ABSTRACT

ASYMPTOTIC DISTRIBUTIONS IN SOME NON-REGULAR STATISTICAL PROBLEMS

by B. L. S. Prakasa Rao

As the title indicates, we consider here two different problems. The first problem deals with estimation of distributions with unimodal density and estimation of distributions with monotone failure rate. The second problem deals with the estimation of the location of the cusp of a continuous density.

Recently Marshall and Proschan (Ann. Math. Statist. **36**, 69-77) have derived the maximum likelihood estimates for distributions with monotone failure rate and they have shown that these estimators are consistent. In Chapter 2, we obtain the asymptotic distribution of these estimators using the results of Chernoff in his paper on the estimation of mode. The estimation problem is reduced at first to that of a stochastic process and the asymptotic distribution is obtained by means of theorems on convergence of distributions of stochastic processes. Similar results are obtained for distributions with unimodal densities in Chapter 1.

Under the usual regularity conditions on the density, it is well known that the maximum likelihood estimator is consistent, asymptotically normal, and asymptotically efficient. Unfortunately, these conditions are not satisfied for distributions like double-exponential with location

parameter θ. Daniels, in his paper in the fourth Berkeley Symposium, has shown that there exist modified maximum likelihood estimators which are asymptotically efficient for the family of densities $f(x,\theta)=C\exp\{-|x - \theta|^k\}$, where x and θ range over $(-\infty,\infty)$ and $\frac{1}{2} < k < 1$. In Chapter 3, we show that hyper-efficient estimators exist for θ when $0 < k < \frac{1}{2}$ and θ is restricted to a finite interval for a wider class of densities. We relate its asymptotic distribution to the distribution of the position of the maximum for a non-stationary Gaussian process. The estimation problem is reduced to that of a stochastic process and the asymptotic distribution is obtained by using theorems on convergence of distributions of stochastic processes in $C[0,1]$.

ASYMPTOTIC DISTRIBUTIONS IN SOME NON-REGULAR STATISTICAL PROBLEMS

By

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To my grand-parents and parents

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CHAPTER 0

INTRODUCTION

As the title of the thesis indicates appropriately, we consider here two different problems. The first problem deals with estimation of distributions with unimodal density and estimation of distributions with monotone failure rate. The second problem deals with the estimation of the location of the cusp of a continuous density.

Grenander [10] derived the maximum likelihood estimators for distributions with unimodal density and for distributions with monotone failure rate. He did not derive the asymptotic distributions for these estimators. It is interesting to note that the maximum likelihood estimators can also be derived by methods used in Brunk [2] or in van Eeden [19]. Recently Marshall and Proschan [14] showed that the maximum likelihood estimator is consistent. In Chapters 1 and 2, we derive the asymptotic distributions of the estimators in both cases. Even though we do not obtain their distributions explicitly, we show that they are related to a solution of the heat equation as was done in Chernoff [4] in the case of the estimation of the mode.

Under the usual regularity conditions on the density, it is well known that maximum likelihood estimator is consistent and asymptotically normal. See Cramer [6], Kulldorf [12], Gurland [11], etc. Their estimators are also asymptotically efficient. In certain cases like double exponential distribution with location parameter θ, these regularity conditions are not satisfied. Daniels [7] has shown that there exist modified maximum likelihood estimators (M.L.E.) which are asymptotically efficient for the family of densities

$$f(x,\theta) = \frac{1}{2\Gamma(1+\frac{1}{k})} \exp\{-|x-\theta|^k\}, \ \frac{1}{2} < k < 1 ,$$

$$-\infty < x < \infty, \quad -\infty < \theta < \infty .$$

Recently, Huber has generalized Daniels' results in his paper presented at the fifth Berkeley Symposium and he has shown that M.L.E. is consistent and asymptotically normal for cusps of order between $\frac{1}{2}$ and 1. We show in Chapter 3 that hyper-efficient estimators exist, when the exponent k lies between 0 and $\frac{1}{2}$ and θ is in a finite interval, for a wider class of densities. We relate its asymptotic distribution to the distribution of the position of the maximum for a non-stationary Gaussian process. In fact, it can be shown that Bayes estimators for smooth prior densities for θ are also hyper-efficient for the above class of densities and asymptotically the estimation problem is equivalent to estimation of location parameter for a non-stationary Gaussian process. It should be mentioned here that some of the Bayes estimators are asymptotically better than the M.L.E.

CHAPTER 1

ESTIMATION OF A UNIMODAL DENSITY

1.1 Introduction:

Given a set of observations $X_1, \ldots, X_n$ from a common distribution F, it is natural to estimate F by the usual empirical distribution function in the absence of additional information. However, one would not use such an estimate if there is some a priori information about the distribution F. In this chapter, we shall investigate the problem of estimation when F is known to be unimodal. Grenander [10] derived the maximum likelihood estimator for f, where f is the density of F. Even though it is well known that the maximum likelihood estimator (M.L.E.) of f is consistent, we shall give a proof for completeness. We shall relate its asymptotic distribution to a solution of a heat equation as was done by Chernoff [4] in the case of the estimation of the psuedo-mode.

Section 1.2 deals with the maximum likelihood estimation of the density. The consistency of the M.L.E. is proved in Section 1.3. Some results related to the asymptotic properties of the M.L.E. are obtained in Section 1.4. In Section 1.5, the estimation problem is reduced to that of a stochastic process. We obtain the asymptotic distribution of the M.L.E. in Section 1.6.

1.2 Maximum likelihood estimation of the density:

We shall assume that the distribution $F(x)$ is absolutely continuous with density f which is unimodal with known mode μ. If μ is the mode and $F = \alpha F_+ + (1 - \alpha)F_-$ where F_+ is the conditional distribution on $[\mu,\infty)$ and F_- is the conditional distribution on $(-\infty, \mu)$, then it can be shown that the M.L.E. of F is $\hat{\alpha}\hat{F}_+ + (1 - \hat{\alpha})\hat{F}_-$ where $\hat{\alpha}$ is the sample proportion on $[\mu,\infty)$ and $\hat{F}_+$ and $\hat{F}_-$ are the M.L.E.'s of the conditional distributions F_+ and F_- respectively. Let f_+ and f_- denote the densities of F_+ and F_- respectively and let $\hat{f}_+$ and $\hat{f}_-$ denote their M.L.E.'s. We shall show later on that for any $\xi \geq \mu$, $[\hat{f}_+(\xi) - f_+(\xi)] = O_p(n^{-1/3})$ and for any $\xi < \mu$, $[\hat{f}_-(\xi) - f_-(\xi)] = O_p(n^{-1/3})$. Since $\hat{\alpha} - \alpha = O_p(n^{-1/2})$ we get that for any ξ, $\hat{f}(\xi) - f(\xi) = O_p(n^{-1/3})$. Therefore it is sufficient to obtain the M.L.E. of f_+. Let us assume that $\mu = 0$ without loss of generality. Therefore $F(x) = 0$ for $x < 0$. Since F is unimodal, f is non-increasing for $x \geq 0$.

Suppose $X_1 \leq \cdots \leq X_n$ are n observations obtained by ordering a random sample of size n from the population with unknown distribution F. Let $\mathcal{F}$ denote the class of unimodal distributions F. Let $X_0 = 0$. Let

$$L(F) = \sum_{i=1}^{n} \log f(X_i) \tag{1.2.1}$$

be the logarithm of the likelihood for $F \in \mathcal{F}$. For any $F \in \mathcal{F}$, define F^* to be the distribution with density

$$f^*(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ Cf(X_i) & \text{for } X_{i-1} < x \leq X_i,\ 1 \leq i \leq n \\ 0 & \text{for } x > X_n \end{cases} \qquad (1.2.2)$$

where C is a normalizing constant. It is easy to see that $C \geq 1$ and $L(F^*) = C^n L(F)$. Therefore, $L(F) \leq L(F^*)$. In other words, the M.L.E. $\hat{f}_n(x)$ of $f(x)$ will be a step function with steps at the order statistics $X_1, X_2, \ldots, X_n$.

Hence the problem of maximizing $L(F)$ for $F \in \mathcal{F}$ reduces to the problem of determining numbers $f_1, f_2, \ldots, f_n$ such that

(i) $f_1 \geq f_2 \geq \cdots \geq f_n$,

(ii) $X_1 f_1 + (X_2 - X_1) f_2 + \ldots + (X_n - X_{n-1}) f_n = 1$, and

(iii) $\prod_1^n f_i$ is maximal. (1.2.3.)

This has been done in Grenander [10]. It can also be done as an application of results obtained by Brunk [2] or van Eeden [19].

This yields for the M.L.E. of $f(x)$,

$$\hat{f}_n(x) = \begin{cases} \hat{f}_n(X_{i+1}) & \text{for } X_i < x \leq X_{i+1},\ 0 \leq i \leq n-1 \\ 0 & \text{for } x \leq X_0 \text{ or } x > X_n \end{cases} \qquad (1.2.4)$$

where

$$\hat{f}_n(X_i) = \underset{n \geq v \geq i}{\text{Max}} \ \underset{0 \leq u \leq i-1}{\text{Min}} \ \frac{v - u}{n(X_v - X_u)} \,. \qquad (1.2.5)$$

The estimator $\hat{f}_n(x)$ can also be written in the form

$$\hat{f}_n(x) = \begin{cases} \underset{v > x}{\text{Sup}} \ \underset{u < x}{\text{Inf}} \ \dfrac{F_n(v) - F_n(u)}{v - u} & \text{for } X_0 < x \leq X_n \\ 0 & \text{otherwise.} \end{cases}$$

(1.2.6)

In other words, <u>the M.L.E.</u> $\hat{f}_n(x)$ <u>is the slope of the concave majorant of empirical distribution</u> F_n <u>at x</u>.

<u>1.3 Consistency of the maximum likelihood estimator</u>:

<u>Theorem 1.3.1</u>.

<u>For every</u> x

$\hat{f}_n(x) \to f(x)$ <u>in probability</u> as $n \to \infty$.

<u>Proof</u>: If $x < 0$, then

$$\hat{f}_n(x) = 0 \text{ for all } n \text{ and } f(x) = 0.$$

Therefore $\hat{f}_n(x) \to f(x)$ in probability.

Let $x \geq 0$. Let $\hat{F}_n(x)$ denote the smallest concave majorant of $F_n(x)$. For any $\varepsilon > 0$,

$$P[\sqrt{n} \sup_{-\infty < x < \infty} |F_n(x) - F(x)| < \varepsilon] \to \ell(\varepsilon) \quad \text{as } n \to \infty \tag{1.3.1}$$

where

$$\ell(\varepsilon) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2\varepsilon^2 j^2}$$

by the Kolmogorov-Smirnov theorem.

Let us choose $\delta > 0$. Then there exists an integer $N_0(\delta)$ such that for every $n > N_0(\delta)$,

$$P[F_n(x) < F(x) + \varepsilon n^{-1/2} \text{ and } F_n(x) > F(x) - \varepsilon n^{-1/2}$$
$$\text{for all } x] > \ell(\varepsilon) - \frac{\delta}{2} \text{ by (1.3.1)}.$$

Since $\hat{F}_n(x) \geq F_n(x)$ for all x by the definition of $\hat{F}_n$,

$F_n(x) > F(x) - \varepsilon n^{-1/2}$ for all x

$$\Rightarrow \hat{F}_n(x) > F(x) - \varepsilon n^{-1/2} \text{ for all } x. \tag{1.3.2}$$

Since $\hat{F}_n(x)$ is the smallest concave majorant of $F_n(x)$ and

$F(x)$ is concave,

$$F_n(x) < F(x) + \varepsilon n^{-1/2} \quad \text{for all } x$$

$$\Rightarrow \hat{F}_n(x) < F(x) + \varepsilon n^{-1/2} \quad \text{for all } x. \qquad (1.3.3)$$

From (1.3.1) - (1.3.3) it follows that for $n > N_0(\delta)$,

$$P[\hat{F}_n(x) \leq F(x) + \varepsilon n^{-1/2} \quad \text{for all } x \text{ and}$$

$$\hat{F}_n(x) > F(x) - \varepsilon n^{-1/2} \quad \text{for all } x] > \ell(\varepsilon) - \frac{\delta}{2}.$$

Therefore, for all $n > N_0(\delta)$,

$$P[\frac{\hat{F}_n(x + n^{-1/4}) - \hat{F}_n(x)}{n^{-1/4}} \leq \frac{F(x + n^{-1/4}) - F(x)}{n^{-1/4}} + 2\varepsilon n^{-1/4}$$

and

$$\frac{\hat{F}_n(x - n^{-1/4}) - \hat{F}_n(x)}{-n^{-1/4}} > \frac{F(x - n^{-1/4}) - F(x)}{-n^{-1/4}} - 2\varepsilon n^{-1/4} \text{ for all } x]$$

$$> \ell(\varepsilon) - \frac{\delta}{2}. \qquad (1.3.4)$$

Let $\zeta > 0$. Since $f(x)$ exists for all x, ther exists an integer $N_1(\zeta)$ such that for every $n > N_1(\zeta)$,

$$\left| \frac{F(x + n^{-1/4}) - F(x)}{n^{-1/4}} - f(x) \right| < \zeta,$$

and

$$\left| \frac{F(x - n^{-1/4}) - F(x)}{-n^{-1/4}} - f(x) \right| < \zeta. \qquad (1.3.5)$$

Now (1.3.4), (1.3.5) together imply that for every

$$n > \text{Max}(N_0(\delta), N_1(\zeta)),$$

$$P[\frac{\hat{F}_n(x + n^{-1/4}) - \hat{F}_n(x)}{n^{-1/4}} < f(x) + \zeta + 2\varepsilon n^{-1/4}$$

and

$$\frac{\hat{F}_n(x - n^{-1/4}) - \hat{F}_n(x)}{-n^{-1/4}} > f(x) - \zeta - 2\varepsilon n^{-1/4}] > \ell(\varepsilon) - \frac{\delta}{2} .$$

Now, since $\hat{f}_n(x)$ is the left-hand derivative of $\hat{F}_n(x)$ at x, it follows that for every $n > \text{Max}\ (N_0(\delta),\ N_1(\delta))$,

$$P[|\hat{f}_n(x) < f(x) + \zeta + 2\varepsilon n^{-1/4} \text{ and } \hat{f}_n(x) > f(x) - \zeta - 2\varepsilon n^{-1/4}] > \ell(\varepsilon) - \frac{\delta}{2} .$$

In other words

$$P[\hat{f}_n(x) - f(x)| < \zeta + 2\varepsilon n^{-1/4}] > \ell(\varepsilon) - \frac{\delta}{2} \qquad (1.3.6)$$

for every $n > \text{Max}\ (N_0(\delta), N_1(\zeta))$. We choose ε such that $\ell(\varepsilon) > 1 - \frac{\delta}{2}$ and n such that

$$2\varepsilon n^{-1/4} < \zeta \text{ or equivalently } n > (\frac{2\varepsilon}{\zeta})^4 = N_2(\zeta, \delta) .$$

Hence, from (1.3.6), it follows that for every

$$n > \text{Max}(N_0(\delta), N_1(\zeta), N_2(\zeta,\delta)),$$

$$P[|\hat{f}_n(x) - f(x)| < 2\zeta] > 1 - \delta .$$

Therefore,

$$\hat{f}_n(x) \rightarrow f(x) \text{ in probability as } n \rightarrow \infty .$$

1.4 Some results related to the asymptotic properties of the maximum likelihood estimator:

Before we proceed to obtain the asymptotic distribution of M.L.E. $\hat{f}_n(x)$, we shall prove some lemmas which simplify the problem. We shall assume that f is differentiable at the point x and that $f'(x)$ is different from zero.

Let $f^*_{n,c}(x)$ denote the slope of the concave majorant of F_n restricted to the interval $[x - 2cn^{-1/3}, x + 2cn^{-1/3}]$, evaluated at x. We shall now prove that

Lemma 1.4.1.

There is a function ϕ such that

(i) $\overline{\lim_n}\, P[f^*_{nc}(x) \neq \hat{f}_n(x)] \leq \phi(c)$

and

(ii) $\phi(c) \to 0$ as $c \to \infty$.

Proof: It is enough to prove that

$$\underline{\lim}_n\, P[F_n(y) \leq F_n(x + cn^{-1/3}) - (x + cn^{-1/3} - y)(f(x) - An^{-1/3})$$

$$\text{for all } y \leq x \text{ and all } y \geq x + 2cn^{-1/3};$$

$$F_n(y) \leq F_n(x - cn^{-1/3}) + (y - x + cn^{-1/3})(f(x) + An^{-1/3})$$

$$\text{for all } y \geq x \text{ and all } y \leq x - 2cn^{-1/3}]$$

$$\geq \psi(c,A) \tag{1.4.1}$$

where $\psi(c,A) \to 1$ as $c \to \infty$ and $A = -cf'(x)$.

We shall show that for $A = -cf'(x)$,

$$\lim_{c\to\infty} \underline{\lim}_n\, P[F_n(y) \leq F_n(x+cn^{-1/3}) - (x+cn^{-1/3}-y)(f(x)-An^{-1/3})$$

$$\text{for all } y \leq x \text{ and all } y \geq x + 2cn^{-1/3}] = 1. \tag{1.4.2}$$

In a similar way, it can be shown that

$$\lim_{c\to\infty} \underline{\lim}_n\, P[F_n(y) \leq F_n(x-cn^{-1/3}) + (y-x+cn^{-1/3})(f(x)+An^{-1/3})$$

$$\text{for all } y \geq x \text{ and all } y \leq x - 2cn^{-1/3}] = 1. \tag{1.4.3}$$

(1.4.2) and (1.4.3) together imply (1.4.1) which in turn proves the lemma.

We note that $A > 0$ since $f'(x) < 0$. Let us obtain a lower bound for

$$p \equiv P[F_n(y) \leq F_n(x+cn^{-1/3}) - (x+cn^{-1/3}-y)(f(x)-An^{-1/3})$$

$$\text{for all } y \leq x]. \tag{1.4.5}$$

Let

$$I_n(x) = n[F_n(x + cn^{-1/3}) - cn^{-1/3}(f(x) - An^{-1/3}) - F_n(x)]$$

$$= n[F(x + cn^{-1/3}) - F(x) - cn^{-1/3}(f(x) - An^{-1/3})]$$

$$+ n[\{F_n(x + cn^{-1/3}) - F_n(x)\} - \{F(x + cn^{-1/3}) - F(x)\}]$$

$$= n[F(x) + cn^{-1/3}f(x) + \tfrac{1}{2}c^2n^{-2/3}f'(x) + o(n^{-2/3})$$

$$- F(x) - cn^{-1/3}f(x) + cAn^{-2/3}]$$

$$+ n[\{F_n(x + cn^{-1/3}) - F_n(x)\} - \{F(x + cn^{-1/3}) - F(x)\}]$$

$$= n^{1/3}[\tfrac{1}{2}c^2f'(x) + cA + o(1)]$$

$$+ n[\{F_n(x + cn^{-1/3}) - F_n(x)\} - \{F(x + cn^{-1/3}) - F(x)\}]$$

$$= n^{1/3}B_n + c^{1/2}n^{1/3}[f(x)]^{1/2}V_n \tag{1.4.6}$$

where

(i) $B_n = \frac{1}{2}c^2f'(x) + cA + o(1) = -\frac{c^2}{2}f'(x) + o(1)$

and

(ii) $V_n = c^{-1/2}[f(x)]^{-1/2}n^{2/3}[\{F_n(x + cn^{-1/3}) - F_n(x)\}$

$$- \{F(x + cn^{-1/3}) - F(x)\}]. \tag{1.4.7}$$

Obviously,

$$E(V_n) = 0,$$

and

$$\mathrm{Var}(V_n) = \frac{n[F(x + cn^{-1/3}) - F(x)][1 + \theta(n^{-1/3})]}{cn^{2/3}f(x)}$$

$$= \frac{n^{1/3}}{cf(x)}[cn^{-1/3}f(x) + 0(n^{-1/3})][1 + o(n^{-1/3})]$$

$$= 1 + o(1).$$

Therefore from (1.4.6) it follows that

$$I_n(x) = n^{1/3}B_n + c^{1/2}n^{1/3}[f(x)]^{1/2}V_n$$

where

$$E(V_n) = 0 \text{ and } \mathrm{Var}(V_n) = 1 + o(1). \tag{1.4.8}$$

Let $0 < \lambda < 1$. By Chebyshev's inequality

$$P[I_n(x) > -\lambda\frac{c^2}{2}f'(x)n^{1/3}] = P[B_n + c^{1/2}[f(x)]^{1/2}V_n > -\lambda\frac{c^2}{2}f'(x)]$$

$$= P[V_n > \{(1-\lambda)\frac{c^2}{2}f'(x) + o(1)\}\frac{1}{c^{1/2}[f(x)]^{1/2}}]$$

$$\geq 1 - \frac{[1 + o(1)]cf(x)}{[(1 - \lambda)\frac{c^2}{2}f'(x) + o(1)]^2}. \tag{1.4.9}$$

As $n \to \infty$,

$$\frac{[1 + o(1)]cf(x)}{[(1 - \lambda)\frac{c^2}{2}f'(x) + o(1)]^2} \to \frac{cf(x)}{[(1 - \lambda)\frac{c^2}{2}f'(x)]^2}$$

$$= \frac{4f(x)}{(1 - \lambda)^2c^3[f'(x)]^2}.$$

Let

$$Q(c) = \frac{4f(x)}{(1 - \lambda)^2c^3[f'(x)]^2}.$$

From (1.4.9), it follows that there exists an integer N_1 such that for every $n > N_1$,

$$P[I_n(x) > -\lambda\frac{c^2}{2}f'(x)n^{1/3}] \geq 1 - \frac{3}{2}Q(c). \tag{1.4.10}$$

From (1.4.5), we have

$$p = P[F_n(x + cn^{-1/3}) - (x + cn^{-1/3} - y)(f(x) - An^{-1/3}) - F_n(y) \geq 0$$

$$\text{for all } y \leq x]$$

$$= P[\{F_n(x+cn^{-1/3})-F_n(y)\} - (x + cn^{-1/3}- y)(f(x)-An^{-1/3}) \geq 0 \quad \text{for all } y \leq x]$$

$$= P[n\{F_n(x+cn^{-1/3})- F_n(x)\} - cn^{2/3}(f(x) - An^{-1/3}) \geq n\{F_n(y)-F_n(x)\} - n(y - x)(f(x)-An^{-1/3}) \text{ for all } y \leq x]$$

$$= P[I_n(x) \geq n[\{F_n(y)-F_n(x)\} - (y - x)(f(x)-An^{-1/3})] \quad \text{for all } y \leq x]$$

$$\geq P[n\{F_n(y) - F_n(x)\} - n(y - x)(f(x)-An^{-1/3}) \leq -\lambda\frac{c^2}{2}f'(x)n^{1/3} \quad \text{for all } y \leq x] - \frac{3}{2}Q(c) \qquad (1.4.11)$$

for every $n > N_1$ by (1.4.10).

Let $F^*(y)$ be the distribution defined by its density,

$$f^*(y) = \begin{cases} 0 & \text{for } y < -a \\ f(x) & \text{for } -a \leq y \leq x \\ f(y) & \text{for } y > x \end{cases} \qquad (1.4.12)$$

where a is chosen so as to make $F^*(y)$ a distribution function.

Since $F^*(y) \geq F(y)$ for all y,

$$P_F[n\{F_n(y)-F_n(x)\} - n(y - x)(f(x)-An^{-1/3}) \leq -\lambda\frac{c^2}{2}f'(x)n^{1/3} \quad \text{for all } y \leq x]$$

$$\geq P_{F^*}[n\{F_n(y)-F_n(x)\} - n(y - x)(f(x)-An^{-1/3}) \leq -\lambda\frac{c^2}{2}f'(x)n^{1/3} \quad \text{for all } y \leq x] \qquad (1.4.13)$$

where P_F denotes the probability when F is the underlying distribution.

(1.4.11) and (1.4.13) imply that for every $n > N_1$,

$$p \geq P_{F^*}[n\{F_n(x)-F_n(y)] - n(x-y)(f(x)-An^{-1/3}) \geq \lambda\frac{c^2}{2}f'(x)n^{1/3}$$
$$\text{for all } y \leq x] - \frac{3}{2}Q(c)$$

$$= P_{F^*}[n\{F_n(x)-F_n(y)\} - n(x-y)(f(x)-An^{-1/3}) - \lambda\frac{c^2}{2}f'(x)n^{1/3} \geq 0$$
$$\text{for all } y \leq x] - \frac{3}{2}Q(c). \quad (1.4.14)$$

Let $Z(B,t)$ for $t \geq 0$ be distributed as

$$N(tq) - nt(f(x) - An^{-1/3}) - n^{1/3}\lambda\frac{c^2}{2}f'(x) \quad (1.4.15)$$

where $N(q)$ is a Poisson process with parameter

$q = [nf(x) - Bn^{1/2}]$ and B is a constant > 0.

From (1.4.13) we have for all $n > N_1$,

$$p \geq P_{F^*}[n\{F_n(x)-F_n(y)\} - n(x-y)(f(x)-An^{-1/3}) - n^{1/3}\lambda\frac{c^2}{2}f'(x) \geq 0$$
$$\text{for all } y \leq x] - \frac{3}{2}Q(c)$$

$$= E_{F^*}\{P_{F^*}[n\{F_n(x)-F_n(y)\} - n(x-y)(f(x)-An^{-1/3})$$
$$-n^{1/3}\lambda\frac{c^2}{2}f'(x) \geq 0 \text{ for all } y \leq x] \,|\, F_n(x)\} - \frac{3}{2}Q(c)$$

$$= E_{F^*}\{P[Z(B,t) \geq 0 \text{ for all } t \text{ such that } 0 \leq t \leq x+a] \,|\, N((x+a)q)$$
$$= nF_n(x)]\} - \frac{3}{2}Q(c).$$

Let

$$T_n = nF_n(x) - n(x+a)(f(x) - An^{-1/3}) - n^{1/3}\lambda\frac{c^2}{2}f'(x).$$

From our earlier remarks it follows that for every $n > N_1$,

$$p \geq E_{F^*}\{P[Z(B,t) \geq 0 \text{ for } 0 \leq t \leq x+a] \,|\, Z(B,x+a) = T_n\}$$
$$- \frac{3}{2}Q(c)$$

$$\geq E_{F^*}\{P[Z(B,t) \geq 0 \text{ for all } t \geq 0] \,|\, Z(B,x+a) = T_n\} - \frac{3}{2}Q(c).$$
$$(1.4.16)$$

Now

$$P[Z(B,t) \geq 0 \text{ for all } t \geq 0] \leq$$

$$E_{F*}[P\{Z(B,t) \geq 0 \text{ for all } t \geq 0\} | Z(B,x+a) = T_n]$$

$$+ P_{F*}[Z(B,x+a) < T_n].$$

Therefore from (1.4.16), it follows that

$$p \geq P[Z(B,t) \geq 0 \text{ for all } t \geq 0] - P_{F*}[Z(B,x+a) < T_n] - \frac{3}{2}Q(c)$$

for all $n > N_1$. (1.4.17)

Now

$$P_{F*}[Z(B,x+a) < T_n] = P_{F*}[N(q(x+a)) < nF_n(x)].$$

By Chebyshev's inequality

$$P_{F*}[N(q(x+a)) < nF_n(x)] \leq \frac{F(x)(1-F(x))}{B^2}.$$

Let us now choose an $\varepsilon > 0$. Then there exists a constant $B_0 > 0$ such that

$$P_{F*}[N(q(x+a)) < nF_n(x)] < \varepsilon$$

where $q = nf(x) - B_0 n^{1/2}$.

Hence, from (1.4.17), we have for all $n > N_1$

$$p \geq P[Z(B_0,t) \geq 0 \text{ for all } t \geq 0] - \frac{3}{2}Q(c) - \varepsilon. \quad (1.4.18)$$

It is obvious that for any real number μ

$$E[\exp\{\mu Z(B_0,t)\}] = \exp\{tq(e^{\mu}-1) - \mu tr - \mu\lambda n^{1/3}\frac{c^2}{2}f'(x)\}$$

where

$$q = nf(x) - B_0 n^{1/2},$$

and

$$r = nf(x) - An^{2/3}.$$

Therefore

$$E[\exp\{\mu Z(B_0,t) - tq(e^{\mu}-1) + \mu tr + \lambda\mu n^{1/3}\frac{c^2}{2}f'(x)\}] = 1$$

for all μ.

Let

$$T = \inf \{t : Z(B_0, t) = 0\}.$$

By Wald's fundamental identity in the continuous parameter case, (See A. Dvoretzky, J. Kiefer and J. Wolfwitz [9]) it follows that

$$E[\exp\{\mu z(B_0, T) - T\{q(e^{\mu}-1) - \mu r\} + \lambda\mu n^{1/3}\frac{c^2}{2}f'(x)\}] = 1.$$

Let us choose μ_0 such that $q(e^{\mu_0} - 1) = \mu_0 r$.

Now we have

$$E[\exp\{\mu_0 Z(B_0, t) + \lambda\mu_0 n^{1/3}\frac{c^2}{2}f'(x)\}] = 1.$$

This implies that

$$P[Z(B_0, T) = 0] \leq \exp\{-\lambda\mu_0 n^{1/3}\frac{c^2}{2} f'(x)\}$$

In other words,

$$P[Z(B_0, t) \geq 0 \text{ for all } t \geq 0]$$

$$\geq 1 - \exp\{-\lambda\mu_0 n^{1/3}\frac{c^2}{2} f'(x)\} \quad (1.4.19)$$

From (1.4.18) and (1.4.19), we have

$$p \geq 1 - \exp\{-\lambda\mu_0 n^{1/3}\frac{c^2}{2} f'(x)\} - \frac{3}{2}Q(c) - \varepsilon \quad (1.4.20)$$

for every $n > N_1$.

Since $q(e^{\mu_0} - 1) = \mu_0 r$ where $q = nf(x) - B_0 n^{1/2}$ and $r = nf(x) - An^{2/3}$,

$$\mu_0 = \frac{-2An^{-1/3}}{f(x)} + o(n^{-1/3}). \quad (1.4.21)$$

From (1.4.20) and (1.4.21), it follows that for every $n > N_1$

$$p \geq 1 - \exp\{-\lambda[\frac{-2An^{-1/3}}{f(x)} + o(n^{-1/3})]n^{1/3}\frac{c^2}{2} f'(x)\} - \varepsilon - \frac{3}{2}Q(c)$$

$$= 1 - \exp\{-\lambda[\frac{2cf'(x)}{f(x)} + o(1)]\frac{c^2}{2} f'(x)\} - \varepsilon - \frac{3}{2}Q(c). \quad (1.4.22)$$

Therefore,

$$\underline{\lim_{n}}\ p \geq 1 - \exp\{-\lambda c^3 \frac{f'^2(x)}{f(x)}\} - \varepsilon - \frac{3}{2} Q(c).$$

Since ε is arbitrary, we have

$$\underline{\lim_{n}}\ p \geq 1 - \exp\{-\lambda\ c^3 \frac{f'^2(x)}{f(x)}\} - \frac{3}{2} Q(c).$$

Let us now take limit as $c \to \infty$.

Since $Q(c) \to 0$ as $c \to \infty$,

$$\lim_{c}\ \underline{\lim_{n}}\ p = 1. \tag{1.4.23}$$

Let

$$s = P[F_n(y) \leq F_n(x + cn^{-1/3}) - (x + cn^{-1/3} - y)(f(x) - An^{-1/3})$$
$$\text{for all } y \geq x + 2c\, n^{-1/3}].$$

It can be shown, by the same methods which were used in proving (1.4.23), that

$$\lim_{c}\ \underline{\lim_{n}}\ s = 1. \tag{1.4.24}$$

(1.4.23) and (1.4.24) together prove (1.4.2).

<u>Lemma 1.4.2</u>.

<u>Suppose that</u> $\{X_{nc}\}$, $\{X_n\}$ <u>are collections of random variables such that</u>

(i) $\lim\limits_{c\to\infty}\ \overline{\lim\limits_{n\to\infty}}\ P[X_{nc} \neq X_n] = 0$

(ii) $\lim\limits_{c\to\infty}\ P[X_c \neq X] = 0$

<u>and</u>

(iii) X_{nc} <u>converges to</u> X_c <u>in law as</u> $n \to \infty$ <u>for every</u> c.

Then

X_n converges to X in law.

Proof:

Let $L(X,Y)$ denote the Levy distance between the distribution functions of X and Y.

Since $L(X,Y) \leq P[X \neq Y]$, we have

$$\text{(i)} \quad \lim_{c\to\infty} \overline{\lim_{n\to\infty}} L(X_{nc}, X_n) = 0$$

and $$\tag{1.4.25}$$

$$\text{(ii)} \quad \lim_{c\to\infty} L(X_c, X) = 0 \ .$$

Since L is a metric,

$$0 \leq L(X_n,X) \leq L(X_n,X_{nc}) + L(X_{nc},X_c) + L(X_c,X).$$

Taking limit as $n\to\infty$, we have for any fixed c

$$0 \leq \overline{\lim_n} L(X_n,X) \leq \overline{\lim_n} L(X_n,X_{nc}) + \overline{\lim_n} L(X_{nc},X_c) + L(X_c,X)$$

$$= \overline{\lim_n} L(X_n,X_{nc}) + L(X_c,X)$$

since $\overline{\lim_n} L(X_{nc},X_c) = 0$ by (iii) of the hypothesis. (Convergence in Levy distance is equivalent to convergence in law.) So we have for any c

$$0 \leq \overline{\lim_n} L(X_n,X) \leq \overline{\lim_n} L(X_n,X_{nc}) + L(X_c,X).$$

The expression in the right hand side of the above inequality is equal to zero by (1.4.25). Therefore, $\lim_n L(X_n,X) = 0$. In otherwords X_n converges to X in law.

As a consequence of lemmas 1.4.1, 1.4.2, it follows that it is enough to find the asymptotic distribution of

$f^*_{n,c}(x)$ as $n\to\infty$ and then prove a result analogous to that in lemma 1.4.1 for limiting random variables in order to obtain the asymptotic distribution of $\hat{f}_n(x)$.

1.5 Reduction to a problem in stochastic processes:

In this section, we shall reduce the problem of calculating the asymptotic distribution of the slope of the concave majorant of $F_n(Y)$ over $[\xi-2c\,n^{-1/3},\ \xi+2c\,n^{-1/3}]$ at $Y=\xi$ to the corresponding problem of a Wiener process over $[-2c,2c]$ after suitable normalization. We assume that f is differentiable at ξ with $f'(\xi)\neq 0$. Let us now consider $F_n(\xi+\delta)-F_n(\xi)$ for δ in $[-2cn^{-1/3},2cn^{-1/3}]$.

Now

$$
\begin{aligned}
F_n(\xi+\delta)-F_n(\xi) &= [F(\xi+\delta)-F(\xi)] + \{[F_n(\xi+\delta)-F(\xi+\delta)]-[F_n(\xi)-F(\xi)]\} \\
&= \delta f(\xi) + \frac{\delta^2}{2} f'(\xi)[1+o(1)] + \\
&\qquad \{[F_n(\xi+\delta) - F(\xi+\delta)]-[F_n(\xi) - F(\xi)]\} \\
&= \delta\, f(\xi) - D\delta^2[1+o(1)] + n^{-1/2}\, Y_n(\delta) \qquad (1.5.1)
\end{aligned}
$$

where

$$
Y_n(\delta) = n^{1/2}\,\{[F_n(\xi+\delta) - F(\xi+\delta)] - [F_n(\xi) - F(\xi)]\} \qquad (1.5.2)
$$

and

$$
D = \frac{-f'(\xi)}{2} > 0 . \qquad (1.5.3)
$$

Let $\alpha_n(\xi)+\delta\beta_n(\xi)$ denote the tangent to the concave majorant of $n^{-1/2}Y_n(\delta) - D\delta^2\ [1+o(1)]$ at $\delta=0$. In other words, $\beta_n(\xi)$ is the slope of the concave majorant of $n^{-1/2}\,Y_n(\delta) - D\delta^2\ [1+o(1)]$ at $\delta=0$.

From (1.5.1), we note that

$$f^*_{n,c}(\xi) = f(\xi) + \beta_n(\xi). \tag{1.5.4}$$

We are now interested in determining the limiting distribution of $\beta_n(\xi)$ after suitable normalization.

Let

$$\delta = r_n\zeta \text{ where } r_n = [fD^{-2}n^{-1}]^{1/3} \text{ and } f = f(\xi). \tag{1.5.5}$$

Let

$$\begin{aligned} W_n(\zeta) &= \frac{\bar{n}^{1/2}\, Y_n(\delta)}{r_n^2\, D} \\ &= \bar{n}^{1/2}[fD^{-2}n^{-1}]^{-2/3}\, D^{-1}\, Y_n(\delta) \\ &= n^{1/6}\, \bar{f}^{2/3}\, D^{1/3}\, Y_n(\delta). \end{aligned} \tag{1.5.6}$$

Let $\alpha_n = \alpha_n(\xi)$ and $\beta_n = \beta_n(\xi)$.

Let us now consider

$$\bar{n}^{1/2}\, Y_n(\delta) - D\delta^2[1 + o(1)] - \alpha_n - \beta_n\delta$$

$$= [\bar{n}^{1/2}\, Y_n(\delta) - r_n^2\zeta^2 D - \alpha_n - \beta_n r_n\zeta - r_n^2\,\zeta^2\, o(1)]$$

$$= r_n^2 D\left[\frac{\bar{n}^{1/2} Y_n(\delta)}{r_n^2\, D} - \zeta^2 - \frac{\alpha_n}{r_n^2\, D} - \frac{\beta_n\zeta}{r_n\, D} - \zeta^2 o(1)\right] \tag{1.5.7}$$

$$= r_n^2 D\left[W_n(\zeta) - \left(\zeta + \frac{\beta_n}{2r_n D}\right)^2 - \left(\frac{\alpha_n}{r_n^2 D} - \frac{\beta_n^2}{4r_n^2 D^2}\right) - \zeta^2 o(1)\right]. \tag{1.5.8}$$

From (1.5.7), we observe that $\frac{\beta_n}{r_n D}$ is the slope of the concave majorant at $\zeta = 0$ of the process

$$X_n(\zeta) = W_n(\zeta) - \zeta^2[1 + o(1)] \tag{1.5.9}$$

on $[-q,q]$ where $q = \frac{2cD^2}{f}$.

Let $D[\alpha,\beta]$ denote the space of all functions on the interval $[\alpha,\beta]$ with discontinuities of first kind and let us introduce the convergence in $D[\alpha,\beta]$ by J_1 - topology (see Sethuraman [18]).

Let $W(\zeta)$ be the Wiener process over $[-q,q]$. It is obvious that the trajectories of the process $W_n(\zeta)$ belong to $D[-q,q]$ with probability one. It is well known that the process $W(\zeta)$ had trajectories in $C[-q,q]$ with probability one and $C[-q,q]$ is a closed subset of $D[-q,q]$.

Let μ_n be the distribution induced by the process W_n on $D[-q,q]$. Let μ be the distribution induced by the process W on $D[-q,q]$. Our aim is to prove that μ_n converges to μ weakly. We shall prove some lemmas which lead to the result.

<u>Lemma 1.5.1</u>. <u>For any</u> ζ <u>in</u> $[-q,q]$, $W_n(\zeta)$ <u>is asymptotically normal with mean</u> 0 <u>and variance</u> $|\zeta|$.

<u>Proof:</u> By definition

$$W_n(\zeta) = n^{1/6}\bar{f}^{2/3}D^{1/3}\ Y_n(\delta)$$

$$= n^{1/6}\bar{f}^{2/3}D^{1/3}\{[F_n(\xi+\delta) - F_n(\xi)] - [F(\xi+\delta)-F(\xi)]\}n^{1/2}.$$

Obviously

$$E[W_n(\zeta)] = 0,$$

and

$$\mathrm{Var}[W_n(\zeta)] = n^{1/3}\bar{f}^{4/3}\ D^{2/3}\ n\tfrac{1}{n}[F(\xi+\delta) - F(\xi)][1+o(1)]$$

$$= n^{1/3}\ \bar{f}^{4/3}\ D^{2/3}\ |\delta|\ f(\xi)[1 + o(1)]$$

$$= n^{1/3}\ \bar{f}^{4/3}\ D^{2/3}\ r_n\ |\zeta|\ f(\xi)\ [1 + o(1)]$$

$$= |\zeta| \; [1 + o(1)],$$

by (1.5.5).

Therefore by the central limit theorem for independent and identically distributed random variables, we get that $W_n(\zeta)$ is asymptotically normal with mean 0 and variance $|\zeta|$.

Remarks:

In a similar way, it can be shown that for any collection $\zeta_1,\cdots,\zeta_k$ such that $|\zeta_i| \le q$, the joint distribution of $[W_n(\zeta_1),W_n(\zeta_2),\ldots,W_n(\zeta_k)]$ converges to a multivariate normal distribution with mean 0 and variance-covariance matrix

$$(\delta(\zeta_i,\zeta_j) \min(|\zeta_i|,|\zeta_j|))$$

where $\delta(a,b)$ is defined by

$$\delta(a,b) = \begin{cases} 1 \text{ if a,b are of the same sign} \\ 0 \text{ otherwise.} \end{cases}$$

The next lemma consists of showing that the processes $W_n(\zeta)$ satisfy an equicontinuity condition.

Lemma 1.5.2.

For any $\zeta_1<\zeta_2<\zeta_3$ in $[-q,q]$,

$$E[|W_n(\zeta_1) - W_n(\zeta_2)|^2 |W_n(\zeta_2) - W_n(\zeta_3)|^2] \le C|\zeta_3 - \zeta_1|^2$$

where C is a constant independent of n.

Proof:

From the definition

$$E[|W_n(\zeta_1) - W_n(\zeta_2)|^2 |W_n(\zeta_2) - W_n(\zeta_3)|^2]$$

$$= E\left[\left\{\frac{n^{-1/2} Y_n(\delta_1)-n^{-1/2}Y_n(\delta_2)}{r_n^2 D}\right\}^2 \left\{\frac{n^{-1/2}Y_n(\delta_2)-n^{-1/2}Y_n(\delta_3)}{r_n^2 D}\right\}^2\right]$$

$$= \frac{n^{-2}}{r_n^8 D^4} E[|Y_n(\delta_1) - Y_n(\delta_2)|^2 |Y_n(\delta_2) - Y_n(\delta_3)|^2]$$

$$\leq \frac{C_0 n^{-2}}{r_n^8 D^4} |\delta_3 - \delta_1|^2$$

where C_0 is a constant independent of n, by Chenstov [3],

$$= \frac{C_0 n^{-2}}{r_n^8 D^4} r_n^2 |\zeta_3 - \zeta_1|^2$$

$$= \frac{C_0 n^{-2}}{[fD^{-2}n^{-1}]^{6/3} D^4} |\zeta_3 - \zeta_1|^2$$

by (1.5.5),

$$= \frac{C_0}{f^2} |\zeta_3 - \zeta_1|^2. \tag{1.5.10}$$

Let $C = C_0 f^{-2}$. From (1.5.10), we have

$$E[|W_n(\zeta_1) - W_n(\zeta_2)|^2 |W_n(\zeta_2) - W_n(\zeta_3)|^2] \leq C |\zeta_3 - \zeta_1|^2$$

where C is independent of n.

We shall now state a theorem connected with convergence of distributions of stochastic processes on $D[\alpha,\beta]$.

<u>Theorem 1.5.3.</u>

<u>Suppose</u> $\{X_n\}$ <u>is a sequence of stochastic processes on</u> $D[\alpha,\beta]$ <u>such that</u>

(i) <u>for any</u> t_i, $1 \leq i \leq k$ <u>in</u> $[\alpha,\beta]$, <u>the joint distribution of</u> $[X_n(t_1),\ldots,X_n(t_k)]$ <u>converges to the joint distribution of</u> $[X(t_1),\ldots,X(t_k)]$

<u>and</u>

(ii) <u>for any</u> $t_1 < t_2 < t_3$ <u>in</u> $[\alpha,\beta]$

$$E[|X_n(t_1)-X_n(t_2)|^{\gamma_1}|X_n(t_2)-X_n(t_3)|^{\gamma_2}|] \leq C|t_3 - t_1|^{1+\gamma_3}$$

<u>where</u> $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ <u>and</u> $C > 0$ <u>are independent of</u> n.

<u>Let</u> $\{\nu_n\}$ <u>and</u> ν <u>be the distributions induced by</u> $\{X_n\}$ <u>and</u> $\{X\}$ <u>respectively on</u> $D[\alpha,\beta]$. <u>Then</u> ν_n <u>converges to</u> ν <u>weakly</u>.

<u>Proof</u>:

See Sethuraman [18].

As a consequence of lemmas 1.5.1 and 1.5.2 and the remarks made after lemma 1.5.1, it follows as a particular case of theorem 1.5.3 that the distribution μ_n converges weakly to the distribution μ.

Furthermore $\zeta^2[1 + o(1)]$ converges to ζ^2 uniformly in ζ since ζ is in $[-q,q]$. Hence, by a simple extension of Slutsky's theorem for processes, it follows that

<u>Theorem 1.5.4</u>.

<u>The sequence of processes</u> $X_n(\zeta) = W_n(\zeta) - \zeta^2[1 + o(1)]$ <u>on</u> $[-q,q]$ <u>converges in distribution to the process</u>

$$X(\zeta) = W(\zeta) - \zeta^2$$

<u>where</u> $W(\zeta)$ <u>is the Wiener process on</u> $[-q,q]$.

<u>1.6 Asymptotic distribution of the maximum likelihood estimator</u>:

For any $x \in D[-q,q]$, let $g(x)$ denote the slope of the concave majorant of $x(\zeta)$ at $\zeta = 0$. It is easy to see that if $x_n \to x$ in J_1-topology and x is continuous, then $x_n \to x$ in the supremum norm topology. (See Sethuraman [18] pp. 129). But, if $x_n \to x$ in the supremum norm topology and concave majorant of x has unique slope at $\zeta = 0$, then $g(x_n) \to g(x)$. Further, it is well known that the process X is continuous with probability one. Therefore g is a functional on $D[-q,q]$ whose set of discontinuities has probability zero with respect to the distribution of X. Further X_n converges in distribution to X by Theorem 1.5.4. Therefore the distribution of $g(X_n)$ converges weakly to the distribution of $g(X)$. Hence we have the following lemma.

Lemma 1.6.1.

Let $f(x)$ be a unimodal density. Let $f^*_{n,c}(\xi)$ denote the slope of the concave majorant of $F_n(y)$ on $[\xi - 2cn^{-1/3}, \xi + 2cn^{-1/3}]$ at $y = \xi$. Further suppose that $f'(\xi)$ exists and is non-zero.

Then

$$\frac{n^{1/3}\,[f^*_{n,c}(\xi) - f(\xi)]}{[fD]^{1/3}}$$

is distributed asymptotically as the slope of the concave majorant of the process

$$W(t) - t^2, \quad -q \leq t \leq q$$

at $t = 0$ where $W(t)$ is the Wiener process with mean 0 and variance 1 per unit t, $W(0) = 0$; $D = \frac{-f'(\xi)}{2}$; $f = f(\xi)$ and $q = 2cf^{-1}D^2$.

Proof: This theorem follows from the remarks made in (1.5.4) and (1.5.9).

The next lemma shows that the slope of the concave majorant of the process X over $[-q,q]$ at $\zeta = 0$ and the slope of the concave majorant of the process X over $(-\infty,\infty)$ at $\zeta = 0$ are essentially the same.

Lemma 1.6.2:

The probability, that the slope of the concave majorant of the process $X(\zeta) = W(\zeta) - \zeta^2$ over $[-q,q]$ at $\zeta = 0$ is different from the slope of the concave majorant of the process $X(\zeta) = W(\zeta) - \zeta^2$ on $(-\infty,\infty)$ at $\zeta = 0$, tends to zero as $q \to \infty$.

Proof:

For any a, let P_c denote the probability that there exist points $u < a-c$ and $v > a+c$ such that $L(u,v,x) \geq X(a)$ where $L(u,v,x)$ denotes the line joining $(u,X(u))$ and $(v,X(v))$. It is obvious that P_c is independent of a.

Let us choose a to be zero.

Then P_c = P(there exist points $u < -c$ or $v > c$ such that

$$L(u,v,x) \geq X(0) = 0). \qquad (1.6.1)$$

We notice that for any ζ,

$$X(\zeta) \leq W(\zeta) + c^2 - 2c|\zeta|. \qquad (1.6.2)$$

Therefore,

$$P_c \leq 2P[X(\zeta) \geq 0 \quad \text{for some } \zeta > c]$$

$$\leq 2P[W(\zeta) \geq 2c\zeta - c^2 \quad \text{for some } \zeta > c]$$

$$= 2E[P\{W(\zeta) \geq 2c\zeta - c^2 \quad \text{for some } \zeta > c\}|W(c)]$$

$$= 2\int_{-\infty}^{\frac{1}{4}c^2} P\{W(\zeta) \geq 2c\zeta - c^2 \text{ for some } \zeta > c \mid W(c)=x\} d\Phi(x)$$

$$+2\int_{\frac{1}{4}c^2}^{\infty} P\{W(\zeta) \geq 2c\zeta - c^2 \text{ for some } \zeta > c \mid W(c)=x\} d\Phi(x)$$

where Φ is normal with mean 0 and variance c,

$$\leq 2\int_{-\infty}^{\frac{1}{4}c^2} P\{W(\zeta)-W(c) \geq 2c\zeta-c^2-x \text{ for some } \zeta > c \mid W(c)=x\} d\Phi(x)$$

$$+ 2\int_{\frac{1}{4}c^2}^{\infty} d\Phi(x)$$

$$= 2\int_{-\infty}^{\frac{1}{4}c^2} P\{W(t) \geq 2c\,t + c^2 - x \text{ for some } t > 0 \mid W(0)=x\} d\Phi(x)$$

$$+ 2\int_{\frac{1}{4}c^2}^{\infty} d\Phi(x)$$

since $W(\zeta)$ is a stationary process with independent increments,

$$= 2\int_{-\infty}^{\frac{1}{4}c^2} P\{W'(t) \geq 2ct + c^2-2x \text{ for some } t > 0 \mid W'(0)=0\} d\Phi(x)$$

$$+ 2\int_{\frac{1}{4}c^2}^{\infty} d\Phi(x)$$

where $W'(t)$ is a Wiener process with $W'(0) = 0$.

But

$$P\{W'(t) \geq 2ct + c^2 - 2x \text{ for some } t > 0 \mid W'(0) = 0\}$$

$$= \exp\{8\,cx - 4\,c^3\}.$$

This can be proved by means of Wald's fundamental identity.

Therefore,

$$P_c \leq 2\int_{-\infty}^{\frac{1}{4}c^2} \exp\{8cx - 4c^3\} \frac{1}{\sqrt{2\pi c}}\, dx$$

$$+ 2\int_{1/4c^2}^{\infty} \exp\{\frac{-x^2}{2c}\} \cdot \frac{1}{\sqrt{2\pi c}} dx$$

$$\leq 2 \exp\{-2c^3\} + \frac{2}{\sqrt{2\pi c}} \frac{\exp\{-\frac{c^3}{32}\}}{(1/4\ c)} .$$

Taking limit at $c \to \infty$, we get that

$$P_c \to 0 \text{ as } c \to \infty . \tag{1.6.3}$$

Now let p be the probability that the slope of the concave majorant of X on $[-q,q]$ at $\zeta = 0$ differs from the slope of the concave majorant of X on $(-\infty,\infty)$ evaluated at $\zeta = 0$. Then

$p \leq P$[There exist points $u_1 < 0$, $v_1 > 2c$ such that $L(u_1,v_1,x) \geq X(c)$ or that there exists points $u_2 < -2c, v_2 > 0$ such that

$L(u_2,v_2,x) \geq X(-c)$]

$\leq 2\ P_c$

by remarks made earlier.

Therefore, by (1.6.3), $p \to 0$ as $q \to \infty$. (1.6.4)

(1.6.4) proves the lemma.

In view of lemmas 1.4.1, 1.4.2 and 1.6.1, 1.6.2, we obtain the following theorem.

<u>Theorem 1.6.3</u>.

<u>Let</u> $f(x)$ <u>be a unimodal density</u>. <u>Let</u> $\hat{f}_n(\xi)$ <u>denote the M.L.E. of</u> $f(\xi)$ <u>based on</u> n <u>observations</u>. <u>Further suppose that</u> f <u>is differentiable at</u> ξ <u>with non-zero derivative</u>. <u>Then</u>

$n^{1/3}[fD]^{-1/3}[\hat{f}_n(\xi) - f(\xi)]$ <u>is asymptotically distributed as the slope of the concave majorant of the process</u>

$$W(t) - t^2, \qquad -\infty < t < \infty$$

at $t = 0$, where $W(t)$ is the Wiener process on $(-\infty, \infty)$ with $W(0) = 0$ and mean 0, variance 1 per unit t, $D = -\frac{f'(\xi)}{2}$, $f = f(\xi)$.

The next step consists of deriving the distribution of the slope of the concave majorant of the process $W(\zeta) - \zeta^2$ over $(-\infty, \infty)$ at $\zeta = 0$. It seems to be impossible to obtain an explicit evaluation of the distribution. We shall show that it is related to a solution of a heat equation as was done by Chernoff [4] in the case of the distribution of the location of the maximum for the process $W(\zeta) - \zeta^2$ over $(-\infty, \infty)$.

Let $\alpha + \beta\zeta$ denote the tangent to the concave majorant of the process $W(\zeta) - \zeta^2$ at $\zeta = 0$. We are now interested in obtaining the distribution of β. Let $h(B)$ denote the value of ζ for which

$$W(\zeta) - (\zeta + B)^2 \tag{1.6.5}$$

is maximized over $(-\infty, \infty)$.

Consider

$$W(\zeta) - \zeta^2 - \alpha - \beta\zeta = W(\zeta) - (\zeta + \tfrac{\beta}{2})^2 - (\alpha - \tfrac{\beta^2}{4}). \tag{1.6.6}$$

By Theorem 1 of Section 4 in Chernoff [4], the probability density function of $h(B)$ is $\psi(\zeta - B)$ where

$$\psi(\zeta) = \frac{1}{2} U_x(\zeta^2, \zeta) U_x(\zeta^2, -\zeta) \tag{1.6.7}$$

where U_x is the partial derivative of $U(x, \zeta)$ with respect to x, $U(x, \zeta)$ being a solution of the heat equation

$$\frac{1}{2} U_{xx} = - U_\zeta \tag{1.6.8}$$

subject to the boundary conditions

$$\text{and} \quad \begin{array}{ll} \text{(i)} & U(x,\zeta) = 1 \quad \text{for } x \geq \zeta^2 \\ \text{(ii)} & U(x,\zeta) \to 0 \quad \text{as } x \to -\infty. \end{array} \qquad (1.6.9)$$

We note that

$$\text{Prob}[B - \delta B \leq \beta \leq B + \delta B]$$

$$= \text{Prob}[h(\tfrac{B - \delta B}{2}) \leq 0,\ h(\tfrac{B + \delta B}{2}) \geq 0]$$

$$= \text{Prob}[h(\tfrac{B - \delta B}{2}) \leq 0] - \text{Prob}[h(\tfrac{B + \delta B}{2}) \leq 0]$$

$$= \int_{-\infty}^{0} \psi(\zeta - \tfrac{B - \delta B}{2})d\zeta - \int_{-\infty}^{0} \psi(\zeta - \tfrac{B + \delta B}{2})d\zeta$$

$$= \int_{-\infty}^{-\frac{(B - \delta B)}{2}} \psi(\zeta)d\zeta - \int_{-\infty}^{-\frac{(B + \delta B)}{2}} \psi(\zeta)d\zeta$$

$$= \int_{-(\frac{B + \delta B}{2})}^{-(\frac{B - \delta B}{2})} \psi(\zeta)d\zeta.$$

Therefore the density of β is

$$\frac{1}{2}\,\psi(-\tfrac{\beta}{2}). \qquad (1.6.10)$$

we note that ψ is symmetric from (1.6.7).

Hence we have the following theorem.

<u>Theorem 1.6.4</u>.

<u>The probability density function of</u> β, viz. <u>the slope of the concave majorant of the process</u> $W(\zeta) - \zeta^2$ <u>at</u> $\zeta = 0$ <u>where</u> $W(\zeta)$ <u>is a two-sided Wiener-Levy process with mean</u> 0 <u>and variance</u> 1 <u>per unit, is</u>

$$\frac{1}{2}\,\psi(\tfrac{\beta}{2})$$

<u>where</u> ψ <u>is defined in</u> (1.6.7).

Combining the results in Theorems 1.6.3 and 1.6.4, we have the following final result.

Theorem 1.6.5.

Let $f(x)$ be a unimodal density. Let $\hat{f}_n(\xi)$ denote the M.L.E. of $f(\xi)$ based on n observations. Further suppose that f is differentiable at ξ with non-zero derivative $f'(\xi)$. Then the asymptotic distribution of

$$n^{1/3}\left[-\frac{f(\xi)f'(\xi)}{2}\right]^{-1/3}[\hat{f}_n(\xi) - f(\xi)]$$

has the density

$$\frac{1}{2}\,\psi\left(\frac{\beta}{2}\right)$$

where ψ is defined in (1.6.7).

CHAPTER 2

ESTIMATION FOR DISTRIBUTIONS WITH MONOTONE FAILURE RATE

2.1 Introduction :

In this chapter, we shall investigate a problem analogous to the problem treated in Chapter 1. We shall now suppose that the distribution F has the monotone failure rate r. (definitions are given in 2.2). Suppose $X_1, \ldots, X_n$ are n independent observations from F. Grenander [10] and Marshall and Proschan [14] have obtained the maximum likelihood estimator (M.L.E.) of r and the latter showed that these estimators are consistent. We shall obtain the asymptotic distribution of the M. L. E. as a function of a solution of a heat equation as was done by Chernoff [4] in the case of estimation of mode. Methods used in the chapter are similar to those in Chapter 1 and therefore, proofs are given only at places where they seem to be necessary.

We mention here that Murthy [15] has obtained some estimators of failure rate which are consistent and asymptotically normal. He does not assume a priori that the failure rate is monotone and his estimators are based on the choice of "window". Watson and Leadbetter [20] have also obtained similar estimators.

We shall give proofs only for the case of distributions with increasing railure rate (IFR). Results in the case of distributions with decreasing failure rate (DFR) are analogous to those in the case of IFR and we shall mention them in section 2.7.

Sections 2.2 and 2.3 deal with definition and properties of distributions with monotone failure rate. Some results related to the asymptotic properties of the M.L.E. of r are given in Section 2.4. The problem is reduced to that of a stochastic process in Section 2.5. The asymptotic distribution of the M. L. E. is obtained in Section 2.6.

2.2 Definition and properties of distributions with monotone failure rate:

Let F be a distribution function with density f.

The failure rate r of F is defined for x such that $F(x) < 1$ by

$$r(x) \equiv \frac{f(x)}{1 - F(x)} \, . \tag{2.2.1}$$

Let $R(x) \equiv -\log (1 - F(x))$. It is easily seen that R is convex on the support of F if and only if r is non-decreasing and that R is concave on the support of F if and only if r is non-increasing. We say that F is an IFR (increasing failure rate) distribution or a DFR (decreasing failure rate) distribution according as r is non-decreasing or non-increasing. Properties of distributions with monotone failure rate are discussed in Barlow, Marshall and Proschan [1].

2.3 Maximum likelihood estimation for increasing failure rate distributions:

Suppose F is an IFR with failure rate r. Let $X_1 \le X_2 \le \cdots \le X_n$ be an ordered sample from F. Let $\mathcal{F}$ be the class of IFR distributions. It is not possible to obtain the maximum likelihood estimator for $F \in \mathcal{F}$ directly by maximizing the likelihood

$$L(F) = \prod_{i=1}^{n} f(X_i) \text{ , since}$$

$f(X_n)$ can be arbitrarily large for $F \in \mathcal{F}$. Therefore, we consider a sub-family $\mathcal{F}^M$ of $\mathcal{F}$ consisting of distributions $F(x)$ with $r \le M$, obtaining $\sup_{F \in \mathcal{F}^M} \prod_1^n f(X_i) \le M^n$. There is a unique distribution $\hat{F}_n^M \in \mathcal{F}^M$ for which the supremum is attained. The failure rate $\hat{r}_n^M$ of $\hat{F}_n^M$ converges to a failure rate $\hat{r}_n$ as $M \to \infty$ for argument $x < X_n$. For $x \ge X_n$, $\hat{r}_n^M = M$ for all M and therefore $\hat{r}_n^M \to \infty$ as $M \to \infty$. This estimator $\hat{r}_n$, which is infinite for $x \ge X_n$, is called the M. L. E. of r.

From the results of Grenander [10] or as an application of van Eeden [19], the estimator $\hat{r}_n$ can be derived and it is given by

$$\hat{r}_n(x) = \begin{cases} 0 & \text{for } x < X_1 \\ \hat{r}_n(X_i) & X_i \le x < X_{i+1}, 1 \le i \le n-1 \\ \infty & x \ge X_n \end{cases}$$

where

$$\hat{r}_n(X_i) = \min_{v \ge i+1} \max_{u \le i} \{[v-u][\sum_{j=u}^{v-1} (n-j)(X_{j+1}-X_j)]^{-1}\}.$$

Marshall and Proschan [14] showed that this estimator is consistent.

The estimator $\hat{r}_n$ can also be written in the form

$$\hat{r}_n(x) = \inf_{v>x} \ \sup_{u<x} \ \frac{F_n(v) - F_n(u)}{\int_u^v [1-F_n(y)]\,dy} \tag{2.3.1}$$

where $F_n(x)$ is the empirical distribution function. In fact

$$[\hat{r}_n(X_i)]^{-1} = \sup_{v \geq \frac{i+1}{n}} \ \operatorname{Inf}_{u \leq \frac{i}{n}} \ \frac{\Phi_n(v) - \Phi_n(u)}{v-u} \tag{2.3.2}$$

where

$$\Phi_n(\tfrac{j}{n}) = \int_0^{X_j} [1 - F_n(x)]\,dx.$$

Let $\hat{\Phi}_n$ be the concave majorant of Φ_n. Then, from (2.3.2), it follows that $[\hat{r}_n(x)]^{-1}$ <u>is the slope of the concave majorant</u> $\hat{\Phi}_n$ <u>at</u> $F_n(x)$.

2.4 Some results related to the asymptotic properties of the maximum likelihood estimator:

Let $\hat{r}_n(x)$ denote the M.L.E. of r at x. Let $\frac{1}{r^*_{n,c}(x)}$ denote the slope of the concave majorant of Φ_n at $F_n(x)$, when the argument of Φ_n is restricted to the interval $[F(x) - cn^{-1/3}, F(x) + cn^{-1/3}]$. It can be shown, by methods analogous to those used in Section 1.4 of Chapter 1, that

Lemma 2.4.1.

$$\lim_{c\to\infty} \overline{\lim_{n}} \; P[r^*_{n,c}(x) \neq \hat{r}_n(x)] = 0.$$

Let ξ be such that $0 < F(\xi) < 1$.

We shall now obtain some asymptotic expansions of $\Phi_n(y)$ for y in the interval $[F(\xi) - cn^{-1/3},\ F(\xi) + cn^{-1/3}]$. We shall assume that

$$\begin{array}{ll} \text{(i)} & f \text{ is differentiable,} \\ \text{(ii)} & f' \text{ is continuous, and} \\ \text{(iii)} & \text{the failure rate } r \text{ is differentiable at} \\ & \text{the point } \xi \text{ and } r'(\xi) \text{ is non-zero.} \end{array} \qquad (2.4.1)$$

(For any function h, h' denotes the derivative of h).

As a consequence of assumptions made above, it follows that for x in a sufficiently small neighborhood of ξ ,

(i) $f(x)$ is bounded away from zero,

(ii) $r(x)$ is bounded away from zero, and

(iii) $\frac{f'(x)}{f(x)}$ is bounded.

Suppose that

$$\begin{array}{ll} \text{(i)} & f(x) \geq \gamma, \\ \text{(ii)} & r(x) \geq \alpha, \text{ and} \\ \text{(iii)} & \left|\frac{f'(x)}{f(x)}\right| \leq k \end{array} \qquad (2.4.2)$$

for x in that neighborhood of ξ .

Let $F_n(\xi) = \eta_n$. Let $F(\xi) = \eta$. It is well known that $\eta_n - \eta = O_p(n^{-1/2})$.

Let

$$U_j = F(X_{j+1}) - F(X_j) - E[F(X_{j+1}) - F(X_j) \mid X_j] \qquad (2.4.3)$$

where X_i, $1 \leq i \leq n$ are the order statistics and $E[Y|X]$

denotes the conditional expectation of Y given X.

It is easy to see that

$$U_j = F(X_{j+1}) - F(X_j) - \lambda(X_j) \tag{2.4.5}$$

where
$$\lambda(X_j) = \frac{1 - F(X_j)}{n - j + 1} .$$

We shall obtain the necessary asymptotic expansions in a series of lemmas which will be combined at the end to give the final result.

We mention here that the approximations which are of the order O_p are all satisfied uniformly for δ in $[-c,c]$ in the following lemmas. Let

$$a = [\eta n] \text{ and } b = [\eta n + \delta n^{2/3}], \tag{2.4.6}$$

where $0 < \eta < 1$ and $-c \leq \delta \leq c$.

<u>Lemma 2.4.2</u>.

$$n[\Phi_n(\tfrac{b}{n}) - \Phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} (n-j) \left\{\frac{\lambda(X_j)+U_j}{f(X_j)}\right\} + O_p(n^{-1/3}) .$$

<u>Proof</u>:

By definition of U_j in (2.4.5), we have

$$F(X_{j+1}) = F(X_j) + \lambda(X_j) + U_j .$$

Therefore, $X_{j+1} = F^{-1}[F(X_j) + \lambda(X_j) + U_j]$.

Expanding $F^{-1}(Y)$ by Taylor's theorem up to second order terms, we get that

$$X_{j+1} = X_j + [\lambda(X_j) + U_j] \left.\frac{dF^{-1}(Y)}{dY}\right|_{Y = F(X_j)}$$

$$+ 1/2[\lambda(X_j) + U_j]^2 \left.\frac{d^2F^{-1}(Y)}{dY^2}\right|_{Y = \theta_j} \tag{2.4.7}$$

where θ_j lies between $F(X_j)$ and $F(X_{j+1})$.

It is easy to see that

$$\frac{dF^{-1}(Y)}{dY}\Bigg|_{Y = F(X_j)} = \frac{1}{f(X_j)}$$

and

$$\frac{d^2F^{-1}(Y)}{dY^2}\Bigg|_{Y = \theta_j} = \frac{-f'(\zeta_j)}{f^3(\zeta_j)} \qquad (2.4.8)$$

where

$$\zeta_j = F^{-1}(\theta_j).$$

By definition,

$$n[\Phi_n(\tfrac{b}{n}) - \Phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} (n-j)(X_{j+1} - X_j)$$

$$= \sum_{j=a}^{b-1} \left\{\frac{\lambda(X_j) + U_j}{f(X_j)}\right\} (n-j)$$

$$-1/2 \sum_{j=a}^{b-1} \frac{f'(\zeta_j)}{f^3(\zeta_j)}(n-j)\{\lambda(X_j) + U_j\}^2 \qquad (2.4.9)$$

by (2.4.7) and (2.4.8).

Now for n sufficiently large, we have by (2.4.2)

$$E\left|\sum_{j=a}^{b-1} \frac{f'(\zeta_j)}{f^3(\zeta_j)} (n-j) \{\lambda(X_j) + U_j\}^2\right|$$

$$\leq \frac{k}{\gamma^2}\sum_{j=a}^{b-1} E[(n-j)(\lambda(X_j) + U_j)]^2$$

$$= \frac{k}{\gamma^2}\sum_{j=a}^{b-1} (n-j)\, E[F(X_{j+1}) - F(X_j)]^2$$

$$= \frac{k}{\gamma^2} \frac{2}{(n+1)(n+2)} \sum_{j=a}^{b-1} (n-j)$$

$$\leq \frac{k}{\gamma^2} \frac{2}{n^2} (b-a)\, n = 0\,(n^{-1/3}) \qquad (2.4.10)$$

since $b-a = 0(n^{2/3})$.

(2.4.9) and (2.4.10) together prove that

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} (n-j)\{\frac{\lambda(X_j)+U_j}{f(X_j)}\} + 0_p(n^{-1/3}).$$

Lemma 2.4.3.

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} + \sum_{j=a}^{b-1} \frac{1}{r(X_j)} + 0_p(n^{-1/3}).$$

Proof: By Lemma 2.4.2,

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} [\frac{\lambda(X_j)+U_j}{f(X_j)}](n-j)+0_p(n^{-1/3})$$

$$= \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} + \sum_{j=a}^{b-1} \frac{(n-j)}{(n-j+1)}\{\frac{1-F(X_j)}{f(X_j)}\} + 0_p(n^{-1/3})$$

$$= \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} + \sum_{j=a}^{b-1} \frac{1}{r(X_j)}$$

$$- \sum_{j=a}^{b-1} \frac{1}{(n-j+1)} \frac{1}{(r(X_j)} + 0_p(n^{-1/3}). \quad (2.4.11)$$

Now

$$E\left|\sum_{j=a}^{b-1} \frac{1}{(n-j+1)} \frac{1}{r(X_j)}\right| \leq \alpha \sum_{j=a}^{b-1} \frac{1}{(n-j+1)} \quad \text{for } n \text{ large by (2.4.2)}$$

$$\leq \alpha \int_{n-a}^{n-b+2} \frac{1}{x}\,dx$$

$$= \alpha \log \frac{n-b+2}{n-a} = 0(n^{-1/3}) \quad (2.4.12)$$

since $b-a = 0(n^{2/3})$ and $\eta < 1$.

From (2.4.11) and (2.4.12), we have

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} + \sum_{j=a}^{b-1} \frac{1}{r(X_j)} + 0_p(n^{-1/3}).$$

Lemma 2.4.4.

$$\sum_{j=a}^{b-1} \frac{1}{r(X_j)} = \frac{b-a}{r[z(\eta)]} - \frac{r'[z(\eta)]}{r^2[z(\eta)]} \frac{1}{f[z(\eta)]} \frac{(b-a)^2}{n} + 0_p(n^{1/6})$$

where $Z(t) = F^{-1}(t)$ for $0 \leq t \leq 1$.

Proof:

Let $Z_j = Z(\frac{j}{n})$.

Now $X_j - Z_j = \dfrac{F(X_j) - F(Z_j)}{f(\alpha_j)}$ for some α_j between X_j and Z_j

and f is bounded away from zero.

By the Kolmorov-Smirnov theorem,

$$\sup_j |F(X_j) - F(Z_j)| = 0_p(n^{-1/2}).$$

Therefore

$$\sup_j |X_j - Z_j| = 0_p(n^{-1/2}). \tag{2.4.13}$$

Since

$$\frac{1}{r(X_j)} - \frac{1}{r(Z_j)} = \frac{(-r')(\zeta_j)}{r^2(\zeta_j)} (X_j - Z_j)$$

for some ζ_j between X_j and Z_j, we have

$$\sum_{j=a}^{b-1} \frac{1}{r(X_j)} - \sum_{j=a}^{b-1} \frac{1}{r(Z_j)} = - \sum_{j=a}^{b-1} \frac{r'(\zeta_j)}{r^2(\zeta_j)}(X_j - Z_j)$$

$$= 0_p(n^{-1/2} \cdot n^{2/3}) = 0_p(n^{1/6})$$

by (2.4.13) and the fact that $\frac{r'}{r^2}$ is bounded.

In other words,

$$\sum_{j=a}^{b-1} \frac{1}{r(X_j)} = \sum_{j=a}^{b-1} \frac{1}{r(Z_j)} + 0_p(n^{1/6}) . \tag{2.4.14}$$

By Taylor's Theorem, we have

$$\frac{1}{r(Z_j)} = \frac{1}{r[z(\eta)]} + (\frac{j}{n} - \eta)\{- \frac{r'[Z(\eta)]}{r^2[Z(\eta)]} \frac{1}{f[Z(\eta)]}\}$$
$$+ (\frac{j}{n} - \eta)\, o(1),$$

which implies that

$$\sum_{j=a}^{b-1} \frac{1}{r(Z_j)} = \frac{b-a}{r[Z(\eta)]} - \frac{r'[Z(\eta)]}{r^2[Z(\eta)]} \frac{1}{f[Z(\eta)]} \sum_{j=a}^{b-1} (\frac{j}{n} - \eta)$$
$$+ \sum_{j=a}^{b-1} (\frac{j}{n} - \eta)\, o(1) \tag{2.4.15}$$
$$= \frac{b-a}{r[Z(\eta)]} - \frac{r'[Z(\eta)]}{r^2[Z(\eta)]} \frac{1}{f[Z(\eta)]} \frac{(b-a)^2}{n} + o(n^{1/3})$$

since $b-a = 0(n^{2/3})$.

From (2.4.14) and (2.4,15) we get that

$$\sum_{j=a}^{b-1} \frac{1}{r(X_j)} = \frac{b-a}{r[Z(\eta)]} - \frac{r'[Z(\eta)]}{r^2[Z(\eta)]} \frac{1}{f[Z(\eta)]} \frac{(b-a)^2}{n} + 0_p(n^{1/6}).$$

<u>Lemma 2.4.5</u>.

$$\sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} = \frac{1}{f[Z(\eta)]} \sum_{j=a}^{b-1} (n-j)U_j + 0_p(n^{1/6}) .$$

<u>Proof:</u>

By the Kolmogorov-Smirnov Theorem, it can be shown as before that

$$\frac{1}{f(X_j)} = \frac{1}{f(Z_j)} + 0_p(\frac{1}{\sqrt{n}}). \tag{2.4.16}$$

Therefore

$$\sum_{j=a}^{b-1}{}' \frac{(n-j)U_j}{f(X_j)} = \sum_{j=a}^{b-1}{}' \frac{(n-j)U_j}{f(Z_j)} + \sum_{j=a}^{b-1}{}'(n-j)U_j \; O_p(\frac{1}{\sqrt{n}}). \quad (2.4.17)$$

But

$$\{E\,|(n-j)U_j O_p(\frac{1}{\sqrt{n}})|\}^2 \le E[(n-j)U_j]^2 E[O_p(\frac{1}{\sqrt{n}})]^2$$

$$= O(\frac{1}{n}) \text{ uniformly in } j. \quad (2.4.18)$$

Therefore (2.4.17) and (2.4.18) imply that

$$\sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j} = \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(Z_j)} + O_p(n^{1/6}) \quad (2.4.19)$$

$$\text{since } b-a = O(n^{2/3}).$$

But $\dfrac{1}{f(Z_j)} = \dfrac{1}{f[Z(\eta)]} + (\frac{j}{n} - \eta)\;[-\dfrac{f'[Z(\eta)]}{f^2[Z(\eta)]} + o(1)]$.

So we have,

$$\sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(Z_j)} = \sum_{j=a}^{b-1} \frac{(n-j)U_j}{f[Z(\eta)]} + \sum_{j=a}^{b-1} U_j (\frac{j}{n} - \eta)[-\frac{f'(Z(\eta))}{f^2(Z(\eta))} + o(1)]\,(n-j)$$

$$= \frac{1}{f[Z(\eta)]} \sum_{j=a}^{b-1} (n-j)U_j + Z_n \quad (2.4.20)$$

where

$$Z_n = \sum_{j=a}^{b-1} (\frac{j}{n} - \eta)[-\frac{f'(Z(\eta))}{f^2(Z(\eta))} + o(1)]\,(n-j)U_j.$$

Since $E(U_j) = 0$ for each j, $E(Z_n) = 0$.

But $\operatorname{Var}(Z_n) \le \dfrac{k^2}{\gamma^2} \displaystyle\sum_{j=a}^{b-1} (\frac{j}{n} - \eta)^2 (n-j)^2 \operatorname{Var}(U_j)$

by (2.4.2)

$$\le \frac{dk^2}{\gamma^2} \sum_{j=a}^{b-1} (\frac{j}{n} - \eta)^2 \frac{(n-j)^2}{n^2} \quad \text{since } \operatorname{Var}(U_j) \le \frac{d}{n^2}$$

for some constant d

$$= O(1) \qquad \text{since } b-a = O(n^{2/3}).$$

Therefore $Z_n = 0_p(1)$.

Now (2.4.19), (2.4.20) together imply that

$$\sum_{j=a}^{b-1} \frac{(n-j)U_j}{f(X_j)} = [\sum_{j=a}^{b-1} (n-j)U_j]\frac{1}{f[Z(\eta)]} + 0_p(n^{1/6}) + 0_p(1)$$

$$= [\sum_{j=a}^{b-1} (n-j)U_j]\frac{1}{f[Z(\eta)]} + 0_p(n^{1/6}).$$

From lemmas (2.4.3) - (2.4.5), we have the following theorem.

Theorem 2.4.6.

Let ξ be such that $0 < F(\xi) < 1$ and let $F(\xi) = \eta$.

Suppose that $-c \leq \delta \leq c$.

Let $a = [\eta n]$ and $b = [\eta n + \delta n^{2/3}]$.

Then

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})]$$

$$= \frac{n^{1/3}}{r[Z(\eta)]} W_n(\delta) - \frac{\delta^2 n^{1/3}}{2} \frac{1}{f[Z(\eta)]} \frac{r'[Z(\eta)]}{r^2[Z(\eta)]}$$

$$+ \frac{\delta n^{2/3}}{r[Z(\eta)]} + 0_p(n^{1/6}),$$

where

$$W_n(\delta) = \sum_{j=a}^{b-1} n^{2/3} \frac{(n-j+1)}{(n-a)} U_j .$$

Proof:

By lemmas (2.4.3) - (2.4.5), we have

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})]$$

$$= \frac{1}{f[Z(\eta)]} \sum_{j=a}^{b-1} (n-j)U_j + \frac{b-a}{r[Z(\eta)]}$$

$$- \frac{(b-a)^2}{2n} B(\eta) + 0_p(n^{1/6}) \qquad (2.4.21)$$

where

$$B(\eta) = \frac{1}{f[Z(\eta)]} \frac{r'[Z(\eta)]}{r^2[Z(\eta)]}. \tag{2.4.22}$$

But $\quad |(b-a) - \delta n^{2/3}| \leq 1.$

Therefore, from (2.4.21), we have

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \frac{(n-a)}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)U_j}{(n-a)}$$

$$+ \frac{\delta n^{2/3}}{r[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2} B(\eta) + 0_p(n^{1/6})$$

$$= \frac{n-\eta n}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} U_j + \frac{\eta n-a}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} U_j$$

$$+ \frac{\delta n^{2/3}}{r[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2} B(\eta) + 0_p(n^{1/6})$$

$$= \frac{n-\eta n}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} U_j + T_n + \frac{\delta n^{2/3}}{f[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2} B(\eta) + 0_p(n^{1/6}), \tag{2.4.23}$$

where

$$T_n = \frac{\eta n-a}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} U_j.$$

Obviously

$$E|T_n| \leq \frac{1}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} E|U_j| = 0(n^{-1/3})$$

since $b-a = 0(n^{2/3})$ and $E|U_j| = 0(n^{-1})$.

Therefore, form (2.4.23), we have

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \frac{n[1-\eta]}{f[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j)}{(n-a)} U_j$$

$$+ \frac{\delta n^{2/3}}{r[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2} B(\eta) + 0_p(n^{1/6})$$

$$= \frac{n}{r[Z(\eta)]} \sum_{j=a}^{b-1} \frac{(n-j+1)}{(n-a)} U_j - R_n$$

$$+ \frac{\delta n^{2/3}}{r[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2} B(\eta) + 0_p(n^{1/6}), \quad (2.4.24)$$

where

$$R_n = \frac{n}{r[Z(\eta)]} \sum_{j=a}^{b-1} \frac{1}{n-a} U_j .$$

Obviously

$$E|R_n| = 0(n^{-1/3}) \text{ since } E\ |U_j| = 0(n^{-1}).$$

Therefore, from (2.4.24), it follows that

$$n[\Phi_n(\tfrac{b}{n}) - \Phi_n(\tfrac{a}{n})] = \frac{n^{1/3}}{r[Z(\eta)]} W_n(\delta) - \frac{\delta^2 n^{1/3}}{2} B(\eta) + \frac{\delta n^{2/3}}{r[Z(\eta)]} + 0_p(n^{1/6})$$

where

$$W_n(\delta) = \sum_{j=a}^{b-1} n^{2/3} \frac{(n-j+1)}{(n-a)} U_j . \quad (2.4.25)$$

Remark: Since $r(x)$ is non-decreasing and since $r'(\xi) \neq 0$, we have $B(\eta) > 0$.

2.5 Reduction to a problem in stochastic processes:

In this section, we shall reduce the problem of calculating the asymptotic distribution of the slope of the concave majorant of $\Phi_n(Y)$ over $[F(\xi) - cn^{-1/3}, F(\xi) + cn^{-1/3}]$ at $Y = F_n(\xi)$ to the corresponding problem of a Wiener process over $[-c,c]$ after suitable normalization. We shall assume that the conditions in (2.4.1) are satisfied.

Let $F_n(\xi) = \eta_n$ and $a = [\eta n]$, $b = [\eta n + \delta n^{2/3}]$, and $F(\xi) = \eta$ where $-c \leq \delta \leq c$.

By theorem 2.4.6,

$$n[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \frac{n^{1/3}W_n(\delta)}{r[Z(\eta)]} - \frac{\delta^2 n^{1/3}}{2}B(\eta) + \frac{\delta n^{2/3}}{r[Z(\eta)]} + 0_p(n^{1/6}),$$

where $B(\eta)$ and $W_n(\delta)$ are defined in (2.4.22) and (2.4.25). Therefore,

$$n^{2/3}[\phi_n(\tfrac{b}{n}) - \phi_n(\tfrac{a}{n})] = \frac{W_n(\delta)}{r[Z(\eta)]} - \frac{\delta^2}{2}B(\eta) + \frac{\delta n^{1/3}}{r[Z(\eta)]} + 0_p(n^{-1/6}). \tag{2.5.1}$$

Let $\alpha_n(\eta) + \delta\,\beta_n(\eta)$ denote the tangent to the concave majorant of

$$\frac{W_n(\delta)}{r[Z(\eta)]} - \frac{\delta^2}{2}B(\eta) + 0_p(n^{-1/6})$$ over $[-c,c]$, at $\delta_n = (\eta_n - \eta)n^{1/3}$.

We notice that

$$n^{1/3}\Big[\frac{1}{r^*_{n,c}[Z(\eta)]} - \frac{1}{r[Z(\eta)]}\Big] = \beta_n(\eta) \tag{2.5.2}$$

where $[r^*_{n,c}[Z(\eta)]]^{-1}$ denotes the slope of the concave majorant of $\phi_n(Y)$ over

$[\eta - cn^{-1/3},\ \eta + cn^{-1/3}]$ at $Y = \eta_n$.

We are now interested in determining the limiting distribution of $\beta_n(\eta)$.

Let $\delta = \lambda\zeta$ where

$$\lambda = \Big\{\frac{r[Z(\eta)]\ B(\eta)}{2}\Big\}^{-2/3}. \tag{2.5.3}$$

Let $V_n(\zeta) = \lambda^{-1/2}\,W_n(\delta)$. (2.5.4)

Let us now consider

$$\frac{W_n(\delta)}{r[Z(\eta)]} - B(\eta)\frac{\delta^2}{2} - \alpha_n(\eta) - \beta_n(\eta)\delta + 0_p(n^{-1/6})$$

$$= \frac{\lambda^{1/2}V_n(\zeta)}{r[Z(\eta)]} - \frac{B(\eta)\lambda^2\zeta^2}{2} - \alpha_n(\eta) - \beta_n(\eta)\lambda\zeta + 0_p(n^{-1/6})$$

$$= \frac{\lambda^{1/2}}{r[Z(\eta)]}[V_n(\zeta) - \zeta^2 - \frac{2\alpha_n(\eta)}{\lambda^2 B(\eta)} - 2\,\frac{\beta_n(\eta)\cdot\zeta}{\lambda B(\eta)}] + 0_p(n^{-1/6}) \quad (2.5.5)$$

by (2.5.3),

$$= \frac{\lambda^{1/2}}{r[Z(\eta)]}\,[V_n(\zeta) - (\zeta + \frac{\beta_n(\eta)}{\lambda\,B(\eta)})^2 - (\frac{2\alpha_n(\eta)}{\lambda^2 B(\eta)} - \frac{\beta_n^2(\eta)}{\lambda^2 B^2(\eta)})]$$

$$+ 0_p(n^{-1/6}). \quad (2.5.6)$$

From (2.5.5), we notice that $\frac{2\beta_n(\eta)}{\lambda B(\eta)}$ is the slope of the concave majorant at $\zeta = \zeta_n = \lambda^{-1}(\eta_n - \eta)n^{1/3}$ of the process

$$X_n(\zeta) = V_n(\zeta) - \zeta^2 + 0_p(n^{-1/6}) \quad (2.5.7)$$

on $[-q,q]$ where $q = \frac{c}{\lambda}$.

Notice that $\zeta_n = 0_p(n^{-1/6})$. (2.5.8)

Let $X(\zeta) = W(\zeta) - \zeta^2$, where $W(\zeta)$ is the Wiener process on $[-q,q]$.

Let $D[\alpha,\beta]$ denote the space of all functions on the interval $[\alpha,\beta]$ with discontinuities of first kind and let us introduce the convergence in $D[\alpha,\beta]$ by J_1-topology. (See Sethuraman [18]).

Let $W(\zeta)$ be the Wiener process over $[-q,q]$. It is obvious that the trajectories of the process $V_n(\zeta)$ belong to $D[-q,q]$ with probability one. It is well known that the process $W(\zeta)$ has trajectories in $C[-q,q]$ with probability one and $C[-q,q]$ is a closed subset of $D[-q,q]$.

Let μ_n be the distribution induced by the process V_n on $D[-q,q]$. Let μ be the distribution induced by the process W on $D[-q,q]$.

Our aim is to prove that μ_n converges to μ weakly. We shall prove some lemmas which lead to the result.

Lemma 2.5.1.

For any δ in $[-c,c]$, $W_n(\delta)$ is asymptotically normal with mean 0 and variance $|\delta|$.

Proof:

Let $\Delta_1,\ldots,\Delta_{n+1}$ be independent random variables each with the exponential distribution ne^{-nx}, $x \geq 0$. Let

$$D_n = \Delta_1 + \ldots + \Delta_{n+1}. \qquad (2.5.9)$$

Then $Z_i = \dfrac{\Delta_1 + \ldots + \Delta_i}{D_n}$, $1 \leq i \leq n$ form order statistics of a sample of size n form the uniform distribution on $[0,1]$. (2.5.10)

From (2.4.25), we have

$$W_n(\delta) = n^{2/3} \sum_{j=a}^{b-1} \frac{n-j+1}{n-a} U_j$$

$$= n^{2/3} \sum_{j=a}^{b-1} \frac{n-j+1}{n-a} \left[F(X_{j+1}) - F(X_j) - \frac{1-F(X_j)}{n-j+1}\right].$$

Therefore, from (2.5.10), it follows that $W_n(\delta)$ has the same distribution as

$$W_n^*(\delta) = \frac{n^{-1/3}}{1-(\frac{a}{n})} \sum_{j=a}^{b-1} (n-j+1)\left[\frac{\Delta_{j+1}}{D_n} - \frac{1-\frac{\Delta_1+\ldots+\Delta_j}{D_n}}{n-j+1}\right]$$

$$= \frac{n^{-1/3}}{D_n(1-\frac{a}{n})} \sum_{j=a}^{b-1} (n-j+1)\left[\Delta_{j+1} - \frac{\Delta_{j+1}+\ldots+\Delta_{n+1}}{n-j+1}\right]$$

$$= \frac{n^{-1/3}}{D_n(1-\frac{a}{n})} \sum_{j=a}^{b-1} (n-j+1)\left[(\Delta_{j+1} - \tfrac{1}{n}) - \left(\frac{\Delta_{j+1}+\ldots+\Delta_{n+1}}{n-j+1} - \tfrac{1}{n}\right)\right]. \quad (2.5.11)$$

Let

(i) $$A_n(\delta) = \frac{n^{-1/3}}{1-\frac{a}{n}} \sum_{j=a}^{b-1} (n-j+1)(\Delta_{j+1} - \tfrac{1}{n})$$

and

(ii) $$B_n(\delta) = \frac{n^{-1/3}}{1-\frac{a}{n}} \sum_{j=a}^{b-1} \left(\Delta_{j+1}+\ldots+\Delta_{n+1} - \frac{n-j+1}{n}\right). \quad (2.5.12)$$

Then

$$W_n^*(\delta) = \frac{A_n(\delta)}{D_n} - \frac{B_n(\delta)}{D_n}. \quad (2.5.13)$$

Since $E(\Delta_j) = \frac{1}{n}$ for $1 \leq j \leq n+1$,

$$E(B_n(\delta)) = 0. \quad (2.5.14)$$

Now

$$\mathrm{Var}(B_n(\delta)) = \frac{n^{-2/3}}{(1-\frac{a}{n})^2} \mathrm{Var}\left[\sum_{j=a}^{b-1} (\Delta_{j+1} + \ldots + \Delta_{n+1})\right]$$

$$= \frac{n^{-2/3}}{(1-\frac{a}{n})^2} \mathrm{Var}\left[\sum_{j=a}^{b-1} (j-a)\Delta_j + (b-a)\sum_{j=b}^{n+1} \Delta_j\right]$$

$$= \frac{n^{-2/3}}{(1-\frac{a}{n})^2} \left[\sum_{j=a+1}^{b-1} \frac{(j-a)^2}{n^2} + (b-a)^2 \frac{(n-b)}{n^2}\right]$$

since Δ_j are independent and $\operatorname{Var}(\Delta_j) = \frac{1}{n^2}$.

Therefore

$$\operatorname{Var}(B_n(\delta)) = \frac{n^{-2/3}}{n^2(1-\frac{a}{n})^2}\left[\sum_{j=1}^{b-a-1} j^2 + (b-a)^2(n-b)\right]$$

$$= \frac{1}{n^{8/3}(1-\frac{a}{n})^2}\left[\frac{(b-a-1)(b-a)\{2(b-a-1)+1\}}{6} + (b-a)^2(n-b)\right]$$

$$\leq \frac{1}{n^{8/3}(1-\frac{a}{n})^2}\left[\frac{(b-a)^3}{3} + (b-a)^2 n\right].$$

Since $b-a = 0(n^{2/3})$, the term on the right hand side tends to zero. Therefore, from (2.5.14), it follows that $B_n(\delta) \to 0$ in probability.

Since D_n converges to 1 in probability, by Slutsky's theorem (See Cramer [6]),

$$\frac{B_n(\delta)}{D_n} \to 0 \text{ in probability.} \tag{2.5.15}$$

Let $f_n(t)$ be the characteristic function of $A_n(\delta)$. Then,

$$f_n(t) = E[\exp\{i\frac{tn^{-1/3}}{1-\frac{a}{n}}\sum_{j=a}^{b-1}[(n-j+1)(\Delta_{j+1} - \frac{1}{n})]\}]$$

$$= \exp\{-i\frac{tn^{-4/3}}{1-\frac{a}{n}}\sum_{j=a}^{b-1}(n-j+1)\}\prod_{j=a}^{b-1}\frac{n}{n-i[\frac{(n-j+1)tn^{-1/3}}{1-\frac{a}{n}}]}$$

since Δ_j's are independent and exponentially distributed.

Therefore

$$\log f_n(t) = -\frac{itn^{-4/3}}{(1-\frac{a}{n})} \sum_{j=a}^{b-1} (n-j+1) + \sum_{j=a}^{b-1} \log n$$

$$- \sum_{j=a}^{b-1} \log[n - \{i \frac{(n-j+1)}{1-\frac{a}{n}} tn^{-1/3}\}]$$

$$= \frac{-itn^{-4/3}}{1-\frac{a}{n}} \sum_{j=a}^{b-1} (n-j+1) - \sum_{j=a}^{b-1} \log[1 - \frac{it(n-j+1)n^{-4/3}}{1-\frac{a}{n}}]$$

$$= -\frac{t^2n^{-8/3}}{2(1-\frac{a}{n})^2} \sum_{j=a}^{b-1} (n-j+1)^2 + o(1)$$

$$= \frac{-t^2n^{-8/3}}{2(1-\frac{a}{n})^2} \sum_{j=a}^{b-1} [(n-a)^2 + (a-j+1)^2 + 2(n-a)(a-j+1)] + o(1)$$

$$= -\frac{t^2n^{-8/3}}{2(1-\frac{a}{n})^2} (b-a)(n-a)^2 + o(1)$$

$$= -\frac{t^2n^{-8/3}}{2} n^2|\delta|n^{2/3} + o(1)$$

$$= -\frac{t^2}{2} |\delta| + o(1).$$

Therefore, $f_n(t) \to \exp\{-\frac{t^2}{2} |\delta|\}$ as $n \to \infty$.

Hence by the continuity theorem for characteristic functions, it follows that

$A_n(\delta)$ is asymptotically normal with mean 0 and variance $|\delta|$. (2.5.16)

Then, by Slutsky's theorem (See Cramer [6]),

$\frac{A_n(\delta)}{D_n}$ is asymptotically normal with mean 0 and variance $|\delta|$ (2.5.17) since D_n converges to 1 in probability.

From (2.5.13), (2.5.15) and (2.5.17), we get that $W_n(\delta)$ is asymptotically normal with mean 0 and variance $|\delta|$.

Lemma 2.5.2.

For any ζ in $[-q,q]$, $V_n(\zeta)$ is asymptotically normal with mean 0 and variance $|\zeta|$.

Proof: This lemma follows immediately from lemma 2.5.1, since by definition

$$V_n(\zeta) = \lambda^{-1/2}\, W_n(\delta).$$

Remark: In a similar manner, it can be shown that for any collection $\zeta, \ldots, \zeta_k$ such that $\zeta_i \in [-q,q]$, the joint distribution of $[V_n(\zeta_1), \ldots, V_n(\zeta_k)]$ converges to the multivariate normal distribution with mean 0 and variance - covariance matrix $(\delta(\zeta_i,\zeta_j), \min(|\zeta_i|,|\zeta_j|))$ where

$$\delta(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha,\beta \text{ are of the same sign} \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma proves that the processes $\{D_n V_n(\zeta)\}$ on $[-q,q]$ (D_n is defined in (2.5.9)) satisfy an equi-continuity condition.

Lemma 2.5.3.

For any ζ_1,ζ_2 in $[-q,q]$,

$$E|D_nV_n(\zeta_1) - D_nV_n(\zeta_2)|^4$$

$$\leq C|\zeta_1 - \zeta_2|^2 + |\zeta_1 - \zeta_2|\, o(1)$$

where C is a constant independent of n.

Proof: We have

$$E|D_nV_n(\delta_1) - D_nV_n(\delta_2)|^4$$
$$= \lambda^{-2} E|D_nW_n(\zeta_1) - D_nW_n(\zeta_2)|^4$$
$$= \lambda^{-2} E|\{A_n(\delta_1) - B_n(\delta_1)\} - \{A_n(\delta_2) - B_n(\delta_2)\}|^4$$

where $A_n(\delta)$, $B_n(\delta)$ are defined in (2.5.12),

$$\leq 8\lambda^{-2}[E|A_n(\delta_1) - A_n(\delta_2)|^4 + E|B_n(\delta_1) - B_n(\delta_2)|^4] \quad (2.5.18)$$

by the elementary inequality

$$E|X + Y|^4 \leq 8[E|X|^4 + E|Y|^4].$$

Let

$$b_1 = [\eta n + \delta_1 n^{2/3}] \; ; \; b_2 = [\eta n + \delta_2 n^{2/3}]. \quad (2.5.19)$$

Let us first compute

$$E|A_n(\delta_1) - A_n(\delta_2)|^4$$
$$= \frac{n^{-4/3}}{(1-\frac{a}{n})^4} E\left|\sum_{j=b_1}^{b_2-1} (n-j+1)(\Delta_{j+1} - \frac{1}{n})\right|^4$$
$$= \frac{n^{-4/3}}{[1-\frac{a}{n}]^4}\left[\sum_{j=b_1}^{b_2-1} (n-j+1)^4 \frac{9}{n^4}\right.$$
$$\left. + 6 \sum_{\substack{i=b_1 \\ i\neq j}}^{b_2-1} \sum_{j=b_1}^{b_2-1} (n-i+1)^2 (n-j+1)^2 \frac{1}{n^4}\right], \quad (2.5.20)$$

since Δ_j are independent and

$$E(\Delta_j) = \frac{1}{n}; \; Var(\Delta_j) = \frac{1}{n^2}; \; E(\Delta_j - \frac{1}{n})^4 = \frac{9}{n^4} \quad (2.5.21)$$

for all j such that $1 \leq j \leq n+1$.

From (2.5.20), we have

$$E|A_n(\delta_1) - A_n(\delta_2)|^4 \le \frac{9n^{-4/3}}{n^4(1-\frac{a}{n})^4}\left[\sum_{j=b_1}^{b_2-1}(n-j+1)^2\right]^2$$

$$\le \frac{9n^{-4/3}}{n^4(1-\frac{a}{n})^4}\left[(b_2-b_1)n^2\right]^2$$

$$\le \frac{9n^{-4/3}}{(1-\eta)^4 n^4}(\delta_2-\delta_1)^2 n^{4/3}n^4$$

$$= C_1|\delta_2-\delta_1|^2 . \qquad (2.5.22)$$

Let us now compute

$$E|B_n(\delta_1) - B_n(\delta_2)|^4$$

$$= \frac{n^{-4/3}}{(1-\frac{a}{n})^4} E\left|\sum_{j=b_1}^{b_2-1}\{\Delta_{j+1}-\frac{1}{n}) + \ldots + (\Delta_{n+1}-\frac{1}{n})\}\right|^4$$

$$= \frac{n^{-4/3}}{(1-\frac{a}{n})^4} E\left|\sum_{j=1}^{b_2-b_1} j(\Delta_{b_1+j}-\frac{1}{n})+(b_2-b_1)\sum_{j=b_2+1}^{n+1}(\Delta_j-\frac{1}{n})\right|^4$$

$$\le \frac{8n^{-4/3}}{(1-\frac{a}{n})^4}\left[E\left|\sum_{j=1}^{b_2-b_1} j(\Delta_{b_1+j}-\frac{1}{n})\right|^4+(b_2-b_1)^4 E\left|\sum_{j=b_2+1}^{n+1}(\Delta_j-\frac{1}{n})\right|^4\right]$$

$$= \frac{8n^{-4/3}}{(1-\frac{a}{n})^4}\left[\left\{\sum_{j=1}^{b_2-b_1}\frac{9j^4}{n^4} + 6\sum_{\substack{j=1\\ i\neq j}}^{b_2-b_1}\sum_{i=1}^{b_2-b_1}\frac{i^2j^2}{n^4}\right\}\right.$$

$$\left. + (b_2-b_1)^4\left\{\sum_{j=b_2+1}^{n+1}\frac{9}{n^4} + 6\sum_{\substack{i\\ i\neq j}}\sum_{j}\frac{1}{n^4}\right\}\right]$$

$$\le \frac{72n^{-4/3}}{n^4(1-\frac{a}{n})^4}\left[\left\{\sum_{j=1}^{b_2-b_1} j^2\right\}^2 + (b_2-b_1)^4(n-b_2)^2\right]$$

$$\le \frac{72n^{-4/3}}{n^4(1-\frac{a}{n})^4}\left[(b_2-b_1)^2 n^4 + (b_2-b_1)^4 n^2\right]$$

$$\leq \frac{72n^{-4/3}}{n^4(1-\frac{a}{n})^4} [|\delta_2-\delta_1|^2 n^{4/3} n^4 + |\delta_2-\delta_1|^4 n^{8/3} n^2]$$
by (2.5.19),

$$= \frac{72}{(1-\frac{a}{n})^4} [|\delta_2-\delta_1|^2 + |\delta_2-\delta_1| n^{-2/3} 8c^3]$$
since $|\delta_2-\delta_1| \leq 2c$,

$$\leq C_2|\delta_2-\delta_1|^2 + |\delta_2-\delta_1| o(1), \qquad (2.5.23)$$

where $C_2 = \dfrac{72}{(1-\eta)^4}$.

Combine (2.5.8), (2.5.22) and 2.5.23) we get that

$$E|D_nV_n(\zeta_1)-D_nV_n(\zeta_2)|^2 \leq 8\lambda^{-2}[C_1|\delta_2-\delta_1|^2 + C_2|\delta_2-\delta_1|^2 + |\delta_2-\delta_1|o(1)]$$
$$= C|\zeta_2-\zeta_1|^2 + |\zeta_2-\zeta_1| o(1)$$

where C is a constant independent of n.

This proves lemma 2.5.3.

<u>Remarks</u>:

Since D_n converges to 1 in probability, $D_nV_n(\zeta)$ is asymptotically normal with mean 0 and variance $|\zeta|$ by lemma 2.5.2. Further from the remarks made at the end of lemma 2.5.2, it follows that the joint distribution of $[D_nV_n(\zeta_1), \ldots , D_nV_n(\zeta_k)]$ is asymptotically multivariate normal with mean 0 and variance - covariance matrix.

$$(\delta(\zeta_i,\zeta_j) \min (|\zeta_i|,|\zeta_j|))$$

where δ is defined by

$$\delta(a,b) = \begin{cases} 1 & \text{if } a,b \text{ are of the same sign} \\ 0 & \text{otherwise.} \end{cases}$$

When $D_nV_n(\zeta)$ is represented in terms of exponentials as in lemma 2.5.1, we note that $D_nV_n(\zeta)$ is a process with independent increments.

We shall now state a theorem connected with convergence of distributions of stochastic processes with independent increments in $D[\alpha,\beta]$.

Theorem 2.5.4.

Let X_n be a sequence of processes with independent increments on $D[\alpha,\beta]$ and X be a process on $D[\alpha,\beta]$ such that

(i) *for any t_i, $1 \leq i \leq k$ in $[\alpha,\beta]$ the joint distribution of $[X_n(t_1), \ldots, X_n(t_k)]$ converges to the joint distribution of $[X(t_1), \ldots, X(t_k)]$, and*

(ii) *there exist constants $\gamma > 0$, $C > 0$ independent of n such that for every $t_1, t_2 \in [\alpha,\beta]$,*

$$E|X_n(t_1) - X_n(t_2)|^\gamma \leq C|t_1-t_2|^2 + |t_2-t_1|\, o(1).$$

Let ν_n and ν be the distributions induced by X_n and X respectively on $D[\alpha,\beta]$. Then ν_n converges to ν weakly.

Proof: From the condition (ii) of the hypothesis it follows that

$$E|X_n(t_1) - X_n(t_2)|^\gamma \leq A|t_1-t_2|$$

for all n and for $t_1, t_2 \in [\alpha,\beta]$ such that $|t_1-t_2| < 1$ where A is a constant independent of n.

Therefore, for any $\lambda > 0$,

$$P\{|X_n(t_1) - X_n(t_2)| > \lambda\} \leq \frac{A|t_2-t_1|}{\lambda^\gamma} \leq \frac{A\delta}{\lambda^\gamma}$$

for all $|t_2-t_1| \leq \delta < 1$.

Let $\Psi(\delta,\lambda) = \frac{A\delta}{\lambda^\gamma}$. We note that $\Psi(\delta,\lambda) \to 0$ as $\delta \to 0$. Now from the remarks made on page 140 of Sethuraman [18], it follows that the distribution ν_n converges weakly to ν.

As a consequence of lemma 2.5.3 and the remarks made at the end of the lemma, we get that the sequence of distributions induced by the processes $D_n V_n(\zeta)$ on $D[-q,q]$ converges weakly to the distribution μ induced by the process $W(\zeta)$ on $D[-q,q]$.

Since D_n converges to 1 in probability, the following theorem can be obtained by Slutsky's theorem generalized to processes. (Rubin [16]).

Theorem 2.5.5.

The sequence of processes $V_n(\zeta)$ *on* $[-q,q]$ *converges in distribution to the process* $W(\zeta)$ *on* $[-q,q]$.

Furthermore

$\zeta^2 + o_p(n^{-1/6})$ converges to ζ^2 uniformly in ζ, since ζ belongs to a finite interval and o_p is uniform for $\zeta \in [-q,q]$.

Hence the process $X_n(\zeta) = V_n(\zeta) - \zeta^2 + o_p(n^{-1/6})$ converges in distribution to the process

$$X(\zeta) = W(\zeta) - \zeta^2 \quad \text{on} \quad [-q,q],$$

where $W(\zeta)$ is the Wiener process on $[-q,q]$.

2.6 Asymptotic distribution of the maximum likelihood estimator for increasing failure rate distributions:

In view of the result obtained at the end of section 2.5

and the lemmas 2.4.1, 1.4.2, 1.6.2, the following final result can be obtained by methods analogous to those used in Section 1.6 of Chapter 1.

Theorem 2.6.1.

Let $F(x)$ *be a distribution with non-decreasing failure rate* $r(x)$. *Suppose that* $\hat{r}_n(\xi)$ *is the M.L.E. of* $r(\xi)$ *based on* n *observations. Further assume that the conditions in* (2.4.1) *are satisfied. Then the asymptotic distribution of*

$$n^{1/3}\left[\frac{r'(\xi)\{r(\xi)\}^{-4}}{2f(\xi)}\right]^{-1/3}\left[\frac{1}{\hat{r}_n(\xi)}-\frac{1}{r(\xi)}\right]$$

has density

$$\frac{1}{2}\,\Psi\left(\frac{\beta}{2}\right)$$

where Ψ *is defined in* (1.6.7).

2.7 Asymptotic distribution of the maximum likelihood estimator for decreasing failure rate distributions:

In this section, we shall give results for distributions with decreasing failure rate. Let $F(x)$ be a DFR distribution with failure rate $r(x)$. Let $\hat{r}_n(x)$ denote the M.L.E. of $r(x)$. It was shown by Marshall and Proschan [14] that the estimate $\hat{r}_n(x)$ is consistent and $[\hat{r}_n(x)]^{-1}$ is the slope of the convex minorant of $\Phi_n(Y)$ at $Y = F_n(x)$, where

$$\Phi_n\left(\frac{j}{n}\right) = \int_0^{x_j}[1 - F_n(y)]\,dy,$$

X_j being the order statistics of a random sample of size n

and F_n is the empirical distribution.

The following theorem can be proved by methods analogous to those used in the case of IFR.

Theorem 2.7.1.

Let $F(x)$ be a distribution with non-increasing failure rate $r(x)$. Let $\hat{r}_n(\xi)$ denote the M.L.E. of $r(x)$ at $x = \xi$. Further suppose that conditions in (2.4.1) are satisfied. Then

$$n^{1/3}\left[\{r(\xi)\}^{-2}\frac{C(\xi)}{2}\right]^{-1/3}\left[\frac{1}{\hat{r}_n(\xi)} - \frac{1}{r(\xi)}\right]$$

is asymptotically distributed as the slope of the convex minorant of the process $W(t) + t^2$, $-\infty < t < \infty$ at $t = 0$ where $W(t)$ is the two-sided Wiener process with mean 0 and variance 1 per unit t and

$$C(\xi) = -\frac{1}{f(\xi)}\frac{r'(\xi)}{r^2(\xi)}.$$

From Chernoff [4], we have the following theorem.

Theorem 2.7.2

The probability density function of $\hat{\zeta}$, the value of ζ which minimizes $W(\zeta) + \zeta^2$ where $W(\zeta)$ is the two-sided Wiener process with mean 0 and variance 1 per unit ζ is

$$\Psi(\zeta)$$

where Ψ is defined in (1.6.7).

From theorems 2.7.1 and 2.7.2, we have the following result for DFR distributions.

Theorem 2.7.3.

Let $F(x)$ be a distribution with non-increasing failure rate $r(x)$. Suppose that $\hat{r}_n(\xi)$ is the M.L.E. of $r(\xi)$ based on n observations. Further assume that the conditions in (2.4.1) are satisfied. Then the asymptotic distribution of

$$n^{1/3}\left[\frac{-r'(\xi)\{r(\xi)\}^{-4}}{2\,f(\xi)}\right]^{-1/3}\left[\frac{1}{\hat{r}_n(\xi)} - \frac{1}{r(\xi)}\right]$$

has the density

$$1/2\ \Psi(\tfrac{\beta}{2})$$

where Ψ is defined in (1.6.7).

Finally we conjecture that similar results can be obtained for the asymptotic distribution for estimates of

$$T(x) = \Phi[F(x)]f(x)$$

when T is monotone and Φ has a special known form.

CHAPTER 3

ESTIMATION OF THE LOCATION OF THE CUSP OF A CONTINUOUS DENSITY

3.1 Introduction:

Chernoff and Rubin [5] and Rubin [17] investigated the problem of estimation of the location of a discontinuity in density in their papers in the third and fourth Berkeley symposiums respectively. They have shown that the maximum likelihood estimator is hyper-efficient under some regularity conditions on the density and that asymptotically the estimation problem is equivalent to that for a non-stationary process with unknown center of non-stationarity. Daniels [7] has obtained an asymptotically efficient estimator of θ (modified maximum likelihood estimator) for the family of densities

$$f(x,\theta) = \frac{1}{2\Gamma(1+\frac{1}{\lambda})} \exp\{-|x-\theta|^{\lambda}\},\ \frac{1}{2} < \lambda < 1.$$

In this chapter, we shall obtain a hyper-efficient estimator for θ where θ is a parameter determining the family of densities $f(x,\theta)$ given by

$$\log f(x,\theta) = \begin{cases} \varepsilon(x,\theta)|x-\theta|^{\lambda} + g(x,\theta) & \text{for } |x| \leq A \\ g(x,\theta) & \text{for } |x| > A \end{cases} \tag{3.1.1}$$

where

(i) $\varepsilon(x,\theta) = \begin{cases} \beta(\theta) & \text{if } x < \theta \\ \gamma(\theta) & \text{if } x > \theta, \end{cases}$

(ii) $0 < \lambda < 1/2$, and (3.1.2)

(iii) $\theta \in (\alpha,\beta)$ where $-A < \alpha < \beta < A$.

We shall prove that hyper-efficient estimators, among them the maximum likelihood estimator (M.L.E.), exist for θ under some regularity conditions and that asymptotically the estimation problem is equivalent to the estimation of the location parameter for a non-stationary Gaussian process.

We obtain some results related to the asymptotic properties of the M.L.E. in Section 3.2. The estimation problem is reduced to that of a stochastic process in Section 3.3. The asymptotic distribution of the M.L.E. is obtained in Section 3.4. Section 3.5 contains the evaluation of integrals encountered in Section 3.2.

3.2 Some results related to the asymptotic properties of the maximum likelihood estimator:

Since our interest centers around obtaining the asymptotic distribution of the M.L.E. of θ, we can assume, without loss of generality, that the true value of the parameter is zero.

We shall assume that the following regularity conditions are satisfied by $f(x,\theta)$.

(i) For each $\theta \neq 0 \in [\alpha,\beta]$, there corresponds a $\delta(\theta) > 0$ such that

$$E_0[\sup_{|\phi-\theta| \leq \delta(\theta)}[\log f(x,\phi) - \log f(x,0)]] < 0. \quad (3.2.1)$$

(ii) For every $\theta \in [\alpha,\beta]$,

$$\frac{\partial g(x,\theta)}{\partial\theta}, \ \frac{\partial^2 g(x,\theta)}{\partial\theta^2} \text{ exist and}$$

$$E_0[\,|[\frac{\partial g(x,\theta)}{\partial\theta}]_{\theta=0}|\,] < \infty \text{ and } E_0[\,|\frac{\partial^2 g(x,\theta)}{\partial\theta^2}|\,] \leq k < \infty$$

for all θ. (3.2.2)

(iii) $$E_0[\frac{\partial \log f(x,\theta)}{\partial\theta}|_{\theta=0}] = 0. \qquad (3.2.3)$$

(iv) $\beta(\theta)$ and $\gamma(\theta)$ are twice differentiable at all θ with bounded derivatives. (3.2.4)

(v) $|f(x,0) - f(0,0)| \leq K\,|x|^{\lambda}$ for all $x \in [-A,A]$ for some constant K. (3.2.5)

We would like to mention here that even if the density is given by

$$\log f(x,\theta) = \begin{cases} \varepsilon(x,\theta)|x-\theta|^{\lambda} + g(x,\theta) & \text{for } -A \leq x \leq B \\ g(x,\theta) & \text{for } x < -A \text{ and } x > B \end{cases}$$

where conditions (3.1.2) and (3.2.1)-(3.2.5) are satisfied, it can be reduced to the form (3.1.1) by suitably modifying the function $g(x,\theta)$ and the conditions (3.1.2) and (3.2.1)-(3.2.5) can be shown to be satisfied by the new density very easily.

Let X_i, $1 \leq i \leq n$ be n independent and identically distributed observations from $f(x,\theta)$.

Let $\hat{\theta}_n$ denote the M.L.E. of θ.

<u>Lemma 3.2.1</u>.

<u>The M.L.E.</u> $\hat{\theta}_n$ <u>is strongly consistent under condition</u> (3.2.1).

<u>Proof</u>:

Let $S(\theta,\delta_\theta)$ denote the interval $(\theta-\delta_\theta,\ \theta+\delta_\theta)$ where δ_θ is given for each θ by condition (3.2.1). Choose any number $\eta > 0$.

Let

$$L_k(\theta) = \sum_{i=1}^{k} \log f(X_i, \theta). \tag{3.2.6}$$

Let $\Omega = [\theta: \alpha \leq \theta \leq \beta] \cap [\theta: |\theta| \geq \eta]$.

We notice that $\bigcup_{\theta \in \Omega} S(\theta, \delta_\theta) \supset \Omega$ and Ω is compact. Therefore, there exists a finite set $\theta_1, \ldots, \theta_M$ in Ω such that

$$\bigcup_{i=1}^{M} S(\theta_i, \delta_{\theta_i}) \supset \Omega. \tag{3.2.7}$$

Now

$$P_0[\bigcup_{k \geq n} \{|\hat{\theta}_k| > \eta\}] \leq P_0[\bigcup_{k \geq n} \{\sup_{\theta \in \Omega} L_k(\theta) > L_k(0)\}]$$

$$\leq \sum_{i=1}^{M} P_0[\bigcup_{k \geq n} \{\sup_{\theta \in S(\theta_i, \delta_{\theta_i})} L_k(\theta) > L_k(0)\}] \tag{3.2.8}$$

by (3.2.7).

Choose an $\varepsilon > 0$.

Since $E\{\sup_{\theta \in S(\theta_i, \delta_{\theta_i})} [\log f(X, \theta) - \log f(X, 0)]\} < 0$

by condition (3.2.1), it follows by the Strong law of large numbers, that there exists an integer $N(\theta_i, \varepsilon)$ such that

$$P[\bigcup_{k \geq n} \sup_{\theta \in S(\theta_i, \delta_{\theta_i})} \sum_{j=1}^{k} [\log f(X_j, \theta) - \log f(X_j, 0)] < 0]$$

$$> 1 - \frac{\varepsilon}{M}.$$

for every $n > N(\theta_i, \varepsilon)$, $i = 1, \ldots, M$.

Since

$$\sup_{\theta \in S(\theta_i, \delta_{\theta_i})} \sum_{j=1}^{n} [\log f(X_j,\theta) - \log f(X_j,0)]$$

$$\leq \sum_{j=1}^{n} \sup_{\theta \in S(\theta_i, \delta_{\theta_i})} [\log f(X_j,\theta) - \log f(X_j,0)],$$

we get that

$$P[\bigcup_{k \geq n} \left\{ \sup_{\theta \in S(\theta_i, \delta_{\theta_i})} L_k(\theta) > L_k(0) \right\}] < \frac{\varepsilon}{M}$$

for every $n > \text{Max}[N(\theta_i,\varepsilon),\ i = 1, \ldots, M]$.

Therefore, from (3.2.7) and (3.2.8) we get that

$$P[\bigcup_{k \geq n} \{|\hat{\theta}_k| > \eta\}] < M \frac{\varepsilon}{M} = \varepsilon$$

for every $n > \text{Max}(N(\theta_i,\varepsilon),\ 1 \leq i \leq M)$.

In other words,

$\hat{\theta}_n$ is strongly consistent.

Let us now consider the log-likelihood ratio

$L_n(\theta) - L_n(0)$ where L_n is defined in (3.2.5).

We have

$$L_n(\theta) - L_n(0) = \sum_{i=1}^{n}{}_1 [\varepsilon(X_i,\theta)|X_i-\theta|^\lambda - \varepsilon(X_i,0)|X_i|^\lambda]$$

$$+ \sum_{i=1}^{n} [g(X_i,\theta) - g(X_i,0)]$$

where Σ_1 denotes that the sum is extended over those X_i such that $|X_i| \leq A$.

Let $g'(X,\theta) = \frac{\partial g(X,\theta)}{\partial\theta}$ and $g''(X,\theta) = \frac{\partial^2 g(X,\theta)}{\partial\theta^2}$.

Then by Taylor's theorem (in view of (3.2.2)),

$$L_n(\theta) - L_n(0) = \sum_{i=1}^{n} [\varepsilon(X_i,\theta)|X_i-\theta|^{\lambda} - \varepsilon(X_i,0)|X_i|^{\lambda}]$$

$$+ \theta \sum_{i=1}^{n} g'(X_i,0)$$

$$+ \frac{\theta^2}{2} \int_0^1 (1-t) \sum_{i=1}^{n} g''(X_i,\theta t)dt. \qquad (3.2.9)$$

$$= \sum_{i=1}^{n} \Psi(X_i,\theta) + n\theta\, E[g'(X,0)] + \sqrt{n}\,\theta\, W_n + n\frac{\theta^2}{2} V_n \qquad (3.2.10)$$

where

(i) $\Psi(X,\theta) = \begin{cases} \varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,0)|X|^{\lambda} & \text{for } |X| \leq A \\ 0 & \text{otherwise,} \end{cases}$

(ii) $W_n = [\frac{1}{\sqrt{n}} (\sum_{i=1}^{n} g'(X_i,0) - n\, E(g'(X,0))]$,

and

(iii) $V_n = \frac{1}{n} \sum_{i=1}^{n} \int_0^1 g''(X_i,\theta t)(1-t)dt. \qquad (3.2.11)$

Since $g'(X_i,0)$ are i.i.d. random variables, condition (3.2.2) implies that W_n is asymptotically normal with mean 0 and finite variance by the central limit theorem for i.i.d. random variables.

Since $g''(X_i,\theta t)$ are i.i.d. random variables, by condition (3.2.2), we get that

$$V_n = 0_p(1).$$

Therefore, from (3.2.10), we have

$$L_n(\theta) - L_n(0)$$

$$= \sum_{i=1}^{n} \Psi(X_i,\theta) + n\,\theta\, E[g'(X,0)]$$

$$+ \sqrt{n}\,\theta\, 0_p(1) + n\frac{\theta^2}{2}\, 0_p(1).$$

The lemmas which we will prove next lead to the calculation of $E_0[L_n(\theta) - L_n(0)]$, $Var_0[L_n(\theta) - L_n(0)]$ and $Var_0[L_n(\theta) - L_n(\phi)]$.

Lemma 3.2.2.

For any $\theta, \phi \in [\alpha, \beta]$,

$$E_0[\Psi(X,\theta) - \Psi(X,\phi)]^2 \leq B\ |\theta - \phi|^{2\lambda+1}$$

where B is a constant independent of θ and ϕ,

and

$$E_0[\Psi(X,\theta) - \Psi(X,\phi)]^2 = |\theta-\phi|^{2\lambda+1}\ [2C\ f(0,0) + o(1)]$$

as $\theta \to 0$ and $\phi \to 0$

where

$$C = \frac{\Gamma(\lambda+1)\Gamma(1/2 - \lambda)}{2^{2\lambda+1}\sqrt{\pi}\ (2\lambda+1)}[\beta^2(0) + \gamma^2(0) - 2\beta(0)\gamma(0)\cos \pi\lambda] \tag{3.2.12}$$

Proof:

Let us assume without loss of generality that $\theta > \phi$.

By the definition of $\Psi(X,\theta)$,

$$E_0[\Psi(X,\theta) - \Psi(X,\phi)]^2$$

$$= \int_{-A}^{A} [\varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}]^2\ f(X,0)dX$$

$$= T_1 + T_2 \tag{3.2.13}$$

where

(i) $$T_1 = \int_{-A}^{A}[\varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}]^2 f(0,0)dX$$

and

(ii) $$T_2 = \int_{-A}^{A}[\varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}]^2 [f(X,0)-f(0,0)]dX. \tag{3.2.14}$$

Let $\eta = \theta - \phi$.

By condition (3.2.5),

$$|T_2| \leq K\int_{-A}^{A}[\varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}]^2|X|^{\lambda}dX$$

$$\leq 2K\ (T_3 + T_4) \tag{3.2.15}$$

where

$$\text{(i)}\quad T_3 = \int_{-A}^{A}[\varepsilon(X-\eta,\phi)|X-\theta|^{\lambda}-\varepsilon(X,\phi)|X-\phi|^{\lambda}]^2\ |X|^{\lambda}\,dX$$

and

$$\text{(ii)}\quad T_4 = \int_{-A}^{A}\ |X-\theta|^{2\lambda}|X|^{\lambda}[\varepsilon(X-\eta,\phi) - \varepsilon(X,\theta)]^2dX \tag{3.2.16}$$

For $X < \theta$, $\varepsilon(X,\theta) - \varepsilon(X-\eta,\phi) = \beta(\theta) - \beta(\phi)$

and for $X > \theta$, $\varepsilon(X,\theta) - \varepsilon(X-\eta,\phi) = \gamma(\theta) - \gamma(\phi)$.

Therefore, integrand of T_4 is of the order $|\theta-\phi|^2 O(|X-\theta|^{2\lambda}|X|^{\lambda})$ since $\beta(\theta)$ and $\gamma(\theta)$ have bounded derivatives.

Since θ belongs to a finite interval, it follows that

$$T_4 = |\theta-\phi|^2\ O(1). \tag{3.2.17}$$

Let

$$\text{(i)}\quad T_5 = \int_{-A}^{A}\ [\varepsilon(X-\eta,\phi)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}]^2\ f(0,0)dX$$

and

$$\text{(ii)}\quad T_6 = \int_{-A}^{A}\ [\{\varepsilon(X,\theta)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}\} -\{\varepsilon(X-\eta,\phi)|X-\theta|^{\lambda} - \varepsilon(X,\phi)|X-\phi|^{\lambda}\}]^2 f(0,0)dX. \tag{3.2.18}$$

From the inequality

$$|[\int[u(x) + v(x)]^2dx]^{1/2} - [\int u^2(x)dx]^{1/2}| \leq [\int v^2(x)dx]^{1/2},$$

it follows that

$$|T_1^{1/2} - T_5^{1/2}| \le T_6^{1/2}. \qquad (3.2.19)$$

From (3.2.18), we have

$$T_6 = \int_{-A}^{A} [\varepsilon(X,\theta) - \varepsilon(X-\eta,\phi)]^2 |X-\theta|^{2\lambda} f(0,0)dX$$

$$= |\theta-\phi|^2\ O(1), \qquad (3.2.20)$$

since θ belongs to a finite interval and $\beta(\theta)$ and $\gamma(\theta)$ have bounded derivatives.

Let us now consider T_5. We have

$$T_5 = \eta^{2\lambda+1} f(0,0) \int_{-A/\eta}^{A/\eta} [\varepsilon(\eta Y-\eta,\phi)|Y - \frac{\theta}{\eta}|^{\lambda} - \varepsilon(\eta Y,\phi)|Y - \frac{\phi}{\eta}|^{\lambda}]^2 dY$$

by the substitution $X = \eta Y$,

$$= \eta^{2\lambda+1} f(0,0) \int_{-\frac{A-\phi}{\eta}}^{\frac{A-\phi}{\eta}} [\varepsilon(\eta Z+\phi-\eta,\phi)\,|Z-1|^{\lambda} - \varepsilon(\eta Z+\phi,\phi)\,|Z|^{\lambda}]^2 dZ$$

by the substitution $Y = Z + \frac{\phi}{\eta}$,

$$= \eta^{2\lambda+1} f(0,0) \int_{-\frac{A-\phi}{\eta}}^{\frac{A-\phi}{\eta}} [h(Z-1)\,|Z-1|^{\lambda} - h(Z)\,|Z|^{\lambda}]^2\ dZ$$

where $h(Z) = \begin{cases} \beta(\phi) & \text{if } Z < 0 \\ \gamma(\phi) & \text{if } Z > 0. \end{cases}$

Therefore,

$$T_5 = \eta^{2\lambda+1} f(0,0) \int_{-\infty}^{\infty} [h(Z-1)\,|Z-1|^{\lambda} - h(Z)\,|Z|^{\lambda}]^2\ dZ$$

$$- \eta^{2\lambda+1} f(0,0) \int_{Z \notin (\frac{-A-\phi}{\eta},\ \frac{A-\phi}{\eta})} [h(Z-1)\,|Z-1|^{\lambda} - h(Z)\,|Z|^{\lambda}]^2\ dZ$$

$$= \eta^{2\lambda+1} f(0,0)\ \ 2C(\phi) - \eta^{2\lambda+1} f(0,0) \int [h(Z-1)|Z-1|^{\lambda} - h(Z)|Z|^{\lambda}]^2 dZ$$

$$Z \notin \left(\frac{-A-\phi}{\eta}, \frac{A-\phi}{\eta}\right) \qquad (3.2.21)$$

$$\text{where } C(\Phi) = \frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\lambda)}{2^{2\lambda+1}\sqrt{\pi}(2\lambda+1)} [\beta^2(\Phi) + \gamma^2(\Phi) - 2\beta(\Phi)\gamma(\Phi)\cos \pi\lambda] \qquad (3.2.22)$$

(The integral will be evaluated in Section 3.5, lemma 3.5.2).

Since θ and Φ belong to a finite interval, it follows from (3.2.21), that for any θ and Φ in $[\alpha,\beta]$

$$T_5 = \eta^{2\lambda+1}\ O(1) \qquad (3.2.23)$$

and for θ and Φ in $[\alpha,\beta]$ such that $\Phi \to 0$ and $\eta \to 0$

$$T_5 = 2\ C(0)\ \eta^{2\lambda+1} f(0,0) + \eta^{2\lambda+1} o(1). \qquad (3.2.24)$$

Let us now consider T_3.

We notice that

$$|T_3| \leq |A|^{\lambda} \int_{-A}^{A} [\varepsilon(X-\eta,\Phi)|X-\theta|^{\lambda} - \varepsilon(X,\Phi)|X-\Phi|^{\lambda}]^2 dX$$

$$\leq \frac{|A|^{\lambda}}{f(0,0)} T_5 \ .$$

Therefore, from (3.2.23), it follows that for any θ and Φ in the interval $[\alpha,\beta]$,

$$T_3 = \eta^{2\lambda+1}\ O(1). \qquad (3.2.25)$$

On the other hand, for θ and Φ such that

$\eta \to 0$ and $\Phi \to 0$, let us evaluate T_3.

$$\text{Now } T_3 = \eta^{2\lambda+1} \int_{\frac{-A-\Phi}{\eta}}^{\frac{A-\Phi}{\eta}} [\varepsilon(\eta Z+\Phi-\eta,\Phi)|Z-1|^{\lambda} - \varepsilon(\eta Z+\Phi,\Phi)|Z|^{\lambda}]^2 |\Phi+\eta Z|^{\lambda} dZ$$

Let $Q_{\Phi,\eta}(Z)$ denote the integrand in the right hand side. We observe that as Φ and $\eta \to 0$, the range of integration tends to $(-\infty,\infty)$. We note that

$$Q_{\phi,\eta}(Z) = \begin{cases} \beta^2(\phi)[|Z-1|^\lambda - |Z|^\lambda]^2 |\phi+\eta Z|^\lambda & \text{for } Z < 0, \\ \gamma^2(\phi)[|Z-1|^\lambda - |Z|^\lambda]^2 |\phi+\eta Z|^\lambda & \text{for } Z > 1, \\ [\beta(\phi)|Z-1|^\lambda - \gamma(\phi)|Z|^\lambda]^2 |\phi+\eta Z|^\lambda & \text{for } 0 < Z < 1. \end{cases}$$

By condition (3.2.4), $\beta(\phi)$ and $\gamma(\phi)$ are bounded.

Let $\quad C_1 = \sup_{\phi\in[\alpha,\beta]} \{|\beta(\phi)|, \ |\gamma(\phi)|\}.$

Therefore the integrand $Q_{\phi,\eta}(Z)$ is bounded by

$$Q(Z) = \begin{cases} C_1^2[|Z-1|^\lambda - |Z|^\lambda]^2 |A|^\lambda & \text{for } Z < 0, \\ C_1^2[|Z-1|^\lambda - |Z|^\lambda]^2 |A|^\lambda & \text{for } Z > 1, \\ C_1^2[|Z-1|^{2\lambda} + C_1^2|Z|^{2\lambda} + 2C_1^2|Z-1|^\lambda|Z|^\lambda]\,|A|^\lambda & \\ & \text{for } 0 < Z < 1, \end{cases}$$

for all ϕ and η.

Further $Q(Z)$ is integrable over $(-\infty,\infty)$ since $0 < \lambda < 1/2$, and $Q_{\phi,\eta}(Z) \to 0$ as $\phi \to 0$ and $\eta \to 0$. Therefore by the bounded convergence theorem,

$$\int_{-\frac{A-\phi}{\eta}}^{\frac{A-\phi}{\eta}} Q_{\phi,\eta}(Z) \quad dZ \to 0 \quad \text{as} \quad \phi \to 0 \quad \text{and} \quad \eta \to 0.$$

In other words, for θ and ϕ such that $\eta \to 0$ and $\phi \to 0$,

$$T_3 = \eta^{2\lambda+1} \, o(1). \tag{3.2.26}$$

From (3.2.19) and (3.2.20), it follows that

$$T_1^{1/2} = T_5^{1/2} + |\theta-\phi| \; O(1).$$

Therefore, from (3.2.23), we have

$$T_1 = T_5 + |\theta-\phi|\, T_5^{1/2}\, 0(1) + |\theta-\phi|^2\, 0(1)$$

$$= T_5 + |\theta-\phi|^{\lambda+3/2}\, 0(1) + |\theta-\phi|^2\, 0(1), \qquad (3.2.27)$$

Let us first consider the general case when θ and ϕ are any numbers in $[\alpha,\beta]$.

Now

$$E_0[\Psi(X,\theta)-\Psi(X,\phi)]^2 = T_1 + T_2 \qquad \text{by (3.2.13)},$$

$$\leq T_5+|\theta-\phi|^{\lambda+3/2}\, 0(1)+|\theta-\phi|^2\, 0(1)+2K(T_3+T_4)$$

by (3.2.15) and (3.2.27),

$$\leq \eta^{2\lambda+1}\, 0(1)+\eta^{\lambda+3/2}\, 0(1)+\eta^2\, 0(1)$$

$$+ 2K\{\eta^{2\lambda+1}\, 0(1)+\eta^{\lambda+3/2}\, 0(1)+\eta^2\, 0(1)\}$$

by (3.2.23), (3.2.25) and (3.2.17),

$$= \eta^{2\lambda+1}[0(1)+\eta^{1/2-\lambda}\, 0(1)+\eta^{1-2\lambda}\, 0(1)]$$

$$= \eta^{2\lambda+1}\, 0(1) \qquad (3.2.28)$$

since $0 < \lambda < 1/2$ and η is in a finite interval.

Suppose in addition that θ and ϕ approach 0.

Then

$$E_0[\Psi(X,\phi)-\Psi(X,\phi)]^2 = T_1 + T_2$$

$$= T_5+|\theta-\phi|^{\lambda+3/2}\, 0(1)+|\theta-\phi|^2\, 0(1)$$

$$+ \eta^{2\lambda+1}\, o(1) + \eta^2\, 0(1)$$

by (3.2.26), (3.2.27), (3.2.15) and (3.2.17)

$$= \eta^{2\lambda+1}[2C\ f(0,0)+o(1)]+\eta^{\lambda+3/2}\, 0(1)$$

$$+ \eta^2\, 0(1) + \eta^{2\lambda+1}\, o(1) \qquad \text{by (3.2.24)},$$

$$= \eta^{2\lambda+1}[2C\ f(0,0)+o(1)+\eta^{1/2-\lambda}\ O(1)$$

$$+ \eta^{1-2\lambda}O(1)]$$

$$= \eta^{2\lambda+1}[2C\ f(0,0)+ o(1)]\ , \tag{3.2.29}$$

since $0 < \lambda < 1/2$.

(3.2.18) and (3.2.29) together prove the lemma.

Let

$$\Phi(X,\theta) = \begin{cases} \theta\ \varepsilon(X,0)\ \lambda\ \mathrm{Sgn}\ X\ |X|^{\lambda-1} & \text{for } |X| \leq A \\ 0 & \text{Otherwise} \end{cases} \tag{3.2.30}$$

<u>Lemma 3.2.3</u>.

$$E_0[\Psi(X,\theta)+\phi(X,\theta)] = |\theta|^{1+2\lambda}[-C+o(1)]\,f(o,o) + \int_{-A}^{A}[\varepsilon(X,\theta)-\varepsilon(X-\theta,0)]\,|X-\theta|^{\lambda}f(X,0)dX$$

<u>where</u> $o(1)$ <u>is in</u> θ <u>and</u> C <u>is given by</u> (3.2.12).

<u>Proof</u>:

Let us suppose that $\theta > 0$.

By the definition of Ψ and ϕ,

$$E_0[\Psi(X,\theta) + \phi(X,\theta)]$$

$$= \int_{-A}^{A}[\varepsilon(X,\theta)\,|X-\theta|^{\lambda}-\varepsilon(X,0)\,|X|^{\lambda}$$

$$+ \theta\ \varepsilon(X,0)\ \lambda\ \mathrm{Sgn}\ X|X|^{\lambda-1}]\,f(X,0)dX$$

$$= T_1 + T_2 \tag{3.2.31}$$

where

(i) $$T_1 = \int_{-A}^{A}[\varepsilon(X-\theta,0)\,|X-\theta|^{\lambda}-\varepsilon(X,0)\,|X|^{\lambda}$$

$$+ \theta\ \varepsilon(X,0)\ \lambda\ \mathrm{Sgn}\ X|X|^{\lambda-1}]\,f(X,0)dX$$

and

$$\text{(ii)} \quad T_2 = \int_{-A}^{A} [\varepsilon(X,\theta) - \varepsilon(X-\theta,0)]\,|X-\theta|^{\lambda} f(X,0)\,dX \quad (3.2.32)$$

We now have

$$T_1 = |\theta|^{1+\lambda} \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\,\varepsilon(X,0)\,\mathrm{Sgn}\,X|X|^{\lambda-1}]\,f(X\theta,0)\,dX$$

$$= |\theta|^{1+\lambda}(T_3 + T_4)$$

where

$$\text{(i)} \quad T_3 = \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\,\varepsilon(X,0)\,\mathrm{Sgn}\,X|X|^{\lambda-1}]\,f(0,0)\,dX$$

and

$$\text{(ii)} \quad T_4 = \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\,\varepsilon(X,0)\,\mathrm{Sgn}\,X\;|X|^{\lambda-1}]\,[f(X\theta,0)-f(0,0)]\,dX, \quad (3.2.33)$$

we have

$$T_3 = f(0,0)\int_{-\infty}^{\infty} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda} + \lambda\varepsilon(X,0)\mathrm{Sgn}\,X|X|^{\lambda-1}]\,dX$$

$$- f(0,0)\int_{|X|\geq A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\,\varepsilon(X,0)\,\mathrm{Sgn}\,X\;|X|^{\lambda-1}]\,dX$$

$$= -f(0,0)\int_{|X|\geq A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\,\varepsilon(X,0)\,\mathrm{Sgn}\,X\;|X|^{\lambda-1}]\,dX$$

by lemma 3.5.1 of Section 3.5.

Since the integrand on the right hand side of the above equality is of the order $0(|X|^{\lambda-2})$,

$$T_3 = 0(1)\;|\theta|^{1-\lambda}. \quad (3.2.34)$$

Let us now evaluate T_4. Let $f = f(0,0)$

From (3.2.33), we have

$$T_4 = \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\, \varepsilon(X,0) \text{ Sgn } X \;\; |X|^{\lambda-1}]\, \varepsilon(X,0)|X\theta|^{\lambda} f\, dX$$

$$+ \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\, \varepsilon(X,0) \text{ Sgn } X \;\; |X|^{\lambda-1}]$$

$$[f(X\theta,0)-f(0,0)-f\varepsilon(X,0)|X\theta|^{\lambda}] \; dX. \quad (3.2.35)$$

Let

(i) $$T_5 = f|\theta|^{\lambda} \int_{-\infty}^{\infty} [\varepsilon(X-1),0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\, \varepsilon(X,0) \text{ Sgn } X \;\; |X|^{\lambda-1}]\, \varepsilon(X,0)|X|^{\lambda} \, dX,$$

(ii) $$T_6 = -f|\theta|^{\lambda} \int_{|X| \geq A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\, \varepsilon(X,0) \text{ Sgn } X \;\; |X|^{\lambda-1}] \;\; \varepsilon(X,0)|X|^{\lambda} \, dX,$$

and

(iii) $$T_7 = \int_{-A/\theta}^{A/\theta} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda}$$

$$+ \lambda\, \varepsilon(X,0) \text{ Sgn } X \;\; |X|^{\lambda-1}]$$

$$[f(X\theta,0) - f(0,0) - f\varepsilon(X,0) \;\; |X\theta|^{\lambda}] \; dX.$$

We have

$$T_4 = T_5 + T_6 + T_7. \quad (3.2.36)$$

By lemma 3.5.4 of Section 3.5,

$$T_5 = -Cf|\theta|^{\lambda} . \quad (3.2.37)$$

Since the integrand of T_6 is of the order $0(|X|^{2\lambda-2})$,

$$T_6 = |\theta|^{\lambda} \, |\theta|^{1-2\lambda} O(1). \qquad (3.2.38)$$

Let us now evaluate T_7.

We notice that the range of integration tends to $(-\infty, \infty)$ as $\theta \to 0$.

Furthermore for each X,

$$\frac{f(X\theta,0) - f(0,0) - f_\delta(X,0)|X\theta|^{\lambda}}{|\theta|^{\lambda}} \to 0 \quad \text{as} \quad \theta \to 0$$

and the integrand of T_7 is bounded by $C_1 \, |X|^{\lambda}|\theta|^{\lambda}$ for each X, and is bounded by $C_2 \, |X|^{\lambda-2}$ for X large for some constants C_1 and C_2.

Therefore, by the bounded convergence theorem,

$$T_7 = |\theta|^{\lambda} o(1). \qquad (3.2.39)$$

From (3.2.34) - (3.2.39), we have

$$T_3 + T_4 = |\theta|^{1-\lambda} \, O(1) - Cf|\theta|^{\lambda} + |\theta|^{\lambda} \, o(1) \qquad (3.2.40)$$

Therefore,

$$T_1 = |\theta|^{1+\lambda}[-Cf|\theta|^{\lambda} + |\theta|^{\lambda} \, o(1) + |\theta|^{1-\lambda} \, O(1)]$$

$$= |\theta|^{1+2\lambda}[-Cf + o(1)].$$

Now, from (3.2.31) and (3.2.32), we have

$$E_0[\Psi(X,\theta) + \Phi(X,\theta)]$$

$$= |\theta|^{1+2\lambda}[-Cf + o(1)] + \int_{-A}^{A}[\varepsilon(X,\theta) - \varepsilon(X-\theta,0)] \, |X-\theta|^{\lambda} f(X,0) \, dX. \qquad (3.2.41)$$

Now, if $\theta \to 0$, then

$$E_0[\Psi(X,\theta) + \Phi(X,\theta)] = |\theta|^{1+2\lambda}[-Cf + o(1)]$$

$$+ \int_{-A}^{A}[\varepsilon(X,\theta) - \varepsilon(X-\theta,0)] \, |X-\theta|^{\lambda} f(X,0) \, dX \qquad (3.2.42)$$

since $0 < \lambda < 1/2$, (3.2.42) proves lemma 3.2.3.

Using the results obtained in lemmas 3.2.2 and 3.2.3, we shall compute $E_0(L_n(\theta) - L_n(0)]$, $\mathrm{Var}_0(L_n(\theta) - L_n(0)$ and $\mathrm{Var}_0\ (L_n(\theta) - L_n(\phi))$ in the following lemmas 3.2.4 and 3.2.5.

Lemma 3.2.4.

$$E_0[L_n(\theta) - L_n(0)] = -\, n\, Cf|\theta|^{1+2\lambda}[1 + o(1)]$$

where $o(1)$ is in θ and C is given in (3.2.12), and in general, for any $\theta \in [\alpha,\beta]$,

$$E_0[L_n(\theta) - L_n(0)] \le -n\ H\ |\theta|^{1+2\lambda}$$

where H is a constant independent of n and θ.

Proof:

Let us assume that $\theta > 0$.

We have

$$\begin{aligned}
E_0[L_n(\theta) - L_n(0)] &= n\ E[\log f(X,\theta) - \log f(X,0)] \\
&= n\ E[\Psi(X,\theta)] + n\ E[g(X,\theta) - g(X,0)] \\
&= n\ E[\Psi(X,\theta)] + n\ E[\theta\ g'(X,0) \\
&\quad + \frac{\theta^2}{2}\int_0^1 (1-t)g''(X,\theta t)\ dt] \\
&= n\ E[\Psi(X,\theta) + \theta\ g'(X,0)] + n\ \theta^2\ 0(1) \\
&\qquad\qquad \text{by condition (3.2.2)}, \\
&= n\ E[\Psi(X,\theta) + \Phi(X,\theta)] + n\ \theta^2\ 0(1) \\
&\quad + n\ E[\theta\ g'(X,0) - \Phi(X,\theta)]. \qquad (3.2.43)
\end{aligned}$$

Where Φ is defined in (3.2.30).

Let

(i) $$T_1 = E[\Psi(X,\theta)+\Phi(X,\theta)] - \int_{-A}^{A}[\varepsilon(X,\theta)-\varepsilon(X-\theta,0)]\ |X-\theta|^{\lambda}f(X,0)dX$$

and

(ii) $T_2 = E[\theta\, g'(X,0) - \phi(X,\theta)]$

$$+ \int_{-A}^{A} [\varepsilon(X,\theta) - \varepsilon(X-\theta,0)]\, |X-\theta|^{\lambda}\, f(X,0)\, dX. \tag{3.2.44}$$

From (3.2.43), we have

$$E_0[L_n(\theta) - L_n(0)] = n\, T_1 + n\, T_2 + n\, \theta^2\, O(1). \tag{3.2.45}$$

From (3.2.3), we have

$$E_0\left\{\frac{\partial \log f(X,\theta)}{\partial\theta}\bigg|_{\theta=0}\right\} = 0.$$

Therefore,

$$\int_{-A}^{A} [\varepsilon'(X,0)|X|^{\lambda} - \lambda\, \varepsilon(X,0)\, \text{Sgn}\, X\, |X|^{\lambda-1}]\, f(X,0)\, dX$$

$$+ \int_{-\infty}^{\infty} g'(X,0)\, f(X,0)\, dX = 0.$$

In other words,

$$E[\theta\, g'(X,0) - \phi(X,\theta)] = -\theta \int_{-A}^{A} \varepsilon'(X,0)|X|^{\lambda}\, f(X,0)\, dX.$$

From (3.2.44), it follows that

$$T_2 = \int_{-A}^{A} \{[\varepsilon(X,\theta) - \varepsilon(X-\theta,0)]\, |X-\theta|^{\lambda} - \theta\varepsilon'(X,0)|X|^{\lambda}\} f(X,0)\, dX.$$

We note that $T_2 = \theta^2\, O(1)$ since θ is in a finite interval and $\beta(\theta), \gamma(\theta)$ have bounded second derivatives.

Therefore from (3.2.45), it follows that

$$E_0[L_n(\theta) - L_n(0)] = n\, T_1 + n\, \theta^2\, O(1).$$

By lemma 3.2.3, as θ approaches zero,

$$T_1 = |\theta|^{1+2\lambda}[-C + o(1)]\, f.$$

Therefore, if $\theta \to 0$, then

$$E_0[L_n(\theta) - L_n(0)] = n\ |\theta|^{1+2\lambda}[-Cf + o(1)] \qquad (3.2.46)$$

since $0 < \lambda < 1/2$.

In other words there exists a number $\eta > 0$ such that

$$E_0[L_n(\theta) - L_n(0)] \leq -n\frac{Cf}{2}|\theta|^{1+2\lambda} \qquad (*)$$

for all θ such that $\theta \in [\alpha,\beta]$ and $|\theta| < \eta$.

Let us now consider the set

$$\Omega = [\theta: \alpha \leq \theta \leq \beta] \cap [\theta: |\theta| \geq \eta].$$

Since this is a compact set, there exists a finite set $\theta_1, \ldots, \theta_m$ in Ω such that

$$\bigcup_{i=1}^{m} S(\theta_i, \delta_{\theta_i}) \supseteq \Omega$$

where $S(\theta_i, \delta_{\theta_i})$ denotes the interval $(\theta_i - \delta_{\theta_i}, \theta_i + \delta_{\theta_i})$ and δ_{θ_i} is given by condition (3.2.1).

Therefore

$$\sup_{\theta\in\Omega} E_0[\log f(X,\theta) - \log f(X,0)]$$

$$\leq \sup_{1 \leq i \leq m} E\left[\sup_{\theta\in S(\theta_i,\delta_{\theta_i})} \{\log f(X,\theta) - \log f(X,0)\}\right]$$

$$< 0$$

by condition (3.2.1).

Since $|\theta| \geq \eta > 0$, we have now

$$\sup_{\theta\in\Omega} E_0\left[\frac{\log f(X,\theta) - \log f(X,0)}{|\theta|^{1+2\lambda}}\right] < 0 .$$

Let $D = -\sup_{\theta\in\Omega} E_0\left\{\frac{\log f(X,\theta) - \log f(X,0)}{|\theta|^{1+2\lambda}}\right\}$

Notice that $D > 0$.

Then for every $\theta \in \Omega$,

$$E_0[L_n(\theta) - L_n(0)] = n\, E[\log f(X,\theta) - \log f(X,0)]$$

$$\leq - nD|\theta|^{1+2\lambda} .$$

This, together with (*), implies that

$$E_0[L_n(\theta) - L_n(0)] \leq - n\, H\, |\theta|^{1+2\lambda} \tag{3.2.47}$$

for every θ in $[\alpha,\beta]$,

where H is greater than zero.

Lemma 3.2.5.

For any θ *and* ϕ *in the interval* $[\alpha,\beta]$,

$$\mathrm{Var}_0[L_n(\theta) - L_n(\phi)] \leq n\, Q\, |\theta-\phi|^{2\lambda+1}$$

where Q *is a constant independent of* θ,ϕ *and* n,

and

$$\mathrm{Var}_0[L_n(\theta) - L_n(0)] = 2\, nC\, f(0,0)\, |\theta|^{2\lambda+1}(1 + o(1))$$

where $o(1)$ *is in* θ *and* C *is given by* (3.2.12).

Proof:

Since X_i, $1 \leq i \leq n$ are i.i.d. random variables

$$\mathrm{Var}_0[L_n(\theta) - L_n(0)]$$

$$= n\, \mathrm{Var}_0[\log f(X,\theta) - \log f(X,0)] . \tag{3.2.48}$$

Let us now compute for $\theta \to 0$,

$$E_0[\log f(X,\theta) - \log f(X,0)]^2$$

$$= E_0[\Psi(X,\theta) + g(X,\theta) - g(X,0)]^2$$

$$= E_0[\Psi(X,\theta)]^2 + E_0[g(X,\theta) - g(X,0)]^2$$

$$+ 2\, E_0[\Psi(X,\theta)\{g(X,\theta) - g(X,0)\}] .$$

$$= T_1 + T_2 + 2T_3 \tag{3.2.49}$$

where

(i) $T_1 = E_0[\Psi(X,\theta)]^2$,

(ii) $T_2 = E_0[g(X,\theta) - g(X,0)]^2$,

and

(iii) $T_3 = E_0[\Psi(X,\theta)\{g(X,\theta) - g(X,0)\}]$.

Note that $T_2 = \theta^2\ 0(1)$ by condition (3.2.2) and

$T_1 = |\theta|^{2\lambda+1}\ 0(1)$ by lemma 3.2.2.

Further

$|T_3| \leq \sqrt{T_1 T_2}$. Therefore $T_3 = |\theta|^{\frac{3+2\lambda}{2}}\ 0(1)$.

Therefore, from (3.2.49), we have for $\theta \to 0$

$$
\begin{aligned}
& E_0[\log f(X,\theta) - \log f(X,0)]^2 \\
& = [2\ c\ f(0,0) + o(1)]\,|\theta|^{2\lambda+1} + |\theta|^{\frac{3+2\lambda}{2}}\ 0(1) + |\theta|^2 0(1) \\
& = |\theta|^{2\lambda+1}[2\ c\ f(0,0) + o(1)] \qquad (3.2.50)
\end{aligned}
$$

since $0 < \lambda < 1/2$.

Now

$$
\begin{aligned}
& \mathrm{Var}_0[\log f(X,\theta) - \log f(X,0)] \\
& = E_0[\log f(X,\theta) - \log f(X,0)]^2 - \{E_0[\log f(X,\theta) - \log f(X,0)]\}^2 \\
& = |\theta|^{2\lambda+1}[2c\ f(0,0) + o(1)] - \{-c|\theta|^{1+2\lambda}[1 + o(1)]\}^2 f^2
\end{aligned}
$$

by (3.2.50) and lemma 3.2.4,

$$
= |\theta|^{2\lambda+1}[2c\ f(0,0) + o(1)].
$$

Therefore, from (3.2.48), we have

$$
\begin{aligned}
& \mathrm{Var}_0[L_n(\theta) - L_n(0)] \\
& = n\ |\theta|^{2\lambda+1}\ 2c\ f(0,0)[1 + o(1)] \qquad (3.2.51)
\end{aligned}
$$

Let us now consider $\mathrm{Var}_0[L_n(\theta) - L_n(\Phi)]$ for any θ, Φ in $[\alpha,\beta]$.

Obviously

$$\mathrm{Var}_0[L_n(\theta) - L_n(\phi)]$$

$$\leq n\, E_0[\log f(X,\theta) - \log f(X,\ )]^2$$

$$= n\, E_0[\{\Psi(X,\theta) - \Psi(X,\phi)\} + \{g(X,\theta) - g(X,\phi)\}]^2$$

$$\leq 2n\ \{E_0[\Psi(X,\theta) - \Psi(X,\phi)]^2 + E[g(X,\theta) - g(X,\phi)]^2\}$$

$$\leq 2n\ \{B|\theta-\phi|^{2\lambda+1} + |\theta-\phi|^2\, 0(1)\}$$

by condition (3.2.2) and lemma 3.2.2,

$$\leq n\, Q\ |\theta-\phi|^{2\lambda+1} \tag{3.2.52}$$

where Q is some constant since θ and ϕ belong to a finite interval. (3.2.51) and (3.2.52) prove lemma 3.2.5.

We shall now prove a theorem which enables us to conclude that the probability, that the maximum of $L_n(\theta) - L_n(0)$ is attained outside the interval $[-K\, n^{-\frac{1}{1+2\lambda}}, K\, n^{-\frac{1}{1+2\lambda}}]$, approaches zero for K sufficiently large. More precisely,

Theorem 3.2.6.

There exists $\eta > 0$ such that

$$\lim_{\tau\to\infty} \overline{\lim_{n}}\, P_0\Big[\sup_{|\theta|>\tau n^{-\frac{1}{1+2\lambda}}} \frac{M_n(\theta)}{n|\theta|^{2\lambda+1}} \geq -\eta\Big] = 0$$

where $M_n(\theta) = L_n(\theta) - L_n(0)$.

Proof:

Since $M_n(\theta)$ is continuous in θ, it is enough to prove that

$$\lim_{\tau\to\infty} \overline{\lim_{n}}\, P_0\Big[\sup_{|\theta_{ijk}|>\tau n^{-\frac{1}{1+2\lambda}}} \frac{M_n(\theta)}{n|\theta_{ijk}|^{2\lambda+1}} \geq -\eta\Big] = 0 \tag{3.2.53}$$

for some set $\{\theta_{ijk}\}$ dense in $\{\theta : |\theta| > \tau n^{-\frac{1}{1+2\lambda}}\}$.

Let $\theta_{ijk} = \tau n^{-\frac{1}{1+2\lambda}} 2^{i+\frac{k}{2^j}}$ for $i = 0,1,2,\ldots$

$j = 0,1,2,\ldots$

$k = 0,1,2,\ldots,2^j-1$. (3.2.54)

Obviously θ_{ijk} is dense in $\{\theta: \theta > \tau n^{-\frac{1}{1+2\lambda}}\}$.

We shall prove (3.2.53) when θ ranges over

$$\{\theta: \theta > \tau n^{-\frac{1}{1+2\lambda}}\}.$$

The proof is analogous when θ ranges over

$$\{\theta: \theta < -\tau n^{-\frac{1}{1+2\lambda}}\}.$$

Let $\gamma = 2\lambda + 1$.

Let us now define

$$T_n(\theta_{ijk}) = M_n(\theta_{ijk}) - E[M_n(\theta_{ijk}) + nH\,\theta^\gamma_{ioo}], \quad (3.2.55)$$

where H is defined in lemma 3.2.4.

Since $\theta^\gamma_{ioo} \leq \theta^\gamma_{ijk}$ and

$$E[M_n(\theta_{ijk})] \leq -nH\,\theta^\gamma_{ijk} \leq -nH\,\theta^\gamma_{ioo},$$

it follows that

$$\frac{T_n(\theta_{ijk})}{\theta^\gamma_{ioo}} \geq \frac{M_n(\theta_{ijk})}{\theta^\gamma_{ijk}}. \quad (3.2.56)$$

Therefore

$$P_0[\sup_{\theta_{ijk} > \tau n^{-\frac{1}{\gamma}}} \frac{M_n(\theta_{ijk})}{n\theta^\gamma_{ijk}} \geq -\eta]$$

$$\leq P_0[\sup_{\theta_{ijk} > \tau n^{-\frac{1}{\gamma}}} \frac{T_n(\theta_{ijk})}{n\theta^\gamma_{ioo}} \geq -\eta] \quad (3.2.57)$$

From (3.2.55),

$$E_0[T_n(\theta_{ioo})] = -nH\ \theta_{ioo}^{\gamma} = -\ H\ \tau^{\gamma}2^{i\gamma}$$

and

$$E_0[T_n(\theta_{i,j,2k+1}) - T_n(\theta_{i,j-1,k})] = 0. \qquad (3.2.58)$$

Now

$$\text{Var}[T_n(\theta_{ioo})] = \text{Var}[M_n(\theta_{ioo})]$$

$$\leq n\ Q\ \theta_{ioo}^{\gamma} = Q\ \tau^{\gamma}2^{i\gamma}, \qquad (3.2.59)$$

Let us now compute

$$\text{Var}_0[T_n(\theta_{i,j,2k+1}) - T_n(\theta_{i,j-1,k})]$$

$$= \text{Var}_0[M_n(\theta_{i,j,2k+1}) - M_n(\theta_{i,j-1,k})]$$

$$\leq n\ Q|\theta_{i,j,2k+1} - \theta_{i,j-1,k}|^{\gamma} \quad \text{by lemma 3.2.5,}$$

$$\leq Q\ \tau^{\gamma}\ 2^{i\gamma}\ \left|2^{\frac{2k+1}{2^j}} - 2^{\frac{k}{2^{j-1}}}\right|^{\gamma}$$

$$= Q\ \tau^{\gamma}\ 2^{i\gamma}\ 2^{\frac{2k+1}{2^j}\gamma}\ |1 - 2^{-j}|^{\gamma}$$

$$\leq Q\ \tau^{\gamma}\ 2^{(i+1)^{\gamma}}\ [\frac{\log 2}{2^j}]^{\gamma}$$

$$= Q\ \tau^{\gamma}\ (2\log 2)^{\gamma}\ 2^{(i-j)^{\gamma}}. \qquad (3.2.60)$$

We observe that

$$T_n(\theta_{ioo}) \leq -n\zeta\ \theta_{ioo}^{\gamma}$$

and

$$T_n(\theta_{i,j,2k+1}) - T_n(\theta_{i,j-1,k}) \leq nP_j\theta_{ioo}^{\gamma}$$

$$\text{for all}\quad i,j \text{ and } k = 0,1,\ldots,2^{j-1}-1$$

imply that

$$T_n(\theta_{ijk}) \leq -n\eta\theta^{\gamma}_{ioo} \text{ for all } i$$

provided

$$-\zeta + \sum_{j=1}^{\infty} P_j \leq -\eta. \tag{3.2.61}$$

We shall choose $\zeta > 0$ and sequence P_j suitably at the end so as to satisfy the condition (3.2.61).

Therefore, $P_0[\sup_{\theta_{ijk}>\tau n} - \frac{1}{1+2\lambda} \frac{T_n(\theta_{ijk})}{n\theta^{\gamma}_{ioo}} \geq -\eta]$

$$\leq \sum_{i=0}^{\infty} P[T_n(\theta_{ioo}) \geq -n\zeta\, \theta^{\gamma}_{ioo}]$$

$$+ \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{2^{j-1}-1} P[T_n(\theta_{i,j,2k+1}) - T_n(\theta_{i,j-1,k}) \leq nP_j\theta^{\gamma}_{ioo}]$$

$$\leq \sum_{i=0}^{\infty} \frac{Q\,\tau^{\gamma}2^{i\gamma}}{[H-\zeta]^2\tau^{2\gamma}2^{2i\gamma}} + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{2^{j-1}\, Q\tau^{\gamma}2^{i\gamma}(2\log 2)^{\gamma}2^{-j\gamma}}{P_j^2\, \tau^{2\gamma}\, 2^{2i\gamma}}$$

by (3.2.58), (3.2.59), (3.2.60) and Chebyshev's inequality,

$$= \frac{Q}{(H-\zeta)^2\tau^{\gamma}} \sum_{i=0}^{\infty} 2^{-i\gamma}$$

$$+ \frac{Q}{\tau^{\gamma}}[\sum_{i=0}^{\infty} 2^{-i\gamma}] \frac{(2\log 2)^{\gamma}}{2} \sum_{j=1}^{\infty} 2^{j(1-\gamma)} \quad P_j^{-2}$$

$$= \frac{Q}{\tau^{\gamma}}(\frac{1}{1-2^{-\gamma}})[\frac{1}{(H-\zeta)^2} + \frac{(2\log 2)^{\gamma}}{2} \sum_{j=1}^{\infty} 2^{j(1-\gamma)}\, P_j^{-2}] \tag{3.2.62}$$

Let us choose $0 < \zeta < H$ and $P_j = 2^{-\frac{\lambda j}{2}}\delta$ where $\delta > 0$ and

$$\eta = \zeta - \sum_{1}^{\infty} P_j = \zeta - \frac{\delta\cdot 2^{-\lambda/2}}{1-2^{-\lambda/2}}.$$

Then, from (3.2.72).

$$P_0\left[\sup_{\theta_{ijk} > \tau n^{\frac{1}{1+2\lambda}}} \frac{T_n(\theta_{ijk})}{n\theta_{ioo}^{\gamma}} \geq -\eta\right]$$

$$\leq \frac{Q}{\tau^{\gamma}} \frac{1}{(1-2^{-\gamma})}\left[\frac{1}{(H-\zeta)^2} + \frac{(2\log 2)^{\gamma}}{2^{\lambda+1}\delta^2} \frac{1}{1-2^{-\lambda}}\right]. \tag{3.2.63}$$

Therefore, by (3.2.57)

$$\overline{\lim_{n}} \quad P_0\left[\sup_{\theta_{ijk} > \tau n^{-\frac{1}{\gamma}}} \frac{M_n(\theta_{ijk})}{n\theta_{ijk}^{\gamma}} \geq -\eta\right]$$

$$\leq \frac{Q}{\tau^{\gamma}} \frac{1}{1-2^{-\gamma}}\left[\frac{1}{(H-\zeta)^2} + \frac{(2\log 2)^{\gamma}}{2^{\lambda+1}\delta^2} \frac{1}{1-2^{-\lambda}}\right]$$

where $\zeta < H$, $\delta > 0$ and $\eta = \zeta - \dfrac{\delta\, 2^{-\lambda/2}}{1-2^{-\lambda/2}}$ and $\gamma = 2\lambda+1$.

Taking limits as $\tau \to \infty$, we get that

$$\overline{\lim_{\tau}} \; \overline{\lim_{n}} \; P_0\left[\sup_{\theta_{ijk} > \tau n^{-\frac{1}{\gamma}}} \frac{M_n(\theta_{ijk})}{n\theta_{ijk}^{\gamma}} \geq -\eta\right] = 0$$

since $\gamma > 0$.

This proves theorem 3.2.6 in view of the remarks made at the beginning of the proof.

3.3 Reduction to a problem in stochastic processes:

We shall reduce now the problem of determining the asymptotic distribution of $\hat{\theta}_n$ or equivalently the asymptotic distribution of the maximum of $M_n(\theta)$ to that of a limiting process. In view of theorem 3.2.6, we can restrict our attention to intervals of the type $[-\tau n^{-\frac{1}{\gamma}}, \tau n^{-\frac{1}{\gamma}}]$ where $\tau > 0$ and $\gamma = 2\lambda + 1$ in order to locate the maximum of $M_n(\theta)$. For $\tau > 0$,

let $X_n(\zeta) = M_n(n^{-\frac{1}{-1+2\lambda}}\zeta)$ for $\zeta \in [-\tau,\tau]$, and $X(\zeta)$ be the continuous normal non-stationary process on $[-\tau,\tau]$ with

$$E[X(\zeta)] = -C|\zeta|^{\gamma} f(0,0),$$

$$\mathrm{Var}[X(\zeta)] = 2\,C\,f(0,0)\;|\zeta|^{\gamma},$$

and $$\mathrm{Cov}[X(\zeta_1),X(\zeta_2)] = C\;f(0,0)[\,|\zeta_1|^{\gamma}+|\zeta_2|^{\gamma}-|\zeta_1-\zeta_2|^{\gamma}],\tag{3.3.1}$$

where $C = \dfrac{\Gamma(\lambda+1)\;\Gamma(1/2-\lambda)}{2^{2\lambda+1}\sqrt{\pi}\;(2\lambda+1)}\,[\beta^2(0) + \gamma^2(0) - 2\beta(0)\gamma(0)\cos\pi\lambda]$.

Let

$$A_n(\zeta) = X_n(\zeta) - E(X_n(\zeta))$$

and $$A(\zeta) = X(\zeta) - E(X(\zeta)).\tag{3.3.2}$$

<u>Theorem 3.3.1</u>.

<u>For any</u> $\zeta \in [-\tau,\tau]$, $A_n(\zeta)$ <u>is asymptotically normal with mean</u> 0 <u>and variance</u> $2C\;f(0,0)\;|\zeta|^{\gamma}$.

<u>Proof</u>:

By definition

$$A_n(\zeta) = M_n(\theta) - E[M_n(\theta)]$$

where

$$\theta = n^{-\frac{1}{1+2\lambda}}\,\zeta.$$

Let

$$A_n(\zeta) = B_n(\zeta) + C_n(\zeta)$$

where

$$B_n(\zeta) = \sum_{i=1}^{n}{}_1 \, [\varepsilon(X_i,\theta)|X_i-\theta|^{\lambda} - \varepsilon(X_i,0)|X_i|^{\lambda}]$$

$$- n\, E[\psi(X,\theta)]$$

and

$$C_n(\zeta) = \sum_{i=1}^{n}{}' \, [g(X_i,\theta) - g(X_i,0)] - n\, E[g(X,\theta) - g(X,0)]$$

where Σ_1 denotes that the sum is extended over only those X_i's for which $|X_i| \leq A$.

Obviously

$$E_0(C_n(\zeta)) = 0$$

and

$$\begin{aligned} \mathrm{Var}_0(C_n(\zeta)) &\leq n\, E_0[g(X,\theta) - g(X,0)]^2 \\ &\leq k\, n\, \theta^2 \qquad \text{by condition 3.2.2} \\ &= k\, n\, n^{-\frac{2}{1+2\lambda}} \zeta^2 \\ &= k\, n^{\frac{2\lambda-1}{2\lambda+1}} \zeta^2 \, . \end{aligned}$$

Since $0 < \lambda < 1/2$, it follows that

$$\mathrm{Var}_0\, (C_n(\zeta)) \to 0 \quad \text{as} \quad n \to \infty .$$

Therefore,

$C_n(\zeta)$ converges to 0 in probability as $n \to \infty$.

Hence $A_n(\zeta)$ and $B_n(\zeta)$ have the same asymptotic distribution by Slutsky's Theorem.

Let F_n^* be the distribution function of $B_n(\zeta)$ and $\Phi(x)$ be the normal distribution with mean 0 and variance 1.

Let

$$Y = \Psi(X,\theta)$$

where

$$\theta = n^{-\frac{1}{1+2\lambda}}\zeta \quad \text{and} \quad \Psi \quad \text{is as defined in (3.2.11).}$$

Then

$$B_n(\zeta) = \sum_{i=1}^{n} [Y_i - E(Y_i)]$$

where Y_i are i.i.d as Y.

By the normal approximation theorem, (See Loeve [13], pp. 288)

$$|F_n^*(x) - \Phi(x)| \leq \frac{C_0}{[\mathrm{Var}\ B_n(\zeta)]^{3/2}} \ n\, E|Y - E(Y)|^3.$$

where C_0 is a numerical constant.

It can be easily shown by methods analogous to those used in lemma 3.2.5, that

$$E|Y - E(Y)|^3 \leq C_1|\theta|^{3\lambda+1}$$

where C_1 is a constant independent of θ.

$$\text{Therefore,}\quad |F_n^*(x) - \Phi(x)| \leq \frac{C_0\ n\ C_1\ n^{-\frac{3\lambda+1}{1+2\lambda}}}{\{2c\ f(0,0)|\zeta|^{\gamma}[1+o(1)]\}^{3/2}}|\zeta|^{3\lambda+1}$$

$$= \frac{C_0C_1|\zeta|^{3\lambda+1}}{\{2c\ f(0,0)|\zeta|^{\gamma}[1+o(1)]\}^{3/2}}\ n^{-\frac{\lambda}{1+2\lambda}}.$$

This term tends to zero as $n \to \infty$ since $\lambda > 0$.

Therefore, $\dfrac{B_n(\zeta)}{\sqrt{\mathrm{Var}(B_n(\zeta))}}$ is asymptotically normal with mean 0 and variance 1. But we note that

$$\mathrm{Var}\ (B_n(\zeta)) = 2c\ f(0,0)|\zeta|^{\gamma}[1 + o(1)],$$

from the proof of lemma 3.2.5.

This establishes that $B_n(\zeta)$ is distributed asymptotically as normal with mean 0 and variance $2C\, f(0,0)|\zeta|^{\gamma}$. Therefore, $A_n(\zeta)$ is distributed asymptotically as normal with mean 0 and variance $2C\, f(0,0)|\zeta|^{\gamma}$.

Remark: By the normal approximation theorem again, it can be shown that for any real numbers $a_1, \ldots, a_k$ and $\zeta_1, \zeta_2, \ldots, \zeta_k$ in $[-\tau,\tau]$,

$$\sum_{i=1}^{k} a_n A_n(\zeta_i) \text{ is asymptotically normal with mean } 0$$

and variance

$$2C\, f(0,0)[\sum_{i=1}^{k} a_i^2|\zeta_i|^{\gamma} + \sum_{\substack{i=1 \\ i<j}}^{k} \sum_{j=1}^{k} a_i\, a_j\{|\zeta_i|^{\gamma}+|\zeta_j|^{\gamma}-|\zeta_i-\zeta_j|^{\gamma}\}]$$

The next theorem shows that the process $A_n(\zeta)$ on $[-\tau,\tau]$ satisfy an equi-continuity condition.

Theorem 3.3.2.

For any ζ_1,ζ_2 in $[-\tau,\tau]$,

$$E|A_n(\zeta_1) - A_n(\zeta_2)|^2 \leq Q|\zeta_1 - \zeta_2|^{\gamma}$$

where Q is a constant independent of n,ζ_1,ζ_2 and $\gamma = 2\lambda + 1$.

Proof:

By the definition of $X_n(\zeta)$

$$E[A_n(\zeta_1) - A_n(\zeta_2)]^2 = \mathrm{Var}_0[M_n(\theta_1) - M_n(\theta_2)]$$

where $\theta_1 = n^{-\frac{1}{\gamma}}\zeta_1$ and $\theta_2 = n^{-\frac{1}{\gamma}}\zeta_2$.

But from lemma 3.2.5,

$$\mathrm{Var}_0\ [M_n(\theta_1) - M_n(\theta_2)] \le n\, Q\ |\theta_1 - \theta_2|^{\gamma}.$$

Therefore

$$E[A_n(\zeta_1) - A_n(\zeta_2)]^2 \le nQ\ n^{-1}|\zeta_1-\zeta_2|^{\gamma}$$
$$= Q\ |\zeta_1 - \zeta_2|^{\gamma}.$$

From Kolmogorov's theorem, (see Doob [8]) the process $A_n(\zeta)$ has trajectories in $C[-\tau,\tau]$ by theorem 3.3.2.

We shall now state a theorem connected with convergence of distributions of stochastic processes on $C[a,b]$, where $C[a,b]$ denotes the space of continuous functions on $[a,b]$ with supremum norm topology.

Theorem 3.3.3.

Let X_n *be a sequence of stochastic processes on* $C[a,b]$ *and* X *be another process on* $C[a,b]$ *such that*

(i) *for any* $t_i \in [a,b]$, $1 \le i \le k$, *the joint distribution of* $[X_n(t_1, \ldots, X_n(t_k)]$ *converges to the joint distribution of* $[X(t_1), \ldots, X(t_k)]$, *and*

(ii) *there exist constants* $A,B,C, > 0$ *independent of* n *such that for every* n

$$E|X_n(t_1) - X_n(t_2)|^A < C|t_1 - t_2|^{1+B}.$$

Then the sequence of processes X_n converge in distribution to the process X.

For a proof of the above theorem, see theorem 2.4 of Sethuraman [18].

Theorem 3.3.1 and the remarks made at the end of its proof together with theorem 3.3.2 imply the following result in view of theorem 3.3.3.

Theorem 3.3.4

The processes $A_n(\zeta)$ on $[-\tau,\tau]$ converge in distribution to the process $A(\zeta)$ on $[-\tau,\tau]$.

Therefore we have

Theorem 3.3.5

The process $X_n(\zeta)$ on $[-\tau,\tau]$ converge in distribution to the process $X(\zeta)$ on $[-\tau,\tau]$.

Proof:

Since $E[X_n(\zeta)] = -C\ |\zeta|^\gamma[1 + o(1)]f$ where $o(1)$ is uniform for $\zeta \in [-\tau,\tau]$ as $n \rightarrow \infty$ by lemma 3.2.4, and since

$E[X(\zeta)] = -Cf|\zeta|^\gamma$, it follows that

$$E[X_n(\zeta)] \rightarrow E[X(\zeta)] \quad \text{as} \quad n \rightarrow \infty$$

uniformly for ζ in the interval $[-\tau,\tau]$.

Therefore, by an extension of Slutsky's theorem to stochastic processes (See Rubin [16]), it follows from theorem 3.3.4, that the process $A_n(\zeta) + E[X_n(\zeta)]$ converges in distribution to the process $A(\zeta) + E[X(\zeta)]$.

In other words,

The process $X_n(\zeta)$ converges in distribution to the process $X(\zeta)$ on $[-\tau,\tau]$.

For any $x \in C[-\tau,\tau]$, let $g(x)$ be the value of t that maximizes $x(t)$ over $[-\tau,\tau]$. Obviously $g(x)$ is a continuous functional in the supremum norm topology on $C[-\tau,\tau]$, provided x has a unique maximum.

Therefore by theorem 3.3.5, <u>the distribution of</u> $g(X_n(\zeta))$ <u>converges to the distribution of</u> $g(X(\zeta))$ for $\zeta \in [-\tau,\tau]$. Hence we have the following theorem.

<u>Theorem 3.3.6</u>.

<u>The distribution of the position of the maximum of</u> $M_n(\theta)$ <u>over</u> $[-\tau n^{-\frac{1}{\gamma}}, \tau n^{-\frac{1}{\gamma}}]$ <u>converges to the distribution of the position of the maximum of non-stationary Gaussian process</u> $X(\zeta)$ <u>defined in</u> (3.3.1) <u>over</u> $[-\tau,\tau]$.

The next theorem proves that the process $X(\zeta)$ over $(-\infty,\infty)$ has its maximum in a finite interval with probability one.

<u>Theorem 3.3.7</u>

$$\text{Prob}\left[\limsup_{|\tau|\to\infty} \frac{X(\tau)}{C|\tau|^{\gamma}f} \leq -1\right] = 1$$

where C is given in (3.2.12).

<u>Proof</u>:

We shall first prove that

$$\text{Prob}\left[\limsup_{\tau\to+\infty} \frac{X(\tau)}{C\tau^{\gamma}f} \leq -1\right] = 1.$$

Define $A(\tau) = X(\tau) - E[X(\tau)]$

$$= X(\tau) + C|\tau|^{\gamma}f. \tag{3.3.3}$$

Let

$$Z_0 = \sup_{1 \leq \tau \leq 2} A(\tau) \quad \text{and}$$

$$Z_n = \sup_{2^n \leq \tau \leq 2^{n+1}} A(\tau) \quad \text{for } n = 0,1,2, \ldots$$

and
$$U = \sup_{1 \leq \tau \leq 2} |A(\tau) - A(1)| \tag{3.3.4}$$

Since $A(\tau)$ is normally distributed with mean 0 and variance $2C\, f(0,0)|\tau|^{\gamma}$ and covariance of $A(\tau_1)$ and $A(\tau_2)$ is $C\, f(0,0)[|\tau_1|^{\gamma} + |\tau_2|^{\gamma} - |\tau_1-\tau_2|^{\gamma}]$, it follows that Z_n and $Z_0\, 2^{\frac{n\gamma}{2}}$ have identical distributions. Therefore, for any $\varepsilon > 0$,

$$P[Z_n > \varepsilon\, 2^{n\gamma}] = P[Z_0 > \varepsilon\, 2^{\frac{n\gamma}{2}}]. \tag{3.3.5}$$

Let $k = C\, f(0,0)$.

We note that $k > 0$ and $1 < \gamma < 2$.

Since $A(\tau)$ is continuous on any finite interval with probability one and since dyadic rationals are dense in $[1,2]$,

$$U = \sup_{\substack{2^{\ell} + 1 \leq s \leq 2^{\ell+1} \\ \ell = 0,1,2, \ldots}} \left|A\left(\frac{s-1}{2^{\ell}}\right) - A\left(\frac{s}{2^{\ell}}\right)\right|$$

with probability one.

Therefore

$$U \leq \sum_{\ell=1}^{\infty} T_{\ell} \quad \text{with probability one} \tag{3.3.6}$$

where
$$T_{\ell} = \sup_{2^{\ell} + 1 \leq s \leq 2^{\ell+1}} \left|A\left(\frac{s-1}{2^{\ell}}\right) - A\left(\frac{s}{2^{\ell}}\right)\right|.$$

Now for any $a > 0$ and $1 > r > 0$

$$P[T_{\ell} > ar^{\ell}]$$
$$= P\left[\sup_{2^{\ell} + 1 \leq s \leq 2^{\ell+1}} \left|A\left(\frac{s-1}{2^{\ell}}\right) - A\left(\frac{s}{2^{\ell}}\right)\right| > ar^{\ell}\right]$$

$$\leq \sum_{s=2^{\ell}+1}^{2^{\ell+1}} P[|A(\frac{s-1}{2^{\ell}}) - A(\frac{s}{2^{\ell}})| > ar^{\ell}]$$

$$= (2^{\ell+1} - 2^{\ell}) \quad P(|A(\frac{s-1}{2^{\ell}}) - A(\frac{s}{2^{\ell}})| > ar^{\ell})$$

for some s since $A(\frac{s-1}{2^{\ell}}) - A(\frac{s}{2^{\ell}})$ are i.i.d as normal with mean 0 and variance $k\ 2^{1-\ell\gamma}$.

Therefore, by Chebyshev's inequality

$$P[T_{\ell} > ar^{\ell}] \leq 2^{\ell} \frac{2^{1-\ell\gamma}}{a^2 r^{2\ell}} k. \tag{3.3.7}$$

Now, from (3.3.6), we have

$$\text{Prob}[U > \frac{a}{1-r}] \leq \text{Prob}\ [T_{\ell} > ar^{\ell} \quad \text{for some } 1 \leq \ell < \infty]$$

$$\leq \sum_{\ell=1}^{\infty} P[T_{\ell} > ar^{\ell}]$$

$$\leq \sum_{\ell=0}^{\infty} 2^{\ell} \frac{2^{1-\ell\gamma}}{a^2 r^{2\ell}} k$$

by (3.3.7),

$$= \frac{2}{a^2} k \frac{1}{(1-\frac{2^{1-\gamma}}{r^2})} .$$

Let $r = 2^{-\frac{\gamma}{4}}$.

Therefore,

$$P[U > \frac{a}{1-2^{-\frac{\gamma}{4}}}] \leq \frac{2}{a^2} k \frac{1}{1-2^{1-\frac{\gamma}{2}}} .$$

Equivalently

$$P[U > a] \leq \frac{D}{a^2} \tag{3.3.8}$$

where D is a constant.

Therefore

$$E(U) = \int_0^{\infty} P(U > a)da$$

$$= \int_0^{D} P[U > a]da + \int_D^{\infty} P[U > a]da$$

$$\leq D + \int_D^{\infty} \frac{D}{a^2} da$$

by (3.3.9),

$$\leq D + 1 < \infty \ . \qquad (3.3.9)$$

Since

$$|Z_0| \leq |A(1)| + \sup_{1 \leq \tau \leq 2} |A(\tau) - A(1)|$$

$$= |A(1)| + U,$$

$$E|Z_0| \leq E(|A(1)|) + E(U) < \infty$$

by (3.3.9).

Let $E|Z_0| = J$.

Now

$$\sum_{n=0}^{\infty} P[Z_n > \varepsilon\ 2^{n\gamma}] = \sum_{n=0}^{\infty} P[Z_0 > \varepsilon\ 2^{n\gamma/2}]$$

by (3.3.5),

$$\leq \sum_{n=0}^{\infty} \frac{J}{\varepsilon} \frac{1}{2^{n\gamma/2}} \qquad \text{by (3.3.9)}$$

and Chebyshev's inequality,

$$= \frac{J}{\varepsilon} \cdot \frac{1}{1-2^{-\gamma/2}} \qquad \text{since } 1 < \gamma < 2.$$

Therefore

$$\sum_{n=0}^{\infty} P[Z_n > \varepsilon\ 2^{n\gamma}] < \infty \qquad \text{for all } \varepsilon > 0.$$

Then, Borel-Cantelli lemma implies that

$$P[Z_n > \varepsilon\, 2^{n\gamma} \text{ infinitely often}] = 0 \quad \text{for every} \quad \varepsilon > 0.$$

In other words

$$\text{Prob}\,[\limsup_{n} \frac{Z_n}{2^{n\gamma}} \leq 0] = 1. \tag{3.3.10}$$

Since $\dfrac{A(\tau)}{\tau^\gamma} \leq \dfrac{Z_n}{2^{n\gamma}}$ if $2^n \leq \tau \leq 2^{n+1}$,

it follows that,

$$\text{Prob}[\limsup_{\tau \to \infty} \frac{A(\tau)}{\tau^\gamma} \leq 0] = 1.$$

Since $A(\tau) = X(\tau) + C\tau^\gamma f$ for $\tau > 0$,

we have

$$\text{Prob}\,[\limsup_{\tau \to +\infty} \frac{X(\tau)}{C\tau^\gamma f} \leq -1] = 1. \tag{3.3.11}$$

Similarly we can prove that

$$\text{Prob}\,[\limsup_{\tau \to -\infty} \frac{X(\tau)}{C|\tau|^\gamma f} \leq -1] = 1. \tag{3.3.12}$$

(3.3.11) and (3.3.12) together prove the theorem 3.3.7.

3.4 Asymptotic distribution of the maximum likelihood estimator :

From theorems 3.2.6, 3.3.6, and 3.3.7, we get the following final theorem.

Theorem 3.4.1 .

Consider the family of densities $f(x,\theta)$ given by

$$\log f(x,\theta) = \begin{cases} \varepsilon(x,\theta)|x-\theta|^\lambda + g(x,\theta) & \text{for } |x| \leq A \\ g(x,\theta) & \text{for } |x| > A \end{cases}$$

where (i) A is a finite number

(ii) $0 < \lambda < 1/2$

(iii) $\varepsilon(x,\theta) = \begin{cases} \beta(\theta) & \text{if } x < \theta \\ \gamma(\theta) & \text{if } x > \theta \end{cases}$

and (iv) θ belongs to a finite interval (α,β) satisfying the regularity conditions (3.2.1) - (3.2.5).

Let $\hat{\theta}_n$ denote the M.L.E. of θ based on n independent observations of $f(x,\theta)$. Let θ_0 denote the true value of θ. Then

$$n^{\frac{1}{1+2\lambda}}[\hat{\theta}_n - \theta_0] \quad \text{has a limiting distribution}$$

and it is the distribution of the position of the maximum of the non-stationary Gaussian process $X(\tau)$ on $(-\infty,\infty)$ with

$$E[X(\tau)] = -C\,|\tau|^{2\lambda+1} f(\theta_0,\theta_0) \quad \text{and}$$

$$\mathrm{Cov}[X(\tau_1),X(\tau_2)] = C\, f(\theta_0,\theta_0)\,[\,|\tau_1|^{2\lambda+1}+|\tau_2|^{2\lambda+1}-|\tau_1-\tau_2|^{2\lambda+1}]$$

where $C = \dfrac{\Gamma(\lambda+1)\Gamma(1/2-\lambda)}{2^{2\lambda+1}\sqrt{\pi}\,(2\lambda+1)}\,[\beta^2(\theta_0)+\gamma^2(\theta_0)-2\beta(\theta_0)\gamma(\theta_0)\cos\pi\lambda]$.

In other words, the M.L.E. $\hat{\theta}_n$ is a hyper-efficient estimator since

$$1/2 < \frac{1}{1+2\lambda} < 1 \quad \text{for} \quad 0 < \lambda < 1/2.$$

In fact, by analogous methods, it can be shown that Bayes estimators for θ, for smooth prior densities, are also hyper-efficient and asymptotically the Bayes estimation of θ is equivalent to the estimation of the location parameter for a non-stationary Gaussian process.

3.5 Evaluation of integrals:

We shall now evaluate the integrals encountered in Section 3.2.

Lemma 3.5.1.

Let $H(\lambda) \equiv \int_{-\infty}^{\infty} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda} + \lambda\varepsilon(X,0)\operatorname{Sgn} X|X|^{\lambda-1}]\,dX.$

Then $H(\lambda) = 0$ for $0 < \lambda < 1/2$.

Proof:

Since the integrand is of the order $0(|X|^{\lambda-2})$ the integral $H(\lambda) < \infty$ for $0 < \lambda < 1/2$. Let us now compute

$$\int_{-A}^{A} [\varepsilon(X-1,0)|X-1|^{\lambda} - \varepsilon(X,0)|X|^{\lambda} + \lambda\varepsilon(X,0)\ \operatorname{Sgn} X|X|^{\lambda-1}]\,dX$$

$$= \int_{-A}^{0} [\beta(0)(1-x)^{\lambda} - \beta(0)(-x)^{\lambda} - \lambda\beta(0)(-x)^{\lambda-1}]\,dx$$

$$+ \int_{0}^{1} [\beta(0)(1-x)^{\lambda} - \gamma(0)x^{\lambda} + \lambda\gamma(0)x^{\lambda-1}]\,dx$$

$$+ \int_{1}^{A} [\gamma(0)(x-1)^{\lambda} - \gamma(0)x^{\lambda} + \lambda\gamma(0)x^{\lambda-1}]\,dx$$

$$= \beta(0)\int_{1}^{A+1} y^{\lambda}dy - \beta(0)\int_{0}^{A} y^{\lambda}dy - \lambda\beta(0)\int_{0}^{A} y^{\lambda-1}dy$$

$$+ \beta(0)\int_{0}^{1} y^{\lambda}dy - \gamma(0)\int_{0}^{1} x^{\lambda}dx + \lambda\gamma(0)\int_{0}^{1} x^{\lambda-1}dx$$

$$+ \gamma(0)\int_{0}^{A-1} x^{\lambda}dx - \gamma(0)\int_{1}^{A} x^{\lambda}dx + \lambda\gamma(0)\int_{1}^{A} x^{\lambda-1}dx$$

$$= \beta(0)\int_{A}^{A+1} y^{\lambda}dy - \lambda\beta(0)\int_{0}^{A} y^{\lambda-1}dy - \gamma(0)\int_{A-1}^{A} y^{\lambda}dy + \lambda\gamma(0)\int_{0}^{A} x^{\lambda-1}dx$$

$$= \beta(0)\left[\frac{(A+1)^{\lambda+1} - A^{\lambda+1}}{\lambda+1} - A^{\lambda}\right] - \gamma(0)\left[\frac{A^{\lambda+1} - (A-1)^{\lambda+1}}{\lambda+1} - A^{\lambda}\right].$$

Since $0 < \lambda < 1/2$,

$$\frac{(A+1)^{\lambda+1} - A^{\lambda+1}}{\lambda+1} - A^{\lambda} \to 0 \quad \text{as} \quad A \to \infty$$

and

$$\frac{A^{\lambda+1} - (A-1)^{\lambda+1}}{\lambda + 1} - A^{\lambda} \to 0 \quad \text{as} \quad A \to \infty .$$

Therefore $H(\lambda) = 0$ for all λ such that $0 < \lambda < 1/2$.

Lemma 3.5.2.

Let $h(y) = \varepsilon(y)|y|^{\lambda}$ for all y

where

$$\varepsilon(y) = \begin{cases} \beta & \text{for } y < 0 \\ \gamma & \text{for } y > 0. \end{cases}$$

Then for any τ_1, τ_2 and $0 < \lambda < 1/2$,

$$R(\tau_1,\tau_2) \equiv \int_{-\infty}^{\infty} [h(y-\tau_1)-h(y)][h(y-\tau_2)-h(y)]\,dy$$

$$= C[|\tau_1|^{2\lambda+1} + |\tau_2|^{2\lambda+1} - |\tau_1-\tau_2|^{2\lambda+1}] \quad (3.5.1)$$

where

$$C \equiv \frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\lambda)}{2^{2\lambda+1}\sqrt{\pi}(2\lambda+1)} [\beta^2+\gamma^2-2\beta\gamma \cos \pi\lambda] . \quad (3.5.2)$$

Proof:

Since the integrand of $R(\tau_1,\tau_2)$ is of the order $|Y|^{2\lambda-2}$ for Y sufficiently large and since $0 < \lambda < 1/2$, the integral $R(\tau_1,\tau_2)$ is finite.

Define

$$h_{\alpha}(y) = \varepsilon(y)\ |y|^{\lambda}e^{-\alpha|y|} \quad \text{for} \quad \alpha > 0 . \quad (3.5.3)$$

Let us now consider

$$|h_{\alpha}(y-\tau_1)-h_{\alpha}(y)|$$

$$= \left|\varepsilon(y-\tau_1)\,|y-\tau_1|^{\lambda}e^{-\alpha|y-\tau_1|} - \varepsilon(y)\,|y|^{\lambda}e^{-\alpha|y|}\right|$$

$$= \left|\varepsilon(y)\,|y-\tau_1|^{\lambda}e^{-\alpha|y-\tau_1|} - \varepsilon(y)\,|y|^{\lambda}e^{-\alpha|y|}\right|$$

for $|y| > |\tau_1|$,

$$\leq \text{Max}\ (|\beta|,|\gamma|)\left|e^{-\alpha|y-\tau_1|}\{|y-\tau_1|^{\lambda} - |y|^{\lambda}e^{-\alpha|y|+\alpha|y-\tau_1|}\}\right|$$

$$= \text{Max}\ (|\beta|,|\gamma|)\ e^{-\alpha|y-\tau_1|}\left|\{|y-\tau_1|^{\lambda} - |y|^{\lambda}\ e^{-\alpha|\tau_1|}\}\right|. \tag{3.5.4}$$

Let $C_0 = \text{Max}\ (|\beta|,|\gamma|)$.

For $|y| > |\tau_1|$, we have from (3.5.4),

$$|h_\alpha(y-\tau_1) - h_\alpha(y)|$$

$$\leq C_0\ e^{-\alpha|y-\tau_1|}\left|(|y-\tau_1|^{\lambda}-|y|^{\lambda})-|y|^{\lambda}(e^{-\alpha|\tau_1|}-1)\right|$$

$$\leq C_0\ \lambda|\tau_1|\,|y|^{\lambda-1}+C_0|y|^{\lambda}(e^{\alpha|\tau_1|}-1)e^{-\alpha|y|}. \tag{3.5.5}$$

For $|y| > \text{Max}\ (|\tau_1|,|\tau_2|)$,

$$|h_\alpha(y-\tau_1)-h_\alpha(y)|\,|h_\alpha(y-\tau_2)-h_\alpha(y)|$$

$$\leq \{C_0\ \lambda|\tau_1|\,|y|^{\lambda-1} + C_0|y|^{\lambda}(e^{\alpha|\tau_1|}-1)e^{-\alpha|y|}\}$$

$$\{C_0\ \lambda|\tau_2|\,|y|^{\lambda-1} + C_0|y|^{\lambda}(e^{\alpha|\tau_2|}-1)e^{-\alpha|y|}\}.$$

$$\leq C_0^2\ \lambda^2\ |\tau_1\tau_2|\,|y|^{2\lambda-2} + C_0^2|y|^{2\lambda}\alpha^2|\tau_1\tau_2|e^{-2\alpha|y|}$$

$$+ 2C_0^2\ \lambda\ \alpha|\tau_1\tau_2|\,|y|^{2\lambda-1}\ e^{-\alpha|y|}. \tag{3.5.6}$$

Let us observe that for $0 < \lambda < 1/2$,

$$\text{(i)}\quad \int_{\text{Max}(|\tau_1|,|\tau_2|)}^{\infty} |y|^{2\lambda-2} < \infty\ ,$$

$$\text{(ii)}\quad \int_{-\infty}^{\infty} |y|^{2\lambda}\alpha^2\ e^{-2\alpha|y|}\ dy = \frac{\Gamma(1+2\lambda)}{2^{1+2\lambda}}\ \alpha^{1-2\lambda} < \infty\ ,$$

and

$$\text{(iii)}\quad \int_{-\infty}^{\infty} \alpha|y|^{2\lambda-1}\ e^{-\alpha|y|}dy = \Gamma(2\lambda)\ \alpha^{1-2\lambda} < \infty. \tag{3.5.7}$$

Therefore, from (3.5.6), we get that

$$\int_{-\infty}^{\infty} |h_\alpha(y-\tau_1) - h_\alpha(y)|\,|h_\alpha(y-\tau_2) - h_\alpha(y)|dy < \infty.$$

In particular, we get that

$$h_\alpha(y-\tau) - h_\alpha(y) \in L_2(R)$$

for every τ , where $L_2(R)$ denote the set of square integrable functions on the real line.

From (3.5.6) and (3.5.7), we observe that for any $A > \max(|\tau_1|, |\tau_2|)$

$$\int_{|y|>A} |h_\alpha(y-\tau_1) - h_\alpha(y)|\,|h_\alpha(y-\tau_2) - h_\alpha(y)|\; dy$$

$$\leq C_1\; A^{2\lambda-1} + C_2\; \alpha^{1-2\lambda}, \tag{3.5.8}$$

where C_1 and C_2 are constants.

Let $\theta_\alpha(y) = [h_\alpha(y-\tau_1) - h_\alpha(y)][h_\alpha(y-\tau_2)-h_\alpha(y)]$.

Obviously $\theta_\alpha(y) \to \theta_0(y)$ as $\alpha \to 0$ for each y.

Let us consider

$$\left|\int_{-\infty}^{\infty} \theta_\alpha(y)dy - \int_{-\infty}^{\infty} \theta_0(y)dy\right|$$

$$\leq \int_{|y|\leq A} |\theta_\alpha(y)-\theta_0(y)|dy + \int_{|y|>A} |\theta_\alpha(y)|dy + \int_{|y|>A} |\theta_0(y)|dy$$

$$\leq \int_{|y|\leq A} |\theta_\alpha(y)-\theta_0(y)|dy + 2C_1A^{2\lambda-1} + C_2\; \alpha^{1-2\lambda}$$

by (3.5.8).

Choose an $\varepsilon > 0$.

Since $0 < \lambda < 1/2$, we can choose a number A_0 such that

$$A_0^{2\lambda-1} < \frac{\varepsilon}{2C_1} .$$

Therefore

$$\left| \int_{-\infty}^{\infty} \theta_\alpha(y)dy - \int_{-\infty}^{\infty} \theta_0(y)dy \right| \le \int_{|y|\le A_0} |\theta_\alpha(y)-\theta_0(y)|dy+\varepsilon+C_2\alpha^{1-2\lambda}$$

By the bounded convergence theorem

$$\int_{|y|\le A_0} |\theta_\alpha(y) - \theta_0(y)|dy \to 0 \quad \text{as} \quad \alpha \to 0 .$$

Therefore there exists an $\alpha_0 > 0$ such that

for $\alpha > \alpha_0$,

$$\left| \int_{-\infty}^{\infty} \theta_\alpha(y)dy - \int_{-\infty}^{\infty} \theta_0(y)dy \right| \le 3\varepsilon$$

In other words

$$\lim_{\alpha\to 0} \int_{-\infty}^{\infty} \theta_\alpha(y)dy = \int_{-\infty}^{\infty} \theta_0(y)dy . \tag{3.5.9}$$

Let $\hat{h}_\alpha(t)$ denote the Fourier transform of $h_\alpha(y)$.

we have

$$\begin{aligned}\hat{h}_\alpha(t) &= \int_{-\infty}^{\infty} h_\alpha(y)\, e^{ity}\, dy \\ &= \int_0^{\infty}\gamma\, y^\lambda\, e^{-(\alpha-it)y}dy+\int_0^{\infty}\beta\, y^\lambda\, e^{-(\alpha+it)y}\, dy \\ &= \frac{\gamma\,\Gamma(1+\lambda)}{(\alpha-it)^{1+\lambda}} + \frac{\beta\,\Gamma(1+\lambda)}{(\alpha+it)^{1+\lambda}} .\end{aligned} \tag{3.5.10}$$

Let $g_\alpha(t,\tau)$ be the Fourier transform of $h_\alpha(y-\tau)-h_\alpha(y)$.

Now

$$\begin{aligned}g_\alpha(t,\tau) &\equiv \int_{-\infty}^{\infty} [h_\alpha(y-\tau)-h_\alpha(y)]\, e^{ity}dy \\ &= \int_{-\infty}^{\infty} h_\alpha(y-\tau)e^{ity}dy - \int_{-\infty}^{\infty} h_\alpha(y)e^{ity}\, dy\end{aligned}$$

$$= \left[\frac{\gamma\ \Gamma(1+\lambda)}{(\alpha-it)^{1+\lambda}} + \frac{\beta\ \Gamma(1+\lambda)}{(\alpha+it)^{1+\lambda}}\right](e^{it\tau}-1) \qquad (3.5.11)$$

by (3.5.10).

By Parseval's theorem,

$$\int_{-\infty}^{\infty}[h_\alpha(y-\tau_1) - h_\alpha(y)][h_\alpha(y-\tau_2) - h_\alpha(y)]\ dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_\alpha(t,\tau_1)\ \overline{g_\alpha(t,\tau_2)}\ dt$$

$$= \frac{1}{2\pi}\Gamma^2(1+\lambda)\int_{-\infty}^{\infty}\left\{\frac{\gamma}{(\alpha-it)^{1+\lambda}} + \frac{\beta}{(\alpha+it)^{1+\lambda}}\right\}(e^{it\tau_1}-1)$$

$$\left\{\frac{\gamma}{(\alpha+it)^{1+\lambda}} + \frac{\beta}{(\alpha-it)^{1+\lambda}}\right\}(e^{-it\tau_2}-1)dt$$

$$= \frac{1}{2\pi}\Gamma^2(1+\lambda)\int_{-\infty}^{\infty}\left[\frac{\gamma^2+\beta^2}{(\alpha^2+t^2)^{1+\lambda}} + \gamma\beta\left\{\frac{1}{(\alpha-it)^{2+2\lambda}} + \frac{1}{(\alpha+it)^{2+2\lambda}}\right\}\right]$$

$$[e^{it(\tau_1-\tau_2)}-e^{it\tau_1}-e^{-it\tau_2}+1]\ dt \qquad (3.5.12)$$

Therefore,

$$R(\tau_1,\tau_2) \equiv \int_{-\infty}^{\infty}[h(y-\tau_1) - h(y)][h(y-\tau_2) - h(y)]dy$$

$$= \int_{-\infty}^{\infty}\lim_{\alpha\to 0}\{[h_\alpha(y-\tau_1)-h_\alpha(y)][h_\alpha(y-\tau_2)-h_\alpha(y)]dy$$

$$= \lim_{\alpha\to 0}\int_{-\infty}^{\infty}[h_\alpha(y-\tau_1)-h_\alpha(y)][h_\alpha(y-\tau_2)-h_\alpha(y)]dy$$

$$= \lim_{\alpha\to 0}\frac{1}{2\pi}\Gamma^2(1+\lambda)\int_{-\infty}^{\infty}\left[\frac{\gamma^2+\beta^2}{(\alpha^2+t^2)^{1+\lambda}}\right.$$

$$\left.+\ \gamma\beta\left\{\frac{1}{(\alpha-it)^{2+2\lambda}} + \frac{1}{(\alpha+it)^{2+2\lambda}}\right\}\right]$$

$$[e^{it(\tau_1-\tau_2)} - e^{it\tau_1} - e^{-it\tau_2} + 1]dt$$

by (3.5.12),

$$= \frac{1}{2\pi} \Gamma^2(1+\lambda) \int_{-\infty}^{\infty} \lim_{\alpha \to 0} \left[\frac{\gamma^2 + \beta^2}{(\alpha^2+t^2)^{1+\lambda}} \right.$$

$$\left. + \gamma\beta \left\{ \frac{1}{(\alpha-it)^{2+2\lambda}} + \frac{1}{(\alpha+it)^{2+2\lambda}} \right\} \right]$$

$$[e^{it(\tau_1-\tau_2)} - e^{it\tau_1} - e^{-it\tau_2} + 1]dt$$

by the bounded convergence theorem,

$$= \frac{1}{2\pi} \Gamma^2(1+\lambda) \int_{-\infty}^{\infty} \frac{1}{|t|^{2+2\lambda}} \{\gamma^2 + \beta^2 - 2\gamma\beta \cos \pi\lambda\}$$

$$\{e^{it(\tau_1-\tau_2)} - e^{it\tau_1} - e^{-it\tau_2} + 1\}dt$$

$$= \frac{1}{2\pi} \Gamma^2(1+\lambda) (\gamma^2+\beta^2 - 2\beta\gamma \cos \pi\lambda)$$

$$\int_{-\infty}^{\infty} \frac{1}{|t|^{2+2\lambda}} \{e^{it(\tau_1-\tau_2)} - e^{it\tau_1} - e^{-it\tau_2} + 1\}dt \,. \tag{3.5.13}$$

Let $\tau_3 = \tau_1 - \tau_2$.

For any $a > 0$, define

$$G(a,\varepsilon) \equiv \int_{-\infty}^{\infty} |t|^{a-1}(e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1)e^{-\varepsilon|t|}\, dt$$

$$= \Gamma(a)\left[\frac{1}{(\varepsilon-i\tau_3)^a} + \frac{1}{(\varepsilon+i\tau_3)^a} - \frac{1}{(\varepsilon-i\tau_1)^a} - \frac{1}{(\varepsilon+i\tau_1)^a} \right.$$

$$\left. - \frac{1}{(\varepsilon+i\tau_2)^a} - \frac{1}{(\varepsilon-i\tau_2)^a} + \frac{2}{\varepsilon^a} \right]$$

Since the integrand in $G(a,\varepsilon)$ is of the order $|t|^{a-1+2}$, the integral can be defined for $a > -1$.

In other words, for $a > -1$

$$G(a,\varepsilon) = \Gamma(a)\left[\frac{1}{(\varepsilon-i\tau_3)^a} + \frac{1}{(\varepsilon+i\tau_3)^a} - \frac{1}{(\varepsilon-i\tau_1)^a} - \frac{1}{(\varepsilon+i\tau_1)^a}\right.$$

$$\left. - \frac{1}{(\varepsilon+i\tau_2)^a} - \frac{1}{(\varepsilon-i\tau_2)^a} + \frac{2}{\varepsilon^a}\right] \qquad (3.5.14)$$

In particular, the above equality is true for $a = -(1+2\lambda)$.

Let $\eta = -(1+2\lambda)$.

Since

$$\left| |t|^{\eta-1} (e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1)\, e^{-\varepsilon|t|} \right|$$

$$\leq |t|^{\eta-1} \left|e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1\right|$$

and $\int |t|^{\eta-1} \left|e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1\right| dt < \infty$,

it follows by the bounded convergence theorem,

$$\lim_{\varepsilon\to 0} G(\eta,\varepsilon) = \int_{-\infty}^{\infty} |t|^{\eta-1} (e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1)\,dt.$$

Therefore, from (3.5.14), we have

$$\int_{-\infty}^{\infty} |t|^{\eta-1} (e^{it\tau_3} - e^{it\tau_1} - e^{-it\tau_2} + 1)\,dt$$

$$= -2\Gamma(\eta) \sin \pi\lambda \left[|\tau_3|^{-\eta} - |\tau_1|^{-\eta} - |\tau_2|^{-\eta}\right]$$

$$= 2\Gamma(\eta) \sin \pi\lambda \left[|\tau_1|^{1+2\lambda} + |\tau_2|^{1+2\lambda} - |\tau_1-\tau_2|^{1+2\lambda}\right] \qquad (3.5.15)$$

Therefore, from (3.5.13) and (3.5.15), we have

$$R(\tau_1,\tau_2) = \frac{1}{2\pi} \Gamma^2(1+\lambda)(\gamma^2+\beta^2-2\beta\gamma \cos\pi\lambda)\, 2\Gamma(-1-2\lambda) \sin \pi\lambda$$

$$\cdot \left[|\tau_1|^{1+2\lambda} + |\tau_2|^{1+2\lambda} - |\tau_1-\tau_2|^{1+2\lambda}\right]. \qquad (3.5.16)$$

Let

$$C = \frac{1}{2\pi} \Gamma^2(1+\lambda) 2\Gamma(-1-2\lambda) \sin\pi\lambda\, (\gamma^2+\beta^2-2\beta\gamma \cos \pi\lambda) \qquad (3.5.17)$$

$$= \frac{\Gamma^2(1+\lambda)}{\pi} \left[\frac{-\Gamma(-2\lambda)}{2\lambda+1}\right] \sin\pi\lambda\, (\gamma^2+\beta^2-2\beta\gamma \cos \pi\lambda)$$

by the formula $\Gamma(x+1) = x\,\Gamma(x)$,

$$= -\frac{\Gamma^2(1+\lambda)}{\pi}\left[\frac{\Gamma(-\lambda)\Gamma(-\lambda+\frac{1}{2})}{2^{2\lambda}\,2\sqrt{\pi}\,(2\lambda+1)}\right]\sin\pi\lambda\,(\gamma^2+\beta^2-2\beta\gamma\cos\pi\lambda)$$

by the duplication formula $\dfrac{\Gamma(x)\Gamma(x+\frac{1}{2})}{\Gamma(2x)}\;2^{2x} = 2\sqrt{\pi}$,

$$= \frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\lambda)}{2^{2\lambda+1}\sqrt{\pi}\,(2\lambda+1)}\,[\beta^2+\gamma^2-2\beta\gamma\cos\pi\lambda] \tag{3.5.18}$$

by the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$.

Combining (3.5.16) and (3.5.17), we have

$$R(\tau_1,\tau_2) = C[\,|\tau_1|^{1+2\lambda} + |\tau_2|^{1+2\lambda} - |\tau_1-\tau_2|^{1+2\lambda}] \tag{3.5.19}$$

where

$$C = \frac{\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\lambda)}{2^{2\lambda+1}\sqrt{\pi}\,(2\lambda+1)}\,[\beta^2+\gamma^2-2\beta\gamma\cos\pi\lambda]\,. \tag{3.5.20}$$

<u>Lemma 3.5.3</u>.

<u>For any</u> τ

$$\int_{-\infty}^{\infty}[\varepsilon(y-\tau)\,|y-\tau|^{\lambda} - \varepsilon(y)\,|y|^{\lambda}]^2 dy = 2C|\tau|^{\gamma'} \tag{3.5.21}$$

<u>where</u> $\gamma' = 2\lambda+1$ <u>and</u> C <u>is defined in</u> (3.5.20).

<u>Proof</u>:

Note that the integral is $R(\tau,\tau)$ where R is defined in the previous lemma.

<u>Lemma 3.5.4</u>.

$$Q(\lambda) \equiv \int_{-\infty}^{\infty}[\varepsilon(x-1)\,|x-1|^{\lambda}-\varepsilon(x)\,|x|^{\lambda}+\lambda\,\mathrm{sgn}\,x\,|x|^{\lambda-1}\varepsilon(x)]\,\varepsilon(x)\,|x|^{\lambda}dx$$

$$= -C$$

where C is defined in (3.5.20).

Proof:

Since the integrand of $Q(\lambda)$ is of the order $0(|x|^{2\lambda-2})$, and since $0 < \lambda < 1/2$, $Q(\lambda)$ is finite.

$$Q(\lambda) = \int_{-\infty}^{0} [\beta(1-x)^{\lambda} - \beta(-x)^{\lambda} - \lambda(-x)^{\lambda-1}\beta]\ \beta(-x)^{\lambda}\,dx$$

$$+ \int_{0}^{1} [\beta(1-x)^{\lambda} - \gamma x^{\lambda} + \lambda x^{\lambda-1}\gamma]\ \gamma\, x^{\lambda}\,dx$$

$$+ \int_{1}^{\infty} [\gamma(x-1)^{\lambda} - \gamma x^{\lambda} + \lambda x^{\lambda-1}\gamma]\ \gamma x^{\lambda}\,dx$$

$$= \beta^2 \int_{0}^{\infty} [(Y+1)^{\lambda} - Y^{\lambda} - \lambda Y^{\lambda-1}]\ Y^{\lambda}\,dY$$

$$+ \gamma^2 \int_{1}^{\infty} [(x-1)^{\lambda} - x^{\lambda} + \lambda x^{\lambda-1}]\ x^{\lambda}\,dx$$

$$+ \beta\gamma \int_{0}^{1} (1-x)^{\lambda} x^{\lambda} dx + \gamma^2 \int_{0}^{1} \lambda\, x^{2\lambda-1}\,dx - \gamma^2 \int_{0}^{1} x^{2\lambda}\,dx$$

$$= \beta^2 \int_{0}^{\infty} [(Y+1)^{\lambda} - Y^{\lambda} - \lambda Y^{\lambda-1}]\ Y^{\lambda}\,dY$$

$$+ \gamma^2 \int_{1}^{\infty} [(x-1)^{\lambda} - x^{\lambda} + \lambda x^{\lambda-1}]\ x^{\lambda}\,dx$$

$$+ \beta\gamma\ B(\lambda+1,\lambda+1) + \gamma^2 \frac{\lambda}{2\lambda} - \frac{\gamma^2}{2\lambda+1}$$

$$= \beta^2 \int_{0}^{\infty} [(Y+1)^{\lambda} - Y^{\lambda} - \lambda Y^{\lambda-1}]\ Y^{\lambda}\,dY$$

$$+ \gamma^2 \int_{1}^{\infty} [(x-1)^{\lambda} - x^{\lambda} + \lambda x^{\lambda-1}]\ x^{\lambda}\,dx$$

$$+ \beta\gamma\ B(\lambda+1,\lambda+1) + \frac{\gamma^2(2\lambda-1)}{2(2\lambda\;1)}\ . \qquad (3.5.22)$$

Let us now compute

$$\int_0^{\infty} [(Y+1)^{\lambda} - Y^{\lambda} - \lambda Y^{\lambda-1}] \, Y^{\lambda} \, dY. \qquad (3.5.23)$$

Let $G(\varepsilon) \equiv \int_0^{\infty} [(Y+1)^{\lambda} - (Y+\varepsilon)^{\lambda} - \lambda(Y+\varepsilon)^{\lambda-1}] \, Y^{\lambda} \, dY.$ (3.5.24)

For any $\alpha' > -1$ and $\beta' < -1/2$,

$$\begin{aligned}\int_0^{\infty} Y^{\alpha'} (\varepsilon+Y)^{\beta'} \, dY &= \varepsilon^{1+\alpha'+\beta'} \int_0^{\infty} Y^{\alpha'} (1+Y)^{\beta'} \, dY \\ &= \varepsilon^{1+\alpha'+\beta'} \int_0^{1} u^{\alpha'} (1-u)^{-2\beta'-2} \, du \\ &= \varepsilon^{1+\alpha'+\beta'} B(\alpha'+1, -2\beta'-1).\end{aligned}$$

Therefore, for $-1 < \lambda < -1/2$,

$$\begin{aligned}G(\varepsilon) = B(\lambda+1, -2\lambda-1) &- \varepsilon^{1+2\lambda} B(\lambda+1, -2\lambda-1) \\ &- \lambda \varepsilon^{2\lambda} B(\lambda+1, -2\lambda+1).\end{aligned}$$

Since $G(\varepsilon)$ is analytic for $\lambda < 1/2$, it follows that

$$\begin{aligned}G(\varepsilon) = B(\lambda+1, -2\lambda-1) &- \varepsilon^{1+2\lambda} B(\lambda+1, -2\lambda-1) \\ &- \lambda \varepsilon^{2\lambda} B(\lambda+1, -2\lambda+1) \qquad (3.5.25)\end{aligned}$$

for all $\lambda < 1/2$ and in particular for $0 < \lambda < 1/2$.

Furthermore the integrand of $G(\varepsilon)$ is bounded uniformly in ε by an integrable function since $0 < \lambda < 1/2$.

We shall now take limit as $\varepsilon \to 0$.

By bounded convergence theorem, it follows that

$$\int_0^{\infty} [(Y+1)^{\lambda} - Y^{\lambda} - \lambda Y^{\lambda-1}] \, Y^{\lambda} \, dy = B(\lambda+1, -2\lambda-1). \qquad (3.5.26)$$

Similarly, we can show that

$$\int_1^{\infty} [(x-1)^{\lambda} - x^{\lambda} + \lambda x^{\lambda-1}]\ x^{\lambda}\ dx = B(\lambda+1,\ -2\lambda-1) - \frac{\gamma^2(2\lambda-1)}{2(2\lambda+1)}\ .$$

Therefore, from (3.5.22), we have

$$\begin{aligned} Q(\lambda) &= [\beta'^2\ B(\lambda+1,\ -2\lambda-1) + \gamma'^2\ B(\lambda+1,\ -2\lambda-1) \\ &\qquad + \beta'\gamma'\ B(\lambda+1,\lambda+1)] \qquad (3.5.27) \\ &= -C\ . \end{aligned}$$

(3.5.27) proves the lemma.

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