CERTAIN EXPANSIONS IN A COMPLEX VARIABLE

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CERTAIN EXPANSIONS IN A COMPLEX VARIABLE

by

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CERTAIN EXPANSIONS IN A COMPLEX VARIABLE

INTRODUCTION

In a paper published in 1931 Flora Streetman and L. R. Ford developed a polynomial expansion in a complex variable.* The series obtained by them provides a formula for the analytic continuation of a function, analytic at the origin, beyond the limits of the circle of convergence of the Maclaurin's series. They introduced the construction of major and minor circles for use in the definition of a region of convergence.

In this paper we obtain an expansion which provides a method for the analytic continuation of a function, having finite singularities, beyond the limits of the ring of convergence of the Laurent's series. We make use of the idea of major and minor circles in obtaining regions of convergence. Some new theorems are proved on the interrelation of these circles.

The nature of the expansion and its region of convergence is discussed and is illustrated for a particular function. Two generalizations of the development are considered; the first, expansion about any point in the plane, is quite obvious; the second is an extension to functions of two or more complex variables.

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* Flora Streetman and L. R. Ford, A Certain Polynomial Expansion, American Mathematical Monthly, Vol. 38 (1931), pp. 198 - 201.

1. Major and minor circles. For a given point z in the complex plane the major circle is designated by M_z and is defined as the circle with center at the point $-hz$ and with radius $(1+h)|z|$, where h is an arbitrary positive constant. Note that the point z lies on the circumference of the circle and that the origin is included within the circle for any choice of h.

The minor circle of z is designated by m_z and is defined as the circle with center $\frac{h}{1+2h}z$ and radius $\frac{1+h}{1+2h}|z|$, h being a positive constant. Again z lies on the minor circle, and the origin is in its interior for any finite choice for h.

An important theorem connecting major and minor circles is

Theorem I. If z_1 is outside, on, or inside the minor circle of z_2, then z_2 is respectively inside, on, or outside the major circle of z_1.

Let $z_1 = x + iy$, and $z_2 = m + in$.

We are given that

$$(1) \quad \left| z_1 - \frac{h}{1+2h} z_2 \right| \gtreqless \frac{1+h}{1+2h} |z_2| .$$

We wish to prove that

$$|z_2 + h z_1| \lesseqgtr (1+h)\,|z_1| .$$

Using the values for z_1 and z_2, we can write (1) in the form

$$\left| x + iy - \frac{h}{1+2h}(m+in) \right| \gtreqless \frac{1+h}{1+2h} |m + in| ,$$

or as

(2) $$(1+2h)^2x^2 - 2xhm(1+2h) + h^2m^2 + (1+2h)^2y^2 - 2yhn(1+2h) + h^2n^2 \gtreqless (1+2h)(m^2+n^2) + h^2(m^2+n^2) .$$

Dividing by $(1+2h)$ and adding $h^2(x^2+y^2)$ to both sides, we have from (2)

(3) $$(1+h)^2(x^2+y^2) \gtreqless (m+hx)^2 + (n+hy)^2 .$$

Taking the square root of both sides of (3), we obtain

(4) $$|z_2 + hz_1| \lesseqgtr (1+h)|z_1| ,$$

as was to be proved.

The following two theorems are stronger than theorem I and are useful in later work.

<u>Theorem II.</u> <u>If</u> z_1 <u>is outside the minor circle of</u> z_2 <u>a distance</u> η, <u>then</u> z_2 <u>is inside the major circle of</u> z_1 <u>a distance greater than or equal to</u> η.

We are given that

(5) $$\left| z_1 - \frac{h}{1+2h} z_2 \right| = \frac{1+h}{1+2h} |z_2| + \eta .$$

We wish to prove that

$$|z_2 + hz_1| \leq (1+h)|z_1| - \eta .$$

From (5) we can write

$$\sqrt{\left(x - \frac{h}{1+2h}m\right)^2 + \left(y - \frac{h}{1+2h}n\right)^2} = \frac{1+h}{1+2h}\sqrt{m^2+n^2} + \eta ,$$

which, squared, becomes

$$\left(x - \frac{h}{1+2h} m\right)^2 + \left(y - \frac{h}{1+2h} n\right)^2$$
$$= \left(\frac{1+h}{1+2h}\right)^2 (m^2+n^2) + 2\eta\left(\frac{1+h}{1+2h}\right)\sqrt{m^2+n^2} + \eta^2.$$

Using the same procedure as in theorem I we arrive at

$$(1+h)^2(x^2+y^2)$$
$$= (m+hx)^2 + (n+hy)^2 + 2\eta(1+h)\sqrt{m^2+n^2} + \eta^2(1+2h).$$

This can be written in the form

$$(6)\quad (m+hx)^2 + (n+hy)^2 = (1+h)^2(x^2+y^2) - 2\eta(1+h)\sqrt{x^2+y^2} + \eta^2$$
$$-2\eta(1+h)\sqrt{m^2+n^2} + 2\eta(1+h)\sqrt{x^2+y^2} - \eta^2(2+2h).$$

We now show that the quantity composed of the last three terms of (6) is negative. That is

$$-2\eta(1+h)\sqrt{m^2+n^2} + 2\eta(1+h)\sqrt{x^2+y^2} - 2\eta^2(1+h) \leq 0,$$

which amounts to showing that

$$|z_1| \leq |z_2| + \eta .$$

This is evident since we know

$$|z_1| \leq \frac{1+h}{1+2h}|z_2| + \eta + \frac{h}{1+2h}|z_2|,$$
$$= |z_2| + \eta .$$

It then follows from (6) that

$$|z_2 + hz_1|^2 \leq (1+h)^2|z_1|^2 - 2\eta(1+h)|z_1| + \eta^2 ,$$

which gives us upon taking the square root

$$|z_2 + hz_1| \leq (1+h)|z_1| - \eta ,$$

as was to be proved.

Theorem III. If z_2 is inside the major circle of z_1 a distance ξ, then z_1 is outside the minor circle of z_2 a distance greater than or equal to $\dfrac{\xi}{1+2h}$.

We are given

(7) $$|z_2 + hz_1| = (1+h)|z_1| - \xi .$$

We wish to prove

$$\left|z_1 - \frac{h}{1+2h} z_2\right| \geq \frac{1+h}{1+2h}|z_2| + \frac{\xi}{1+2h} .$$

From equation (7) we can write

(8) $$(m+hx)^2 + (n+hy)^2$$
$$= (1+h)^2(x^2+y^2) - 2\xi(1+h)\sqrt{x^2+y^2} + \xi^2 .$$

Expanding (8), subtracting $h^2(x^2+y^2)$ from each side, multiplying through by $(1+2h)$, and adding to each side $h^2(m^2+n^2)$, we obtain

$$\big((1+2h)x - hm\big)^2 + \big((1+2h)y - hn\big)^2 - 2\xi(1+h)(1+2h)\sqrt{x^2+y^2}$$
$$+ \xi^2(1+2h)^2 = (m^2+n^2)(1+h)^2 .$$

Dividing by $(1+2h)^2$ and transposing, we have

(9) $$\left(x - \frac{h}{1+2h}m\right)^2 + \left(y - \frac{h}{1+2h}n\right)^2$$
$$= \left(\frac{1+h}{1+2h}\right)^2(m^2+n^2) + \frac{2\xi(1+h)}{(1+2h)}\sqrt{x^2+y^2} - \frac{\xi^2}{(1+2h)} .$$

Equation (9) can be written in the form

(10) $$\left|z_1 - \frac{h}{1+2h}z_2\right|^2 = \left(\frac{1+h}{1+2h}\right)^2|z_2|^2 + \frac{2\xi(1+h)}{(1+2h)^2}|z_2| + \frac{\xi^2}{(1+2h)^2}$$
$$+ 2\xi\frac{1+h}{1+2h}|z_1| - \frac{2\xi(1+h)}{(1+2h)^2}|z_2| - \frac{2\xi^2(1+h)}{(1+2h)^2} .$$

We now show that the quantity composed of the last three terms of (10) is positive. That is

$$\frac{2\xi(1+h)}{(1+2h)}|z_1| - \frac{2\xi(1+h)}{(1+2h)^2}|z_2| - \frac{2\xi^2(1+h)}{(1+2h)^2} \geq 0 ,$$

which means

(11) $$(1+h)|z_1| + h|z_1| - |z_2| - \xi \geq 0 .$$

Since by hypothesis

$$(1+h)|z_1| = |z_2 + hz_1| + \xi > |z_2| + \xi ,$$

upon substituting in (11) we have the desired result

$$\cancel{|z_2|} + \cancel{\xi} + h|z_1| - \cancel{|z_2|} - \cancel{\xi} \geq 0 .$$

It is now evident from equation (10) that

$$\left|z_1 - \frac{h}{1+2h}z_2\right| \geq \frac{1+h}{1+2h}|z_2| + \frac{\xi}{1+2h} ,$$

as was to be proved.

One more theorem which will be useful is

Theorem IV. If $|z_1| > |z_2|$, then M_{z_1} contains m_{z_2} completely.

We designate by t_2 the points on m_{z_2}.

It then follows that

$$\left| t_2 - \frac{h}{1+2h} z_2 \right| = \frac{1+h}{1+2h} |z_2| \geq |t_2| - \frac{h}{1+2h} |z_2| \,,$$

which allows us to write

$$(12) \quad |t_2| \leq \frac{1+h}{1+2h} |z_2| + \frac{h}{1+2h} |z_2| = |z_2| \,.$$

Designate by t_1 the points on M_{z_1}.

From the definition of M_{z_1} we have

$$|t_1 + h z_1| = (1+h) |z_1| \leq |t_1| + h |z_1| \,,$$

which gives

$$(13) \quad |t_1| \geq (1+h) |z_1| - h |z_1| = |z_1| \,.$$

Therefore from (12) and (13) it follows that

$$|t_2| \leq |z_2| < |z_1| \leq |t_1| \,,$$

and

$$|t_2| < |t_1| \,,$$

as was to be proved.

2. <u>The expansion for a function analytic in a region.</u> We shall now consider any function, $f(z)$, which is single-valued and analytic in and on the contour of a region S. By analytic is meant that the function is defined and has derivatives at each point. The region S is such that it is bounded by two regular curves about the origin, C_1 and C_2, where every point on C_1 is less in absolute value than any point on C_2.

Construct the major circles for all points t_2 on C_2 and the minor circles for all points t_1 on C_1. Since $|t_1| < |t_2|$ for all t_1 and t_2, then by theorem IV all the minor circles, m_{t_1}, are completely inside all the major circles, M_{t_2}. Therefore there exists a region, R, bounded on the outside by the major circles, M_{t_2}, and on the inside by the minor circles, m_{t_1}, such that R encircles the origin. Also R is contained in S.

For any z in R we have by Cauchy's integral formula

$$(14) \quad f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(t-z)}\,dt - \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t-z)}\,dt,$$

$$= \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(t-z)}\,dt + \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(z-t)}\,dt.$$

For the first integral of (14) we use the expansion

$$(15) \quad \frac{1}{t-z} = \frac{1}{(1+h)t - (z+ht)},$$

$$= \frac{1}{t+ht}\left\{1 + \frac{z+ht}{t+ht} + \left(\frac{z+ht}{t+ht}\right)^2 + \cdots\cdots + \left(\frac{z+ht}{t+ht}\right)^2 + \cdots\cdots\right\}.$$

Since z is in R it is inside the major circle of every t on C_2. The distance of z from the center of the major circle of any t on C_2 is at least some value, ξ, less than the radius of the major circle. We have, then

$$|z+ht| \leq (1+h)|t| - \xi,$$

or

$$(16) \quad \left|\frac{z+ht}{t+ht}\right| \leq 1 - \frac{\xi}{(1+h)|t|} = r_1 < 1.$$

Relation (16) holds for all values of t on C_2.

It follows that each term in expansion (15) is less than or equal to the corresponding term in the convergent constant term series $\displaystyle\sum_{n=0}^{\infty} \frac{r_1^{\,n}}{(1+h)|t'|}$,

where $|t'|$ is the maximum value of $|t|$ on C_2. By the Weierstrass

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* E. J. Townsend, Functions of a Complex Variable, p. 75. (Hereafter referred to as Townsend.)

test expansion (15) is uniformly convergent in t on C_2 for a chosen z in R.* Since $f(t)$ is bounded for t on C, the uniform convergence is not affected if the series is multiplied by $f(t)$. We can then integrate term by term and obtain

$$(17)\quad \int_{C_2} \frac{f(t)}{(t-z)}\,dt = \sum_{n=0}^{\infty} \int_{C_2} \frac{(z+ht)^n}{(t+ht)^{n+1}} f(t)\,dt\ ,$$

For the second integral of (14) we use the expansion

$$(18)\quad \frac{1}{(z-t)} = \frac{1}{(1+h)z-(t+hz)}\ ,$$

$$= \frac{1}{z+hz}\left\{1+\left(\frac{t+hz}{z+hz}\right)+\left(\frac{t+hz}{z+hz}\right)^2+\cdots\left(\frac{t+hz}{z+hz}\right)^n+\cdots\right\}.$$

Since z is in R it is outside the minor circle of every t on C_1 by at least some distance η. Then by theorem II any t on C_1 is inside the major circle of z by at least η. This means

$$|t+hz| + \eta \leq |z+hz|\ ,$$

or

$$(19)\quad \left|\frac{t+hz}{z+hz}\right| \leq 1-\frac{\eta}{(1+h)|z|} = r_2 < 1\ .$$

Relation (19) holds for all values of t on C_1.

Again by the Weierstrass test, series (18) is uniformly convergent for t on C_1 and any chosen z in R. Multiplying (18) by $f(t)$ which is bounded on C_1 and integrating the series term by term, we have

$$(20)\quad \int_{C_1} \frac{f(t)}{z-t}\,dt = \sum_{n=0}^{\infty} \int_{C_1} \frac{(t+hz)^n}{(z+hz)^{n+1}} f(t)\,dt\ ,$$

* Townsend, p. 220.

Making use of series (17) and (20), we can now write equation (14) in the form

$$(21)\quad f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{C_2} \frac{(z+ht)^n}{(t+ht)^{n+1}} f(t)\,dt + \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{C_1} \frac{(t+hz)^n}{(z+hz)^{n+1}} f(t)\,dt \,.$$

Since $f(z)$ is analytic throughout the region S, the contours of integration, C_1 and C_2, may be deformed into any regular curve, C, lying completely in S and encircling the origin. For any z in R we can write

$$(22)\quad f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{C} \frac{(z+ht)^n}{(t+ht)^{n+1}} f(t)\,dt + \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{C} \frac{(t+hz)^n}{(z+hz)^{n+1}} f(t)\,dt \,.$$

3. <u>The expansion for functions with singularities.</u> We now consider any function, $F(z)$, having a denumerable number of finite singularities. It is desired to apply expansion (22) to such a function in a properly defined region.

Select any one of the singular points which is not a limit point of singularities of greater absolute value. Let us call this point α. For all points, $\ldots, \gamma, \beta, \alpha$, which are less than or equal to α in magnitude, construct the minor circles, $\ldots, m_\gamma, m_\beta, m_\alpha$. For the remaining finite singular points, a, b, c, $\ldots$, construct the major circles, M_a, M_b, M_c, - - - -, Since we have established that $- - - -, |\gamma|, |\beta|, |\alpha| < |a|, |b|, |c|, - - - -$, then by theorem IV $- - - -, m_\gamma, m_\beta, m_\alpha$ are all completely inside M_a, M_b, M_c, - - - -. Therefore, there exists a domain, D, bounded by the minor circles on the inside and by the major circles on the outside in such a way that D encircles the origin. Figure 1, page 12, illustrates such a region.

By the nature of the definition of the region, D, we see that $F(z)$ is analytic in D but not on the boundary. If expansion (22) is to hold for any z in D there must exist a region, σ, containing z and encircling the origin such that $F(z)$ is analytic in and on the boundary of σ and such that z is outside the minor circles of all points of the inner boundary of σ and inside the major circles of all points of the outer boundary of σ. Figure 2, page 13, displays the constructions used in the definition of this region σ.

Pick any point z in the domain D. Construct the major circle, M_z, of z. Since z is outside the minor circles, $- - - -, m_\gamma, m_\beta, m_\alpha,$ by at least some distance η, then by theorem II the points, $- - - -, \gamma, \beta, \alpha$, are all inside M_z by at least η. However, M_z may cut some or all of the circles, $- - - -, m_\gamma, m_\beta, m_\alpha$.

There exist circles, $- - - -, \overline{m_\gamma}, \overline{m_\beta}, \overline{m_\alpha}$, concentric with $- - - -, m_\gamma, m_\beta, m_\alpha$ respectively with slightly larger radii which do not include the point z. Circles with centers

$$- - - -, \frac{h}{1+2h}\gamma, \frac{h}{1+2h}\beta, \frac{h}{1+2h}\alpha,$$

and with radii

$$- - - -, \frac{1+h}{1+2h}|\gamma| + \frac{\eta}{2}, \frac{1+h}{1+2h}|\beta| + \frac{\eta}{2}, \frac{1+h}{1+2h}|\alpha| + \frac{\eta}{2},$$

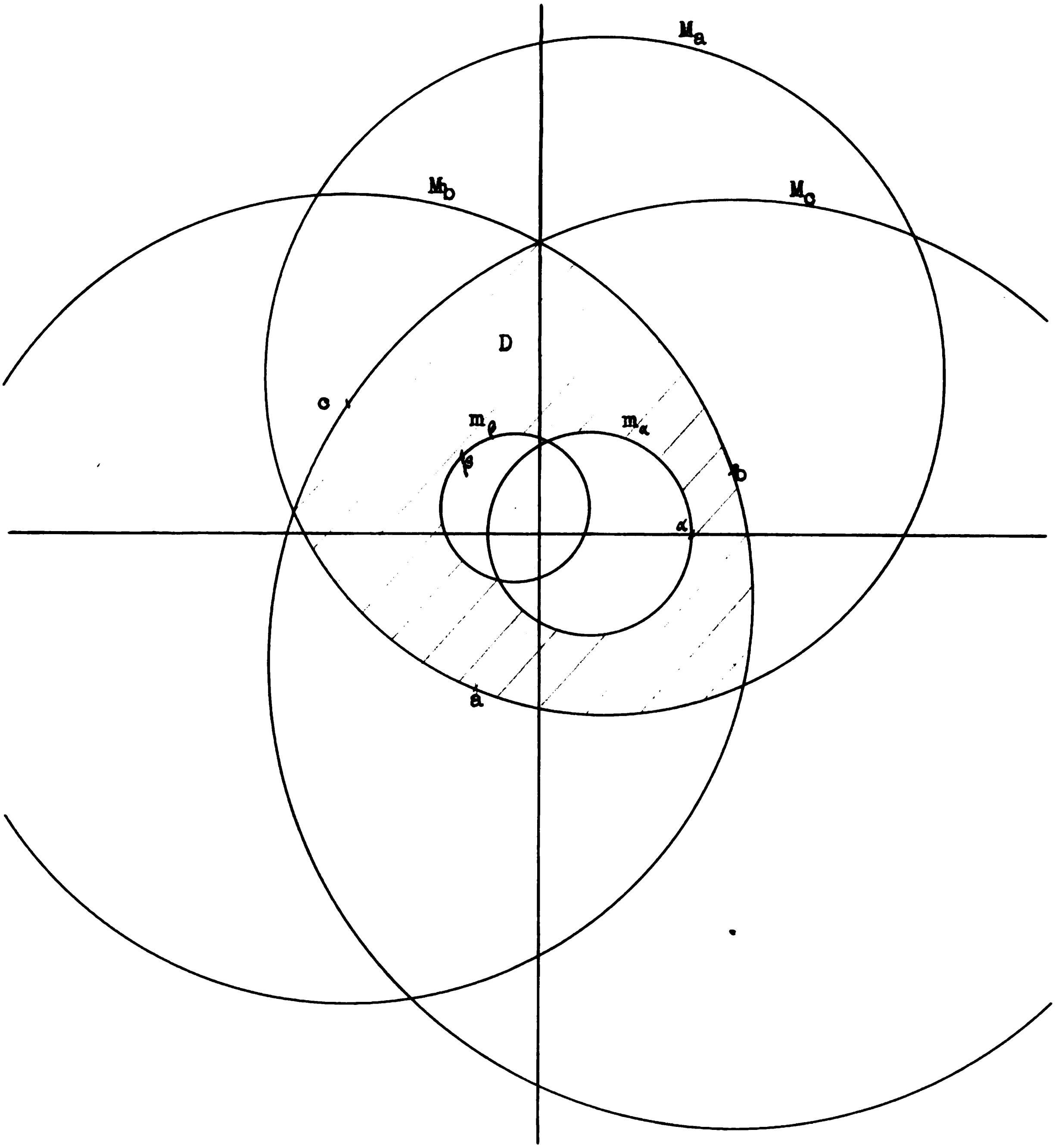

Figure 1. The domain, D ; singular points, ρ, α , a, b, c; h = 1 .

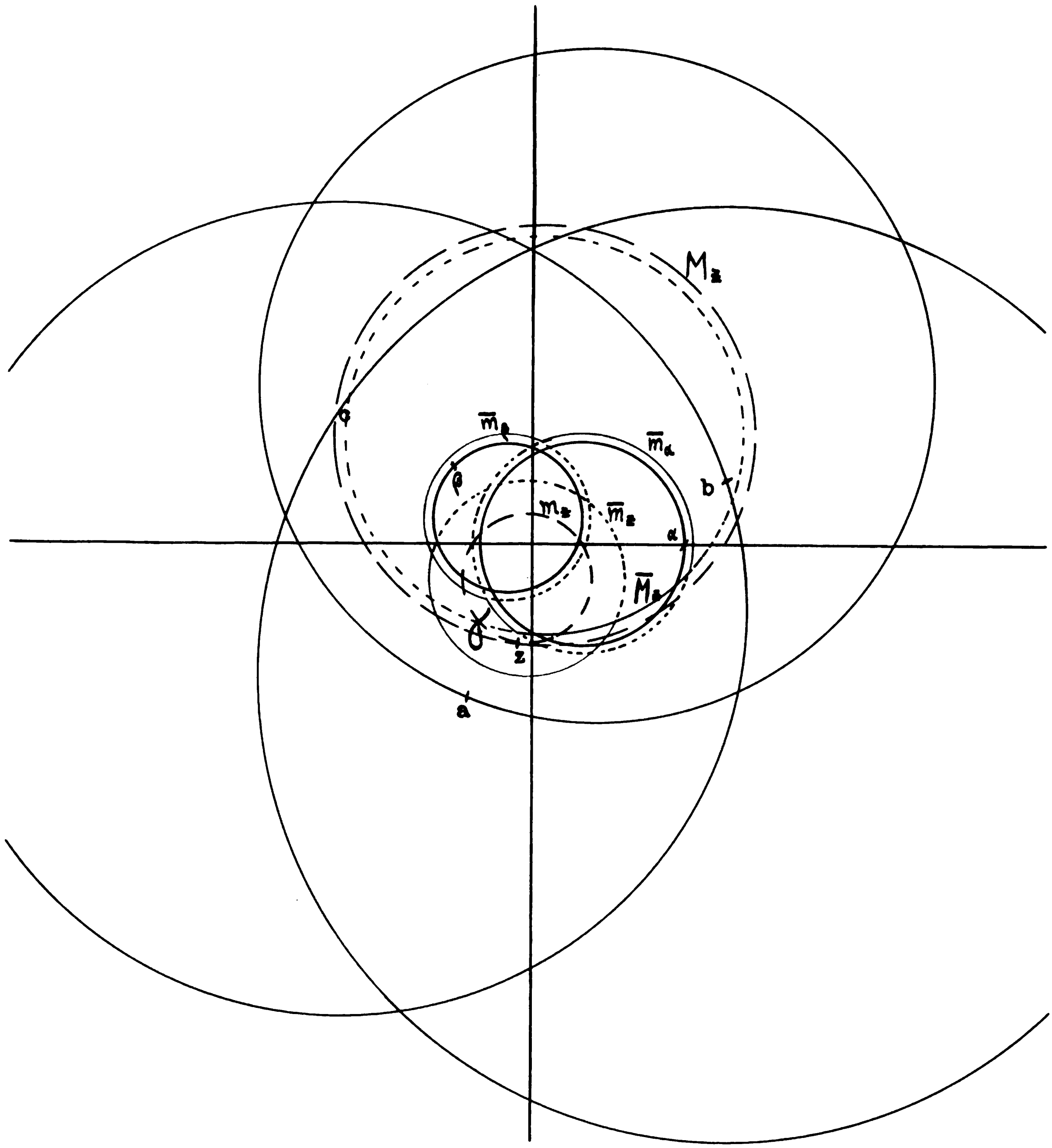

Figure 2. Definition of the region, σ.

respectively have this property. Likewise there is a circle, $\overline{M_z}$, concentric with M_z but of slightly smaller radius which still includes $----, \gamma, \beta, \alpha$ in its interior. A circle with its center at $-hz$ and with radius $(1+h)|z| - \frac{\eta}{2}$ possesses this property.

The arcs of the circumferences of the circles, $----, \overline{m_\gamma}, \overline{m_\beta}, \overline{m_\alpha}$, exterior to each other plus the part of $\overline{M_z}$ intersecting these circles form a regular curve, C_1, which is inside M_z. There are no singular points on C_1. It follows that $F(z)$ is analytic on C_1, and by theorem III, that z is outside the minor circle of every point of C_1 by at least $\frac{\eta}{2(1+2h)}$.

Next construct the minor circle, m_z, of z. Since z is in R, it is inside the major circles, $M_a, M_b, M_c, ----$, by at least some distance ξ. Then by theorem III the singular points, $a, b, c, ----$, are outside m_z by at least $\frac{\xi}{(1+2h)}$.

There exists a circle, $\overline{m}_z$, concentric with m_z but of slightly larger radius which does not include $a, b, c, ----$. A circle with center $\frac{h}{1+2h} z$ and radius $\frac{1+h}{1+2h}|z| + \frac{\xi}{2(1+2h)}$ possesses this property.

Denote by C_2 the curve formed by the arc of $\overline{m}_z$ exterior to C_1 plus the part of C_1 exterior to $\overline{m}_z$. Then C_2 is completely outside the minor circle of z and has no singular points on it. Therefore, $F(z)$ is analytic along C_2, and z is inside the major circle of every point of C_2 by at least $\frac{\xi}{2(1+2h)}$.

The region, σ bounded by C_1 and C_2 and including the common part of C_1 and C_2 encircling the origin, satisfies the condition fulfilled by the region, S, used in developing equation (22). The point z fulfils the conditions for region R used previously since it is outside the minor circles of C_1 and inside the major circles of C_2. However, since z was chosen as any point in the domain, D, the conditions necessary for the use of expansion (22) are satisfied for any point in D.

We can now write for any z in D and any regular curve C in D around the origin

$$(23)\quad F(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{(z+ht)^n}{(t+ht)^{n+1}} F(t)\,dt$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{(t+hz)^n}{(z+hz)^{n+1}} F(t)\,dt \,.$$

4. <u>Discussion of the expansion.</u> We denote the two series in the expansion of $F(z)$ by

$$(24)\quad A = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{(z+ht)^n}{(t+ht)^{n+1}} F(t)\,dt,$$

and

$$(25)\quad B = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{(t+hz)^n}{(z+hz)^{n+1}} F(t)\,dt,$$

so that

$$F(z) = A + B$$

In the general term of series A we make the substitution

$$\phi(t) = (z+ht)^n \; F(t) \; .$$

We have then for the general term

$$(26) \quad \frac{1}{(1+h)^{n+1}} \; \frac{1}{2\pi i} \int_C \frac{\phi(t)}{t^{n+1}} \, dt \; .$$

Since from Cauchy's integral formula

$$\phi^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{\phi(t)}{(t-\alpha)} \, dt ,$$

we can write (26) in the form

$$\frac{1}{n!\,(1+h)^{n+1}} \; \phi^{(n)}(0) = \frac{1}{n!\,(1+h)^{n+1}} \left\{ \frac{d^n}{dt} \; (z+ht)^n F(t) \right\}_{t=0} .$$

Thus it is evident that each term of series A can be written as a polynomial in z .

In considering series B , we write the general term as

$$\frac{1}{(z+hz)^{n+1}} \; \frac{1}{2\pi i} \int_C (t+hz)^n \; F(t) \; dt \; .$$

Examination of the integrand shows that z occurs only with integral powers from the binomial, $(t+hz)^n$, the highest power being z^n. Results of integration by residues or otherwise will then give a polynomial in z which, combined with the coefficient $\frac{1}{(z+hz)^{n+1}}$ gives for the general term a rational fractional function of z .

It is of particular interest to note the effect upon series (23) and upon the region of convergence of the choice of different values for h .

First we consider the case for $h = 0$. Substitution of this value in equation (2 3) gives us

$$
\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{z^n}{t^{n+1}} F(t)\, dt \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{t^n}{z^{n+1}} F(t)\, dt , \\
&= \sum_{n=0}^{\infty} \frac{z^n}{2\pi i}\int_C \frac{1}{t^{n+1}} F(t)\, dt \\
&\quad + \sum_{n=-1}^{-\infty} \frac{z^n}{2\pi i}\int_C \frac{1}{t^{n+1}} F(t)\, dt , \\
&= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2\pi i}\int_C \frac{1}{t^{n+1}} F(t)\, dt\right) z^n = \sum_{n=-\infty}^{+\infty} \alpha_n z^n .
\end{aligned}
$$

The final form of this two-way series is identified as the regular Laurent's expansion of $F(z)$ about the origin.* Notice also that the region of convergence is now defined as the ring of convergence of the Laurent's series. It is bounded on the inside by the circle with center at the origin and radius $|\alpha|$. It is bounded on the outside by the circle with center at the origin and radius the minimum of $|a|, |b|, |c|$, - - - -. This is evident since the minor circle of α with center at $\frac{h}{1+2h}\alpha$ and radius $\frac{1+h}{1+2h}|\alpha|$ becomes, for $h = 0$, the circle with center at the origin, radius $|\alpha|$. Likewise the major circle of a, having center $-ha$ and radius $(1+h)|a|$, becomes the circle with center at the origin and radius $|a|$.

* Townsend, p. 278.

Next we observe the effect of assigning large values to h. The region of convergence, D, is increased in size with larger h. This is true since the minor circles, - - - -, $m_\gamma, m_\beta, m_\alpha$, become smaller and the major circles, M_a, M_b, M_c, - - - -, become larger. Consider, for example, the minor circle, m_α. Since

$$\lim_{h=\infty} \frac{h}{1+2h}\,\alpha \quad = \frac{1}{2}\,\alpha \,,$$

and

$$\lim_{h=\infty} \frac{1+h}{1+2h}\,|\alpha| \quad = \frac{1}{2}\,|\alpha| \,,$$

then m_α approaches as a limiting position for increasing h the circle with center $\frac{1}{2}\alpha$ and radius $\frac{1}{2}|\alpha|$. Likewise for the major circle, M_a, of a, we have

$$\lim_{h=\infty} (-\,h\,a) \quad = \infty \quad ,$$

and

$$\lim_{h=\infty} (1+h)\,|a| \quad = \infty \quad ,$$

This shows that the major circles increase in size without limit for increasing h. Thus, in its limiting form, the domain, D, has an inner boundary composed of the circles through the origin and the singular points, - - - -, γ, β, α. Its outer boundary is composed of straight lines through the singular points, a, b, c, - - - -, perpendicular to the radius vector of that point. Figure 3, page 19, illustrates this limiting position for D in comparison with the Laurent's ring.

To study the effect of different choices for h on the expansion itself, we notice that in series (14) and (18) on which (23) depends, the ratios determining the convergence depend directly on h.

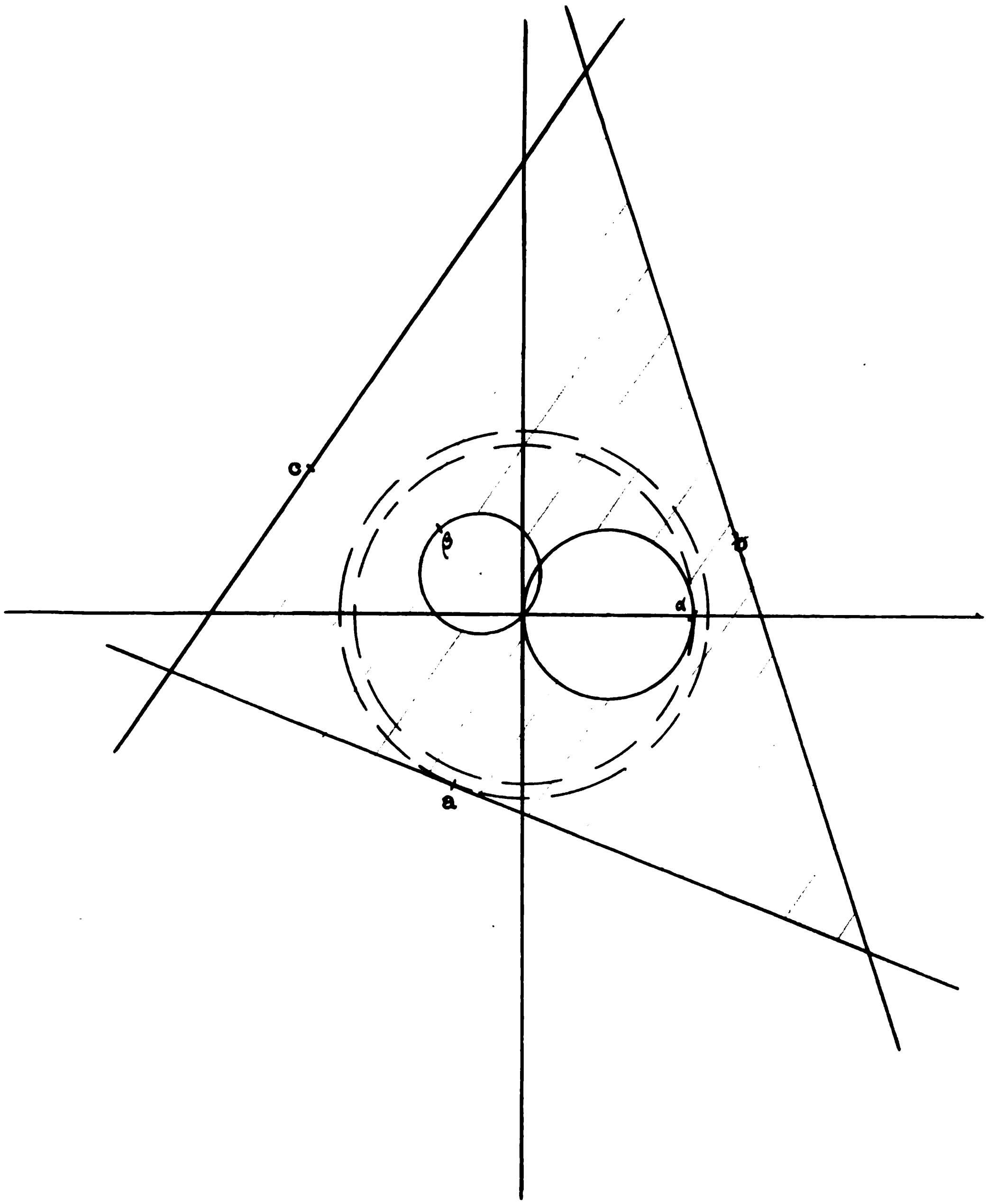

Figure 3. Limiting position of D , and the Laurent's ring.

In (14) we note that

$$r_1 = 1 - \frac{\xi}{(1+h)|t'|} ,$$

and in (18) that

$$r_2 = 1 - \frac{\eta}{(1+h)|z|} .$$

It is evident that the two series converge most rapidly for $h = 0$ and less rapidly as h is increased in value. Here we see that the convergence of the Laurent's series, where $h = 0$, is more rapid than the convergence of (23) for any other choice for h. The advantage of using (23) lies in the fact that the region of convergence is enlarged for larger h, and we have a means of analytic continuation of the function into a region beyond the limits of the Laurent's ring.

5. <u>An application.</u> Let us consider the function,

$$F(z) = \frac{1}{(z-1)(z-2)} ,$$

having singularities at $z = 1$ and $z = 2$. We shall write the expansion for the function for any z in the region bounded by the minor circle of $z = 1$ and the major circle of $z = 2$. Figure 4, page 21, displays such a region for $h = \frac{1}{2}$.

From equation (23) the expansion in general form is

$$(27) \quad F(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(z+ht)^n \, dt}{(1+h)^{n+1} t^{n+1} (t-1)(t-2)}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(t+hz)^n \, dt}{(z+hz)^{n+1} (t-1)(t-2)} .$$

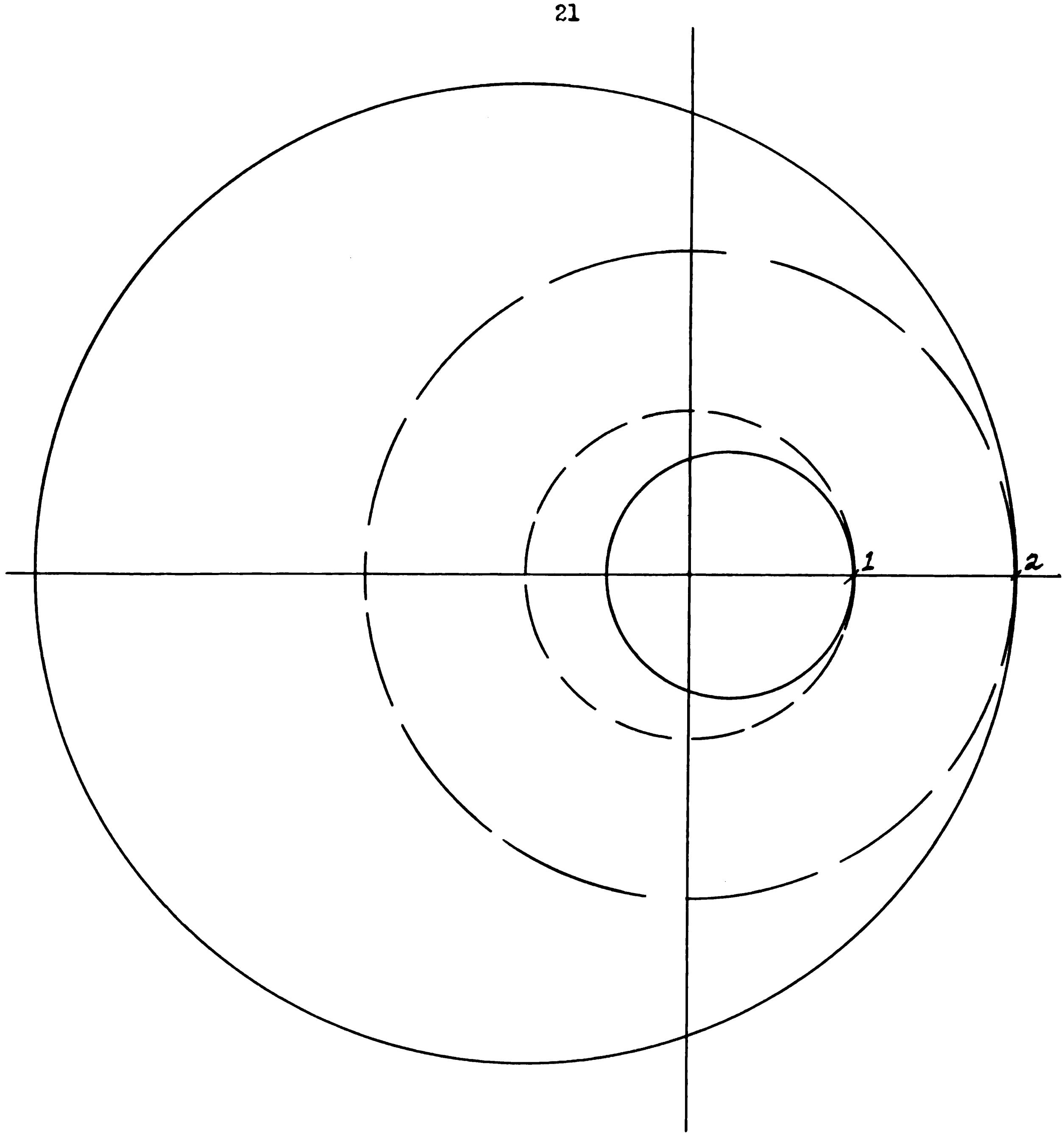

Figure 4. Region of convergence for $F(z) = \frac{1}{(z-1)(z-2)}$, h = $\frac{1}{2}$, and the Laurent's ring.

Consider the general term of the first summation which is

(28) $$\frac{1}{2\pi i}\int_C \frac{(z+ht)^n \quad dt}{(1+h)^{n+1}\, t^{n+1}\,(t-1)\,(t-2)} \quad .$$

To evaluate this integral we make use of the theory of residues.* The integrand function of (28) has a pole of order one at $t = 1$, and a pole of order $(n+1)$ at $t = 0$. The value of the term will equal the sum of the residues at these two points.

The residue at $t = 1$ can be obtained as the coefficient of the term involving $\frac{1}{t-1}$ in the Laurent's expansion about the point, $t = 1$, of the integrand function,

$$\frac{(z+ht)^n}{(1+h)^{n+1}\, t^{n+1}(t-1)(t-2)} \quad .$$

Write the function in the form

$$\frac{1}{(1+h)^{n+1}} \times \frac{1}{(t-1)} \times R(t) \;,$$

where

$$R(t) = \frac{(z+ht)^n}{t^{n+1}\,(t-2)} \quad .$$

Expanding $R(t)$ in powers of $(t-1)$ in a Taylor's series, we find the constant term is

$$R(1) = -(z+h)^n \quad .$$

Thus, the coefficient of $\frac{1}{(t-1)}$ for the integrand function and, therefore, the residue at $t = 1$ becomes

(29) $$\frac{-(z+h)^n}{(1+h)^{n+1}} \quad .$$

---

* Townsend, pp. 284 - 289.

The residue at $t = 0$ can be obtained as the coefficient of the term involving $\frac{1}{t}$ in the Laurent's expansion of the integrand function about the origin. Write the function in the form

$$\frac{1}{(1+h)^{n+1}} \times \frac{1}{t^{n+1}} \times Q(t) ,$$

where

$$Q(t) = \frac{(z+ht)^n}{2-3t+t^2} .$$

We need the coefficient of the term t^n in the expansion of $Q(t)$. To determine this coefficient we write

$$(z+ht)^n = (2-3t+t^2)(A_0 + A_1 t + \cdots + A_n t^n + \cdots).$$

Equating coefficients of like powers of t, we obtain

$$\begin{aligned}
2A_0 &= z^n , \\
2A_1 - 3A_0 &= \binom{n}{1} z^{n-1} h , \\
2A_2 - 3A_1 + A_0 &= \binom{n}{2} z^{n-2} h^2 , \\
2A_3 - 3A_2 + A_1 &= \binom{n}{3} z^{n-3} h^3 , \qquad (30) \\
&\vdots \\
2A_{n-1} - 3A_{n-2} + A_{n-3} &= \binom{n}{n-1} z h^{n-1} , \\
2A_n - 3A_{n-1} + A_{n-2} &= h^n , \\
2A_{n+1} - 3A_n + A_{n-1} &= 0 , \\
&\vdots
\end{aligned}$$

We wish to find A_n , the coefficient of t^n . From (30) we find

$$A_0 = \frac{z^n}{2} ,$$

$$A_1 = \frac{\binom{n}{1} z^{n-1} h}{2} + \frac{3}{2}\left(\frac{z^n}{2}\right) ,$$

$$A_2 = \frac{\binom{n}{2} z^{n-2} h^2}{2} + \frac{3}{2} A_1 - \frac{1}{2} A_0 ,$$

$$= \frac{\binom{n}{2} z^{n-2} h^2}{2} + \frac{3}{4}\binom{n}{1} z^{n-1} h + \frac{7}{8} z^n ,$$

$$A_3 = \frac{\binom{n}{3} z^{n-3} h^3}{2} + \frac{3}{2} A_2 - \frac{1}{2} A_1 ,$$

$$= \frac{\binom{n}{3} z^{n-3} h^3}{2} + \frac{3}{4}\binom{n}{2} z^{n-2} h^2 + \frac{7}{8}\binom{n}{1} z^{n-1} h + \frac{15}{16} z^n .$$

Assume that the coefficient A_k , $k \leq n$, is given by

$$A_k = \frac{\binom{n}{k} z^{n-k} h^k}{2} + \frac{3}{4}\binom{n}{k-1} z^{n-k+1} h^{k-1}$$

$$+ \frac{7}{8}\binom{n}{k-2} z^{n-k+2} h^{k-2} + ----$$

$$+ \frac{2^k - 1}{2^k}\binom{n}{1} z^{n-1} h + \frac{2^{k+1}-1}{2^{k+1}} z^n .$$

We know from (30) the relationship

$$(31) \quad A_{k+1} = \frac{\binom{n}{k+1} z^{n-k-1} h^{k+1}}{2} + \frac{3}{2} A_k - \frac{1}{2} A_{k-1} .$$

By substitution in (31) for A_k and A_{k+1}, we obtain

$$A_{k+1} = \frac{\binom{n}{k+1} z^{n-k+1} h^{k+1}}{2} + \frac{3}{4}\binom{n}{k} z^{n-k} h^{k}$$

$$+ \frac{9}{8}\binom{n}{k-1} z^{n-k+1} h^{k-1} + \frac{21}{16}\binom{n}{k-2} z^{n-k+2} h^{k-2} + \text{----}$$

$$+ \frac{3}{2} \times \frac{2^{k}-1}{2^{k}} \binom{n}{1} z^{n-1} h + \frac{3}{2} \times \frac{2^{k+1}-1}{2^{k+1}} z^{n}$$

$$- \frac{1}{4}\binom{n}{k-1} z^{n-k+1} h^{k-1} - \frac{3}{8}\binom{n}{k-2} z^{n-k+2} h^{k-2} \text{----}$$

$$- \frac{1}{2} \times \frac{2^{k-1}-1}{2^{k-1}} \binom{n}{1} z^{n-1} h - \frac{1}{2} \times \frac{2^{k}-1}{2^{k}} z^{n} .$$

Collecting terms, we have

$$A_{k+1} = \frac{\binom{n}{k+1} z^{n-k+1} h^{k+1}}{2} + \frac{3}{4}\binom{n}{k} z^{n-k} h^{k} \tag{32}$$

$$+ \frac{7}{8}\binom{n}{k-1} z^{n-k+1} h^{k-1} + \frac{15}{16}\binom{n}{k-2} z^{n-k+2} h^{k-2} + \text{----}$$

$$+ \left\{\frac{3(2^{k}-1)}{2^{k+1}} - \frac{2^{k-1}-1}{2^{k}}\right\} \binom{n}{1} z^{n} h$$

$$+ \left\{\frac{3(2^{k+1}-1)}{2^{k+2}} - \frac{2^{k}-1}{2^{k+1}}\right\} z^{n} .$$

However, in the last two terms of (32) we see that

$$\left\{\frac{3\,(2^{k}-1)}{2^{k+1}} \quad - \quad \frac{2^{k-1}-1}{2^{k}}\right\}$$

$$= \frac{(2^{2}-1)(2^{k}-1) \;-\; 2^{k}+2}{2^{k+1}}$$

$$= \frac{2(2^{k+1}-1) \;-\; (2^{k+1}-1)}{2^{k+1}}$$

$$= \frac{2^{k+1}-1}{2^{k+1}} \qquad ,$$

and that

$$\left\{\frac{3\,(2^{k+1}-1)}{2^{k+2}} \;-\; \frac{2^{k}-1}{2^{k+1}}\right\} \;=\; \frac{2^{k+2}-1}{2^{k+2}} \qquad .$$

Then we can write

$$A_{k+1} = \frac{\binom{n}{k+1} z^{n-k-1} h^{k+1}}{2} + \frac{3}{4}\binom{n}{k} z^{n-k} h^{k} + \cdots$$

$$+ \frac{2^{k+1}-1}{2^{k+1}}\binom{n}{1} z^{n-1} h + \frac{2^{k+2}-1}{2^{k+2}}\, z^{n} \qquad ,$$

which in summation notation becomes

$$A_{k+1} = \sum_{i=0}^{k+1} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} \qquad .$$

Since this derived value for A_{k+1} agrees with the assumed value for A_k , $k \leq n$, the proof by mathematical induction is complete and we have

$$A_{n} = \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} \qquad .$$

This gives us the value of the residue at $t = 0$ which can be written as

(33) $$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} \quad .$$

The sum of the residues at $t = 0$ and $t = 1$ is, then, from (29) and (33)

$$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \frac{(z+h)^n}{(1+h)^{n+1}}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \right\}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \left(\frac{2^{n-i+1}-1-2^{n-i+1}}{2^{n-i+1}} \right) \right\}$$

$$= \frac{1}{2(1+h)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{z}{2}\right)^{n-i} h^{i}$$

$$= \frac{\left(\frac{z}{2}+h\right)^n}{2(1+h)^{n+1}} = -\frac{(z+2h)^n}{(2+2h)^{n+1}} \quad .$$

From this it follows for the first summation of (27) that

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{(z+ht)^n \, dt}{(1+h)^{n+1} \, t^{n+1} (t-1)(t-2)} = -\sum_{n=0}^{\infty} \frac{(z+2h)^n}{(2+2h)^{n+1}} \quad .$$

For the second summation in (27), consider the general integral term,

$$\frac{1}{2\pi i} \int_C \frac{(t+hz)^n \, dt}{(z+hz)^{n+1} (t-1)(t-2)} \quad .$$

This gives us the value of the residue at $t = 0$ which can be written as

(33) $$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} \; .$$

The sum of the residues at $t = 0$ and $t = 1$ is, then, from (29) and (33)

$$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \frac{(z+h)^{n}}{(1+h)^{n+1}}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \right\}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \left(\frac{2^{n-i+1}-1-2^{n-i+1}}{2^{n-i+1}} \right) \right\}$$

$$= \frac{1}{2(1+h)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{z}{2}\right)^{n-i} h^{i}$$

$$= \frac{\left(\frac{z}{2}+h\right)^{n}}{2(1+h)^{n+1}} = -\frac{(z+2h)^{n}}{(2+2h)^{n+1}} \; .$$

From this it follows for the first summation of (27) that

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C} \frac{(z+ht)^{n} \; dt}{(1+h)^{n+1} \, t^{n+1} (t-1)(t-2)} = -\sum_{n=0}^{\infty} \frac{(z+2h)^{n}}{(2+2h)^{n+1}} \; .$$

For the second summation in (27), consider the general integral term,

$$\frac{1}{2\pi i} \int_{C} \frac{(t+hz)^{n} \; dt}{(z+hz)^{n+1} (t-1)(t-2)} \; .$$

This gives us the value of the residue at $t = 0$ which can be written as

(33) $$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} .$$

The sum of the residues at $t = 0$ and $t = 1$ is, then, from (29) and (33)

$$\frac{1}{(1+h)^{n+1}} \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \frac{(z+h)^{n}}{(1+h)^{n+1}}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \frac{2^{n-i+1}-1}{2^{n-i+1}} \binom{n}{i} z^{n-i} h^{i} - \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \right\}$$

$$= \frac{1}{(1+h)^{n+1}} \left\{ \sum_{i=0}^{n} \binom{n}{i} z^{n-i} h^{i} \left(\frac{2^{n-i+1}-1-2^{n-i+1}}{2^{n-i+1}} \right) \right\}$$

$$= \frac{1}{2(1+h)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} \left(\frac{z}{2}\right)^{n-i} h^{i}$$

$$= \frac{\left(\frac{z}{2}+h\right)^{n}}{2(1+h)^{n+1}} = -\frac{(z+2h)^{n}}{(2+2h)^{n+1}} .$$

From this it follows for the first summation of (27) that

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{C} \frac{(z+ht)^{n} \, dt}{(1+h)^{n+1} \, t^{n+1} (t-1)(t-2)} = - \sum_{n=0}^{\infty} \frac{(z+2h)^{n}}{(2+2h)^{n+1}} .$$

For the second summation in (27), consider the general integral term,

$$\frac{1}{2\pi i} \int_{C} \frac{(t+hz)^{n} \, dt}{(z+hz)^{n+1} (t-1)(t-2)} .$$

The integrand function has a pole of order one at $t = 1$. To find the residue, which is the coefficient of the term $\frac{1}{(t-1)}$ in the Laurent's expansion, we write

$$\frac{(t+hz)^n}{(z+hz)^{n+1}(t-1)(t-2)} = \frac{1}{(z+hz)^{n+1}} \times \frac{1}{(t-1)} \times P(t) ,$$

where

$$P(t) = \frac{\left[(hz+1)+(t-1)\right]^n}{-1+(t-1)} .$$

The constant term of the expansion of $P(t)$ by division is $-(hz+1)^n$ which gives us for the residue at $t = 1$,

(34) $$-\frac{(hz+1)^n}{(z+hz)^{n+1}} .$$

It follows from (34) that we can express the second summation as

$$\sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_C \frac{(t+hz)^n \, dt}{(z+hz)^{n+1}(t-1)(t-2)} = -\sum_{n=0}^{\infty} \frac{(hz+1)^n}{(z+hz)^{n+1}} .$$

The final form for the expansion of the function

$$F(z) = \frac{1}{(z-1)(z-2)}$$

becomes

(35) $$F(z) = -\sum_{n=0}^{\infty} \frac{(z+2h)^n}{(2+2h)^{n+1}} - \sum_{n=0}^{\infty} \frac{(hz+1)^n}{(z+hz)^{n+1}} .$$

This can be written in expanded form in a two-way series,

$$F(z) = \text{-----} - \frac{(z+2h)^n}{(2+2h)^{n+1}} \text{-----} - \frac{(z+2h)}{(2+2h)^2} - \frac{1}{(2+2h)}$$

$$- \frac{1}{(z+hz)} - \frac{(hz+1)}{(z+hz)^2} \text{-----} - \frac{(hz+1)^n}{(z+hz)^{n+1}} \text{-----} .$$

We note here that for $h = 0$ the expansion resolves into

$$F(z) = - \cdots - \frac{z^h}{2^{h+1}} - \cdots - \frac{z}{2^2} - \frac{1}{2}$$

$$- \frac{1}{z} - \frac{1}{z^2} - \cdots \frac{1}{z^n} - \cdots ,$$

which is the Laurent's expansion in the ring about the origin for any z such that $1 < |z| < 2$.* Observe also that by the discussion on page 18, for a sufficiently large choice of h the expansion (35) will hold for any finite z where $z = x + iy$, such that

$$\left| z - \frac{1}{2} \right| \geq r_1 > \frac{1}{2} ,$$

and

$$x \leq r_2 < 2 .$$

6. Expansion about a point not the origin. The foregoing work has been concerned with an expansion which holds in a defined region that encircles the origin. The extension to an expansion holding in a defined region about any other point, z_0, in the complex plane is immediate.

It is first necessary to set up definitions of the major and minor circles with respect to the point z_0 . For any point z the major circle will be defined with center at $(1+h) z_0 - h z$ and with radius $(1+h)|z - z_0|$. The minor circle of z will be taken with center at $\left(\frac{1+h}{1+2h}\right) z_0 - \frac{h}{1+2h} z$,

and radius $\left(\frac{1+h}{1+2h}\right) |z - z_0|$.

* Townsend, p. 283.

Major and minor circles defined in this manner have the same relation to the point z_0 as the original major and minor circles had with the origin. Also theorems I, II, III, and IV carry over directly.

The procedure is parallel to that used in deriving expansion (22). We consider a function, $f(z)$, single valued and analytic in and on the contour of a region S' about the point z_0. This region is bounded by two curves, C_1' and C_2', such tha t every point of C_1' is nearer to z_0 than any point of C_2'. Construct the major circles for all points of C_2' and the minor circles for all points of C_1'. The region, R', thus defined lies in S' and encircles z_0.

For any z in R' we have again by Cauchy's formula that

$$f(z) = \frac{1}{2\pi i}\int_{C_2'} \frac{f(t)}{(t-z)}\,dt + \frac{1}{2\pi i}\int_{C_1'} \frac{f(t)}{(z-t)}\,dt \; .$$

For the first integral we use the series

$$\frac{1}{t-z} = \frac{1}{(1+h)(t-z_0) - [z-z_0 + h(t-z_0)]} ,$$

$$= \frac{1}{(1+h)(t-z_0)}\left\{1 + \frac{(z-z_0)+h(t-z_0)}{(1+h)(t-z_0)} + \cdots + \left[\frac{(z-z_0)+h(t-z_0)}{(1+h)(t-z_0)}\right]^n + \cdots\right\} .$$

This series is uniformly convergent in t on C_2' for any z in R'. For the second integral use the series

$$\frac{1}{z-t} = \frac{1}{(1+h)(z-z_0) - [t-z_0 + h(z-z_0)]} ,$$

$$= \frac{1}{(1+h)(z-z_0)}\left\{1 + \frac{(t-z_0)+h(z-z_0)}{(1+h)(z-z_0)} + \cdots + \left[\frac{(t-z_0)+h(z-z_0)}{(1+h)(z-z_0)}\right]^n + \cdots\right\} .$$

which is also uniformly convergent in t on C_1' for z in R'.

It follows then that the expansion for $f(z)$ can be written in the form

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{[(z-z_0) + h(t-z_0)]^n}{(1+h)^{n+1}(t-z_0)^{n+1}} f(t)\,dt$$

$$+ \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{[(t-z_0) + h(z-z_0)]^n}{(1+h)^{n+1}(z-z_0)^{n+1}} f(t)\,dt .$$

The integration is taken along any regular curve C which lies in S' and encircles the point z_0.

The extension of this expansion to hold in a region defined by the major and minor circles of the singular point of a function follows exactly the same steps used in section 3 to justify equation (23).

7. <u>Extension to functions of several variables.</u> Consider a function, $F(z_1, z_2, - - - , z_n) = F(z)$, of n independent complex variables, such that $F(z)$ is analytic in and on the boundary of a $2n$ dimensional domain, B. By analytic is meant that the function is defined for every point of B and that the partial derivatives, mixed and iterated, exist. This domain, B, is a generalized poly-cylinder, which is a region of points, $(z_1, z_2, - - - , z_n)$, defined by z_1 in $A^{(1)}$, z_2 in $A^{(2)}$, $- - -$, z_n in $A^{(n)}$, where $A^{(j)}$, $j = 1, 2, - - -, n$, are regions on the respective coordinate planes bounded by regular

curves, $C^{(j)}$.* Then by Cauchy's extended integral formula, for any point, z, in B we can write

$$(36)\quad F(z) = \left(\frac{1}{2\pi i}\right)^n \int_{C^{(1)}} dt_1 \int_{C^{(2)}} dt_2 - - - \int_{C^{(n)}} \frac{F(t)\, dt_n}{\prod_{i=1}^{n} (t_i - z_i)},$$

where $t_1, t_2, - - - , t_n$ are the parameters of integration taken in a positive sense along $C^{(j)}$.** For a simpler notation we write (36) as

$$F(z) = \left(\frac{1}{2\pi i}\right)^n \int_C \frac{F(t)}{(t-z)} dt .$$

We now define B by choosing $A^{(j)}$ as regions on the respective coordinate planes such that each $A^{(j)}$ is bounded by two regular curves, $C_1^{(j)}$ and $C_2^{(j)}$, about the origin, where the absolute value of any point on $C_1^{(j)}$ is less than the absolute value of every point on the corresponding $C_2^{(j)}$. We next define regions, $R^{(j)}$, in the respective coordinate planes such that each $R^{(j)}$ is bounded on the inside by the minor circles of $C_1^{(j)}$ and on the outside by the major circles of $C_2^{(j)}$. These regions, each one being completely within the corresponding $A^{(j)}$, define a generalized polycylinder, E, which is contained in B. For any point, z, contained in E, we have by Cauchy's extended integral formula

$$(37)\quad F(z) = \left(\frac{1}{2\pi i}\right)^n \int_{C_2} \frac{F(t)\, dt}{(t-z)} + \left(\frac{1}{2\pi i}\right)^n \int_{C_1} \frac{F(t)\, dt}{(z-t)} .$$

For the first of the integrals in (37) we can write

$$\frac{1}{t-z} = \frac{1}{\prod_{i=1}^{n}(t_i - z_i)} = \frac{1}{(t_1 - z_1)} \times \frac{1}{(t_2 - z_2)} \times - - - \times \frac{1}{(t_n - z_n)} .$$

* S. Bochner, Functions of Several Complex Variables, p. 161.
** Goursat, Mathematical Analysis, Vol. II, Part I, p. 225.

By means of expansion (15), each fraction can be expanded in the form

$$\frac{1}{(t_i - z_i)} = \frac{1}{(1+h_i)t_i}\left(1 + \left(\frac{z_i + h_i t_i}{(1+h_i)t_i}\right) + \cdots + \left(\frac{z_i + h_i t_i}{(1+h_i)t_i}\right)^n + \cdots\right).$$

If we form a generalized Cauchy product of these n series, we have

$$\frac{1}{(t - z)} = \frac{1}{\prod_{i=1}^{n}(1+h_i)t_i} \sum_{k=0}^{\infty} p_k ,$$

where

$$p_k = \sum_{\alpha_{n-1}=0}^{k} \sum_{\alpha_{n-2}=0}^{\alpha_{n-1}} \cdots \sum_{\alpha_1=0}^{\alpha_2} \left(\frac{z_1 + h_1 t_1}{(1+h_1)t_1}\right)^{\alpha_1} \left(\frac{z_2 + h_2 t_2}{(1+h_2)t_2}\right)^{\alpha_2 - \alpha_1} \cdots \left(\frac{z_n + h_n t_n}{(1+h_n)t_n}\right)^{k - \alpha_{n-1}}.$$

Since z is in E and t_j is on $C^{(j)}$ for each $j = 1, 2, \cdots, n$, then we have by (16)

$$\left(\frac{|z_j + h_j t_j|}{(1+h_j)|t_j|}\right) \leq r_j < 1 , \quad (j) .$$

Choosing ρ equal to the maximum r_j then, since there are $\frac{(n+k-1)!}{(n-1)!\,k!}$ terms in p_k with the sum of the exponents being k, we can write

$$p_k \leq \frac{(n+k-1)!}{(n-1)!\,k!} \rho^k .$$

Thus, series (38) is dominated by the series,

$$\frac{1}{\prod_{i=1}^{n}(1+h_i)t_i} \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!\,k!} \rho^k .$$

Using the ratio test for convergence, we obtain

$$\lim_{k=\infty} \frac{(n+k-1)!}{(n-1)!\,k!} \frac{(n-1)!\,(k-1)!}{(n+k-2)!} \frac{\rho^k}{\rho^{k+1}} =$$

$$\lim_{k=\infty} \frac{(n+k-1)}{k} \rho = \rho < 1 .$$

Therefore, series (39) is convergent and it follows that series (38) is uniformly convergent for t_j on $C^{(j)}$ and z in E.

We have considered (38) as a series in which the terms are from a Cauchy product. In fact the k^{th} term consists of $\frac{(n+k-1)!}{(n-1)!\,k!}$ terms of the form

(40) $$\left(\frac{z_1+h_1t_1}{(1+h_1)t_1}\right)^{\beta_1}\left(\frac{z_2+h_2t_2}{(1+h_2)t_2}\right)^{\beta_2} \times --- \times \left(\frac{z_n+h_nt_n}{(1+h_n)t_n}\right)^{\beta_n} .$$

Since (39) is a positive term series, we can express the series by summing each ρ^k, $\frac{(n+k-1)!}{(n-1)!\,k!}$ times.* The series in this form with terms, ρ^k, is still convergent and still dominates the series obtained from (38) by writing it as a sum of terms of the type in (40). This amounts to removing the parentheses from the Cauchy product terms and then rearranging terms as desired. Multiplying this series from (38) by $F(z)$ and integrating n times, we obtain for the first integral of (37)

(41) $$\int_{C_2} \frac{F(t)\,dt}{(t-z)} = \sum_{\beta_1\cdots\beta_n=0}^{\infty} \int_{C_2} \prod_{i=1}^{n} \left(\frac{(z_i+h_it_i)^{\beta_i}}{(t_i+h_it_i)^{\beta_i+1}}\right) F(t)\,dt .$$

* Pierpont, Functions of a Complex Variable, pp. 66 - 67.

In a like manner for the second integral of (37) we can write

$$\frac{1}{(z-t)} = \frac{1}{\prod_{i=1}^{n}(z_i - t_i)} = \frac{1}{(z_1-t_1)} \times \frac{1}{(z_2-t_2)} \times \cdots \times \frac{1}{(z_n-t_n)} .$$

Using the expansion of (18) for each of these fractions and forming a general Cauchy product, we have again

$$(42)\qquad \frac{1}{(z-t)} = \frac{1}{\prod_{i=1}^{n}(1+h_i)z_i} \sum_{k=0}^{\infty} g_k \;,$$

where

$$g_k = \sum_{\alpha_{n-1}=0}^{k} \sum_{\alpha_{n-2}=0}^{\alpha_{n-1}} \cdots \sum_{\alpha_1=0}^{\alpha_2} \left(\frac{t_1+h_1 z_1}{(1+h_1) z_1}\right)^{\alpha_1} \left(\frac{t_2+h_2 z_2}{(1+h_2) z_2}\right)^{\alpha_2-\alpha_1} \times \cdots \times \left(\frac{t_n+h_n z_n}{(1+h_n) z_n}\right)^{k-\alpha_{n-1}} .$$

Since t_j is on $C_1^{(j)}$ and z is in E, we have from (19) that

$$\frac{|t_j + h_j z_j|}{(1+h_j)|z_j|} \leq r_j < 1 \;, \quad (j) .$$

This again is sufficient to show that (42) is dominated by a convergent positive term series and, therefore, is uniformly convergent. Then for t_j on $C_1^{(j)}$ and z in E, we obtain

$$(43)\qquad \int_{C_1} \frac{F(t)\,dt}{(z-t)} = \sum_{\beta_1 \cdots \beta_n = 0}^{\infty} \int_{C_1} \prod_{i=1}^{n} \left(\frac{(t_i + h_i z_i)^{\beta_i}}{(z_i + h_i z_i)^{\beta_i + 1}}\right) F(t)\,dt .$$

If each $C_1^{(j)}$ and $C_2^{(j)}$ is deformed into a regular curve, $C^{(j)}$, which lies in $A^{(j)}$ and encircles the origin on that respective coordinate plane, then from (41) and (43) we can substitute in (37) and obtain for any z in E

(44) $$F(z) = \sum_{\beta_1 \cdots \beta_n = 0}^{\infty} \left(\frac{1}{2\pi i}\right)^n \int_C \prod_{i=1}^{n} \left(\frac{(z_i + h_i t_i)^{\beta_i}}{(t_i + h_i t_i)^{\beta_i + 1}}\right) F(t)\, dt$$
$$+ \sum_{\beta_1 \cdots \beta_n = 0}^{\infty} \left(\frac{1}{2\pi i}\right)^n \int_C \prod_{i=1}^{n} \left(\frac{(t_i + h_i z_i)^{\beta_i}}{(z_i + h_i z_i)^{\beta_i + 1}}\right) F(t)\, dt.$$

In considering the extension of (44) to functions of several variables possessing a denumerable number of singularities, it will be necessary to restrict the study to those functions whose singularities are due only to the occurrence of the variables separately and not to the occurrence of two or more variables together. This restriction allows us to confine our attention to the coordinate planes in defining regions in terms of the singular points. These regions in turn define a generalized polycylinder for the domain of convergence.

We take a function, $F(z)$, possessing the singular points, $a_{11}, a_{12}, --, a_{1n}, --; a_{21}, a_{22}, --, a_{2n}, --; a_{31}, a_{32}, ---, a_{3n}, --; ----; a_{n1}, a_{n2}, --, a_{nn}, --$. The first subscript denotes the variable from which that set of singularities arises. The method of defining and proving a region of convergence on each coordinate plane follows that of section 3. Such a region, $D^{(j)}$, in the z_j plane has an inner boundary composed of the minor circles of a_{jK} and all singular points a_{ji} such that $|a_{ji}| \leq |a_{jK}|$. Its outer boundary is formed by the major circles of all singular points such that $|a_{ji}| \geq |a_{jK}|$. The only restriction on the choice of a_{jK} is that it can not be the limit point of a sequence of singular points which are of greater absolute value than it is. These regions, $D^{(j)}$, $j = 1,2,- - -,n$, define a generalized polycylinder, D, such that for any z in D, expansion (44) will hold.

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