AN APPROXIMATE SOLUTION FOR AIR-COUPLED RAYLEIGH WAVES PROPOGATING ACROSS A VERTICAL BOUNDARY

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ABSTRACT

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By

Walter Ray Turpening

An approximate solution to the problem of the propogation of air-coupled Rayleigh waves across a vertical boundary between two materials is obtained by solving the matrix of coefficients for the boundary and source conditions. Three cases are studied: 1) the boundary between two solids, 2) the boundary between a fluid and a solid with the source over the fluid and 3) the boundary between a solid and a fluid with the source over the solid. Gaussian Elimination is used to solve the matrices and determine the coefficients of the associated displacement potentials.

The potentials in turn are used to find the expressions for the surface displacements as functions of the distance from ground zero and from the boundary. It is observed that compared with the displacements for a single material the boundary introduces attenuation and as increase in oscillation of the associated Bessel functions for the region beyond the boundary.

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NOTATION

ρ	density
α_i	P-wave velocity
β_i	S-wave velocity
ϕ_i	dilatation potential
ψ_i	rotational potential
r, θ, z	cylindrical coordinates
p_{ij}	stress (i-th face, j-th direction)
k	radial wave number
κ	axial wave number
q	radial displacement
w	vertical displacement
ω	angular frequency
λ, μ	constants of Lamé
$J_0(kr)$, $J_1(kr)$	Bessel's functions (0-th and 1-st orders)

It will be assumed that the potentials being used will be functions of r, z and time (t). The notation being used follows that of Jardetzky and Press (1952) where the axial wave numbers are defined as:

$$\gamma_0^2 = k^2 - \frac{\omega^2}{\alpha_0^2} \qquad z < 0$$

$$\gamma_1^2 = k^2 - \frac{\omega^2}{\alpha_1^2}$$

$$\gamma_2^2 = k^2 - \frac{\omega^2}{\beta_1^2} \qquad z > 0,\ r < r_o$$

$$\gamma_3^2 = k^2 - \frac{\omega^2}{\alpha_2^2}$$

$$\gamma_4^2 = k^2 - \frac{\omega^2}{\beta_2^2} \qquad z > 0,\ r > r_o$$

INTRODUCTION

This thesis is concerned with the phenomenon of Rayleigh waves coupled to atmospheric compressional waves and the effect of the change in surface material on the coupled Rayleigh waves. We will limit oursleves here to the case of the boundary between the two materials being perpendicular to the horizontal surface. Page Figure 1. illustrates the physical situation.

Lamb (1932) showed that air-coupled Rayleigh waves are generated in a dispersive medium when an air wave passes over the surface of the medium if the phase velocity of the Rayleigh wave at some frequency matches the speed of sound in air. Therefore, the characteristics of air-coupled Rayleigh waves are constant frequency and a phase velocity of sound in air.

Jardetzky and Press (1952) developed the theory of air-coupled Rayleigh waves for the case of a solid surface layer overlying a solid half space. The thoery adds a branch to the dispersion curves for Rayleigh waves which illustrates the frequency dependence of air-coupling.

Harkrider and Flinn (1970) considered the problem of a nuclear blast at high altitudes and the associated Rayleigh waves generated. Crustal structure and its effect on the

Rayleigh waves at teleseismic distances were of primary interest. A realistic isothermally layered atmosphere, three different continental models and one oceanic model were considered. Their results showed that the oceanic structure gives rise to an order of magnitude higher vertical component versus the same conditions over continental models. Taking anelastic attenuation into account reverses the results at long distances.

Much work has been done in the area of interpreting the information from the experimental records of sonic booms. Goforth and McDonald (1969, 1970), Espinosa, et al., (1967), Oliver and Isacks (1962) all considered the experimental problem of air-coupling.

Other works too numerous to mention here are listed in the bibliography.

THE PROBLEM

We will be considering three different cases of the same configuration shown in Figure 1. The general case will be the situation of two solids in contact with their difference being the physical properties.

The second case is of prime interest. Material I is a fluid and material II is a solid. Since the fluid has a shear modulus of zero ($\mu = 0$) the shear wave is nonexistant, therefore, the fluid does not support Rayleigh wave type propogation and no air-coupled Rayleigh wave occurs. This case is used to determine the generation of the air-coupled waves as a function of distance from the boundary. Interest in this case arises from the problem of recording an air shot near a river, lake or other body of water.

Case three reverses the fluid in case two and will be used to study the reflection of the air generated waves.

In all cases the main interest will be the radial and axial displacements of the solid surface as a function from ground zero.

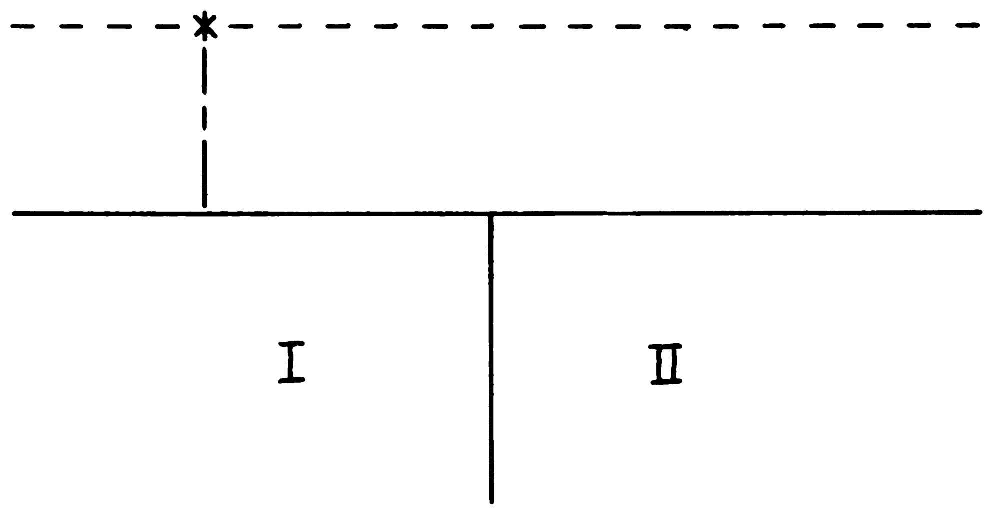

Figure 1.

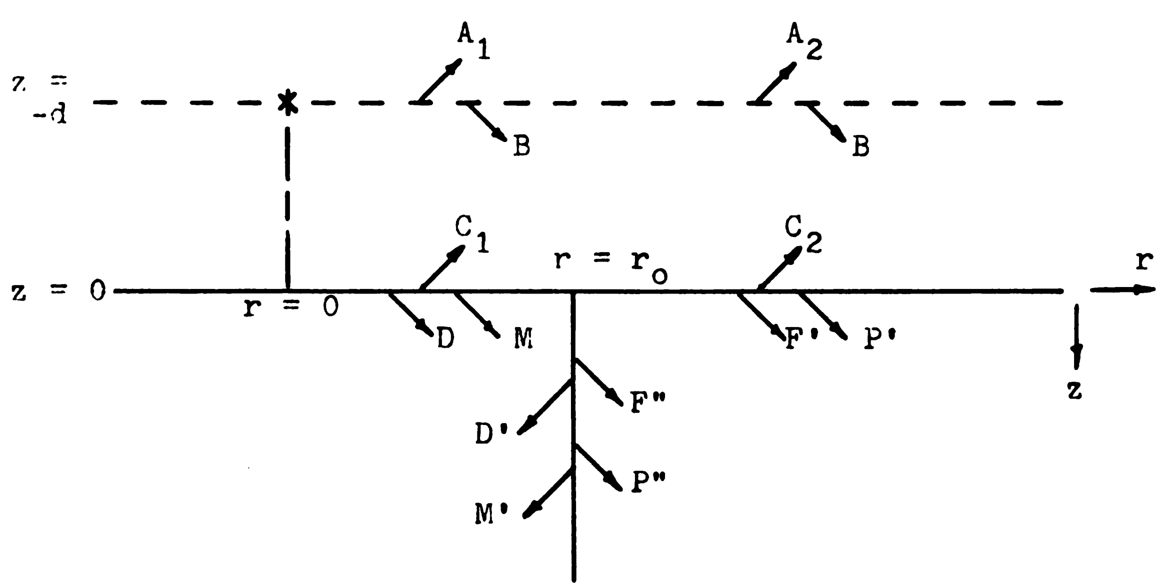

Figure 2.

THE SOLUTION

The Theory

Following the method of Lamb (1904) and Jardetzky and Press (1952) the horizontal and vertical components of displacement can be written:

$$q = \frac{\partial\phi}{\partial r} + \frac{\partial^2\psi}{\partial z\partial r} \tag{1}$$

$$w = \frac{\partial\phi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} + \frac{\omega^2}{\beta^2}\psi. \tag{2}$$

The stress components written:

$$\begin{aligned} p_{zz} &= \lambda\nabla^2\phi + 2\mu\frac{\partial w}{\partial z} \\ p_{zr} &= \mu\left(\frac{\partial q}{\partial z} + \frac{\partial w}{\partial r}\right) \\ p_{rr} &= \lambda\nabla^2\phi + 2\mu\frac{\partial q}{\partial r} \end{aligned} \tag{3}$$

with

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2}$$

where r and z are cylindrical coordinates.

In general each ϕ_i and ψ_i must be a solution to the reduced form of the wave equations:

$$\left(\nabla^2 + \frac{\omega^2}{\alpha_i^2}\right)\phi_i = 0 \tag{4a}$$

and

$$(\nabla^2 + \frac{\omega^2}{\beta_i^2})\psi_i = 0 \qquad (4b)$$

Equations of the form given above were studied by Bessel and the solution is the well known form of the Bessel function:

$$\Phi = A\exp(i\omega t \pm \gamma z)J_o(kr) \qquad (5)$$

The factor $\exp(i\omega t)$ will be assumed when not written to save space.

To completely describe the functions, ϕ_i and ψ_i above for some particular problem the boundary conditions and any initial conditions must be imposed. Following are the stress and displacement conditions that describe the problem:

at $z = 0$, $r < r_o$

$$\begin{aligned}(p_{zz})_1 &= (p_{zz})_o \\ (p_{zr})_1 &= 0 \\ w_1 &- w_o\end{aligned} \qquad (6)$$

at $z = 0$, $r > r_o$

$$\begin{aligned}(p_{zz})_2 &= (p_{zz})_o \\ (p_{zr})_2 &= 0 \\ w_1 &- w_o\end{aligned} \qquad (7)$$

at $z > 0$, $r = r_o$

$$(p_{zz})_2 = (p_{zz})_1$$
$$(p_{zr})_2 = (p_{zr})_1 \qquad (8)$$
$$w_2 - w_1$$
$$q_2 = q_1$$

Following Jardetzky and Press (1952) the plane through the source $z = -d$ introduces the conditions of continuity of pressure across the plane and the discontinuity of the vertical component of velocity, given by:

$$\rho_o \frac{\partial \phi_o'}{\partial t} = \rho_o \frac{\partial \phi_o''}{\partial t}$$

and $\qquad (9)$

$$\frac{\partial \phi_o'}{\partial z} - \frac{\partial \phi_o''}{\partial z} = 2YJ_o(kr)$$

respectively.

Using displacement potentials in the form of equation (5) the response materials and air can be described with respect to the source. Below are the potentials to be used for each of the specified regions:

$$\phi_o' = AJ_o(kr)\exp(\gamma_o z) \qquad z < -d \qquad (10)$$

$$\phi_o'' = (B\exp(-\gamma_o z)+C\exp(\gamma_o z))J_o(kr) \qquad -d < z < 0 \quad (11)$$

$$\phi_1 = (DJ_o(kr)+D'J_o(k(r_o-r)))\exp(-\gamma_1 z) \qquad (12)$$

$$\psi_1 = (MJ_o(kr)+M'J_o(k(r_o-r)))\exp(-\gamma_2 z) \qquad (13)$$

$$\phi_2 = FJ_o(k(r-r_o))\exp(-\gamma_3 z) \qquad (14)$$

$$\psi_2 = PJ_o(k(r-r_o))\exp(-\gamma_4 z) \qquad (15)$$

with F' and F" ⟶ F and P' and P" ⟶ P

At this point we must note that equations (12-15) are approximations to the solution displacement potentials. Assuming that the boundary is reasonably flat compared to the curvature of the cylindrical wave front the reflected waves D' and M' can be cylindrical waves propogating from the boundary as indicated by the $J_o(k(r_o-r))$ terms. Also in the exact case we must have the radial wave numbers equal across a horizontal surface and axial wave numbers equal across a vertical boundary. Since we will be considering the surface displacements the second approximation for the wave numbers can be neglected to symplify the analysis.

Figure 2 gives the boundaries and regions associated with each amplitude denoted by the primed and unprimed capital letters.

Matrix Formulation--General Case

By substituting the potentials given in equations (10-15) into the boundary conditions given by equations (6-9) the following equations can be obtained to solve for the amplitude functions.

$$DJ_o(kr)\left(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2\right)+D'J_o(k(r_o-r))\left(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2\right)$$

$$+MJ_o(kr)(-\gamma_2k^2)+M'J_o(k(r_o-r))(-\gamma_2k^2) \quad (16)$$

$$+(B+C_1)J_o(kr)\left(\lambda_o\frac{\omega^2}{\alpha_o^2}\right) = 0$$

$$DJ_o(kr)(-\gamma_1)+D'J_o(k(r_o-r))(-\gamma_1)+MJ_o(kr)(k^2)$$

$$+M'J_o(k(r_o-r))(k^2)+(B-C_1)J_o(kr)(\gamma_o) = 0 \quad (17)$$

$$DJ_1(kr)(2k\gamma_1)+D'J_1(k(r_o-r))(-2k\gamma_1)+M(-J_1(kr))\left(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}\right)$$

$$+M'J_1(k(r_o-r))\left(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}\right) = 0 \quad (18)$$

$$FJ_o(k(r-r_o))\left(-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2\right)+PJ_o(k(r-r_o))(-\gamma_4k^2)$$

$$+(B+C_2)J_o(kr)\left(\lambda_o\frac{\omega^2}{\alpha_o^2}\right) = 0 \quad (19)$$

$$FJ_o(k(r-r_o))(-\gamma_3)+PJ_o(k(r-r_o))(k^2)$$

$$+(B-C_2)J_o(kr)(\gamma_o) = 0 \quad (20)$$

$$FJ_1(k(r-r_o))(2\gamma_3k\mu_2)$$

$$+PJ_1(k(r-r_o))\left(2\gamma_4^2+\frac{\omega^2}{\beta_2^2}\right)(-k\mu_2) = 0 \quad (21)$$

$$D\exp(-\gamma_1 z)(2\mu_1\gamma_1 k)J_1(kr_o) + M\exp(-\gamma_2 z)(2\gamma_2^2+\frac{\omega^2}{\beta_1^2})(-kJ_1(kr_o)) = 0 \qquad (22)$$

$$D\exp(-\gamma_1 z)(\gamma_1 kJ_1(kr_o))+M\exp(-\gamma_2 z)(\gamma_2 kJ_1(kr_o)) = 0 \qquad (23)$$

$$D\exp(-\gamma_1 z)(-\gamma_1 J_o(kr_o))+D'\exp(-\gamma_1 z)(-\gamma_1) + M\exp(-\gamma_2 z)k^2 J_o(kr_o)+M'\exp(-\gamma_2 z)(k^2) + F\exp(-\gamma_3 z)(-\gamma_3)+P\exp(-\gamma_4 z)(-k^2) = 0 \qquad (24)$$

$$\begin{aligned} &D\exp(-\gamma_1 z)\left[J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2} - 2\mu_1 k^2)+2\mu_1(\frac{k}{r_o})J_1(kr_o)\right] \\ &+D'\exp(-\gamma_1 z)\left[-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1(-k^2+\frac{k}{2})\right] \\ &+M\exp(-\gamma_2 z)\left[J_o(kr_o)(2\mu_1\gamma_2 k^2)-2\mu_1\gamma_2\frac{k}{2r_o}J_1(kr_o)\right] \\ &+M'\exp(-\gamma_2 z)\left[-2\mu_1\gamma_2(-k^2+\frac{k}{2})\right] \\ &+F\exp(-\gamma_3 z)\left[\lambda_2\frac{\omega^2}{\alpha_2^2} - 2\mu_2(-k^2+\frac{k}{2})\right] \\ &+P\exp(-\gamma_4 z)\left[2\mu_2\gamma_4(-k^2+\frac{k}{2})\right] = 0 \end{aligned} \qquad (25)$$

$$(A_1-C_1)\exp(-\gamma_o d)-B\exp(\gamma_o d) = 0 \qquad (26)$$

$$(A_1-C_1)\exp(-\gamma_o d)+B\exp(\gamma_o d) = \frac{2Y}{\gamma_o} \qquad (27)$$

$$(A_2-C_2)\exp(-\gamma_o d)-B\exp(\gamma_o d) = 0 \qquad (28)$$

$$(A_2-C_2)\exp(-\gamma_o d)+B\exp(\gamma_o d) = \frac{2Y}{\gamma_o} \qquad (29)$$

These equations can be reduced in number by eliminating four of the coefficients through the last four equations. These are associated with the conditions at the source plane. The result is:

$$B = \frac{Y\exp(-\gamma_0 d)}{\gamma_0}$$

$$C_1 = A_1 - \frac{Y\exp(\gamma_0 d)}{\gamma_0} \tag{30}$$

$$C_2 = A_2 - \frac{Y\exp(\gamma_0 d)}{\gamma_0}$$

Using the above results we can write an augmented matrix of the coefficients. The lower case letters denote the coefficients that are funtions of the wave numbers and the constants of the various media. At this point we can note that the amplitudes of the potentials will be determined in terms of the amplitude of the source as given by the discontinuity of vertical velocity. Jardetzky and Press (1952) note that this difficulty can be generalized using the Fourier-Bessel integral by taking Y = k dk and integrating with respect to k from 0 to infinity. Proceeding in this direction by first solving the augmented matrix by Gaussian Elimination (this method can be found in any text on numerical methods, for example: Conte, p. 155ff). This method triangularizes the matrix, then by back substituting, each of the amplitudes is determined.

Using the following relations obtained from the equations for the boundary conditions:

$$M = \frac{D\exp(\gamma_2 z - \gamma_1 z)}{\gamma_2}$$

$$P = F\left(\frac{2\gamma_3}{2\gamma_4^2 + \frac{\omega^2}{\beta_2^2}}\right) \tag{31}$$

with $$Y_1 = Y_7 = YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o \frac{\omega^2}{\alpha_o^2}}{\gamma_o}$$

$$Y_2 = Y_6 = -YJ_o(kr)2\cosh(\gamma_o d)$$

we can rearrange the augmented matrix of coefficients into the following:

$$\begin{matrix} a_7 & d_7 & d_7' & m_7' & & & Y_7 \\ a_6 & d_6 & d_6' & m_6' & & & Y_6 \\ & d_5 & d_5' & m_5' & & & \\ & d_4 & d_4' & m_4' & f_4 & & \\ & d_3 & d_3' & m_3' & f_3 & & \\ & & & & f_2 & a_2 & Y_2 \\ & & & & f_1 & a_1 & Y_1 \end{matrix} \tag{32}$$

Without boring the reader with all the steps in the method we will give the resulting triangularized matrix. For those interested the details of the reduction are given in the appendix.

$$
\begin{array}{ccccccc}
a_7 & d_7 & d'_7 & m'_7 & & & Y_7 \\
 & \hat{d}_6 & \hat{d}'_6 & \hat{m}'_6 & & & \hat{Y}_6 \\
 & & \hat{d}'_5 & \hat{m}'_5 & & & \hat{Y}_5 \\
 & & & \hat{\hat{m}}'_4 & f_4 & & \hat{\hat{Y}}_4 \\
 & & & & \hat{f}_3 & & \tilde{Y}_3 \\
 & & & & & a_2 & \hat{Y}_2 \\
 & & & & & a_1 & \hat{Y}_1
\end{array}
\qquad (33)
$$

The various symbols used on the elements of the matrix are explained in the appendix. Now if we make all of the back substitutions the amplitudes of the displacement potentials are determined in terms of the Lamé constants of the particular mediums, wave numbers, and the source. The reduced potentials become:

$$
\begin{aligned}
\phi'_0 &= YJ_0(kr)\exp(\gamma_0 z)\bar{A}_1 \\
&= YJ_0(kr)\exp(\gamma_0 z)\bar{A}_2
\end{aligned}
\qquad (34)
$$

$$
\begin{aligned}
\phi''_0 &= YJ_0(kr)\left[\frac{\exp(-\gamma_0(z+d))}{\gamma_0} + \exp(\gamma_0 z)(\bar{A}_1 - \frac{\exp(\gamma_0 d)}{\gamma_0})\right] \\
&= YJ_0(kr)\left[\frac{\exp(-\gamma_0(z+d))}{\gamma_0} + \exp(\gamma_0 z)(\bar{A}_2 - \frac{\exp(\gamma_0 d)}{\gamma_0})\right]
\end{aligned}
\qquad (35)
$$

$$
\phi_1 = YJ_0(kr)\exp(-\gamma_1 z)(-2\exp(-\gamma_0 d))\left[\Delta + J_0(k(r_0-r))\Delta'\right] \qquad (36)
$$

$$\psi_1 = YJ_o(kr)\exp(-\gamma_2 z)(-2\exp(-\gamma_o d)) \left[\Delta \frac{\exp(z(\ _2 - \ _1))}{2} + M' J_o(k(r_o - r))\right] \tag{37}$$

$$\phi_2 = YJ_o(kr)J_o(k(r-r_o))\exp(-\gamma_3 z)(-2\exp(-\gamma_o d))\Gamma \tag{38}$$

$$\psi_2 = YJ_o(kr)J_o(k(r-r_o))\exp(-\gamma_4 z)(-2\exp(-\gamma_o d))\frac{2\gamma_3}{2\gamma_4^2 + \frac{\omega^2}{\beta_2^2}}\Gamma \tag{39}$$

The capital letters used here are functions of the constants that arise from the reduction and solution of the augmented matrix. Each is shown in detail in the appendix.

Matrix Formulation--Case 2

As was stated in the outline of the problem this case is one of the generation of an air-coupled Rayleigh wave in media II of Figure 1 with media I being a fluid. We can proceed with the same method as that was used in the general case and forming an augmented matrix from the equations for the boundary conditions as was done in the general case.

Without hesitation we can write the boundary conditions using the same notation:

$z = 0,\ r < r_o$

$$(p_{zz})_1 = (p_{zz})_o \qquad w_1 - w_o \tag{40}$$

$z = 0,\ r > r_o$

$$(p_{zz})_2 = (p_{zz})_o$$

$$(p_{zr})_2 = 0 \qquad (41)$$

$$w_2 - w_o$$

$z > 0,\ r = r_o$

$$(p_{rr})_2 = (p_{rr})_1$$

$$(p_{zr})_2 = 0 \qquad (42)$$

$$q_2 = q_2$$

$z = -d$

$$\rho_o \frac{\partial \phi_o'}{\partial t} = \rho_o \frac{\partial \phi_o''}{t}$$

$$\frac{\partial \phi_o'}{\partial z} - \frac{\partial \phi_o''}{\partial z} = 2YJ_o(kr) \qquad (43)$$

Again we can use displacement potentials to describe the response of the media in question to the source. Remaining in cylindrical coordinates we have the potentials of the form of equation 5:

$$\phi_o' = AJ_o(kr)\exp(\gamma_o z) \qquad (44)$$

$$\phi_o'' = (B\exp(-\gamma_o z) + C\exp(\gamma_o z))J_o(kr) \qquad (45)$$

$$\phi_1 = (DJ_o(kr) + D'J_o(k(r_o-r)))\exp(-\gamma_1 z) \qquad (46)$$

$$\phi_2 = FJ_o(k(r-r_o))\exp(-\gamma_3 z) \qquad (47)$$

$$\psi_2 = PJ_o(k(r-r_o))\exp(-\gamma_4 z) \qquad (48)$$

The reader is referred to Figure 3 for the relative position of the amplitudes of equations (44-48) denoted by the capital letters.

Similar to the method of the general case we insert the potentials of equations (44-48) into the boundary conditions and rearranging to the following equations.

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2}) + D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2}) + (B+C_1)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) = 0 \quad (49)$$

$$DJ_o(kr)(-\gamma_1) + D'J_o(k(r_o-r))(-\gamma_1) + (B-C_1)J_o(kr)(\gamma_o) = 0 \quad (50)$$

$$De\mathrm{xp}(-\gamma_1 z)J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2}) + D'\exp(-\gamma_1 z)(-\lambda_1\frac{\omega^2}{\alpha_1^2})$$

$$+ F\exp(-\gamma_3 z)(\lambda_2\frac{\omega^2}{\alpha_2^2} - 2\mu_2(-k^2+ \frac{k}{2}))+P\exp(-\gamma_4 z) \quad (51)$$

$$(2\mu_2\gamma_4(-k^2+\frac{k}{2})) = 0$$

$$DJ_1(kr_o)\exp(-\gamma_1 z) + D'\exp(-\gamma_1 z) + F\exp(-\gamma_3 z) + P\exp(-\gamma_4 z)(-\gamma_4)= 0 \quad (52)$$

$$F\exp(-\gamma_3 z)(2\mu_2 k\gamma_3) + P\exp(-\gamma_4 z)(-\mu_2 k(2\gamma_4^2+\frac{\omega^2}{\beta_2^2})) = 0 \quad (53)$$

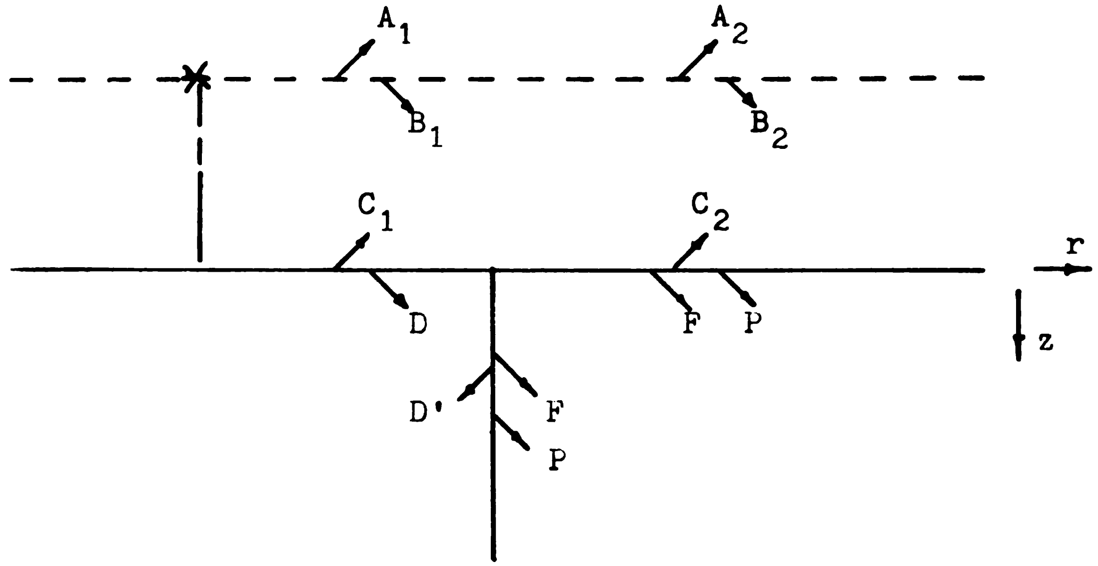

Figure 3

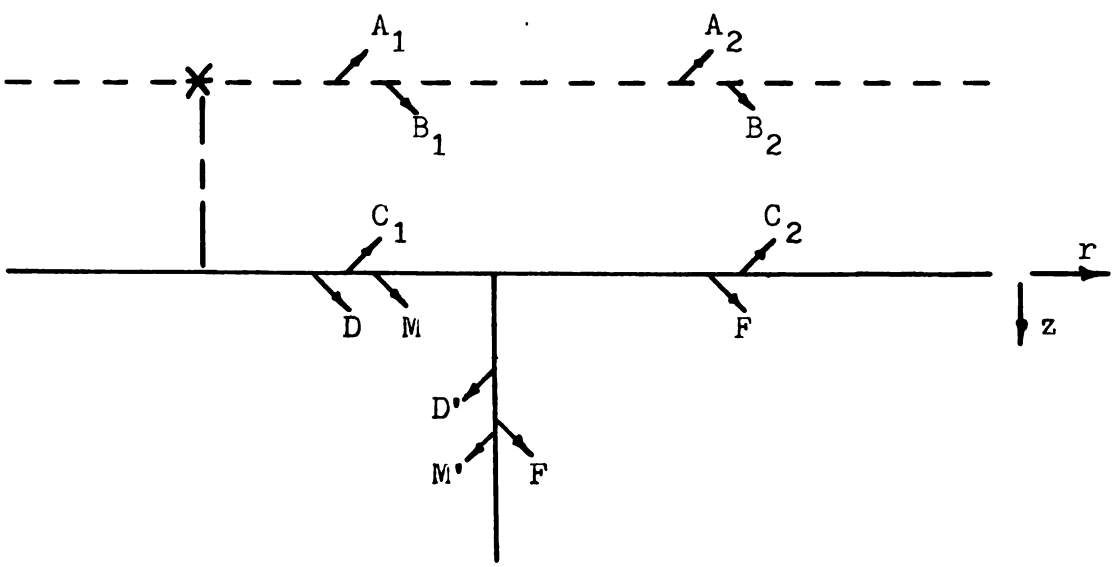

Figure 4

$$FJ_1(k(r-r_o))(2\mu_2 k\gamma_3)$$

$$+ PJ_1(k(r-r_o))(-\mu_2 k(2\gamma_4^2 + \frac{\omega^2}{\beta_2^2})) = 0 \tag{54}$$

$$FJ_o(k(r-r_o))(-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2) + PJ_o(k(r-r_o))(-\gamma_4 k^2)$$

$$+ (B+C_2)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) = 0 \tag{55}$$

$$FJ_o(k(r-r_o))(-\gamma_3) + PJ_o(k(r-r_o))(k^2)$$

$$+ (B-C_2)J_o(kr)(\gamma_o) = 0 \tag{56}$$

with the same source equations as in the general case, equations (26-29). Reducing the number of equations as we did in the general case we have:

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2}) + D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2}) + A_1J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2})$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o} \tag{57}$$

$$DJ_o(kr)(-\gamma_1) + D'J_o(k(r_o-r))(-\gamma_1) + A_1J_o(kr)(-\gamma_o)$$

$$= -2YJ_o(kr)\cosh(\gamma_o d) \tag{58}$$

$$D\exp(-\gamma_1 z)J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2})+D'\exp(-\gamma_1 z)(-\lambda_1\frac{\omega^2}{\alpha_1^2})$$

$$+ F\left[\exp(-\gamma_3 z)(\lambda_2\frac{\omega^2}{\alpha_2^2} - 2\mu_2(-k^2+\frac{k}{2}))+\exp(-\gamma_4 z)\right. \qquad (59)$$

$$\left.(\frac{4\mu_2\gamma_3\gamma_4(-k^2+\frac{k}{2})}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}})\right] = 0$$

$$DJ_o(kr_o)\exp(-\gamma_1 z)+D'\exp(-\gamma_1 z)$$

$$+ F\left[\exp(-\gamma_3 z)-\exp(\gamma_4 z)(\frac{2\gamma_3\gamma_4}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}})\right] = 0 \qquad (60)$$

$$FJ_o(k(r-r_o))(-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2 - \frac{2\gamma_4\gamma_3 k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}})+A_2J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2})$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o} \qquad (61)$$

$$FJ_o(k(r-r_o))\left[-\gamma_3+\frac{2\gamma_3 k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right. + A_2J_o(kr)(-\gamma_o)$$

$$= -YJ_o(kr)2\cosh(\gamma_o d) \qquad (62)$$

with

$$B = \frac{Y\exp(-\gamma_o d)}{\gamma_o} \qquad C_1 = A_1 - \frac{Y\exp(\gamma_o d)}{\gamma_o}$$

$$C_2 = A_2 - \frac{Y\exp(\gamma_o d)}{\gamma_o} \qquad\qquad P = F\left(\frac{2\gamma_3}{2\gamma_4^2 + \frac{\omega^2}{\beta_2^2}}\right)$$

Forming the augmented matrix of coefficients and applying the Gaussian elimination method we have the following triangularized matrix:

$$\begin{matrix} a_1 & d_1 & d_1' & & & Y_1 \\ & \hat{d}_2 & \hat{d}_2' & & & \hat{Y}_2 \\ & & \hat{d}_3' & f_3 & & \hat{Y}_3 \\ & & & \hat{f}_4 & & \hat{\hat{Y}}_4 \\ & & & & a_5 & \hat{Y}_5 \\ & & & & a_6 & \hat{Y}_6 \end{matrix} \tag{63}$$

Back substitution obtains the amplitudes which we use to get the following displacement potentials.

$$\phi_o' = YJ_o(kr)\exp(\gamma_o z)\frac{A_1}{\lambda_o\frac{\omega^2}{\alpha_o^2}}$$

$$= YJ_o(kr)\exp(\gamma_o z)\frac{\bar{A}_1}{(\lambda_o\frac{\omega^2}{\alpha_o^2} - \gamma_o)} \tag{64}$$

$$\phi_o'' = YJ_o(kr)\,\frac{\exp(-\gamma_o(z+d))}{\gamma_o} + \exp(\gamma_o z)\left(\frac{\bar{A}_1}{\lambda_o\frac{\omega^2}{\alpha_o^2}} - \frac{\exp(\gamma_o d)}{\gamma_o}\right)$$

$$= YJ_o(kr)\,\frac{\exp(-\gamma_o(z+d))}{\gamma_o} + \exp(\gamma_o z)\left(\frac{\bar{A}_2}{\lambda_o\frac{\omega^2}{\alpha_o^2} - \gamma_o} - \frac{\exp(\gamma_o d)}{\gamma_o}\right) \tag{65}$$

$$\phi_1 = YJ_o(kr)\exp(-\gamma_1 z)(-2\exp(-\gamma_o d))\left[\Delta + \Delta' J_o(k(r_o - r))\right] \tag{66}$$

$$\phi_2 = YJ_o(kr)J_o(k(r-r_o))\exp(-\gamma_3 z)(-2\exp(-\gamma_o d))\Gamma \tag{67}$$

$$\psi_2 = YJ_o(kr)J_o(k(r-r_o))\exp(-\gamma_4 z)(-2\exp(-\gamma_o d))\Gamma \quad \frac{2\gamma_3}{2\gamma_4^2 + \frac{\omega^2}{\beta_2^2}} \tag{68}$$

Matrix Formulation -- Case 3

Without being repetitive we can proceed as before and write directly the results of similar calculations that were done in the previous cases. Keeping in mind that a fluid does not support shear wave propogation and therefore no Rayleigh waves. The boundary conditions for this case are as follows:

$z = 0,\ r < r_o$

$$(p_{zz})_1 = (p_{zz})_o$$
$$(p_{zr})_1 = 0 \qquad (69)$$
$$w_1 = w_o$$

$z = 0,\ r > r_o$

$$(p_{zz})_2 = (p_{zz})_o$$
$$w_2 = w_o \qquad (70)$$

$z > 0,\ r = r_o$

$$(p_{rr})_1 = (p_{rr})_2$$
$$(p_{zr})_1 = 0 \qquad (71)$$
$$q_1 = q_2$$

$z = -d$

$$\rho_o \frac{\partial \phi_o'}{\partial t} = \rho_o \frac{\partial \phi_o''}{\partial t}$$
$$\frac{\partial \phi_o'}{\partial z} - \frac{\partial \phi_o'''}{\partial z} = 2YJ_o(kr) \qquad (72)$$

The displacement potentials to be used here are as follows:

$$\phi_o' = AJ_o(kr)\exp(\gamma_o z) \qquad (73)$$

$$\phi_o'' = (B\exp(-\gamma_o z)+C\exp(\gamma_o z))J_o(kr) \qquad (74)$$

$$\phi_1 = (DJ_o(kr)+D'J_o(k(r_o-r)))\exp(-\gamma_1 z) \qquad (75)$$

$$\psi_1 = (MJ_o(kr)+M'J_o(k(r_o-r)))\exp(-\gamma_2 z) \qquad (76)$$

$$\phi_2 = FJ_o(k(r-r_o))\exp(-\gamma_3 z) \tag{77}$$

The reader is referred to Figure 4 for the relative position of the potentials. Inserting the potentials into the boundary conditions we have the following equations.

$$\begin{aligned} &DJ_o(kr)\left(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2\right) + D'J_o(k(r_o-r))\left(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2\right) \\ &\quad + MJ_o(kr)(-\gamma_1 k^2) + M'J_o(k(r_o-r))(-\gamma_2 k^2) \\ &\quad + (B+C_1)J_o(kr)\left(\lambda_o\frac{\omega^2}{\alpha_o^2}\right) = 0 \end{aligned} \tag{78}$$

$$\begin{aligned} &DJ_o(kr)(-\gamma_1) + D'J_o(k(r_o-r))(-\gamma_1) + MJ_o(kr)(k^2) \\ &\quad + M'J_o(k(r_o-r))(k^2) + (B-C_1)J_o(kr)(-\gamma_o) = 0 \end{aligned} \tag{79}$$

$$\begin{aligned} &DJ_1(kr)(2k\gamma_1) + D'J_1(k(r_o-r))(-2\gamma_1 k) \\ &\quad + MJ_1(kr)\left(-2k\gamma_2^2 + \frac{\omega^2}{\beta_1^2}\right) + M'J_o(k(r_o-r))\left(2k\gamma_2^2 + \frac{\omega^2}{\beta_1^2}\right) = 0 \end{aligned} \tag{80}$$

$$FJ_o(k(r-r_o))\left(-\lambda_1\frac{\omega^2}{\alpha_2^2}\right) + (B+C_2)J_o(kr)\left(\lambda_o\frac{\omega^2}{\alpha_o^2}\right) = 0 \tag{81}$$

$$FJ_o(k(r-r_o))(-\gamma_3) + (B-C_2)J_o(kr)(\gamma_o) = 0 \tag{82}$$

$$F\exp(-\gamma_3 z)(\lambda_2\frac{\omega^2}{\alpha_2^2}) + D\exp(-\gamma_1 z)(J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2} - 2k^2\mu_1)$$

$$+J_1(kr_o)2\mu_1\frac{k}{r_o}) + D'\exp(-\gamma_1 z)$$

$$(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1(-k^2+ \frac{k}{2})) \quad (83)$$

$$+ M\exp(-\gamma_2 z)(-2\mu_1\gamma_2)(-k^2 J_o(kr_o)+ \frac{k}{r_o}J_1(kr_o))$$

$$+ M'\exp(-\gamma_2 z)(-2\mu_1\gamma_2)(-k^2- \frac{k}{2}) = 0$$

$$D\exp(-\gamma_1 z)(-2\mu_1\gamma_1) + M\exp(-\gamma_2 z)(2\gamma_2^2+ \frac{\omega^2}{\beta_1^2}) = 0 \quad (84)$$

$$D\exp(-\gamma_1 z) + M\exp(-\gamma_2 z)(-\gamma_2) = 0 \quad (85)$$

Reducing the order of the equations we have the following

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2-k^2) + D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2}+2\mu_1\gamma_1^2)$$

$$+ M'J_o(k(r_o-r))(-\gamma_2 k^2)+A_1 J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) \quad (86)$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o}$$

$$DJ_o(kr)(\frac{k^2}{\gamma_2} -\gamma_1) + D'J_o(k(r_o-r)) + M'J_o(k(r_o-r))(k^2)$$

$$+A_1 J_o(kr)(-\gamma_o) = -YJ_o(kr)2\cosh(\gamma_o d) \quad (87)$$

$$DJ_1(kr)(2k\gamma_1-2k\gamma_2-\frac{\omega^2}{\beta_1^2\gamma_2}) + D'J_1(k(r_o-r))(-2k\gamma_1)$$

$$+ M'J_1(k(r_o-r))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}) = 0 \tag{88}$$

$$Fexp(-\gamma_3 z)(\lambda_2\frac{\omega^2}{\alpha_2^2})+Dexp(-\gamma_1 z)J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2})$$

$$+ D'exp(-\gamma_1 z)(-\lambda_1\frac{\omega^2}{\alpha_1^2} +2\mu_1(-k^2-\frac{k}{2})) \tag{89}$$

$$+ M'exp(-\gamma_2 z)(-2\mu_1\gamma_2)(-k^2-\frac{k}{2}) = 0$$

$$FJ_o(k(r-r_o))(-\lambda_2\frac{\omega^2}{\alpha_2^2}) + A_2J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2})$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o} \tag{90}$$

$$FJ_o(k(r-r_o))(-\gamma_3) + A_2J_o(kr)(-\gamma_o)$$

$$= -YJ_o(kr)2\cosh(\gamma_o d) \tag{91}$$

with

$$B = Yexp\frac{(-\gamma_o d)}{\gamma_o}$$

$$C_1 = A_1-(\frac{Yexp(\gamma_o d)}{\gamma_o})$$

$$C_2 = A_2 - \left(\frac{Y\exp(\gamma_o d)}{\gamma_o}\right)$$

$$M = \frac{D\exp(-\gamma_1 z)}{\gamma_2\exp(-\gamma_2 z)}$$

Writing into an augmented matrix of the coefficients and solving we have the displacement potentials:

$$\phi_o' = YJ_o(kr)\exp(\gamma_o z)\frac{\bar{A}_1}{(\lambda_o\frac{\omega^2}{\alpha_o^2})}$$

$$= YJ_o(kr)\exp(\gamma_o z)\frac{\bar{A}_2}{(\lambda_o\frac{\omega^2}{\alpha_o^2} - \gamma_o)} \tag{92}$$

$$\phi_o'' = YJ_o(kr)\,\frac{\exp(-\gamma_o(z+d))}{\gamma_o} + \exp(\gamma_o z)$$

$$\left(\frac{\bar{A}_1}{\lambda_o\frac{\omega^2}{\alpha_o^2}} + \frac{\exp(\gamma_o d)}{\gamma_o}\right)$$

$$= YJ_o(kr)\,\frac{\exp(-\gamma_o(z+d))}{\gamma_o} \tag{93}$$

$$+ \exp(\gamma_o z)\left(\frac{\bar{A}_2}{\lambda_o\frac{\omega^2}{\alpha_o^2} - o} + \frac{\exp(\gamma_o d)}{\gamma_o}\right)$$

$$\phi_1 = YJ_o(kr)\exp(-\gamma_1 z)\left[\Delta + \Delta' J_o(k(r_o - r))\right] \tag{94}$$

$$\psi_1 = YJ_o(kr)\exp(-\gamma_2 z)\left[\Delta \frac{\exp(\gamma_2 z - \gamma_1 z)}{\gamma_2} + J_o(k(r-r_o))M'\right] \tag{95}$$

$$\phi_2 = YJ_o(kr)\exp(-\gamma_3 z)\Gamma J_o(k(r-r_o)) \tag{96}$$

Surface Displacements-

By substituting the potentials derived in the preceeding sections into equations 1 and 2, setting z = 0 the surface displacements can be written as follows for each of the cases:

General Case--

$$q_1 = Y2k\exp(-\gamma_o d)\Big\{ J_1(kr)\left(1 - \frac{\gamma_2}{\gamma_o}\right) + \left[J_1(kr)J_o(k(r_o-r)) - J_o(kr)J_1(k(r_o-r))\right](\Delta' + M')\Big\} \tag{97}$$

$$w_1 = YJ_o(kr)(-2\exp(-\gamma_o d)) \Big\{\left(-\gamma_1 + \frac{\gamma_1^2}{\gamma_o} + \frac{1}{\gamma_o}\right)\Delta - \left[\gamma_1\Delta' + (1+\gamma_2^2)M'\right] J_o(k(r_o-r))\Big\} \tag{98}$$

$$q_2 = Y2k\exp(-\gamma_o d)\left(1 - \frac{2\gamma_4\gamma_3}{2\gamma_4^2 + \frac{\omega^2}{\beta_2^2}}\right) \Gamma\left\{J_1(kr)J_o(k(r-r_o)) + J_o(kr)J_1(k(r-r_o))\right\} \tag{99}$$

$$w_2 = YJ_o(kr)J_o(k(r-r_o))(-2\exp(-\gamma_o d))$$

$$\Gamma\left\{-\gamma_3 + \frac{2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\left(\gamma_4^2+\frac{\omega^2}{\beta_2^2}\right)\right\} \tag{100}$$

Case 2--

$$q_1 = Y2k\exp(-\gamma_o d)$$

$$\left\{\Delta J_1(kr)+\Delta'\left[J_1(kr)J_o(k(r_o-r))-J_o(kr)J_1(k(r_o-r))\right]\right\} \tag{101}$$

$$w_1 = YJ_o(kr)2\ {}_1\exp(-\gamma_o d)\left\{\Delta + \Delta' J_o(k(r_o-r))\right\} \tag{102}$$

$$q_2 = Y2k\exp(-\gamma_o d)\left(1-\frac{2\gamma_4\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right)$$

$$\Gamma\left\{J_1(kr)J_o(k(r-r_o))+J_o(kr)J_1(k(r-r_o))\right\} \tag{103}$$

$$w_2 = YJ_o(kr)J_o(k(r-r_o))(-2\exp(-\gamma_o d)$$

$$\Gamma\left\{-\gamma_3+\frac{2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\left(\gamma_4^2+\frac{\omega^2}{\beta_2^2}\right)\right\} \tag{104}$$

Case 3--

$$q_1 = Y2k\exp(-\gamma_o d)\left\{ J_1(kr)\left(1-\frac{\gamma_2}{\gamma_o}\right)\right.$$

$$\left.+\left[J_1(kr)J_o(k(r_o-r))-J_o(kr)J_1(k(r_o-r))\right](\Delta'+M')\right\} \tag{105}$$

$$w_1 = YJ_o(kr)(-2\exp(-\gamma_o d))$$

$$\left\{\left(-\gamma_1+\frac{\gamma_1^2}{\gamma_o}+\frac{1}{\gamma_o}\right)\Delta-\left[\gamma_1\Delta'+(1+\gamma_2^2)M'\right]J_o(k(r_o-r))\right\} \tag{106}$$

$$q_2 = Y2k\exp(-\gamma_0 d) \quad (107)$$

$$\Gamma\left\{J_1(kr)J_0(k(r-r_0)) + J_0(kr)J_1(k(r-r_0))\right\}$$

$$w_2 = YJ_0(kr)J_0(k(r-r_0))(-2\exp(-\gamma_0 d))\Gamma(-\gamma_3) \quad (108)$$

For equations 97, 101, 105 the reader is referred to Figure 7, equations 98, 102, 106 to Figure 6, equations 99, 103, 107 to Figure 8 and equations 100, 104, 108 to Figure 5.

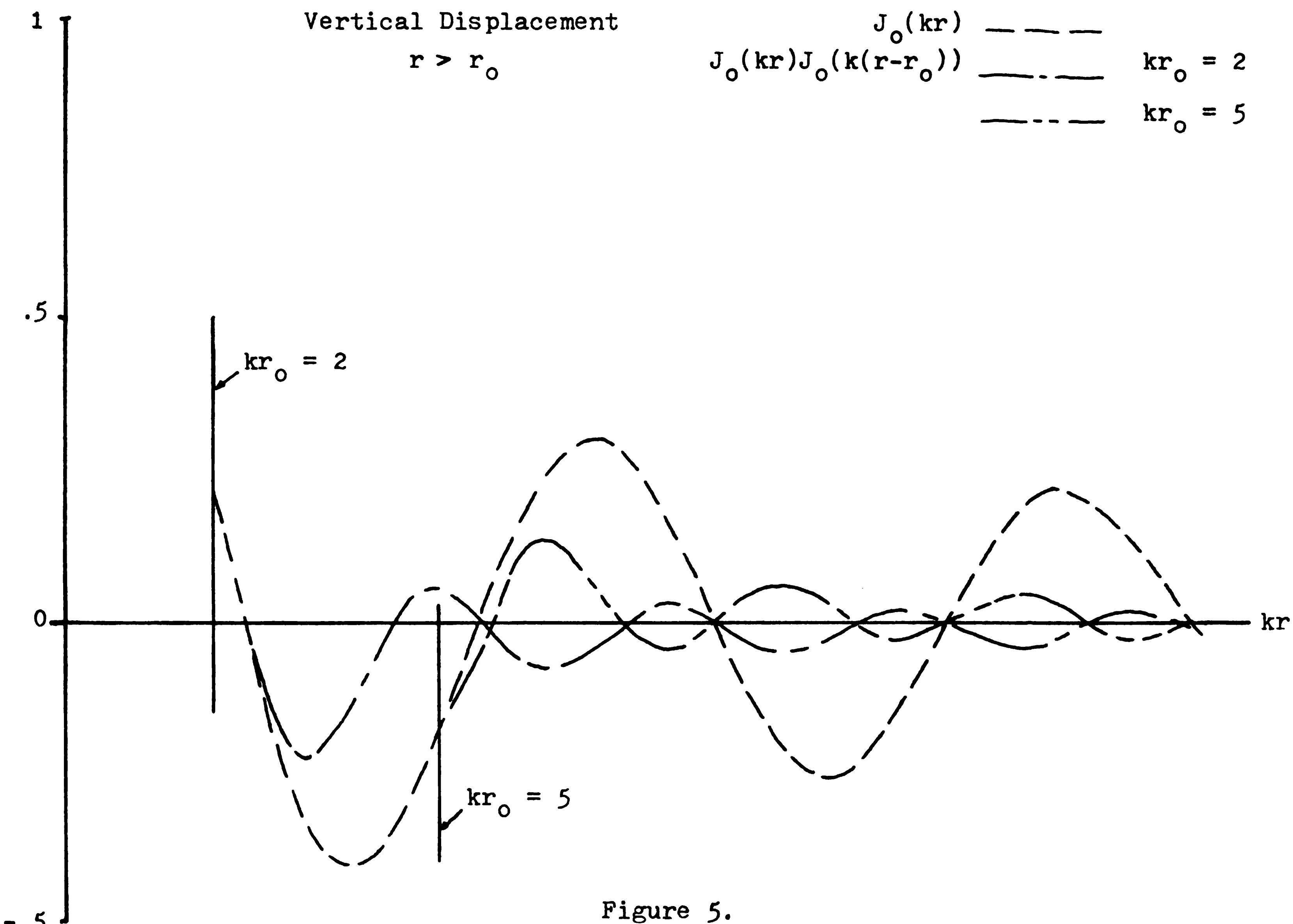
Vertical Displacement
r > r$_o$
J$_o$(kr)
J$_o$(kr)J$_o$(k(r-r$_o$))
kr$_o$ = 2
kr$_o$ = 5
1
.5
0
-.5
kr
kr$_o$ = 2
kr$_o$ = 5

Figure 5.

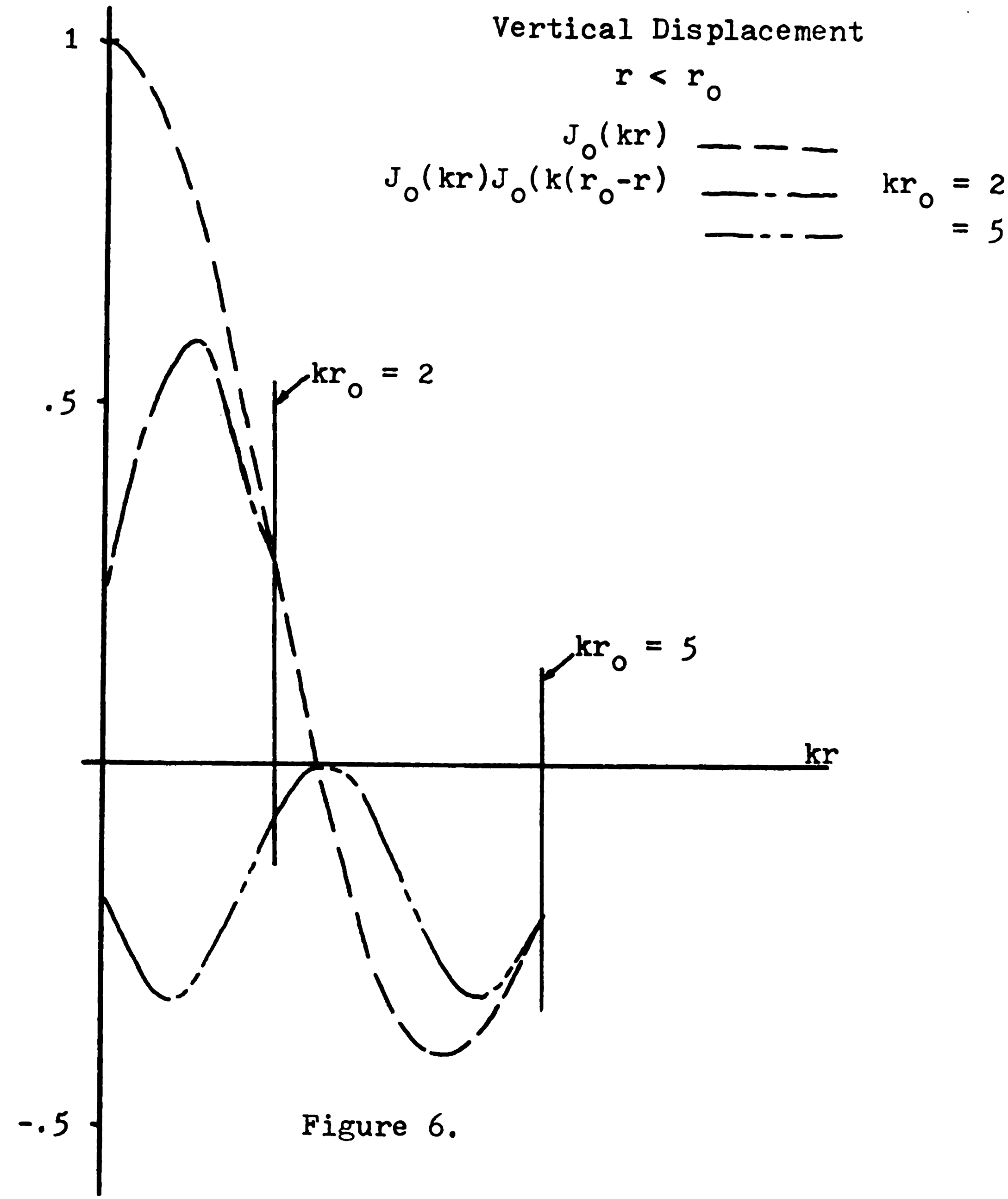
Vertical Displacement
r < r_o
J_o(kr)
J_o(kr)J_o(k(r_o-r)
kr_o = 2
= 5
kr_o = 2
kr_o = 5
kr
1
.5
-.5

Figure 6.

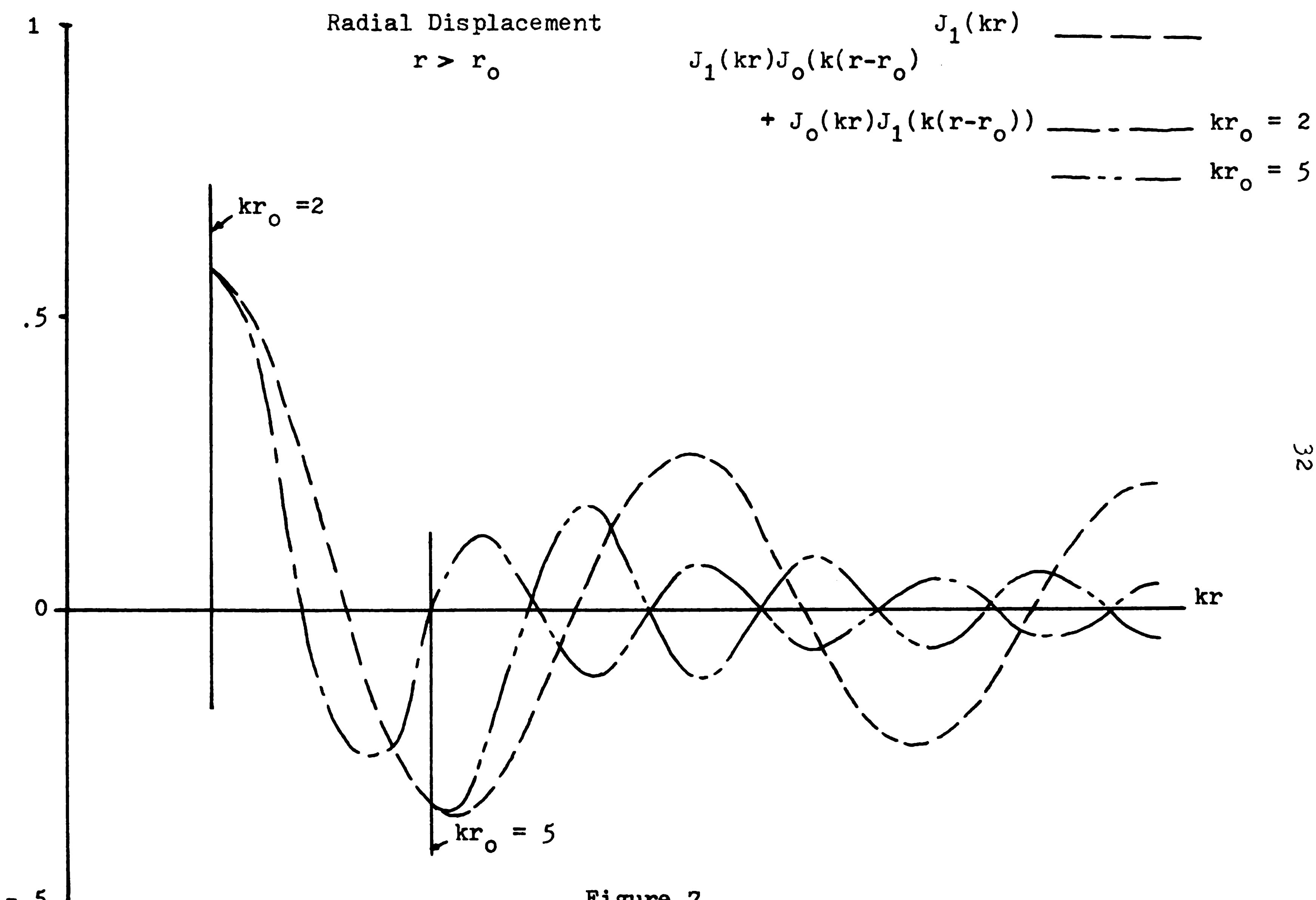
Radial Displacement
$r > r_o$
$J_1(kr)$
$J_1(kr)J_o(k(r-r_o)$
$+ J_o(kr)J_1(k(r-r_o))$
$kr_o = 2$
$kr_o = 5$
1
.5
0
-.5
kr
$kr_o = 2$
$kr_o = 5$

Figure 7.

DISCUSSION AND CONCLUSION

In this thesis we are looking at the surface displacements as a function of distance from ground zero and from the vertical boundary. If we consider the problem of a single material the vertical displacement has a distance dependence of $J_o(kr)$ and the radial displacement has the distance dependence of $J_1(kr)$. Introducing a second material with a vertical boundary produces an additional factor in the potentials that indicates a wave starting at the boundary and propogating outward. The new factor is $J_o(k(r-r_o))$. The distance dependence factor associated with the vertical and radial displacements are $\left[J_o(kr)\ J_o(k(r-r_o))\right]$ and $\left[J_1(kr)\ J_o(k(r-r_o)) + J_o(kr)\ J_1(k(r-r_o))\right]$ respectively.

Refering to Figures 5 and 7 the effects of a boundary are clearly illustrated. The first effect is the distinct attenuation of the wave as compared with $J_o(kr)$ and $J_1(kr)$ functions. Secondly there is an increase in oscillations per unit distance beyond the boundary in relation to the single material case for the same absolute distance from ground zero.

Obviously only two frequencies are illustrated in the figures but there seems to be no reason to not expect the same relative effect to be exibited for all frequencies. It appears that the boundary may have the effect of distorting the wave to a more complex form.

Within the boundary, Figures 6 and 8 illustrate similar features of attenuation of the waves and increased oscillation per unit distance. However, as has been stated earlier in this thesis the conditions and potentials are not completely exact and the results as shown may be even more complex when studied more closely.

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APPENDIX

APPENDIX

GENERAL CASE

Given the following equations for the boundary conditions for the interfaces of the solids:

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2)+D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2)$$

$$+MJ_o(kr)(-\gamma_2k^2)+M'J_o(k(r_o-r))(-\gamma_2k^2)$$

$$+(B+C_1)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) = 0$$

$$DJ_o(kr)(-\gamma_1)+D'J_o(k(r_o-r))(-\gamma_1)+MJ_o(kr)(k^2)$$

$$+M'J_o(k(r_o-r))(k^2)+(B-C_1)J_o(kr)(\gamma_o) = 0$$

$$DJ_1(kr)(2k\gamma_1)+D'J_1(k(r_o-r))(-2k\gamma_1)+M(-J_1(kr))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2})$$

$$+M'J_1(k(r_o-r))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}) = 0$$

$$FJ_o(k(r-r_o))(-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2)+PJ_o(k(r-r_o))(-\gamma_4k^2)$$

$$+(B+C_2)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_2^2}) = 0$$

$$FJ_o(k(r-r_o))(-\gamma_3)+PJ_o(k(r-r_o))(k^2)$$

$$+(B-C_2)J_o(kr)(\gamma_o) = 0$$

$$FJ_1(k(r-r_o))(2\gamma_3 k\mu_2)$$

$$+PJ_1(k(r-r_o))(2\gamma_4^2+ \frac{\omega^2}{\beta_2^2})(-k\mu_2) = 0$$

$$Dexp(-\gamma_1 z)(2\mu_1\gamma_1 k)J_1(kr_o)$$

$$+Mexp(-\gamma_2 z)(2\gamma_2^2+ \frac{\omega^2}{\beta_1^2})(-kJ_1(kr_o)) = 0$$

$$Dexp(-\gamma_1 z)(\gamma_1 kJ_1(kr_o))+Mexp(-\gamma_2 z)(\gamma_2 kJ_1(kr_o)) = 0$$

$$Dexp(-\gamma_1 z)(-\gamma_1 J_o(kr_o))+D'exp(-\gamma_1 z)(-\gamma_1)$$

$$+Mexp(-\gamma_2 z)k^2 J_o(kr_o)+M'exp(-\gamma_2 z)(k^2)$$

$$+Fexp(-\gamma_3 z)(-\gamma_3)+Pexp(-\gamma_4 z)(-k^2) = 0$$

$$Dexp(-\gamma_1 z)\left[J_o(kr_o)(-\lambda_1\frac{\omega^2}{\alpha_1^2} - 2\mu_1 k^2)+2\mu_1(\frac{k}{r_o})J_1(kr_o)\right]$$

$$+D'exp(-\gamma_1 z)\left[-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1(-k^2+ \frac{k}{2})\right]$$

$$+Mexp(-\gamma_2 z)\left[J_o(kr_o)(2\mu_1\gamma_2 k^2)-2\mu_1\gamma_2\frac{k}{2r_o}J_1(kr_o)\right]$$

$$+M'exp(-\gamma_2 z)\left[-2\mu_1\gamma_2(-k^2+ \frac{k}{2})\right]$$

$$+F\exp(-\gamma_3 z)\left[\lambda_2\frac{\omega^2}{\alpha_2^2} - 2\mu_2(-k^2+\frac{k}{2})\right]$$

$$+P\exp(-\gamma_4 z)\left[2\mu_2\gamma_4(-k^2+\frac{k}{2})\right] = 0$$

From the conditions for the source plane we have:

$$B = \frac{Y\exp(\gamma_o d)}{\gamma_o}$$

$$C = A-(\frac{Y\exp(\gamma_o d)}{\gamma_o}$$

where the relation between A and C is the same over either material and kept separate by the subscripts 1 or 2. The differences arise from the different materials that reflect the wave associated with C (i.e. the -z propogating term written $\exp(\gamma_o z)$). Making the substitutions for B and C_1 and C_2 we have the following:

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2)+D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2} + 2\mu_1\gamma_1^2)$$

$$MJ_o(kr)(-\gamma_2 k^2)+M'J_o(k(r_o-r))(-\gamma_2 k^2)$$

$$+(B+C_1)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) = 0$$

$$DJ_o(kr)(-\gamma_1)+D'J_o(k(r_o-r))(-\gamma_1)+MJ_o(kr)(k^2)$$

$$+M'J_o(k(r_o-r))(k^2)+(B-C_1)J_o(kr)(\gamma_o) = 0$$

$$DJ_1(kr)(2k\gamma_1)+D'J_1(k(r_o-r))+M(-J_1(kr))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2})$$

$$+M'J_1(k(r_o-r))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}) = 0$$

$$FJ_o(k(r-r_o))(-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2) + PJ_o(k(r-r_o))(-\gamma_4 k^2)$$

$$+(B+C_2)J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) = 0$$

$$FJ_o(k(r-r_o))(-\gamma_3)+PJ_o(k(r-r_o))(k^2)+(B-C_2)J_o(kr)(\gamma_o) = 0$$

$$F(-2\gamma_3)+P(2\gamma_4^2+\frac{\omega^2}{\beta_2^2}) = 0$$

$$D(-2\mu_1\gamma_1)\exp(-\gamma_1 z)(-kJ_1(kr_o))$$

$$+M(2\gamma_2^2+\frac{\omega^2}{\beta_2^2})\exp(-\gamma_2 z)(-kJ_1(kr_o)) = 0$$

$$DJ_o(kr_o)(-\gamma_1\exp(-\gamma_1 z))+D'(-\gamma_1\exp(-\gamma_1 z))$$

$$+MJ_o(kr_o)\exp(-\gamma_1 z)k^2+M'\exp(-\gamma_2 z)k^2$$

$$+F(-\gamma_3\exp(-\gamma_3 z))+P(-k^2\exp(-\gamma_4 z)) = 0$$

$$D(-kJ_1(kr_o))\exp(-\gamma_1 z)+M(-kJ_1(kr_o))(-\gamma_2\exp(-\gamma_2 z)) = 0$$

$$D\exp(-\gamma_1 z)\Big[\lambda_1 J_o(kr_o)(-\frac{\omega^2}{\alpha_1^2})+2\mu_1(-k^2J_o(kr_o)+\frac{k}{r_o}J_1(kr_o))\Big]$$

$$+D'\exp(-\gamma_1 z)\Big[\lambda_1(-\frac{\omega^2}{\alpha_1^2})+2\mu_1(-k^2+\frac{k}{2})\Big]$$

$$+M\exp(-\gamma_2 z)\Big[2\mu_1(-\gamma_2)(-k^2J_o(kr_o)+\frac{k}{r_o}J_1(kr_o))\Big]$$

$$+M'\exp(-\gamma_2 z)\Big[2\lambda_1(-\gamma_2)(-k^2+\frac{k}{2})\Big]$$

$$= F\exp(-\gamma_3 z)\Big[\lambda_2(-\frac{\omega^2}{\alpha_2^2})+2\mu_2(-k^2+\frac{k}{2})\Big]$$

$$+P\exp(-\gamma_4 z)\Big[2\mu_2(-\gamma_4)(-k^2+\frac{k}{2})\Big]$$

From these equations we can get the following relations between D and M and between F and P:

$$M = \frac{D\exp(\gamma_2 z-\gamma_1 z)}{\gamma_2}$$

$$P = F\Big(\frac{2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\Big)$$

and we further reduce the order of the equations to:

$$FJ_o(k(r-r_o))\left[-\lambda_2\frac{\omega^2}{\alpha_2^2} + 2\mu_2\gamma_3^2 - \frac{\gamma_4 k^2 2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right]$$

$$+A_2J_o(kr)\left(\lambda_o\frac{\omega^2}{\alpha_o^2}\right) \qquad 1$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o}$$

$$FJ_o(k(r-r_o))\left[-\gamma_3+\frac{2\gamma_3k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right] +A_2J_o(kr)(-\gamma_o) \qquad 2$$

$$= YJ_o(kr)\cosh(\gamma_o d)$$

$$F\Bigg[\exp(-\gamma_3z)\left(\lambda_2\frac{\omega^2}{\alpha_2^2} - 2\mu_2\left(-k^2+\frac{k}{2}\right)\right)$$

$$+\exp(-\gamma_4z)\frac{2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\left(-2\mu_2\gamma_4\left(-k^2+\frac{k}{2}\right)\right)\Bigg] \qquad 3$$

$$+D\exp(-\gamma_1z)J_o(kr_o)\left(-\lambda_1\frac{\omega^2}{\alpha_1^2}\right)+D'\exp(-\gamma_1z)\Bigg[-\lambda_1\frac{\omega^2}{\alpha_1^2}$$

$$+2\mu_1\left(-k^2+\frac{k}{2}\right)\Bigg]+M'\exp(-\gamma_2z)\left[2\mu_1\gamma_2\left(-k^2+\frac{k}{2}\right)\right] = 0$$

$$F\left[\exp(-\gamma_3z)(-\gamma_3)-\frac{2\gamma_3k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\exp(-\gamma_4z)\right]$$

$$+DJ_o(kr_o)\exp(-\gamma_1 z)\left[\frac{k^2}{\gamma_2}-\gamma_1\right]+D'\exp(-\gamma_1 z)(-\gamma_1)$$

$$+M'\exp(-\gamma_2 z)(k^2) = 0 \qquad 4$$

$$DJ_1(kr)(2k(\gamma_1-\gamma_2)-\frac{\omega^2}{\gamma_2\beta_1^2})+D'J_1(k(r_o-r))(-2k\gamma_1)$$

$$+M'J_1(k(r_o-r))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2}) = 0 \qquad 5$$

$$DJ_o(kr)(\frac{k^2}{\gamma_2}-\gamma_1)+D'J_o(k(r_o-r))(-\gamma_1)+M'J_o(k(r_o-r)$$

$$+A_1J_o(kr)(-\gamma_o) = -YJ_o(kr)\ 2\cosh(\gamma_o d) \qquad 6$$

$$DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2}-k^2+2\mu_1\gamma_1^2)+D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2}+2\mu_1\gamma_2^2)$$

$$+M'J_o(k(r_o-r))(-\gamma_2k^2)+A_1J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2}) \qquad 7$$

$$= YJ_o(kr)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o}2\sinh(\gamma_o d)$$

rewriting the terms of the equations in similar order we have:

$$A_2J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2})+FJ_o(k(r-r_o))\left[-\lambda_2\frac{\omega^2}{\alpha_2^2}+2\mu_2\gamma_3^2-\frac{2\gamma_3\gamma_4k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right]$$

$$= YJ_o(kr)2\sinh(\gamma_o d)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o} \tag{1}$$

$$A_2J_o(kr)(-\gamma_o)+FJ_o(k(r-r_o))\left[-\gamma_3+\frac{2\gamma_3k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}}\right.$$

$$= YJ_o(kr)\cosh(\gamma_o d) \tag{2}$$

$$DJ_o(kr_o)\exp(-\gamma_1 z)(-\lambda_1\frac{\omega^2}{\alpha_1^2})+D'\exp(-\gamma_1 z)\left[-\lambda_1\frac{\omega^2}{\alpha_1^2}+2\mu_1(-k^2+\frac{k}{2})\right]$$

$$+M'\exp(-\gamma_2 z)\left[2\mu_1\gamma_2(-k^2+\frac{k}{2})\right]$$

$$+F\left[\exp(-\gamma_3 z)(\lambda_2\frac{\omega^2}{\alpha_2^2}-2\mu_2(-k^2+\frac{k}{2}))\right. \tag{3}$$

$$\left.+\exp(-\gamma_4 z)(\frac{2\gamma_3}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}})(-2\mu_2\gamma_4(-k^2+\frac{k}{2}))\right] = 0$$

$$DJ_o(kr_o)\exp(-\gamma_1 z)(\frac{k}{\gamma_2}-\gamma_1)+D'\exp(-\gamma_1 z)(-\gamma_1)$$

$$+M'\exp(-\gamma_2 z)(k^2) \tag{4}$$

$$+F\left[\exp(-\gamma_3 z)(-\gamma_3)\exp(-\gamma_4 z)(\frac{2\gamma_3k^2}{2\gamma_4^2+\frac{\omega^2}{\beta_2^2}})\right] = 0$$

$$DJ_1(kr)(2k(\gamma_1-\gamma_2)-\frac{\omega^2}{\gamma_2\beta_1^2})+D'J_1(k(r_o-r))(-2k\gamma_1)$$

$$+M'J_1(k(r_o-r))(2k\gamma_2^2+\frac{\omega^2}{\beta_1^2} = 0 \qquad 5$$

$$A_1J_o(kr)(-\gamma_o)+DJ_o(kr)(\frac{k^2}{\gamma_2}-\gamma_1)+D'J_o(k(r_o-r))(-\gamma_1)$$

$$+M'J_o(k(r_o-r)) = -YJ_o(kr)2\cosh(\gamma_o d) \qquad 6$$

$$A_1J_o(kr)(\lambda_o\frac{\omega^2}{\alpha_o^2})+DJ_o(kr)(-\lambda_1\frac{\omega^2}{\alpha_1^2}-k^2+2\mu_1\gamma_1^2)$$

$$+D'J_o(k(r_o-r))(-\lambda_1\frac{\omega^2}{\alpha_1^2}+2\mu_1\gamma_2^2) \qquad 7$$

$$+M'J_o(k(r_o-r))(-\gamma_1k^2) = YJ_o(kr)\frac{\lambda_o\frac{\omega^2}{\alpha_o^2}}{\gamma_o}2\sinh(\gamma_o d)$$

Using lower case letters to denote the coefficients of the amplitudes we can write the previous equations in a more compact form:

$$A_2a_1 + Ff_1 = Y_1$$

$$A_2a_2 + Ff_2 = Y_2$$

$$Dd_3 + D'd_3' + M'm_3' + Ff_3 = 0$$

$$Dd_4 + D'd_4' + M'm_4' + Ff_4 = 0$$

$$Dd_5 + D'd_5' + M'm_5' = 0$$

$$A_1a_6 + Dd_6 + D'd_6' + M'm_6' = Y_6$$

$$A_1a_7 + Dd_7 + D'd_7' + M'm_7' = Y_7$$

Obviously the above system of equations is not homogeneous. The right side of some of the equations contains a term that is associated with the source. The method of solving these simultanous equations for the amplitudes specifically is by solving the nonhomogeneous matrix of amplitude coefficients, in other words an augmented matrix of coefficients. The matrix is written:

$$\begin{array}{ccccccc}
a_7 & d_7 & d_7' & m_7' & & & Y_7 \\
a_6 & d_6 & d_6' & m_6' & & & Y_6 \\
 & d_5 & d_5' & m_5' & & & \\
 & d_4 & d_4' & m_4' & f_4 & & \\
 & d_3 & d_3' & m_3' & f_3 & & \\
 & & & & f_2 & a_2 & Y_2 \\
 & & & & f_1 & a_1 & Y_1
\end{array}$$

At this point we can turn to numerical analysis for a method to solve the augmented matrix. Much work in this field of mathematics has gone into various ways to solve high order matrices with all sorts of manipulations to side step problems that arise from using high speed computers. There are two basic problems that bother these computers, 1) multiplying by infinity and 2) dividing by zero, or numbers very large or small respectively. Therefore, one must take care in choosing his method when using our handy machines. In the problem here we are trying to develope an algerbraic expression for the solution, thus we can use an extension of the old three equation--three unknown method from algerbra, Gaussian Elimination. One other factor is also helpful in our choice, the order of our matrix is only 7.

Elements of the matrix to the left and below the principle diagonal are eliminated by multiplying an above row by the appropriate constant and subtracting the two rows. An illustration is given below:
(Using the first three rows of the matrix for the illustration.)

$$\begin{matrix} a_7 & d_7 & d'_7 & m'_7 & & Y_7 \\ a_6 & d_6 & d'_6 & m'_6 & & Y_6 \\ & d_5 & d'_5 & m'_5 & & \end{matrix}$$

$$\begin{matrix} a_7 & d_7 & d'_7 & m'_7 & & Y_7 \\ & \hat{d}_6 & \hat{d}'_6 & \hat{m}'_6 & & \hat{Y}_6 \\ & d_5 & d'_5 & m'_5 & & \end{matrix}$$

with

$$\hat{d}_6 = d_6 - d_7 \; a^*_6$$

$$\hat{d}'_6 = d'_6 - d'_7 \; a^*_6$$

$$\hat{m}'_6 = m'_6 - m'_7 \; a^*_6$$

$$\hat{Y}_6 = Y_6 - Y_7 a_6^*$$

and $$a_6^* = \frac{a_6}{a_7}$$

then

$$\begin{matrix} a_7 & d_7 & d_7' & m_7' & Y_7 \\ & \hat{d}_6 & \hat{d}_6' & \hat{m}_6' & \hat{Y}_6 \\ & & \hat{d}_5' & \hat{m}_5' & \hat{Y}_5 \end{matrix}$$

with $$\hat{d}_5' = d_5' - \hat{d}_6' d_5^*$$

$$\hat{m}_5' = m_5' - \hat{m}_6' d_5^*$$

$$\hat{Y}_5 = -\hat{Y}_6 d_5'^*$$

and $$d_5^* = \frac{d_5}{\hat{d}_6}$$

With back substitution in the original triangularized matrix for a few amplitudes we have: (to illustrate the complex nature)

$$A_1 = \frac{Y}{\lambda_o \frac{\omega^2}{\alpha_o^2}} A_1$$

with

$$\mathcal{A}_1 = J_o(kr)\Big[\frac{\lambda_o \frac{\omega^2}{\alpha_o^2}}{\gamma_o} 2 \sinh(\gamma_o d) + 2\exp(-\gamma_o d)\big[\Delta d_7 + \Delta' d_7' + M' m_7'\big]\Big]$$

and

$$C_1 = \frac{Y}{\lambda_o \frac{\omega^2}{\alpha_o^2}} \mathcal{A}_1 - \frac{Y}{\gamma_o} \exp(\gamma_o d)$$

similarily

$$A_2 = \frac{Y}{\lambda_o \frac{\omega^2}{\alpha_o^2} - \gamma_o} \mathcal{A}_2$$

with

$$\mathcal{A}_2 = J_o(kr)\left[2\sinh(\gamma_o d)\frac{\lambda_o \frac{\omega^2}{\alpha_o^2}}{\gamma_o} - 2\cosh(\gamma_o d)\right.$$
$$\left. - J_o(kr)(-2\exp(-\gamma_o d))(d^* \frac{(f_1 - f_2)}{f_8})\right]$$