ON THE SYNTHESIS OF N-TERMINAL R-NETWORK AND POLYGON TO STAR TRANSFORMATION

Thesis for the Degree of M. S.
MICHIGAN STATE UNIVERSITY

Chung-Lei Wang

1961

ON THE SYNTHESIS OF N-TERMINAL R-NETWORK AND POLYGON TO STAR TRANSFORMATION

BY

Chung-Lei Wang

AN ABSTRACT

Submitted to the College of Engineering
Michigan State University of Agriculture and
Applied Science in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

Department of Electrical Engineering

1961

Approved: ______________________

ABSTRACT

In this thesis synthesis of n-terminal R-networks is considered. It is assumed that the terminal representation of the n-terminal R-network is given by both the terminal graph and the terminal equations. The object of this thesis is to calculate the element values of the R-network directly from the given specifications. The method represented in this thesis does not require the tree transformation which, generally, is necessary to use in other existing methods.

As a second topic, in this thesis transformation of a given polygon R-network into a star R-network is discussed and a set of necessary and sufficient conditions for the existance of such transformation is established.

ON THE SYNTHESIS OF N-TERMINAL R-NETWORK AND POLYGON TO STAR TRANSFORMATION

BY

Chung-Lei Wang

A THESIS

Submitted to the College of Engineering
Michigan State University of Agriculture and
Applied Science in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

Department of Electrical Engineering

1961

ACKNOWLEDGMENT

The author is indebted to his major professor, Dr. H. E. Koenig, and to Dr. Yilmaz Tokad, his thesis advisor, for their helpful advice and guidance throughout the preparation of this thesis. The research leading to this thesis was supported by the National Science Foundation under project G-9735.

CONTENTS

1. INTRODUCTION

Synthesis of n-terminal R-networks are discussed in the literature by various authors[1-4]. If a complete representation of n-terminal R-network is given in terms of a "terminal graph" and the corresponding set of "terminal equations[5]", the synthesis problem can easily be solve by transforming the given terminal graph into a Lagrangian tree. The element values are then detected directly from the corresponding terminal equations. (See e.g., reference 3).

If the terminal graph is not given it must be determined as a first step in the synthesis procedure. Cederbaum[1] presents a purely algebraic method for this determination while Guillemin[2] and Boiorce and Çivalleri[4] have developed "searching" procedures. Brown and Tokad[3] also consider this problem in certain special cases.

If the Coefficient Matrix in the terminal equations contain some off diagonal entries which are equal to zero, the above methods do not give a unique solution. To apply these procedures it is necessary to replace the zero entries by +1 or -1.

In the first part of this thesis the synthesis of n-terminal R-networks is considered, assuming the terminal graph is known. A direct method for calculating the element values from the given terminal equations is presented. The given terminal equations are put in a new form

$$\mathcal{A}\mathcal{G} = \mathcal{Y}$$

It is shown that $\mathcal{A}$ can be written easily from the given terminal graph. Since $\mathcal{A}^{-1}$ always exists, the column matrix, $\mathcal{G}$, which contains the element values of the R-network, can be found by direct solution of a set of simultaneous linear algebraic equations.

This new rearrangement of terminal equations offers the following advantages over other procedures:

(a) Eliminates the tree transformation process;

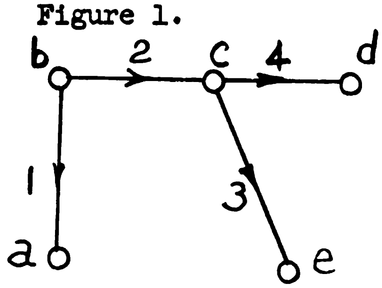
(b) In a large R-networks where it is desire to use computering facilities the above form is ideally suited.

In the second part of the thesis the existance of the polygon to star transformation is considered along with application to synthesis. Necessary and sufficient conditions for the existance of such a transformation are stated in terms of two types of terminal representation.

2. A NEW METHOD FOR THE SYNTHESIS OF N-TERMINAL R-NETWORKS

2.1 Formulation

Consider the terminal representations of an R-network given in Figure 1.



Terminal Graph

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ i_3(t) \\ i_4(t) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ \cdot & G_{22} & G_{23} & G_{24} \\ \cdot & \cdot & G_{33} & G_{34} \\ \cdot & \cdot & \cdot & G_{44} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix}$$

Terminal Equations

Figure 1

The coefficient matrix $\mathcal{G}$ is the product of three matrices of the form

$$\mathcal{G} = \mathcal{S}\mathcal{G}_e\mathcal{S}^T \tag{1}$$

where $\mathcal{S}$ is the submatrix of the corresponding cut-set matrix and $\mathcal{G}_e$ is a diagonal matrix containing element conductance values.

If the order of $\mathcal{G}$ is n x n [(n+1) - terminal R-network], then R-network containes a maximum of $m = \frac{n(n+1)}{2}$ elements, i.e., $\mathcal{S}$ is of order n x m and $\mathcal{G}_e$ has m rows and columns.

For a complete representation the terminal graph (TG) and terminal equations (TE) are both given and the $\mathcal{S}$ matrix can be written easily from the TG. For simplicity let us consider the example in Figure 1. Since the R-network has only the vertices which are the terminal vertices it is complete network as shown in Figure 2. If the elements of the corresponding R-network are labeled as indicated in Figure 2, the relevant submatrix $\mathcal{S}$ of the cut-set matrix reads

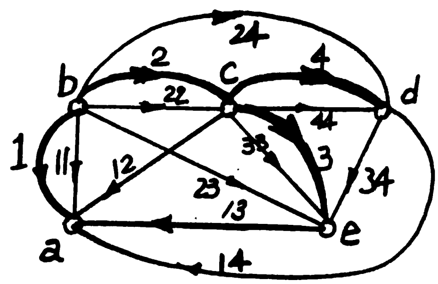


Figure 2

$$\mathcal{S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 11 & 22 & 33 & 44 & 12 & 13 & 14 & 23 & 24 & 34 \end{array} \\ \left[\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & -1 \end{array}\right] \end{array} \quad (2)$$

The number of equations relating the entries of $\mathcal{G}$ and $\mathcal{G}_e$ that can be obtained is equal to m. If the entries of $\mathcal{G}_e$ are considered as unknown and the entries of $\mathcal{G}$ are known then there are exactly m unkown in m equations. Let these equations be arranged in the form

$$\mathcal{A} \begin{bmatrix} g_{11} \\ g_{22} \\ \vdots \\ g_{m-1,m} \end{bmatrix} = \begin{bmatrix} G_{11} \\ G_{22} \\ \vdots \\ G_{n-1,n} \end{bmatrix} \quad (3)$$

where g_{ij}'s are the element conductance values and G_{ij}'s are the entries in $\mathcal{G}$ and $\mathcal{A}$ is a square matrix to be determined from $\mathcal{S}$. In order to obtain $\mathcal{A}$ from $\mathcal{S}$, let us write the $\mathcal{S}$ matrix in the form

$$\mathcal{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & ---- & s_{1m} \\ \vdots & & & & \\ s_{n1} & s_{n2} & s_{n3} & ---- & s_{nm} \end{bmatrix} \quad (4)$$

and

$$\mathcal{G}_e = \begin{bmatrix} g_1 & & & & \\ & g_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & g_m \end{bmatrix}$$

where the subscripts are consistent with the previous conventions, i.e., $g_1 = g_{11}$, $g_2 = g_{22}$, ..., $g_m = g_{m-1,m}$. Equation (1) can now be written as

$$\mathcal{G} = \begin{bmatrix} s_{11} & s_{12} & --- & s_{1m} \\ \vdots & & & \\ s_{n1} & s_{n2} & --- & s_{nm} \end{bmatrix} \begin{bmatrix} g_1 & & & \\ & g_2 & & \\ & & \cdot & \\ & & & \cdot g_m \end{bmatrix} \begin{bmatrix} s_{11} & --- & s_{n1} \\ s_{12} & & s_{n2} \\ \vdots & & \vdots \\ s_{1m} & & s_{nm} \end{bmatrix}$$

$$= \begin{bmatrix} s_{11}g_1 & s_{12}g_2 & --- & s_{1m}g_m \\ & & & \\ s_{n1}g_1 & s_{n2}g_2 & --- & s_{nm}g_m \end{bmatrix} \begin{bmatrix} s_{11} & --- & s_{n1} \\ s_{12} & & s_{n2} \\ \vdots & & \vdots \\ s_{1m} & & s_{nm} \end{bmatrix} \quad (5)$$

We can consider the rows of matrix $\mathcal{S}$ as vectors (row matrix) i.e., $\bar{\alpha}_i = [s_{i1}, s_{i2}, \ldots, s_{im}]$. Therefore $\mathcal{S}$ matrix has the form

$$\mathcal{S} = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \vdots \\ \bar{\alpha}_m \end{bmatrix} \quad (6)$$

Let us define a new set of vectors (row matrices) by use of $\bar{\alpha}_i$'s as follows,

$$\bar{k}_{ij} = \bar{\alpha}_i \times \bar{\alpha}_j = \bar{\alpha}_j \times \bar{\alpha}_i = \bar{k}_{ji}$$

where $\bar{k}_{ij}$ is given by

$$\bar{k}_{ij} = [s_{i1}s_{j1}, s_{i2}s_{j2}, \; . \; . \; . \; . \; , \; s_{im}s_{jm}] \tag{7}$$

With this new notation equation (5) can be written as

$$\mathcal{Y} = \begin{bmatrix} \bar{k}_{11}g & \bar{k}_{12}g & \cdots & \bar{k}_{1n}g \\ \bar{k}_{21}g & \bar{k}_{22}g & \cdots & \bar{k}_{2n}g \\ \vdots & \vdots & & \vdots \\ \bar{k}_{n1} & \bar{k}_{n2}g & \cdots & \bar{k}_{nn}g \end{bmatrix} \tag{8}$$

$$\text{with } g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix} = \bar{g}^T \tag{9}$$

Therefore, $\bar{k}_{ij}g$ is the scaler (or dot) product of two vectors, i.e.

$$\bar{k}_{ij}g = \sum_{\lambda=1}^{m} s_{i\lambda} \; s_{j\lambda} \; g_{\lambda} \tag{10}$$

Equation (8) now can be rewritten by considering only the diagonal and, say, upper off-diagonal entries. The lower off-diagonal entries will give the same equations. Hence, one arrangement of the entries of $\mathcal{Y}$ yields.

$$\begin{bmatrix} G_{11} \\ G_{22} \\ \vdots \\ G_{nn} \\ \vdots \\ G_{n-1,n} \end{bmatrix} = \begin{bmatrix} \bar{k}_{11}g \\ \bar{k}_{22}g \\ \vdots \\ \bar{k}_{nn} \\ \vdots \\ \bar{k}_{n-1,}g \end{bmatrix} = \begin{bmatrix} \bar{k}_{11} \\ \bar{k}_{22} \\ \vdots \\ \bar{k}_{nn} \\ \vdots \\ \bar{k}_{n-1,n} \end{bmatrix} g \tag{11}$$

or form (11)

$$\mathcal{A}\begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_m \end{bmatrix} = \begin{bmatrix} G_{11} \\ G_{22} \\ \cdot \\ \cdot \\ \cdot \\ G_{n-1,n} \end{bmatrix} \tag{12}$$

where

$$\mathcal{A} = \begin{bmatrix} \bar{k}_{11} \\ \bar{k}_{12} \\ \cdot \\ \cdot \\ \cdot \\ \bar{k}_{n-1,n} \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_1 & x & \bar{\alpha}_1 \\ \bar{\alpha}_2 & x & \bar{\alpha}_2 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \bar{\alpha}_{n-1} & x & \bar{\alpha}_n \end{bmatrix} \tag{13}$$

This result represents an algorithm for writing $\mathcal{A}$ from $\mathcal{S}$.

Example: As an example consider Figure 2, the $\mathcal{S}$ matrix of which is given by equation (2).

For this example we have

$$\bar{\alpha}_1 = (1, 0, 0, 0, 1, 1, 1, 0, 0, C)$$
$$\bar{\alpha}_2 = (0, 1, 0, 0,-1,-1,-1, 1, 1, 0)$$
$$\bar{\alpha}_3 = (0, 0, 1, 0, 0,-1, 0, 1, 0, 1)$$
$$\bar{\alpha}_4 = (0, 0, 0, 1, 0, 0,-1, 0, 1,-1)$$

From equation (7) we have

$$\bar{k}_{11} = (1, 0, 0, 0, 1, 1, 1, 0, 0, 0)$$
$$\bar{k}_{22} = (0, 1, 0, 0, 1, 1, 1, 1, 1, 0)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\bar{k}_{34} = (0, 0, 0, 0, 0, 0, 0, 0, 0,-1)$$

Hence, the rows of $\mathcal{A}$ matrix are obtained (see equation 13).

Properties of the $\mathcal{A}$ matrix

Let the $\mathcal{A}$ matrix be partitioned in the form

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 \\ \hdashline \mathcal{A}_2 \end{bmatrix} = \begin{bmatrix} \bar{k}_{11} \\ \bar{k}_{22} \\ \vdots \\ \bar{k}_{nn} \\ \hdashline \bar{k}_{12} \\ \vdots \\ \bar{k}_{n-1,n} \end{bmatrix} \tag{14}$$

and consider the properties of

$$\mathcal{A}_1 = \begin{bmatrix} \bar{k}_{11} \\ \bar{k}_{22} \\ \vdots \\ \bar{k}_{nn} \end{bmatrix} \text{ and } \mathcal{A}_2 = \begin{bmatrix} \bar{k}_{12} \\ \bar{k}_{13} \\ \vdots \\ \bar{k}_{n-1,m} \end{bmatrix} \text{ separately.} \tag{15}$$

Every row in $\mathcal{A}_1$ is of the form

$$\bar{k}_{ii} = \bar{\alpha}_i \times \bar{\alpha}_i = (s_{i1}^2, s_{i2}^2, \dots, s_{im}^2) = (|s_{i1}|, |s_{i2}|, \dots |s_{im}|) \tag{16}$$

i.e., every entry of $\bar{k}_{ii}$ is equal to the corresponding squared entry of $\bar{\alpha}_i$. Therefore we have:

The entries of $\mathcal{A}_1$ are either 0 or +1.

To establish additional properties let $|\bar{\alpha}_i|$ be defined as the vector (row matrix),

$$|\bar{\alpha}_i| = [\,|s_{i1}|, |s_{i2}|, \dots, |s_{im}|\,] \tag{17}$$

and the matrix $|\mathcal{S}|$ as

$$|\mathcal{S}| = \begin{bmatrix} |\bar{\alpha}_1| \\ |\bar{\alpha}_2| \\ \vdots \\ |\bar{\alpha}_n| \end{bmatrix}$$

then $\mathcal{Q}_1$ can be written as the form

$$\mathcal{Q}_1 = \begin{bmatrix} \bar{k}_{11} \\ \bar{k}_{12} \\ \vdots \\ \bar{k}_{nn} \end{bmatrix} = \begin{bmatrix} |\bar{\alpha}_1| \\ |\bar{\alpha}_2| \\ \vdots \\ |\bar{\alpha}_n| \end{bmatrix}$$

Therefore we have the important result:

The $\mathcal{Q}_1$ matrix is obtained from $\mathcal{S}$ by simply changing the signs of all its negative entries.

Let us consider now the other submatrix of $\mathcal{Q}$ i.e.,

$$\mathcal{Q}_2 = \begin{bmatrix} \bar{k}_{12} \\ \bar{k}_{13} \\ \vdots \\ \bar{k}_{1n} \\ \bar{k}_{23} \\ \vdots \\ \bar{k}_{n-1,n} \end{bmatrix} \quad (20)$$

It can be seen from equations (2) and (4) that if $i \neq j$ and $1 \leq i, j \leq n$, then $s_{i\lambda} = 0$ for $\lambda = 1, 2, \ldots, i-1, i+1, \ldots n$.

This implies that in equation (7) the first n entries are equal to zero under the above conditions (i.e., $i \neq j$ and $1 \leq i, j \leq n$).

Therefore we have a second important result:

The $\mathcal{A}_2$ matrix has the following form:

$$\mathcal{A}_2 = \begin{bmatrix} \overset{(n)}{0} & \overset{\frac{n(n-1)}{2}}{\mathcal{K}} \end{bmatrix} \frac{n(n-1)}{2}$$

where $\mathcal{K}$ is a square matrix of order

$$\frac{n(n-1)}{2}$$

With the above results the $\mathcal{A}$ matrix in equation (13) can be written as follows:

$$\mathcal{A} = \left[\begin{array}{c|c} \multicolumn{2}{c}{|\delta|} \\ \hline 0 & \mathcal{K} \end{array}\right] = \left[\begin{array}{c|c} U & |\delta_1| \\ \hline 0 & \mathcal{K} \end{array}\right] \tag{21}$$

In Section 2.2 it is shown that $\mathcal{K}^{-1}$ exists and therefore $\mathcal{A}^{-1}$ always exists and can be calculated as follows:

$$\mathcal{A}^{-1} = \left[\begin{array}{c|c} U & -|\delta_1|\,\mathcal{K}^{-1} \\ \hline 0 & \mathcal{K}^{-1} \end{array}\right] \tag{22}$$

i.e., to invert $\mathcal{A}$ it is necessary to know only the inverse of $\mathcal{K}$.

Existance of $\mathcal{A}^{-1}$ permits us to write equation (12) or (3) as

$$\begin{bmatrix} g_{11} \\ g_{22} \\ \vdots \\ g_{m-1,m} \end{bmatrix} = \mathcal{A}^{-1} \begin{bmatrix} G_{11} \\ G_{22} \\ \vdots \\ G_{n-1,n} \end{bmatrix} \tag{23}$$

Therefore from the given terminal presentation in the form of the terminal graph and coefficient matrix $\mathcal{Y}$, $\mathcal{A}$ can be written directly and the element values of the R-network can be found directly by equation (23).

2.2 The Existence of $\mathcal{K}^{-1}$ and Related Properties

Consider the $\mathcal{Q}$ matrix (21) in Section 2.1. It has already been indicated in Section 2.1 that existence of $\mathcal{Q}^{-1}$ is implied by the existence of $\mathcal{K}^{-1}$.

The proof that $\mathcal{K}^{-1}$ exists is based on the induction method. Let us consider an R-network whose terminal graph consisting of two elements as it is shown in Figure 1 (For a simplest TG, which is a single line-segment, the order of $\mathcal{K}$ matrix is zero, hence this case is omitted).

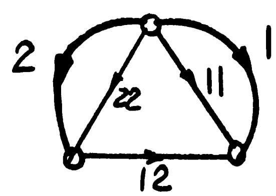


Figure 1

The submatrix of the cut-set matrix is

$$\mathcal{S} = \begin{matrix} 1 \\ 2 \end{matrix}\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] = \left[\mathcal{U}_2 \mid \mathcal{S}_1\right]$$

From equation (21) in Section 2.1

$$\mathcal{Q} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & -1 \end{array}\right] = \left[\begin{array}{c|c} \mathcal{U}_1 & \mathcal{S}_1 \\ \hline 0 & \mathcal{K}_2 \end{array}\right]$$

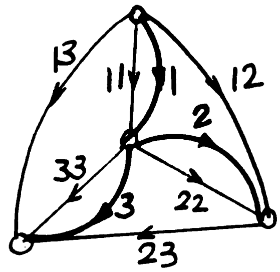
Therefore

$$\det \mathcal{K}_2 = -1 = -1 \neq 0$$

i.e. $\mathcal{K}_2$ is a non-singular matrix.

For the terminal graph consisting of three elements we have two possible configurations;

a) If the terminal graph is in the form of a <u>Lagrangian tree</u> then from Figure 2 we have



$$\mathcal{S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 12 & 13 & 23 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Figure 2

Therefore

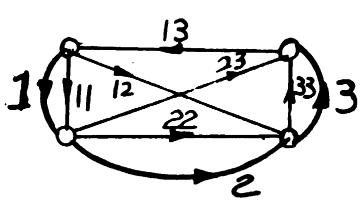
$$\mathcal{A} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

which gives

$$\det \mathcal{K}_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 \neq 0$$

and $\mathcal{K}_3^{-1}$ exists.

b) If the terminal graph is in the form of a path then from Figure 3 we have



$$\mathcal{S} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 12 & 13 & 23 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \end{array}$$

Figure 3

and

$$\mathcal{A} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore

$$\det \mathcal{K}_3 = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

i.e., $\mathcal{K}_3^{-1}$ exists.

Note here that labeling of the elements of the terminal graph does not effect the existence of $\mathcal{K}_3^{-1}$. This can be seen as follows: Suppose we have interchanged the labeling of any two elements of the terminal graph, say 2 and 3. This changes will effect to the $\mathcal{S}$ matrix in such a way that only the second and third row of $\mathcal{S}$ are interchanged. This interchange will be reflected only as an interchanging in the same rows of $\mathcal{K}_3$ (in this case first and second rows of $\mathcal{K}_3$ are interchanged). Therefore det $\mathcal{K}_3$ is not alter except in sign.

Changing the labeling of the R graph elements does not alter the det $\mathcal{K}_3$ except in sign, since it corresponds to a change between the columns of the matrix i.e., changes between the columns of $\mathcal{K}_3$ matrix. It is obvious that the above property applies also for a terminal graph having more than three elements.

Proceeding with the induction proof, assume that $\mathcal{K}_n^{-1}$ exists for a terminal graph having n elements. We shall show that $\mathcal{K}_{n+1}^{-1}$

also exists.

Consider an R-network whose system graph is represented by G_n and the terminal graph by T_n, as in Figure 4(a).

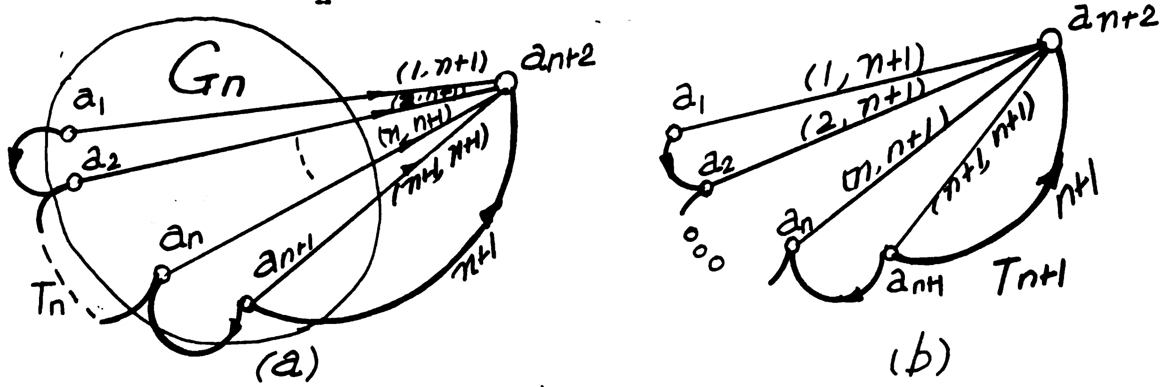


Figure 4

Suppose a new vertex a_{n+2} is added to the graph and all the vertices a_i $(i = 1, 2, \ldots n+1)$ are connected to this vertex by a set of new elements (1, n+1), (2, n+1),...., (n+1, n+1), also a new element (n+1) is added to the terminal graph T_n. By doing this we have obtained again a complete R-graph for which the terminal graph (T_{n+1}) contains (n+1) elements. Let the $\mathcal{S}$ matrix for G_n and T_n be written as

$$\mathcal{S} = \begin{matrix} 1 \\ \vdots \\ n \end{matrix} \begin{bmatrix} \mathcal{U} & \mathcal{S}_1 \end{bmatrix}$$

then for the new graph G_{n+1} and T_{n+1} the matrix will have the form

$$\mathcal{S} = \begin{matrix} \\ 1 \\ \vdots \\ n \\ n+1 \end{matrix} \begin{matrix} (n) & (1) & \left(\frac{n(n-1)}{2}\right) & \text{New elements } (n) \\ \left[\begin{matrix} \mathcal{U} \\ \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ \\ 1 \end{matrix} & \begin{matrix} \mathcal{S}_1 \\ \\ \mathcal{S}_{21} \end{matrix} & \left.\begin{matrix} \mathcal{S}_{12} \\ \\ \mathcal{S}_{22} \end{matrix}\right] \end{matrix} \quad (1)$$

Since the last cut-set corresponding to the element (n+1) contains only the element (1, n+1), ..., (n+1, n+1), the last row of $\mathcal{S}$ has the properties such that $\mathcal{S}_{21} = 0$ and $\mathcal{S}_{22} = [\,1 \;\; 1 \;\; 1 \ldots 1\,]$. Since the element orientations of the new elements in G_{n+1} are chosen as in Figure 4(b) the entries of $\mathcal{S}_{22}$ are all +1. If we change some of the orientations of these elements the corresponding entries of $\mathcal{S}_{22}$ becomes -1. It will be seen later that changes in the orientations of these elements are immaterial as far as the inverse of $\mathcal{K}_{n+1}$ is concerned.

Now from equation (15) in Section 2.1 we can construct the $\mathcal{K}_{n+1}$ matrix. It is easy to see that

$$\mathcal{K}_{n+1} = \left[\begin{array}{c|c} \mathcal{K}_n & \mathcal{L}_{12} \\ \hline 0 & \mathcal{K}_{22} \end{array}\right]$$

with $\mathcal{K}_{22} = \mathcal{S}_{12}$ since $\mathcal{S}_{22} = [\ 1\ \ 1\ \ 1\ \ldots\ 1]$.

If we can show that $\mathcal{S}_{12}$ is nonsingular then $\mathcal{K}_{22}^{-1}$ exists. This implies that $\mathcal{K}_{n+1}$ is nonsingular, since by elementary transformation matrix equivalent to $\mathcal{K}_{n+1}$ can be obtained in the form

$$\begin{bmatrix} \mathcal{K}_n & 0 \\ 0 & \mathcal{K}_{22} \end{bmatrix}$$

which has an inverse.

Now the problem is reduced to show that $\mathcal{S}_{12}$ is nonsingular. From equation (1) it can be seen that $\mathcal{S}_{12}$ is a square matrix of order n. Let us consider the R-network in Figure 4(a). For this network we can disregard G_n and consider only T_n since T_n and a set of corresponding terminal equations constitute terminal representation of G_n. Therefore, we have a system graph containing $T_n \cup (n+1) = T_{n+1}$ and the elements (1, n+1), --- (n+1,n+1) as in Figure 4(b). If T_{n+1} is chosen as a tree in this graph, then the cut-set matrix is of form

$$\mathcal{C} = \begin{array}{c} 1 \\ 2 \\ \vdots \\ n+1 \end{array} \left[\begin{array}{c|c} \mathcal{U}_{n+1} & \overset{(1,n+1)\ldots(n+1,n+1)}{\mathcal{C}_{12}} \end{array} \right] = \left[\begin{array}{c|c|c} \mathcal{U}_{n+1} & \mathcal{S}_{12} & \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline & 1 \quad 1 \ldots 1 & 1 \end{array} \right] \tag{3}$$

On the other hand, the chord-set in Figure 4(b) also constitute a tree of the graph, i.e., all through and across variables of T_{n+1} can be expressed in terms of the corresponding variables for the elements (1,n+1), ..., (n+1,n+1), and vice versa. In other words considering matrix (3) we have

$$\left[\mathcal{U} \quad \mathcal{C}_{12} \right] \begin{bmatrix} \mathcal{J}_T \\ \mathcal{J}_c \end{bmatrix} = 0 \tag{4}$$

If T_{n+1} is taken as chord set then we have

$$\left[\mathcal{U} \quad \mathcal{D}_{12} \right] \begin{bmatrix} \mathcal{J}_c \\ \mathcal{J}_T \end{bmatrix} = 0 \tag{5}$$

Equations (4) and (5) give

$$\mathcal{J}_T = -\mathcal{C}_{12} \mathcal{J}_c$$

$$\mathcal{J}_c = -\mathcal{D}_{12} \mathcal{J}_T$$

respectively which imply that

$$\mathcal{C}_{12} \mathcal{D}_{12} = \mathcal{D}_{12} \mathcal{C}_{12} = \mathcal{U}$$

Hence

$\mathcal{C}_{12} = \mathcal{D}_{12}^{-1}$ or $\mathcal{D}_{12} = \mathcal{C}_{12}^{-1}$, therefore $[\mathcal{S}_{12}]^{-1}$ exists and the proof that $\mathcal{S}_{12}$ is a nonsingular matrix is established.

Example 1: Six-Vertex Terminal Graph

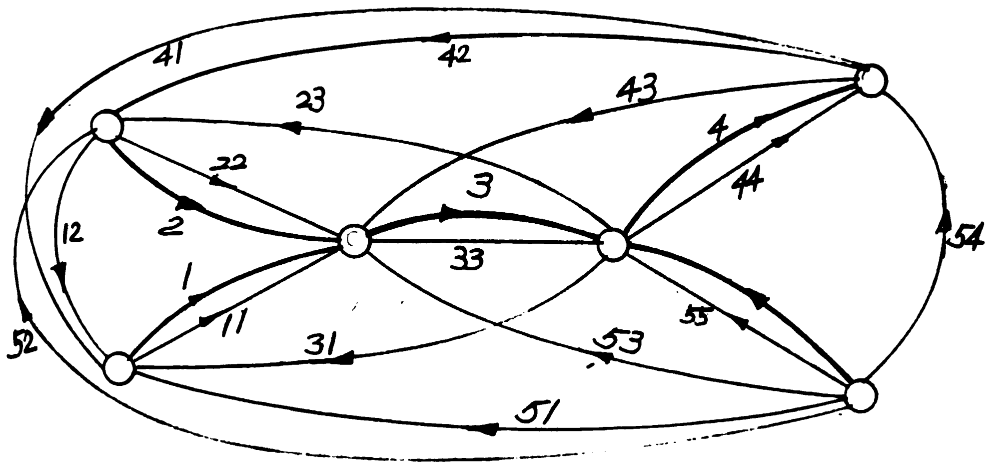


Figure 5

For Figure 5 the matrix is

$$\mathcal{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with the columns headed, left to right:

1	2	3	4	5	12	31	32	41	42	43	51	52	53	54

from which

$$\mathcal{K}_5 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: Six-Vertex Path Terminal Graph

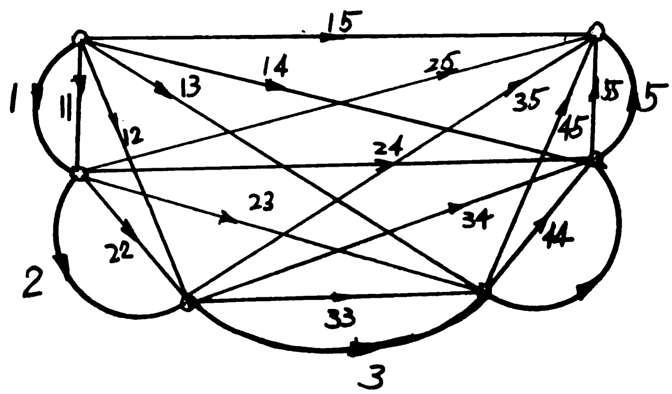


Figure 6

For the δ matrix we have

$$\delta = \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 12 & 13 & 23 & 14 & 24 & 34 & 15 & 25 & 35 & 45 \end{array} \\ \left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \end{array}$$

and

$$K_5 = \left[\begin{array}{cccccccccc} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

General form of K-matrix and its inverse.

One can choose the labeling of the elements in the graph G_n

(-corresponds to R-network) such that the $\mathcal{K}$ matrix assumes a simple form. We already know that $\mathcal{K}_{n+1}$ can be written as in equation (3) by labeling the new elements in G_{n+1} as discussed earlier i.e.,

$$\mathcal{K}_{n+1} = \begin{bmatrix} \mathcal{K}_n & \mathcal{L}_{n+1} \\ 0 & \mathcal{K}^{(n+1)} \end{bmatrix} \qquad (6)$$

where $\mathcal{K}^{(n+1)}$ is used to replace $\mathcal{K}_{22}$ in equation (3). If we do the same thing for the vertices in G_n we would have e.g.,

$$\mathcal{K}_n = \begin{bmatrix} \mathcal{K}_{n-1} & \mathcal{L}_n \\ 0 & \mathcal{K}^{(n)} \end{bmatrix}$$

Continuing this labeling process, $\mathcal{K}_n$ then can be put in the form

$$\mathcal{K}_n = \begin{bmatrix} \mathcal{K}^{(1)} & \mathcal{L}_{12} & \mathcal{L}_{13} \cdots & \mathcal{L}_{1n} \\ 0 & \mathcal{K}^{(2)} & \mathcal{L}_{23} \cdots & \mathcal{L}_{2n} \\ 0 & 0 & \mathcal{K}^{(3)} & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \mathcal{L}_{n-1,n} \\ 0 & 0 & 0 & \mathcal{K}^{(n)} \end{bmatrix}$$

where the order of $\mathcal{K}^{(i)}$ is exactly equal to i. Also it has been established that $\mathcal{K}^{(i)}$ has an inverse

The inverse of $\mathcal{K}_n$ will be in the form

$$\mathcal{K}_n^{-1} = \begin{bmatrix} \mathcal{K}^{(1)^{-1}} & \mathcal{P}_{12} & \cdots & \mathcal{P}_{1n} \\ 0 & \mathcal{K}^{(2)^{-1}} & \mathcal{P}_{23} & \mathcal{P}_{2n} \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 \cdots & \mathcal{P}_{n-1,n} \\ 0 & 0 & 0 \quad 0 & \mathcal{K}^{(n)^{-1}} \end{bmatrix}$$

Where $\mathcal{P}_{ij}$ can be calculated in terms of $\mathcal{K}^{(i)}$ and $\mathcal{L}_{ij}$, but these expressions for $\mathcal{P}_{ij}$ are not convenient to calculate the inverse of $\mathcal{K}_n$, since the complexity of these expressions increase as the indices i and j increase. For example the expressions for some of the $\mathcal{P}_{ij}$ in terms of $\mathcal{K}^{(i)^{-1}}$ and $\mathcal{L}_{ij}$ are.

$$\mathcal{P}_{12} = -\mathcal{K}_1^{-1}\mathcal{L}_{12}\mathcal{K}_2^{-1}$$

$$\mathcal{P}_{13} = -\mathcal{K}_1^{-1}\mathcal{L}_{13}\mathcal{K}_3^{-1} + \mathcal{K}_1^{-1}\mathcal{L}_{12}\mathcal{K}_2^{-1}\mathcal{L}_{23}\mathcal{K}_3^{-1}$$

$$\mathcal{P}_{14} = -\mathcal{K}_1^{-1}\mathcal{L}_{14}\mathcal{K}_4^{-1} + \mathcal{K}_1^{-1}\mathcal{L}_{12}\mathcal{K}_2^{-1}\mathcal{L}_{24}\mathcal{K}_4^{-1} + \mathcal{K}_1^{-1}\mathcal{L}_{13}\mathcal{K}_3^{-1}\mathcal{L}_{34}\mathcal{K}_4^{-1} - \mathcal{K}_1^{-1}\mathcal{L}_{12}\mathcal{K}_2^{-1}\mathcal{L}_{23}\mathcal{K}_3^{-1}\mathcal{L}_{34}\mathcal{K}_4^{-1}$$

$$\mathcal{P}_{23} = -\mathcal{K}_2^{-1}\mathcal{L}_{23}\mathcal{K}_3^{-1}$$

$$\mathcal{P}_{24} = -\mathcal{K}_2^{-1}\mathcal{L}_{24}\mathcal{K}_4^{-1} + \mathcal{K}_2^{-1}\mathcal{L}_{23}\mathcal{K}_3^{-1}\mathcal{L}_{34}\mathcal{K}_4^{-1}$$

The Labeling of Graph Elements

In order to put $\mathcal{K}_n$ in the form indicated in equation 7 the labeling of the elements of the terminal graph as well as the R-graph elements must be chosen carefully.

a) Labeling of the Terminal graph elements

Guillemin[10] has described a method by which a tree can be assumed to "grow". This procedure can be used effectively here. To illustrate, let us consider the tree in Figure 7(a).

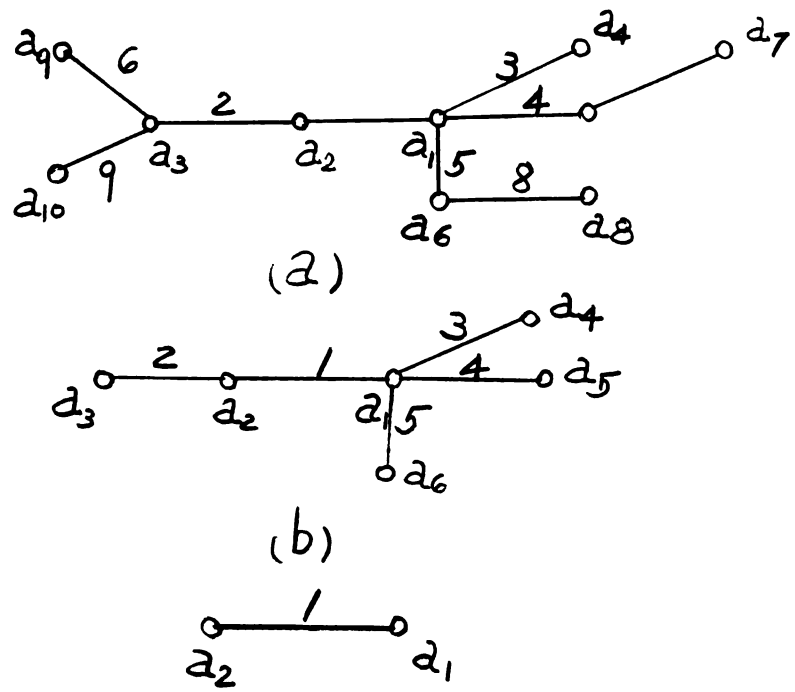


Figure 7

In Guillemin's terminology Figure 7(c) is called a "one-year-old tree", Figure 7(b) is called a "two-year-old tree". Therefore the tree in Figure 7(a) is a "three-year-old tree". The labeling of the latter is obtained by first labeling Figures 7(b) and (c).

b) Labeling of the R-graph elements.

To label the R-graph elements the following procedure is used.

1. Consider the end element of the terminal graph with the labeling of highest number. (This number in Figure 7(a) is 9).

2. Since the element considered in (1) is an end element, consider the vertex of this element which is incident to this element only (in Figure 9(a) this vertex is a_{10}).

3. Consider the elements between the vertices a_{10} and a_1, a_2, a_3,..... a_9 let us temporarily indicate these elements by the symbol (a_{10}, a_i)

where i = 1, 2, ..., 9.

4. Consider vertices $a_i (i \neq 10)$ which is incident to only one element (e.g., a_7, a_8, a_4, a_9) and the corresponding elements (a_{10}, a_7), (a_{10}, a_8), ... (a_{10}, a_9). Label these elements as (g_j). Where j is the labeling of the end elements at which the selected vertices are incident. For example (a_{10}, a_7) is labeled as (g_7) while element (a_{10}, a_4) is labeled as (g_3).

5. Remove all the labeled elements and the corresponding end elements. Repeat the same thing for the new end elements.

6. After labeling all the elements which are incident to the vertex considered in (2), remove all these elements and the element considered in (1).

7. Repeat the same thing for the second highest labeled element and continue to this process to establish a labeling for all the elements of the R-graph.

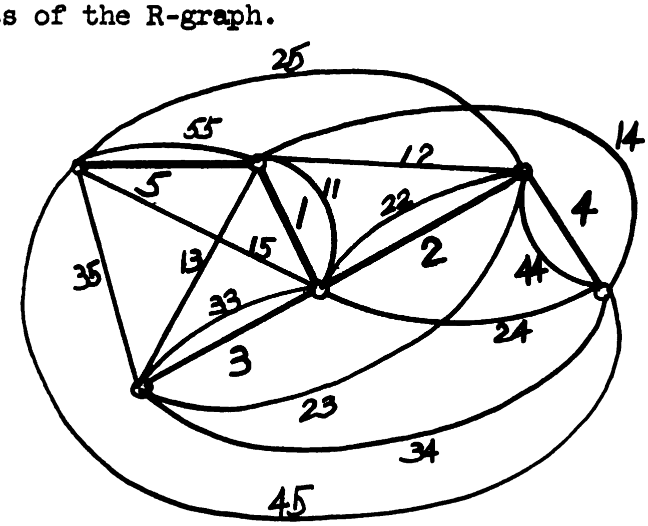


Figure 8

3. POLYGON TO STAR TRANSFORMATION

3.1 Necessary and Sufficient Conditions

In this section the transformation of a polygon-connected network into an equivalent star-connected network is considered. A set of necessary and sufficient conditions for the existence of such a transformation is presented in terms of the relationships between the entries of a matrix defined later. These conditions are also interpreted in terms of the elements of the given polygon-connected network. For the sake of simplicity in the proof only R-networks are considered. However, the result applies more generally. An alternate statement of necessary and sufficient conditions is also stated.

Consider a polygonal R-network having the nodes $A_1, A_2, \dots A_n$ and the element conductance values G_{ij}, $(i, j = 1, 2, \dots n; i \neq j)$, where G_{ij} corresponds to the element between the nodes A_i and A_j. To characterize the properties of this R-network, an additional isolated node, A_{n+1}, is chosen as the reference node. A star-like tree (Lagrangian tree) terminal graph[5], T, having A_i's $(i = 1, 2, \dots, n)$ as its end vertices is selected. This terminal graph is given in Figure 1, and the corresponding terminal equations are given in (1) and (2)

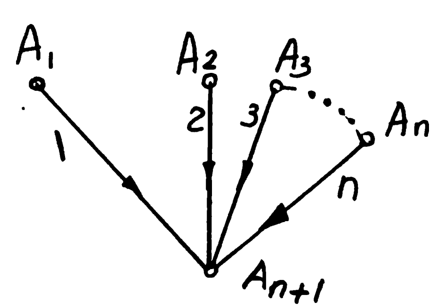


Figure 1

$$\begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdot \\ \cdot \\ \cdot \\ i_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix} \tag{1}$$

or

$$\mathcal{I}(t) = \mathcal{A}\mathcal{V}(t) \tag{2}$$

If all the elements of T are oriented toward (or away from) A_{n+1}, it is well known[6] that all the off diagonal entries in the coefficient matrix, A, in Figure 1, are negative and their magnitudes are equal to the conductance values of the elements, i.e., $-a_{ij} = G_{ij} (i \neq j)$. The diagonal entries, a_{ii}, are equal to the sum of the conductance values of all the elements incident to node A_j, i.e., $a_{ii} = \sum_{j=1}^{n} G_{ij}$. More specifically, since $a_{ii} = -\sum_{\substack{j=1 \\ j \neq 1}}^{n} a_{ij}$, then the coefficient matrix, A, is strictly dominant.*

Consider a star-connected R-network having the terminal nodes $B_1, B_2, \ldots, B_n$ and the element conductance values $g_i (i = 1, 2, \ldots, n)$. Where g_i corresponds to the element between the nodes B_i and the center node B_{n+1}. An additional isolated node B_{n+2} is chosen as the reference node and a starlike tree terminal graph, T_1, having B_i $(i = 1, 2, \ldots, n+1)$ as its end nodes is selected. It can be seen easily that coefficient matrix appearing in the terminal equations for this star-connected R-network is as follows

---

* For matrix A since dominantcy condition is satisfied with equality sign, $\sum_{j=1}^{n} a_{ij} = 0$, the word "strictly cominant" is used. A strictly dominant matrix actually is an "indefinite admittance" matrix[7] or an "equicofactor matrix[8]."

$$\left[\begin{array}{ccccccc|c}
g_1 & & & & & & & -g_1 \\
 & g_2 & & & & & & -g_2 \\
 & & \cdot & & & & & \cdot \\
 & & & \cdot & & & & \cdot \\
 & & & & \cdot & & & \cdot \\
 & & & & & \cdot & & \cdot \\
 & & & & & & g_n & -g_n \\
\hline
-g_1 & & -g_2 & & & & -g_n & \Sigma
\end{array}\right] \qquad (3)$$

where $\Sigma = \sum_{k=1}^{n} g_k$. In order to characterize the properties of this R-network at the terminal nodes, B_i ($i = 1, 2, \ldots, n$), corresponding to a terminal graph, T_2, having the same form as T in Figure 1, the current variable $i_{n+1}(t)$ is set equal to zero. Therefore, from Eq. (3) we obtain the coefficient matrix of the terminal equations corresponding to T_2.

$$\begin{bmatrix} g_1 & & & & \\ & g_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & g_n \end{bmatrix} - \frac{1}{\Sigma} \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_n \end{bmatrix} [g_1 \; g_2 \; \cdot \cdot \cdot \; g_n] \qquad (4)$$

Since terminal graphs T and T_2 are identical, the two networks are equivalent when the matrices in Eqs. (1) and (4) are identical, i.e.,

$$\mathcal{A} = \mathcal{Y} \qquad (5)$$

or

$$a_{ij} = -\frac{g_i g_j}{\Sigma} \quad (i \neq j) \qquad (6)$$

and

$$a_{ii} = g_i - \frac{g_i^{\,2}}{\Sigma} \qquad (7)$$

The following relation is easily established from Eq. (6)

$$\frac{a_{ij}}{a_{1j}} = \frac{g_i}{g_1} = \alpha_{i-1} \qquad (i = 1, 2, \ldots, n;\ \alpha_o \equiv 1) \tag{8}$$

where α_{i-1} is a positive real constant. Since $-a_{ij} = G_{ij}$ $(i \neq j)$ Eq. (4) is actually Rosen's theorem[9], i.e., Eq. (6) expresses the element values of the polygon-connected network in terms of the element values of the star-connected network. (Star to polygon transformation).

To establish the inverse relation, i.e., the polygon to star transformation, solve Eq. (8) for g_i

$$g_i = \alpha_{i-1}\, g_1 \quad (i = 1, 2, \ldots, n) \tag{9}$$

Hence, if the element value g_1 is obtained the values of all other g_i follow from Eq. (9).

Consider Eq. (7) for $i = 1$ and substitute for Σ the identity $\Sigma = g_1\,(1 + \alpha_1 + \ldots + \alpha_{n-1})$. We have then

$$g_1 = \frac{a_{11}}{1 - \frac{1}{\sigma}} \tag{10}$$

where $\sigma = 1 + \alpha_1 + \ldots + \alpha_{n-1}$. Since $\sigma > 1$, and $a_{11} < 0$, Eq. (10) implies $g_1 > 0$. From Eq. (9) we also have in general $g_i > 0$.

From the above discussion, the following theorem can be stated:

Theorem: For a given n-node R-network to have an equivalent n-branch star R-network the necessary and sufficient conditions are: [assuming the terminal representations of this network is given by the terminal graph as shown in Fig. 1 and the terminal equations (1)]

$$\frac{a_{ij}}{a_{1j}} = \alpha_{i-1} \qquad \begin{matrix} i = 1, \ldots, n; & \alpha_o \equiv 1 \\ j = 2, \ldots, n; & i \neq j \end{matrix} \tag{11}$$

where α_{i-1} is a positive real constant.

Proof: Necessity: Follows immediately from the fact that $g_i > 0$ in Eq. (8).

Sufficiency: Given $\alpha_{i-1} > 0$, we have from Eq. (10), $g_1 > 0$ and from Eq. (9), $g_i > 0$, i.e., there exists a unique star equivalent.

The condition (11) implies the following property of the polygon connected network:

> Let A_i and A_j be any two nodes of the network. The ratio of the conductance of the elements between each one of these nodes and any other node A_k (for k = 1, 2, ..., n; $k \neq k$, $k \neq j$) must be the same.

Take any vertex as reference node as shown in Fig. 2 by the heavy line.

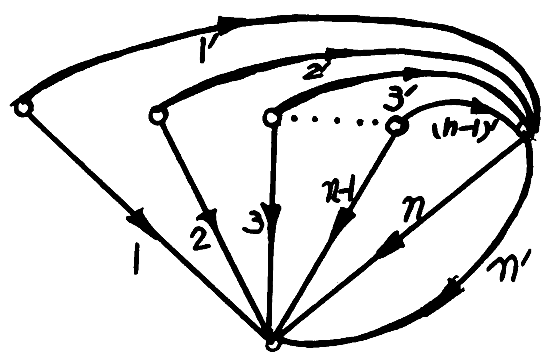


Figure 2

Denote the new terminal equation in matrix notation by

$$\mathcal{J}' = \mathcal{a}'\mathcal{V}' \qquad (12)$$

where

$$\mathcal{J}' = \begin{bmatrix} i'_1 \\ i'_2 \\ \vdots \\ i'_{n-1} \\ \hline i'_n \end{bmatrix} \quad \mathcal{V}' = \begin{bmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_{n-1} \\ \hline v'_n \end{bmatrix}$$

To establish the coefficient matrix $\mathcal{a}'$, apply a tree transformation to take the original set of voltages to the new set.

$$\mathcal{V} = \mathcal{T}\mathcal{V}'$$

From the circuit equations of Fig. 2 we have

$$\mathcal{T} = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & & 1 & 1 \\ \hline 0 & \cdots & \cdots & \cdots & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \mathcal{U} & A \\ \hline 0 & 1 \end{array}\right]$$

It is well-known that

$$\mathcal{I}' = \mathcal{T}^t \mathcal{I} \qquad (13)$$

Hence from Equations (12) and (13)

$$\mathcal{I}' = \mathcal{T}^t \mathcal{I} = \mathcal{T}^t \mathcal{a} \mathcal{V} = \mathcal{T}^t \mathcal{a} \mathcal{T} \mathcal{V}'$$

or $$\mathcal{a}' = \mathcal{T}^t \mathcal{a} \mathcal{T} \qquad (14)$$

Let the $\mathcal{a}$ matrix be written in the partitional form

$$\mathcal{a} = \begin{bmatrix} \mathcal{a}_{11} & \mathcal{a}_{12} \\ \mathcal{a}_{12}^t & \mathcal{a}_{22} \end{bmatrix} \qquad (15)$$

where $$\mathcal{a}_{11} = \begin{bmatrix} g_1 - \frac{g_1^2}{\Sigma} & -\frac{g_1 g_2}{\Sigma} & \cdots & -\frac{g_1 g_{n-1}}{\Sigma} \\ -\frac{g_1 g_2}{\Sigma} & g_2 - \frac{g_2^2}{\Sigma} & \cdots & -\frac{g_2 g_{n-1}}{\Sigma} \\ \vdots & \vdots & & \vdots \\ -\frac{g_{n-1} g_n}{\Sigma} & -\frac{g_2 g_{n-1}}{\Sigma} & \cdots & g_{n-1} - \frac{g_{n-1}^2}{\Sigma} \end{bmatrix} \qquad (16)$$

$$\mathcal{A}_{12} = \begin{bmatrix} -\dfrac{g_1 g_n}{\Sigma} \\ -\dfrac{g_2 g_n}{\Sigma} \\ -\dfrac{g_{n-1} g_n}{\Sigma} \end{bmatrix}; \qquad \mathcal{A}_{22} = g_n - \frac{g_n^2}{\Sigma}$$

Thus for the coefficient matrix in Equation (14) we have

$$\mathcal{A}' = T^t \mathcal{A} T = \begin{bmatrix} U & 0 \\ A^t & 1 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12}' & \mathcal{A}_{22} \end{bmatrix} \begin{bmatrix} U & A \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{11} A + \mathcal{A}_{12} \\ A^t \mathcal{A}_{11} + \mathcal{A}_{12}^t & (A^t \mathcal{A}_{11} + \mathcal{A}_{12}^t) A + A^t \mathcal{A}_{12} + \mathcal{A}_{22} \end{bmatrix}$$

where

$$\begin{bmatrix} g_1 - \dfrac{g_1^2}{\Sigma} & -\dfrac{g_1 g_2}{\Sigma} & \cdots & -\dfrac{g_1 g_{n-1}}{\Sigma} \\ -\dfrac{g_1 g_2}{\Sigma} & g_2 - \dfrac{g_2^2}{\Sigma} & \cdots & -\dfrac{g_2 g_{n-1}}{\Sigma} \\ \vdots & \vdots & & \vdots \\ -\dfrac{g_1 g_{n-1}}{\Sigma} & -\dfrac{g_2 g_{n-1}}{\Sigma} & \cdots & g_{n-1} - \dfrac{g_{n-1}^2}{\Sigma} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \begin{bmatrix} -\dfrac{g_1 g_n}{\Sigma} \\ -\dfrac{g_2 g_n}{\Sigma} \\ \vdots \\ -\dfrac{g_{n-1} g_n}{\Sigma} \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similarly $A^t \mathcal{A}_{11} + \mathcal{A}_{12}^t \equiv [0 \;\; 0 \; \ldots \; 0]$

$$A^t \mathcal{A}_{12} + \mathcal{A}_{22} \equiv [0]$$

Equation (12) now reads

$$\begin{bmatrix} i'_1 \\ i'_2 \\ \vdots \\ i'_{n-1} \\ \hline i'_n \end{bmatrix} = \left[\begin{array}{c|c} \mathcal{A}_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{n-1} \\ \hline V_n \end{bmatrix}$$

and the last equation in the set of no concern, i.e. the n^{th} element can be deleted from the terminal representation giving

$$\begin{bmatrix} i'_1 \\ i'_2 \\ \vdots \\ i'_{n-1} \end{bmatrix} = \mathcal{A}_{11} \begin{bmatrix} V'_1 \\ V'_2 \\ \vdots \\ V'_{n-1} \end{bmatrix}$$

$\mathcal{A}_{11}$ is a submatrix of $\mathcal{A}$, and is a dominant matrix but not strictly dominant in the sense discussed earlier. An equivalent statement of the theorem for a given n-terminal complete polygon using an n-terminal Langrangian tree can now be given. The necessary and sufficient condition on the conductance matrix of an n-terminal polygon such that the given polygon has an equivalent star network are:

(1) Except for the diagonal element, all the ratios of the off-diagonal elements in the same row to the corresponding elements in the first row must be the same, e.g.,

$$\frac{a_{23}}{a_{13}} = \frac{a_{24}}{a_{14}} = \dots = \frac{a_{2,n-1}}{a_{1,n-1}} = \alpha_1$$

(2) The ratio of any row sum to the first row sum must give the same ratio as the condition in (1)

e.g.

$$\frac{\sum_{i=1}^{n-1} a_{2i}}{\sum_{j=1}^{n-1} a_{1j}} = \alpha_1$$

If the above conditions are satisfied we can use Equation (10) to calculate the corresponding star element values. From Equation (16) we can determine α_1, α_2 ... α_{n-2} as before and α_{n-1} is obtained from the ratio of any column sum to the corresponding absolute value of the element value in the first row.

$$\text{e.g.}\quad \alpha_{n-1} = \frac{\sum_{j=1}^{n-1} a_{j2}}{a_{12}}$$

3.2 Applications

Example 1

Does the polygon in Figure 3 have an equivalent star network?

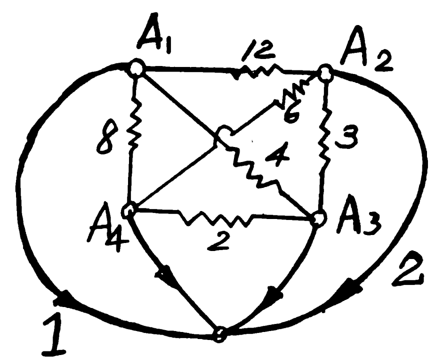


Figure 3

To use the criterion of Theorem 1 derive the conductance matrix using an 5 vertex terminal graph with the 5th vertex taken as the common vertex. Then we have

$$\mathcal{a} = \begin{bmatrix} 24 & -12 & -4 & -8 \\ -12 & 21 & -3 & -6 \\ -4 & -3 & 9 & -2 \\ -8 & -6 & -2 & 16 \end{bmatrix}$$

It is easy to see that the condition of Theorem 1 is satisfied and the element values in the equivalent 5 vertex star configuration are

$$g_1 = \frac{a_{11}}{1 - \frac{1}{\sigma}}$$

where $a_{11} = 24$, α_1, $\frac{3}{4}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$

$$\sigma = 1 + \frac{3}{4} + \frac{1}{4} + \frac{1}{2} = \frac{5}{2}$$

or $$g_1 = \frac{24}{1 - \frac{2}{5}} = \frac{24}{\frac{3}{5}} = 40$$

$$g_2 = \alpha_1 g_1 = \frac{3}{4} \times 40 = 30$$

$$g_3 = \alpha_2 g_1 = \frac{1}{4} \times 40 = 10$$

$$g_4 = \alpha_3 g_1 = \frac{1}{2} \times 40 = 20$$

and the equivalent star network is given in Figure 4.

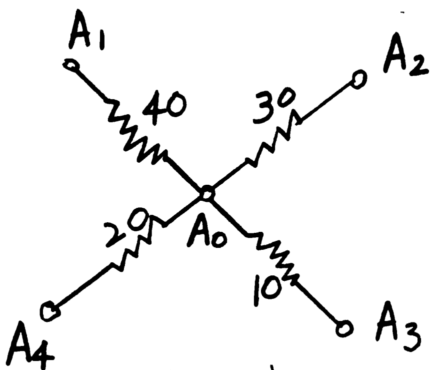


Figure 4

Example 2

Does the polygon-connected network in Figure 5 have an equivalent star-connected network?

Using additional vertex A_6 as reference node then the conductance matrix.

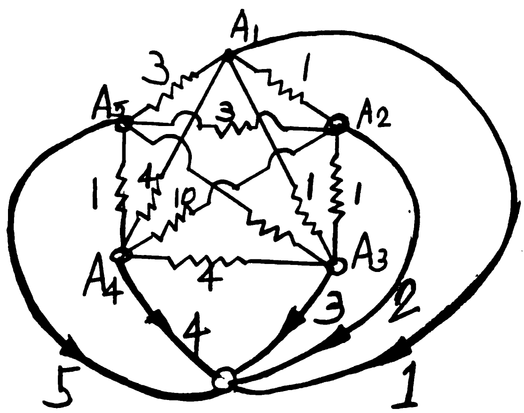


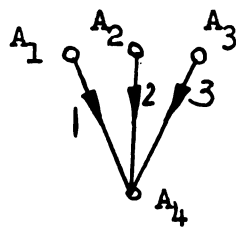
Figure 5

$$a = \begin{bmatrix} 9 & -1 & -1 & -4 & -3 \\ -1 & 15 & -1 & -10 & -3 \\ -1 & -1 & 8 & -4 & -2 \\ -4 & -10 & -4 & 30 & -12 \\ -3 & -3 & -2 & -12 & 20 \end{bmatrix}$$

We see immediately that the ratios between the elements of the first and the second rows are not the same and we need not go further to conclude that there is no equivalent star-connected network for this given polygon-connected network

Example 3

Given Terminal Graph and Terminal Equation as shown in Figure 6. Can this be synthesized as a star-connected R-network?



Terminal Graph

$$\begin{bmatrix} i_1 \\ i_1 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10.5 & -6 & -1.5 \\ -6 & 12 & -2 \\ -1.5 & -2 & 4.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Terminal Equation

Figure 6

In this condition we simply use the alternate theorem. Since condition (1) and (2) of this theorem is satisfied there exists a 5-vertex star network the element values of which are given by Equation (10).

$$g_1 = \frac{a_{11}}{1 - \frac{1}{\sigma}}$$

now $a_{11} = 10.5$

$$\alpha_1 = \frac{-2}{-1.5} = \frac{4}{3}$$

$$\alpha_2 = \frac{-2}{-6} = \frac{1}{3}$$

$$\alpha_3 = \frac{12-6-2}{1 - 61} = \frac{4}{6} = \frac{2}{3}$$

$$\sigma = 1 + \frac{4}{3} + \frac{1}{3} + \frac{2}{3} = \frac{10}{3}$$

$$g_1 = \frac{10.5}{1-\frac{3}{10}} = \frac{105}{7} = 15$$

$$g_2 = \alpha_1 g_1 = \frac{4}{3} \times 15 = 20$$

$$g_3 = \alpha_3 g_1 = \frac{1}{3} \times 15 = 5$$

$$g_4 = \alpha_3 g_1 = \frac{2}{3} \times 15 = 10$$

The star network is given in Figure 7.

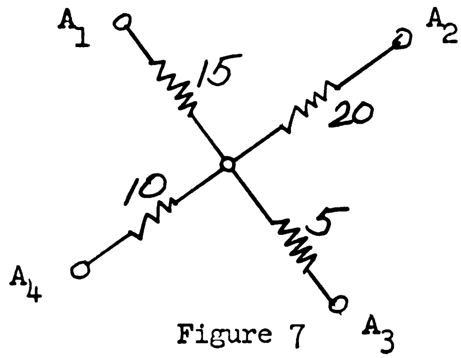


Figure 7

REFERENCES

1. I. Cederbaum, " Applications of Matrix Algebra to Network Theory" IRE Trans. on Circuit Theory, Vol-CT-6, Special Suppl. PP 127-137, May, 1959
2. E. A. Guillemin, "On the Analysis and Synthesis of Single-Element Kind Networks" IRE Trans. on Circuit Theory, Vol-CT-T, September, 1960.
3. D. P. Brown and Y. Tokad "On the Synthesis of R Network" IRE Trans. on Circuit Theory, Vol-CT-8, March, 1961
4. G. Biorci and P.P. Civalleri, "On the Synthesis of Resistive N-Port Network" IRE Trans. on Circuit Theory, Vol-CT-8, March 1961.
5. H. E. Koenig and W. A. Blackwell, Electromechanical Systems Theory, McGraw-Hill Book Company, Inc., 1961.
6. P. Slepian and L. Weinberg, "Applications of Paramount and Dominant Matrices,"Proc. Natl. Electronics Conference, Vol. 14, pp. 611-630, October, 1958.
7. J. Shekel, "Voltage Reference Node," Wireless Engineers, Vol. 31, pp. 6-10; January, 1954.
8. G. E. Sharpe and B. Spain, "On the Solution of Networks by Means of the Equicofactor Matrix," IRE Trans. on Circuit Theory, Vol. CT-7, pp. 230-239, September, 1960.
9. A. Rosen, "A New Network Theorem," IEE Journal, Vol. 62, pp. 91-198, 1924.
10. E. A. Guillemin, "How to Grow your Own Trees from Given Cut-Set or Tei-set Matrices", IRE Trans. on Circuit Theory, Vol. CT-6, pp. 110-126; May, 1959.