

[6] Recall, in Lemma 5.3, we assume $A>z_1>2$.

$\underline{\psi}'(z) \leq \frac{1}{2A} \cdot \frac{p}{1+\eta} \cdot \frac{\varepsilon^p}{\sqrt{M}\kappa}$. So

$$\left|\frac{z\underline{\psi}'}{\underline{\psi}}\right| \leq \frac{2A \cdot \frac{1}{2A} \cdot \frac{p}{1+\eta} \cdot \frac{\varepsilon^p}{\sqrt{M}\kappa}}{\frac{1}{1+\eta} \cdot \frac{\varepsilon^p}{\sqrt{M}\kappa}} = p.$$

- For $z - z_1 > A \geq z_1$, we have $1 < \frac{z}{z - z_1} < 2$, and consequently,

$$\begin{aligned}
&\left|\frac{z\underline{\psi}'}{\underline{\psi}}\right| \\
\leq &\frac{\frac{z}{z-z_1} \cdot \frac{1}{1+\eta} \cdot \left(1 - \frac{\varepsilon^p}{\sqrt{M}\kappa}\right) \frac{1}{\sqrt{2\pi}} \cdot e^{-\Lambda^2/2} \cdot Y}{\frac{1}{1+\eta} \cdot \frac{\varepsilon^p}{\sqrt{M}\kappa}} \\
\leq &\frac{2 \cdot \left(1 - \frac{\varepsilon^p}{\sqrt{M}\kappa}\right) \frac{1}{\sqrt{2\pi}} \cdot 1 \cdot Y}{\frac{\varepsilon^p}{\sqrt{M}\kappa}} \\
\overset{(1)}{=} &\frac{2 \cdot \left(\frac{\sqrt{M}\kappa - \varepsilon^p}{\sqrt{M}\kappa}\right) \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} \cdot \frac{p\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p}}{\frac{\varepsilon^p}{\sqrt{M}\kappa}} \\
= &p,
\end{aligned}$$

(1) holds since $Y = \frac{\sqrt{2\pi}}{2} \frac{p\varepsilon^p}{\sqrt{M}\kappa - \varepsilon^p}$.

This completes the proof for (iii), hence the proof for Lemma 5.3.

□

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