DEVELOPMENT OF REFINED MATHEMATICAL PROGRAMMING METHODS FOR INDUSTRIAL ENGINEERING PROBLEMS

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DEVELOPMENT OF REFINED MATHEMATICAL PROGRAMMING METHODS FOR INDUSTRIAL ENGINEERING PROBLEMS

By

Robert W. Metzger

AN ABSTRACT

Submitted to the College of Engineering
Michigan State University of Agriculture and
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ABSTRACT

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Mathematical programming is presented in a broader concept than linear programming and as one facet of the broader field of Operations Research.

The methods of mathematical programming namely; the modified distribution method, Vogel's approximation method and the simplex method are presented in an easy to understand step by step manner, and further illustrated via typical though simple problems. Situations of degeneracy, unequal supply and demand, and other miscellaneous restrictions are included in the discussion of the distribution methods. Degeneracy and types of algebraic relationships are included in the discussion of the simplex method.

Two larger problems, a production planning and a manufacturing problem are presented and solved. The problems are formulated and solved in a logical step by step manner to further illustrate the application of mathematical programming in solving industrial problems. The production planning problem is the first of its kind to be presented in its entirety.

Mathematical notation is simplified and minimized. No attempt is made to prove the methods and their various theorems but liberal references are provided for the student who wishes to pursue the subject.

The thesis is summarized by discussing the advantages, prerequisites and limitations of mathematical programming as well as typical problem areas. In the conclusions several possible areas for further research are presented and discussed.

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MATHEMATICAL PROGRAMMING APPLIED TO INDUSTRIAL ENGINEERING PROBLEMS

I. INTRODUCTION

Mathematics has always been a useful tool to industrial engineers. Yet, until recently the full capabilities of certain mathematical techniques in solving industrial engineering problems have not been realized. This thesis presents a new and powerful tool called mathematical programming which offers a new perspective in solving many industrial problems.

Mathematical programming is one facet of the larger field of operations research. Operations research refers to the application of the scientific methods of the physical sciences to analyzing and solving complex business problems. Operations research, as such, had its beginning during World War II when scientists were employed by the British armed forces to apply their scientific knowledge to problems of strategy and tactics.

In many respects it is unfortunate that the work of an early operations researcher, Thomas A. Edison, fell into obscurity. Mr. Edison, while chairman of the Naval Consulting Board during World War I, became involved in various tactical problems.[1] It is interesting to note how many of his developments had to be rediscovered to reach their full usefulness in World War II. Operations research was more effective in World War II primarily due to organization. The Naval Consulting Board was a group of civilian advisors reporting to a civilian, the Secretary of the Navy. Whereas operations research groups in World War II usually worked for and reported to an operational command. Therein lies the chief reason for the wider use and general acceptance of operations research results by command personnel during World War II.

During the post war period, operations research personnel, usually called operations analysts, entered industry to apply their military proven methods

[1] Scott, Lloyd N. Naval Consulting Board of the United States, Government Printing Office, Washington. 1920. See also Wm. F. Whitmore, Edison and Operations Research, Journal of the Operations Research Society of America. Vol. 1, No. 2, February, 1953. P. 83-5.

to solving industrial problems. Unfortunately many of the results derived by the early operations research groups were little more than results that could have been achieved by the application of good common sense. This factor plus the often highly sophisticated and complicated notation and writing of operations analysts caused management to summarily dismiss operations research as just a new name for the same thing industry has been doing by committee for years. This explains the generally slow acceptance of operations research in American industry.

Operations research actually differs from anything industry has done in the past, chiefly in method and approach to a problem. Operations research requires an exacting statement of the problem. All the variables and factors with their proper interrelationships must be included. The tools of operations research include some very advanced mathematical techniques which make precise formulation a prime requisite. When this becomes an established habit, then the new viewpoint of operations research is possible in attacking industrial problems.

This thesis will be limited to that portion of operations research called mathematical or linear programming. The term mathematical programming will be used throughout the thesis since it implies a broader concept than linear programming.

Mathematical programming has been defined from time to time as follows:

> ". . . a method for picking a best choice when choices exist. . . . a formal method of calculating the best solution to a problem or situation where many solutions or management decisions are possible, depending on certain limiting conditions." [2]

> ". . . a number of new procedures which make it possible for management to solve a wide variety of important company problems much faster, more easily, and more accurately than ever before." [3]

> ". . . is used to find the optimum relationship between a number of interdependent variables where the interrelationships are algebraically linear." [4]

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[2] Ferguson, Robert O., Linear Programming, American Machinst Special Report No. 389. McGraw-Hill Company, 1955.

[3] Henderson, A. and Schlaifer, R. Mathematical Programming: Better Information for Better Decision Making, Harvard Business Review. V. 32, No. 3. May-June, 1954. Pp. 73-100.

[4] Goland, M., and Koenigsberg, E. Operations Research: Scientific Approach to Management, Chemical Week. McGraw-Hill Company, May 21, 1955.

". . . the theory by which activities -- independently variable and subject to certain restrictions -- related to each other in linear fashion . . . are arranged to obtain maximum results."[5]

". . . methods of solving a general class of optimization problems dealing with the interaction of many variables subject to certain restraining conditions."[6]

In essence then mathematical programming consists of several methods used to find the optimum combination of variables interrelated in linear expressions. In general the number of variables exceeds the number of significant expressions.

This thesis will consider the two primary methods of mathematical programming, namely the Distribution and Simplex methods.

The Distribution methods will be considered first, since these are the easiest to understand and use. These methods are applicable to problems of product distribution and to transport problems. Hence the name distribution methods. It is important to note though, that these methods can be successfully applied to other types of problems. However, in these cases the problem can be abstracted into a type of distribution problem.

The Distribution methods are, in reality, three methods. They are:

1. Basic Distribution Method or "Stepping Stone" method.

2. Modified Distribution Method.

3. Vogel's Approximation Method.

One of the earliest approaches to solving distribution problems using formal mathematical methods appeared in 1941.[7] During World War II significant improvements in the solution methods occurred. In 1951

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[5] Ibid.

[6] Arnoff, E. L. The Application of Linear Programming to Production Engineering and Scheduling, ASME Paper No. 54-A-223 presented at the Annual Meeting, November 28-December 3, 1954.

[7] Hitchcock, Frank L. The Distribution of a Product from Several Sources to Numerous Localities, Journal of Mathematics and Physics, Vol. 20, 1941; Pp. 224-230.

considerable advance was given to operations research and mathematical programming by the Cowles Commission[8] and particularly to the solution of distribution problems by G. B. Dantzig[9] and T. C. Koopmans.[10] The Basic Distribution Method is properly credited to G. B. Dantzig. However, the methods were still difficult for a non-mathematician to understand.

The "stepping stone" method devised by W. W. Cooper and A. Charnes[11] essentially presented Dantzig's method. However, the terminology was such that the "stepping stone" method could be very easily understood by the average person. The solution methods were somewhat improved in 1954[12] and further refined to the Modified Distribution Method which appeared in 1955.[13]

The Modified Distribution Method, while it is based upon the "stepping stone" method, so improved the computation procedure that it has supplanted the "stepping stone" method. Therefore, the "stepping stone" method will not be discussed here.

Vogel's Approximation Method[14] permits a much better initial solution than can usually be had by any other means. Vogel's Approximation Method with the modified distribution method permits more rapid hand computation of problems heretofore only practical to solve with electronic computors.

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[8] Koopmans, T. C., Activity Analysis of Production and Allocation, ed., Cowles Commission monograph 13; John Wiley and Sons, Inc., 1951.

[9] Ibid. Part 4, Chapter XXIII.

[10] Ibid. Part 2, Chapter XIV.

[11] Transportation Scheduling by Linear Programming. W. W. Cooper and A. Charnes. Proceedings of the Conference on Operations Research in Marketing; Case Institute, January, 1953. See also "The Stepping Stone Method of Explaining Linear Programming Calculations in Transportation Problems." A. Charnes and W. W. Cooper, Management Science, Vol. 1, No. 1, October, 1954.

[12] Henderson and Schlaifer. Op. cit.

[13] Ferguson, R. O. Op. cit.

[14] Credited to Mr. W. R. Vogel, Conference Leader, Ordnance Management Engineering Training Program, Rock Island Arsenal, Rock Island, Illinois, Material as yet unpublished.

The second primary method in mathematical programming, the simplex method,[15] while it is not quite as simple as the name may imply, is the more general method of mathematical programming. The simplex method will be presented with the geometric interpretation of the typical linear programming problem. The simplex method is applicable to a wider range of problems than are the distribution methods.

Two typical industrial problems, production planning and product allocation are presented. One serves to illustrate an application of the modified distribution method while the other is solved via the simplex method.

The production planning problem presented herein is the first such problem recorded in written form. The material presented includes the step by step approach to such a problem. Information previously published discusses the general case of the production planning problem and nowhere has a specific problem been developed in its entirety as the one presented here.

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[15] Charnes, A., Cooper, W. W., and Henderson, A. An Introduction to Linear Programming. John Wiley and Sons, New York. 1953. P. 74.

II. MATHEMATICAL PROGRAMMING METHODS

The Modified Distribution Method, Vogel's Approximation Method, and the Simplex Method will be presented here. The mathematics will be presented along with suitable terminology. The methods themselves are presented here in terms of a general statement of the problem. The two subsequent sections of the thesis develop and present typical problems solvable with these methods.

Modified Distribution Method

The Modified Distribution Method, more popularly known as the MODI method, is applicable to solving problems of product distribution from several sources or factories to various destinations, either warehouses or customers. First, the general distribution problem will be formulated, and then the Modified Distribution method will be developed.

If we let:

x_{ij} = number of product dispersed from the i'th source to the j'th destination

c_{ij} = cost per unit to distribute from the i'th source to the j'th destination

S_i = the supply (in units) at the i'th source

D_j = the demand (in units) at the j'th destination

i = (1, 2, . . . m) the sources

j = (1, 2, . . . n) the destinations

then the distribution problem may be formulated in two sets of relations.

$$\sum_{j=1}^{n} x_{ij} \leq S_i$$

$$\sum_{i=1}^{m} x_{ij} \leq D_j$$

$$x_{ij} \geq 0$$

Briefly this means that what is distributed from the i'th source must be less-than or equal-to the available supply at that source. Similarily the product received by the j'th destination must be less-than or equal-to the demand at that destination.

The objective is obviously to disperse the product at a minimum cost. Hence the objective is:

$$\sum_{\substack{i=1\\j=1}}^{\substack{n\\m}} c_{ij}\, x_{ij} = \text{minimum}$$

Thus the distribution problem is expressed in a system of relationships with an objective function (total cost) to be minimized.

At this point, for the sake of clarity, the discussion will consider the particular case where supply equals demand. The more general case of unequal supply and demand will be considered later. The problem is then to minimize

$$\sum_{\substack{i=1\\j=1}}^{\substack{n\\m}} c_{ij}\, x_{ij}$$

subject to the following

$$\sum_{i=1}^{m} x_{ij} = S_i$$

$$\sum_{j=1}^{n} x_{ij} = D_j$$

Obviously then

$$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$$

The problem can be organized into a matrix array m by n in size. This organization, commonly called a route table is as follows:

Table I. Distribution Matrix (Route Table)

To / From		Destinations				
	1	2		n-1	n	Supply
Sources						
1	x_{11}	x_{12}		$x_{1,n-1}$	x_{1n}	S_1
2	x_{21}	x_{22}		$x_{2,n-1}$	x_{2n}	S_2
m	x_{m1}	x_{m2}		$x_{m,n-1}$	x_{mn}	S_m
Demand	D_1	D_2		D_{n-1}	D_n	$\sum D_j = \sum S_i$

The computational method then determines the values of the x_{ij}'s that will minimize costs and still satisfy the demand. In doing this, reference must be made to the costs c_{ij}'s which can be most easily presented in the form of a cost table. This is done in the basic distribution or "stepping stone" method. The Modified Distribution Method combines both the distribution matrix and the cost table into one tabular presentation thusly:

Table II. Distribution Matrix
(Route and Cost Tables Combined)

To / From	Destinations 1	2		n	Supply
Sources 1	x_{11} c_{11}	c_{12}		x_{1n} c_{1n}	S_1
2	x_{21} c_{21}	x_{22} c_{22}		x_{2n} c_{2n}	S_2
m	x_{m1} c_{m1}	x_{m2} c_{m2}		x_{mn} c_{mn}	S_m
Demand	D_1	D_2		D_n	$\sum S_i = \sum D_j$

In this manner all the relevant information is readily usable in one tabular presentation.

A Problem

The Modified Distribution Method can best be presented via a simple problem. Consider an organization with warehouses in three locations, Flint, Janesville, and St. Louis. An order dispatcher must determine how to most economically distribute a product from these three warehouses (sources) to four customers (destinations). The warehouses have the following quantities of this product in stock:

Warehouse Location	Supply
Flint	150 Units
Janesville	40 Units
St. Louis	80 Units
	270 Units

The four customers have the following requirements:

Customer Location	Demand
Chicago	90 Units
Cleveland	70 Units
Dayton	50 Units
Minneapolis	60 Units
	270 Units

It costs $.10 per unit per mile to transport this product. The problem is then -- what warehouse should ship how much product to what customers so that the total distribution costs are minimum?

The first step is to determine the distribution costs per unit (c_{ij}'s) from each of the three warehouses to each of the four customers. The mileage between warehouses and customers can be obtained from a map or suitable route tables and is as follows:

Table III. Mileage Chart

From \ To	Chicago	Cleveland	Dayton	Minneapolis
Flint	270	230	310	690
Janesville	100	450	400	320
St. Louis	300	540	350	570

Since it costs $.10 per unit per mile to distribute this product then the distribution costs per unit will be as follows:

Table IV. Shipping Costs Per Unit (c_{ij}'s)

To / From	Chicago	Cleveland	Dayton	Minneapolis
Flint	$27.00	$23.00	$31.00	$69.00
Janesville	$10.00	$45.00	$40.00	$32.00
St. Louis	$30.00	$54.00	$35.00	$57.00

Mathematically the problem can be expressed as follows:

x_{ij} = number of units dispersed from the i'th warehouse to the j'th customer

c_{ij} = cost per unit to ship from the i'th warehouse to the j'th customer

S_i = supply (in units) at the i'th warehouse

D_j = requirement (in units) of the j'th customer

$i = (1, 2, 3)$

$j = (1, 2, 3, 4)$

The problem is then to minimize the distribution costs

$$+27x_{11} + 23x_{12} + 31x_{13} + 69x_{14} + 10x_{21} + 45x_{22} + 40x_{23} + 32x_{24}$$

$$+30x_{31} + 54x_{32} + 35x_{33} + 57x_{34} = \text{minimum}$$

subject to the following restraints:

$$S_1 = 150 = x_{11} + x_{12} + x_{13} + x_{14}$$

$$S_2 = 40 = x_{21} + x_{22} + x_{23} + x_{24}$$

$$S_3 = 80 = x_{31} + x_{32} + x_{33} + x_{34}$$

and

$$D_1 = 90 = x_{11} + x_{21} + x_{31}$$

$$D_2 = 70 = x_{12} + x_{22} + x_{32}$$

$$D_3 = 50 = x_{13} + x_{23} + x_{33}$$

$$D_4 = 60 = x_{14} + x_{24} + x_{34}$$

This information can be organized into a distribution matrix as shown in Table V.

Table V. Distribution Matrix

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27	-23	-31	-69	150
Janesville	-10	-45	-40	-32	40
St. Louis	-30	-54	-35	-57	80
Demand	90	70	50	60	270 \ 270

It is important to note here that the costs are expressed as negative numbers. The techniques of mathematical programming are algebraically maximizing methods. Algebraically maximizing a negative quantity will minimize the absolute value of that quantity. Hence to minimize, the objective function is preceded by a negative sign. Mathematically this is:

$$\min f(x) = \max -f(x)$$

The first step is to establish an initial solution. This initial solution may be an arbitrary one or very carefully obtained by inspection. Obviously the better the initial solution the fewer the steps required to obtain the optimum answer.

For this discussion a rather arbitrary initial solution commonly called the northwest corner initial solution will be used. The northwest corner solution is one of arbitrary assignments beginning at the upper left corner of the matrix and assigning consecutively to the right and down until all assignments have been made. The northwest corner initial solution for this problem is shown in Table VI.

Table VI. Northwest Corner Initial Solution

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27 (90)	-23 (60)	-31	-69	150
Janesville	-10	-45 (10)	-40 (30)	-32	40
St. Louis	-30	-54	-35 (20)	-57 (60)	80
Demand	90	70	50	60	270 / 270

Several parts of Table VI can be named for easy reference. The supply and demand figures are commonly referred to as rim conditions. The circled numbers (assignments) are called stones. The squares containing a circled number (an assignment) will be called stone squares. The squares with no circled numbers (where $x_{ij} = 0$) will be called water squares. This terminology is derived directly from the "stepping stone" method. Basically then the problem is to move about the stones (circled assignments) until the distribution costs are minimum.

If row and column designations are used rather than specific customer locations then the discussion will be more general. The common designation is R_i for row and K_j for column with the subscript denoting the respective row and column. Thus for this problem the row designations are R_1, R_2 and R_3 and the column designations are K_1, K_2, K_3 and K_4 respectively.

The initial step is to compute the various R and K values. To do this one value must be assumed. In every case R_1 will be set equal to zero. This is rather arbitrary and it makes no particular difference what value is assumed for any R or K. However for consistancy here R_1 will always be zero.

With R_1 established then the remainder of the R and K values may be computed by the following relationship:

$$R_i + K_j = c_{ij} \text{ (at a stone square)}$$

Several R and K values will be computed to clarify this. Since there is a stone (assignment) at R_1K_1 (see Table VI) then K_1 can be computed. Therefore:

$$R_1 + K_1 = c_{11}$$

There is also a stone at R_1K_2. Therefore:

$$R_1 + K_2 = c_{12}$$

and similarly

$$R_2 + K_2 = c_{22}$$

$$R_2 + K_3 = c_{23}$$

$$R_3 + K_3 = c_{33}$$

$$R_3 + K_4 = c_{34}$$

This exhausts the occurrence of stone squares and will permit all the R and K values to be computed. If the initial solution is improperly established or the problem is degenerate the R and K values cannot be computed. This situation will be discussed later. The statement $R_1 + K_4 = c_{14}$ is not true since R_1K_4 is not a stone square.

Note that the subscript notation in the preceding equations will double check. The subscripts for c are first row and then column. In every case the R subscript and the first subscript of c are the same and the K subscript and the second subscript of c are the same. This guarantees that the cost considered is the one at the point of intersection.

The respective costs can be read directly from the matrix and are:

$c_{11} = -27$ $\qquad c_{23} = -40$

$c_{12} = -23$ $\qquad c_{33} = -35$

$c_{22} = -45$ $\qquad c_{34} = -57$

Since R_1 is zero the successive row and column values may be calculated from the preceeding six equations. The R and K values are:

$R_1 = 0$ $\qquad$ $K_1 = -27$

$R_2 = -22$ $\qquad$ $K_2 = -23$

$R_3 = -17$ $\qquad$ $K_3 = -18$

$K_4 = -40$

It does not hold true that the R and K values will always be negative. They may be positive, negative or zero depending upon the problem. The distribution matrix with the R and K values included is shown in Table VII.

Table VII. Initial Solution
R_i and K_j Values Established

To / From	$K_1 = -27$	$K_2 = -23$	$K_3 = -18$	$K_4 = -40$	Supply
$R_1 = 0$	(90) -27	(60) -23	-31	-69	150
$R_2 = -22$	-10	(10) -45	(30) -40	-32	40
$R_3 = -17$	-30	-54	(20) -35	(60) -57	80
Demand	90	70	50	60	270 / 270

Practice at computing R_i and K_j values will show that it can usually be done mentally rather than writing out the equations.

With all the R's and K's computed the next step is to evaluate each water square. This is accomplished in the following manner:

$$R_i + K_j - c_{ij} \text{ (COST)} =$$

If the result is positive then no improvement possibility exists in that water square. If the result is negative (-) then further improvement is possible.

Each water square can then be evaluated as follows:

Water Square			Improvement
R_1K_3	$R_1 + K_3 - c_{13}$	=	
	$0 + (-18) - (-31)$	= +13	No
R_1K_4	$R_1 + K_4 - c_{14}$	=	
	$0 + (-40) - (-69)$	= +29	No
R_2K_1	$R_2 + K_1 - c_{21}$	=	
	$(-22) + (-27) - (-10)$	= -39	Yes
R_2K_4	$R_2 + K_4 - c_{24}$	=	
	$-22 + (-40) - (-32)$	= -30	Yes
R_3K_1	$R_3 + K_1 - c_{31}$	=	
	$-17 + (-27) - (-30)$	= -14	Yes
R_3K_2	$R_3 + K_2 - c_{32}$	=	
	$-17 + (-23) - (-54)$	= +14	No

This indicates that three water squares show improvement possibility. Any of these three could be selected to improve the solution. However if the water square with the most negative evaluation is selected then the best solution will generally be obtained most rapidly.

Water square R_2K_1 has the most negative evaluation and will be used to improve the solution.

The next step determines the changes that must occur to improve the solution by making square R_2K_1 a stone square. For every unit that will be assigned to square R_2K_1 (shipment from Janesville to Chicago in this particular problem) one unit must be removed from square R_1K_1, one unit added to square R_1K_2 and one unit removed from square R_2K_2. If this were designated by plus and minus signs it would appear as in Table VIII. The "X" in square R_2K_1 merely serves as a reminder that this is the water square being considered.

Table VIII. Changes in Initial Assignments

To / From	$K_1 = -27$	$K_2 = -23$	$K_3 = -18$	$K_4 = -40$	Supply
$R_1 = 0$	- (90) -27	+ (60) -23	-31	-69	150
$R_2 = -22$	+ X -10	- (10) -45	(30) -40	-32	40
$R_3 = -17$	-30	-54	(20) -35	(60) -57	80
Demand	90	70	50	60	270 / 270

Another way to look at the changes that must occur in the solution is as follows:

Starting with the selected water square and moving horizontally or vertically (as a rook moves in chess) trace a closed path, stepping only on stones, that returns to the selected water square. The arrows in Table VIII show this path. Assign alternate (+) and (-) signs along this path beginning with a (+) in the water square. Note that at every stone square stepped on a right angle turn was made.

If the distribution matrix has been properly formulated then one and only one such closed path exists for any water square.

The next step is to determine how many units may be moved. It makes good sense to change as many units as possible. The largest amount that can be changed is the smallest stone at a negative place (-) in the closed path. In this case ten units is the most that can be changed. This quantity is then added or subtracted from the other squares in the closed path according to the assigned plus or minus signs.

Table VIIIa. Changes in Initial Assignments

From \ To	$K_1 = -27$	$K_2 = -23$	$K_3 = -18$	$K_4 = -40$	Supply
$R_1 = 0$	- -27 (80) ~~90~~	+ -23 (70) ~~60~~	-31	-69	150
$R_2 = -22$	+ -10 (10)	- -45 ~~10~~	-40 (30)	-32	40
$R_3 = -17$	-30	-54	-35 (20)	-57 (60)	80
Demand	90	70	50	60	270 \ 270

The changes are then shown in Table VIIIa. Note that the initial assignments have been crossed out and the new assignments made. Note also that the rim conditions (supply and demand figures) are still met. Making this change essentially involved adding and subtracting the same quantity from each of two rows and two columns, thereby leaving the rim conditions unchanged.

This improved solution may be checked to determine the amount of improvement attained.

Initial N. W. Corner Solution

Square	No. Units	Cost/Unit	Total Cost
R_1K_1	90	\$27.00	\$2,430.00
R_1K_2	60	23.00	1,380.00
R_2K_2	10	45.00	450.00
R_2K_3	30	40.00	1,200.00
R_3K_3	20	35.00	700.00
R_3K_4	60	57.00	3,420.00
			\$9,580.00

First Improved Solution

Square	No. Units	Cost/Unit	Total Cost
R_1K_1	80	\$27.00	\$2,160.00
R_1K_2	70	23.00	1,610.00
R_2K_1	10	10.00	100.00
R_2K_3	30	40.00	1,200.00
R_3K_3	20	35.00	700.00
R_3K_4	60	57.00	3,420.00
			\$9,190.00

This improved solution shows a savings of \$390.00 over the initial N. W. Corner Solution. Note that this amount is exactly equal to the number of units moved times the water square evaluation for square R_1K_2.

R_1K_2	Water square evaluation	-39
	Number of units moved	10
	Net savings by improvement	\$390.00

Hence the water square evaluation can be used to predict the amount of improvement step by step.

These are essentially the steps required in the Modified Distribution Method. This iterative process is continued until all water square evaluations are positive or zero, at which time the optimum solution is at hand. The problem discussed here had balanced rim conditions and was not degenerate. These two situations will be considered now.

Unequal Supply and Demand. When supply exceeds demand or vice versa, it is desirable to allow the mathematics to determine which source retains some of its supply or which customer must be short on his order. To do this arbitrarily might defeat the solution to the problem.

This condition is easily accomplished if we consider the same problem as above with the supply at Janesville increased to 55 units. The problem then appears as in Table IX. The northwest corner solution indicates that 15 units will always remain in the St. Louis warehouse.

Table IX. Distribution Problem
Unequal Supply and Demand

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27 (90)	-23 (60)	-31	-69	150
Janesville	-10	-45 (10)	-40 (45)	-32	55
St. Louis	-30	-54	-35 (5)	-57 (60)	80
Demand	90	70	50	60	285 \ 270

It would be better if the problem were set up to allow the mathematics to determine which warehouse retains a portion of its supply. This can be accomplished by inserting a dummy customer with a demand of 15 units. This will balance the rim conditions.

Table X. Distribution Problem
Balanced Supply and Demand

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Dummy	Supply
Flint	-27 (90)	-23 (60)	-31	-69	0	150
Janesville	-10	-45 (10)	-40 (45)	-32	0	55
St. Louis	-30	-54	-35 (5)	-57 (60)	0 (15)	80
Demand	90	70	50	60	15	285 \ 285

The cost factors for the dummy customer are zero, since the dummy customer represents a fictitious demand for units that will in reality remain in a warehouse. In certain situations it may be desirable to insert inventory costs for the various warehouses in the dummy column rather than zeros. The problem will then minimize distribution and inventory costs.

Now that the problem is set up the modified distribution method may be used to obtain the optimum answer.

Much the same approach is used when demand exceeds supply. Here a dummy warehouse (row) is established and the problem developed in much the same manner as discussed above.

Occasionally it may be impossible to ship from a warehouse to a certain customer. When this situation arises then the cost of shipment between the two is considered as -M. The symbol -M being defined as so large that it dominates all else in the problem. Hence that shipment is mathematically factored out of the problem.

This added technique permits a wide range of management restrictions to be considered in a problem.

Degeneracy. This occurs in mathematical programming problems when the problem begins to cycle (i. e., return to the same solution), or when an infinite number of steps are required to obtain an answer or when the solution method begins to collapse before an optimum answer is obtained. Obviously, all of these conditions can occur when errors are present in the work. However they can also occur due to the nature of the problem.

In distribution problems degeneracy is most apparent when the rim conditions are similar. Consider the previous problem with slight modification in rim conditions as shown in Table XI.

TABLE XI. DISTRIBUTION MATRIX
A DEGENERATE PROBLEM

Table XI. Distribution Matrix

A Degenerate Problem

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27	-23	-31	-69	160
Janesville	-10	-45	-40	-32	40
St. Louis	-30	-54	-35	-57	80
Demand	90	70	40	80	280 / 280

The initial northwest corner solution would then be as follows:

Table XII. Degenerate Distribution Problem
An Initial Solution

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27 (90)	-23 (70)	-31	-69	160
Janesville	-10	-45	-40 (40)	-32	40
St. Louis	-30	-54	-35	-57 (80)	80
Demand	90	70	40	80	280 / 280

The degeneracy would be evident when attempting to establish R_i and K_j values. If $R_1 = 0$ is assumed then K_1 and K_2 can be determined. Here it is impossible to proceed further. Note that the stair step pattern of the northwest corner solution is missing. Actually the problem matrix (Table XII) can be partitioned into three parts as it now stands.

The degeneracy can be resolved by inserting zero stones (0) wherever needed. This has no effect upon the real problem but does afford a useful gimmick to permit solution of the problem.

TABLE XIII. DEGENERATE DISTRIBUTION PROBLEM

DEGENERACY RESOLVED

Table XIII. Degenerate Distribution Problem
Degeneracy Resolved

From \ To	Chicago	Cleveland	Dayton	Minneapolis	Supply
Flint	-27 (90)	-23 (70)	-31	-69	160
Janesville	-10	-45 (0)	-40 (40)	-32	40
St. Louis	-30	-54	-35 (0)	-57 (80)	80
Demand	90	70	40	80	280 / 280

Now the remaining R_i and K_j values may be computed.

Actually there is considerable lattitude as to the placement of the zero stones. However only the correct number of stones may be used. It can be seen from the previous problems that m + n - 1 represents the number of stones or assignments. If less than m + n - 1 stones exist then the problem is degenerate.[16] This applies similarly for any step in the solution method. When more than m + n - 1 stones exist usually it implies that an error has been made.

Alternate Solutions. The modified distribution method permits very rapid evaluation of alternate solutions by way of the water square evaluations. The economics of alternate courses of action can be quickly evaluated for both equally optimum or less-than-optimum solutions. The water square evaluations predict the per unit increase or decrease in costs for any change in the solution. Obviously then when one or more zero water square evaluations exist in a final solution then many equally optimum alternate solutions exist for the problem.

If for some reason management says that a particular solution cannot be carried out, then an alternate solution can be developed. The difference in costs resulting from such a decision is readily available and management has then a concrete cost value for that decision. This presents some very interesting and most useful information for management.

Summary. The modified distribution method presents a very simple and useful means to solving distribution type problems. It is significant to note that, while costs were used throughout the discussion, wherever the word cost appeared the word profit could be inserted. Obviously the rational executive wishes to maximize profits. Hence profits would be expressed as positive numbers. This handling of costs as negative and profit as positive numbers involves exactly the same mathematics to minimize costs or maximize profits.

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[16] See Dantzig, G. B. Application of the Simplex Method to a Transportation Problem in Activity Analysis of Production and Allocation, p. 360.

Distance and time are two other factors that can conceivably be used in distribution type problems.

It is entirely possible to have problems with both cost and profit elements. Here the mathematics is unchanged and the optimum solution will both maximize profits and minimize costs.

The steps of the modified distribution method can be summarized as follows:

1. Set up the problem distribution matrix.
2. Balance the rim conditions (supply and demand).
3. Establish initial solution.
   a. N. W. corner solution.
   b. Inspection solution.

   Note: The number of stones = m + n - 1
4. Establish R and K values. $R_1 = 0$
   R + K = Cost or profit at an intersecting stone square.
   a. Resolve degeneracy as may be necessary with zero stones (0).
5. Evaluate each water square.
   R + K - (Cost or profit of water square)
   a. If above is positive (+) then no further improvement is possible.
   b. If above is negative (-) then there is further improvement possible.
6. Select the water square with the greatest negative value.
   a. Establish a closed path (as a rook moves in chess) from this water square via stone squares back to the same water square. There will be one and only one such path.
   b. Establish alternate plus (+) and minus (-) signs on this path starting with a plus (+) in the particular water square. This should still permit balanced rim conditions to exist.

7. Determine the amount to be placed in the selected water square as the smallest stone at a negative place on the closed path. Place that smallest stone in the selected water square.

8. Make the desired changes by adding or subtracting that selected amount from every stone in the path.

9. Re-establish R and K values as may be necessary.

10. Repeat steps 5 through 10 until all water square values are plus (+) or zero (0).

Interpretation of the water square evaluations:

1. A positive number indicates the per unit increase in cost or reduction in profit that would result when a stone for one unit is placed in the water square and the necessary adjustments are made in the program so as not to violate the rim conditions.

2. A negative number indicates the per unit decrease in cost or increase in profit that would result when a stone for one unit is placed in the water square and the necessary adjustments are made in the program so as not to violate the rim conditions.

3. When one or more water square values are zero and the remaining water square values are all positive numbers (greater than zero) there exists one or more equally optimum alternate solutions to the problem.

4. Mathematically speaking if two equally optimum alternate solutions exist then an infinite number of equally optimum alternate solutions exist for the problem.

Vogel's Approximation Method[17]

Vogel's approximation method is a technique for developing an initial solution to distribution problems. In most cases this method will develop a much better solution than could be developed by inspection. The work required to successfully solve distribution type problems is materially reduced when a better initial solution is obtained.

[17] Credited to Mr. W. R. Vogel. This material will be included in Mathematical Programming for Industrial and Systems Engineers by N. V. Reinfeld and W. R. Vogel (in preparation).

Vogel's approximation method can be applied when the distribution matrix is established and the rim conditions balanced. The steps of the method are:

1. For each row and column determine the difference between the two most algebraically maximum cost or profit elements.

2. Select that row or column with the greatest difference and assign as many units as possible to the square associated with the most algebraically maximum element (lowest cost or highest profit square).

3. Cross out that row or column that has been completely utilized or satisfied.

4. Redetermine the differences as in step 1, neglecting the row(s) or column(s) crossed out.

5. Repeat steps 2 through 4 until all assignments have been made.

6. Check the assignments and improve the solution with the modified distribution method.

Several supplementary steps may be required when the same difference occurs in two rows or columns or in a row and column. These steps are:

7. When the same difference occurs in a row and a column and the cost (or profit) element at the junction is the lowest (or highest) then assign as much as possible in that square. If the junction is not the lowest cost or highest profit then assign in either the row or column wherever the lowest cost or highest profit prevails.

8. When the same difference is obtained for two or more rows or columns then assign wherever the lowest cost or highest profit element prevails.

It is possible to recognize and resolve degeneracy when applying Vogel's approximation method.

9. If, when an assignment is made both a row and column are fulfilled simultaneously the problem is degenerate. Resolve degeneracy by placing a zero stone (0) in a remaining square in either the row or column.

Experience with Vogel's approximation method indicates that the optimum answer is usually obtained immediately in smaller distribution type problems. In larger problems it is safe to estimate at least seventy-five per cent of the work is eliminated by using this method to obtain an initial solution.

It is felt by the author that Vogel's approximation method is superior to the method offered by Mr. H. S. Houthakker.[18] Mr. Houthakker's method is limited to a class of problems possessing a unique cost pattern. Vogel's approximation method can be applied equally well to any type distribution problem.

The Simplex Method

The origin of the Simplex Method is properly credited to G. B. Dantzig.[19] Various extensions have been made notably by Beale, Charnes, Cooper, Lemke, Orden and Wolfe which have made the simplex method rather mechanical and reasonably simple to understand and use.

The simplex method is a method of algebraic manipulations used to solve systems of linear equations where one solution of an infinite number of solutions is desired. The method is much like Gauss' scheme (Doolittle's method) for solving systems of linear equations. Several other methods notably the relaxation method exist for solving systems of linear equations but these are beyond the scope of this discussion.

This section will present the Simplex Method much like that offered by Charnes and Cooper[20] except that the terminology employed here will be less rigorous. The proof of the method and its various theorems will not be presented here for three reasons: First, the proofs are well developed in numerous publications, notably in the work by Dantzig and Charnes and Cooper; secondly, an understanding of the theorems and their proofs is unnecessary to effectively use the Simplex Method; third, the author lacks the high degree of mathematical maturity required to successfully accomplish such a venture without virtually copying from other authors.

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[18] Houthakker, H. C. "On the Numerical Solution of the Transportation Problem", Journal of the Operations Research Society of America, Volume 3, Number 2. May, 1955. Pages 210-214.

[19] Koopmans, et al. Chapter XXI, pages 339-347. Op. cit.

[20] Charnes, Cooper, Henderson. Op. cit.

The Simplex Method can best be illustrated via a typical manufacturing problem. The general mathematical statement of the problem will be developed then a specific problem will be solved.

If we let

j = (1, 2, --- n) the commodities to be produced

i = (1, 2, --- m) the machine tools or manufacturing processes

x_{ij} = number of units of the j'th commodity produced on the i'th manufacturing process

t_{ij} = time required to manufacture one unit of the j'th commodity on the i'th manufacturing process

T_i = total available time on the i'th manufacturing process

P_{ij} = the profit derived from the sale of one unit of the j'th commodity that was produced by the i'th manufacturing process

and an assumption is made that all that is produced can be sold, then the rational executive would want to manufacture the commodities so as to realize the greatest possible profit. Mathematically this would be stated: to determine the x_{ij}'s so as to maximize

$$p_{11} x_{11} + p_{12} x_{12} + \cdots + p_{ij} x_{ij} + \cdots + p_{mn} x_{mn}$$

subject to the following:

$$t_{11} x_{11} + t_{12} x_{12} \cdots + t_{1j} x_{1j} + \cdots t_{1n} x_{1n} \leq T_1$$

$$t_{21} x_{21} + t_{22} x_{22} \cdots + t_{2j} x_{2j} + \cdots t_{2n} x_{2n} \leq T_2$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$t_{m1} x_{m1} + t_{m2} x_{m2} + \cdots + t_{mj} x_{mj} + \cdots t_{mn} x_{mn} \leq T_m$$

This set of relationships could also be expressed:

$$\sum_{\substack{i=1 \\ j=1}}^{\substack{n \\ m}} t_{ij}\, x_{ij} \leq T_i$$

In essence the above says to determine how much of each commodity to manufacture so that the maximum profit can be realized and at the same time not exceed the available time on the various manufacturing processes. The Simplex Method can accomplish this.

A Problem. Consider a specific problem much like the above general case.

A manufacturer wishes to determine how to produce two products (A and B) so as to realize the maximum total profit from the sale of the products. Both products are made in two processes (I and II). It takes 7 hours in process I and 4 hours in process II to manufacture 100 units of product A. It requires 6 hours in process I and 2 hours in process II to manufacture 100 units of product B. Process I can handle 84 hours of work and process II can take 32 hours of work in the schedule period.

If the profit is \$11.00 per 100 units for product A and \$4.00 per 100 units for product B then how much of product A and B should be manufactured to realize the maximum profit. It is assumed that whatever is produced may be sold and that setup time on the two processes is negligible.

While this is a relatively simple problem it will serve to illustrate the simplex method. In terms of the previous symbolism the problem can be expressed thusly,

j = 1, 2 the commodities to be produced
1 = Product A, 2 = Product B

i = 1, 2 the manufacturing processes
1 = Process I, 2 = Process II

x_1 = number of units (x100) commodity A produced

x_2 = number of units (x100) of commodity B produced

t_{ij} = time to manufacture one unit (x100) of the j'th commodity on the i'th process

T_i = total available time on the i'th process

p_j = the profit derived from the sale of one unit (x100) of the j'th commodity

The problem is then to determine the value of x_1 and x_2 so as to maximize

$$11\ x_1 + 4\ x_2$$

subject to the following:

$$7\ x_1 + 6\ x_2 \leq 84$$

$$4\ x_1 + 2\ x_2 \leq 32$$

In order to simplify the notation, since this is a two variable problem, let

$$x_1 = x$$

$$x_2 = y$$

then the problem is to maximize

$$11\ x + 4\ y \qquad (2.1)$$

subject to

$$7\ x + 6\ y \leq 84 \qquad (2.2)$$

$$4\ x + 2\ y \leq 32 \qquad (2.3)$$

It can be seen that expression (2.1) represents the profit or objective function. Expressions (2.2 and 2.3) represent the manufacturing time for each of the two processes.

The first step is to make equations of the above inequalities. This is accomplished with the addition of slack variables. A slack variable can be of any size (as long as it is positive) and merely causes equality to exist. This is accomplished as follows:

$$7x + 6y + W_1 = 84 \qquad (2.4)$$

$$4x + 2y + W_2 = 32 \qquad (2.5)$$

In the real physical problem the slack variables W_1 and W_2 would represent idle equipment time on process I and II respectively. If no profit or loss were associated with these slack variables then the profit function (2.1) can be amended to:

$$11x + 4y + 0 \cdot W_1 + 0 \cdot W_2 = \max \qquad (2.6)$$

It is possible to consider the burden costs for idle equipment if that is desired. For simplicity here no costs will be associated with idle equipment time.

Note that the problem as it now stands involves four variables in two equations. Mathematically this means that an infinite number of solutions exist.

The second step is to organize the equations in matrix form to begin the first simplex tableau. Table XIV illustrates the equations in matrix form.

Table XIV. Problem Equations in Matrix Form

	x	y	W_1	W_2
84	7	6	1	0
32	4	2	0	1

The simplex method requires an initial solution as did the distribution methods. Here the initial solution is the trivial or worst possible solution.

In this case the initial solution would be to produce nothing, having wholly idle time on the equipment, and realize no profit. This is shown in Table XV.

Table XV. Initial or Trivial Solution

		x	y	W_1	W_2
W_1	84	7	6	1	0
W_2	32	4	2	0	1

Now it is necessary to evaluate this solution to serve as a guide for further improvement. To accomplish this the objective (profit) function must be included. The objective function is written in a row above the position of the variables.

At this point it would be well to name some of the parts of the matrix for easier reference. This is shown in Table XVI.

Table XVI. Initial Solution Objective Function Included

			11	4	0	0	Objective row
			x	y	W_1	W_2	Variable row
0	W_1	84	7	6	1	0	
0	W_2	32	4	2	0	1	
			Body of the matrix		Identity of the matrix		

A new row called the index row can now be developed along the bottom of the matrix. The numbers in this row are developed by the formula:

Index number = $\sum$ (number in the column X the corresponding number in the objective column) - number in the objective row at the head of the column.

This is used to develop index numbers in the constant column and the body and identity of the matrix. The index number in the constant column would be:

$$\text{Index number (Constant col.)} = \left[84 \cdot 0 + 32 \cdot 0\right] - 0$$

$$= 0$$

The index number for the first column of the body of the matrix is:

$$\text{Index number} = (7 \cdot 0 + 2 \cdot 0) - 11$$

$$= -11$$

The completed index row is shown in Table XVII. This illustrates the first simplex tableau with all the various parts named.

Table XVII. Initial Simplex Tableau

Objective column	Variable column	Constant column					
			11	4	0	0	Objective row
			x	y	W_1	W_2	Variable row
0	W_1	84	7	6	1	0	The problem equations
0	W_2	32	4	2	0	1	
		0	-11	-4	0	0	Index row

Body: columns x, y; Identity: columns W_1, W_2

It will be noted that the index row in this problem is the negative of the objective row. This holds true only when all the slack variables have zeros in the objective column.

The initial solution is then:

$x = 0$

$y = 0$

$W_1 = 84$

$W_2 = 16$

Total Profit = 0

The index numbers point to possible improvement in the solution. In essence \$11.00 profit (represented by the -11) is being lost by not producing any x and \$4.00 profit is being lost by not producing any y. Since x shows the highest profit potential (the largest negative number) it is selected to improve the solution. The column in which x appears may then be called the key column.

Obviously to bring x into the solution (have it appear in the variable column) either W_1 or W_2 must be removed from the solution. The variable that is to be removed from the solution is determined by dividing the numbers in the constant column by the corresponding positive non-zero numbers in the key column. The row with the smallest quotient becomes the key row and the variable in that row is removed from the solution. This can be verified by standard algebraic methods as the change that will permit the largest x that still satisfies the system of relationships.

The number at the intersection of the key column and key row is called the key number. This is illustrated in Table XVIII.

Table XVIII. Initial Simplex Tableau
Key Row and Column Indicated

			11	4	0	0
			x	y	W_1	W_2
0	W_1	84	7	6	1	0
0	W_2	32	4	2	0	1
		0	-11	-4	0	0

Key row

Key Column

Key Number

Now the improved solution can be developed. This entails developing a new tableau or mathematically it is changing the basis of the matrix.

The key row is divided by the key number and appears in the same position as the main row of the new tableau. The variable (x) and its objective number (11) of the key column replaces the variable (W_2) and its objective number (0) of the key row. The remaining variables in the variable column and their objective numbers remain unchanged.

Table XIX. Main Row of New Tableau

				11	4	0	0	
				x	y	W_1	W_2	
Tableau I	0	W_1	84	7	6	1	0	
	0	W_2	32	4	2	0	1	Key row
			0	-11	-4	0	0	

Tableau II	0	W_1						
	11	x	8	1	1/2	0	1/4	Main row

For the sake of brevity the variable and objective rows are not rewritten in succeeding tableaus.

The remaining numbers of the new tableau in the constant column, body, identity and index row, are determined by the formula:

$$\text{New no.} = \text{Old no.} - \left[\frac{\text{Corres. No. of key row} \times \text{Corres. No. of key column}}{\text{Key number}} \right]$$

The number in row one in the constant column would be determined as follows:

$$\text{New number} = 84 - \frac{16 \times 7}{2}$$

$$= 84 - 56$$

$$= 28$$

Applying the same formula would develop the second tableau as shown in Table XX.

Table XX. Simplex Tableaus I and II
Optimum Solution

Tableau I

			11	4	0	0
			x	y	W_1	W_2
0	W_1	84	7	6	1	0
0	W_2	32	4	2	0	1
		0	-11	-4	0	0

Tableau II

0	W_1	28	0	5/2	1	-7/4
11	x	8	1	1/2	0	1/4
		88	0	3/2	0	11/4

The solution presented in tableau II is the optimum solution because all the numbers in the index row are either positive or zero. If any negative numbers still existed in the index row the entire process would be repeated, and a further improved solution would be obtained.

The optimum answer to the problem is

$x = 8$

$y = 0$

$W_1 = 28$

$W_2 = 0$

Maximum Profit = \$88.00

In terms of the initial problem statement the answer is:

Product A - produce 800 units

Product B - produce 0 units

Process I - will have 28 hours unutilized time

Process II - will be 100% utilized

Profit = \$88.00

This optimum answer is easily verified. Any other solution will yield a lower profit.

It is of interest to note that an economic significance can be attached to the index row numbers in the final tableau.

Any numbers under the body of the matrix (3/2 for y in this case) represents the reduction in the objective function per unit of that variable introduced into the solution. In the case of the above problem the total profit would be reduced by \$1.50 per "y" introduced into the solution. To introduce one "y" would mean removing some "x" (in this case .5x) hence the reduced profit. This answer would be:

$x = 7.5$

$y = 1$

$W_1 = 27.5$

$W_2 = 0$

Profit = 7.5 (11) + 1 (4) = \$86.50

which can be easily verified with the initial equations.

Any numbers in the index row under the identity (11/4 for W_2 in this case) can be considered to represent opportunity profit. This means that an increase of one "W_1" in the initial problem (i. e., relaxing the restriction on the second process by one hour) would permit an increase of 11/4 or $2. 75 in the profit function. This can serve as a guide for equipment expansion or replacement.

It is necessary to have some means of checking the work since errors can easily occur. One method of checking involves checking the numbers in the index row. The same method used to establish the index row in the first tableau can be used to check the index row of succeeding tableaus. This check, however, is of limited value since it cannot detect an error in a row where the objective number is zero.

A better checking device is to employ a check column. The numbers in the check column are established in the first tableau as equal to the sum of the numbers in the respective row. This includes the numbers in the constant column and everything to the right. In succeeding tableaus the check column is handled as a part of the problem matrix, just like the index row, and will always represent the sum of the numbers in the respective rows. If the check column does not tally then an error has occurred and can easily be found and corrected.

The check column, while it does add another column to the matrix array, is by far the easiest and best method for checking the accuracy of the work.

Of course as Charnes and Cooper[21] mention, the all important check though it may seem trite is in assessing the resulting solution(s) in terms of its meaningfulness and practicability as a course of action to the problem at hand.

It is well to note several useful short cuts in developing a new tableau. These are:

1. A "+1" appears at the intersection of a row and column containing the same variable. All other numbers of that column are zero including the number in the index row.

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[21] Charnes, Cooper, Henderson, Ibid, page 18.

2. If there is a zero in the key row of the previous tableau or main row of the new tableau all figures in the column in which the zero appears are unchanged and thus repeated in the new tableau.

3. If there is a zero in the key column, that row containing the zero is unchanged and thus repeated in the new tableau.

These short cuts can be easily verified by the previous formula for developing the numbers of the new tableau. This will tend to substantially reduce the hand calculation time.

Degeneracy. It is possible to have degeneracy in much the same manner as in distribution problems. However, the resolution of degeneracy is a little more involved in the simplex method.

Degeneracy can be recognized when determining the key row. Degeneracy exists when two or more rows yield the same smallest quotient. In this case a choice must be made. If the wrong choice is made the problem may begin to cycle and hence never reach an optimum.

The degeneracy is resolved as follows:[22]

1. Divide each element in the "tied" rows by its number in the key column.

2. Compare the quotients obtained term by term from left to right only in the identity of the matrix array.

3. At the first place where the quotients are unequal the tie is broken.

4. Select that row with the algebraically smaller ratio.

This is a perfectly general procedure and will resolve degeneracy very effectively.

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[22] Charnes, Cooper, Henderson, Ibid, pages 20-25. This is the basis for the above discussion and includes a very comprehensive appraisal of degeneracy.

Several sources have indicated a rule of thumb method for resolving degeneracy much like the north west corner rule used in distribution problems. It says to select the uppermost row if two rows are tied or to select the left column if two columns are tied. This may work; however, it yields exactly the opposite choice in the first tableau of Charnes and Cooper's nut mix problem.[23] This author solved the nut mix problem with this opposite choice and obtained the same answer in the same number of steps. However, the validity of the northwest rule to resolve degeneracy in the simplex method is most certainly open to serious question.

Types of Relationships. In mathematical programming problems the less-than-or-equal-to ($\leq$) relationship is not the only type that will be encountered. It is possible to have some approximately-equal-to ($\cong$) and greater-than-or-equal-to ($\geq$) relationships. Each of these must be handled differently.

Consider the relationship:

$$5x + 3y \leq 50$$

Add a slack variable and it becomes an equation:

$$5x + 3y + W = 50$$

This is in suitable form to be put into the simplex matrix.

Consider

$$5x + 3y \cong 50$$

In this situation a slack variable is both added and subtracted to form the equation

$$5x + 3y - W_1 + W_2 = 50$$

[23] Charnes, Cooper, Henderson. Ibid, page 10.

The slack variables are included in the objective function preceded by a "-1". This will tend to minimize the slack variables and make

$$5x + 3y \rightarrow 50$$

In this case the "W_1" would appear in the body of the matrix and the "W_2" in the identity.

Consider

$$5x + 3y \geq 50$$

which would be the same as

$$-5x - 3y \leq -50$$

This seems suitable except that all the numbers in the constant column must be positive in order to yield meaningful solutions. If a slack variable were subtracted from the first expression an equation would result as follows:

$$5x + 3y - W = 50$$

This cannot be placed into the simplex matrix as yet. The simplex matrix array requires that the identity be square (i. e., same number of rows as columns) and further, that it be a positive unit diagonal (i. e., positive 1's appearing in a diagonal from upper left to lower right).

As the expression stands now there is no variable that can be placed in the identity. Here an artificial variable can be employed.

$$5x + 3y - W + U = 50$$

A factor -M is attached to "U" in the objective function, since equality existed before "U" was added, and since "U" has no economic significance in the real problem. This "-M" factor, defined as so large that it dominates all else in the problem, automatically assures that "U" will have no value in the optimum solution. This means that the variable "U" is in essence a computational gimmick that permits this type of relationship to be included in the simplex method. The variables "x", "y", and "W" would appear in the body of the matrix. The variable "U" would appear in the identity.

Geometric Interpretation. It is possible to attach a geometric interpretation to the mathematical programming problem and the simplex method.

In the previous problem the relationships were

$$7x + 6y \leq 84 \qquad (2.4)$$

$$4x + 2y \leq 32 \qquad (2.5)$$

These linear relationships can be plotted as shown in the graph (Figure 1).

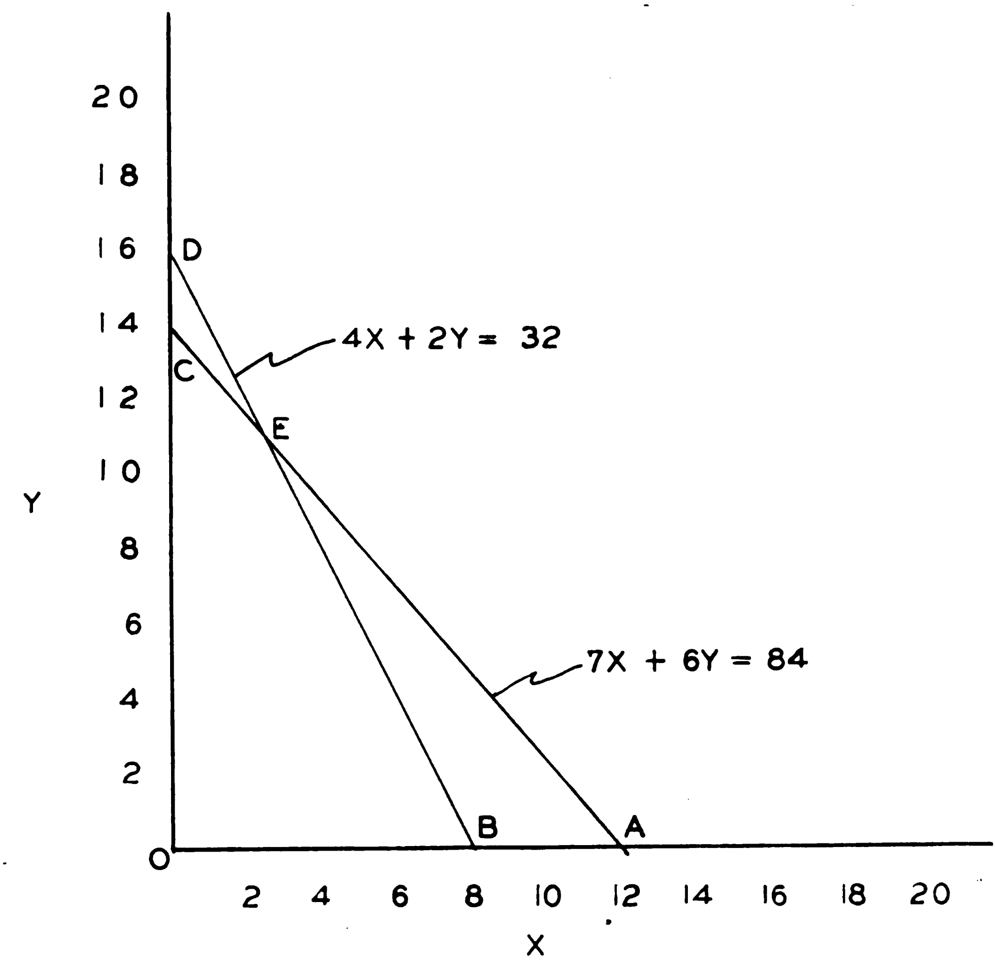

FIG. 1 PROBLEM EQUATIONS GRAPH

The line AEC represents possible solutions to the equation,

$$7x + 6y = 84$$

The area AECO represents the area of possible solutions to the inequality,

$$7x + 6y \leq 84$$

The line DEB represents possible solutions to the equation,

$$4x + 2y = 32$$

The area DEBO represents the area of possible solutions to the inequality,

$$4x + 2y \leq 32$$

Point E is the simultaneous solution to the set of equations,

$$7x + 6y = 84$$

$$4x + 2y = 32$$

The area CEBO represents the area of possible solutions to the system of inequalities,

$$7x + 6y \leq 84$$

$$4x + 2y \leq 32$$

and consequently is the area of possible solutions to the problem. The profit function must be included in order to find that solution which yields the maximum profit.

The profit function

$$11x + 4y$$

can be represented by a line with a $\frac{-11}{4}$ slope.

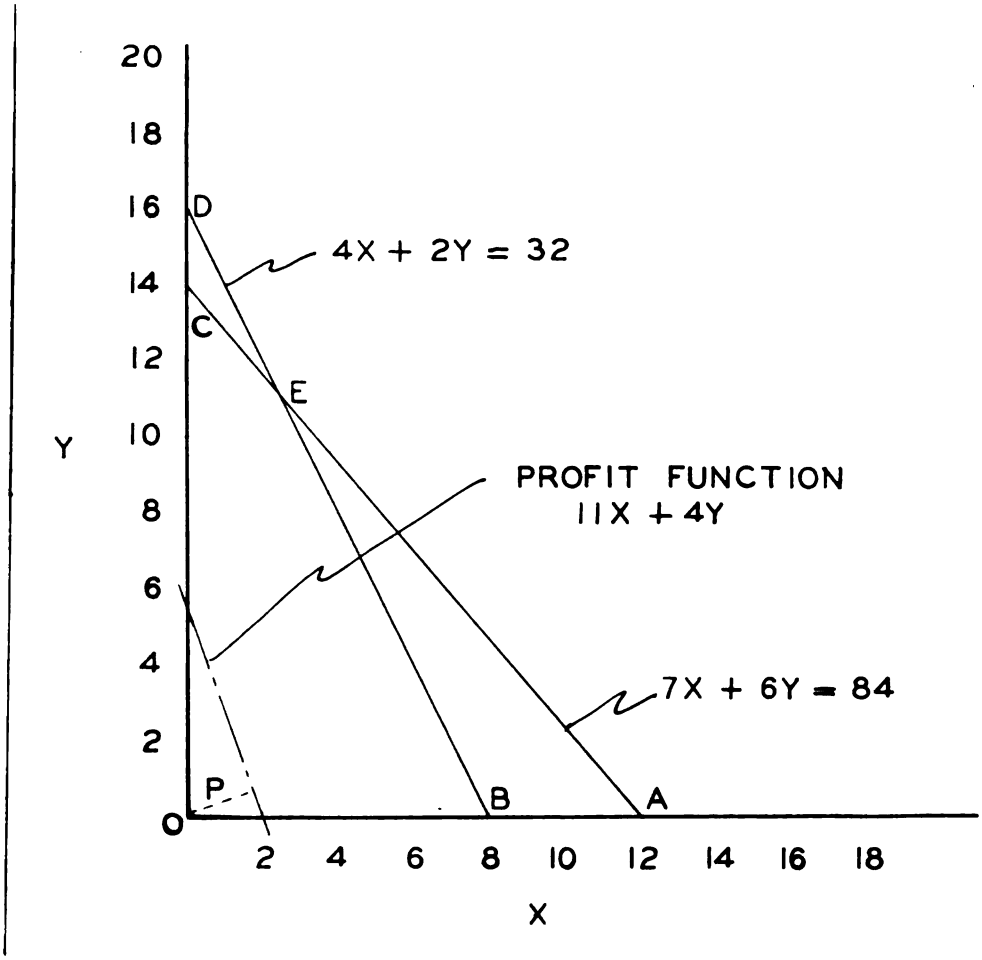

Fig. 2 Problem Equations Graph
Profit Function Included

The distance the line moves from the origin, represented by ray P, is equivalent to an increase in the value of the profit function. The maximum profit is obtained when the profit function moves to the farthest extreme point of the possible solutions space (area CEBO in this problem).

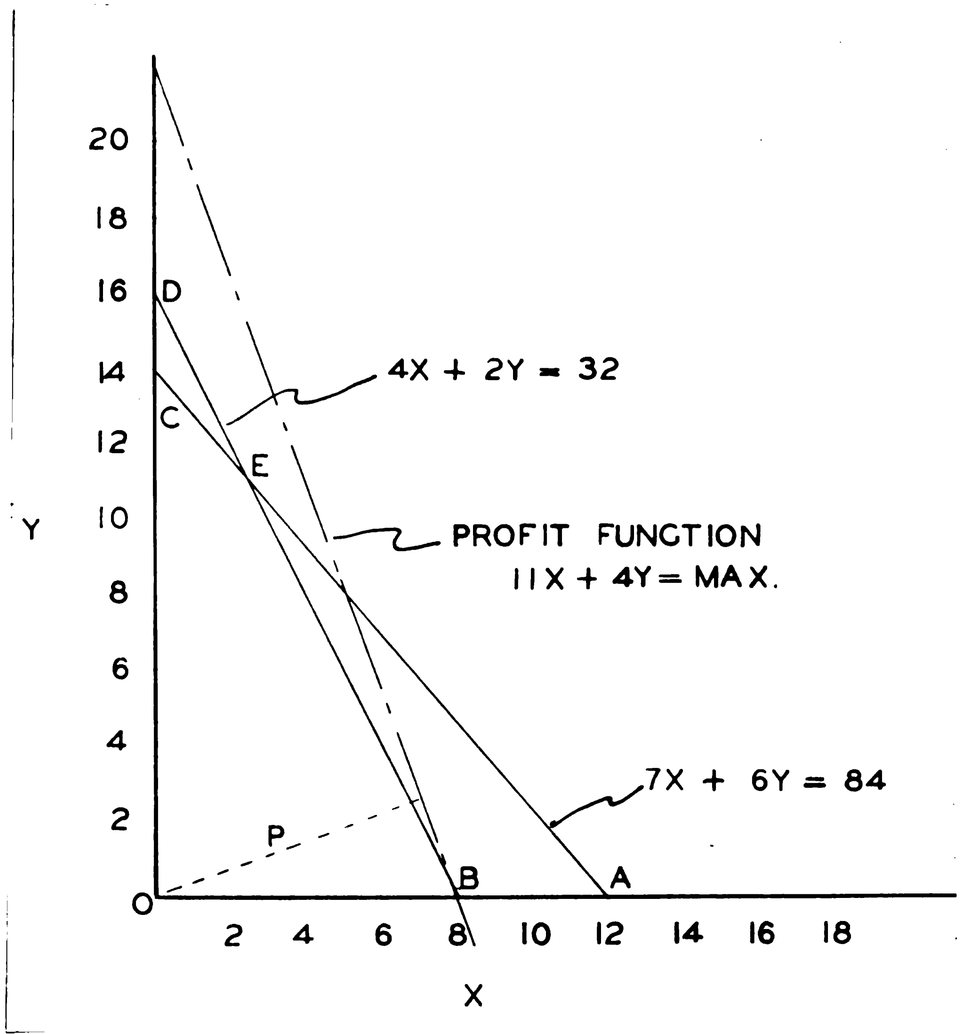

Fig. 3. Optimum Solution

Figure 3 will verify the previously derived solution. The value of the profit function can be obtained by scaling the ray P and computing the total profit.

This graphical method can quickly provide the answer to similar two variable problems. However the accuracy of this graphical approach depends upon the accuracy of the graph construction.

The important point in the geometric interpretation is that it holds true regardless of the number of variables. In essence the linear programming problem in "n" variables can be thought of as forming a convex polyhedral cone in an "n" dimension space. The functional can be thought of as representing a hyperplane in this "n" dimension space. The simplex method then proceeds to move the functional from an extreme point to an adjacent extreme point until an optimum has been obtained (i. e., the functional is moved as far from the origin as possible and still intersects the convex polyhedral cone).[24] In mathematical programming problems the optimum always lies on a boundary and never within the solutions space.

Summary

The simplex method can be summarized in ten steps:

1. Develop the equations, including slack and artificial variables as may be required.

2. Form the matrix, and establish the check column.

   a. The numbers in the constant column (except in the index row) must be positive.

   b. The identity of the matrix must be square and must be a positive unit diagonal.

   c. The check column is the algebraic sum of all numbers in the row, including the constant column.

3. Put the coefficients of corresponding variables of the objective equation above the proper columns.

4. Develop index row values.

   a. Multiply numbers of each column by corresponding numbers in the objective column and total; from this sum subtract the number in the objective row at a top of the column. The value so found is placed in the index row.

[24] Charnes, Cooper, Henderson. Ibid, part II, pages 41-62.

5. Select key column.

   a. The most negative index number locates the key column. In case of a tie, select either column.

6. Select key row.

   a. Divide the numbers of constant column by the corresponding positive non-zero numbers of the key column. Select that row with the smallest quotient. In case of a tie, see discussion on degeneracy below.

7. Select key number.

   a. At intersection of key row and key column.

8. Develop new tableau.

   a. Main row of the new tableau is in the same position as the key row of the previous tableau. The main row is developed by dividing the numbers of the key row by the key number. The key column of the previous tableau appears in the new tableau with a "+1" at the junction of the main row, all other numbers in that column are "0".

   b. The variable and its objective number from the key column replaces the variable and its objective number from the key row and becomes the objective and variable for the main row of the new tableau.

   c. The remaining variables and their objective numbers associated with other rows and columns remain unchanged and occupy the same position in the new tableau.

   d. The remaining numbers in the identity, body, constant column, check column, and index row are determined by the following formula:

$$\text{New Number} = \text{Old Number} - \frac{\text{Corresponding No. of Key Row} \times \text{Corres. No. of Key Col.}}{\text{Key number}}$$

e. Check the accuracy of the new tableau by examination of the check column.

(1) If an error appears, use step 4 to check numbers of index row. This will isolate the error by column and row.

f. Short cuts.

(1) These short cuts can easily be verified by the equation in 8d preceding. A +1 appears at the intersection of the row and column containing the same variable. All other numbers of that column are zero including the number in the index row.

(2) If there is a "0" in the key row of the previous tableau or main row of new tableau, all figures in the column in which the "0" appears are unchanged and thus repeated in the new tableau.

(3) If there is a "0" in the key column, that row containing the "0" is unchanged and thus repeated in the new tableau.

9. If all the new index row numbers are "0" or positive, the best solution has been reached. If negative numbers occur, a new tableau is formed by repeating steps 5 through 9.

10. The best solution is reached when the index row contains no negative numbers in the body and identity of the matrix. The interpretation is as follows:

a. The number in the index row of the constant column gives the value of the objective equation for that solution.

b. The other numbers of the constant column are the values of the corresponding variables.

c. The numbers in the index row, below the body of the tableau, represent the amount of reduction in the objective equation if one unit of that variable is introduced into the solution with the corresponding changes in the other variables.

d. The numbers in the index row, below the identity of the tableau, represent the "opportunity profit" or the increase which would occur in the objective function if one more unit of that variable were available (i. e., if the restriction were relaxed by one more unit).

e. Equally optimal alternate solutions are evidenced by a zero in the index row under a variable not in the solution.

In reality the simplex method is not as difficult as it is time consuming and tedious. Actually the effort in applying mathematical programming must be directed to defining and refining the problem statement, setup of the simplex matrix and finally interpreting the final results and passing them on to management for action. Applying the simplex method is in essence a means to an end. The simplex method is another addition to the industrial engineer's tool kit -- a very powerful addition.

11. Degeneracy occurs when a tie occurs in selecting the key row (step 6 above). Degeneracy may be resolved as follows:

    a. Divide each number in the "tied" rows by the number in the key column.

    b. Compare the quotients obtained, column by column from left to right in the identity of the matrix array.

    c. At the first column where the quotients are unequal the tie is broken.

    d. Select as the key row that row with the algebraically smaller quotient.

III. A PRODUCTION PLANNING PROBLEM

At present the function of production planning in industry is a somewhat inexact activity that strives by various means to develop a broad overall producing plan for an organization. This is management's grand strategy planning. In developing such a plan an incomplete picture of the cost relationships usually exists and in many cases it is difficult to obtain a factual comparison between two or more production plans.

The production planning usually begins with what data is available and works toward a manufacturing plan. This involves consideration of various tangible and intangible factors and their effects upon the manufacturing plan and vice versa. The application of mathematical programming can greatly assist the production planning activity.

With mathematical programming the lowest cost producing plan can be determined.[25] This lowest cost plan can then be tempered as necessary by the several intangible factors such as employee relations, community relations and other management policies. The resulting production plan should certainly be the best plan possible.

This section will develop, solve and analyze a typical production planning problem.

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[25] The author is indebted to Mr. N. V. Renfield of Executive Services, Cleveland, Ohio who suggested this general approach to the production planning problem. A similar approach may be found in: Harrison, Jr., Joseph O. Linear Programming and Operations Research, J. F. McCloskey and F. N. Trefethen (eds.) Operations Research for Management. P. 231-33, Johns Hopkins Press, Baltimore, 1954, and Bowman, Edward H. Production Scheduling by the Transportation Method of Linear Programming, Journal of the Operations Research Society of America, V. 4, No. 1, February, 1956. Pages 100-103.

The Problem

An organization, producing a variety of home workshop machines, desires a manufacturing plan to be prepared for a combination disk and belt sanding machine. The sales forecast and all pertinent cost and capacity data are available. The planning is best accomplished by working with man-hour data because this is the most reliable measure of producing capacity.

The problem is complicated to some extent because the manufacturing plans are already prepared for all products except this sanding machine. This means that a portion of the total factory capacity is already taken up in producing these other products. Therefore, the available capacity shows a considerable fluctuation during the year.

The problem is then to develop the best, not necessarily lowest cost, producing plan to meet the sales forecast for the sanding machine.

Data

The following data is representative of data required to develop a manufacturing plan:

1. Forecasted sales in units:

January	12, 500	July	25, 000
February	7, 500	August	27, 500
March	17, 500	September	32, 500
April	22, 500	October	30, 000
May	17, 500	November	22, 500
June	20, 000	December	15, 000

Total - 250, 000 units [26]

[26] This forecast is by no means to be construed as representative of the actual demand for this type of product. This is merely an illustrative problem.

2. The factory capacity already planned for the other products is as follows:

January	46, 000 man-hours	July	48, 000 man-hours
February	53, 000	August	41, 000
March	38, 000	September	30, 000
April	44, 000	October	48, 000
May	44, 000	November	48, 000
June	42, 000	December	49, 000

3. The factory normally works two shifts of 40 hours per week, 52 weeks per year. The two shifts are of equal size.

4. In a normal two shift working day the total plant capacity is 3520 man-hours.

5. Each unit of the sanding machine requires 1. 5 man-hours of direct labor. This includes necessary plant efficiency allowances.

6. The average direct labor costs are:

First Shift	\$1. 70 per hour
Second Shift	\$1. 75 per hour

7. The average factory burden is determined as 200% of the direct labor costs.

8. Each unit requires \$10. 00 in direct materials.

9. Overtime is paid at time and one-half for work in excess of 8. 0 hours in one day and 40. 0 hours in one week. Overtime is paid at double time for Sundays and holidays and for work in excess of 8. 0 hours on a Saturday.

10. Storage facilities are sufficient and cost \$0. 20 per square foot per month. Each finished unit occupies 1. 3 square feet of floor space but they can be stacked four units high.

11. Inventory charges are 20 per cent per annum on the inventory investment. Inventory charges are usually figured monthly.

12. Finished stock inventory at the end of the producing year must not exceed 500 units.

13. Production and distribution are such that the total units produced in January are available for sale in January. The lead time from manufacturing to the consumer is zero.

This is the type of data that should be available to the production planners. The initial step of the problem is then to organize the data in some form that is adaptable to analysis. This is accomplished for the straight time producing capacity for the calendar year 1957 as illustrated in Figure 4, page 84 in the Appendix, and summarized in Table XXII, page 90 in the Appendix.

Table XXII indicates a shortage in straight time producing capacity of 10,296 units or 15,444 man-hours. Since sufficient straight time producing capacity does not exist it will be necessary to consider overtime production capacities. Management must specify the desired maximum overtime consistent with their policies. For this problem maximum time and one-half overtime is considered as eight hours per Saturday and two hours per week day. Maximum double time overtime includes eight hours per Sunday. This is accomplished as shown in Figure 5, page 85 in the Appendix, and summarized in Table XXIII, page 91 in the Appendix.

The cost information is then calculated in terms of cost per unit for direct material, direct labor, factory burden, storage (floor space), and inventory costs as shown in Figures 6 and 7, pages 86 and 87 in the Appendix. The storage and inventory costs are summarized in Table XXIV, page 92 in the Appendix. Inventory and storage costs are calculated monthly and added to the inventory investment. This, in effect, causes a cumulative or compound inventory charge.

Formulation

Prior to considering the specific problem at hand the general statement of the production planning problem will be developed and discussed.

If we let:

P_i = number of units of producing capacity in the i'th month

Q_j = number of units of forecasted sales for the j'th month

x_{ij} = number of units produced in the i'th month and sold in the j'th month

a_{ij} = cost per unit to produce in the i'th month and inventory for sales demand in the j'th month

$i = (1, 2, 3, --- 12)$

$j = (1, 2, 3, --- 12)$

then the production planning problem can be formulated in two sets of relationships. The first set (restrictions) says that what is produced in the i'th month and sold in that month and succeeding months is less-than-or equal to the production capacity for the i'th month. Mathematically this is:

$$\sum_{j=i}^{12} x_{ij} \leq P_i \qquad (3.1)$$

The second set of relationships (equations) considers the sales demand. What is sold in the j'th month must naturally be produced in that month or preceding months (since back orders are excluded from consideration). Mathematically this can be stated as follows:

$$\sum_{i=1}^{j} x_{ij} = Q_j \qquad (3.2)$$

The objective function is to minimize the producing and storage costs and can be stated thus:

$$\sum_{\substack{i=1 \\ j=1}}^{12} (a_{ij}\, x_{ij}) = \text{minimum} \qquad (3.3)$$

The total producing capacity is then:

$$\sum_{i=1}^{12} P_i = P_{total} \qquad (3.3)$$

and the total sales forecast is then:

$$\sum_{j=1}^{12} Q_j = Q_{total} \qquad (3.4)$$

where:

$$P_{total} \geq Q_{total}$$

The total producing capacity (P_{total}) is always greater than the sales demand (Q_{total}). This may always be obtained by considering overtime producing capacity and/or subcontracting. The latter can conceivably present an unlimited potential capacity.

The summation limits of (3. 1) and (3. 2) factor out the possibility of back orders, i. e., February's production cannot be used to fulfill the January sales demand. Back orders may be considered if management can provide factual cost information. However, this general statement of the problem will not allow back orders.

Since the problem at hand must include overtime production of at least 10,296 units (see Table XXII, page 90 in Appendix) this should be factored into the statement of the problem. For the sake of simplicity, subcontracting will not be considered at this time.

In order to include this overtime consideration in the problem formulation the symbols must be expanded as follows:

x_{ij} = number of units produced at straight time in the i'th month and sold in the j'th month

x'_{ij} = number of units produced at time and one-half overtime in the i'th month and sold in the j'th month

x''_{ij} = number of units produced at double time in the i'th month and sold in the j'th month

a_{ij} = cost per unit to produce at straight time in the i'th month and inventory for sales demand in the j'th month

a'_{ij} = cost per unit to produce at time and one-half overtime in the i'th month and inventory for sales demand in the j'th month

a''_{ij} = cost per unit to produce at double time in the i'th month and inventory for sales demand in the j'th month

P_i = number of units of producing capacity in the i'th month

Q_j = number of units of forecasted sales for the j'th month

j = (1, 2, 3, - - - 12)

i = (1, 2, 3, - - - 12)

The new form of the first set of restrictions involving producing capacities can now be expanded to include the overtime consideration as follows:

$$\sum_{j=i}^{12} (x_{ij} + x'_{ij} + x''_{ij}) \leq P_i \qquad (3.5)$$

The second set of relationships (equations) expressing the sales demand can also be expanded thus:

$$\sum_{i=1}^{j} (x_{ij} + x'_{ij} + x''_{ij}) = Q_j \qquad (3.6)$$

and the expanded form of the objective function is now:

$$\sum_{\substack{i=1 \\ j=1}}^{12} (a_{ij}\, x_{ij} + a'_{ij}\, x'_{ij} + a''_{ij}\, x''_{ij}) = \text{minimum} \qquad (3.7)$$

This objective function (3.7) can be simplified by deducting direct material, straight time (direct) labor and factory burden costs, since these are common to all units produced. The problem objective is then simplified to minimize the labor overtime premium, storage and inventory costs which are the basic variables in this problem.

The general production planning problem is thus represented by a mathematical model containing 12 linear equations and 12 linear inequalities involving 234 variables. At this point the problem would seem insurmountable since the simplex setup would result in a 24 by 258 matrix array (24 equations or rows plus 234 variables or columns plus a 24 x 24 identity).

The production planning problem lends itself to solution with the distribution methods. The factory capacities month by month can be considered as sources of supply while the sales forecast can be considered as demand. With this analogy it is possible to visualize a distribution matrix with monthly factory capacities for the rows and the monthly sales forecast as the columns or vice versa. The distribution matrix form for this production planning problem appears as Table XXV, page 94 in Appendix.

The production planning problem is now represented in a 13 by 36 distribution matrix array. This organization certainly makes the problem appear more easily solvable and yet every relationship of the general statement of the problem (3. 5, 3. 6, 3. 7) remains intact in the problem. The producing capacity (3. 5) is represented as the columns and the sales demand (3. 6) is represented as the rows. The simplified form of the cost factors (a_{ij}, a'_{ij}, a''_{ij} representing overtime premium, storage and inventory costs) appears as the cost factors in the respective i'th columns and j'th rows. The quantities x_{ij}, x'_{ij}, x''_{ij} are to be determined subject to the rim conditions.

Note that the rim conditions have been balanced with the dummy row (row 13) which represents fictious demand for the product. In the real physical situation this represents unutilized producing capacity.

Note also that the cost factors for about half of the matrix are -M. Since back orders are not considered practical in this problem the -M cost (defined as so large that it dominates all else in the problem) accomplishes this restriction. If management allowed back orders then a suitable cost factor could be used. However, such a cost would be difficult if not impossible to obtain.

Optimum Solution

The production planning problem can now be solved by any of the distribution methods. Vogel's Approximation Method (VAM) and subsequent improvement with the Modified Distribution Method yielded the answer in several steps[27] by hand. The problem can also be solved with any of the

[27] This problem required the VAM plus six steps with the MODI method for solution. This required 10 hours. However, it is felt by the author that the solution could be accomplished in fewer steps and a shorter time period. This was verified by resolving the problem requiring the VAM + two steps with the MODI method. This was accomplished in approximately 3 hours.

digital computors. However, the cost of machine computation over hand computation for a problem of this size might be difficult to justify.

The final optimum solution matrix is shown in Table XXVI, page 95 in the Appendix and is also summarized in Figures 8 and 9, pages 88 and 89 in the Appendix. This solution certainly appears to be a practicable one.

Analysis of the Optimum Solution

It is now possible to analyze the optimum (minimum cost) solution in light of the various intangible considerations and management policies.

Two points about the problem should be borne in mind. First, this problem assumes a zero lead time from manufacturing to consumer sales. While this is seldom if ever the real case a suitable lead time can easily be included in the problem formulation. Second, the inventory charges are cumulative and calculated monthly for a full month's storage in inventory. In an actual situation this may vary somewhat and could conceivably reduce the inventory charges.

It can be seen from the final optimum solution (page 91 in the Appendix) that a FIFO (first-in-first-out) inventory system is in effect. This is partly due to the compound inventory charges that were used. In general a FIFO inventory system will be more economical than a LIFO (last-in-first-out) system when inventory charges are compounded.

Summary

Now for probably the first time management has a truly complete analysis of a production plan. Alternatives brought about by any intangible factors can easily be analyzed in light of the effect upon costs.

However, it is necessary here to present a word or two of caution. This type of solution does not and cannot include costs associated with labor fluctuations and turnover and other such non-linear items (costs that do not vary directly with the production quantities). Nor can it include costs associated with less than full budgeted utilization of equipment and facilities. Then too, the production plan is only as good as the sales forecast. This means that no matter how well a production plan is developed it is poor if based upon an ill conceived sales forecast.

In a real situation the greatest difficulty is not in obtaining the numerical solution but in obtaining meaningful costs, capacities, and requirements.

If desired, alternative production plans can be based upon some established upper and lower limits of the sales forecast. Comparison of these alternative plans for the upper limit, the mean, and the lower limit of the sales forecast will provide management with a guide as to the flexibility of the production plan with a variable sales forecast. This additional information can be very useful in such situations.

The problem can be expanded to include planning for the desired inventories period by period. Here the problem matrix takes on more rows as the various inventories are included. In many situations this would be the more desired formulation of the problem.

The production planning may be accomplished on a continuous basis, either monthly or quarterly, for the next twelve-month period. This type of continuous planning means that a twelve-month plan is always available. While this requires more work it means that more and better information is available to management.

A once-per-year planning cannot include changes in sales demand from the original forecast. Continuous planning can successfully accomplish such changes. Less risk of sub-optimization (optimizing one year's plan at a sacrifice in the next year's plan) would result in continuous planning.

The 13 x 36 distribution matrix array of the production planning problem can be used as a standard form by most manufacturers. The problem setup can be reduced to a routine and the solution can be accomplished either with a computor or properly trained clerks.

The production planning problem may be expanded to include several products. It is necessary, in considering such a problem, to express the production and sales in some common unit such as man-hours or equivalent units or in terms of one hour's production as suggested by Mr. Bowman.[28] While this would increase the size (number of columns) of the distribution matrix it still remains a solvable problem.

This application illustrates how time periods can be successfully incorporated within the framework of a linear programming problem. A useful extension of this can result in better analysis of product distribution to various warehousing locations. In situations where factory output and/or customer demand fluctuate through several periods a better product distribution can result when considering several periods rather than just one at a time.

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[28] Ibid, pg. 101-2.

IV. A MANUFACTURING PROBLEM

In most manufacturing situations a number of courses of action are possible. Usually the multi-facet nature of the problem defies solution by inspection or intuition. Such a manufacturing problem will be considered here.

The Problem

A manufacturer receives orders for two products (A and B). The customers require 200 units of product A and 300 units of product B. Both products are manufactured in two operations.

The first operation is performed in Process I and it requires two hours and four hours per unit to produce products A and B respectively. The second operation can be performed in either Process II or III. It requires four hours per unit of product A and seven hours per unit of product B produced in Process II. It requires ten hours per unit of product A and twelve hours per unit of product B manufactured in Process III.

There are 1700 hours on Process I, 1000 hours on Process II and 3000 hours on Process III available in the schedule period. An additional 500 hours is available on Process II in overtime.

The labor and burden costs are \$3.00, \$3.00 and \$2.00 per hour on Processes I, II and III respectively. The overtime on Process II increases the costs to \$4.50 per hour.

If no penalty is assumed for idle machine time then the problem is to determine how to manufacture the products so the overall costs are a minimum.

Problem Formulation

If we let

x_1 = number of units of product A manufactured in Process I and at straight time in Process II

x_2 = number of units of product A manufactured in Process I and at overtime in Process II

x_3 = number of units of product A manufactured in Process I and Process III

x_4 = number of units of product B manufactured in Process I and at straight time in Process II

x_5 = number of units of product B manufactured in Process I and at overtime in Process II

x_6 = number of units of product B manufactured in Process I and Process III

the formulation is then

(Process I)	$2x_1 + 2x_2 + 2x_3 + 4x_4 + 4x_5 + 4x_6 \leq 1700$	(4.1)
(Process II Straight Time)	$4x_1 + 7x_4 \leq 1000$	(4.2)
(Process II Overtime)	$4x_2 + 7x_5 \leq 500$	(4.3)
(Process III)	$10x_3 + 12x_6 \leq 3000$	(4.4)
(Product A)	$x_1 + x_2 + x_3 = 200$	(4.5)
(Product B)	$x_4 + x_5 + x_6 = 300$	(4.6)

Multiplying the time by the cost per hour will yield the following objective (cost) relationship:

$$-18x_1 - 24x_2 - 26x_3 - 33x_4 - 43.5x_5 - 36x_6 = \text{Minimum} \qquad (4.7)$$

The problem is thus stated in four inequalities (4.1 - 4.4) and two equations (4.5 and 4.6) and the objective (cost) function (4.7) is to be minimized. The four inequalities represent process time on the three processes. The two equations (4.6 and 4.7) insure that the required amount of each product is made. The objective function represents the cost (in dollars) to produce the two products by the various processes.

The relationships must be prepared for solution by the simplex method. This is accomplished by adding slack (w) and artificial (U) variables as follows:

$$2x_1 + 2x_2 + 2x_3 + 4x_4 + 4x_5 + 4x_6 + W_1 = 1700 \qquad (4.8)$$

$$4x_1 + 7x_4 + W_2 = 1000 \qquad (4.9)$$

$$4x_2 + 7x_5 + W_3 = 500 \qquad (4.10)$$

$$10x_3 + 12x_6 + W_4 = 3000 \qquad (4.11)$$

$$x_1 + x_2 + x_3 + U_1 = 200 \qquad (4.12)$$

$$x_4 + x_5 + x_6 + U_2 = 300 \qquad (4.13)$$

The objective function now becomes:

$$-18x_1 - 24x_2 - 26x_3 - 33x_4 - 43.5x_5 - 36x_6 + 0.W_1 + 0.W_2 + 0.W_3 + 0.W_4 - -MU_1 - MU_2 = \text{Minimum} \qquad (4.14)$$

The slack variables (W_1 - W_4) were added to make equations of the inequalities. These can be thought of as representing idle process time. The slack variables have zero weight in the objective function because no penalty is assumed for idle process time.

The artificial variables (U_1 and U_2) are included to form a square identity or basis for the simplex method. The -M cost factor (defined as so large that it dominates all else in the problem) is attached to the artificial variables to assure that they will be zero, since equality exists without them. The problem is now ready for the simplex method.

The initial simplex tableau is shown in Table XXI.

Table XXI. Initial Simplex Matrix

			-18	-24	-26	-33	-43.5	-36	0	0	0	0	-M	-M		Check	
			x_1	x_2	x_3	x_4	x_5	x_6	W_1	W_2	W_3	W_4	U_1	U_2			
0	W_1	1700	2	2	2	4	4	4	1							1719	
0	W_2	1000	4			7				1						1012	
0	W_3	500		4			7				1					512	
0	W_4	3000			10			12				1				3023	
-M	U_1	200	1	1	1								1			204	
-M	U_2	300				1	1	1						1		304	
		-500M	-M + 18	-M + 24	-M + 26	-M + 33	-M + 43.5	-M + 36	0	0	0	0	0	0		-506M +180.5	

All the relationships are included. The initial or trivial solution, the index row, and check column are shown.

Solution

This problem was solved in five iterations, the details of which are shown in Table XXVII (page 96 in the Appendix, fold out for reference). The optimum solution, in this case lowest cost solution in terms of the formulation, is as follows:

$x_1 = 200$	$W_1 = 100$
$x_2 = 0$	$W_2 = 0$
$x_3 = 0$	$W_3 = 350$
$x_4 = \frac{200}{7}$	$W_4 = 0$
$x_5 = \frac{150}{7}$	$U_1 = 0$
$x_6 = 250$	$U_2 = 0$

Minimum Cost = $14, 475. 00

These answers can be seen in Tableau V, Table XXVII and can be easily verified by substitution in the original equations (4. 8 through 4. 14).

Process I

$$2x_1 + 2x_2 + 2x_3 + 4x_4 + 4x_5 + 4x_6 + W_1 = 1700 \quad (4.8)$$

$$2(200) + 2(0) + 2(0) + 4\left(\frac{200}{7}\right) + 4\left(\frac{150}{7}\right) + 100 = 1700$$

$$1700 = 1700$$

Process II Straight Time

$$4x_1 + 7x_4 + W_2 = 1000 \quad (4.9)$$

$$4(200) + 7\left(\frac{200}{7}\right) + 0 = 1000$$

$$1000 = 1000$$

Process II Overtime

$$4x_2 + 7x_5 + W_3 = 500 \qquad (4.10)$$

$$4(0) + 7\left(\frac{150}{7}\right) + 350 = 500$$

$$500 = 500$$

Process III

$$10x_3 + 12x_6 + W_4 = 3000 \qquad (4.11)$$

$$10(0) + 12(250) + 0 = 3000$$

$$3000 = 3000$$

$$x_1 + x_2 + x_3 = 200 \qquad (4.12)$$

$$200 + 0 + 0 = 200$$

$$200 = 200$$

$$x_4 + x_5 + x_6 = 300 \qquad (4.13)$$

$$\frac{200}{7} + \frac{150}{7} + 250 = 300$$

$$300 = 300$$

$$-18x_1 - 24x_2 - 26x_3 - 33x_4 - 43.5x_5 - 36x_6 - MU_1 - MU_2 = \text{Minimum} \qquad (4.14)$$

$$-18(200) - 24(0) - 26(0) - 33\left(\frac{200}{7}\right) - 43.5\left(\frac{150}{7}\right)$$

$$-36(250) - M\cdot 0 - M\cdot 0 = \text{Minimum}$$

$$14475 = \text{Minimum}$$

Analysis of the index row will provide some useful information about alternate solutions (refer to Tableau V, Table XXVII, page 96 in the Appendix).

The index number for the second column is zero. This indicates that "x_2" could be introduced into the solution without increasing the total cost. This would provide an equally optimum alternate solution. The equally optimum solution is shown in Table XXVIII, page 97 in the Appendix. It is obtained by introducing "x_2" in place of "x_5" from Tableau V, Table XXVII.

The alternate solution is:

$x_1 = 162.5$	$W_1 = 100$
$x_2 = 37.5$	$W_2 = 0$
$x_3 = 0$	$W_3 = 350$
$x_4 = 50$	$W_4 = 0$
$x_5 = 0$	$U_1 = 0$
$x_6 = 250$	$U_2 = 0$

Minimum Cost = $14, 475. 00

This can be easily verified by the initial problem equations. The index row also provides useful information about less-than-optimum alternate solutions. The cost function would be increased by $8. 25 per unit of "$x_3$" introduced into the solution.

Opportunity profit is shown by the index numbers under the identity of the matrix. These numbers can be interpreted as follows:

> The cost function would be reduced by $1. 50 per each additional hour (W_2) on Process II at straight time.
> The cost function would be reduced by $0. 625 per additional hour (W_4) on Process III.

In other words, cost would be reduced if the restraints were relaxed (available time increased).

Summary

The problem discussed here is a typical example of many manufacturing problems. The simplex method was employed here for illustrative purposes. In an actual plant situation a suitable approximation method would probably be used. This would not necessarily result in the very lowest cost solution but it would tend to reduce the combined computation and production cost.

As with any computational method the economics of the problem and the method must be considered. If such a problem were encountered once per quarter the simplex method would seem justified. If once per week a suitable approximation method can be employed. If the problem comes up several times per day then either an approximation method or a high speed computor can be used to obtain the necessary answers.

In many cases the solution is not as significant as the side results obtained in the index row. These side results, for probably the first time, provide management with factual comparisons between processes and products. Then too, the requirement of an exacting statement of the problem often uncovers other problems which heretofore were unsolved. In many cases merely the attempt to apply mathematical programming will bring many otherwise hidden problems to light.

V. SUMMARY

The material presented in this thesis is designed for the engineer. Liberal references through the text and in the bibliography will provide adequate material for the student interested in research and development in this field. While a wealth of material has been published to date, a really adequate primer in mathematical programming as yet does not exist, although several are in preparation.

The mathematical programming type problems can best be summarized in terms of what must exist in the problem to apply the mathematical programming methods.

There must be:

1. A number of choices or ways of taking action.
2. An efficiency (or cost) differential between the possible choices.
3. A set of restrictions or upper limits, i.e., that which cannot be exceeded.
4. A set of requirements or lower limits, i.e., that which must be accomplished (often implicit in the problem statement).
5. An objective or policy statement, i.e., the goal to aim at; maximum profit, minimum costs, etc.
6. An interrelationship of the variables in significant expressions.
7. A common unit of measure.

The preceding are necessary though not necessarily sufficient prerequisites for the mathematical programming problem.

Some of the industrial applications of mathematical programming are:

1. Production allocation and scheduling.
2. Distribution and shipping.

3. Market research -- locating outlets, warehouses, etc.
4. Salary and job evaluation.
5. Blending -- oils, gasolines, alloys, etc.
6. Product mix problems.
7. Materials handling (non-automated).
8. Production planning.

The mathematical programming approach to solving the above problems is much superior to many of the present intuitive methods employed today. It is interesting however to discover that often the intuitive solution by an experienced person will be very close to the optimum solution obtained by mathematical programming. This however does not discount the value of the mathematical formulation and solution, since any individual with only nominal experience can always obtain the very best answer with mathematical programming whereas years of experience are usually required to develop any valid intuitive method.

Mathematical programming was limited initially to static analysis. As developments progressed the production planning problem was presented as the first example of dynamic analysis (incorporating more than one period of time) with mathematical programming. The planning problem presented in this thesis is the first such problem shown in its entirety. It is possible now to solve certain dynamic problems with mathematical programming. In fact, a problem such as production planning can be made more dynamic by analysis and solution periodically, perhaps monthly or quarterly. This approach can accommodate changes or variations in product demand and available capacity (inputs to the problem) as time progresses. While this presents more work it provides management with more and better answers to an ever changing problem.

Much the same type of dynamic analysis can be applied to problems of product distribution. This applies particularly well when the demand varies from period to period. In fact, the danger of sub-optimization (optimizing one problem or a part of a problem at a disproportionate sacrifice in another problem or another part of the same problem) can only be minimized by dynamic analysis in problems of this type.

Usually the most difficult task in mathematical programming is obtaining the necessary factual information. Often the digging for the required information will uncover previously hidden problems which when exposed can be easily solved. It is not inconceivable that the gains derived in solving these previously hidden problems may far outweigh the gains from mathematical programming. In fact, the problem to be solved may be reduced to where the solution is obvious thereby making mathematical programming unnecessary for solution. This however does not discount the value of mathematical programming since it involves more of a philosophy of problem approach rather than merely the solution methods discussed in the thesis. In the final analysis anything that forces a critical look at what is being done is of value to an organization.

Mathematical programming is not a panacea for solving all industrial problems in that it has some very serious limitations. The biggest limitation is that the relationships of the variables must be linear. Some work has been done to circumvent this restriction[29] however only limited progress has been made to date.

Non-linear elements such as setup times in a manufacturing problem cannot be included in a mathematical programming model. To date a problem like this may be handled by arbitrarily setting aside a portion of the available equipment time for setup and solving the problem using run time only. The setup time required by the solution is then checked against the time allotted previously for setup. Any discrepancies are corrected by reallocating setup time and resolving the problem. This technique, while it yields a usable

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29 See Saline, L. E. Quadratic Programming of Interdependent Activities for Optimum Performance, A. S. M. E. Paper No. 54-A-58, November, 1954. 21 pp.

answer, is very questionable from the standpoint - is the answer truly optimum. The danger of sub-optimization exists in this type of problem approach. However until non-linear elements can be treated more scientifically this is about the only approach that can be made to such problems.

Mathematical programming requires the variables to be related in meaningful expressions. This is often difficult though not impossible to obtain. However, much difficulty will usually be encountered in establishing the objective function. This is usually difficult for management to pin point. The objective may be maximum equipment utilization, minimum cost, maximum profit, maximum number of pieces, etc. Each of these objectives may yield a different solution to the same model. The problem used to illustrate the simplex method (page 33) is just such a model where different solutions may be obtained for maximum profit, maximum equipment utilization and maximum number of pieces produced. If management cannot clearly envision and state the objective then mathematical programming is stopped before it has even begun.

It is well to point out here that digital computors may be used for solving mathematical programming problems. The IBM type 701 computor can solve distribution problems up to $m \times n = 3000$, and simplex problems up to 50 x 100 in size. The newer IBM 704 and 705 will handle even larger problems, primarily due to larger memory capacity.

In general computors will be employed either when the problem is very large and an appropriate approximation is not available or of suitable accuracy, or where the problem must be solved on a repetitive basis and the answer is required rather quickly to be of use. In some cases a computor may be used as a periodic check when an approximation method is regularly used.

This thesis contains recommended improvements primarily in the terminology of mathematical programming. The presentation here has been in non-mathematical terms. Throughout the thesis the mathematics is minimized though not completely ignored. The thesis presents the methods for the first time reduced to an easy to understand step by step procedure. This will permit more emphasis to be properly placed on problem formulation and the interpretation of the answers rather than on the methods employed. Actually the methods of mathematical programming are little more than a means of cranking out an answer. The effort then must be channeled to the remaining work in defining and solving a problem.

The thesis was limited to discussing industrial applications of mathematical programming. However, many problems other than industrial can be solved provided they have the above mentioned prerequisites.

VI. CONCLUSIONS

Mathematical programming is presented here in an easy to understand, step by step method. This thesis records for the first time in one writing all the methods of mathematical programming in simplified terminology.

Mathematical programming applied to production planning as discussed in this thesis offers a new approach of scientific analysis to problems of this type. The production planning problem exemplifies mathematical programming as a tool for dynamic analysis of distribution type problems. Heretofore the general production planning problem has been presented as an example of dynamic analysis. This thesis presents a typical problem fully developed in order to completely illustrate the phases of work required for such an analysis. This analysis can be even more dynamic if solved for example every month for the succeeding twelve month period. This would accommodate variations in capacity, forecast, and sales and would provide better answers for management. This problem also presents mathematical programming in its proper perspective: As a tool of analysis that in many situations yields an initial solution that may have to be modified in light of other intangible and/or non-linear considerations. However, this does not discount the value of mathematical programming since, in addition to the optimum solution, much very useful information regarding alternate choices of action is revealed. In the case of production planning, mathematical programming is used to minimize the tangible cost factors such as labor premium, inventory and storage costs. However, intangible costs associated with labor turnover and low equipment utilization cannot be included within the framework of mathematical programming analysis.

Mathematical programming applied to the manufacturing problem presents a scientific approach to optimizing work allocation. While the problem discussed here considered unlimited sales potential (usually an impractical assumption) and no loss or penalty associated with idle equipment time, these factors can be modified by a forecasted sales potential and idle equipment charges (unabsorbed burden) and included in the analysis. The real significance of this problem lies in the fact that for the first time costs are obtained that reflect the interaction of the various jobs competing for the available machines. At the same time profit opportunities are presented that reflect the interrelationship of the equipment efficiencies in light of the products manufactured.

Research is continually being conducted in this field but this thesis points toward several possible research studies. Pure and applied research is required to expand the applications of mathematical programming and to develop means of better coping with the problem of non-linear elements. Developments in both these directions should improve and expand the usefulness of mathematical programming in industry.

Research could be undertaken to determine a mathematical basis for Vogel's approximation method. This method was intuitively developed but it seems that some logical mathematical basis exists. Once this is obtained it may be possible to apply it toward obtaining a better initial solution in the simplex method. This type of research can greatly expand the use of mathematical programming and has the potential to materially reduce computation time and cost.

This thesis can be considered as a primer in mathematical programming and can serve as initial training for an individual interested in this field. The references in the text and bibliography are provided for those interested in further development in both mathematical programming and operations research.

APPENDIX

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Month

January, 1957

Regular Working Days

22

Total Plant Capacity

3520 Man-hours/day x 22 days = 77, 440 Man-hours

Capacity not available 46, 000 Man-hours

Capacity available 31, 440 Man-hours

$$\frac{31,440 \text{ Man-hours}}{1.5 \text{ Man-hours/Unit}} = 20,960 \text{ Units}$$

Required Capacity (Sales Forecast)

12, 500 Units

18, 750 Man-hours

Difference between available and required capacity

+ 12, 690 Man-hours

+ 8, 460 Units (Surplus over required capacity)

Number of employes

$$\frac{31,440 \text{ Man-hours}}{22(8) \text{ Hours/man}} = 179 \text{ employes}$$

Figure 4. Straight Time Production Data
Sample Calculations

Month

January, 1957

Regular Work Days

22

Overtime Work Days

Saturdays - 4
Sundays - 4

Number of employes

179

Units at time and one-half overtime

22 days at 2 hours/day	= 44 hours
4 Saturdays at 8 hours/day	= 32 hours
Total	76 hours
179 men at 76 hours	= 13,604 Man-hours
$\frac{13,604 \text{ man-hours}}{1.5 \text{ man-hours/unit}}$	= 9,069 Units

Units at double time overtime

4 Sundays at 8 hours/day	= 32 hours
179 men at 32 hours	= 5,728 man-hours
$\frac{5,728 \text{ man-hours}}{1.5 \text{ man-hours/unit}}$	= 3,819 units

Figure 5. Overtime Production Data
Sample Calculations

Straight Time Production Costs		
Direct Material	$10.00	
Direct Labor 1.5 man-hours/unit x $1.725/man-hour	2.588	
Manufacturing burden (200% direct labor)	5.176	
Total unit cost		$17.764
Time and one-half overtime Production Costs		
Direct Material	$10.00	
Direct labor (including overtime) 1.5 man-hours/unit x 1.5 ($1.725/man-hour)	3.882	
Manufacturing burden*	5.176	
Total unit cost		$19.058
Double time overtime Production Costs		
Direct material	$10.00	
Direct labor 1.5 man-hours/unit x 2 ($1.725)	5.176	
Manufacturing burden*	5.176	
Total unit cost		$20.352

Figure 6. Unit Cost Calculations

*Note: Manufacturing burden rate would actually be lower at other than straight time. For simplicity the same number of dollars of burden is used. This reflects the lower rate.

Unit Cost			\$17.764
One Month's Storage			
Inventory Charges			
.20/year (1 year / 12 months) \$17.764	=	.296	
Floor space Costs			
1.3 square foot / 4 units (\$.20/sq. ft./month)	=	.065	
Total inventory and floorspace charges			.361
Total unit cost (including one month store)			\$18.125

Figure 7. Storage and Inventory Charges

Sample Calculations

Note: Inventory charges for the second month are based upon the unit cost of \$18.125. Subsequent inventory charges are based upon the unit cost including applicable storage charges.

January	Produce 20, 960 units	Sell 12, 500 in January Sell 7, 500 in February Sell 960 in March
February	Produce 11, 600 units	Sell 11, 600 in March
March	Produce 23, 946 units	Sell 4, 940 in March Sell 19, 006 in April
April	Produce 19, 946 units	Sell 3, 494 in April Sell 16, 452 in May
May	Produce 22, 293 units	Sell 1, 048 in May Sell 20, 000 in June Sell 1, 245 in July
June	Produce 18, 933 units	Sell 18, 933 in July
July	Produce 19, 627 units	Sell 4, 822 in July Sell 14, 805 in August
August	Produce 24, 293 units	Sell 12, 695 in August Sell 11, 598 in September
September	Produce 26, 933 units	Sell 20, 902 in September Sell 6, 031 in October
October	Produce 21, 973 units	Sell 21, 872 in October Sell 101 in November
	*Produce 2, 097	Sell 2, 097 in October
November	Produce 14, 933 units *Produce 7, 466	Sell 14, 933 in November Sell 7, 466 in November
December	Produce 14, 267 units *Produce 733	Sell 14, 267 in December Sell 733 in December

Figure 8. Final Optimum Solution

*At overtime.

PRODUCTION PLAN COST SUMMARY		
Manufacturing Costs		
Direct material costs 250, 000 units @ $10. 00	$2, 500, 000. 00	
Direct labor costs 250, 000 units @ 1. 5 man-hours/unit @ $1. 725/hour	646, 875. 00	
Manufacturing burden @ 200% direct labor	1, 293, 750. 00	
Total manufacturing costs		$4, 440, 625. 00
Additional Costs		
Overtime premiums	$ 13, 323. 02	
Inventory charges	38, 632. 29	
Floor space charges	8, 478. 34	
Total additional costs		$ 60, 433. 65
Total costs and charges		$4, 501, 058. 65
Average unit cost = $18. 005		

Figure 9. Final Solution
Cost Summary

TABLE XXII

1957: DATA FOR PRODUCTION PLAN

STRAIGHT TIME PRODUCTION CAPACITY

	Days	Sat.	Sun.	Total Capacity	Cap. Not Available	Available Capacity		Required Capacity		Difference		No. Empl.	Cum. Inv.
				Man Hrs.	Man Hrs.	Man Hrs.	Units	Man Hrs.	Units	Man Hrs.	Units		
January	22	4	4	77, 440	46, 000	31, 440	20, 960	18, 750	12, 500	+12, 690	+ 8, 460	179	+ 8, 460
February	20	4	4	70, 400	53, 000	17, 400	11, 600	11, 250	7, 500	+ 6, 150	+ 4, 100	108	+12, 560
March	21	5	5	73, 920	38, 000	35, 920	23, 946	26, 250	17, 500	+ 9, 670	+ 6, 446	214	+19, 006
April	21	4	4	73, 920	44, 000	29, 920	19, 946	33, 750	22, 500	- 3, 830	- 2, 554	178	+16, 452
May	22	4	4	77, 440	44, 000	33, 440	22, 293	26, 250	17, 500	+ 7, 190	+ 4, 793	192	+21, 245
June	20	5	5	70, 400	42, 000	28, 400	18, 933	30, 000	20, 000	- 1, 600	- 1, 067	178	+20, 178
July	22	4	4	77, 440	48, 000	29, 440	19, 627	37, 500	25, 000	- 8, 060	- 5, 373	167	+14, 805
August	22	5	4	77, 440	41, 000	36, 440	24, 293	41, 250	27, 500	- 4, 810	- 3, 207	207	+11, 598
September	20	4	5	70, 400	30, 000	40, 400	26, 933	48, 750	32, 500	- 8, 350	- 5, 567	253	+ 6, 031
October	23	4	4	80, 960	48, 000	32, 960	21, 973	45, 000	30, 000	-12, 040	- 8, 027	179	- 1, 996
November	20	5	4	70, 400	48, 000	22, 400	14, 933	33, 750	22, 500	-11, 350	- 7, 567	140	- 9, 563
December	20	4	5	70, 400	49, 000	21, 400	14, 267	22, 500	15, 000	- 1, 100	- 733	122	-10, 296
Total							239, 704		250, 000		-10, 296		-10, 296

TABLE XXIII

1957: DATA FOR PRODUCTION PLAN

OVERTIME PRODUCTION CAPACITY*

	Number Employed	Work Days	Time and one-half overtime				Double time overtime			
			Saturday	Hours	Man Hours	Units	Sunday	Hours	Man Hours	Units
January	179	22	4	76	13, 604	9, 069	4	32	5, 728	3, 819
February	108	20	4	72	7, 756	5, 170	4	32	3, 456	2, 304
March	214	21	5	82	17, 548	11, 699	5	40	8, 560	5, 706
April	178	21	4	74	13, 172	8, 781	4	32	5, 696	3, 797
May	192	22	4	76	14, 592	9, 728	4	32	6, 144	4, 096
June	178	20	5	80	14, 240	9, 493	5	40	7, 120	4, 746
July	167	22	4	76	12, 692	8, 461	4	32	5, 344	3, 562
August	207	22	5	84	17, 388	11, 592	4	32	6, 624	4, 416
September	253	20	4	72	18, 216	12, 077	5	40	10, 120	6, 746
October	179	23	4	78	13, 962	9, 308	4	32	5, 728	3, 818
November	140	20	5	80	11, 200	7, 466	4	32	4, 480	2, 986
December	122	20	4	72	8, 784	5, 876	5	40	4, 880	3, 253

*Overtime determined as follows:

Time and one-half -- 8 hours per Saturday plus two hours per week day.
Double time -- 8 hours per Sunday.

TABLE XXIV

UNIT COSTS FOR PRODUCTION AND STORAGE

	Cost at straight time	Difference from U. C. = 17. 764	Cost at time and one-half overtime	Difference from U. C. = 17. 764	Cost at double time overtime	Difference from U. C. = 17. 764
	17. 764		19. 058	1. 294	20. 352	2. 588
First Month Storage						
Inventory cost	. 296		. 318		. 339	
Floor space cost	. 065		. 065		. 065	
	18, 125	. 361	19. 441	1. 677	20. 756	2. 992
Second Month Storage						
Inventory cost	. 302		. 324		. 366	
Floor space cost	. 065		. 065		. 065	
	18. 492	. 728	19. 830	2. 066	21. 187	3. 423
Third Month Storage						
Inventory cost	. 308		. 331		. 353	
Floor space cost	. 065		. 065		. 065	
	18. 865	1. 101	20. 226	2. 462	21. 605	3. 841
Fourth Month Storage						
Inventory cost	. 314		. 337		. 343	
Floor space cost	. 065		. 065		. 065	
	19. 244	1. 480	20. 628	2. 864	22. 013	4. 249
Fifth Month Storage						
Inventory cost	. 321		. 344		. 368	
Floor space cost	. 065		. 065		. 065	
	19. 630	1. 866	21. 037	3. 273	22. 446	4. 682
Sixth Month Storage						
Inventory cost	. 327		. 351		. 374	
Floor space cost	. 065		. 065		. 065	
	20. 022	2. 258	21. 453	3. 689	22. 885	5. 121
Seventh Month Storage						
Inventory cost	. 333		. 358		. 381	
Floor space cost	. 065		. 065		. 065	
	20. 420	2. 656	21. 876	4. 112	23. 331	5. 567

	Cost at straight time	Difference from U.C. = 17.764	Cost at time and one-half overtime	Difference from U.C. = 17.764	Cost at double time overtime	Difference from U.C. = 17.764
	20.420	2.656	21.876	4.112	23.331	5.567
<u>Eighth Month Storage</u>						
Inventory cost	.340		.365		.389	
Floor space cost	.065		.065		.065	
	20.825	3.061	22.306	4.542	23.785	6.021
<u>Ninth Month Storage</u>						
Inventory cost	.347		.372		.396	
Floor space cost	.065		.065		.065	
	21.237	3.473	22.743	4.979	24.246	6.482
<u>Tenth Month Storage</u>						
Inventory cost	.354		.379		.404	
Floor space cost	.065		.065		.065	
	21.656	3.892	23.187	5.423	24.715	6.951
<u>Eleven Month Storage</u>						
Inventory cost	.361		.386		.412	
Floor space cost	.065		.065		.065	
	22.082	4.318	23.638	5.874	25.192	7.428

LOWEST COST PRODUCING PLAN FOR SANDING MACHINE FOR 1957

	Jan. Straight Time	Jan. 1-1/2 Overtime	Jan. 2 Overtime	Feb. Straight Time	Feb. 1-1/2 Overtime	Feb. 2 Overtime	Mar. Straight Time	Mar. 1-1/2 Overtime	Mar. 2 Overtime	Apr. Straight Time	Apr. 1-1/2 Overtime	Apr. 2 Overtime
JAN.	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M	-M	-M	-M
FEB.	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M
MAR.	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M
APR.	-1,101	-2,462	-3,841	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588
MAY	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841	-.728	-2,066	-3,423	-.361	-1,677	-2,992
JUNE	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841	-.728	-2,066	-3,423
JULY	-2,258	-3,689	-5,121	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841
AUG.	-2,656	-4,112	-5,567	-2,258	-3,689	-5,121	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249
SEPT.	-3,061	-4,542	-6,021	-2,656	-4,112	-5,567	-2,258	-3,689	-5,121	-1,866	-3,273	-4,682
OCT.	-3,473	-4,979	-6,482	-3,061	-4,542	-6,021	-2,656	-4,112	-5,567	-2,258	-3,689	-5,121
NOV.	-3,892	-5,423	-6,951	-3,473	-4,979	-6,482	-3,061	-4,542	-6,021	-2,656	-4,112	-5,567
DEC.	-4,318	-5,874	-7,428	-3,892	-5,423	-6,951	-3,473	-4,979	-6,482	-3,061	-4,542	-6,021
DUMMY	0	0	0	0	0	0	0	0	0	0	0	0
PRODUCING CAPACITY	20,960	9,069	3,819	11,600	5,170	2,304	23,946	11,698	5,706	19,946	8,781	3,797

	May Straight Time	May 1-1/2 Overtime	May 2 Overtime	June Straight Time	June 1-1/2 Overtime	June 2 Overtime	July Straight Time	July 1-1/2 Overtime	July 2 Overtime	Aug. Straight Time	Aug. 1-1/2 Overtime	Aug. 2 Overtime
JAN.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
FEB.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
MAR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
APR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
MAY	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M	-M	-M	-M
JUNE	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M
JULY	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M
AUG.	-1,101	-2,462	-3,841	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588
SEPT.	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841	-.728	-2,066	-3,423	-.361	-1,677	-2,992
OCT.	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841	-.728	-2,066	-3,423
NOV.	-2,258	-3,689	-5,121	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249	-1,101	-2,462	-3,841
DEC.	-2,656	-4,112	-5,567	-2,258	-3,689	-5,121	-1,866	-3,273	-4,682	-1,480	-2,864	-4,249
DUMMY	0	0	0	0	0	0	0	0	0	0	0	0
PRODUCING CAPACITY	22,293	9,728	4,096	18,933	9,493	4,746	19,627	8,461	3,562	24,293	11,592	4,416

	Sept. Straight Time	Sept. 1-1/2 Overtime	Sept. 2 Overtime	Oct. Straight Time	Oct. 1-1/2 Overtime	Oct. 2 Overtime	Nov. Straight Time	Nov. 1-1/2 Overtime	Nov. 2 Overtime	Dec. Straight Time	Dec. 1-1/2 Overtime	Dec. 2 Overtime	Demand Forecast
JAN.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	12,500
FEB.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	7,500
MAR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	17,500
APR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	22,500
MAY	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	17,500
JUNE	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	20,000
JULY	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	25,000
AUG.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	27,500
SEPT.	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M	-M	-M	-M	32,500
OCT.	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M	-M	-M	-M	30,000
NOV.	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588	-M	-M	-M	22,500
DEC.	-1,101	-2,462	-3,841	-.728	-2,066	-3,423	-.361	-1,677	-2,992	0	-1,294	-2,588	15,000
DUMMY	0	0	0	0	0	0	0	0	0	0	0	0	147,673
PRODUCING CAPACITY	26,933	12,077	6,746	21,973	9,308	3,818	14,933	7,466	2,986	14,267	5,876	3,253	397,673

LOWEST COST PRODUCING PLAN FOR SANDING MACHINE FOR 1957

(Each cell: cost / allocation where an allocation is printed.)

	Jan. Straight Time	Jan. 1-1/2 Overtime	Jan. 2 Overtime	Feb. Straight Time	Feb. 1-1/2 Overtime	Feb. 2 Overtime	Mar. Straight Time	Mar. 1-1/2 Overtime	Mar. 2 Overtime	Apr. Straight Time	Apr. 1-1/2 Overtime	Apr. 2 Overtime
JAN.	0 / 12,500	-1.294	-2.588	-M	-M	-M	-M	-M	-M	-M	-M	-M
FEB.	-.361 / 7,500	-1.677	-2.992	0	-1.294	-2.588	-M	-M	-M	-M	-M	-M
MAR.	-.728 / 960	-2.066	-3.423	-.361 / 11,600	-1.677	-2.992	0 / 4,940	-1.294	-2.588	-M	-M	-M
APR.	-1.101	-2.462	-3.841	-.728	-2.066	-3.423	-.361 / 19,006	-1.677	-2.992	0 / 3,494	-1.294	-2.588
MAY	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841	-.728	-2.066	-3.423	-.361 / 16,452	-1.677	-2.992
JUNE	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841	-.728	-2.066	-3.423
JULY	-2.258	-3.689	-5.121	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841
AUG.	-2.656	-4.112	-5.567	-2.258	-3.689	-5.121	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249
SEPT.	-3.061	-4.542	-6.021	-2.656	-4.112	-5.567	-2.258	-3.689	-5.121	-1.866	-3.273	-4.682
OCT.	-3.473	-4.979	-6.482	-3.061	-4.542	-6.021	-2.656	-4.112	-5.567	-2.258	-3.689	-5.121
NOV.	-3.892	-5.423	-6.951	-3.473	-4.979	-6.482	-3.061	-4.542	-6.021	-2.656	-4.112	-5.567
DEC.	-4.318	-5.874	-7.428	-3.892	-5.423	-6.951	-3.473	-4.979	-6.482	-3.061	-4.542	-6.021
DUMMY	0	0 / 9,069	0 / 3,819	0	0 / 5,170	0 / 2,304	0	0 / 11,699	0 / 5,706	0	0 / 8,781	0 / 3,797
PRODUCING CAPACITY	20,960	9,069	3,819	11,600	5,170	2,304	23,946	11,699	5,706	19,946	8,781	3,797

	May Straight Time	May 1-1/2 Overtime	May 2 Overtime	June Straight Time	June 1-1/2 Overtime	June 2 Overtime	July Straight Time	July 1-1/2 Overtime	July 2 Overtime	Aug. Straight Time	Aug. 1-1/2 Overtime	Aug. 2 Overtime
JAN.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
FEB.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
MAR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
APR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M
MAY	0 / 1,048	-1.294	-2.588	-M	-M	-M	-M	-M	-M	-M	-M	-M
JUNE	-.361 / 20,000	-1.677	-2.992	0	-1.294	-2.588	-M	-M	-M	-M	-M	-M
JULY	-.728 / 1,245	-2.066	-3.423	-.361 / 18,933	-1.677	-2.992	0 / 4,822	-1.294	-2.588	-M	-M	-M
AUG.	-1.101	-2.462	-3.841	-.728	-2.066	-3.423	-.361 / 14,805	-1.677	-2.992	0 / 12,695	-1.294	-2.588
SEPT.	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841	-.728	-2.066	-3.423	-.361 / 11,598	-1.677	-2.992
OCT.	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841	-.728	-2.066	-3.423
NOV.	-2.258	-3.689	-5.121	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249	-1.101	-2.462	-3.841
DEC.	-2.656	-4.112	-5.567	-2.258	-3.689	-5.121	-1.866	-3.273	-4.682	-1.480	-2.864	-4.249
DUMMY	0	0 / 9,728	0 / 4,096	0	0 / 9,493	0 / 4,746	0	0 / 8,461	0 / 3,562	0	0 / 11,592	0 / 4,416
PRODUCING CAPACITY	22,293	9,728	4,096	18,933	9,493	4,746	19,627	8,461	3,562	24,293	11,592	4,416

	Sept. Straight Time	Sept. 1-1/2 Overtime	Sept. 2 Overtime	Oct. Straight Time	Oct. 1-1/2 Overtime	Oct. 2 Overtime	Nov. Straight Time	Nov. 1-1/2 Overtime	Nov. 2 Overtime	Dec. Straight Time	Dec. 1-1/2 Overtime	Dec. 2 Overtime	Demand Forecast
JAN.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	12,500
FEB.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	7,500
MAR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	17,500
APR.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	22,500
MAY	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	17,500
JUNE	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	20,000
JULY	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	25,000
AUG.	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	-M	27,500
SEPT.	0 / 20,902	-1.294	-2.588	-M	-M	-M	-M	-M	-M	-M	-M	-M	32,500
OCT.	-.361 / 6,031	-1.677	-2.992	0 / 21,872	-1.294 / 2,097	-2.588	-M	-M	-M	-M	-M	-M	30,000
NOV.	-.728	-2.066	-3.423	-.361 / 101	-1.677	-2.992	0 / 14,933	-1.294 / 7,466	-2.588	-M	-M	-M	22,500
DEC.	-1.101	-2.462	-3.841	-.728	-2.066	-3.423	-.361	-1.677	-2.992	0 / 14,267	-1.294 / 733	-2.588	15,000
DUMMY	0	0 / 12,077	0 / 6,746	0	0 / 7,211	0 / 3,818	0	0	0 / 2,986	0	0 / 5,143	0 / 3,253	147,673
PRODUCING CAPACITY	26,933	12,077	6,746	21,973	9,308	3,818	14,933	7,466	2,986	14,267	5,876	3,253	397,673

I

			-18	-24	-26	-33	-43.5	-36	0	0	0	0	-M	-M			ck.
			x_1	x_2	x_3	x_4	x_5	x_6	w_1	w_2	w_3	w_4	u_1	u_2			
0	w_1	1700	2	2	2	4	4	4	1	0	0	0	0	0			1719
0	w_2	1000	4	0	0	7	0	0	0	1	0	0	0	0			1012
0	w_3	500	0	4	0	0	7	0	0	0	1	0	0	0			512
0	w_4	3000	0	0	10	0	0	12	0	0	0	1	0	0			3023
-M	u_1	200	1	1	1	0	0	0	0	0	0	0	1	0			204
-M	u_2	300	0	0	0	1	1	1	0	0	0	0	0	1			304
		-500M	-M +18	-M +24	-M +26	-M +33	-M +43.5	-M +36	0	0	0	0	0	0			-506M +108.5

II

0	w_1	1300	0	0	0	4	4	4	1	0	0	0	-2	0			1311
0	w_2	200	0	-4	-4	7	0	0	0	1	0	0	-4	0			196
0	w_3	500	0	4	0	0	7	0	0	0	1	0	0	0			512
0	w_4	3000	0	0	10	0	0	12	0	0	0	1	0	0			3023
→ -18	x_1	200	1	1	1	0	0	0	0	0	0	0	1	0			204
-M	u_2	300	0	0	0	1	1	1	0	0	0	0	0	1			304
		-300M +3600	0	+6	+8	-M +33	-M +43.5	-M +36	0	0	0	0	M -18	0			-302M -3491.5

III

0	w_1	$\frac{8300}{7}$	0	$\frac{16}{7}$	$\frac{16}{7}$	0	4	4	1	$-\frac{4}{7}$	0	0	$\frac{2}{7}$	0			1199
→ -33	x_4	$\frac{200}{7}$	0	$-\frac{4}{7}$	$-\frac{4}{7}$	1	0	0	0	$\frac{1}{7}$	0	0	$-\frac{4}{7}$	0			28
0	w_3	500	0	4	0	0	7	0	0	0	1	0	0	0			512
0	w_4	3000	0	0	10	0	0	12	0	0	0	1	0	0			3023
-18	x_1	200	1	1	1	0	0	0	0	0	0	0	1	0			204
-M	u_2	$\frac{1900}{7}$	0	$\frac{4}{7}$	$\frac{4}{7}$	0	1	1	0	$-\frac{1}{7}$	0	0	$\frac{4}{7}$	1			276
		$-\frac{1900M}{7}$	0	$-\frac{4M}{7}$	$-\frac{4M}{7}$	0	-M +43.5	-M +36	0	$+\frac{M}{7}$	0	0	$\frac{3M}{7}$	0			-274M
		$-\frac{31800}{7}$		$+\frac{174}{7}$	$+\frac{188}{7}$					$-\frac{33}{7}$			$+\frac{6}{7}$				-4415.5

IV

0	w_1	$\frac{1300}{7}$	0	$\frac{16}{7}$	$-\frac{22}{21}$	0	4	0	1	$-\frac{4}{7}$	0	$-\frac{1}{3}$	$\frac{2}{7}$	0			$\frac{574}{3}$
-33	x_4	$\frac{200}{7}$	0	$-\frac{4}{7}$	$-\frac{4}{7}$	1	0	0	0	$\frac{1}{7}$	0	0	$-\frac{4}{7}$	0			28
0	w_3	500	0	4	0	0	7	0	0	0	1	0	0	0			512
→ -36	x_6	250	0	0	$\frac{5}{6}$	0	0	1	0	0	0	$\frac{1}{12}$	0	0			$\frac{3023}{12}$
-18	x_1	200	1	1	1	0	0	0	0	0	0	0	1	0			204
-M	u_2	$\frac{150}{7}$	0	$\frac{4}{7}$	$\frac{11}{42}$	0	1	0	0	$-\frac{1}{7}$	0	$-\frac{1}{12}$	$\frac{4}{7}$	1			$\frac{289}{12}$
		$-\frac{150M}{7}$	0	$-\frac{4M}{7}$	$\frac{11M}{42}$	0	-M +43.5	0	0	$\frac{M}{7}$	0	$\frac{M}{12}$	$\frac{3M}{7}$	0			$-\frac{265M}{12}$
		$-\frac{94800}{7}$		$+\frac{174}{7}$	$-\frac{132}{42}$					$-\frac{33}{7}$		-3	$+\frac{6}{7}$				-13484.5

V

0	w_1	100	0	0	0	0	0	0	1	0	0	0	-2	-4			95
-33	x_4	$\frac{200}{7}$	0	$-\frac{4}{7}$	$-\frac{4}{7}$	1	0	0	0	$\frac{1}{7}$	0	0	$-\frac{4}{7}$	0			28
0	w_3	350	0	0	$\frac{11}{6}$	0	0	0	0	1	1	$\frac{7}{12}$	-4	-7			$\frac{4121}{12}$
-36	x_6	250	0	0	$\frac{5}{6}$	0	0	1	0	0	0	$\frac{1}{12}$	0	0			$\frac{3023}{12}$
-18	x_1	200	1	1	1	0	0	0	0	0	0	0	1	0			204
→ -43.5	x_5	$\frac{150}{7}$	0	$\frac{4}{7}$	$-\frac{11}{42}$	0	1	0	0	$-\frac{1}{7}$	0	$-\frac{1}{12}$	$\frac{4}{7}$	1			$\frac{289}{12}$
		-14475	0	0	$8\frac{1}{4}$	0	0	0	0	1.5	0	.625	M -24	M -43.5			2M - $\frac{58128.5}{4}$

TABLE XXVII SIMPLEX SOLUTION TO PRODUCT ALLOCATION PROBLEM

V-A

			-18	-24	-26	-33	-43.5	-36	0	0	0	0	-M	-M			
			x_1	x_2	x_3	x_4	x_5	x_6	w_1	w_2	w_3	w_4	u_1	u_2			CK.
0	w_1	100	0	0	0	0	0	0	1	0	0	0	-2	-4			95
-33	x_4	50	0	0	$-\frac{5}{6}$	1	1	0	0	0	0	$-\frac{1}{12}$	0	1			$\frac{625}{12}$
0	w_3	350	0	0	$\frac{11}{6}$	0	0	0	0	1	1	$\frac{7}{12}$	-4	-7			$\frac{4121}{12}$
-36	x_6	250	0	0	$\frac{5}{6}$	0	0	1	0	0	0	$\frac{1}{12}$	0	0			$\frac{3023}{12}$
-18	x_1	1625	1	0	$\frac{35}{24}$	0	$-\frac{7}{4}$	0	0	$\frac{1}{4}$	0	$\frac{7}{48}$	0	$-\frac{7}{4}$			$\frac{7769}{48}$
→ -24	x_2	37.5	0	1	$-\frac{11}{24}$	0	$\frac{7}{4}$	0	0	$-\frac{1}{4}$	0	$-\frac{7}{48}$	1	$\frac{7}{4}$			$\frac{2023}{48}$
		-14,475	0	0	8.25	0	0	0	0	1.5	0	.625	M -24	M -43.5			2M -14532.125

Date Due

Aug 23 '57			
Dec 6 '57			
Feb 18 '58			
NOV 10 1961			
OCT 23 1962			
MAY 11 1964			