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METRIC DIFFERENTIAL GEOMETRY
ON A CONOID

THESIS FOR THE DEGREE OF M. A.
David Francis Randolph
1933

Geometry, Differential

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METRIC DIFFERENTIAL GEOMETRY ON A CONOID

A Thesis
Submitted to the Faculty
of
MICHIGAN STATE COLLEGE
of
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In Partial Fulfillment of the
Requirements for the Degree
of
Master of Arts

by
David Francis Randolph

1933

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CONTENTS

1. Introduction	1.
2. Defining Differential Equations	2.
3. Fundamental Coefficients and Forms	8.
4. Asymptotic Curves on a Conoid	13.
5. The Quadric of Lie	16.
6. Normal Polar Reciprocal Quadric	19.
7. The Normal Quadric	24.
8. Chasle's Correlation	26.
9. The Parametric Osculating Ruled Surface R_v	28.
10. The Focal Conoid	32.
11. The Osculating Conoid	39.
12. Quadric Conoids	41.

METRIC DIFFERENTIAL GEOMETRY ON A CONOID

1. INTRODUCTION

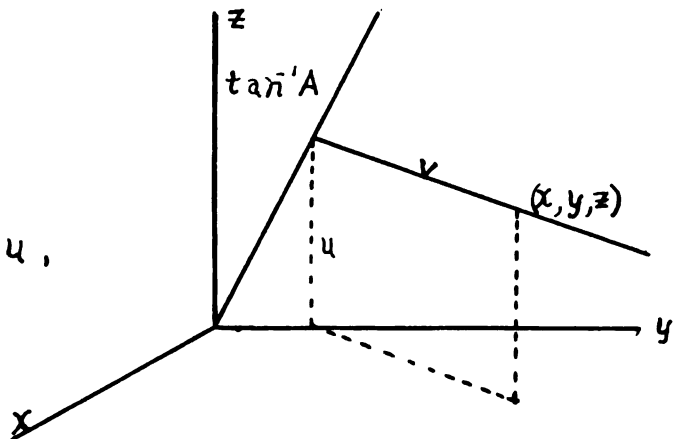
It is the purpose of this paper to discuss some of the euclidian metrical properties of ruled surfaces belonging to a linear congruence. As is well known, such surfaces have as their flecnod curves two straight lines*. For the purpose of our study we shall choose one of these lines as the ideal line in the xy -plane, and the second lying in the yz -plane and passing through the origin. Such surfaces, from the euclidian point of view, are conoids.

To repeat, a conoid is a one parameter family of lines, the generators, which always intersect a given line, the directrix, and are parallel to a fixed plane, the directing plane.

* E. J. Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Leipzig, B. G. Teubner, 1906, p. 168. Hereafter referred to as Wilczynski, Geometry.

We may write the parametric equations of a conoid as follows:

$$\begin{aligned} x &= v \ell, \\ (1) \quad y &= v m + A u, \\ z &= u, \end{aligned}$$



wherein u is the distance of the generator from the directing plane; v is the distance of the point, (x, y, z) from the directrix, measured along the generator;

A is the tangent of the angle which the directrix makes with the z -axis; and ℓ and m are the direction cosines of the generators. Both ℓ and m are functions of only.

The curves $u = \text{const.}$ are the generators. The curves $v = \text{const.}$ are the curves cutting the generators at a constant distance from the directrix. The equation of the directrix is, of course, $v = 0$. If the directrix is perpendicular to the directing plane, the conoid is called a right conoid. It is evident that a necessary and sufficient condition that a conoid be a right conoid is $A = 0$.

3. DEFINING DIFFERENTIAL EQUATIONS

The parametric equations of the conoid in homogeneous coordinates may be written in the form

$$(2) \quad x_1 = v \ell, \quad x_2 = v m + A u, \quad x_3 = u, \quad x_4 = 1.$$

From (2) we may determine the coefficients of the following defining differential equations

$$(3) \quad \begin{aligned} x_{uu} &= a_1 x_{uv} + b_1 x_u + c_1 x_v + d_1 x, \\ x_{vv} &= a_2 x_{uv} + b_2 x_u + c_2 x_v + d_2 x. \end{aligned}$$

wherein

$$(4) \quad \begin{aligned} a_1 &= \frac{v(\ell m'' - m \ell'')}{\ell m' - m \ell'}, \quad c_1 = - \frac{v(\ell' m'' - m' \ell'')}{\ell m' - m \ell'}, \\ b_1 &= d_1 = a_2 = b_2 = c_2 = d_2 = 0. \end{aligned}$$

Since $\ell^2 + m^2 = 1$, and if we let $\Delta = \ell m' - m \ell'$ we may show that

$$(5) \quad \Delta = -\frac{\ell'}{m} = \frac{m'}{\ell}, \quad \Delta' = \ell m'' - \ell'' m, \quad \Delta^3 = \ell' m'' - m' \ell''.$$

We may now rewrite (3) in the simple form

$$(6) \quad \begin{aligned} \chi_{uu} &= \frac{v\Delta'}{\Delta} \chi_{uv} - v\Delta^2 \chi_v, \\ \chi_{vv} &= 0. \end{aligned}$$

Another set of defining differential equations may be found by choosing two curves on the surface, say $v = v_1$ and $v = v_2 \neq v_1$. The parametric equations of these curves in homogeneous coordinates are respectively

$$(7) \quad \begin{aligned} x_1 &= v_1 \ell, \quad x_2 = v_1 m + A u, \quad x_3 = u, \quad x_4 = 1; \\ y &= v_2 \ell, \quad y_2 = v_2 m + A u, \quad y_3 = u, \quad y_4 = 1. \end{aligned}$$

It is well known* that the pairs of functions (x, y) of (7) satisfy a system of differential equations of the

* Wilczynski, Geometry, p. 126.

form

$$(8) \quad x'' + p_{11} x' + p_{12} y' + q_{11} x + q_{12} y = 0,$$

$$y'' + p_{21} x' + p_{22} y' + q_{21} x + q_{22} y = 0,$$

wherein

$$x' = \frac{dx}{du}, \quad x'' = \frac{d^2x}{du^2}, \quad \text{etc.}$$

We find the coefficients of (8) to be

$$p_{11} = -\frac{\Delta' V_1}{\Delta(V_1 - V_2)}, \quad p_{12} = \frac{\Delta' V_1}{\Delta(V_1 - V_2)}, \quad q_{11} = \frac{\Delta^2 V_1}{V_1 - V_2}, \quad q_{12} = -\frac{\Delta^2 V_1}{V_1 - V_2},$$

$$p_{21} = -\frac{\Delta' V_2}{\Delta(V_2 - V_1)}, \quad p_{22} = \frac{\Delta' V_2}{\Delta(V_2 - V_1)}, \quad q_{21} = \frac{\Delta^2 V_2}{V_2 - V_1}, \quad q_{22} = -\frac{\Delta^2 V_2}{V_2 - V_1}.$$

The defining differential equations known as the
"Gauss Equations" are*

* L. P. Eisenhart, Differential Geometry of Curves and Surfaces, New York, Ginn and Company, 1909, p. 154.
Hereafter referred to as Eisenhart, Geometry.

$$\begin{aligned}
 \chi_{uu} &= \{', '\} \chi_u + \{', ''\} \chi_v + D X, \\
 (9) \quad \chi_{uv} &= \{', ''\} \chi_u + \{', '''\} \chi_v + D' X, \\
 \chi_{vv} &= \{'', ''\} \chi_u + \{'', '''\} \chi_v + D'' X,
 \end{aligned}$$

wherein

$$\left\{ \begin{matrix} ' & ' \\ ' & ' \end{matrix} \right\} = \frac{V^2 \Delta \Delta' + A V m' + A V m \Delta^2}{1 + (V \Delta + A \ell)^2}, \quad D X = - \frac{V \Delta' m}{1 + (V \Delta + A \ell)^2},$$

$$\begin{aligned}
 \left\{ \begin{matrix} ' & ' \\ ' & '' \end{matrix} \right\} &= \frac{A m' (V^2 \Delta^2 + 2 A V m' + A^2 + 1) - A m (V^2 \Delta \Delta' + A V m'')}{1 + (V \Delta + A \ell)^2} \\
 &\quad - \frac{(V^2 \Delta^2 + 2 A V m' + A^2 + 1) (V^2 \Delta^2 + A m')}{1 + (V \Delta + A \ell)^2},
 \end{aligned}$$

$$\left\{ \begin{matrix} ' & '' \\ ' & '' \end{matrix} \right\} = \frac{V \Delta^2 + A m'}{1 + (V \Delta + A \ell)^2}, \quad \left\{ \begin{matrix} ' & '' \\ ' & '' \end{matrix} \right\} = - \frac{A m (V \Delta^2 + A m')}{1 + (V \Delta + A \ell)^2},$$

$$\left\{ \begin{matrix} '' & '' \\ ' & '' \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} '' & '' \\ ' & '' \end{matrix} \right\} = - \frac{A m m'}{1 + (V \Delta + A \ell)^2}, \quad D'' X = 0,$$

$$D' X = - \frac{\Delta m}{1 + (V \Delta + A \ell)^2}.$$

Let us make the transformation of curvilinear coordinates

$$\bar{u} = \Phi(u, v) = u, \quad \bar{v} = \Psi(u, v).$$

Then the differential equation (3) becomes

$$\begin{aligned}
 (10) \quad \chi_{\bar{u}\bar{u}} &= \bar{a}_1 \chi_{\bar{u}\bar{v}} + \bar{b}_1 \chi_{\bar{u}} + \bar{c}_1 \chi_{\bar{v}} + \bar{d}_1 \chi, \\
 \chi_{\bar{v}\bar{v}} &= \bar{a}_2 \chi_{\bar{u}\bar{v}} + \bar{b}_2 \chi_{\bar{u}} + \bar{c}_2 \chi_{\bar{v}} + \bar{d}_2 \chi,
 \end{aligned}$$

wherein

$$\begin{aligned}
 (11) \quad \bar{a}_1 &= a \psi_v - 2 \psi_u, \quad \bar{c}_2 = - \frac{\psi_{vv}}{\psi_v^2}, \\
 \bar{c}_1 &= c \psi_v + a \psi_{uv} - \psi_{uu} + \psi_{vv} \left(\frac{\psi_u^2}{\psi_v^2} - a \frac{\psi_u}{\psi_v} \right), \\
 \bar{b}_1 &= \bar{d}_1 = \bar{a}_2 = \bar{b}_2 = \bar{d}_2 = 0,
 \end{aligned}$$

wherein the values of a and c are given by (6) as

$$a = \frac{v \Delta'}{\Delta}, \quad c = -v \Delta^2.$$

Therefore (10) may be rewritten in the form

$$\begin{aligned}
 (12) \quad \chi_{\bar{u}\bar{u}} &= \left(\frac{v \Delta'}{\Delta} \psi_v - 2 \psi_u \right) \chi_{\bar{u}\bar{v}} \\
 &\quad + \left[\frac{v \Delta}{\Delta} \psi_{uv} - v \Delta^2 \psi_v - \psi_{uu} + \psi_{vv} \left(\frac{\psi_u^2}{\psi_v^2} - a \frac{\psi_u}{\psi_v} \right) \right] \chi_{\bar{v}}, \\
 \chi_{\bar{v}\bar{v}} &= - \frac{\psi_{vv}}{\psi_v^2} \chi_{\bar{v}}.
 \end{aligned}$$

3. FUNDAMENTAL COEFFICIENTS AND FORMS

It is well known that the functions defined as follows*:

$$(13) \quad \begin{aligned} A &= |y_u, z_v| = -m, & B &= |z_u, x_v| = l, \\ C &= |x_u, y_v| = -(V\Delta + A l). \end{aligned}$$

are proportional to the direction cosines of the normal to the surface at the point (x, y, z) . If we let $A^2 + B^2 + C^2 = H^2$, then the direction cosines X, Y, Z of the normal to the conoid may be written in the form

$$(14) \quad \begin{aligned} X &= \frac{A}{H} = -\frac{m}{1 + (V\Delta + A l)^2}, & Y &= \frac{B}{H} = \frac{l}{1 + (V\Delta + A l)^2}, \\ Z &= \frac{C}{H} = -\frac{V\Delta + A l}{1 + (V\Delta + A l)^2}. \end{aligned}$$

Consider, on a conoid, any curve C whose equation is of the form $\varphi(u, v) = 0$. Then the element of arc

* Eisenhart, Geometry, p. 114.

of C is given by

$$(15) \quad (ds)^2 = f = E du^2 + 2F du dv + G dv^2.$$

The differential form (15) is known as the first fundamental form, and E , F , and G as the first fundamental coefficients.* The latter are

$$E = x_u^2 + y_u^2 + z_u^2 = v^2 \Delta^2 + 2AVm' + A^2 + 1,$$

$$(16) \quad F = x_u x_v + y_u y_v + z_u z_v = Am,$$

$$G = x_v^2 + y_v^2 + z_v^2 = 1.$$

Hence the first fundamental form for the conoid is

$$(17) \quad f = (v^2 \Delta^2 + 2AVm + A^2 + 1) du^2 + 2Am du dv + dv^2.$$

It can be shown that

$$EG - F^2 = H^2 = A^2 + B^2 + C^2.$$

The net defined by $f = 0$ is called the minimal net.

* Eisenhart, Geometry, p. 70.

The minimal net is imaginary on all real surfaces.
 Therefore the minimal net is imaginary on a conoid.

The quadratic differential form

$$(18) \quad \Phi = D du^2 + 2 D' du dv + D'' dv^2$$

is called the second fundamental form, and D , D' , and D'' are the second fundamental coefficients.* The second fundamental coefficients are defined by the formulas

$$(19) \quad HD = |x_{uu}, y_u, z_v|, HD' = |x_{uv}, y_u, z_v|, HD'' = |x_{vv}, y_u, z_v|.$$

Therefore the second fundamental form for the conoid is

$$(20) \quad \Phi = \frac{v \Delta'}{\sqrt{1 + (v\Delta + A\ell)^2}} du^2 + \frac{2\Delta}{\sqrt{1 + (v\Delta + A\ell)^2}} du dv.$$

The differential equation of the asymptotic curves in curvilinear coordinates is $\Phi = 0$. The asymptotic curves may be found by quadratures. A discussion of the asymptotic curves will be given in the next article.

* Eisenhart, Geometry, p. 115.

We shall now find the condition that the curves $V = \text{const.}$, or the curves that cut the generators at a constant distance from the directrix, be plane curves. The equations of these curves are

$$x = v \ell, \quad y = v m + A u, \quad z = u,$$

wherein V is constant.

A necessary and sufficient condition* that a curve be a plane curve is that its torsion be zero. The condition that the torsion of a curve C_u on any surface be zero is **

$$| x_u, y_{uu}, z_{uuu} | = 0.$$

For the conoid this condition becomes

$$\ell'' m''' - m'' \ell''' = 0.$$

* Eisenhart, Geometry, p. 16.

** Eisenhart, Geometry, p. 17.

We may integrate this differential equation and obtain

$$m = c_1 l + c_2 u + c_3.$$

Since $l^2 + m^2 = 1$ and $l = \sin \alpha$ we may write the last equation in the form

$$\alpha = \gamma - \arccos [(c_1 u + c_3) \sin \gamma],$$

$$\beta = \frac{\pi}{2} - \alpha.$$

wherein γ , c_1 and c_3 are independent constants. Hence a necessary and sufficient condition that the curves $V = \text{Const.}$ be plane curves is that the above relations hold.

A necessary and sufficient condition* that a surface be a developable surface is

$$D D'' - D'^2 = 0.$$

For the conoid this condition becomes $\Delta = 0$.

* Eisenhart, Geometry, p. 156.

By use of (5), and integrating, we find that the necessary and sufficient conditions that a conoid be a developable surface are

$$l = c_1, \quad m = c_2.$$

Under these conditions a conoid is a plane surface.

Since plane surfaces are of little interest to us, in this paper, we shall hereafter assume $\Delta \neq 0$.

4. THE ASYMPTOTIC CURVES ON A CONOID

The differential equation defining the asymptotic curves on a conoid is

$$(21) \quad \sqrt{\Delta}' du^2 + 2\Delta du dv = 0.$$

Evidently, $u = \text{const.}$ is one of the sets of solutions. Therefore the generators form one family of asymptotic curves on a conoid. Removing the factor du and separating the variables we find

$$\frac{\Delta'}{\Delta} du + \frac{2}{\sqrt{\Delta}} dv = 0.$$

Integrating we obtain the family of curved asymptotics on the conoid

$$(22) \quad V^2 \Delta = C.$$

Necessary and sufficient conditions* that the asymptotic curves on a surface be parametric are

$$D = D'' = 0.$$

For the conoid these conditions are

$$V \Delta' = 0.$$

The curve $V = 0$ is, as we have seen, the directrix.

The condition $\Delta' = 0$ implies that

$$\Delta = -\frac{l'}{m} = \frac{m'}{l} = C.$$

Integrating we find

$$(23) \quad l = \sin(c_1 u + c_2), \quad m = \cos(c_1 u + c_2).$$

* Eisenhart, Geometry, p. 129.

This is the condition that a conoid be a helicoid, since a helicoid* is a surface generated by a line which is rotated about a fixed line as axis, and at the same time translated in the direction of the axis with a velocity which is in constant ratio with the velocity of rotation. If the values (23) be substituted in (1), it is readily seen that the above definition is satisfied. Therefore a necessary and sufficient condition that the curves which intersect the generators at a constant distance from the directrix be asymptotic curves is that the conoid be a helicoid.

A necessary and sufficient condition** that the tangents to the asymptotic curves separate the tangents to the minimal curves harmonically, is that the harmonic invariant I of the first and second fundamental forms vanish. For a conoid

$$I = ED'' - 2FD' + GD = V\Delta - 2Am\Delta.$$

The condition $I = 0$, $\Delta' \neq 0$ implies $V = 2Am\Delta/\Delta'$. Therefore on a conoid, which is not a helicoid, there is

* Eisenhart, Geometry, p.146.

** Eisenhart, Geometry, P. 129.

an unique curve along the points of which the asymptotic tangents separate the minimal tangents harmonically. On a right conoid the curve is the directrix. The condition $\mathcal{I} \equiv 0$ implies $\Delta' = 0$, $A = 0$, therefore the only minimal conoid is a right conoid and a helicoid.

5. THE QUADRIC OF LIE

The quadric of Lie or the osculating quadric along a fixed generator, is the surface generated by the tangents to the curved asymptotics at the points where they cross the fixed generator.

By (22) the equation of the curved asymptotics on a conoid are

$$(22 \text{ bis}) \quad \sqrt{2} \Delta = C.$$

A point on the tangent line to a curve at a point (x, y, z) is of the form

$$(24) \quad \xi = x + t(x_u + x_v \frac{dv}{du}), \quad \eta = y + t(y_u + y_v \frac{dv}{du}), \quad \zeta = z + t(z_u + z_v \frac{dv}{du}).$$

The coordinates of the point (x, y, z) are given by (1).

Moreover from (21) we find the relation $\frac{dv}{du} = - \frac{\sqrt{2} \Delta'}{2 \Delta}.$

Therefore a point on the tangent to the curved asymptotic through the point (x, y, z) has the coordinates

$$(25) \quad \xi = v\ell + vt\left(\ell' - \frac{\ell\Delta'}{2\Delta}\right), \quad \eta = vm + t\left(m' - \frac{m\Delta'}{2\Delta}\right) + A(u+t), \quad \zeta = u+t.$$

We may rewrite (25) in the form

$$(26) \quad \bar{\xi} = v(\ell + Rt), \quad \bar{\eta} = v(m + Pt) + At, \quad \bar{\zeta} = t,$$

wherein we have made the following translation

$$\bar{\xi} = \xi, \quad \eta = \bar{\eta} - Au, \quad \bar{\zeta} = \zeta - u,$$

and have adopted the notation

$$(27) \quad R = \ell' - \frac{\ell\Delta'}{2\Delta}, \quad P = m' - \frac{m\Delta'}{2\Delta}.$$

The coordinates $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ in (26) are functions of v and t only. Therefore we may find the locus of this point by eliminating the parameters v and t . We obtain

$$(28) \quad AR\bar{\zeta}^2 + P\bar{\xi}\bar{\zeta} - R\eta\bar{\zeta} + m\bar{\xi} - \ell\bar{\eta} + A\ell\bar{\zeta} = 0.$$

This is a second degree equation, hence the locus of the tangents to the curved asymptotics at points of a fixed

generator is a quadric surface. This quadric is called the quadric of Lie. For the right conoid the equation of the quadric of Lie becomes

$$(29) \quad P \bar{\xi} \bar{\xi} - R \bar{\eta} \bar{\xi} + m \bar{\xi} - l \bar{\eta} = 0 .$$

To determine the nature of the quadric of Lie, we shall find its intersection with the ideal plane. If we put the equation of the quadric in homogeneous coordinates $(\bar{\xi}, \bar{\eta}, \bar{\xi}, \bar{\omega})$ equation (28) becomes

$$(30) \quad A R \bar{\xi}^2 + P \bar{\xi} \bar{\xi} - R \bar{\eta} \bar{\xi} + m \bar{\xi} \bar{\omega} - l \bar{\eta} \bar{\omega} + A l \bar{\xi} \bar{\omega} = 0 .$$

We find the intersection of the quadric of Lie with the ideal plane $\bar{\omega} = 0$ to be the two straight lines

$$(31) \quad \bar{\xi} = 0, \bar{\omega} = 0; A R \bar{\xi} + P \bar{\xi} - R \bar{\eta} = 0, \bar{\omega} = 0 .$$

Therefore the quadric of Lie for a conoid is a hyperbolic paraboloid.*

* L. W. Dowling, Projective Geometry, New York, McGraw-Hill Book Company, 1917, p. 110.

6. THE POLAR RECIPROCAL OF THE NORMALS TO THE QUADRIC OF LIE OF A RIGHT CONOID

The polar reciprocal of any line with respect to the quadric of Lie, is the line of intersection of the two tangent planes at the points where the given line intersects the quadric of Lie.

In particular, the equation of the normal to the conoid at points of a fixed generator is

$$(32) \quad \xi = v\ell + mt, \quad \eta = vm - \ell t, \quad \zeta = \Delta v t,$$

wherein the trihedral of reference is the same as the one used in developing equation (28). To determine the values of t for the points in which the normal intersects the quadric of Lie, we shall solve equations (29) and (32) simultaneously. We find

$$(33) \quad P v \Delta (v \ell + m t) t - R v \Delta t (v m - \ell t) + m (v \ell + m t) - \ell (v m - \ell t) = 0.$$

If we use the following relation

$$(34) \quad P \ell - R m \equiv \Delta, \quad P m + R \ell \equiv -\frac{\Delta'}{2\Delta},$$

we may reduce (33) to the form

$$(35) \quad \sqrt{\Delta'} t^2 - 2(1 + v^2 \Delta') t = 0.$$

We must notice the two cases, $\Delta' = 0$ and $\Delta' \neq 0$. If $\Delta' = 0$ we have only one finite solution, $t = 0$. If $\Delta' \neq 0$ we have the two solutions

$$t = 0, \quad t = \frac{2 + 2v^2 \Delta'}{\sqrt{\Delta'}}.$$

If we use these values of t in (32) and let $(\xi' \eta' \zeta')$ be the point when $t = 0$ and $(\xi'' \eta'' \zeta'')$ be the second point, the two points are

$$(36) \quad \begin{aligned} \xi' &= v \ell, & \eta' &= v m, & \zeta' &= 0; \\ \xi'' &= v \ell + m \left(\frac{2 + 2v^2 \Delta'}{\sqrt{\Delta'}} \right), & \eta'' &= v m - \ell \left(\frac{2 + 2v^2 \Delta'}{\sqrt{\Delta'}} \right), & \zeta'' &= \frac{2\Delta + 2v^2 \Delta^3}{\sqrt{\Delta'}}. \end{aligned}$$

The tangent plane to a quadric surface $f(\xi, \eta, \zeta, \omega) = 0$ at a point (ξ'', η'', ζ'') is given by*

*V. Snyder and C. H. Sisam, Analytic Geometry of Space, New York, Henry Holt, 1914, p. 209.

$$(38) \quad \frac{\partial f}{\partial \xi} \xi'' + \frac{\partial f}{\partial \eta} \eta'' + \frac{\partial f}{\partial \xi} \xi'' + \frac{\partial f}{\partial \omega} = 0.$$

Therefore the equation of the tangent plane at the point (ξ', η', ξ') to the quadric of Lie of the right conoid is

$$(39) \quad (P\xi'' + m)\xi - (R\xi'' + l)\eta + (P\xi'' - R\eta'')\xi + m\xi'' - l\eta'' = 0.$$

Making use of (34) and the value of (ξ'', η'', ξ'') from (36), we may reduce equation (39) to the form

$$(40) \quad \frac{2 + 2v^2\Delta^2}{v\Delta'} \left[v\Delta(P\xi - R\eta) - \frac{\Delta'}{2\Delta} \xi + 1 \right] + m\xi - l\eta + v\Delta\xi = 0.$$

Since (ξ', η', ξ') is the point found by using $t = 0$, it is therefore on the conoid; the equation of the tangent plane to the quadric at (ξ', η', ξ') is

$$(41) \quad m(\xi - \xi') - l(\eta - \eta') + v\Delta(\xi - \xi') = 0.$$

By the use of (36) we may reduce (41) to the form

$$(42) \quad m\xi - l\eta + v\Delta\xi = 0.$$

By examining equations (40) and (42) it is readily seen that they are the equations of two planes of the pencil which has as its axis the line

$$(43) \quad v\Delta P\xi - v\Delta R\eta - \frac{\Delta'}{2\Delta}\zeta + 1 = 0, \quad m\xi - \ell\eta + v\Delta\zeta = 0.$$

Therefore the intersection of the two tangent planes is the line (43). We will call this line the normal polar reciprocal line. Numbers proportional to the direction cosines of this line are found to be

$$(44) \quad K\lambda = \ell\Delta' + 2v^2\Delta^2R, \quad K\mu = m\Delta' + 2v^2\Delta^2P, \quad K\nu = 2\Delta^3.$$

Let us find the point in which the normal polar reciprocal line intersects the fixed generator. This point is the point of intersection of the two tangent planes and the plane determined by the normal to the conoid and the fixed generator of the conoid. The equation of the latter plane is

$$(45) \quad m v \Delta \xi - \ell v \Delta \eta - \zeta = 0.$$

Hence the coordinates of the point of intersection of the polar reciprocal line and the generator may be found by solving equations (43) and (45) simultaneously.

we find the coordinates to be

$$(46) \quad \xi = -\frac{\ell}{V\Delta^2}, \quad \eta = -\frac{m}{V\Delta^2}, \quad \zeta = 0.$$

Therefore the equation of the normal polar reciprocal line is

$$(47) \quad \xi = -\frac{\ell}{V\Delta^2} + d(\ell\Delta' + 2V^2\Delta^3R), \quad \eta = -\frac{m}{V\Delta^2} + d(m\Delta' + 2V^2\Delta^3P), \quad \zeta = 2V\Delta^3d.$$

As the point (x, y, z) moves along the generator the normal polar reciprocal line generates a surface whose parametric equations are (47), with parameters V and d , since u is fixed. Eliminating these parameters we obtain the equation of the locus to be

$$(48) \quad 2\ell m \Delta (\xi^2 - \eta^2) - \Delta' (m^2 \xi^2 + \ell^2 \eta^2 + \zeta^2) \\ + [2\ell m \Delta' + 2\Delta (m^2 - \ell^2)] \xi \eta - 2\Delta \zeta = 0.$$

This surface is, of course, a quadric. We shall call it the normal polar reciprocal quadric.

The condition that the normal polar reciprocal line be perpendicular to the generator is, that the sum of the products of corresponding direction cosines be zero.

This condition is

$$(49) \quad \ell(\ell\Delta' + 2v^2\Delta^3R) + m(m\Delta' + 2v^2\Delta^3P) = 0.$$

Solving we find

$$v = \frac{1}{\Delta}, \quad v = -\frac{1}{\Delta}.$$

Therefore on each generator there exists two points on opposite sides of the directrix and equidistant from it, such that the normals of the surface at these points have polar reciprocals perpendicular to the generator.

7. THE NORMAL QUADRIC

The equation of the normal to the right conoid at the point (x, y, z) on a fixed generator by (32) is

$$(32 \text{ bis}) \quad \xi = v\ell + m\tau, \quad \eta = vm - \ell\tau, \quad \zeta = \tau(v\Delta + A\ell).$$

wherein the trihedral of reference is the same as that used in deriving equation (28). The coordinates (ξ, η, ζ) are functions of v and τ only. Therefore we may find the locus of this point by eliminating these parameters.

We find the equation of the locus to be

$$(50) \quad \Delta m \ell (\xi^2 - \eta^2) + \Delta (m^2 - \ell^2) \xi \eta + A \ell (m \xi - \ell \eta) - \mathcal{Z} = 0.$$

This locus is, of course, a quadric surface. We shall call this quadric the normal quadric. To determine the nature of the normal quadric we will reduce the quadric to standard form*. We find that for proper choice of trihedral of reference the equation of the normal quadric may be written in the form

$$(51) \quad \Delta (3\ell^2 m^2 - 1)^{\frac{1}{2}} (\xi^2 - \eta^2) = -2\mathcal{Z}.$$

Therefore the normal quadric surface is a hyperbolic paraboloid.

The equation of the normal polar reciprocal quadric by (48) is

$$(48bis) \quad \begin{aligned} & 2\ell m \Delta (\xi^2 - \eta^2) - \Delta' (m^2 \xi^2 + \ell^2 \eta^2 + \mathcal{Z}) \\ & + [2\ell m \Delta' + 2\Delta^2 (m^2 - \ell^2)] \xi \eta - 2\Delta \mathcal{Z} = 0. \end{aligned}$$

* Snyder and Sisam, Analytic Geometry of Space, New York, Henry Holt, 1914, p. 79.

The equation of the normal quadric for a right conoid may be determined from (50) to be of the form

$$(52) \quad m \ell \Delta (\xi^2 - \eta^2) + \Delta (m^2 - \ell^2) \xi \eta - \mathcal{J} = 0.$$

It is easily seen that the condition that the two equations (48) and (52) be identical is

$$\Delta' = 0.$$

Therefore the condition that the normal polar reciprocal quadric and the normal quadric on a right conoid be identical is that the conoid be a helicoid.

8. CHASLE'S CORRELATION

The tangent plane to the conoid at the point (x, y, z) is

$$(53) \quad m \xi - \ell \eta + \sqrt{\Delta} (\mathcal{J} - u) = 0.$$

wherein the trihedral of reference is the one used in deriving equation (1). We notice that as $\sqrt{\Delta}$ varies along the

fixed generator equation (53) represents a pencil of planes with the line

$$(54) \quad m\xi - \ell\eta = 0, \quad \mathfrak{I} - u = 0,$$

as axis. For every V there is uniquely determined a plane. Therefore the points of contact and the tangent planes are in one to one correspondence.

Consider a pencil of planes $(a\chi) + \lambda(b\chi) = 0$. The cross ratio of the four planes determined by the parameters $\lambda, \lambda_1, \lambda_2, \lambda_3$ is the cross ratio of the parameters.* Therefore the cross ratio of the four tangent planes determined by the values V_1, V_2, V_3, V_4 used in equation (53) is

$$(55) \quad (V_1, V_3 / V_2, V_4)$$

The cross ratio of four points on a line is the cross ratio of their distances from a fixed point on the line. Since V gives the distance of the point of contact from

R. M. Winger, An Introduction to Projective Geometry, New York, D. C. Heath and Company, 1923 p. 93.

the directrix, it is evident that the cross ratio of the points of contact corresponding to the four planes is

$$(56) \quad (V_1 \ V_3 / V_2 \ V_4) \ .$$

Therefore the tangent planes to a conoid along points of a generator, are projectively related to the corresponding points of contact. This is known as Chasle's Correlation.*

9. THE PARAMETRIC OSCULATING RULED SURFACE R_v

The equations of the tangents to the curve are of the form

$$(57) \quad \xi = v\ell + v\ell't, \quad \eta = Au + vm + t(A + vm'), \quad \zeta = u + t.$$

This surface generated by the tangents as we allow the point to move along a fixed generator is, of course, a ruled surface, R_v . We shall show that the surface R_v cannot be a developable surface. A necessary and

*Wilczynski, Geometry, p. 136.

sufficient condition that a surface be developable is

$$D D'' - D'^2 = 0.$$

The values of D , D' and D'' may be determined by the use of (19). We find them to be

$$(58) \quad D = 0, \quad H D = -\Delta, \quad D'' = 0.$$

Therefore the parametric ruled surface cannot be a developable surface.

Any point on the tangent to the curve $V = \text{Const.}$ at the point (x, y, z) on a right conoid is found from (57) to be

$$(59) \quad \xi = v(\ell + \ell' t), \quad \eta = v(m + m' t), \quad \zeta = u + t.$$

The locus of this point is also a surface R_v . To determine the quadric of Lie for this surface, we must find the equations of the asymptotic curves. The coefficients of the defining differential equation are the values of (58). Therefore the asymptotic curves are the parametric curves defined by the differential equation

$$\Delta dv dt = 0.$$

The curves $V = \text{Const.}$ are straight lines, and generators of the ruled surface R_V . Therefore the quadric of Lie of the surface is determined by the tangents to the curved asymptotes $t = \text{Const.}$. The equation of the tangent line to the curve $t = \text{Const.}$ at the point (ξ, η, ζ) is

$$(60) \quad \bar{\xi} = (\ell + \ell' t)(V + d), \quad \bar{\eta} = (m + m' t)(V + d), \quad \bar{\zeta} = u + t.$$

The coordinates $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ are functions of t and d only since we are keeping V fixed. Therefore the locus of the tangents as the point (ξ, η, ζ) moves along the curve $V = \text{Const.}$ may be found by eliminating the parameters. We find this locus to be a quadric

$$(61) \quad m' \bar{\xi} \bar{\zeta} - \ell' \bar{\eta} \bar{\zeta} + m \bar{\xi} - \ell \bar{\eta} = 0.$$

wherein we have made the translation

$$\xi = \bar{\xi}, \quad \eta = \bar{\eta}, \quad \zeta = \bar{\zeta} - u.$$

The quadric (61) is the quadric of Lie of the ruled surface R_V .

The equation of the quadric of Lie for a right

conoid by (29) is

$$(29 \text{ bis}) \quad P \xi \xi - R \eta \xi + m \xi - \ell \eta = 0.$$

We notice that the condition that the two quadrics (29) and (61) be identical is

$$R = \ell' , \quad P = m' .$$

Since

$$R = \ell' - \frac{\ell \Delta'}{2 \Delta} , \quad P = m' - \frac{m \Delta'}{2 \Delta} ,$$

we readily see that the condition that the two quadrics of Lie be identical is

$$\Delta' = 0 .$$

Therefore the condition that the quadric of Lie of the right conoid be identical to the quadric of Lie of the ruled surface R_v is that the conoid be a helicoid.

10. THE FOCAL CONOID

The equation of the tangent to the curve $V = \text{Const.}$ at the point (x, y, z) by (57) is

$$(57 \text{ bis}) \quad \xi = V(l + l't), \quad \eta = V(m + m't) + A(u + t), \quad \zeta = u + t.$$

We wish to determine a curve $V = V(u)$ such that the tangents to the curves $V = V(u)$ are the same as the tangents to the curves $V = \text{Const.}$. The curves $V = V(u)$ so determined are the curves which correspond to the developables of the congruence formed by the tangents to the curves $V = \text{Const.}$.

As the point (x, y, z) moves along the curve $V = V(u)$ on the conoid the point (ξ, η, ζ) will describe a curve $V = V(u)$ on the surface generated by (ξ, η, ζ) . We will find the direction cosines of the tangents to the curves described by the point (ξ, η, ζ) , in the manner just stated, at the point (ξ, η, ζ) . , then determine the condition that these direction cosines be proportional to the direction cosines of the tangents to the curves $V = \text{Const.}$ on the conoid. When the two sets of direction cosines are proportional, the two lines are identical since they both pass through the point (ξ, η, ζ) . Numbers proportional to the direction cosines to the curves $V = V(u)$ on the

surface generated by (ξ, η, ζ) at that point are

$$\begin{aligned} \frac{d\xi}{du} &= v l' + v l'' t + v l' t' + (l + l' t) \frac{dv}{du}, \\ (62) \quad \frac{d\eta}{du} &= A + v m' + v m'' t + (A + v m') t + (m + m' t) \frac{dv}{du}, \\ \frac{d\zeta}{du} &= 1 + t. \end{aligned}$$

Numbers proportional to the direction cosines of the tangents to the curves $v = \text{Const.}$ on the conoid are

$$(63) \quad v l', \quad A + v m', \quad 1.$$

The conditions that the direction cosines of (62) and (63) be proportional are

$$\begin{aligned} v l'' t + (l + l' t) \frac{dv}{du} &= 0, \\ (64) \quad v m'' t + (m + m' t) \frac{dv}{du} &= 0. \end{aligned}$$

We notice that the equations of (64) are homogeneous in $\frac{dv}{du}$ and 1 , also in t and 1 . The conditions that they have a common solution are

$$(65) \quad t(\Delta' + \Delta^3 t) = 0, \quad \frac{dv}{du} (v \Delta' + \Delta \frac{dv}{du}) = 0.$$

Therefore the conditions that the direction cosines of the

tangents to $V = V(u)$ be proportional to the direction cosines of the tangents to $V = \text{const.}$ are

$$(66) \quad t = 0, \quad t = -\frac{\Delta'}{\Delta^3}, \quad v = c, \quad v = \frac{c}{\Delta}.$$

Let us now examine the conditions (66). If $t = 0$ the focal point on the tangent to $V = \text{Const.}$ at the point (x, y, z) is the point itself. Therefore the locus of the focal points, when $t = 0$, is the original spheroid. If $t = -\frac{\Delta'}{\Delta^3}$ the equation of the locus of the point is of the form

$$(67) \quad \xi = v(\ell - \frac{\ell'\Delta'}{\Delta^3}), \quad \eta = v(m - \frac{m'\Delta'}{\Delta^3}) + A(u - \frac{\Delta'}{\Delta^3}), \quad \zeta = u - \frac{\Delta'}{\Delta^3}.$$

This equation may be simplified by the translation of coordinates

$$(68) \quad \bar{\xi} = \xi, \quad \bar{\eta} = \eta - A(u - \frac{\Delta'}{\Delta^3}), \quad \bar{\zeta} = \zeta - u.$$

Equation (67) may now be written in the form

$$(69) \quad \bar{\xi} = v(\ell - \frac{\ell'\Delta'}{\Delta^3}), \quad \bar{\eta} = v(m - \frac{m'\Delta'}{\Delta^3}), \quad \bar{\zeta} = -\frac{\Delta'}{\Delta^3}.$$

The points whose coordinates are given by (69) are, of

course, the focal points* on the tangents to the curves $V = \text{const.}$. The locus of (69) is a ruled surface, and its equations are similar to equations (1) defining the original conoid. Let us see if (69) is also the equation of a conoid.

The conditions that a surface be a conoid are

1. The surface must be a ruled surface;
2. All generators must pass through a fixed straight line;
3. All generators must be parallel to a given plane.

We have previously stated that the locus of (69) is a ruled surface. If we eliminate the parameter V , we get the straight line

$$(70) \quad \left(m - \frac{m'\Delta'}{\Delta^3}\right) \xi - \left(\ell - \frac{\ell'\Delta'}{\Delta^3}\right) \eta = 0, \quad \zeta + \frac{\Delta'}{\Delta^3} = 0.$$

This line always intersects the z -axis, since it is satisfied by $\xi = 0$ and $\eta = 0$. As we allow u to vary we get a family of lines passing through the z -axis. These lines are the generators of the new surface. The generators lie in the plane $\zeta + \frac{\Delta'}{\Delta^3} = 0$. Therefore they are

parallel to the xy -plane. Hence the locus of the point (ξ, η, ζ) is a conoid, which has the z -axis as directrix, and the xy -plane as directing plane. It is evident that this conoid is a right conoid. Therefore the surface generated by the focal points on the tangents to the curves $v = \text{const.}$ is a right conoid. We shall call this conoid the focal conoid.

A generator of the focal conoid is a curve $u = \text{const.}$ Its equations by (69) are

$$(69 \text{ bis}) \quad \xi = v \left(\ell - \frac{\ell' \Delta'}{\Delta^3} \right), \quad \eta = v \left(m - \frac{m' \Delta'}{\Delta^3} \right), \quad \zeta = -\frac{\Delta'}{\Delta^3},$$

wherein u is fixed. The normal polar reciprocal quadric for a right conoid by (48) is

$$(48 \text{ bis}) \quad \begin{aligned} & 2 \ell m \Delta (\xi^2 - \eta^2) - \Delta' (m^2 \xi^2 + \ell^2 \eta^2 - \zeta) \\ & + [2 \ell m \Delta' + 2 \Delta^2 (m^2 - \ell^2)] \xi \eta - 2 \Delta \zeta = 0. \end{aligned}$$

If we substitute (69) into (48), we find that the generator of the focal conoid will lie on the normal polar reciprocal quadric if and only if

$$\Delta' = 0.$$

But it is evident on examination of (67) that the focal conoid is identical to the original conoid when $\Delta' = 0$.
If

$$(71) \quad \Delta'^2 - \Delta^3 = 0,$$

the generators of the focal conoid are tangent to the normal polar reciprocal quadric, if the given conoid is not a helicoid. The equation (71) may be integrated and written in the form

$$(72) \quad \ell = \sin \left[\frac{c_1(u+c_1)+4}{u+c_1} \right], \quad m = \cos \left[\frac{c_1(u+c_1)+4}{u+c_1} \right].$$

A necessary and sufficient condition that a conoid be a plane surface is that it be a developable surface. We have previously seen that the condition that a surface be developable is

$$D D'' - D'^2 = 0.$$

The condition for the focal conoid is

$$\frac{\Delta \Delta'' - 3 \Delta'^2}{\Delta^4} - 1 = 0.$$

From (73) we find

$$\Delta' = \Delta^3(u+c).$$

If we substitute this value into (69) we find

$$(74) \quad \xi = v[\ell - \ell'(u+c)], \quad \eta = v[m - m'(u+c)], \quad \mathfrak{z} = c.$$

Therefore if the focal conoid is a plane, it is a plane parallel to the xy -plane. From (73) we find by repeated integration that the right conoid, whose focal conoid is a plane surface, is the conoid whose parametric equations are given by (1), wherein

$$A = 0,$$

(75)

$$\ell = \left(\frac{u+c_1}{c_2}\right) \sqrt{1-c_3^2} + c_3 \sqrt{1-\left(\frac{u+c_1}{c_2}\right)^2},$$

$$m = \pm \left[\sqrt{1-c_3^2} \sqrt{1-\left(\frac{u+c_1}{c_2}\right)^2} - c_3 \left(\frac{u+c_1}{c_2}\right) \right].$$

11. THE OSCULATING CONOID

The equation of the osculating plane* of the
curves at the point (x, y, z) is

$$(76) \quad \begin{vmatrix} \xi - x, & \eta - y, & \zeta - z \\ \frac{dx}{du}, & \frac{dy}{du}, & \frac{dz}{du} \\ \frac{d^2x}{du^2}, & \frac{d^2y}{du^2}, & \frac{d^2z}{du^2} \end{vmatrix} = 0.$$

The equation of the osculating plane at (x, y, z) of
 $V = \text{Const.}$ on a right conoid is therefore

$$(77) \quad m'' \xi - l'' \eta - V \Delta^3 \left(\zeta - u + \frac{\Delta'}{\Delta^3} \right) = 0.$$

We notice that for any particular value of V equation
(77) represents one member of the pencil of planes having
as its axis the line

$$(78) \quad m'' \xi - l'' \eta = 0, \quad \zeta - u + \frac{\Delta'}{\Delta^3} = 0.$$

* Eisenhart, Geometry, p. 11.

This line intersects the z-axis, since (78) is satisfied by $\xi = 0, \eta = 0$. As we let u vary we get a family of lines passing through the z-axis. These lines lie in the plane $\zeta = u - \frac{\Delta'}{\Delta^3}$. Therefore the lines are parallel to the xy-plane. Hence we see that this locus satisfies the conditions that a ruled surface be a conoid. The equations of the conoid are

$$(79) \quad \xi = \ell'' t, \quad \eta = m'' t, \quad \zeta = u - \frac{\Delta'}{\Delta^3}.$$

We shall call this conoid the osculating conoid, corresponding to the curves $V = \text{const.}$.

The equations of the focal conoid of a right conoid from (67) are

$$(67 \text{ bis}) \quad \xi = v(\ell - \frac{\ell' \Delta'}{\Delta^3}), \quad \eta = v(m - \frac{m' \Delta'}{\Delta^3}), \quad \zeta = u - \frac{\Delta'}{\Delta^3}.$$

The equations of the osculating conoid are

$$(79) \quad \xi = \ell'' t, \quad \eta = m'' t, \quad \zeta = u - \frac{\Delta'}{\Delta^3}.$$

For any fixed value of u the generators of both of these quadrics lie in the same plane, $\zeta = u - \frac{\Delta'}{\Delta^3}$, parallel to the xy-plane. Also both generators pass through the z-axis.

Therefore the condition that the generators of the two quadrics be identical is that their direction cosines be proportional. The generators are therefore identical if and only if

$$(80) \quad \frac{\ell''}{m''} = \frac{\Delta^3 \ell - \ell' \Delta'}{\Delta^3 m - m' \Delta'} .$$

We may readily show that (80) is an identity in u .
Therefore the osculating conoid corresponding to the curves $v = \text{const.}$ and the focal conoid of a right conoid are identical.

13. QUADRIC CONOIDS

One system of differential equations defining the conoid is the system (6), namely

$$(6 \text{ bis}) \quad \chi_{vv} = \frac{v \Delta'}{\Delta} \chi_{uv} - v \Delta \chi_v, \quad \chi_{vv} = 0 .$$

The solutions of (6) are of the form

$$\chi_1 = \chi_1(u, v), \quad \chi_2 = \chi_2(u, v), \quad \chi_3 = \chi_3(u, v), \quad \chi_4 = \chi_4(u, v).$$

Let us choose any point χ , with coordinates $(\chi_1, \chi_2, \chi_3, \chi_4)$.

corresponding to some fixed value of u and v . On the curve $v = \text{const.}$ choose another point X_1 . The coordinates of the point X_1 may be represented by a Taylor's series of the form

$$X_1 = X + X_u \Delta u + X_{uu} \frac{\Delta u^2}{2!} + X_{uuu} \frac{\Delta u^3}{3!} + \dots$$

The point defined by

$$\frac{X_1 - X}{\Delta u}$$

is any point on the secant line through X and X_1 , since this expression is a linear combination of X and X_1 .

The limiting position of this secant line as X_1 approaches X along the curve $v = \text{const.}$ is the tangent line at X .

The limit is

$$\lim_{\Delta u \rightarrow 0} \left(\frac{X_1 - X}{\Delta u} \right) = X_u$$

Therefore the tangent to the curve $v = \text{const.}$ at the point X is determined by the points X and X_u .

Choose another point X on the curve $v = \text{const.}$.

The coordinates of X_1 may be represented by the series

$$X_1 = X + X_u \Delta u + X_{uu} \frac{\Delta u^2}{2!} + X_{uuu} \frac{\Delta u^3}{3!} + \dots$$

The point defined by

$$2 \left(\frac{X_2 - X - X_u \Delta u}{\Delta u^2} \right),$$

lies in the plane determined by the tangent line of the curve $V = \text{const.}$ at X , and the point X_2 . The limit of the above expression as X_2 approaches X along the curve $V = \text{const.}$ is X_{uu} . Therefore the osculating plane to the curve $V = \text{const.}$ at X is determined by the points X , X_u and X_{uu} .*

Suppose that the parametric curves $u = \text{const}$ and $v = \text{const.}$ are the asymptotic curves on the conoid. Then the osculating planes to the curves $V = \text{const.}$ are identical to the tangent planes at the same point. Any point in the tangent plane may be expressed as a linear combination of the points X , X_u , X_v . Therefore when the asymptotic curves are parametric the coordinates of the point X satisfy differential equations of the form

$$\begin{aligned} X_{uu} &= \alpha X_u + \beta X_v + p X, \\ X_{vv} &= \gamma X_u + \delta X_v + q X. \end{aligned}$$

* E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Chicago, Chicago University Press, 1932, p.11.

Therefore when the parametric curves are the asymptotic curves, the coefficient of χ_{uv} in (6) must be zero. Hence the condition for the conoid is

$$\Delta' = 0.$$

Therefore only when the conoid is a helicoid are the generators and the curves which cut the generators at a constant distance from the directrix asymptotic curves.

Now let us make the following transformation of the curvilinear coordinates u and v :

$$(81) \quad \bar{u} = \varphi(u, v), \quad \bar{v} = \psi(u, v).$$

The differential equations (3) under this transformation assume the form

$$(82) \quad \begin{aligned} \chi_{\bar{u}\bar{u}} &= \bar{a}, \chi_{\bar{u}\bar{v}} + \bar{b}, \chi_{\bar{u}} + \bar{c}, \chi_{\bar{v}} + \bar{d}, \chi, \\ \chi_{\bar{v}\bar{v}} &= \bar{a}_2, \chi_{\bar{u}\bar{v}} + \bar{b}_2, \chi_{\bar{u}} + \bar{c}_2, \chi_{\bar{v}} + \bar{d}_2, \chi. \end{aligned}$$

In order that the new parametric curves $\bar{u} = \text{const.}$ and $\bar{v} = \text{const.}$ be asymptotic curves, we must have

$$(83) \quad \bar{a}_1 = 0, \quad \bar{a}_2 = 0.$$

Suppose that conditions (83) are satisfied. Then the conditions that the asymptotic curves be straight lines are

$$(84) \quad \bar{c}_1 = 0, \quad \bar{b}_2 = 0.$$

When conditions (83) and (84) are satisfied the surface is such that the parametric curves are the asymptotic curves and are straight lines. The surface under these conditions is a quadric, since quadric surfaces are the only doubly ruled surfaces.

In particular let us make the transformation

$$\bar{u} = u, \quad \bar{v} = v^2 \Delta.$$

We obtain the system

$$(85) \quad \chi_{\bar{u}\bar{u}} = \frac{v^2(4\Delta^4 + 3\Delta'^2 - 2\Delta\Delta'')}{2\Delta} \chi_{\bar{v}}, \quad \chi_{\bar{v}\bar{v}} = 0.$$

We observe that in (85) $\bar{a}_1 = 0$ and $\bar{b}_2 = 0$ for all values of the parameters, Therefore the curves $\bar{u} = \text{const.}$ are straight lines and, of course, asymptotic curves.

If we equate to zero the coefficient of $X_{\bar{v}}$ in the first of (85), we obtain the condition that the conoid be a quadric, namely

$$(86) \quad 4 \Delta^4 + 3 \Delta'^2 - 2 \Delta \Delta'' = 0.$$

Equation (86) may be considered as a third order differential equation in ℓ . If we integrate this differential equation, we obtain

$$(87) \quad \tanh(\sin^{-1} \ell) = \frac{c_1 u + c_1 c_2 + 4 c_3}{c_1 c_2 u + c_1 c_2 c_3 + 4}.$$

Let α and β be the angles which the generator makes with the x-axis and y-axis respectively. It follows at once that $\sin^{-1} \ell = \beta$. We may now write (87) in the form

$$(88) \quad \tanh \beta = \frac{a u + b}{c u + d}.$$

wherein

$$a = c_1, \quad b = c_1 c_2 + 4 c_3, \quad c = c_1 c_2, \quad d = c_1 c_2 c_3 + 4.$$

Of course, a, b, c , and d are not independent. The relation existing between them is

$$d = a b c - 4(a^2 - c^2).$$

We may rewrite (88) in the form

$$(89) \quad \beta = \frac{1}{2} \log \frac{(c-a)u + d-b}{(c+a)u + d+b} = \frac{1}{2} \log \frac{pu+r}{qu+s} .$$

wherein

$$p = c-a, \quad q = c+a, \quad r = d-b, \quad s = d+b.$$

The relation existing between the constants of (89) is

$$s = \frac{2r(p^2 - q^2) - 32pq}{p^2 - q^2 + 4} .$$

We may write (89) in the form

$$(90) \quad e^{2\beta} = \frac{pu+r}{qu+s}, \quad \alpha = \frac{\pi}{2} - \beta.$$

Therefore the conoid will be a quadric if and only if the equations (90) hold.

We have seen that if the asymptotic parameters are parametric, the homogeneous coordinates of a point $\mathcal{X}$ on a surface are solutions of differential equations

of the form

$$\begin{aligned} \chi_{uu} &= \alpha \chi_u + \beta \chi_v + p \chi, \\ (91) \quad \chi_{vv} &= \gamma \chi_u + \delta \chi_v + q \chi. \end{aligned}$$

And furthermore if we make the transformation of curvilinear coordinates

$$\bar{u} = u, \quad \bar{v} = v^2 \Delta,$$

that the defining differential equations (3) assumes the form

$$(85 \text{ bis}) \quad \chi_{\bar{u}\bar{u}} = v^2 \left(\frac{4\Delta'' + 3\Delta'^2 - 2\Delta\Delta''}{2\Delta} \right) \chi_{\bar{v}}, \quad \chi_{\bar{v}\bar{v}} = 0.$$

Therefore $\bar{u} = \text{const.}$ and $\bar{v} = \text{const.}$ are the asymptotic curves.

Now let us find the condition that the curved asymptotics belong to linear complexes. The curved asymptotics $\bar{v} = \text{const.}$ belong to linear complexes if and only if *

*O. T. Sullivan, Properties of Ruled Surfaces, Whose Asymptotic Curves Belong to Linear Complexes, Transactions of American Mathematical Society, Vol. 15, 1914, p. 190.

$$(92) \quad \frac{\partial^2}{\partial u \partial v} \log \frac{v^2(4\Delta^4 + 3\Delta'^2 - 2\Delta\Delta'')}{2\Delta} = 0.$$

We may readily verify that equation (92) is satisfied identically for all values of u and v . Therefore on the conoid the curved asymptotics $v^2\Delta = \text{const.}$ belong to linear complexes.

Since, from a projective point of view, a conoid is a ruled surface belonging to a linear congruence, we may state the more general theorem: the curved asymptotics on any ruled surface which belongs to a linear congruence belong to linear complexes.

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