A STUDY OF THE CORRESPONDENCE BETWEEN POINTS OF A SPACE OF THREE DIMENSIONS AND LINES OF A SPACE OF FIVE DIMENSIONS

THESIS FOR THE DEGREE OF M. A.

Margaret Grace Walcott

1932

Hyperspace

Mathematics

ACKNOWLEDGEMENT

To Doctor Vernon Guy Grove without whose aid and encouragement this thesis would not have been completed.

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A Thesis
Submitted to the Faculty
of

MICHIGAN STATE COLLEGE
of
AGRICULTURE AND APPLIED SCIENCE

In Partial Fulfillment of the
Requirements for the Degree
of

Master of Arts

by

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INTRODUCTION

It is the purpose of this paper to discuss the correspondence between lines of a space S_3 of three dimensions, and points of a certain hyperquadric Q_4^2 in a flat space S_5 of five dimensions. The dualistic transformation between points and planes of the same S_3 is well known. However, it is impossible to make points and lines of the same S_3 correspond unless they lie also in the same plane S_2.

The first part of this paper is devoted to a discussion of the general correspondence between lines of an S_3 and points of an S_5. The second part is concerned with the correspondence between certain line loci in S_3 associated with the transformation* C of nets and point loci in S_5.

Let there be given two points x and y, in S_3 with homogeneous projective coordinates (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) respectively. The homogeneous Plücker coordinates of the line l joining x to y may be denoted by $l_1, l_2, l_3, l_4, l_5, l_6$ where

$$\begin{aligned} &l_1 = x_1 y_4 - x_4 y_1, \quad l_2 = x_1 y_2 - x_2 y_1, \quad l_3 = x_1 y_3 - x_3 y_1, \\ &l_4 = x_4 y_2 - x_2 y_4, \quad l_5 = x_4 y_3 - x_3 y_4, \quad l_6 = x_2 y_3 - x_3 y_2. \end{aligned} \tag{1}$$

* V. G. Grove, Transformations of Nets, Transactions of the American Mathematical Society, Vol. 30 (1928), p. 483. Hereafter referred to as Grove, Nets.

These six Plücker coordinates satisfy

(2) $Q_4^2 = \ell_1\ell_6 - \ell_2\ell_5 + \ell_3\ell_4 = 0.$

identically for all values of x_i and y_i, $i = 1, 2, 3, 4$.

We shall interpret the six ordered numbers $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6)$ as the six homogeneous projective coordinates of a point L in an S_5. From (2) the point lies on the hyperquadric Q_4^2. We shall say that the point L is the image in S_5 of the line ℓ in S_3 and conversely that ℓ is the image of L. If follows that every line in S_3 has an image on Q_4^2 in S_5 and every point on Q_4^2 in S_5 has an image in S_3. These statements may be summarized in the following

Principle of Imagery: From any statement or theorem concerning the relative position of the lines of a geometrical configuration in ordinary projective space of three dimensions there can be obtained another statement or theorem concerning the relative position of the points of a geometrical configuration on a hyperquadric in a projective space of five dimensions.

PART I

Consider any four non-coplanar points, x, r, s, y, in an S_3 as forming a local tetrahedron of reference. Let the coordinated of these points with respect to a general tetrahedron of reference be x_i, r_i, s_i, y_i, $i = 1, 2, 3, 4$, respectively.

We may prove that the coordinates of any point z in this S_3 may be expressed in terms of the coordinates of x, r, s, y.

Consider the equations

(3) $\quad z_i = t_1 x_i + t_2 r_i + t_3 s_i + t_4 y_i,$ $\qquad i = 1, 2, 3, 4.$

The four equations (3) may be solved for the four unknowns, t_1, t_2, t_3, t_4 in terms of x_i, r_i, s_i, y_i. These solutions will be unique, since the four points, x, r, s, y are non-coplanar and therefore the determinant $|x, r, s, y| \neq 0$. It is a well known fact that t_1, t_2, t_3, t_4 are the coordinates of z with respect to the local tetrahedron of reference x, r, s, y.

We shall denote the lines joining x to y, x to r, x to s, y to r, y to s, r to s by $\rho, \xi, \eta, \bar\xi, \bar\eta, h$ respectively. The points $\rho, \xi, \eta, \bar\xi, \bar\eta, h$ the images in S_5 of these lines in S_3, form a non degenerate pyramid of reference. The coordinates of any point P in S_5 may be written in the form

(4) $\qquad P = x_1 \rho + x_2 \xi + x_3 \eta + x_4 \bar\xi + x_5 \bar\eta + x_6 h,$

where $x_1, x_2, x_3, x_4, x_5, x_6$ are coordinates of P with the pyramid $\rho, \xi, \eta, \bar\xi, \bar\eta, h$ as local pyramid of reference.

1. Images in S_5 of the Lines of a Linear Congruence in S_3.

The points of ρ joined by lines to the points of h in every possible way form a linear congruence in S_3. We shall find the points in S_5 which correspond to these lines in S_3. We may define the points P_1, on ρ and P_2 on h by the following expressions

$$P_1 = x + \lambda y,$$
$$P_2 = r + \mu s.$$

The Plücker coordinates of the line p joining P_1 to P_2 will be of the form

$$P = (P_1, P_2) = (x + \lambda y,\ r + \mu s)$$

$$= (x, r + \mu s) + \lambda (y, r + \mu s)$$

$$(5) \qquad = (x, r) + \mu (x, s) + \lambda (y, r) + \lambda \mu (y, s)$$

$$= \xi + \mu \eta + \lambda \bar{\xi} + \lambda \mu \bar{\eta}.$$

The local coordinates of the point P defined by (5) are therefore

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = \mu, \quad x_4 = \lambda, \quad x_5 = \lambda \mu, \quad x_6 = 0.$$

Eliminating λ and μ from the above equations, we find that

$$(6) \qquad x_1 = 0, \qquad x_6 = 0, \qquad x_3 x_4 = x_2 x_5.$$

The first two of equations (6) are the equations of two hyperplanes, each determining a space of four dimensions. These hyperplanes intersect in a space of one less or three dimensions. Equations (6) therefore represent a quadric lying in a space of three dimensions. We may, therefore, state the theorem:

<u>The points in S_5 corresponding to the lines of a linear congruence in S_3 lie on two hyperplanes and an ordinary quadric in an S_3 in those hyperplanes.</u>

2. Images in S_5 of the Lines of a Special Linear Complex in S_3.

Consider all of the lines in S_3 intersecting ρ. These lines form a special linear complex. Any point on the line is defined by an expression of the form $x + \lambda y$; any point in the plane determined by y, r, s is defined by an expression of the form $y + \mu r + \nu s$. The Plücker coordinates of the line joining these two points (any line intersecting ρ) are therefore defined by

$$P = (x + \lambda y,\ y + \mu t + \nu s)$$
$$= (x, y) + \mu(x, t) + \nu(x, s) + \lambda\mu(y, t) + \lambda\nu(y, s)$$
$$= \rho + \mu\xi + \nu\eta + \lambda\mu\bar{\xi} + \lambda\nu\bar{\eta}.$$

The local coordinates of the point P in S_5, the image of p in S_3, are therefore

$$x_1 = 1,\ x_2 = \mu,\ x_3 = \nu,\ x_4 = \lambda\mu,\ x_5 = \lambda\nu,\ x_6 = 0.$$

Eliminating λ, μ, ν we find that

(7) $$x_6 = 0 \quad , \quad x_3 x_4 = x_2 x_5 .$$

The first of equations (7) is the equation of a hyperplane. The second is the equation of a hyperquadric. Together they represent a cone with vertex at $\rho\,(1,0,0,0,0,0)$ and the quadric mentioned in section 1 as the base. We may, therefore, state the theorem:

<u>Corresponding to the lines of a special linear complex in S_3 are the points in S_5 which lie on one hyperplane and a cone with a point for its vertex and an ordinary quadric in an S_3 for its **base**.</u>

3. Images in S_5 of the Generators of a Cone in S_3.

Consider the lines joining y to the points on the conic $x_1^2 = K x_2 x_3$, $x_4 = 0$ referred to the local tetrahedron x, t, s, y. We may write the parametric equations of this conic in the form

$$x_1 = K\lambda, \quad x_2 = K, \quad x_3 = \lambda^2, \quad x_4 = 0.$$

Thus the conic $x_1^2 = k\,x_2 x_3$, $x_4 = 0$ is the locus of the point z defined by the expression

$$z = k\lambda x + kr + \lambda^2 s.$$

The Plücker coordinates of the lines p joining y to z are defined by

$$\begin{aligned} p = (y, z) &= (y, k\lambda x + kr + \lambda^2 s) \\ &= -k\lambda(x, y) + k(y, r) + \lambda^2 (y, s) \\ &= -k\lambda\,\rho + k\,\xi + \lambda^2\,\eta. \end{aligned}$$

The local coordinates of the point P, the image of the line p, are therefore

$$x_1 = -k\lambda, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = k, \quad x_5 = \lambda^2, \quad x_6 = 0.$$

Eliminating λ we find that

(8) $$x_2 = 0, \quad x_3 = 0, \quad x_6 = 0, \quad x_1^2 = k\,x_4 x_5.$$

The three hyperplanes $x_2 = 0$, $x_3 = 0$, $x_6 = 0$ intersect in a space of two dimensions. Therefore equations (8) represent an ordinary conic lying in three hyperplanes. We may state the theorem:

<u>The images in S_5 of a ruled cone in S_3 is a point conic.</u>

4. The Images in S_5 of the Lines of a Ruled Surface in S_3.

The points

$$A = a_1 x + a_2 r + a_3 s + a_4 y,$$

$$B = b_1 x + b_2 r + b_3 s + b_4 y,$$

Wherein a_i, b_i are functions of t each describe a curve in S_3. The lines p joining points of A to points of B generate a ruled surface. We shall find the image of this ruled surface in S_5. The Plücker coordinates of p are defined by

$$P = (a_1 x + a_2 r + a_3 s + a_4 y,\ b_1 x + b_2 r + b_3 s + b_4 y)$$

$$= (a_1 b_4 - a_4 b_1)(x, y) + (a_1 b_2 - a_2 b_1)(x, r) + (a_1 b_3 - a_3 b_1)(x, s)$$

$$+ (a_4 b_2 - a_2 b_4)(y, r) + (a_4 b_3 - a_3 b_4)(y, s) + (a_2 b_3 - a_3 b_2)(r, s)$$

$$= (a_1 b_4 - a_4 b_1)\,\rho + (a_1 b_2 - a_2 b_1)\,\xi + (a_1 b_3 - a_3 b_1)\,\eta$$

$$+ (a_4 b_2 - a_2 b_4)\,\bar{\xi} + (a_4 b_3 - a_3 b_4)\,\bar{\eta} + (a_2 b_3 - a_3 b_2)\,\ell .$$

Hence the local coordinates of the image of p in S_5 are

$$(9) \qquad \begin{array}{ll} x_1 = a_1 b_4 - a_4 b_1, & x_2 = a_1 b_2 - a_2 b_1, \\ x_3 = a_1 b_3 - a_3 b_1, & x_4 = a_4 b_2 - a_2 b_4, \\ x_5 = a_4 b_3 - a_3 b_4, & x_6 = a_2 b_3 - a_3 b_2. \end{array}$$

From (9) we find

$$x_1 x_6 - x_2 x_5 + x_3 x_4 = 0.$$

We may, therefore, state the theorem:

<u>The points in S_5 corresponding to the generators of a ruled surface in S_3 lie on a curve on the hyperquadric Q^2_4.</u>

As a special case consider the images of the generators of a regulus. The point rows defined by the points $x + \lambda y$ on ρ and $r + \lambda s$ on ℓ are projectively related. The lines joining these points therefore generate a regulus. We shall find the points in S_5 corresponding to these lines in S_3. The line p in S_3 is defined by the expression

$$p = (x + \lambda y,\ r + \lambda s)$$
$$= (x, r) + \lambda (x, s) + \lambda (y, r) + \lambda^2 (y, s)$$
$$= \xi + \lambda \eta + \lambda \bar{\xi} + \lambda^2 \bar{\eta}.$$

The local coordinates of the image of p are therefore

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = \lambda, \quad x_4 = \lambda, \quad x_5 = \lambda^2, \quad x_6 = 0.$$

Eliminating λ we find that

$$x_1 = 0, \quad x_3 - x_4 = 0, \quad x_6 = 0, \quad x_3 x_4 = x_2 x_5.$$

Therefore, we have the theorem:

<u>The points in S_5 corresponding to the generators of a regulus lie on an ordinary conic in three hyperplanes</u>.

5. Images in S_5 of the Tangents to a Conic in S_3.

The conic, whose equations are

(10) $$x_1^2 = K x_2 x_3, \qquad x_4 = 0,$$

lies in the plane of x, r, s. The parametric equations of this conic are

$$x_1 = K \lambda, \quad x_2 = K, \quad x_3 = \lambda^2, \quad x_4 = 0.$$

A point on the tangent line to the conic has the coordinates

$$x_1' = \kappa, \quad x_2' = 0, \quad x_3' = 2\lambda, \quad x_4' = 0.$$

The expression defining the line p, tangent to the conic whose equations are given by (10), is, therefore,

$$p = (\kappa\lambda x + \kappa a + \lambda^2 a, \ \kappa x + 2\lambda a)$$
$$= -\kappa^2 \xi + \kappa\lambda^2 \eta + 2\kappa\lambda h.$$

The image of the line p has the local coordinates

$$x_1 = 0, \quad x_2 = -\kappa^2, \quad x_3 = \kappa\lambda^2, \quad x_4 = 0, \quad x_5 = 0, \quad x_6 = -2\kappa\lambda.$$

Eliminating λ we find that

(11) $\quad x_1 = 0, \quad x_4 = 0, \quad x_5 = 0, \quad 4 x_2 x_3 = -\kappa x_6^2.$

Equations (11) represent a conic lying in one of the plane faces of the pyramid of reference. It passes through two of the vertices, the intersection of $x_2 = 0$, $x_6 = 0$ and the intersection of $x_3 = 0$, $x_6 = 0$. It is also tangent to the edge $x_6 = 0$. We may state the theorem:

<u>The locus of the points in S_5 corresponding to the tangents to a conic in S_3 is an ordinary conic lying in three hyperplanes.</u>

6. Condition Necessary for a Point P in S_5 to have an Image in S_3.

Consider any point P in S_5 defined by the expression

(12) $$P = t_1 \rho + t_2 \xi + t_3 \eta + t_4 \bar{\xi} + t_5 \bar{\eta} + t_6 h.$$

Consider also any line p in S_3 defined by the expression

(13) $$p = (x + \lambda y + \mu h, \ x + \lambda_1 y + \mu_1 a)$$
$$= (\lambda_1 - \lambda)\rho + \mu \xi + \mu_1 \eta - \lambda_1 \mu \bar{\xi} + \lambda \mu_1 \bar{\eta} + \mu \mu_1 h.$$

The image in S_5 of the line p in S_3 defined by (13) is therefore

(14) $$P = (\lambda_1 - \lambda)\rho + \mu \xi + \mu_1 \eta - \lambda_1 \mu \bar{\xi} + \lambda \mu_1 \bar{\eta} + \mu \mu_1 h.$$

Therefore the condition that p in S_3 be the image of P in S_5

is

(15) $\quad t_1 = \lambda_1 - \lambda, \ t_2 = \mu, \ t_3 = \mu_1, \ t_4 = -\lambda_1\mu, \ t_5 = \lambda\mu_1, \ t_6 = \mu\mu_1.$

Eliminating $\lambda, \lambda_1, \mu, \mu_1$ from (15) we find

$$t_1 t_6 - t_2 t_5 + t_3 t_4 = 0.$$

We may, therefore, state the theorem:

A point in S_5 will have an image in S_3 if and only if it lies on the hyperquadric Q_4^2 in S_5.

The quadric Q_4^2 in S_5 has two distinct types of plane generators. We shall say that a plane generator is of the first kind if it is determined by the images of three concurrent but not coplanar lines in S_3. We shall say that a plane generator is of the second kind if it is determined by the images of three coplanar but not concurrent lines in S_3.

Suppose that the point P is in a plane generator of the first kind, say in the plane determined by ρ, ξ, η. The point P is defined by an expression of the form

(16) $$P = t_1\rho + t_2\xi + t_3\eta.$$

It is obvious that the image of P in S_3 is the line p defined by

$$p = (x, t_1 y + t_2 r + t_3 s).$$

If the coefficients t_i are functions of t, the locus of P in S_5 defined by (16) is a curve lying on a plane generator of the hyperquadric. The locus of the image of P is a cone since it consists of the lines joining x to the points of the curve traced by the point z where z is defined by the expression

$$z = t_1 y + t_2 r + t_3 s.$$

Knowing the type of curve generated by P we can describe the cone which is the image in S_3 of the curve in S_5.

Suppose that the point P is in a plane generator of the second kind, say in the plane determined by the points ξ, η, h.

The point P is defined by an expression of the form

(17) $$P = t_2 \xi + t_3 \eta + t_6 h.$$

Consider also the line p in S_3 defined by the expression

$$p = (x + \lambda r,\ x + \mu s)$$

$$= -\lambda \xi + \mu \eta + \lambda \mu h.$$

If p is the image in S_3 of the point P defined by (17), we must have

$$-\lambda = \alpha t_2, \qquad \mu = \alpha t_3, \qquad \lambda \mu = \alpha t_6$$

Wherein α is a factor of proportionality. We find readily that

$$\alpha = -\frac{t_6}{t_2 t_3}$$

Therefore

$$\lambda = \frac{t_6}{t_3}, \qquad \mu = -\frac{t_6}{t_2}.$$

The line p joining the points defined by

$$t_3 x + t_6 r, \qquad t_2 x - t_6 s$$

is the image of the point P defined by (17). We may state our results in the following theorems:

<u>The image in S_3 of a point curve in a plane generator of the first kind of the hyperquadric is a cone with vertex at the point of intersection of the images of the three points determining that plane generator.</u>

<u>The image in S_3 of a point curve in a plane generator of the second kind of the hyperquadric is a one parameter family of lines in the plane determined by the images of the three points determining that plane generator. We may, therefore, consider the image of the locus of P in such a plane generator as the envelope of this one parameter family of lines.</u>

7. Relation between the Local Coordinates of Points in S_3 and S_5.

Let the local coordinates of points of S_3 be, with respect to the local tetrahedron of reference, x, t, s, y. Thus we may find the local coordinates of a line in S_3, that is the Plücker coordinates referred to x, t, s, y. Using the pyramid $\rho, \xi, \eta, \bar{\xi}, \bar{\eta}, k$ as pyramid of reference in S_5, we may set up a local coordinate system for points in S_5. We shall find the relation between the local coordinates in both systems.

Consider the line in S_3 joining the points $\bar{x}$ and $\bar{y}$ defined by the expressions

$$\bar{x} = x_1 x + x_2 t + x_3 s + x_4 y,$$

$$\bar{y} = y_1 x + y_2 t + y_3 s + y_4 y.$$

The point in S_5 which is the image of this line may be expressed as follows

$$\begin{aligned} P &= (x_1 x + x_2 t + x_3 s + x_4 y,\ y_1 x + y_2 t + y_3 s + y_4 y) \\ &= (x_1 y_4 - x_4 y_1)\rho + (x_1 y_2 - x_2 y_1)\xi + (x_1 y_3 - x_3 y_1)\eta \\ &\quad + (x_4 y_2 - x_2 y_4)\bar{\xi} + (x_4 y_3 - x_3 y_4)\bar{\eta} + (x_2 y_3 - x_3 y_2) k. \end{aligned}$$

Therefore, we find that

$$t_1 = x_1 y_4 - x_4 y_1, \quad t_2 = x_1 y_2 - x_2 y_1, \quad t_3 = x_1 y_3 - x_3 y_1,$$

$$t_4 = x_4 y_2 - x_2 y_4, \quad t_5 = x_4 y_3 - x_3 y_4, \quad t_6 = x_2 y_3 - x_3 y_2.$$

PART II

Let

$$x_i = x_i(u,v), \qquad y_i = y_i(u,v), \qquad i=1, 2, 3, 4.$$

be the parametric equations of the surfaces S_x and S_y. Consider the parametric nets N_x and N_y on S_x and S_y as being in relation C. The functions y_i and x_i satisfy a system of differential equations of the form

$$
\begin{aligned}
&x_{uu} = \alpha x_u + \beta x_v + p x + L y, \\
&x_{uv} = a x_u + b x_v + c x + M y, \qquad\qquad m \neq n \neq 0, \\
(12) \qquad &x_{vv} = \gamma x_u + \delta x_v + q x + N y, \\
&y_u = m x_u + f x + A y, \\
&y_v = n x_v + g x + B y.
\end{aligned}
$$

The coefficients of (12) satisfy the following integrability conditions*

$$
\begin{aligned}
&a_u - \alpha_v + ab - \gamma\beta + c &&= -mM, \\
&b_v - \delta_u + ba - \beta\gamma + c &&= -nM, \\
&b_u - \beta_v + a\alpha + (b-\alpha)b - \delta\beta - p &&= nL, \\
&a_v - \gamma_u + b\gamma + (a-\delta)a - \gamma\alpha - q &&= mN, \\
&c_u - p_v + ap + (b-\alpha)c - q\beta &&= gL - fM, \\
(13) \qquad &c_v - q_u + bq + (a-\delta)c - p\gamma &&= fN - gM, \\
&M_u - L_v + aL + (b-\alpha)M - N\beta &&= BL - AM, \\
&M_v - N_u + bN + (a-\delta)M - \gamma L &&= AN - BM, \\
&m_v + (m-n)a - q &&= mB, \\
&n_u + (n-m)b - f &&= nA, \\
&g_u - f_v + (n-m)c &&= gA - fB, \\
&B_u - A_v + (n-m)M &&= 0.
\end{aligned}
$$

If the fourth and fifth equations of system (12) are differentiated with respect to u and v and the values of x_{uu}, x_{uv}, x_{vv} from the first three equations of (12) are substituted we obtain the following system

*Grove, Nets, p. 484.

$$
\begin{aligned}
&y_{uu} = \bar\alpha\, y_u + \bar\beta\, y_v + \bar p\, y + \bar L\, z, \\
&y_{uv} = \bar a\, y_u + \bar b\, y_v + \bar c\, y + \bar M\, z, \\
(20)\quad &y_{vv} = \bar\gamma\, y_u + \bar\delta\, y_v + \bar q\, y + \bar N\, z, \\
&z_u = \bar m\, y_u + \bar f\, y + \bar A\, z, \\
&z_v = \bar n\, y_v + \bar g\, y + \bar B\, z.
\end{aligned}
$$

wherein

$$\bar\alpha = \alpha + \frac{h}{m} + \frac{m_u}{m} + A, \quad \bar\beta = \frac{m\beta}{n}, \quad \bar p = mL + m\left(\frac{A}{m}\right)_u - A\alpha - A\frac{h}{m} - \frac{mB\beta}{n},$$

$$\bar L = -m\left[\alpha\frac{h}{m} + \beta\frac{g}{n} + \frac{h^2}{m^2} - p - \left(\frac{h}{m}\right)_u\right],$$

$$\bar a = a + \frac{m_v}{m}, \quad \bar b = b + \frac{n_u}{n}, \quad \bar c = mM + m\left(\frac{A}{m}\right)_v - Aa - B\frac{h}{m} - \frac{mb\beta}{n},$$

$$\bar M = -m\left[a\frac{h}{m} + b\frac{g}{n} + \frac{hg}{mn} - c - \left(\frac{h}{m}\right)_v\right],$$

$$\bar\gamma = \frac{n\gamma}{m}, \quad \bar\delta = \delta + \frac{g}{n} + \frac{n_v}{n} + B, \quad \bar q = nN + n\left(\frac{B}{n}\right)_v - B\delta - B\frac{g}{n} - \frac{nA\gamma}{m},$$

$$\bar N = -n\left[\delta\frac{g}{n} + \gamma\frac{h}{n} + \frac{g^2}{n^2} - q - \left(\frac{g}{n}\right)_v\right]$$

$$\bar m = \frac{1}{m}, \quad \bar f = -\frac{A}{m}, \quad \bar A = -\frac{h}{m}, \quad \bar n = \frac{1}{n}, \quad \bar g = -\frac{B}{n}, \quad \bar B = -\frac{g}{n}.$$

1. First derivatives.

The first derivatives of ρ, h etc. are expressible as linear combinations* of ρ, h etc. Since

$$\rho = (z, y),$$

we find that

$$
\begin{aligned}
\rho_u &= (z_u, y) + (z, y_u) \\
&= \left(r - \frac{h}{m}z, y\right) + (z, mr + Ay) \\
&= (r, y) + \left(A - \frac{h}{m}\right)(z, y) + m(z, r) \\
&= \left(A - \frac{h}{m}\right)\rho + m\xi - \bar\xi.
\end{aligned}
$$

* E.P. Lane, The Projective Differential Geometry of Systems of Linear Homogeneous Differential Equations of the first Order, Transactions of the American Mathematical Society, Vol. 30 (1928) p. 785.

In a similar manner we may find the first derivatives of ξ, η, etc. The explicit expressions for these are

$$
\begin{aligned}
\rho_u &= (A - \tfrac{l}{m})\rho + m\xi - \bar\xi, \\
\xi_u &= \alpha\xi + \beta\eta + L\rho, \\
\eta_u &= (b - \tfrac{l}{m})\eta + (a + \tfrac{g}{m})\xi + M\rho + h, \\
\bar\xi_u &= (\bar\alpha - \tfrac{\bar m_u}{\bar m})\bar\xi + \tfrac{\bar m}{n}\bar\beta\bar\eta - \bar m\bar L\rho, \\
\bar\eta_u &= (\bar b - \tfrac{\bar l}{\bar m} - \tfrac{\bar n_u}{\bar n})\bar\eta + \tfrac{\bar n}{m}(\bar a + \tfrac{\bar g}{\bar m})\bar\xi - \bar n\bar M\rho + \tfrac{1}{\bar m}h, \\
h_u &= (\alpha + \tfrac{l}{m} + b)h - \tfrac{\bar M}{m}\xi - M\bar\xi + \tfrac{\bar L}{m}\eta + L\bar\eta,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\rho_v &= (B - \tfrac{g}{n})\rho + n\eta - \bar\eta, \\
\eta_v &= \delta\eta + \gamma\xi + N\rho, \\
\xi_v &= (a - \tfrac{g}{n})\xi + (b + \tfrac{l}{m})\eta + M\rho - h, \\
\bar\eta_v &= (\bar\delta - \tfrac{\bar n_v}{\bar n})\bar\eta + \tfrac{\bar n}{m}\bar\gamma\bar\xi - \bar n\bar N\rho, \\
\bar\xi_v &= (\bar a - \tfrac{\bar g}{n} - \tfrac{\bar m_v}{\bar m})\bar\xi + \tfrac{\bar m}{n}(\bar b + \tfrac{\bar l}{m})\bar\eta - \bar m\bar M\rho - \tfrac{1}{n}h, \\
-h_v &= -(\delta + \tfrac{g}{n} + a)h - \tfrac{\bar M}{m}\eta - M\bar\eta + \tfrac{\bar N}{n}\xi + N\bar\xi.
\end{aligned}
$$

From (21) we find that

$$\rho_u - (A - \tfrac{l}{m})\rho = m\xi - \bar\xi.$$

Therefore the tangents to the curves v = const. on S_ρ intersect the line joining the points ξ and $\bar\xi$ in the covariant point

$$m\xi - \bar\xi.$$

Since the lines ξ and $\bar\xi$ intersect in S_3, the line joining the points ξ and $\bar\xi$ in S_5 lies on the hyperquadric Q_4^2. To the covariant point in S_5 must correspond a covariant line in S_3 with the coordinates

$$
\begin{aligned}
m\xi - \bar\xi &= m(x, r) - (y, r) \\
&= (mx - y, r)
\end{aligned}
$$

Thus the line joins the point r to the point $mx - y$, that is to one of the focal points* of ρ.

By symmetry we find that the tangent to the curve u = const. on S_ρ intersects the line $\eta\bar{\eta}$ in S_5 in the covariant point

$$m\,\eta - \bar{\eta}.$$

This point corresponds to the covariant line in S_3 joining the point s to the other focal point of ρ. We may, therefore, state the theorem:

<u>The tangent line to the parametric curve v = const. (u = const.) on S_ρ at ρ intersects the line $\xi\bar{\xi}$ ($\eta\bar{\eta}$) in a point. The image in S_3 of this point is the line joining x (s) to the focal point of the second (first) rank.</u>**

We also have from (21)

$$\xi_u - \alpha\,\xi = \beta\,\eta + L\,\rho.$$

Thus the tangent to the curve v = const. on S_ξ intersects the line joining η to ρ in the covariant point

$$\beta\,\eta + L\,\rho = (x,\ \beta\,s + L\,y).$$

Therefore the image in S_3 of this point is the line joining x to $\beta s + L y$. This latter point must be a covariant point also. We shall investigate the geometrical significance of this point.

We have from (18)

$$x_{uu} - \alpha\,x_u - p\,x = \beta\,x_v + L\,y$$

$$= \beta\left(s - \frac{g}{u}x\right) + L\,y.$$

Therefore

$$x_{uu} - \alpha x_u - \left(p - \beta\frac{g}{u}\right)x = \beta\,s + L\,y.$$

Thus the point $\beta s + L y$ is the intersection of the osculating

*Grove, <u>Nets</u>, p. 486.

**Ibid, p. 490.

plane to the curve v = const. on $S_{\mathfrak{x}}$ with the line $\mathfrak{y}\mathfrak{s}$. We may, therefore, state the theorem:

<u>The tangent line to the parametric curve v = const. on S_ξ at ξ intersects the line $\eta\rho$ in a point. The image in S_3 of this point is the line joining $\mathfrak{x}$ to the point of intersection of the osculating plane to the parametric curve v = const. on $S_{\mathfrak{x}}$ at $\mathfrak{x}$ with the line $\mathfrak{y}\mathfrak{s}$.</u>

Similar theorems may be stated for the tangent lines to the parametric curves on S_η, $S_{\bar\xi}$, $S_{\bar\eta}$.

From (31) we have

$$\xi_v - (a - \tfrac{g}{m})\xi = (b + \tfrac{l}{m})\eta + M\rho - h,$$

$$\xi_u - \alpha\xi \quad = \beta \quad \eta + L\rho.$$

If $L \neq 0$, that is, if the parametric curve v = const. on $S_{\mathfrak{x}}$ is not asymptotic, we may eliminate ρ between the two equations above. The resulting expression is

$$\xi_v - \frac{M\xi_u}{L} - (a - \tfrac{g}{m} - \tfrac{\alpha M}{L})\xi = (b + \tfrac{l}{m} - \tfrac{M\beta}{L})\eta - h.$$

Thus the covariant point

$$(b + \tfrac{l}{m} - \tfrac{M\beta}{L})\eta - h$$

in S_5 lies in the tangent plane to S_ξ at ξ and on the line ηh. Its image in S_3 is the line with coordinates

$$(b + \tfrac{l}{m} - \tfrac{M\beta}{L})(\mathfrak{x}, \mathfrak{s}) - (\mathfrak{r}, \mathfrak{s}) = [\mathfrak{r} - (b + \tfrac{l}{m} - \tfrac{M\beta}{L})\mathfrak{x}, \mathfrak{s}]$$

$$= [\mathfrak{x}_u - (b - \tfrac{M\beta}{L})\mathfrak{x}, \mathfrak{s}]$$

The image is, therefore, the line joining $\mathfrak{s}$ to the ray* point on the line ξ of the point $\mathfrak{x}$ with respect to the net $N_{\mathfrak{x}}$. We may state the theorem:

* Grove, <u>Nets</u>, p. 489.

<u>The image in S_5 of the line in S_3 joining s to the ray point of the point x with respect to the net N_x is the point of intersection of the tangent plane to S_ξ at ξ with the edge ρh of the pyramid of reference.</u>

We may find similar results by using the tangent planes to S_η, $S_{\bar\xi}$, $S_{\bar\eta}$.

We also have from (21)

$$\eta_u - (b - \tfrac{k}{m})\eta - M\rho = (a + \tfrac{g}{m})\xi + h.$$

Thus the plane determined by the points η_u, η, ρ in S_5 intersects the line ξh in the point

$$(a + \tfrac{g}{m})\xi + h.$$

The image of this point in S_3 is the line joining s to $x_v - ax$. The latter point is the intersection of the line in Green's relation* R to ρ with η. Likewise we may find the point in S_5 which is the image of the line in S_3 joining s to the intersection of the line in Green's relation R to ρ with ξ. Similar results may be obtained using the expressions for $\bar\eta_u$ and $\bar\xi_v$. We may state the theorem:

<u>The plane in S_5 determined by η_u, η, ρ intersects the line ξh in a point. The image of this point in S_3 is the line joining s to the intersection of the line in Green's relation R to ρ with η.</u>

Using the expressions for h_u and h_v from (21) we may eliminate in turn ξ, η, $\bar\xi$, $\bar\eta$ provided $\bar N \neq 0$, $\bar M \neq 0$, $N \neq 0$, $M \neq 0$. We may thus obtain eight expressions of the form

*Grove, <u>Nets</u>, p. 401.

(22)
$$[h_u, h_v, h, \xi] = \frac{\bar{L}N - M\bar{M}}{mN}\eta + \frac{LN - M^2}{N}\bar{\eta}, \quad N \neq 0,$$
$$= \left[\frac{\bar{L}N - M\bar{M}}{mN}x + \frac{LN - M^2}{N}y, s\right]$$

Thus in S_5 the S_3 determined by the tangent plane to h and the point ξ intersects the line joining η and $\bar{\eta}$ in the point
$$\frac{\bar{L}N - M\bar{M}}{mN}\eta + \frac{LN - M^2}{N}\bar{\eta}.$$
This point in S_5 corresponds to the line in S_3 joining s to the point
$$\frac{\bar{L}N - M\bar{M}}{mN}x + \frac{LN - M^2}{N}y.$$
The latter is a covariant point on the line ρ. We shall leave the geometrical interpretation of these points for some future date. If $M = \bar{M} = 0$, that is, if the given nets are F transforms,* expression (22) reduces to

(22a) $$[h_u, h_v, h, \xi] = [\frac{\bar{L}}{mN}x + Ly, s].$$

If the given nets are F transforms we may make a transformation of the form $x = \lambda\bar{x}$ such that it will make $f = g = A = B = 0$, $p = \frac{\bar{L}}{m}$. The first of equations (18) becomes
$$x_{uu} - \alpha x_u - \beta x_v = \frac{\bar{L}}{m}x + Ly.$$
Thus if the given nets are F transforms the intersection of the three space determined by h_u, h_v, h, ξ with the line $\eta\bar{\eta}$ is the point of intersection of the plane determined by x_{uu}, x_u, x_v with the line ρ.

2. Second Derivatives

* L.P. Eisenhart, Transformation of Surfaces, Princeton University Press, (1923) p. 34.

We may also find second derivatives of ρ, ξ, η etc. with respect to u and v. In particular we find

$$
\begin{aligned}
\rho_{uu} &= \left(A-\tfrac{f}{m}\right)\rho_u + \left[A_u + mL - \tfrac{\alpha f}{m} - \tfrac{\beta g}{n} + p - \tfrac{f^2}{m^2}\right]\rho + (m\alpha + m_u)\xi \\
&\quad + m\beta\eta - \left(A + \tfrac{f}{m} + \alpha\right)\bar{\xi} - \beta\bar{\eta}, \\
\rho_{vv} &= \left(B-\tfrac{g}{n}\right)\rho_v + \left[B_v + nN - \tfrac{\delta g}{n} - \tfrac{\gamma f}{m} + q - \tfrac{g^2}{n^2}\right]\rho + (n\delta + n_v)\eta \\
&\quad + n\gamma\xi - \left(B + \tfrac{g}{n} + \delta\right)\bar{\eta} - \gamma\bar{\xi}, \\
\xi_{uu} &= \alpha\,\xi_u + \left(\alpha_u + \beta a + \beta\tfrac{g}{n} + mL\right)\xi + \left(\beta b - \beta\tfrac{f}{m} + \beta_u\right)\eta + \beta h \\
&\quad - L\bar{\xi} + \left(\beta M + AL - \tfrac{Lf}{m} + L_u\right)\rho, \qquad\qquad (23) \\
\rho_{uv} &= \left(B-\tfrac{g}{n}\right)\rho_u + \left[B_u + nM - \tfrac{bg}{n} - \tfrac{af}{m} + c - \tfrac{fg}{mn}\right]\rho \\
&\quad + \left(nb - \tfrac{nf}{m} + n_u\right)\eta + (na + g)\xi, \\
\xi_{uv} &= \alpha\,\xi_v + (\alpha_v + \beta\gamma)\xi + (\beta\delta + B_v + Ln)\eta \\
&\quad + \left(\beta n + LB - \tfrac{Lg}{n} + L_v\right)\rho - L\bar{\eta}.
\end{aligned}
$$

Using the expressions defining ρ_u and ρ_v from (21) we may eliminate $\bar{\eta}$ and $\bar{\xi}$ from the first two of equations (23) obtaining the following expressions

$$
\begin{aligned}
\rho_{uu} + (\ )\rho_u + (\ )\rho_v + (\ )\rho &= (m_u - mA - f)\xi + (m-n)\beta\eta, \\
\rho_{vv} + (\ )\rho_u + (\ )\rho_v + (\ )\rho &= (n_v - nB - g)\eta + (n-m)\gamma\xi,
\end{aligned}
$$

wherein the coefficients of ρ_u, ρ_v, ρ are immaterial for our purposes. We may readily see that ρ_{uu} or ρ_{vv} will be a linear combination of ρ_u, ρ_v, ρ that is the parametric nets v= const. or u = const. on the surface S_ρ at ρ will be asymptotic if

Case I $\qquad m_u - mA - f = 0, \qquad n - m = 0,$

Case II $\qquad n_v - nB - g = 0, \qquad n - m = 0.$

Case III $\qquad m_u - mA - f = 0, \qquad \beta = 0,$

Case IV $\qquad n_v - nB - g = 0, \qquad \gamma = 0.$

If n-m = 0 the nets N_x and N_y are radial transforms*. We

* Grove, Nets, p. 486.

excluded this case at the beginning of the paper. We shall consider cases III and IV. The asymptotic nets on S_τ and $S_{\tau'}$, the focal surfaces of the congruence of lines ρ, are defined by the expressions*

$$(g - n_v + nB)\,du^2 - (n - m)\beta\,dv^2 = 0,$$
$$(f - m_u + mA)\,du^2 - (m - n)\gamma\,dv^2 = 0.$$

For case III these equations reduce to

$$du^2 = 0, \qquad dv^2 = 0.$$

Thus the asymptotic curves on the focal surfaces S_τ and $S_{\tau'}$ form only one one parameter family, ie. these focal surfaces are developable. Case IV gives the same result. We may, therefore, state the theorem:

<u>The parametric nets on S_ρ will be asymptotic if and only if the focal surfaces S_τ and $S_{\tau'}$ are developable.</u>

We may eliminate ℓ from the third of equations (23) by using the value of ξ_v from (21). We find

$$\xi_{uu} + (\ )\xi_u + (\ )\xi_v + (\ )\xi = (2\beta b + \beta_u)\eta + (2\beta M + AL - L\tfrac{\ell}{m} + L_u)\rho - L\bar{\xi}.$$

ξ_{uu} will be a linear combination of ξ_u, ξ_v, ξ if $\beta = L = 0$. However, we must exclude this case since it reduces the second of equations (21) to $\xi_u = d\xi$. This equation indicates that the surface S_ξ is only a curve. ξ_{uu} would also be a linear combination of the lower derivatives of ξ if $L = M = 0,\ 2\beta b = -\beta_u$. This case must also be excluded since it is the condition that the surface S_ξ be a developable. We obtain similar results using the expression defining ξ_{vv}. We may, therefore, state

*Grove, <u>Nets</u>, p. 431.

the theorem:

<u>The parametric nets on S_ξ cannot be asymptotic</u>.

We obtain the same result for the surfaces S_η, $S_{\bar\xi}$, $S_{\bar\eta}$.

Using the expression defining ξ_u from (21) we may eliminate ρ from the third of equations (23) if $L \neq 0$. We obtain the following result

$$\xi_{uu} + (\ )\xi_u + (\ )\xi + (\ )\eta = \beta z - L\bar\xi.$$

Thus a covariant point on the line joining z to $\bar\xi$ in S_5 lies in the S_3 determined by the osculating plane to the curve v = const. and the point η. To this point in S_5 corresponds the line in S_3 joining the point z to the point $\beta z + L y$. The geometrical interpretation of the latter point was given on page 18 of this paper. Similar points may be found by using expressions involving η_{vv}, $\bar\xi_{uu}$, $\bar\eta_{vv}$.

The functions $\bar\eta$ and $\bar\xi$ may be eliminated from the fourth of equations (23). The resulting expression for ρ_{uv} is of the form

$$\rho_{uv} + (\ )\rho_u + (\ )\rho_v + (\ )\rho = (n-m)z - (n-m)(b + t_n)\eta + (n-m)(a + g_u)\xi.$$

It is readily seen that the parametric nets on S_ρ can be conjugate, that is a linear relation can exist between ρ_{uv}, ρ_u, ρ_v, ρ if and only if n-m = 0. This is the condition that N_z and N_y be radial transforms which was mentioned previously. We may, therefore, state the theorem:

<u>The parametric nets on S_ρ cannot be conjugate</u>.

Eliminating η from the last of equations (23) ($\beta \neq 0$) we find

$$\xi_{uv} + (\ )\xi_u + (\ )\xi_v + (\ )\xi = \left[\beta N + L B - \frac{Lg}{u} + L_v - \frac{L}{\beta}(\beta\gamma + \beta_v + uL)\right]\rho - L\bar\eta.$$

Thus a linear relation exists between ξ_{uv}, x_u, x_v and if and only if L = N = 0, that is the parametric net on S_x is asymptotic. We may, therefore, state the theorem:

<u>The parametric nets on the surface S_ξ will be conjugate if and only if the parametric nets on S_x are asymptotic.</u>

Similar theorems may be stated concerning the parametric nets on the surfaces S_η, $S_{\bar{\xi}}$, $S_{\bar{\eta}}$.

3. Certain Congruences in S_5 and their Relation to the Asymptotics on S_x in S_3.

Consider the line joining the points ξ and η in S_5. Any point on this line will have coordinates of the form

$$\tau = \xi + \lambda\eta. \tag{24}$$

The coordinates of any point on the tangent line to the curve traced by the point defined by the expression (24) will be of the form

$$\begin{aligned}\frac{d\tau}{du} &= \xi_u + \xi_v\frac{dv}{du} + \lambda\eta_u + \lambda\eta_v\frac{dv}{du} + \lambda_u\eta + \lambda_v\eta\frac{dv}{du} \\ &= \left[\alpha + \left(a - \frac{g}{m}\right)\frac{dv}{du}\right]\xi + \left[\beta + \lambda_u + \left(b + \frac{f}{m} + \lambda_v\right)\frac{dv}{du}\right]\eta \\ &\quad + \left[L + \lambda M + (M + \lambda N)\frac{dv}{du}\right]\rho + \left(\lambda - \frac{dv}{du}\right)h.\end{aligned} \tag{25}$$

If the tangent line coincides with the line joining ξ to η, that is if the surface generated by τ is a developable, the coefficients of ρ and h in (25) must be zero. In that case we find

$$\begin{aligned} L + \lambda M + (M + \lambda N)\frac{dv}{du} &= 0, \\ \lambda - \frac{dv}{du} &= 0. \end{aligned} \tag{26}$$

Eliminating λ from equations (26) we find

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0. \tag{27}$$

Likewise eliminating $\frac{dv}{du}$ from (26) we find

$$L + 2M\lambda + N\lambda^2 = 0. \tag{23}$$

Thus the line joining the points ξ and η in S_5 generates a congruence, that is the lines may be grouped into two one parameter families of developable surfaces. From (22) we see that the developables of this congruence correspond to the asymptotics* on S_ξ in S_3. We may, therefore, state the theorem: <u>The line joining ξ and η in S_5 generates a congruence. The image in S_5 of the asymptotics on S in S_3 are the curves on S and S which correspond to the developables of the congruence of lines $\xi\eta$.</u>

A similar theorem may be stated for $\xi\bar{\eta}$ and S_η.

*Grove, <u>Nets</u>, p. 435.

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