THE MÖBIUS FUNCTION OF GENERALIZED FACTOR ORDER

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ABSTRACT

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We use discrete Morse theory to determine the Möbius function of posets ordered by generalized factor order. Ordinary factor order on the Kleene closure A^* of a set A is the partial order defined by letting $u \leq w$ if w contains u as a subsequence of consecutive letters. The Möbius function of ordinary factor order was determined by Björner. Using Babson and Hersh's application of Robin Forman's discrete Morse theory to lexicographically ordered chains, we are able to gain new understanding of Björner's result and its proof. We generalize the notion of factor order to take into account a partial order on the alphabet A and, relying heavily on discrete Morse theory, give a formula in the case where each letter of the alphabet covers a unique letter.

For my parents
Mike and Joan Willenbring

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I would like to thank my advisor, Dr. Bruce Sagan, for his dedication and commitment to my education, regardless of his physical proximity to it.

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Introduction

The Möbius function μ is an important invariant providing a bridge amongst various diverse areas of mathematics, such as combinatorics, number theory, and algebraic topology. The Möbius function of factor order was determined by Anders Björner [4]. This research was motivated by a desire to give a proof of Björner's result which provides a deeper explanation of the concepts he used to state his recursive formula, and to generalize these concepts so that they apply to a wider class of posets. These investigations utilize discrete Morse theory, which also provides a useful context in which to consider the topology of posets. The major result is a formula for the Möbius function when factor order is generalized to include a partial ordering of letters, provided each letter covers a unique element. Since it is clear this formula would have been nearly impossible to discover using other techniques of investigating Möbius functions, this paper illustrates the ability of discrete Morse theory to simplify complex combinatorial problems of this nature.

Let A be any set. The *Kleene closure*, A^*, is the set of all finite length words over A. So if w is a word and $w(i)$ is the i^{th} letter in w, then

$$A^* = \{w = w(1)w(2)\dots w(n) : \infty > n \geq 0 \text{ and } w(i) \in A \text{ for all } i\}.$$

The *length* of w, denoted $|w|$, is the number of letters in w. *Ordinary factor order* on A^* is the partial order on A^* defined by letting $u \leq w$ if w contains a subsequence of consecutive letters $w(i+1)w(i+2)\dots w(i+n)$ such that $u(j) = w(i+j)$ for $1 \leq j \leq n = |u|$. When $u \leq w$, we call u a *factor* of w. A word u is *flat* if $u(1) = \ldots = u(n)$, where $n = |u|$.

For example, if $A = \{a, b\}$, then $bbabb$ is an element of A^* of length $|bbabb| = 5$. Factors of $bbabb$ include words such as $bbab$, abb, and bb. Notice the word bb is flat.

A closed interval $[u, w]$ in A^* is the subposet consisting of all $v \in A^*$ satisfying $u \leq v \leq w$. The open interval (u, w) is defined similarly. If the interval $[u, w]$ consists of the two elements u and w, we say w *covers* u and write $w \to u$. The Hasse diagram of a poset P is the graph whose vertices are the elements of P, and in which an edge is found between two vertices w and u if $w \to u$. For example, if $A = \{a, b\}$, the Hasse diagram of $[b, bbabb]$ in A^* is given in Figure 1.

Let $u < w$ be two elements in a poset P. The Möbius function μ is a map from $P \times P$ to the integers defined recursively as follows:

$$\mu(u, u) = 1$$

$$\mu(u, w) = -\sum_{u \leq v < w} \mu(u, v)$$

$$\mu(w, u) = 0.$$

Figure 1 contains the Möbius value $\mu(b, v)$ for each word v in the interval $[b, bbabb]$. Working from the bottom of the diagram to the top, we see that $\mu(b, b) = 1$ by the first condition in the definition. Using the recursive definition, we can calculate $\mu(b, bb) = -\mu(b, b) = -1$. In higher rows, we again use the recursive definition to calculate $\mu(b, abb) = -(\mu(b, bb)+\mu(b, ab)+\mu(b, b)) = 1$, and $\mu(b, babb) = -(\mu(b, abb) + \mu(b, bab) + \mu(b, bb) + \mu(b, ab) + \mu(ba, b) + \mu(b, b)) = 0$.

To state Björner's formula, we need a few more definitions. A *prefix* of a word $w \in A^*$ is a factor of w that includes the first letter of w. Similarly, a *suffix* of w is a factor of w that contains the last letter of w. A prefix or suffix is proper if it is not equal to w. Define the *outer word* $o(w)$ of w to be the longest factor that appears as both a proper prefix and suffix in w.

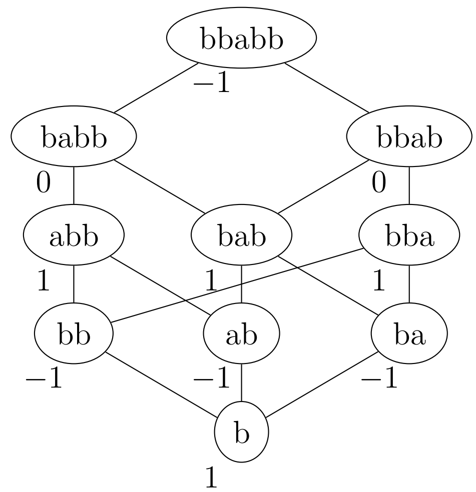

Figure 1: The interval $[b, bbabb]$. The Möbius value $\mu(b, v)$ is given to the lower left of each word v.

Notice that $o(w)$ can be the empty word. Define the *inner word* $i(w)$ of w to be the factor $i(w) = w(2)...w(n-1)$, where $n = |w|$.

For example, prefixes of $bbabb$ include bb and bba, while suffixes include bb and b. Note $o(bbabb) = bb$ and $i(bbabb) = bab$. The word abb has the empty word as its outer word.

The following theorem of Björner gives a formula for the Möbius function in ordinary factor order.

Theorem 1 ([4]). *In ordinary factor order, if $u \leq w$ then*

$$\mu(u, w) = \begin{cases} \mu(u, o(w)) & \text{if } |w| - |u| > 2 \text{ and } u \leq o(w) \not\leq i(w), \\ 1 & \text{if } |w| - |u| = 2,\ w \text{ is not flat, and } u = o(w) \text{ or } u = i(w), \\ (-1)^{|w|-|u|} & \text{if } |w| - |u| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Using this formula, we see that $\mu(b, bbabb) = \mu(b, bb) = -1$. Since b is the outer word of

bab, we get $\mu(b, bab) = 1$ by the second condition. Notice that since $o(babb) = b < ab = i(w)$, $\mu(b, babb) = 0$. The reader is encouraged to verify the remaining values implied by the definition of the Möbius function for the example in Figure 1 are consistent with those found by using this formula. Notice this formula indicates the only possible values for the Möbius function in A^* are -1, 0, and 1.

To prove his result, Björner relied on successively removing irreducible elements in an interval, where an irreducible element is one covered by or covering exactly one element. Björner also considered the Möbius function of subword order [3]. In [7], Sagan and Vatter expanded upon his results in the subword order case. In one proof of their result, they used the technique of critical maximal chains introduced by Babson and Hersh in [1]. This technique uses discrete Morse theory, which was developed by Forman [5]. Since our first goal is to apply the result of Babson and Hersh to reprove Björner's formula, we will introduce the notation and definitions we need to make use of the relevant theorem. For a more complete introduction to discrete Morse theory, see Forman's primer [6].

Discrete Morse theory is an adaptation of Morse theory that can be used to analyze the topology of a simplicial complex. An *abstract simplicial complex* is a set of vertices V and a set K of subsets of V satisfying the following conditions:

$$\text{if } v \in V, \text{ then } \{v\} \in K,$$

and

$$\text{if } \alpha \in K \text{ and } \gamma \subseteq \alpha, \text{ then } \gamma \in K.$$

Each α is called a *simplex*, and if $\alpha < \beta$, α is called a *face* of β. The *dimension* of a simplex α is the number of vertices in α minus 1. Writing α^d will indicate the simplex α has dimension d.

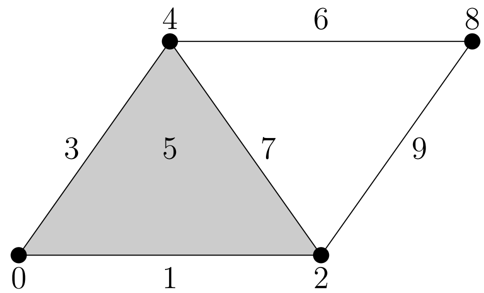

Figure 2: An example of a Morse function on a simplicial complex

Let K be a simplicial complex and assume K has an empty simplex of dimension -1 which is contained in every other simplex. A function $f : K \longrightarrow \mathbb{R}$ is a *discrete Morse function* if every simplex α^d satisfies the following conditions:

$$\#\{\beta^{d+1} > \alpha | f(\beta) \leq f(\alpha)\} \leq 1 \tag{1.1}$$

$$\#\{\gamma^{d-1} < \alpha | f(\gamma) \geq f(\alpha)\} \leq 1 \tag{1.2}$$

We denote the set in (1.1) by α^+ and the set in (1.2) by α_-.

A simplex α is *critical* if $\#\alpha^+ = \#\alpha_- = 0$. Note this definition states that with at most one local exception, a Morse function increases with respect to the dimension of a simplex.

Figure 2 contains an example of a simplicial complex K consisting of 4 simplices of dimension 0, 5 simplices of dimension 1, and 1 simplex of dimension 2. The values of a Morse function f appear next to each simplex. Note that the edge labeled 9 and vertex labeled 0 are the only critical simplices because locally, they are the only simplices for which the Morse function increases with respect to dimension.

One can prove that at most one of the sets α^+ and α_- has size 1. This result is crucial in proving the results concerning discrete Morse functions in this introduction. Note that since $\beta \in \alpha^+$ implies $\alpha \in \beta_-$, it follows that simplices which are not critical come in pairs, and these pairs satisfy $f(\alpha^d) > f(\beta^{d+1})$. A *Morse matching* is a partition of the simplices of K into sets of size one or two such that each one element set contains a critical simplex of a Morse function f and each two element set consists of two non-critical simplices $\alpha^d < \beta^{d+1}$ satisfying $f(\alpha^d) > f(\beta^{d+1})$.

The Morse matching of the simplices for the function shown in Figure 2 is $\{0\}, \{2, 1\}$, $\{4, 3\}, \{7, 5\}, \{8, 6\}, \{9\}$, where, as an abuse of notation, each number refers to the simplex to which it is assigned.

Critical simplices are important from a topological viewpoint because Forman shows in [5] that K is homotopy equivalent to a CW-complex with exactly one cell of dimension d for each critical simplex α of dimension d. Forman also proves the following Theorem in [5].

Theorem 2 (Weak Morse Inequalities)**.** *Let* $\tilde{m}_d$ *be the number of critical simplices of dimension* d, $\tilde{b}_d$ *be the d-th reduced Betti number over the integers, and* $\tilde{\chi}$ *be the reduced Euler characteristic. Then*

$$\tilde{b}_d \leq \tilde{m}_d \text{ for } d \geq -1$$

and

$$\tilde{\chi}(W) = \sum_{d \geq -1} (-1)^d \tilde{m}_d.$$

Discrete Morse theory can be used to find information about the Möbius function of a poset by using the connection between it and the reduced Euler characteristic of the order complex of a poset. A chain in a poset is set of elements $\{v_0, v_1, \ldots, v_n\}$ such that $v_0 > v_1 > \ldots > v_n$. For clarity, we write a chain C as $C : v_0 > v_1 > \ldots > v_n$. Given two elements u and w of a poset P, the order complex $\Delta(u, w)$ is the abstract simplicial complex whose simplices are the chains in the open interval (u, w). An important fact about the Möbius function [8] is

$$\mu(u, w) = \tilde{\chi}(\Delta(u, w)).$$

Therefore, by using discrete Morse theory to investigate the chains of (u, w), it is possible to calculate the Möbius function and obtain additional information about the topology of the order complex.

Given two elements u, w in a poset P, Babson and Hersh have developed a way of finding a Morse matching for $\Delta(u, w)$ which gives a relatively small number of critical simplices. We need to develop a considerable amount of terminology to properly state their result.

Let $C : v_0 \to v_1 \to \ldots \to v_n$ be a chain in a poset P. Since each pair of adjacent elements are related by a cover, C is called a *saturated chain*. The *closed interval* of C from v_i to v_j is the chain $C[v_i, v_j] : v_i \to v_{i+1} \to \ldots \to v_j$. The *open interval* of C from v_i to v_j, $C(v_i, v_j)$, and the half open intervals $C[v_i, v_j)$ and $C(v_i, v_j]$ are defined similarly. The closed interval $C[v_i, v_i]$ consisting of the single element v_i will also be written v_i, but the context will always indicate whether we are referring to the element or the interval. Notice that the interval $[u, w]$ is non-empty when $u \leq w$ in the poset P, while $C[v_i, v_j]$ is non-empty when $v_i \geq v_j$. A chain C of the interval $[u, w]$ is a *maximal chain* if $v_0 = w$ and $v_n = u$.

Given two maximal chains $C : v_0 \to v_1 \to \ldots \to v_n$ and $D : w_0 \to w_1 \to \ldots \to w_n$ in an interval $[u, w]$, we say C and D *agree to index* k if $v_i = w_i$ for all $i \leq k$. We say C and D *diverge from index* k if C and D agree to index k and $v_{k+1} \neq w_{k+1}$.

In Table 1, we have listed the eight maximal chains of $[b, bbabb]$. The first two maximal chains agree to indices 0, 1, and 2. These two chains diverge from index 2.

A total ordering $C_1 < C_2 < \ldots < C_n$ of the maximal chains of an interval is a *poset lexicographic order* if it satisfies the following: suppose $C < D$ and C and D diverge from index

v_0		v_1		v_2		v_3		v_4
$bbabb$	$\longrightarrow$	$babb$	$\longrightarrow$	abb	$\longrightarrow$	bb	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$babb$	$\longrightarrow$	abb	$\longrightarrow$	ab	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$babb$	$\longrightarrow$	bab	$\longrightarrow$	ab	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$babb$	$\longrightarrow$	bab	$\longrightarrow$	ba	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$bbab$	$\longrightarrow$	bab	$\longrightarrow$	ab	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$bbab$	$\longrightarrow$	bab	$\longrightarrow$	ba	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$bbab$	$\longrightarrow$	bba	$\longrightarrow$	ba	$\longrightarrow$	b
$bbabb$	$\longrightarrow$	$bbab$	$\longrightarrow$	bba	$\longrightarrow$	bb	$\longrightarrow$	b

Table 1: A poset lexicographic ordering of the eight maximal chains of $[b, bbabb]$

k; if C' and D' agree to index $k+1$ with C and D, respectively, then $C' < D'$.

To illustrate the definition, let us investigate one case in Table 1. Note the first chain and eighth chain agree to index 0. Since the second chain agrees with the first to index 1, and the seventh chain agrees with the eighth to index 1, we would need to have the second chain appear earlier than the seventh in order for this to be a poset lexicgoraphic order. In fact, the ordering given in Table 1 is a poset lexicographic ordering of the maximal chains of $[b, bbabb]$.

To verify we have given a poset lexicographic ordering of the maximal chains of $[b, bbabb]$, it is easiest to informally discuss how it was created. The formal construction appears in Section 2. To get this ordering, we record the positions l_i, relative to w, removed to get from v_{i-1} to v_i. The sequence $l_1 - l_2 - \ldots - l_n$ is called the chain id of the maximal chain. By lexicographically ordering these chain ids, we get a poset lexicographic order (in terms of Babson and Hersh, this is a chain labeling). To clearly indicate where each l_i comes from, we can zero out the positions removed in each word, creating an embedding of the word v_i into w. If a word consists entirely of one letter (such as bb), removing any letter gives the same word, so by convention, we only allow the first (non-zero) position to be removed.

In Table 2, we give same poset lexicographic order as above with this extra information about

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4
1-2-3-4	$bbabb$	1	$0babb$	2	$00abb$	3	$000bb$	4	$0000b$
1-2-5-3	$bbabb$	1	$0babb$	2	$00abb$	5	$00ab0$	3	$000b0$
1-5-2-3	$bbabb$	1	$0babb$	5	$0bab0$	2	$00ab0$	3	$000b0$
1-5-4-3	$bbabb$	1	$0babb$	5	$0bab0$	4	$0ba00$	3	$0b000$
5-1-2-3	$bbabb$	5	$bbab0$	1	$0bab0$	2	$00ab0$	3	$000b0$
5-1-4-3	$bbabb$	5	$bbab0$	1	$0bab0$	4	$0ba00$	3	$0b000$
5-4-1-3	$bbabb$	5	$bbab0$	4	$bba00$	1	$0ba00$	3	$0b000$
5-4-3-1	$bbabb$	5	$bbab0$	4	$bba00$	3	$bb000$	1	$0b000$

Table 2: The same ordering of the maximal chains of $[b, bbabb]$ with information about the chain ids.

the order. Notice that when only one digit numbers are used for each label l_i, lexicographically ordering the chain ids is equivalent to ordering n digit numbers by size.

Recall we need to consider the chains of the open interval (u, w) because the order complex $\Delta(u, w)$ is defined in terms of the open interval. Note that the maximal chains of (u, w) are in a one to one correspondence with those of $[u, w]$: by removing the first and last vertex from a maximal chain in $[u, w]$, we get a maximal chain in (u, w). Thus, we will continue to list v_0 and v_n in our examples to make it easier to identify the labels l_1 and l_n.

Suppose $C_1 < C_2 < \ldots < C_n$ is a lexicographic ordering of the maximal chains of (u, w). Recall that the chains of (u, w) are the simplices in the order complex $\Delta(u, w)$. Call a simplex contained in C *new* if it is not contained in any C' for $C' < C$. We can inductively define a Morse matching on $\Delta(u, w)$ by extending the matching at the k^{th} step to the new simplices of C. In fact, Babson and Hersh show in [1] that a matching can be constructed in this manner so that during each step k, at most one new simplex is a critical simplex. So, using this process, adding each maximal chain adds at most one critical simplex. We will refer to a maximal chain that contributes a critical simplex as a *critical chain*.

To motivate the next set of definitions, we consider which subchains of a maximal chain C appear in a lexicographically earlier chain, and which ones are new simplices. In a maximal chain C of a poset lexicographic ordering, Babson and Hersh prove that each maximal subchain of C which appears in a lexicographically earlier chain consists of a subchain of C given by a skipping a single interval of consecutive ranks. This is because the poset lexicographic ordering assures that if C and C' diverge from index k, but later are the same at index ℓ, there is some chain $D' < C$, possibly equal to C', such that C and D' are the same outside the interval $C(v_k, v_\ell)$. These skipped intervals are referred to as minimally skipped intervals. Therefore, a simplex in C entirely belongs to a lexicographically earlier chain if it entirely misses any of the minimally skipped intervals. Equivalently, the new simplices of C are those subchains of C which intersect every minimally skipped interval of C non-trivially.

Formally, given an ordering of the maximal chains of $[u, w]$, a non-empty interval (v_i, v_j) is a *skipped interval* of a maximal chain C if

$$C - C(v_i, v_j) \subset C' \text{ for some } C' < C.$$

It is a *minimally skipped interval (MSI)* if it does not properly contain another skipped interval. We write $I(C)$ for the set of all MSIs of a chain C. To find the set $I(C)$, consider each interval $I \subseteq C$ and see if $C - I \subset C'$ for any $C' \subset C$, then throw out any such interval that is nonminimal.

For an example of both MSIs and new simplices, see Table 3. In this table, we have placed brackets around each interval which is minimally skipped, and placed the corresponding brackets into the chain id. For example, the intervals $D[bb, bb]$ and $D[bbab, bba]$ are minimally skipped intervals of chain D with chain id $5 - 4 - 3 - 1$. To see how one determines the minimally skipped intervals, consider the third chain C with chain id $1 - 5 - 2 - 3$. The only intervals I satisfying $C - I \subset C'$ for some $C' < C$ are $C[bab, ab]$, since $C - C[bab, ab]$ is in the first

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4	MSIs
1-2-3-4	$bbabb$	1	$0babb$	2	$00abb$	3	$000bb$	4	$0000b$	
1-2-5[-]3	$bbabb$	1	$0babb$	2	$00abb$	5	$[00ab0]$	3	$000b0$	$[ab, ab]$
1-5[-]2-3	$bbabb$	1	$0babb$	5	$[0bab0]$	2	$00ab0$	3	$000b0$	$[bab, bab]$
1-5-4[-]3	$bbabb$	1	$0babb$	5	$0bab0$	4	$[0ba00]$	3	$0b000$	$[ba, ba]$
5[-]1-2-3	$bbabb$	5	$[bbab0]$	1	$0bab0$	2	$00ab0$	3	$000b0$	$[bbab, bbab]$
5[-]1-4[-]3	$bbabb$	5	$[bbab0]$	1	$0bab0$	4	$[0ba00]$	3	$0b000$	$[bbab, bbab]$ $[ba, ba]$
5-4[-]1-3	$bbabb$	5	$bbab0$	4	$[bba00]$	1	$0ba00$	3	$0b000$	$[bba, bba]$
5[-4-]3[-]1	$bbabb$	5	$[bbab0$	4	$bba00]$	3	$[bb000]$	1	$0b000$	$[bbab, bba]$ $[bb, bb]$

The new simplices, which are in $C \setminus (\cup_{C' < C} C')$, for selected maximal chains C:

$$1-5-2-3: \quad bab, \quad babb-bab, \quad bab-ab, \quad babb-bab-ab$$
$$5-4-3-1: \quad bba-bb, \quad bbab-bb, \quad bbab-bba-bb$$

Table 3: The MSIs of the maximal chains of $[b, bbabb]$.

chain, and $C[bab, bab]$, since $C - C[bab, bab]$ is in the second chain. Since the second interval is contained in the first, the only MSI in the third chain is $C[bab, bab]$. So the new simplices in this chain, which intersect this MSI, are the subchains bab, $babb - bab$, $bab - ab$, and $babb - bab - ab$.

The set of MSIs $I(C)$ *covers* C if its union equals the open interval $C(v_0, v_n)$. This last definition reflects the fact that the order complex of an interval is constructed without the maximum and minimum elements. Notice the set $I(D)$ for the maximal chain D with chain id $5-4-3-1$ covers D, and that the set of MSIs $I(C)$ for any other maximal chain C in the interval $[b, bbabb]$ does not cover C.

Notice $I(C)$ could contain intervals that overlap, that is, intervals with non-empty intersection. We will need to produce a set of disjoint intervals from $I(C)$, which we will call $J(C)$. We construct $J(C) = \{J_1, J_2, \ldots\}$ as follows. Order the intervals of $I(C)$ based on when they are first encountered in C. Thus, I_1 will contain the word v_i of smallest index that appears in any interval in $I(C)$, I_2 will contain the word v_j of smallest index that appears in any interval in $I(C)$ not equal to I_1, etc. Let $J_1 = I_1$. Then consider the intervals $I'_2 = I_2 - J_1$, $I'_3 = I_3 - J_1$, and so forth. Throw out any that are not containment minimal, and pick the first one that remains to be J_2. Continue this process until no intervals remain to add to $J(C)$.

For an example of the difference between $I(C)$ and $J(C)$, we need to consider a new interval. In the interval $[a, abbabb]$, whose maximal chains are found in Table 4, the chain C with chain id $6-5-4-3-2$ has a set of MSIs $I(C)$ in which there is overlap. To construct $J(C)$, we first add the MSI $I_1 = C[abbab, abba]$ of $I(C)$ to $J(C)$. Then we truncate the remaining intervals in $I(C)$. In particular, $I_2 = C[abba, abb]$ becomes $I_2' = C[abb, abb]$, while the chain $I_3 = C[abb, ab]$ does not overlap with I_1, giving $I_3' = I_3$. Now we see that $I_2' \subset I_3'$. So we remove I_3' from the set of intervals under consideration, and add I_2' to $J(C)$. At this point, no intervals of the original set $I(C)$ remain to be considered, so $J(C) = \{I_1, I_2'\}$.

The following theorem of Babson and Hersh gives the connection between $J(C)$ and $\mu(u, w)$. For a description, and example, of the Morse matching of simplices which leads to this Theorem, please see Appendix A.

Theorem 3 ([1])**.** *Let P be a poset and $[u, w]$ be a finite interval in P. For any poset lexicographic order on the maximal chains of $[u, w]$, the above construction can be used to produce a Morse matching which matches all the new simplices of a chain, except possibly one which is critical. The set $J(C)$ for each C has the following properties:*

1. *A maximal chain C is critical if and only if $J(C)$ covers C.*
2. *If C is critical, then the dimension of its critical simplex is*
$$d(C) = \#J(C) - 1.$$
3. *The Möbius value from u to w is*
$$\mu(u, w) = \sum_C (-1)^{d(C)}$$
where the sum is over all critical chains C in $[u, w]$.

$I(C)$ intervals for $[a, abbabb]$

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4	l_5	v_5
1-2-3-6-5	$abbabb$	1	$0bbabb$	2	$00babb$	3	$000abb$	6	$000ab0$	5	$000a00$
1-2-6[-]3-5	$abbabb$	1	$0bbabb$	2	$00babb$	6	$[00bab0]$	3	$000ab0$	5	$000a00$
1-2-6-5[-]3	$abbabb$	1	$0bbabb$	2	$00babb$	6	$00bab0$	5	$[00ba00]$	3	$000a00$
1-6[-]2-3-5	$abbabb$	1	$0bbabb$	6	$[0bbab0]$	2	$00bab0$	3	$000ab0$	5	$000a00$
1-6[-]2-5[-]3	$abbabb$	1	$0bbabb$	6	$[0bbab0]$	2	$00bab0$	5	$[00ba00]$	3	$000a00$
1-6-5[-]2-3	$abbabb$	1	$0bbabb$	6	$0bbab0$	5	$[0bba00]$	2	$00ba00$	3	$000a00$
6[-]1-2-3-5	$abbabb$	6	$[abbab0]$	1	$0bbab0$	2	$00bab0$	3	$000ab0$	5	$000a00$
6[-]1-2-5[-]3	$abbabb$	6	$[abbab0]$	1	$0bbab0$	2	$00bab0$	5	$[00ba00]$	3	$000a00$
6[-]1-5[-]2-3	$abbabb$	6	$[abbab0]$	1	$0bbab0$	5	$[0bba00]$	2	$00ba00$	3	$000a00$
6-5[-]1-2-3	$abbabb$	6	$abbab0$	5	$[abba00]$	1	$0bba00$	2	$00ba00$	3	$000a00$
6[-5[-]4[-]3-]2	$abbabb$	6	$[abbab0$	5	$[abba00]$	4	$[abb000]$	3	$ab0000]$	2	$a00000$

$J(C)$ intervals for $[a, abbabb]$
The set $I(C)$ only changes for the last chain:

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4	l_5	v_5
6[-5-]4[-]3-2	$abbabb$	6	$[abbab0$	5	$abba00]$	4	$[abb000]$	3	$ab0000$	2	$a00000$

Table 4: Comparing $I(C)$, $J(C)$ for the maximal chains of $[a, abbabb]$.

The rest of the paper is organized as follows. In Section 2, we use Theorem 3 to reprove Björner's formula in Theorem 1. We also give a simple proof of his characterization of the topology of posets ordered by factor order using discrete Morse theory. In Section 3, we will consider generalized factor order on the integers, meaning we will allow a partial ordering of the letters to be taken into account. Section 4 considers generalized factor order on rooted forests, allowing us to clearly show that the formula obtained in Section 3 for the Möbius of function of generalized factor on the integers is a generalization of Björner's formula. This is not obvious and in fact two independent proofs are needed to establish the connection. Section 5 discusses open problems related to this work.

Chapter 2

Ordinary Factor Order

Let A be any set. Partially order A^* using ordinary factor order. To get a sense of the structure of the poset A^*, we first consider the covering relations in this poset.

Lemma 4. *A word $w = w(1)\ldots w(n)$ in A^* can only cover the words $w(2)\ldots w(n)$ and $w(1)\ldots w(n-1)$. These two words are distinct unless w is flat, in which case they are equal and flat.*

Proof. Note w covers words of length $|w|-1$. Since the letters of a factor of w must appear consecutively in w, the longest proper suffix $w(2)\ldots w(n)$ and the longest proper prefix $w(1)\ldots w(n-1)$ are the only two words w covers. Should these two words be equal, we have $w(1) = w(2)$, $w(2) = w(3)$, …, and $w(n-1) = w(n)$ so that $w(1) = w(2) = \ldots = w(n)$. This implies that w is flat when it covers a single word. □

We will now formalize some of the concepts introduced informally in the introduction. Suppose there is an element $0 \notin A$. A word $\eta \in (A \cup 0)^*$ is an *expansion* of $u \in A^*$ if $\eta \in 0^*u0^*$, where adjacency denotes concatenation of words and letters. An *embedding* of u into w is an expansion η of u with length $|w|$ such that for all i, either $\eta(i) = w(i)$ or $\eta(i) = 0$. In the latter case, we say $w(i)$ is *reduced to* 0.

Note that the words appearing in Tables 2 through 4 are actually the embeddings of the words v_i. For example, we note that $bb000$ is the prefix embedding of bb into $bbabb$, while $000bb$ is the suffix embedding.

Let $[u, w]$ be an interval in A^*. Let $w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain in $[u, w]$, where the l_i are defined by the corresponding sequence of embeddings of v_i into w, ηv_i, in the sense that

$$\eta v_i(l_i) = 0 \text{ and } \eta v_i(j) = \eta v_{i-1}(j) \text{ when } j \neq l_i.$$

In the case where v_{i-1} is flat, we require l_i to be the smaller of the two possible values. Note that this gives each maximal chain its own unique sequence $l_1 \dots l_n$ which we can use to identify it. We call this sequence a maximal chain's *chain id*. By lexicographically ordering these chain ids, we produce a poset lexicographic order on the maximal chains of $[u, w]$. This is the order we will use to find the MSIs of the maximal chains C, and ultimately the sets $J(C)$ which will allow us to apply Theorem 3. Examples of this ordering were given in Tables 2 through 4.

To facilitate the exposition, we make the following definitions. A *descent* in a maximal chain is a word v_i where $l_i > l_{i+1}$. We say v_i is a *strong descent* if $l_i > l_{i+1} + 1$, and a *weak descent* if $l_i = l_{i+1} + 1$. An *ascent* in a maximal chain is a word v_i where $l_i < l_{i+1}$. In Table 2, the chain $5 - 4 - 1 - 3$ has $v_1 = bbab$ as a weak descent, $v_2 = bba$ as a strong descent, and $v_3 = ba$ as an ascent.

Lemma 5. *Let* $C : v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n$ *be a maximal chain in* $[u, w]$*. If* v_i *is a strong descent, then* v_i *is an MSI in* C*.*

Proof. Suppose v_{i+1} is not flat. Since $l_i - l_{i+1} > 1$, we know that $v_{i-1} = w(l_{i+1})v_{i+1}w(l_i)$ and $v_i = w(l_{i+1})v_{i+1}$. Therefore, if $u = v_{i+1}w(l_i)$, then

$$C' = v_0 \overset{l_1}{\to} \dots \overset{l_2}{\to} v_{i-2} \overset{l_{i-1}}{\to} v_{i-1} \overset{l_{i+1}}{\to} u \overset{l_i}{\to} v_{i+1} \overset{l_{i+2}}{\to} \dots \overset{l_n}{\to} v_n$$

is a lexicographically earlier chain than C. Hence, v_i is a skipped interval in C. Since v_i is an interval consisting of a single element, v_i is an MSI.

If v_{i+1} is flat and v_i is as well, the above argument still applies. If v_{i+1} is flat, but v_i is not, then $u = v_{i+1}w(l_i)$ must be a flat word. So l_i cannot be reduced in u. However, the chain

$$D' = v_0 \stackrel{l_1}{\to} \dots \stackrel{l_2}{\to} v_{i-2} \stackrel{l_{i-1}}{\to} v_{i-1} \stackrel{l_{i+1}}{\to} u \stackrel{l_{i+2}}{\to} v_{i+1} \stackrel{l_{i+3}}{\to} \dots \stackrel{l_n}{\to} v_{n-1} \stackrel{l_{n+1}}{\to} v_n$$

is a lexicographically earlier chain than C because each v_j for $j > i$ is flat. Hence, v_i is an MSI in this case as well. □

Comparing this result in Tables 3 and 4 shows that it accounts for a small proportion of the MSIs in ordinary factor order. The next result, however, gives a great deal of information about the critical chains of ordinary factor order.

Lemma 6. *Let* $C : v_0 \stackrel{l_1}{\to} v_1 \stackrel{l_2}{\to} \dots \stackrel{l_{n-1}}{\to} v_{n-1} \stackrel{l_n}{\to} v_n$ *be a maximal chain in* $[u, w]$*. If* v_i *is an ascent, then it is not contained in any MSI.*

Proof. We will prove this lemma by contradiction. Suppose $C[v_r, v_s]$ is an MSI that contains v_i. Notice that v_i may only be preceded by ascents in this interval because if there are descents, the last one that occurs before v_i would be a strong descent. By Lemma 5, this would be an MSI, contradicting the minimality of $C[v_r, v_s]$. Thus, it suffices to derive a contradiction for $v_i = v_r$, the first ascent in the interval $C[v_r, v_s]$.

Since $C[v_r, v_s]$ is an MSI of $C[w, u]$ if and only if it is an MSI of $C[v_{r-1}, v_{s+1}]$, it suffices to consider the case $r = 1$. However, if $r = 1$ then v_1 being an ascent forces $l_1 = 1$. This implies v_1 appears in all chains preceding C. Therefore, v_r can be removed from any skipped interval in which it appears and that interval will still be skipped, contradicting the fact that $C[v_r, v_s]$

is minimal. □

Notice that Lemma 6 implies that all MSIs of a chain C consist entirely of descents. Thus, only the lexicographically last chain in an interval can possibly be critical. The next lemma covers the two basic cases of MSIs in a chain C. We have already encountered the first case, while the second case is new.

Lemma 7. *Suppose w is not flat and $|w| \geq 2$:*

1. *There are two maximal chains in the interval $[i(w), w]$, and if C is the second chain, then it has a unique MSI, $C(w, i(w))$.*

2. *If $o(w) \not< i(w)$, there are two maximal chains in the interval $[o(w), w]$, and if C is the second chain, then it has a unique MSI, $C(w, o(w))$.*

Proof. The first case follows from Lemma 5. For the second case, once we remove the first or last element, there is only one copy of $o(w)$ left in w. Therefore, there are two maximal chains in the interval $[o(w), w]$: the first chain, which ends at the suffix embedding, and the last chain, which ends at the prefix embedding. Since these two chains share only $o(w)$ and w in common, $C(w, o(w))$ is an MSI, completing the proof. □

We can illustrate this lemma using the intervals $[aa, baab]$ and $[b, baab]$. Note the inner word of $baab$ is aa, so that the maximal chains of $[aa, baab]$ are $baab - aab - aa$ and $baab - baa - aa$, giving $C(baab, aa)$ as an MSI in the second chain. The outer word of $baab$ is b, so that the maximal chains of $[b, baab]$ are $baab - aab - ab - b$ and $baab - baa - ba - b$, giving $C(baab, b)$ as an MSI in the second chain.

There is another way to think about these two types of MSIs that will prove useful moving forward. In the first type, the embedding of $i(w)$ in the critical chain and first chain are the same when $i(w)$ is not flat. In the second type, the embeddings of $o(w)$ in the critical chain and the first chain are different. Notice that if both $i(w)$ and $o(w)$ are flat, then w is flat, $o(w) < i(w)$, and there is a unique maximal chain in every non-empty interval $[u, w]$. Thus, there can be no overlap between these two types of MSI. We will see this observation about same and different embeddings provides a very useful way of determining how MSIs arise, even though it does not apply to MSIs that end at flat words.

Proposition 8 generalizes Lemma 7(2), and the theorem that follows shows that we have identified all cases of MSIs. It will be convenient to adopt the convention that a sequence l_{i+1} consisting of a single label is not decreasing, corresponding to the fact that the interval $C(v_i, v_{i+1})$ is empty and so not an MSI.

Proposition 8. *Let* $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ *be a maximal chain in the interval* $[u, w]$*. Suppose there are* i *and* j *such that* $v_j = o(v_i) \not< i(v_i)$*, and such that the sequence* $l_{i+1}, \dots, l_j$ *is decreasing. Then* $C(v_i, v_j)$ *is an MSI in* C*.*

Proof. Since the sequence $l_{i+1}, \dots, l_j$ is decreasing, v_i can not be flat and $j \geq i + 2$. So Lemma 7(2) implies that $C(v_i, v_j)$ is an MSI in the subchain of C that is its intersection with $[v_j, v_i]$. So there is a lexicographically earlier maximal chain D in $[v_j, v_i]$. Thus, the chain C' formed by replacing the subchain $C[v_i, v_j]$ in C with D yields $C - C(v_i, v_j) \subset C'$. Therefore, $C(v_i, v_j)$ is a skipped interval of C.

To see that it must also be an MSI, note that $o(v_i) \not< i(v_i)$ so that $[v_j, v_i]$ has only two maximal chains and they intersect only at v_i and v_j. So the same must be true of any maximal chain in $[u, w]$ containing v_i and v_j. This forces minimality. □

Theorem 9. *The interval* $C(v_i, v_j)$ *is an MSI of a maximal chain* C *of* $[u, w]$ *if and only if* $C(v_i, v_j) = v_{i+1}$ *and* v_{i+1} *is a strong descent, or* $v_j = o(v_i) \not< i(v_i)$*, where the sequence*

$l_{i+1}, \ldots, l_j$ *is decreasing.*

Proof. The reverse implication follows from Lemma 5 and Proposition 8.

Suppose $C(v_i, v_j)$ is an MSI in C. Note by Lemma 6 the sequence $l_{i+1}, \ldots, l_j$ is a decreasing sequence. If v_{i+1} is a strong descent, then v_{i+1} is an MSI by Lemma 5. This implies $C(v_i, v_j) = v_{i+1}$. If v_{i+1} is not a strong descent then, by containment minimality, none of the descents are strong descents. Also, our sequence is decreasing, so we conclude that $v_i = v_j w(l_j) \ldots w(l_{i+1})$. Thus, v_j is the prefix of the word v_i.

In order for $C(v_i, v_j)$ to be an MSI, there must be at least two embeddings of v_j into v_i, else no maximal chain can differ from C on the interval $[v_i, v_j]$. Let k be the largest index so that v_k contains exactly two copies of v_j. Then v_{k+1} contains only one embedding of v_j, implying that v_j is a suffix of v_k. By the previous paragraph, v_j is also a prefix of v_k. Thus $o(v_k) = v_j$ because if a word longer than v_j was $o(v_k)$, v_{k+1} would have more than one copy of v_j. Similarly, $o(v_k) \not< i(v_k)$. So by Proposition 8, $C(v_k, v_j)$ is an MSI of C. Thus, by containment minimality, it must be the case that $k = i$. □

Theorem 9 completes the characterization of the MSIs in an interval $[u, w]$ in factor order. Notice the definitions of the inner and outer word, which Björner used to state his formula, naturally arise when determining the MSIs. Also, the inequality $u \leq o(w) \not\leq i(w)$ is forced upon us by the poset lexicographic ordering of the maximal chains.

We are now ready to prove Björner's formula using discrete Morse theory. We have broken the proof up into several cases to make it easier to follow.

Theorem 1 ([4]). *In ordinary factor order, if $u \le w$ then*

$$\mu(u,w) = \begin{cases} \mu(u, o(w)) & \text{if } |w|-|u| > 2 \text{ and } u \le o(w) \not\le i(w), \\ 1 & \text{if } |w|-|u| = 2,\ w \text{ is not flat, and } u = o(w) \text{ or } u = i(w), \\ (-1)^{|w|-|u|} & \text{if } |w|-|u| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $[u, w]$ be an interval in ordinary factor order. Suppose first that $|w| - |u| < 2$. Then $u = w$ or $|u| = |w| - 1$. By the definition of the Möbius function, we have $\mu(u, w) = 1$ in the first case and $\mu(u, w) = -1$ in the second case. Thus, the formula for $\mu(u, w)$ holds when $|w| - |u| < 2$.

Now suppose $|w| - |u| = 2$. Then by the Möbius recursion $\mu(u, w) = 0$ if there is one element in the interval (u, w) and $\mu(u, w) = 1$ when there are 2 elements in the interval (u, w). Since w covers at most two elements, these are the only possibilities. If w is flat, then $\mu(u, w) = 0$ since (u, w) contains a single element. If w is not flat and $u = i(w)$, then removing either the first or last letter of w gives us an element in (u, w), implying $\mu(u, w) = 1$. If w is not flat and $u = o(w)$, then removing either the first two letters or last two letters of w gives us u. Thus, (u, w) has 2 elements implying $\mu(u, w) = 1$. If the above cases do not hold, then u is either a prefix or a suffix of w, but not both. In these cases, (u, w) has 1 element implying $\mu(u, w) = 0$. Thus, the formula for $\mu(u, w)$ holds when $|w| - |u| = 2$.

We now turn to the case $|w| - |u| > 2$. We will use Theorem 3 to calculate $\mu(u, w)$ from the critical chains in $[u, w]$. By Lemma 6, the chain id of a critical chain must be decreasing. Since a strong descent is followed by an ascent unless it is the last element in a chain, all the descents must be weak descents except possibly the last one. Also, the only maximal chain in $[u, w]$ that could be critical is the one that is lexicographically last. Call this chain C.

Suppose first that $u \leq o(w) \not\leq i(w)$. We need to show that $o(w)$ is an element in the chain C. Let $k = |w| - |o(w)|$. Since $v_k = o(w)$ is not contained in $i(w)$, the word $o(w)w(k+1)$ cannot be flat even if $o(w)$ is flat. This observation, along with the fact that $u \leq o(w)$, allows us to conclude that $|w|, |w| - 1, \ldots, |w| - (k-1)$ is a valid beginning for the chain id of a maximal chain D. Notice that each of these entries is the largest possible entry that does not already appear in the sequence. Thus, any chain whose chain id differs from the chain id of D in the first k entries is lexicographically earlier than D. So in the chain C, $l_1 = |w|$, $l_2 = |w| - 1,\ldots,$ $l_k = |w| - (k-1)$, and $v_k = o(w)$.

If $u = o(w) \not\leq i(w)$, the previous paragraph implies that the sequence $l_1, \ldots, l_k$ is decreasing. Thus, Theorem 9 implies $C(w, o(w))$ is the only interval in $J(C)$. So by Theorem 3, $\mu(u, w) = 1$. Of course, in this case $\mu(u, o(w)) = \mu(u, u) = 1$ as well, so the formula holds.

Next we consider $u < o(w) \not\leq i(w)$. Since l_k was the largest possible entry remaining, $l_{k+1} < l_k$, implying that $o(w)$ is a descent. Let C' be the restriction of C to the interval $[u, o(w)]$. We will show that $\#J(C) = 2 + \#J(C')$, allowing us to apply Theorem 3 to complete the case $u \leq o(w) \not\leq i(w)$. Since the sequence $l_1, \ldots, l_k$ is decreasing, Theorem 9 implies $C(w, o(w))$ is the first interval in $J(C)$. We claim $o(w)$ is the second interval in $J(C)$. If $o(w)$ is a strong descent, this follows immediately. If $o(w)$ is a weak descent, $v_{k+1} = w(1)w(2)\ldots w(|w| - k - 1) < o(w)$, implying there are at least two copies of v_{k+1} contained in w. Let j be the the largest value such that v_j contains two copies of v_{k+1}. Since in this case $v_1, \ldots, v_j$ are weak descents, $v_{k+1} \not\leq i(v_j)$ because the two copies of v_{k+1} in v_j must be the prefix and suffix embeddings. Furthermore, $o(v_j) = v_{k+1}$ because the prefix with one additional letter,

$o(w)$, appears only once in v_j. Since the sequence $l_{j+1}, \ldots, l_{k+1}$ is decreasing, Theorem 9 implies $C(v_j, v_{k+1})$ is a skipped interval in C. By the process of constructing $J(C)$ from $I(C)$, $o(w) = C(v_j, v_{k+1}) - C(v_0, v_k)$ is the second MSI in $J(C)$, proving the claim. Since $o(w)$ is an MSI consisting of one element, all the remaining intervals in $J(C)$ are contained in the interval $(u, o(w))$. Therefore, $J(C) = J(C') \cup \{C(w, o(w)), o(w)\}$ and $\#J(C) = 2 + \#J(C')$. So by Theorem 3, $\mu(u, w) = \mu(u, o(w))$, proving the formula for $|w| - |u| > 2$ and $u \leq o(w) \not\leq i(w)$.

It remains to consider what happens when $|w| - |u| > 2$ and $u \leq o(w) \not\leq i(w)$ does not hold. To show $\mu(u, w) = 0$, we proceed by contradiction. If $\mu(u, w) \neq 0$ then, by Theorem 3, $J(C)$ must cover C. This implies that $J_1 = C(v_0, v_j)$ is an MSI for some v_j. Recall that Theorem 9 gives two possibilities for MSIs. If $J_1 = v_1$ and v_1 is a strong descent, then since $|w| - |u| > 2$, v_2 is an ascent. This contradicts the fact that C has a decreasing chain id. Alternatively, we must have $v_j = o(w) \not\leq i(w)$. However, since $v_j \geq u$, $u \leq o(w) \not\leq i(w)$, contradicting our assumption that this inequality does not hold. So $\mu(u, w) = 0$, completing the proof. □

By reproving Björner's formula using this technique, it is easy to verify Björner's description of the homotopy type of a poset ordered by factor order.

Theorem 10. *Let $[u, w]$ be an interval in A^*. Then $\Delta(u, w)$ is homotopic to a sphere or is contractible*

Proof. By Forman's fundamental theorem of discrete Morse Theory [5], a simplicial complex with a discrete Morse function is homotopy equivalent to a CW complex with exactly one cell of dimension d for each critical simplex of dimension d (as well as a dimension 0 cell). By Babson and Hersh's theorem for poset lexicographic orders (Theorem 3 and [1]), $\Delta(u, w)$ has a discrete Morse function in which a maximal chain is critical (contributes a critical simplex) if and only if $J(C)$ covers C.

Thus, if $[u, w]$ has no critical chains, it is homotopy equivalent to a CW complex with only the 0-cell. By Lemma 6, a critical chain must have a decreasing chain id, which means only the lexicographically last chain can be critical. So there can be at most one critical chain. This gives us a CW complex with a 0-cell and one other cell, which by Theorem 3 has dimension $\#J(C) - 1$. The unique way to attach this cell to the 0 cell is through a map which is constant on the boundary, resulting in a sphere of dimension $\#J(C) - 1$. □

As a final note on the homotopy type, notice that a critical chain contains at most one MSI caused by a strong descent in $J(C)$. Thus, the dimension of the sphere grows larger as the number of recursive calls to the formula (because of outerwords) increases. So if $A = \{a, b\}$, $u = a$, and w is an alternating word of a's and b's with $|w| = n > 2$, the sphere has dimension $n - 3$.

Chapter 3

Generalized Factor Order on $\mathbb{P}$

Let P be a partially ordered set. Partially order P^* using *generalized factor order*, which is the partial order on P^* defined by letting $u \leq w$ if w contains a subsequence $w(i+1)w(i+2)\ldots w(i+n)$ such that $u(j) \leq w(i+j)$ in P for $1 \leq j \leq n = |u|$. If $u \leq w$, we will again say that u is a *factor* of w. In this section, we will consider generalized factor order on the positive integers $\mathbb{P}$.

In Figure 3 is the Hasse diagram of the interval $[121, 1221]$, ordered by generalized factor order on $\mathbb{P}^*$. To see that $121 < 1221$, let $w = 1221$ and note that $1 \leq w(1)$, $2 \leq w(2)$, and $1 \leq w(3)$. In fact, 121 appears twice in 1221 because $1 \leq w(2)$, $2 \leq w(3)$, and $1 \leq w(4)$. This interval will useful in illustrating some of the ideas in this section. The Möbius values are listed for convenience - notice there is a Möbius value outside of the set $\{-1, 0, 1\}$.

Although we now have two partial orders to work with, the use of inequalities will never be ambiguous because the partial order induced by generalized factor order on words of one letter is the same as the order in P. Notice that when P is an antichain, generalized factor order and ordinary factor order are the same partial order.

Our definition of length for ordinary factor order on A^* can be used in the context of generalized factor order on $\mathbb{P}^*$. Similarly, the concept of an expansion of a word in $(A \cup 0)^*$ can be used for words in $(\mathbb{P} \cup 0)^*$. An *embedding* of a word u into w is an expansion η of u satisfying $\eta \leq w$ for generalized factor order on $(\mathbb{P} \cup 0)^*$. Thus, 0121 and 1210 are embeddings of 121 into 1221.

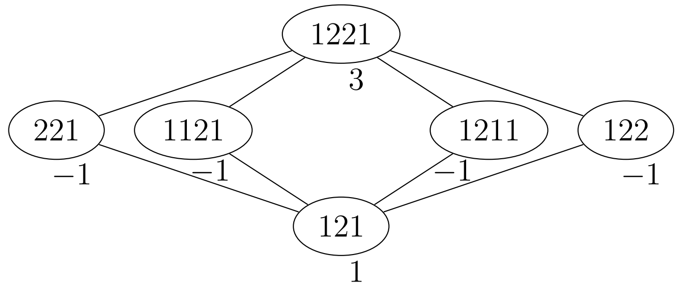

Figure 3: The interval $[121, 1221]$. The Möbius value $\mu(121, v)$ is given below of each word v.

A word is *flat* in $\mathbb{P}^*$ if it is a sequence of 1's. This is a natural refinement of the definition from the antichain case, as in that context every element is minimal, while in $\mathbb{P}$ only 1 is minimal. As in the previous section, we consider the covering relations in $\mathbb{P}^*$ to get a sense of the structure of this poset. This proof of this lemma is similar to that of Lemma 4. *Reducing* a letter $w(i) > 1$ means we are replacing it with $w(i) - 1$. To be precise, *reducing* a letter $w(i) = 1$ means removing it when considering words in $\mathbb{P}^*$, or replacing it with 0 when considering embeddings in $(\mathbb{P} \cup 0)^*$.

Lemma 11. *A word $w = w(1) \dots w(n)$ in $\mathbb{P}^*$ can cover up to n words, each formed by reducing a letter in w by 1. Reducing the letters $w(1)$ and $w(n)$ will always produce a factor, while reducing $w(i)$ for $1 < i < n$ can only produce a new factor if $w(i) \geq 2$. These words are distinct unless w is flat, in which case w only covers one word which is flat.* □

Since a larger example will also be useful in illustrating the ideas in this section, the interval $[2, 2212]$ is found in Figure 4.

It should be noted that whenever a word begins or ends with a sequence of 1's, we could produce the same factor by reducing any 1 from the sequence. However, since such a word is always a prefix or suffix, to assure our embeddings respect generalized factor order on $(\mathbb{P} \cup 0)^*$, 1's can only be reduced if they are at the beginning or end of a word. If a word is flat, our convention

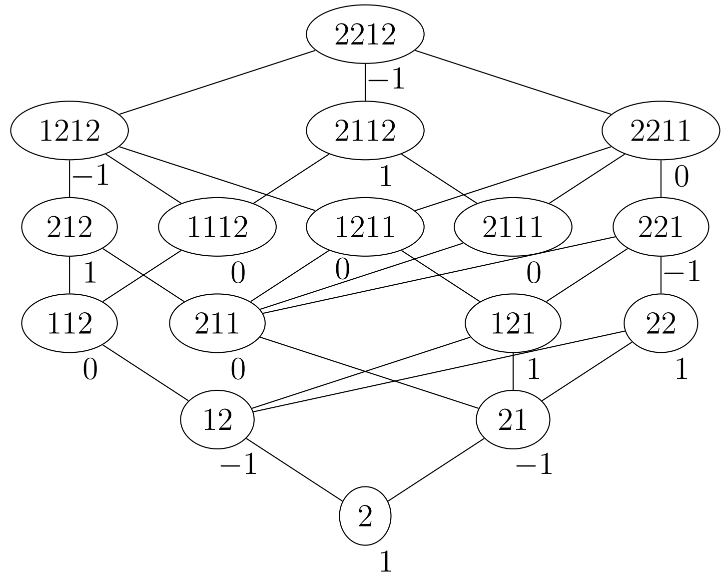

Figure 4: The interval $[2, 2212]$. The Möbius value $\mu(2, v)$ is given below of each word v.

is that only the first 1 can be reduced. For ease, we will say the letter $w(i)$ is *reducible* in the word w if $w(i) > 1$ or $w(i) = 1$ and its position i is consistent with the preceding discussion. Similarly, we say the letter $\eta(i)$ in the embedding of a word v into w is *reducible* if $\eta(i) > 1$, if $\eta(i) = 1$ is the first non-zero letter, or if v is not flat and $\eta(i) = 1$ is the last non-zero letter.

For example, the word 1211, found in the center of Figure 4, has three reducible positions: 1, 2 and 4. Notice reducing position 2 results in the word 1111, which is not in the interval $[2, 2212]$. Also, in a flat word, such as 1111, only the first position is reducible. Similarly, if a flat word is given as an embedding, such as 0110 in 2212, then only the first non-zero position, in this case position 2, is reducible.

Let $[u, w]$ be an interval in $\mathbb{P}^*$. Let $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain in $[u, w]$, where the l_i are defined by the corresponding sequence of embeddings ηv_i in the sense that

$$\eta v_i(l_i) = \eta v_{i-1}(l_i) - 1 \text{ and } \eta v_i(j) = \eta v_{i-1}(j) \text{ when } j \neq l_i.$$

This gives each maximal chain C a chain id $l_1 \ldots l_n$. By lexicographically ordering these chain ids, we produce a poset lexicographic order on the maximal chains of $[u, w]$. We will use this order to find the MSIs of the maximal chains. Examples of a poset lexicographic order were given in Section 1, Tables 1 and 4. Specific examples for this section will be given later in Tables 5 and 6.

Suppose $|u| = |w|$. Let $m_i = w(i) - u(i)$. By Lemma 11, every permutation of the multiset $M_{uw} = \{1^{m_1}, 2^{m_2}, \ldots\}$ is the chain id for a maximal chain in $[u, w]$. Since there is a single embedding of u into w, these permutations account for every maximal chain in $[u, w]$. This implies the same-length case is really a direct product of chains. If $[n]$ is the poset consisting of the integers $1, \ldots, n$, partially ordered by size, a direct product of chains is the well-known poset defined by $P = [n_1] \times [n_2] \times \ldots \times [n_m]$, in which $(i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)$ if $i_k \leq j_k$ for all $1 \leq k \leq m$. We record some relevant results about this poset here for later use. Recall that the *rank function* of any poset P in which every maximal chain has the same length is a map ρ from the elements of P to the non-negative integers defined by $\rho(x) = 0$ if x is minimal, and $\rho(y) = \rho(x) + 1$ if y covers x.

Proposition 12. *If $[u, w] \subset \mathbb{P}^*$ with $|u| = |w|$ and $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \ldots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ is a maximal chain in $[u, w]$, then each descent v_i is an MSI of C and this accounts for all MSIs of C.*

Corollary 13. *Let $[u, w] \subset \mathbb{P}^*$ with $|u| = |w|$. Then*

$$\mu(u, w) = \begin{cases} (-1)^{\rho(u,w)} & \text{if } w(i) - u(i) \leq 1 \text{ for all } 1 \leq i \leq |w|, \\ 0 & \text{otherwise,} \end{cases}$$

where ρ denotes the rank function in $\mathbb{P}^*$ and $\rho(u,w) = \rho(w) - \rho(u)$.

Corollary 14. *Let $[u,w] \subset \mathbb{P}^*$ and $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain in $[u,w]$. If $|v_i| = |v_j|$ and v_k is a descent with $i < k < j$, then v_k is an MSI in C.* $\square$

Let $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain in $[u,w]$. If a 1 is reduced in a word v_i, then $\eta_{v_i}(l_{i+1})$ must be reducible, implying that l_{i+1} corresponds to the first or last non-zero letter in the embedding η_{v_i}. The only other restrictions on reducibility occur when a word is flat. Therefore, we can determine precisely which sequences correspond to chain ids by considering what conditions imply the letters $\eta_{v_i}(l_{i+1})$ are reducible.

Let η be an embedding of u into w. Let $m_i = w(i) - \eta(i)$. Let f denote the position of the first non-zero letter in the embedding η, and ℓ denote the position of the last non-zero letter in the embedding η. We say a permutation of the multiset $M_\eta = \{1^{m_1}, 2^{m_2}, \dots\}$ is *admissible* if the last i appears before the last $i+1$ for all $1 \le i \le f-2$ and the last j before the last $j-1$ for all $\ell + 2 \le j \le |w|$. An admissible permutation is *strongly admissible* if the last $\ell + 1$ appears before either one value from the set $\{f, f+1, \dots, \ell\}$, or 2 copies of a value from the set $\{1, \dots, f-1\}$.

For an example of an admissible permutation, we consider the embedding $\eta = 0200$ in 2212. Here, $M_\eta = \{1^2, 3^1, 4^2\}$, and $f = \ell = 2$. So as long as the 3 appears after the last 4, as in the sequence $1-4-4-1-3$, we get an admissible permutation. For an example of a strongly admissible permutation, we consider the embedding $\eta = 00110$ in 22122. Here, $M_\eta = \{1^2, 2^2, 4^1, 5^2\}$, $f = 3$, and $\ell = 4$. So we need the last 2 to occur after the last 1, and the last 5 to appear before either the last 4, two 1's, or two 2's. Examples include $2-4-5-5-1-1-2$ and $1-1-2-2-5-5-4$.

Proposition 15. *Let η be an embedding of u in w, $m_i = w(i) - \eta(i)$, and $M_\eta = \{1^{m_1}, 2^{m_2}, \dots\}$.*

If u is not flat, then a sequence of numbers is the chain id for a maximal chain in $[u, w]$ ending at η if and only if it is an admissible permutation of the multiset M_η.

If u is flat, then a sequence of numbers is the chain id for a maximal chain in $[u, w]$ ending at η if and only if it is a strongly admissible permutation of the multiset M_η.

Proof. For a maximal chain in the interval $[u, w]$ to finish at the embedding η, its chain id must be a permutation of M_η. The sequence $l_1, \ldots, l_n$ is a chain id in $[u, w]$ if and only if every word in the corresponding sequence $v_1, \ldots v_n$ is a factor of w that contains u as a factor.

If v_i is not flat, 1's can only be reduced at the beginning or end of the word. Thus, when u is not flat, each v_i will be a factor of w that contains u in the embedding corresponding to η if and only if the positions in η smaller than f are reduced to 0 from left to right, and the positions greater than ℓ are reduced to 0 from right to left. The permutations in M_η that satisfy this requirement are the admissible permutations.

If v_i is flat, only the first position of the word can be reduced. Thus, when u is flat, each v_i will be a factor of w that contains u in the embedding corresponding to η if and only if the positions in η smaller than f are reduced to 0 from left to right, the positions greater than ℓ are reduced to 0 from right to left, and all positions greater than ℓ are reduced to 0 before the last 2 in the word is reduced. The permutations in M_η that satisfy this requirement are the strongly admissible permutations. □

Now that we know which permutations of a chain id produce maximal chains, let's take a look at the maximal chains in $[121, 1221]$ and $[2, 2212]$.

In $[121, 1221]$, whose maximal chains are given in Table 5, descents v_i satisfying $l_{i+1} = l_i + 1$ are MSIs. Also, there is an MSI containing an ascent. So this example highlights some of the differences between the case of $\mathbb{P}^*$ and that of an antichain.

Chain Id	v_0	l_1	v_1	l_2	v_2
1-2	1221	1	0221	2	0121
2[-]1	1221	2	[1121]	1	0121
3[-]4	1221	3	[1211]	4	1210
4[-]3	1221	4	[1220]	3	1210

Table 5: The MSIs of the maximal chains of $[121, 1221]$ in $\mathbb{P}^*$.

In $[2, 2212]$, whose maximal chains are given in Table 6, we see that every descent that does not remove two 1's from the back of a word consecutively is an MSI. Furthermore, for every MSI $C(v_i, v_j)$ of length greater than 1, the embedding of v_j into v_i is not found in any previous chain. So viewed from the right perspective, this example highlights some similarities between the case of $\mathbb{P}^*$ and that of an antichain.

As in Section 2, we begin our investigation of the MSIs by determining precisely when a descent corresponds to an MSI. Suppose v_i is a descent of a chain C. Notice that whenever interchanging l_{i+1} and l_i in the label sequence of C produces another maximal chain in $[u, w]$, the new chain is lexicographically earlier than C. This implies v_i is an MSI of C. We will invoke this line of reasoning by saying "l_{i+1} and l_i can be interchanged."

To maintain consistency with Section 2, we will say a descent v_i is a *strong descent* if $v_{i-1} \neq v_{i+1}11$ and we will say v_i is a *weak descent* if $v_{i-1} = v_{i+1}11$.

I(C) for $[2, 2212]$

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4	l_5	v_5
1-1-2-2-3	2212	1	1212	1	0212	2	0112	2	0012	3	0002
1-1-4[-4-]3	2212	1	1212	1	0212	4	[0211	4	0210]	3	0200
1-2[-]1-2-3	2212	1	1212	2	[1112]	1	0112	2	0012	3	0002
1-4[-]1-4-3	2212	1	1212	4	[1211]	1	0211	4	0210	3	0200
1-4-4[-]1-3	2212	1	1212	4	1211	4	[1210]	1	0210	3	0200
1-4[-4-]3[-]1	2212	1	1212	4	[1211	4	1210]	3	[1200]	1	0200
2[-]1-1-2-3	2212	2	[2112]	1	1112	1	0112	2	0012	3	0002
2[-4[-]4-3-]2	2212	2	[2112	4	[2111]	4	2110	3	2100]	2	2000
4[-]1-1-4-3	2212	4	[2211]	1	1211	1	0211	4	0210	3	0200
4[-]1-4[-]1-3	2212	4	[2211]	1	1211	4	[1210]	1	0210	3	0200
4[-]1-4-3[-]1	2212	4	[2211]	1	1211	4	1210	3	[1200]	1	0200
4[-]2[-]4-3-2	2212	4	[2211]	2	[2111]	4	2110	3	2100	2	2000
4-4[-]1-1-3	2212	4	2211	4	[2210]	1	1210	1	0210	3	0200
4-4[-]1-3[-]1	2212	4	2211	4	[2210]	1	1210	3	[1200]	1	0200
4-4[-]2[-]3-2	2212	4	2211	4	[2210]	2	[2110]	3	2100	2	2000
4-4-3[-]1-1	2212	4	2211	4	2210	3	[2200]	1	1200	1	0200
4-4-3[-]2[-]2	2212	4	2211	4	2210	3	[2200]	2	[2100]	2	2000

$J(C)$ intervals for $[a, abbabb]$

The set $I(C)$ only changes for the chain $2-4-4-3-2$:

Chain Id	v_0	l_1	v_1	l_2	v_2	l_3	v_3	l_4	v_4	l_5	v_5
2[-4-]4[-3-]2	2212	2	[2112	4	2111]	4	[2110	3	2100]	2	2000

Table 6: The MSIs of the maximal chains of $[2, 2212]$ in $\mathbb{P}^*$.

Proposition 16. *Suppose $[u, w] \subset \mathbb{P}^*$ and $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ is a maximal chain in $[u, w]$. Suppose v_i is a descent. Then v_i is a length 1 MSI if and only if v_i is strong.*

Proof. We will prove this proposition by considering the difference in length between v_{i-1} and v_{i+1}.

If $|v_{i-1}| = |v_{i+1}|$, v_i is an MSI by Corollary 14.

Suppose $|v_{i-1}| = |v_{i+1}| + 1$. Then either $\eta v_{i-1}(l_{i+1}) = 1$ and l_{i+1} corresponds to the first letter of v_{i-1} or $\eta v_{i-1}(l_i) = 1$ and l_i corresponds to the last letter of v_{i-1}. In the first case, by Proposition 15, l_{i+1} and l_i can be interchanged. In the second case, when v_{i+1} is not flat, Proposition 15 implies l_{i+1} and l_i can be interchanged. If v_{i+1} is flat, $\eta v_{i-1}(l_{i+1}) = 2$ because otherwise v_{i-1} would be flat and l_i could not correspond to the last letter of v_{i-1}. Therefore, if v is a flat sequence of 1's with length $|v_{i-1}|$, then the chain

$$D : w = v_0 \overset{l_1}{\to} \ldots \overset{l_{i-1}}{\to} v_{i-1} \overset{l_{i+1}}{\to} v \overset{l_{i+2}}{\to} v_{i+1} \overset{l_{i+3}}{\to} \ldots \overset{l_n}{\to} v_{n-1} \overset{l_{n+1}}{\to} v_n = u$$

is a lexicographically earlier chain than C in $[u, w]$. Since $C - v_i \subset D$, v_i is a skipped interval of C, implying it is an MSI of C.

Suppose $|v_{i-1}| = |v_{i+1}|+2$. Since v_i is a descent, our restrictions on reducing 1's imply v_{i+1} is not flat and either $v_{i-1} = 1v_{i+1}1$ or $v_{i-1} = v_{i+1}11$. In the first case, by Proposition 15, l_{i+1} and l_i can be interchanged. In the second case, v_i cannot be a length 1 MSI because there is a unique maximal chain in the interval $[v_{i+1}, v_{i-1}]$. □

In the proof of the above result, we showed that unless the word v_{i+1} is flat, all descents v_i which are MSIs have v_{i+1} in the same embedding into v_{i-1} as a previous chain. So suppose $C(v_i, v_j)$ is an MSI of the chain C, and $D < C$ is a chain satisfying $C - C(v_i, v_j) = D - D(v_i, v_j)$. Then if there is a chain $D < C$ satisfying $C - C(v_i, v_j) = D - D(v_i, v_j)$ such that embedding of v_j into v_i in D is the same as C, we should have an MSI which is a strong descent. If every chain $D < C$ satisfying $C - C(v_i, v_j) = D - D(v_i, v_j)$ has an embedding of v_j into v_i different than C, unless v_j is flat, we should have a different type of MSI.

To describe the second type of MSI in generalized factor order on $\mathbb{P}^*$, we need to develop the notion of a "principal factor." If $|u| \leq |w|$ and $u(i) \leq w(i)$ for $1 \leq i \leq |u|$, we call u a *prefix* of w. A *suffix* of w is defined analogously. If $|u| < |w|$, a prefix or suffix is *proper*. If u is both a proper prefix and a proper suffix of w, we say it is an *outer factor* of w. To simplify the language, we will call an outer factor of w not contained in a longer outer factor a *maximal outer factor of* w.

Using these definitions, we see that 21 is a prefix of 2212, 11 is a suffix of 2212, and that 211 and 111 are maximal outer factors of 2212.

From this point forward, if u is a prefix of w, we will often be dealing with the corresponding embedding. If this is the case, will abuse notation and write $u(i) = 0$ in place of introducing η and writing $\eta(i) = 0$. As an example, if $u = 22$ and $w = 2212$, we may write $w(3) > u(3)$, assuming the third position of u is 0.

Let p be a maximal outer factor of w. Suppose p is not flat. Then the *principal index* i of p in w is the smallest index such that $w(i) > p(i)$ and $w(i)$ is reducible. We say p is a *principal factor* of w if the word produced by reducing $w(i)$ by 1 no longer contains p as a suffix.

Intuitively, the principal index i of an outer factor p is the first position of w that can be reduced without removing the prefix embedding of p. The factor becomes a principal factor if reducing i removes the suffix embedding of p from w. Notice the principal index of a principal factor satisfies $i > 1$. Also, since p is not flat and a suffix, $w(i) > 1$ because the letters in the suffix embedding of p greater than 1 necessarily occur later in w than the corresponding letters in the prefix embedding of p.

For our first examples, we consider the principal factors in the intervals $[121, 1221]$ and $[2, 2212]$ given in Tables 5 and 6. Note 121 is a prefix of 1221, and it's principal index is 3. Our results below will show the MSI in the chain $3 - 4$ results from the fact that 121 is a principal factor of 1221. For the second example, the only principal factor of 2212 is 211, as the flat maximal outer factor 111 is excluded from the definition of a principal factor. As for the other words in the interval $[2, 2212]$, 1212 has 12 as a principal factor, 2112 has 2 as a principal factor, 2211 has 211 as a principal factor, 212 has 2 as a principal factor, and 221 has 21 as a principal factor. We point out that when considering the entire set of maximal chains of $[2, 2212]$, each of these principal factors immediately follows exactly one MSI not caused by a strong descent.

Some additional examples may help to further clarify the definition. The principal factors of 12222 are 1211, 1212, 1221, and 1222. The principal factors of 33133 are 3111, 3112, 3113 and 33. The words 3121 and 11211 have no principal factors.

It is important to note that a principal factor has exactly two embeddings in w. If there were a third embedding of a principal factor p, we could extend it by a sequence of 1's to create a suffix of w. This new suffix would also be a prefix since 1 is the minimum of $\mathbb{P}$, implying that p would be contained in a longer outer factor.

Using principal factors, we can identify the second type of MSI in generalized factor order on $\mathbb{P}$.

Proposition 17. *Suppose u is a principal factor of w with principal index i. Let $C : w = v_0 \xrightarrow{i} v_1 \xrightarrow{l_2} \cdots \xrightarrow{l_{n-1}} v_{n-1} \xrightarrow{l_n} v_n = u$ be the lexicographically first chain in $[u, w]$ with $l_1 = i$. Then $C(u, w)$ is an MSI of C.*

Proof. From the definition of principal factor, we conclude the word v_1 contains only the prefix embedding of u. By the definition of outer factor, w contains two embeddings of u, the prefix embedding and the suffix embedding. So by Proposition 15, there is a maximal chain in $[u, w]$ with a chain id beginning with 1. Since the principal index i of u is greater than 1, there exist chains that are lexicographically earlier than C.

Let C' be an arbitrary maximal chain that is lexicographically earlier than C. Then $l'_1 < i$

because C is the lexicographically first chain in $[u, w]$ with $l_1 = i$. From the definition of principal index, we conclude that the word v'_1 in the chain C' does not contain the prefix embedding of u. So v'_1 must contain the suffix embedding of u. Furthermore, since C is the lexicographically first chain with the prefix embedding of u, we reduce the letters at the end last, implying $v_{n-1} = u1$. Since v'_1 does not contain $u1$, the only words common to C and C' are w and u. Since C' was an arbitrary maximal chain in $[u, w]$ with $l'_1 < i$, and C is the first maximal chain in $[u, w]$ with $l_1 = i$, we conclude $C(u, w)$ is an MSI. □

Note that this proposition shows that since 121 is a principal factor of 1221, the chain in the interval $[121, 1221]$ with chain id $3 - 4$ has $C(1221, 121)$ as an MSI (see Table 5). Also, since 211 is a principal factor of 2212, the chain in the interval $[2, 2212]$ with chain id $2-4-4-3-2$ has $C(211, 2212)$ as an MSI (see Table 6). As a last example, since 21 is a principal factor of 221, the chain in the interval $[2, 2212]$ with chain id $4 - 4 - 2 - 3 - 2$ has $C(21, 221)$ as an MSI.

To complete the characterization of the MSIs, we will need a precise description of the lexicographically first chain in an interval $[u, w]$ that contains an embedding η of u. The chain id of this chain, C_η, is the lexicographically first permutation of M_η that is the chain id of a maximal chain. Using Proposition 15, we will describe the structure of C_η when u is not flat. First, it reduces all the letters before the support of the embedding to 0 in order from left to right. Next, it reduces all the letters in the support of the embedding down to the corresponding u-value in order from left to right. In the third step, C_η reduces all letters beyond the support of the embedding to 1 from left to right. Finally, once we reach the end of w, all the 1's beyond the support of the embedding are reduced to 0 from right to left.

For example, the first admissible chain of $[121, 1221]$ ending at the prefix embedding has chain id $3 - 4$. The first admissible chain of $[2, 2212]$ ending at the prefix embedding has chain id $2-4-4-3-2$.

If u is not flat, call C_η the *first admissible* chain ending at η. Recall that a sequence is *unimodal* if it consists of a weakly increasing sequence followed by a weakly decreasing sequence. So a sequence $l_1 \dots l_n$ is unimodal if $l_1 \leq \dots \leq l_i \geq \dots \geq l_n$ for some index i. This discussion implies the following lemma.

Lemma 18. *Suppose $[u, w] \subset \mathbb{P}^*$ and η is an embedding of u into w. Let ℓ be the index of the largest non-zero number in η. If u is not flat, C_η has as its chain id the unique unimodal permutation of M_η with decreasing suffix $|w|, |w|-1, \dots, \ell+1$.* □

To make better use of the Lemma, we introduce the notation 1^m to represent a sequence of m 1's. For example, if considering the prefix embedding of u into w, then $u1^m$, where $m = |w| - |u|$, is the word in C_u which precedes the decreasing suffix $|w|, |w|-1, \dots, \ell+1$.

The following Theorem completes the characterization of the MSIs.

Theorem 19. *Suppose $[u, w] \subset \mathbb{P}^*$ and $C : w = v_0 \xrightarrow{l_1} v_1 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} v_{n-1} \xrightarrow{l_n} v_n = u$ is a maximal chain in $[u, w]$. Then $C(v_i, v_j)$ is an MSI of C if and only if $C(v_i, v_j)$ consists of a single strong descent, or v_j is a principal factor of v_i, l_{i+1} is the principal index of v_j with respect to the embedding ηv_j of v_j into w, and $C[v_i, v_j]$ is the first admissible chain in $[v_j, v_i]$ ending at the prefix-embedding of v_j.*

Proof. The reverse implication follows from Propositions 16 and 17.

Suppose $C(v_i, v_j)$ is an MSI of C. By the definition of a poset lexicographic order, $C(v_i, v_j)$ is an MSI of C if and only if it is an MSI of $C[v_i, v_j]$ in the interval $[v_j, v_i]$. Thus, it suffices to consider the case $w = v_i$ and $u = v_j$. Corollary 14 implies that when $|u| = |w|$, any MSI consists of a single descent. Proposition 16 states that if $C(w, u)$ is an MSI and it consists of a single descent, then $w \neq u11$.

To finish the proof, it suffices to consider the case when $|u| < |w|$ and $C(w, u)$ is an MSI of C that does not consist of a strong descent. Our first goals are to establish that w has a prefix embedding of u and $C[w, u]$ is the first admissible chain ending at the prefix embedding. Let η be the embedding of u into w at the end of the chain C. Note that any descent v_k in C is a weak descent satisfying $v_{k-1} = v_k 1 = v_{k+1} 11$ since otherwise, by Proposition 16, $C(v_{k-1}, v_{k+1})$ would be an MSI, contradicting the minimality of $C(w, u)$. If $k + 1 \neq n$, this forces $l_{k+2} < l_{k+1} = l_k - 1$, implying that v_{k+1} is also a weak descent. By continually applying this idea, we find that any descents contained in $C(w, u)$ occur in a single sequence of weak descents at the end of this interval, and the corresponding labels form a decreasing sequence of consecutive numbers. To see η is the prefix embedding of u into w, suppose for a contradiction that the set M_η contains a 1. Since any descents occur in a sequence at the end of the interval, it follows that $l_1 = 1$ or $l_n = 1$. If $l_n = 1$, then v_{n-1} is descent satisfying $u11 = v_{n-2}$, implying u is the empty word. However, this contradicts Proposition 15 because v_{n-1} is obtained by trimming a 1 from the back of a flat word. Suppose $l_1 = 1$. Then every chain C' lexicographically earlier than C has $l'_1 = 1$ and thus contains v_1, contradicting the fact that $C(w, u)$ is an MSI. Therefore, η is the prefix embedding of u into w, allowing us to write $\eta = u$. Since $|u| < |w|$, we must reduce position $|w|$ by the end of the chain. Note that the label $|w|$ can only be followed by another $|w|$ or the sequence of labels $|w| - 1, \ldots, |u| + 1$, which leads to the sequence of weak descents. It follows that C must contain the word $u1^m$, where $m = |w| - |u|$. This implies u is not flat. So by Lemma 18, C is the first admissible permutation of M_u. Thus, we have shown C is the first admissible chain ending at the prefix-embedding of u.

Next, we will show u is an outer factor of w by showing it is a suffix of w. Recall that

C is not the lexicographically first chain in $[u, w]$ because $C(w, u)$ is an MSI. Let C' be the lexicographically first chain in the interval $[u, w]$. Lemma 18 implies C' contains the word $v_{n-1} = u1$ unless it ends at the suffix embedding of u into w. However, C' cannot contain v_{n-1} as otherwise $C - C(w, v_{n-1}) \subset C'$, contradicting the assumption that $C(w, u)$ is an MSI. Therefore, C' must end at the suffix embedding of u into w. Thus, u is an outer factor of w which is not flat.

To establish that u is a principal factor of w, it remains to show that u is not contained in a longer outer factor, that l_1 satisfies the definition of a principal index, and that reducing $w(l_1)$ removes the suffix embedding.

Suppose u were contained in a longer outer factor of w. Then u would be a prefix of this larger factor, which is a suffix of w. Therefore, there exists an outer factor v of w such that $v = u1^m$, where $m = |v| - |u|$. By Lemma 18, C must contain this word. Since C is the first admissible chain ending at the prefix-embedding of u, $C[w, v]$ is the first admissible chain ending at the prefix-embedding of v. Let D be the lexicographically first chain in the interval $[v, w]$. Since v contains u, it is not flat. So the structure of D is determined by Lemma 18, which implies D ends at the embedding of v into w that is farthest to the right. So D ends at the suffix embedding of v into w. Let $\hat{C} = D[w, v] \cup C[v, u]$. Then $\hat{C}$ is a maximal chain in $[u, w]$ lexicographically earlier than C. But $C - C(w, v) \subset \hat{C}$, contradicting the fact that $C(w, u)$ is an MSI. So u is not contained in a longer outer factor of w.

Lemma 18 implies that l_1 is the first index such that $w(l_1) > u(l_1)$ and $w(l_1)$ is reducible. Suppose v_1, the word produced by reducing $w(l_1)$ by 1, contains the suffix embedding of u. Then u is an outer factor of v_1 and $|v_1| = |w|$. By Lemma 18, $C[v_1, u]$ is the first admissible chain ending at the prefix-embedding of u. Since v_1 also contains the suffix embedding of u, Lemma 18 implies that $C[v_1, u]$ is not the lexicographically first chain in $[u, v_1]$. Let D be the lexicographically first chain in the interval $[u, v_1]$, and $\hat{C} = w \cup D$. Then $\hat{C}$ is a maximal chain in $[u, w]$ lexicographically earlier than C. But $C - C(v_1, u) \subset \hat{C}$, contradicting the fact that $C(w, u)$ is an MSI. Therefore, v_1 does not contain the suffix embedding of u, and we have established that u is a principal factor of w with principal index l_1, completing the proof. □

Theorem 19 shows there are precisely two types of MSIs. We will refer to the first type by saying "an MSI caused by a strong descent," or just by referring to a word v_i as a strong descent. We will refer to the second type by saying "an MSI caused by the principal factor p_{v_i}." This language implies that p_{v_i} is a principal factor of the word v_i, l_{i+1} is its principal index with respect to the embedding η_{v_i} of v_i into w, the interval $C(v_i, p_{v_i})$ is an MSI in the related chain C, and $C[v_i, p_{v_i}]$ is the first admissible chain in $[p_{v_i}, v_i]$ ending at the prefix embedding of p_{v_i}.

Our next goal is to determine precisely which maximal chains are critical chains in an interval in $\mathbb{P}^*$. Our first objective is to consider those critical chains that consist entirely of strong descents.

Proposition 20. *Suppose $[u, w] \subset \mathbb{P}^*$, (u, w) is non-empty, and w is not flat. Suppose η is an embedding of u into w. Then there is a critical chain C in $[u, w]$ ending at η that consists entirely of (strong) descents if and only if $w(i) - \eta(i) \leq 1$ for all i, $\eta(2) \neq 0$, and $\eta(|w| - 1) \neq 0$. Furthermore, these conditions imply $|w| - |u| \leq 2$ and that $[u, w]$ has at most two critical chains consisting entirely of (strong) descents.*

Proof. Suppose C is a critical chain in $[u, w]$ that ends at η and consists entirely of descents. Then $l_1 > l_2 > \ldots > l_{n-1} > l_n$. This implies $w(i) - \eta(i) \leq 1$ for all i as each letter can be reduced at most once. Since 1's may only be reduced at the beginning or end of a word, the decreasing label sequence also implies $\eta(2) \neq 0$ since it is not possible to reduce position 1 before position 2. To finish this implication, suppose for a contradiction that $\eta(|w|-1) = 0$. Since each l_i is distinct and the sequence is decreasing, it follows that $w = v_2 11$ as the last letter must be reduced to zero before the second to last can be. So by Theorem 19, the MSI containing v_1 is not caused by a descent and therefore must be caused by a principal factor $p(w)$. Since the chain id is decreasing, the corresponding principal index l_1 must be $|w|$. By the definition of principal index, $w(|w|) > 1$. This contradicts $w = v_2 11$, implying $\eta(|w|-1) \neq 0$.

Suppose $w(i) - \eta(i) \leq 1$ for all i, $\eta(2) \neq 0$, and $\eta(|w|-1) \neq 0$. The first assumption implies each entry in M_η is distinct. If u is not flat, then the decreasing and $\eta(2) \neq 0$ conditions imply the two conditions in the definition of admissibility. If u is flat, having $w(i) - \eta(i) \leq 1$ implies w consists only of 1's and 2's. So the decreasing permutation of M_η is strongly admissible because once the last 2 has been reduced to a 1, at most an initial 1 remains to be reduced since $\eta(2) \neq 0$. Therefore, Proposition 15 implies that the decreasing permutation of M_η is the chain id of a maximal chain C in $[u, w]$. Since each entry in the chain id is distinct, C consists entirely of descents. Furthermore, since $\eta(|w|-1) \neq 0$, $w \neq v_2 11$. So no descent has the property that $v_{i-1} \neq v_{i+1} 11$ as a decreasing chain id only permits the removal of consecutive 1's from the end of a word at the beginning of the chain. Therefore, by Theorem 19 each is a length 1 MSI in C, implying that $J(C)$ covers $C(w, u)$. Thus, C is a critical chain consisting entirely of descents and ending at η.

To see each interval $[u, w]$ has at most two critical chains consisting entirely of descents, note that $|u| \geq |w| - 2$ because the first and last positions in an embedding satisfying the necessary restrictions are the only ones which can be zero. Whenever $|u| = |w|$, there is also only one such embedding. If $|u| = |w| - 2$ there can only be one such embedding as only the first and last letters of w can be reduced, implying $0u0$ is the only embedding of u into w. If $|u| = |w|-1$, then it is possible the prefix embedding and the suffix embedding satisfy the necessary restrictions. Thus, there are at most two embeddings of u that satisfy the restrictions for a critical chain consisting entirely of descents. Since there is only one weakly decreasing permutation of any set M_η, this completes the proof. □

We note the example $[121, 1221]$ from Table 5 provides a nice illustration of this proposition, as the critical chains whose chain ids are $2-1$ and $4-3$ have MSIs caused by strong descents. As stated in the theorem, we can have at most 2 critical chains consisting entirely of strong descents.

The remaining critical chains must contain at least one MSI caused by a principal factor. So these chains contain a principal factor of w or a principal factor of some v_i with the property that $C(w, v_{i+1})$ consists entirely of descents v_k such that $v_{k-1} \neq v_{k+1}11$. The second possibility is easier to work with if we consider the relationship between the principal factor of v_i and w. This is the content of the next proposition.

Proposition 21. *If a critical maximal chain C in $[u, w]$ does not consist entirely of descents, then it contains a $v_i \in (u, w]$ with a principal factor $p_{v_i} \geq u$ such that $p_{v_i}1^m$ is a maximal outer factor of w for some $m \geq 0$.*

Proof. By Theorem 19, C contains at least one MSI caused by a principal factor. Suppose $C(v_i, p_{v_i})$ is the first such MSI. If p_{v_i} is a principal factor of w, there is nothing to show. If not, then by Theorem 19, $w \neq v_2 11$ and the chain id of C decreases through l_{i+1}. This implies only the last letter of w can be reduced to 0 in η_{v_i}. So v_i and thus p_{v_i} are prefixes of w.

To complete the proof, it suffices to show that $p_{v_i}1^m$ is an outer factor for some $m \geq 0$ because then the theorem holds for $p_{v_i}1^\ell$, where $\ell = |v| - |p_{v_i}|$ and v is a maximal outer factor of w containing pv_i. Recall that pv_i is an outer factor of v_i, which is a prefix of w such that only the last letter of w can be reduced to 0 in ηv_i. Therefore, pv_i or pv_i1 is an outer factor of w so that desired statement holds for $m = 0$ or $m = 1$. □

In order to have a critical chain in $[u, w]$ involving an MSI resulting from a principal factor pv_i, pv_i needs to be contained in a different MSI or pv_i must equal u. By Theorem 19, pv_i could be contained in one of three types of MSI: an MSI caused by a strong descent, an overlapping MSI caused by a principal factor, or an adjacent MSI caused by a principal factor. We will show the last possibility cannot occur.

Proposition 22. *Suppose $C(v_i, pv_i)$ is an MSI of a critical chain C caused by the principal factor pv_i. Then pv_i is contained in an MSI caused by a descent or an overlapping MSI caused by a principal factor.*

Proof. Let $v_{j+1} = pv_i$. Suppose for a contradiction that $C(v_j, pv_j)$ is an MSI caused by a principal factor. Since $C(v_i, v_{j+1})$ is an MSI caused by a principal factor, Theorem 19 implies $C[v_i, v_{j+1}]$ is the first admissible chain ending at the corresponding embedding. So by Lemma 18, $v_j = v_{j+1}1$. However, this implies the value at the principal index l_{j+1} of pv_j in v_j is $v_j(l_{j+1}) = 1$. This contradicts the fact that $v_j(l_{j+1}) > 1$, as pv_j was assumed to be a principal factor of v_j. So $C(v_j, pv_j)$ is not an MSI caused by a principal factor, completing the proof. □

While refining the set of MSIs $I(C)$ to create the set $J(C)$, we reduce any intervals remaining in $I(C)$ that overlap with an earlier MSI and remove any that are no longer containment minimal after this reduction. The next proposition states that overlapping MSIs in the set $I(C)$ of a critical chain must come in pairs.

Proposition 23. *An MSI of a critical chain can overlap with at most one other MSI.*

Proof. Let C be maximal chain of $[u, w]$. Suppose $C(v_{i_1}, v_{j_1}), C(v_{i_2}, v_{j_2}), \ldots, C(v_{i_k}, v_{j_k})$ is a maximal sequence of three or more overlapping MSIs satisfying $i_1 < i_2 < \ldots < i_k$. Since MSIs caused by strong descents always have length one, they cannot be involved in overlapping MSIs. So by Theorem 19, each v_{j_ℓ} is a principal factor of v_{i_ℓ}, $l_{i_\ell+1}$ is the principal index of v_{j_ℓ} with respect to the embedding ηv_{i_ℓ} of v_{i_ℓ} into w, and $C[v_{j_\ell}, v_{j_\ell}]$ is the first admissible chain in $[v_{j_\ell}, v_{j_\ell}]$ ending at the prefix embedding of v_{j_ℓ}. By Lemma 18, any two consecutive MSIs must have unimodal label sequences that overlap. Note each $\eta v_{i_\ell}(l_{i_\ell+1}) > 1$ because $l_{i_\ell+1}$ corresponds to a principal index. So each label sequence must contain at least one entry before reaching its decreasing suffix. Therefore, the overlap between two intervals must start on the weakly increasing portion of the label sequences. Since no letters are removed before reaching the decreasing suffix, this implies $|v_{i_1}| = |v_{i_2}| = \ldots = |v_{i_k}|$. So each MSI $C(v_{i_\ell}, v_{j_\ell})$ includes all of the decreasing suffix of $C(v_{i_1}, v_{j_1})$, and each MSI except the first contains all of the decreasing suffix of $C(v_{i_2}, v_{j_2})$.

Consider the process of refining the set of intervals $I(C)$ to $J(C)$. The first interval in the sequence of overlapping intervals, $C(v_{i_1}, v_{j_1})$, is added to $J(C)$. Any other interval in $I(C)$ has its overlap with this interval removed, and any interval that is no longer containment minimal is discarded. However, each interval $C(v_{i_\ell}, v_{j_\ell})$ left in $I(C)$ contains the truncated portion of $C(v_{i_2}, v_{j_2})$. So $C[v_{j_1}, v_{j_2})$ is added to $J(C)$ and the rest of the intervals are discarded. Thus, v_{j_2} is not in an interval in $J(C)$ and C is not a critical chain. $\square$

We now have enough information to give a nice description of the structure of a critical chain.

Theorem 24. *Let* $C : w = v_0 \xrightarrow{l_1} v_1 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} v_{n-1} \xrightarrow{l_n} v_n = u$ *be a maximal chain of* $[u, w]$. *Then* C *is a critical chain if and only if* $C(w, u)$ *can be written as a sequence of intervals*

$$C[v_{i_1} = v_1, v_{i_2}) \cup C[v_{i_2}, v_{i_3}) \cup \dots \cup C[v_{i_{k-1}}, v_{i_k} = u)$$

where each interval $C[v_{i_j}, v_{i_{j+1}})$ *is one of the following three types:*

1. $C[v_{i_j}, v_{i_{j+1}})$ *is an MSI caused by the strong descent* v_{i_j}.

2. $C[v_{i_j}, v_{i_{j+1}})$ *is an MSI caused by the principal factor* $v_{i_{j+1}}$ *of the word* $v_{i_j - 1}$.

3. *The word* $v_{i_{j+1}}$ *is a principal factor of a word in* $C[v_{i_{j-1}}, v_{i_j})$ *and satisfies* $v_{i_{j+1}} 1^m = v_{i_j}$ *for* $m = |v_{i_j}| - |v_{i_{j+1}}| > 0$. *The value* m *is unique in the sense that no other word satisfies the description of* $v_{i_{j+1}}$ *for another value* m.

Furthermore, type (1) intervals are followed by intervals of type (1) or (2), type (2) intervals are followed by intervals of type (1) or (3), and type (3) intervals are followed by intervals of type (1). Finally, only intervals of type (1) or (2) can begin the decomposition.

Proof. For the forward implication, we need to show that in a critical chain, each interval in the set $J(C)$ is one of the three interval types listed and the order of the intervals respects the ordering restrictions of the proposition. By Theorem 19, each MSI is caused by a strong descent or a principal factor. So intervals of type (1) and type (2) can occur in $J(C)$. Next, since Proposition 23 states that overlapping intervals must occur in pairs, we need to show that the second interval in an overlapping pair is an interval of type (3). Let $C[v_{i_j}, v_{i_{j+1}})$ be the remainder of an interval from $I(C)$ that was reduced in $J(C)$. From the proof of Proposition 23, $C[v_{i_j}, v_{i_{j+1}})$ must be the remainder of the MSI $C(v, v_{i_{j+1}})$ caused by the principal factor

$v_{i_{j+1}}$, where the preceding interval $C[v_{i_{j-1}}, v_{i_j})$ is of type (2) and $|v| = |v_{i_{j-1}}|$. Since 1's in these MSIs can only be trimmed from the back, $v_{i_j} = v_{i_{j+1}} 1^m$ for $m = |v_{i_j}| - |v_{i_{j+1}}|$. To assure this is the second interval in a pair of overlapping intervals in $I(C)$, there cannot be another word in $C[v_{i_j}, u]$ that is a principal factor of a word in $C[v_{i_{j-1}}, v_{i_j})$. In particular, this means there can be no word in $C[v_{i_j}, u]$ satisfying the previous description of $v_{i_{j+1}}$ for another value m. Therefore, $C[v_{i_j}, v_{i_{j+1}})$ is an interval of type (3) and all intervals reduced from overlapping MSIs in $I(C)$ are of this type. Thus, there are no other types of intervals that can occur in $J(C)$. Finally, we need to verify the ordering restrictions. Type (1) intervals cannot be followed by intervals of type (3) because the half-open interval under consideration is empty. Intervals of type (2) cannot be followed by type (2) intervals by Proposition 22. Type (3) intervals cannot be followed by type (2) intervals by Proposition 22, and cannot be followed by type (3) intervals because the value m is unique. And type (3) intervals cannot begin the decomposition because the half-open interval under consideration does not exist. This completes the proof of this implication.

For the backwards implication, since the given set of intervals covers C, we need to show the interval types are always reductions of MSIs in $I(C)$. Type (1) and type (2) intervals are MSIs by Theorem 19. Type (3) intervals must occur after a type (2) interval by definition. Suppose $C[v_{i_j}, v_{i_{j+1}})$ is a type (3) interval so that the word $v_{i_{j+1}}$ is a principal factor of a word v in $[v_{i_{j-1}}, v_{i_j})$. Lemma 18 implies that the label sequence of $C[v_{i_{j-1}}, v_{i_j})$ is unimodal. So Lemma 18 also implies that v, the word which has $v_{i_{j+1}}$ as a principal factor, occurs before the decreasing suffix of the label sequence of $C[v_{i_{j-1}-1}, v_{i_j}]$. Thus, $C[v, v_{i_{j+1}}]$ is the first admissible chain in the corresponding interval $[v_{i_{j+1}}, v]$ with $v_{i_{j+1}}$ in the prefix embedding. So by Theorem 19, $C(v, v_{i_{j+1}})$ is the MSI in $I(C)$ that is reduced to the interval $C[v_{i_j}, v_{i_{j+1}})$.

Furthermore, this reduced interval appears in $J(C)$ because $v_{i_{j+1}} 1^m = v_{i_j}$ for a unique value of m. Indeed, if m were not unique, a different reduced interval would either be contained inside this one or have this interval contained within it; both cases contradict the fact that $J(C)$ covers C. Since the ordering restrictions respect Propositions 22 and 23, this completes the proof of this implication. □

In the interval $[121, 1221]$, the critical chain with chain id $2-1$ consists of a single type (1) interval, $C[1121, 121) = C(1221, 121)$ (see Table 5). In the interval $[2, 2212]$, the critical chain with chain id $2-4-4-3-2$ consists of the type (2) interval $C[2112, 211) = C(2212, 211)$ followed by the type (3) interval $C[211, 2)$, which is truncated from the MSI $C(2112, 2)$ (see Table 6).

Using Theorem 24, we can separate the critical chains of $[u, w]$ into three groups based on what happens after the first type (2) interval in the chain: critical chains with no type (2) intervals, critical chains whose first type (2) interval is the last interval or is followed by a type (1) interval, and critical chains whose first type (2) interval is followed by a type (3) interval. Notice the first group is investigated in Proposition 20. So if we can find the total contribution of all critical chains whose first interval is a given type (2) interval, we will have enough information to write down a recursive formula for the Möbius value.

Let C be a critical chain. If $J(C)$ contains any type (2) intervals, then by Theorem 24 the first type (2) interval in the set must either be the first interval or occur after a sequence of type (1) intervals. Furthermore, if the first type (2) interval is followed by a type (3) interval, Theorem 24 implies the principal factor causing the type (3) interval must be a certain prefix of the principal factor causing the type (2) interval. So to facilitate the exposition, we need to make several new definitions.

Define $w \setminus 1^m$ to be the word that results when m 1's are removed from the suffix of w, or as undefined if w ends in less than m 1's.

Define a word v to be a *base of* w if $v(j) = w(j)$ or $v(j) = w(j) - 1$ for all j and $|v| = |w|$ or $|w| - 1$. Define the *degree* of a base to be the number of indices j for which $v(j) = w(j) - 1$. This way, if a word v_i in a chain C is a base of w, then it is a base of degree i. When $i > 0$, the language "v_i is based in C" will indicate the word v_i of the chain C is a base of w and the labels $l_1, \ldots, l_i$ of C form a decreasing sequence. In Proposition 20, we showed that this condition forces each word $v_1, \ldots, v_{i-1}$ to be a strong descent.

Suppose v_i is a base of w of degree i. Let l be the index of the smallest position satisfying $v_i(l) = w(l) - 1$, or $|w| + 1$ if $i = 0$. We define any word p_{v_i} that is a principal factor of v_i and whose principal index takes a value less than l to be a *principal factor of* w *of degree* i. This definition is an extension of the definition of a principal factor because a principal factor of w satisfies the new definition for $i = 0$. So to maintain consistency in the language, when a degree is not noted in the language or the notation, the assumption will be that the principal factor has degree 0.

In the example $[2, 2212]$ found in Table 6, the bases of 2212 of degree 1 are 1212, 2112, and 2211. The base 2211 of admits 211 as a principal factor of degree 1, but 2112 and 1212 have principal factors with principal indices greater than the smallest position satisfying $v_i(l) = w(l) - 1$. Returning to an example first given after the definition of a principal factor, the degree 2 base 12211 of the word 12222 has 1211 as a principal factor, making 1211 a principal factor of degree 2. In fact, 1211 is a principal factor of 4 bases of 12222: 12211, 12212, 12221. and 12222 itself.

Let p_{v_i} be a principal factor of w of degree i. By Theorem 24, p_{v_i} could cause the first type

(2) interval $C(v_i, p_{v_i})$ in some critical chain C because when v_i is based in C, the condition on the principal index of p_{v_i} implies v_i is a strong descent. Notice p_{v_i} must be an outer factor of w or $w \setminus 1$. The latter is a possibility when $w \setminus 1$ is defined because, by the definition of a base, $v_i(|w|)$ could be 0. Furthermore, the proof of Proposition 21 implies $p_{v_i}1^m$ is a maximal outer factor of w for some $m \geq 0$. Finally, p_{v_i} could be a principal factor of other bases v_j of w.

Suppose p_{v_i} is a principal factor of v_i. Let v_i' be the word that results when the letter in v_i at the principal index of p_{v_i} is reduced by 1. That is, $v_i'(j) = v_i(j) - 1$ when j is the principal index of p_{v_i} and $v_i'(j) = v_i(j)$ for all other indices j. Define the *primary prefix* $x(p_{v_i})$ of a principal factor p_{v_i} to be the factor $p_{v_i} \setminus 1^m$, where m is the smallest positive integer such that $p_{v_i} \setminus 1^m$ is an outer factor of v_i'. So if no $m > 0$ satisfies the restriction, the primary prefix is undefined.

Intuitively, the primary prefix of a principal factor p_{v_i} is the longest proper prefix of p_{v_i} that has two embeddings in v_i' and only differs from p_{v_i} by some number of 1's removed from the back. Notice the primary prefix depends on the word v_i', or perhaps more intuitively, the word v_i and the principal index of p_{v_i} in v_i. In the example $[2, 2212]$, 2 is the primary prefix of the principal factor 211 of 2212. However, in the interval $[2, 2222]$, which contains the previous interval as a subinterval, 21 is the primary prefix of the principal factor 211 of 2222. And in the interval $[2, 2211]$, the principal factor 211 does not have a primary prefix.

The following proposition asserts that the primary prefix is the only word that can cause a type (3) interval after the type (2) interval $C(v_i, p_{v_i})$.

Proposition 25. *Let C be a maximal chain of $[u, w]$ and suppose $C(v_i, p_{v_i})$ is a type (2) interval in the set $J(C)$. Then $C(v_i, p_{v_i})$ is followed by a type (3) interval $C[p_{v_i}, x)$ in $J(C)$ if and only if x is the primary prefix of p_{v_i} and x appears in C.*

Proof. First suppose x is the primary prefix of p_{v_i} and that it appears in C. By definition, x is a maximal outer factor of v_i'. Thus, it is a maximal outer factor of any word in $C(v_i, p_{v_i})$ of which it is an outer factor. Let v_m be the last word in the interval $C(v_i, p_{v_i})$ that contains x as an outer factor. We will show x is a principal factor of v_m with principal index l_{m+1}. Since p_{v_i} is a prefix of v_{m+1}, x is as well. So v_{m+1} no longer contains the suffix embedding of p_{v_i}. Furthermore, since $C[v_i, p_{v_i}]$ is the first admissible chain in $[p_{v_i}, v_i]$ ending at the prefix embedding of p_{v_i}, for all $l_{i+1} < k < l_{m+1}$, either we have $v_m(k) = p_{v_i}(k)$ or we have $v_m(k) = 1$ and $p_{v_i}(k) = 0$. Therefore, for all $l_{i+1} < k < l_{m+1}$, either we have $v_m(k) = x(k)$ or we have $v_m(k) = 1$ and $x(k) = 0$. This implies $C[v_m, x]$ is the first admissible chain in $[x, v_m]$ ending at the prefix embedding of x and $v_m(l_{m+1})$ is the first reducible letter in v_m greater than the corresponding position in the prefix embedding of x. Thus, l_{m+1} satisfies the definition of a principal index, which implies x is a principal factor of v_m and $C(v_m, x)$ is an MSI of C. Since x is a maximal outer factor of v_i', and $C(v_i, p_{v_i})$ is a type (2) interval, no word between p_{v_i} and x can be a principal factor of a word v_k in C. Therefore, $C(v_m, x)$ is reduced to the type (3) interval $C[p_{v_i}, x)$ in $J(C)$, completing the reverse implication.

Now suppose that $C(v_i, p_{v_i})$ is followed by a type (3) interval $C[p_{v_i}, x)$ in the set $J(C)$. Then by Theorem 24, x is a principal factor of a word v in $C(v_i, p_{v_i})$ and $x = p_{v_i} \setminus 1^m$ for some m. From the proof of Theorem 24, we know $|v| = |v_i'|$ and $v \leq v_i'$, implying x is an outer factor of v_i'. By Theorem 24 part (3), it suffices to show that x is a maximal outer factor of v_i'. For a contradiction, suppose x is not a maximal outer factor of v_i'. Then a word of the form $x1^k$ would be a maximal outer factor of v_i' and by the argument in the paragraph above, a principal factor of some word in $C(v_i, p_{v_i})$. Thus, $C(v, x1^k)$ would be an MSI in $I(C)$. This would be reduced to the interval $C[p_{v_i}, x1^k)$, which is contained in $C[p_{v_i}, x)$, implying that $C[p_{v_i}, x)$ could not be in $J(C)$. This is a contradiction. Thus, x is a maximal outer factor of v_i', implying it is the primary prefix of the word p_{v_i}. □

Let $\mu(u,v)$ be the normal Möbius function if u and v are both elements of $\mathbb{P}^*$, or zero if either is undefined. Define the function $\nu(u,v)$ to be

$$\nu(u,v) = \sum_{i \geq 0} \mu(u, v \setminus 1^i).$$

Notice all the terms in the summation will be zero beyond the largest value $i = m$ for which $v \setminus 1^m$ is defined, or the smallest value $i = m$ for which $v \setminus 1^m \leq u$.

We are now ready to consider the contribution of critical chains whose first type (2) interval is a specific interval. This proof is very technical, and we have broken it up into several cases to make it easier to follow.

Proposition 26. *Suppose $C(v_i, p_{v_i})$ is the first type (2) interval of some critical chain C of $[u,w]$. Then p_{v_i} is a principal factor of w of degree i and v_i is based in C. Furthermore, the contribution to the Möbius value $\mu(u,w)$ of all critical chains in $[u,w]$ that have $C(v_i, p_{v_i})$ as the first type (2) interval is*

$$(-1)^i \left(\nu(u, p_{v_i}) - \nu(u, x(p_{v_i})) \right),$$

where $x(p_{v_i})$ is the primary prefix of p_{v_i}.

Proof. Since $C(v_i, p_{v_i})$ is a type (2) interval, Theorem 24 implies p_{v_i} must be a principal factor of v_i. If $i = 0$, p_{v_i} is a principal factor of w of degree 0. If $i \neq 0$, then since C is a critical chain and $C(v_i, p_{v_i})$ is the first type (2) interval, Theorem 24 implies v_i and the words $v_1, \ldots, v_{i-1}$

which precede it must each be contained in a type (1) interval. Since $v_2 11 \neq v_0$, $|v_i| = |w|$ or $|w| - 1$. Also, the i labels preceding v_i are distinct and form a decreasing sequence. Therefore, $v_i(j) = w(j)$ or $v_i(j) = w(j) - 1$. This implies v_i is a base of w, which allows us to conclude v_i is based in C. Since v_i is in a type (1) interval, $l_{i+1} < l_i$, implying p_{v_i} is a principal factor of w of degree i.

To facilitate the discussion, we will first consider the case when p_{v_i} ends with a letter other than 1. Then $p_{v_i} \setminus 1$ is undefined, so $\nu(u, p_{v_i}) = \mu(u, p_{v_i})$. Note that the set $J(C)$ must cover the entire chain $C(w, u)$ for C to be critical, and we already know $J(C)$ covers $C(w, p_{v_i})$ when $C(v_i, p_{v_i})$ is the first type (2) interval. Furthermore, p_{v_i} does not have a primary prefix, so by Theorem 24, $C(v_i, p_{v_i})$ can only be followed by an interval of type (1) in $J(C)$. Recall from the proof of Theorem 19 that the last label in $C(v_i, p_{v_i})$, l_k, is the last non-zero position in $p_{v_i} 1$. So $l_k = |p_{v_i} 1|$ by Lemma 18. Thus, in any critical chain containing the type (2) interval $C(v_i, p_{v_i})$, $l_{k+1} < l_k$. We now consider two subcases depending on whether $u = p_{v_i}$.

First suppose $u < p_{v_i}$. Since p_{v_i} does not end in a 1, it must be a strong descent. So in any maximal chain containing $C[v_i, p_{v_i}]$, p_{v_i} is contained in a type (1) interval. It follows that $J(C[p_{v_i}, u])$ must cover $C(p_{v_i}, u)$, whose first label corresponds to l_{k+1} in C. Since any choice of label l_{k+1} puts p_{v_i} in a type (1) interval, the set of critical chains in $[u, w]$ that contain $C(v_i, p_{v_i})$ are in a one-to-one correspondence with the critical chains of $[u, p_{v_i}]$. The corresponding map between the sets $J(C)$ and $J(C[p_{v_i}, u])$ is given by the addition or subtraction of the $i+2$ intervals $v_1, \ldots, v_i, C(v_i, p_{v_i})$, and p_{v_i}. Therefore, the number of intervals in $J(C)$ is $i + 2 + |J(C[p_{v_i}, u])|$. So by Theorem 3, the total contribution of all these chains to $\mu(u, w)$ is

$$(-1)^{i+2} \mu(u, p_{v_i}) = (-1)^i \mu(u, p_{v_i}).$$

Now suppose $u = pv_i$. The above formula follows from Theorem 3 because $\mu(pv_i, pv_i) = 1$ and there are precisely $i+1$ intervals in $J(C) = J(C([w, pv_i]))$. So in this case, $\mu(u, w) = (-1)^i$. Since pv_i does not have a primary prefix and $\nu(u, pv_i) = \mu(u, pv_i)$, this completes the proof for this case.

Now we consider the case pv_i ends with a 1. If $u = pv_i$, the argument does not change from the one above. If $u < pv_i$, there are two significant differences from the previous discussion. First, while pv_i is still a descent v_k in any maximal chain containing $C(v_i, pv_i)$, it is possible it could be a weak descent. The second difference is pv_i could have a primary prefix, implying that $C(v_i, pv_i)$ could be followed by a type (3) interval in a critical chain. Fortunately, because of the required conditions on the descent pv_i, $C(v_i, pv_i)$ can only be followed by a type (1) interval if $l_{k+1} < |pv_i|$, and a type (3) interval if $l_{k+1} = |pv_i|$. This allows us to consider these two cases separately.

First, suppose $pv_i = v_k$ ends with a 1 and a critical chain C containing the type (2) interval $C(v_i, pv_i)$ has $l_{k+1} < |pv_i|$. Then C can only be critical if $C(v_i, pv_i)$ is followed by a type (1) interval and $C(w, v_i]$ consists of i type (1) intervals. So as above, it follows that C is critical if and only if $J(C[pv_i, u])$ covers $C(pv_i, u)$, whose first label is l_{k+1} in C. Since $l_{k+1} < |pv_i|$, we have a one-to-one correspondence between the critical chains of $[u, w]$ that contain $C(v_i, pv_i)$ and the critical chains of $[u, pv_i]$ whose first label is not $|pv_i|$. Notice that $\mu(u, pv_i)$ could count critical chains of $[u, pv_i]$ whose first label is $|pv_i|$ and first word is $pv_i \setminus 1$. So we must subtract out these critical chains when calculating the contribution to the Möbius value. Since the corresponding map between the sets of critical chains is given by the addition or subtraction of the $i + 2$ intervals $v_1, \ldots, v_i, C(v_i, pv_i)$, and pv_i, it follows from Theorem 3 that the total contribution of all these critical chains to $\mu(u,w)$ is

$$(-1)^{i+2}\left(\mu(u,pv_i)-O(u,pv_i)\right)=(-1)^i(\mu(u,pv_i)-O(u,pv_i)),$$

where $O(u,pv_i)$ is the total contribution to $\mu(u,pv_i)$ of the critical chains of $[u,pv_i]$ for which $v_1'=pv_i\setminus 1$. Since $|pv_i|$ does not satisfy the definition of a principal index, Theorem 24 implies that $pv_i\setminus 1$ can only be in a critical chain of $[u,pv_i]$ if it is in a type (1) interval. If $pv_i\setminus 1^2$ is undefined, we are looking for all critical chains of $[u,pv_i\setminus 1]$. However, if $pv_i\setminus 1^2$ is defined, $O(u,pv_i)$ is the contribution of all critical chains of $[u,pv_i\setminus 1]$ that do not start with the values $|pv_i\setminus 1|$ and $|pv_i\setminus 1^2|$, as this would put $pv_i\setminus 1$ in a non-MSI-causing descent in $[u,pv_i]$. Furthermore, these critical chains contain one less interval than those of $[u,pv_i]$ because they do not contain the interval pv_i. So we need to account for this by taking the negative of the values of the chains in $[u,pv_i\setminus 1]$. Thus,

$$O(u,pv_i)=-\left(\mu(u,pv_i\setminus 1)-O(u,pv_i\setminus 1)\right),$$

since $O(u,pv_i\setminus 1)$ is the total contribution to $\mu(u,pv_i\setminus 1)$ of those critical chains of $[u,pv_i\setminus 1]$ for which $v_1''=pv_i\setminus 1^2$. This recursive definition for $O(u,pv_i)$ terminates when we find an ℓ for which $pv_i\setminus 1^\ell$ is undefined or $pv_i\setminus 1^\ell<u$ because both of these cases imply $O(u,pv_i\setminus 1^\ell)=0$. Therefore,

$$O(u,pv_i)=-\left(\mu(u,pv_i\setminus 1)+\mu(u,pv_i\setminus 1^2)+\ldots+\mu(u,pv_i\setminus 1^{\ell-1})\right).$$

This implies that when $pv_i=v_k$ ends in a 1, the total contribution of all critical chains containing the type (2) interval $C(v_i, pv_i)$ and satisfying $l_{k+1} < |pv_i|$ is

$$(-1)^i \left(\mu(u, pv_i) + \nu(u, pv_i \setminus 1)\right) = (-1)^i \nu(u, pv_i).$$

Our last goal is to consider the case when $pv_i = v_k$ ends with a 1 and a critical chain C containing the type (2) interval $C(v_i, pv_i)$ has $l_{k+1} = |pv_i|$. In this case, the set $J(C)$ must contain i type (1) intervals before $C(v_i, pv_i)$, and a type (3) interval immediately following $C(v_i, pv_i)$. By Proposition 25, the type (3) interval must be caused by the primary prefix of pv_i, $x(pv_i)$. So the type (3) interval is $C[pv_i, x(pv_i))$. Notice that Theorem 24 implies $x(pv_i)$ must be contained in a type (1) interval. In the previous case, note that pv_i is the first word in the interval $[u, w]$ which is not in the intervals of $J(C)$ required for the type (2) interval $C(v_i, pv_i)$, and pv_i must be contained in a type (1) interval. In this case, $x(pv_i)$ is the first word in $[u, w]$ that is not contained in the intervals of $J(C)$ required for the type (3) interval $C[pv_i, x(pv_i))$, and $x(pv_i)$ must be contained in a type (1) interval for C to be critical. Therefore, the argument is very similar, except $x(pv_i)$ takes the place of pv_i. If $u < x(pv_i)$, the major difference is C is now critical if and only if $J(C[x(pv_i), u])$ covers $C(x(pv_i), u)$. So the $i + 3$ intervals $v_1, \ldots, v_i, C(v_i, pv_i), C[pv_i, x(pv_i))$, and $x(pv_i)$ precede the elements of $J(C[x(pv_i), u])$ in the set $J(C)$. Beyond this detail, the argument leading to $(-1)^i \nu(u, pv_i)$ is the same, with $x(pv_i)$ taking the place of pv_i. Therefore, the contribution to $\mu(u, w)$ of the critical chains of $[u, w]$ that contain the type (3) interval $C[pv_i, x(pv_i))$ is

$$(-1)^{i+3} \nu(u, x(v_i)) = (-1)^{i+1} \nu(u, x(pv_i)).$$

Note this formula also holds if $u = x(pv_i)$ because then $\nu(u, x(pv_i)) = \mu(u, u) = 1$ and there are

$i + 2$ intervals in $J(C)$.

Thus, if $x(pv_i)$ is the primary prefix of pv_i, the contribution to the Möbius value of all critical chains that contain the type (2) interval $C(v_i, pv_i)$ as their first type (2) interval is

$$(-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right)$$

because if $x(pv_i)$ is not defined, $\nu(u, x(pv_i)) = 0$. This completes the proof. $\square$

Using Propositions 20 and 26, we are able to write down a formula for $\mu(u, w)$. Recall that ρ denotes the rank function in $\mathbb{P}^*$, and $\rho(u, w) = \rho(w) - \rho(u)$. For simplicity, let $0 \leq t \leq 2$ be the number of critical chains in $[u, w]$ that consist entirely of strong descents, and define

$$d(u, w) = \begin{cases} t(-1)^{\rho(u,w)} & \text{if } \rho(u, w) > 1 \\ (-1)^{\rho(u,w)} & \text{if } \rho(u, w) \leq 1. \end{cases}$$

Theorem 27. *Suppose $u \leq w$ in the poset $\mathbb{P}^*$. Then*

$$\mu(u, w) = d(u, w) + \sum (-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right),$$

where the sum is over all triples $v_i, pv_i, x(pv_i)$ such that pv_i is a principal factor of w of degree i with base v_i and primary prefix $x(pv_i)$.

Proof. First suppose $\rho(u, w) \leq 1$. Then $[u, w]$ cannot have any maximal chains because the open interval is empty or undefined. So both summations contribute 0 to the Möbius value. By definition of the Möbius value, when $\rho(u, w) = 1$, $\mu(u, w) = -1$ and when $\rho(u, w) = 0$, $u = w$ and $\mu(u, u) = 1$. These are the values that are defined in the formula $d(u, w)$.

Now suppose $\rho(u,w) > 1$. If C is a critical chain of $[u,w]$ that consists entirely of strong descents, then $J(C)$ consists of $\rho(u,w)-1$ intervals of length 1. Therefore, by Theorem 3, C contributes

$$(-1)^{\rho(u,w)-2} = (-1)^{\rho(u,w)}$$

to the Möbius value $\mu(u,w)$. By Proposition 20, there are 0, 1, or 2 such chains in any interval $[u,w]$, making the total contribution of all such chains

$$t(-1)^{\rho(u,w)},$$

where t is the total number of these chains. This is the value in the formula for $d(u,w)$. Thus, by adding the contribution of all critical chains that do not consist entirely of strong descents to $d(u,w)$, we will get a formula for the Möbius value.

If a critical chain C in $[u,w]$ does not consist entirely of strong descents, then by Theorem 19, C must contain an interval caused by a principal factor. By Proposition 26, the first such interval is caused by a principal factor of some degree i. Let $C(v_i, p_{v_i})$ be this interval, where i is the degree of the principal factor p_{v_i}, and consider the set S of all critical chains that contain this interval as the first type (2) interval. Then Proposition 26 implies the contribution of the chains in S to the Möbius value is

$$(-1)^i \left(\nu(u, p_{v_i}) - \nu(u, x(p_{v_i}))\right),$$

where $x(p_{v_i})$ is the primary prefix of p_{v_i}. This is precisely the term that appears in the given formula for the triple $v_i, p_{v_i}, x(p_{v_i})$. By Theorem 24, summing over all such triples yields a summation which gives the contribution of all critical chains containing at least one type

(2) interval. Thus, the Möbius value can be found by adding $d(u, w)$ to this last summation, completing the proof. □

We close this section with several example Möbius function calculations using the above formula.

$$\begin{aligned}\mu(121, 1221) &= d(121, 1221) + \nu(121, 121)\\ &= 2 + 1\\ &= 3\end{aligned}$$

$$\begin{aligned}\mu(2, 2212) &= d(2, 2212) + \nu(2, 211) - \nu(2, 2)\\ &= 0 + \mu(2, 211) + \mu(2, 21) + \mu(2, 2) - \mu(2, 2)\\ &= 0 + 0 - 1 + 1 - 1\\ &= -1\\ \mu(2, 3121) &= d(2, 3121) + 0\\ &= 0\end{aligned}$$

For larger examples, it is helpful to collect like terms and eliminate coefficients. In the example below, 3111 has 31 as its primary prefix for one degree 0 and one degree 1 base, while it has 3 as its primary prefix for one degree 1 base.

$$\begin{aligned}\mu(3,33133) =& 0+(1-2+1)\nu(3,3111)-(1-1)\nu(3,31)-(-1)\nu(3,3)\\ &+(1-2+1)\nu(3,3112)+\nu(3,3113)+\nu(3,33)\\ =&\mu(3,3)+\mu(3,3113)+\mu(3,33)\\ =&1+\mu(3,3)+\mu(3,3)\\ =&3\end{aligned}$$

A quick investigation of the formula yields the coefficient for each term $\nu(u,v)$ is dependent on how many times v occurs as a principal factor of odd versus even degree, or how many times v occurs as the primary prefix of principal factors of odd versus even degree. It is not possible for the primary prefix $x = x(p_{v_i})$ to be a principal factor of any degree. Indeed, its principal index is the same as the principal factor $p_{v_i} = x1^m$ of w so that there are at least 3 embeddings of x in w. So we would need to remove the middle embedding from w via strong descents before removing the suffix embedding. This is impossible because the suffix embedding starts at a later index than any other embedding.

Remark. We have noticed that there is often a clear relationship between the odd and even counts, such as a binomial sum, resulting in an unusually high number of terms $\nu(u,v)$ with coefficient zero. However, a precise description of when the coefficients are zero has eluded us. For more comments on this, see the last section on open problems.

Chapter 4

Generalized Factor Order on Trees and Forests

A *tree* is a poset for which the undirected graph underlying its Hasse diagram has no cycles, and a *forest* is a disjoint union of trees. A *rooted tree* T is a tree with a unique minimal element. We will show in this section that our results generalize to the case of a *rooted forest* F, which is a disjoint union of rooted trees.

Suppose T is a rooted tree, and let r be its root. Since T has no cycles and every element s satisfies $s \geq r$, it follows that every element except the root covers a unique element. This is why we are considering generalized factor order on these posets. It can be shown the Möbius function of generalized factor order on T^* is similar to that of $\mathbb{P}^*$, and the proofs leading to the result are nearly identical. For this reason, we have chosen not to consider this case separately from the rooted forest case.

Suppose F is a rooted forest. Like the rooted tree case, in a rooted forest, every nonminimal element covers a unique element. However, there are multiple minimal elements in this poset. This leads us to suspect that we need to combine the results of Section 2 with Theorem 27. While this is largely true, we will see that the definition of principal factor does not translate quite as expected, and that having multiple minimal elements complicates several results from Section 3.

To be consist with Section 2, we say a flat word in the Kleene closure F^* is a sequence of

m's, where m is a minimal element. We say a word is *rooted* if it consists entirely of minimal elements since each minimal element is the root of a tree. Note that a flat word is also rooted. The following lemma states that the covering relations of F^* are analogous to those of $\mathbb{P}^*$. It's proof is similar to that of Lemmas 4 and 11.

Lemma 28. *A word $w = w(1) \ldots w(n)$ in F^* can cover up to n words, each formed by reducing a letter in w to the unique letter it covers, where reducing a minimal element means removing it from the word. Reducing $w(1)$ and $w(n)$ will always produce a factor, while reducing $w(i)$ for $1 < i < n$ can only produce a new factor if $w(i)$ is nonminimal. These words are distinct unless w is flat, in which case w only covers one word which is flat.* □

Note that a minimal element m cannot be reduced unless it is at the beginning or end of a word. We maintain the convention that if a word is flat, only the first m can be reduced. This allows us to maintain the notion of a reducible letter $w(i)$.

Suppose we have a distinguished symbol $\hat{0}$ and $\hat{0} \notin F$. Define $\hat{F}$ to be the poset F with $\hat{0}$ added as the unique minimal element. This allows us to maintain the definition of expansion from the previous sections, that is, a word $\eta \in \hat{F}$ is an expansion of $u \in F$ if $\eta \in \hat{0}^* u \hat{0}^*$.

Let $[u, w]$ be an interval in F^*. Let $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \ldots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain in $[u, w]$, where the l_i are defined by the corresponding sequence of embeddings ηv_i in the sense that

$$\eta v_i(l_i) = s \text{ where } \eta v_{i-1}(l_i) \to s \text{ and } \eta v_i(j) = \eta v_{i-1}(j) \text{ when } j \neq l_i.$$

This gives each maximal chain C a chain id $l_1 \ldots l_n$. This chain id is unique because every element in $\hat{F}$ except $\hat{0}$ covers a unique element. Notice this is the most general class of posets in which every interval ordered by generalized factor order has maximal chains with unique chain ids of this form. By lexicographically ordering the chain ids, we get a poset lexicographic order on the maximal chains of $[u, w]$ which we will use to find the MSIs in the rooted forest case.

Let η be an embedding of u into w. Let $m_i = \rho(\eta(i)) - \rho(w(i))$, where ρ is the rank function in $\hat{F}$. This allows us to maintain the idea of an admissible permutation of the multiset $M_\eta = \{1^{m_1}, 2^{m_2}, \ldots\}$ from the previous section.

By Lemma 28, the characterization of chain ids ending at nonflat words does not change. Unfortunately, since both rooted and unrooted words can cover flat words, it is not possibly to write down a useful characterization of chain ids ending at flat words. However, since flat words can only be reduced to flat words, we will be able to deal with this case separately.

Proposition 29. *Suppose F is a rooted forest, and u and w are two elements in F^* satisfying $u \leq w$. Let η be an embedding of u into w, $m_i = \rho(w(i)) - \rho(\eta(i))$, and $M_\eta = \{1^{m_1}, 2^{m_2}, \ldots\}$.*

If u is not flat, then a sequence of numbers is the chain id for a maximal chain in $[u, w]$ ending at η if and only if it is an admissible permutation of the multiset M_η. □

Since there are multiple minimal elements, we need to reconcile our previous classifications of MSIs in the positive integer and antichain cases. To begin considering MSIs in the new setting, we need to once again identify all intervals $[u, w]$ in which a chain C has $C(w, u)$ as an MSI.

Recall from Section 2 that descents always caused MSIs in the antichain case when they were strong descents, that is, $l_{i+1} < l_i - 1$. In the context of $\mathbb{P}^*$, a strong descent satisfied $v_{n-1} \neq v_{n+1}11$. These conditions are analogous. So we call any descent v_n that does not remove two minimal elements from the back of a word a *strong descent*, that is, $v_{n-1} \neq v_{n+1}mn$ for any minimal elements m and n. The next proposition states that a strong descent causes a length 1 MSI. Since its proof is essentially the same as that of Proposition 16, we omit it.

Proposition 30. *Suppose* $[u, w] \subset F^*$ *and* $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \dots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ *is a maximal chain in* $[u, w]$*. If* v_i *is a strong descent, then* v_i *is a length 1 MSI.* □

Notice this statement is not an if and only if, as in the antichain case outer word MSIs can reduce consecutive letters from the back. Indeed, if 1 and a are minimal elements in F, the interval $[1a, 1a1a]$ contains a length 1 MSI. The presence of length 1 MSIs containing weak descents is one of the key differences between the antichain and positive integer cases.

While the idea of a maximal outer factor still holds in the new context, simple examples reveal that the previous definition of a principal factor will not be sufficient for the forest case. For example, suppose F is disjoint union of two chains, $b \to a$ and $2 \to 1$. Then in the interval $[b, bb1]$, the chain C with chain id 232 has $C(bb1, b)$ as an MSI, even though b is not a suffix of $bb1$. A quick analysis reveals that b behaves like the principal factors from the previous section because the MSI results from a unimodal chain id which is the lexicographically first id leading to the prefix embedding of the word b.

Deeper analysis shows that words causing MSIs in this manner can have multiple embeddings in w. The smallest example we could find of this is in the interval $[21a22, 21a221a22a22]$. In spite of the fact $21a22$ has three embeddings in the larger word, there is a maximal chain in which $C(21a221a22a22, 21a22)$ is an MSI. Fortunately, the definition of a principal factor can still be generalized to fit the new context so that once again, we will have exactly two types of MSIs in F^*.

Let p be a word in F^* and let w be a word in F^* that is not flat. Suppose that p is a prefix of w with other embeddings in w, and no longer prefix containing p has multiple embeddings in w. Then there is a smallest index i, called the principal index of p in w, such that $w(i) > p(i)$ and $w(i)$ is reducible. We say p is a *principal factor* of w if the word produced by reducing $w(i)$

contains only the prefix embedding of p.

Notice this definition accounts for both of the examples given before it. As in the previous cases, the principal index of a principal factor must take a value greater than 1. Before proceeding, it is important to understand why this definition includes both outer words and the Section 3 definition of a principal factor as special cases. In the antichain case, an outer word $o(w)$ of a nonflat word is a principal factor when its principal index is $|w|$ because only 1 and $|w|$ are reducible in the case of an antichain. Since reducing $|w|$ can only remove the suffix embedding of a word, we must have $o(w) \not\leq i(w)$ in order for it to be a principal factor, where $i(w)$ is the inner word. This provides further insight into this condition of Björner's formula.

To see that the definition of a principal factor from Section 3 is generalized by the new one, first note that a principal factor of an unrooted word w cannot be rooted. Indeed, if w contains a nonminimal element, then the principal index of any rooted prefix p will point to the first nonminimal element. So reducing this element cannot possibly remove any embeddings of p. Furthermore, the new definition guarantees p is a suffix in the case of $\mathbb{P}$. If p has an embedding in w which is neither prefix nor suffix, the word $p1$ would have a prefix embedding and another embedding, contradicting the maximality of p. While it is more difficult to identify principal factors when they do not have a suffix embedding, this generalization clearly shows which properties of principal factors cause MSIs.

Proposition 31. *Suppose u is a principal factor of w with principal index i. Let $C : w = v_0 \xrightarrow{i} v_1 \xrightarrow{l_2} \dots \xrightarrow{l_{n-1}} v_{n-1} \xrightarrow{l_n} v_n = u$ be the lexicographically first chain in $[u, w]$ with $l_1 = i$. Then $C(u, w)$ is an MSI of C.*

Proof. Using the same argument from the beginning of the proof of Proposition 17, we conclude there exist chains that are lexicographically earlier than C. Also from this proof, if C' is an arbitrary maximal chain that is lexicographically earlier than C, then $l'_1 < i$ and v'_1 must contain an embedding of u which is not prefix.

Furthermore, since C is the lexicographically first chain ending at the prefix embedding of u, we reduce the minimal letters at the end last, implying $v_{n-1} = um$ for some minimal element m. If v'_1 contained um, u would be contained in a longer prefix with multiple embeddings in w, contradicting the fact that u is a principal factor of w. Thus, the only words common to C and C' are w and u. Since C' was an arbitrary maximal chain in $[u, w]$ with $l'_1 < i$, and C is the first maximal chain in $[u, w]$ with $l_1 = i$, we conclude $C(u, w)$ is an MSI. □

If u is not flat, the *first admissible* chain ending at η, C_η, is the maximal chain whose chain id is the lexicographically first permutation of M_η that is the chain id of a maximal chain. This chain has the same structure it did in the case of $\mathbb{P}$.

Lemma 32. *Suppose $[u, w] \subset \mathbb{P}^*$ and η is an embedding of u into w. Let ℓ be the index of the largest non-zero number in η. If u is not flat, C_η has as its chain id the unique unimodal permutation of M_η with decreasing suffix $|w|, |w| - 1, \ldots, \ell + 1$.* □

The following Theorem completes the characterization of the MSIs and states they are once again caused by strong descents or principal factors.

Theorem 33. *Suppose $[u, w] \subset F^*$, u is not the empty word, and $C : w = v_0 \stackrel{l_1}{\to} v_1 \stackrel{l_2}{\to} \ldots \stackrel{l_{n-1}}{\to} v_{n-1} \stackrel{l_n}{\to} v_n = u$ is a maximal chain in $[u, w]$. Then $C(v_i, v_j)$ is an MSI of C if and only if $C(v_i, v_j)$ consists of a strong descent, meaning $v_i \neq v_j mn$ for any minimal elements m and n, or v_j is a principal factor of v_i, l_{i+1} is the principal index of v_j with respect to the embedding ηv_j, and $C[v_i, v_j]$ is the first admissible chain in $[v_j, v_i]$ ending at the prefix-embedding of v_j.*

Proof. The reverse implication follows from Propositions 30 and 31.

Suppose $C(v_i, v_j)$ is an MSI of C. By the definition of a poset lexicographic order, $C(v_i, v_j)$ is an MSI of C if and only if it is an MSI of $C[v_i, v_j]$ in the interval $[v_j, v_i]$. Thus, it suffices to consider the case $w = v_i$ and $u = v_j$.

Since $|u| = |w|$ implies $[u, w]$ is direct product of chains, Corollary 14 can be translated to the forest case. This implies strong descents are the only MSIs when $|u| = |w|$. However, strong descents are always length 1 MSIs. Therefore, it suffices to consider the case when $|u| < |w|$ and $C(w, u)$ is an MSI of C that does not consist of a strong descent. Our first goals are to establish that w has a prefix embedding of u and $C[w, u]$ is the first admissible chain ending at the prefix embedding.

Let η be the embedding of u into w at the end of the chain C. Note that any descent v_k in C is a weak descent since otherwise, by Proposition 30, $C(v_{k-1}, v_{k+1})$ would be an MSI, contradicting the minimality of $C(w, u)$. If $k+1 \neq n$, this forces $l_{k+2} < l_{k+1} = l_k - 1$, implying that v_{k+1} is also a weak descent. By continually applying this idea, we find that any descents contained in $C(w, u)$ occur in a single sequence of weak descents at the end of this interval, and the corresponding labels form a decreasing sequence of consecutive numbers.

To see η is the prefix embedding of u into w, suppose for a contradiction that the set M_η contains a 1. Since any descents occur in a sequence at the end of the interval, it follows that $l_1 = 1$ or $l_n = 1$. If $l_n = 1$, then since v_{n-1} is a weak descent, u is the empty word. This contradicts our assumptions. Suppose $l_1 = 1$. Then any chain C' lexicographically earlier than C has $l'_1 = 1$ and thus contains v_1, contradicting the fact that $C(w, u)$ is an MSI. Therefore, η is the prefix embedding of u into w, allowing us to write $\eta = u$.

Since $|u| < |w|$, we must reduce position $|w|$ by the end of the chain. Note that the label $|w|$ can only be followed by another $|w|$ or the sequence of labels $|w| - 1, \ldots, |u| + 1$, which leads to a sequence of weak descents. It follows that C must contain the word um for some minimal element m. Since $l_n = |u| + 1$, C reduces the m at the end of $v_{n-1} = um$ to get $v_n = u$. This implies um (and hence w) is not flat. So by Lemma 32, $C[w, um]$ is the first admissible chain ending at the prefix-embedding of um. Since minimal elements can only be removed from the front or back of a word, C is the first admissible chain ending at the prefix-embedding of u.

Since $C(w, u)$ is an MSI, C cannot be the lexicographically first chain in $[u, w]$. Therefore, w must contain another embedding of u in addition to the prefix embedding. This implies that u has a principal index in w.

Next, we will show l_1 is the principal index of u in w. Since C is the first admissible chain ending at the prefix embedding, $w(l_1)$ is the first letter that is reducible and satisfies $w(l_1) > u(l_1)$.

We also need to show that the word v_1 contains only the prefix embedding of u. For a contradiction, suppose v_1 contains another embedding ρ of u besides the prefix embedding. Then there is a chain C' ending at ρ whose chain id begins $l_1 1 \ldots$ and has $v'_1 = v_1$. This chain is thus lexicographically earlier than C and satisfies $C - C(v_1, u) \subset C'$, contradicting the fact that $C(w, u)$ is an MSI. So v_1 contains only the prefix embedding.

It remains to show that there is not a longer prefix containing u with another embedding in w. For a contradiction, suppose there is a longer prefix of w containing u that has multiple embeddings in w. Then um is a prefix of w for some unique minimal element m and has another embedding in w. Let C' be the lexicographically first chain in the interval containing um. Note C also contains um because as the first admissible chain ending at the prefix embedding, $l_n = |u| + 1$. Thus, $C - C(w, um) \subset C'$, implying $C(w, um)$ is a skipped interval. This contradicts the fact that $C(w, u)$ is an MSI. □

We can now easily describe the critical chains that consist entirely of strong descents. Since the proof is very similar to that of Proposition 20, we omit it.

Proposition 34. *Suppose $[u, w] \subset \mathbb{P}^*$, (u, w) is non-empty, and w is not flat. Suppose η is an embedding of u into w. Then there is a critical chain C in $[u, w]$ ending at η that consists entirely of strong descents if and only if $w(i) = \eta(i)$ or $w(i) \to \eta(i)$ for all i, $\eta(2) \neq \hat{0}$, and $\eta(|w| - 1) \neq \hat{0}$. Furthermore, these conditions imply $|w| - |u| \leq 2$ and that $[u, w]$ has at most two critical chains consisting entirely of strong descents.* □

It should be noted that in the antichain case, the fact that $\mu(i(w), w) = 1$ for the inner word $i(w)$ when w is not flat follows directly from this proposition.

As in the case of $\mathbb{P}$, the remaining critical chains must contain at least one MSI caused by a principal factor. So these chains contain a principal factor of w or a principal factor pv_i of some v_i with the property that $C(w, v_{i+1})$ consists entirely of strong descents.

In order to have a critical chain in $[u, w]$ involving an MSI resulting from a principal factor pv_i, pv_i needs to be contained in a different MSI or pv_i must equal u. By Theorem 33, pv_i could be contained in one of three types of MSI: an MSI caused by a strong descent, an overlapping MSI caused by a principal factor, or an adjacent MSI caused by a principal factor. This is where the forest case becomes more complex than the case of $\mathbb{P}$ because the third possibility can happen. However, it can only happen when v_i is a rooted word.

Moving forward, we will have few results that apply to both rooted and unrooted words. However, unrooted words still behave much like they did in the positive integer case, and our proof of Björner's formula will help us understand rooted words in the new context.

Proposition 35. *Suppose $C(v_i, pv_i)$ is an MSI of a critical chain C caused by the principal factor pv_i. If v_i is unrooted, then pv_i is unrooted and is contained in an MSI caused by a strong* *descent or an overlapping MSI caused by a principal factor.*

Proof. If p is rooted prefix of an unrooted rooted v_i, then the principal index of p in v_i contains a nonminimal element. So reducing the letter at the principal index cannot eliminate any embedding of p. Thus, a rooted word cannot be a principal factor of an unrooted word.

Let $v_{j+1} = pv_i$. Suppose for a contradiction that $C(v_j, pv_j)$ is an MSI caused by a principal factor. Since $C(v_i, v_{j+1})$ is an MSI caused by a principal factor, Theorem 33 implies $C[v_i, v_{j+1}]$ is the first admissible chain ending at the corresponding embedding. So by Lemma 32, $v_j = v_{j+1}m$ for some minimal element m. However, this implies the value at the principal index $l_{j+1} = |v_j|$ of pv_j in v_j is m. But then v_j and $v_{j+1} = pv_i$ are rooted words because the principal index of an unrooted principal factor must contain a nonminimal letter. Since pv_i is not rooted, this is a contradiction. □

As stated above, this result does not apply to rooted words. For example, in $[1, 1a1a]$, the rooted principal factor $1a$ is contained in the adjacent MSI caused by the rooted principal factor 1 of $1a1$.

The previous result essentially separates the unrooted words from the rooted ones.

Corollary 36. *If w is not a rooted word and $C(w, u)$ is a critical maximal chain of $[u, w]$, then $C(w, u)$ has no MSIs caused by rooted principal factors.*

Proof. Suppose for a contradiction $C(w, u)$ did contain an MSI caused by an rooted principal factor. By Proposition 35, a principal factor of an unrooted word is also not rooted. Thus, the last unrooted word in $C(w, u)$ must be a strong descent v_k. Since no rooted word is a principal factor of v_k, v_{k+1} must also be a strong descent for it to be contained in an MSI. This implies $l_{k+2} = 1$, so that v_{k+2} cannot be a strong descent MSI. However, the principal index of a principal factor cannot be 1. Thus, v_{k+2} cannot be in an MSI caused by a principal factor. So by Theorem 33, if $u \neq v_{k+2}$, v_{k+2} is not in any MSI, contradicting the fact that $C(w, u)$ is a critical maximal chain. If $u = v_{k+2}$, then we have shown $C(w, u)$ has no MSIs caused by rooted principal factors. $\square$

This allows us to conclude that if w is not rooted, any overlapping MSIs in the set $I(C)$ of a critical chain must come in pairs.

Proposition 37. *If w is not rooted, an MSI of a critical chain can overlap with at most one other MSI.*

Proof. This proof is entirely analogous to that of Proposition 23, with two notable exceptions. First, we need to point out that by Corollary 36, all words involved in the overlapping intervals $C(v_{i_1}, v_{j_1}), C(v_{i_2}, v_{j_2}), \ldots, C(v_{i_k}, v_{j_k})$ are not rooted. Second, this implies each $\eta v_{i_\ell}(l_{i_\ell + 1})$ is nonminimal because $l_{i_\ell + 1}$ corresponds to a principal index of an unrooted word. $\square$

Notice again that this restriction on overlapping MSIs does not apply to rooted words. For example, in the last maximal chain of $[a, a11aa11a]$, the word $a11aa$ is contained in three MSIs. Nevertheless, this chain is critical.

At this point, it is clear that the $J(C)$ structure of the critical chains is essentially the same as in section 3 when a word is not rooted, while the forest case reduces to the ordinary factor order case of section 2 when a word is rooted. To establish the formula, we will also need to update the definitions of primary prefixes and the ν function, and use them to establish the formula when w is not rooted. We have little choice but to establish the formula separately for rooted words.

Theorem 38. *Suppose w is not a rooted word. Let $C : w = v_0 \overset{l_1}{\to} v_1 \overset{l_2}{\to} \ldots \overset{l_{n-1}}{\to} v_{n-1} \overset{l_n}{\to} v_n = u$ be a maximal chain of $[u, w]$. Then C is a critical chain if and only if $C(w, u)$ can be written as a sequence of intervals*

$$C[v_{i_1} = v_1, v_{i_2}) \cup C[v_{i_2}, v_{i_3}) \cup \ldots \cup C[v_{i_{k-1}}, v_{i_k} = u)$$

where each interval $C[v_{i_j}, v_{i_{j+1}})$ is one of the following three types:

1. *$C[v_{i_j}, v_{i_{j+1}})$ is an MSI caused by the strong descent v_{i_j}.*
2. *$C[v_{i_j}, v_{i_{j+1}})$ is an MSI caused by the principal factor $v_{i_{j+1}}$ of the word v_{i_j-1}.*
3. *The word $v_{i_{j+1}}$ is a principal factor of a word in $C[v_{i_{j-1}}, v_{i_j})$ and satisfies $v_{i_{j+1}} m_1 \ldots m_k = v_{i_j}$, where each m_i is a minimal element and $k = |v_{i_j}| - |v_{i_{j+1}}| > 0$. The value k is unique in the sense that no other word satisfies the description of $v_{i_{j+1}}$ for another value k.*

Furthermore, type (1) intervals are followed by intervals of type (1) or (2), type (2) intervals are followed by intervals of type (1) or (3), and type (3) intervals are followed by intervals of type (1). Finally, only intervals of type (1) or (2) can begin the decomposition.

Proof. Since w is not rooted, by Corollary 36, no rooted word can be a principal factor which causes an MSI.

Thus, both implications can be proved as they were in the proof of Theorem 24 by updating the relationship between between v_{i_j} and $v_{i_{j+1}}$. For example, in the forward implication, we have $v_{i_j} = v_{i_{j+1}} m_1 \ldots m_k = v_{i_j}$, where each m_i is a minimal element and $k = |v_{i_j}| - |v_{i_{j+1}}| > 0$, instead of $v_{i_j} = v_{i_{j+1}} 1^m$ for $m = |v_{i_j}| - |v_{i_{j+1}}|$. □

Since Theorem 38 gives essentially the same result as Theorem 24, when w is not rooted, we need only update the appropriate definitions to get the desired formula. However, we will need to handle the case when w is rooted separately before stating the formula.

Define a word v to be a *base of* w if $v(j) = w(j)$ or $w(j) \to v(j)$ for all j and $|v| = |w|$ or $|w| - 1$. Define the *degree* of a base to be the number of indices j for which $w(j) \to v(j)$.

Suppose v_i is a base of w of degree i. Let l be the index of the smallest position satisfying $w(l) \to v_i(l)$, or $|w| + 1$ if $i = 0$. We define any word p_{v_i} that is a principal factor of v_i and whose principal index takes a value less than l to be a *principal factor of* w *of degree* i. As in the previous section, when a degree is not noted in the language or the notation, the assumption will be that the principal factor has degree 0.

Define $w \setminus k$ to be the word that results when k minimal elements are removed from the suffix of w, or as undefined if w ends in less than k minimal elements. Note that unlike the integer case when 1 was the only minimal element, there are multiple minimal elements which could be reduced.

Suppose p_{v_i} is a principal factor of v_i. Let v_i' be the word that results when the letter in v_i at the principal index of p_{v_i} is reduced by 1 rank. That is, $v_i(j) \to v_i'(j)$ when j is the principal index of p_{v_i} and $v_i'(j) = v_i(j)$ for all other indices j. Define the *primary prefix* $x(p_{v_i})$ of a principal factor p_{v_i} to be the longest proper prefix that has at least 2 embeddings in v_i' and satisfies $x(p_{v_i}) = p_{v_i} \setminus k$ for some k. So if no $k > 0$ satisfies the restriction, or no such word has at least two embeddings in v_i', the primary prefix is undefined

Notice the primary prefix definition still makes sense when p_{v_i} is rooted, in which case $i = 0$ and p_{v_0} is an outer word not contained in the inner word. In this case, $x(p_{v_0}) = p_{v_0} \setminus 1$. As was seen in our proof of Björners result, this implies that whenever $u \leq x(p_{v_0})$, p_{v_0} is contained in a length 1 MSI.

The following proposition asserts that the primary prefix is the only word that can cause a type (3) interval after the type (2) interval $C(v_i, p_{v_i})$. While the spirit of the proof is similar to that of Proposition 25, the definition of a primary prefix has changed enough to warrant stating the entire proof.

Proposition 39. *Suppose w is not rooted. Let C be a maximal chain of $[u, w]$ and suppose $C(v_i, p_{v_i})$ is a type (2) interval in the set $J(C)$. Then $C(v_i, p_{v_i})$ is followed by a type (3) interval $C[p_{v_i}, x)$ in $J(C)$ if and only if x is the primary prefix of p_{v_i} and x appears in C.*

Proof. First suppose x is the primary prefix of p_{v_i} and that it appears in C. Let v_ℓ be the last word in the interval $C(v_i, p_{v_i})$ that contains at least two embeddings of x. We will show x is a principal factor of v_ℓ with principal index $l_{\ell+1}$. Since p_{v_i} is a prefix of $v_{\ell+1}$, x is as well. So $v_{\ell+1}$ only contains the prefix embedding of p_{v_i}. Furthermore, since $C[v_i, p_{v_i}]$ is the first admissible chain in $[p_{v_i}, v_i]$ ending at the prefix embedding of p_{v_i}, for all $l_{i+1} < k < l_{\ell+1}$, either we have $v_\ell(k) = p_{v_i}(k)$ or we have $v_\ell(k)$ minimal and $p_{v_i}(k) = 0$. Therefore, for all $l_{i+1} < k < l_{\ell+1}$, either we have $v_\ell(k) = x(k)$ or we have $v_\ell(k)$ minimal and $x(k) = \hat{0}$. This implies $C[v_\ell, x]$ is the first admissible chain in $[x, v_\ell]$ ending at the prefix embedding of x and $v_\ell(l_{\ell+1})$ is the first reducible letter in v_ℓ greater than the corresponding position in the prefix embedding of x. Thus, $l_{\ell+1}$ satisfies the definition of a principal index, and since $v_{\ell+1}$ only contains the prefix embedding of p_{v_i}, x is a principal factor of v_ℓ and $C(v_\ell, x)$ is an MSI of C. Since x is the longest proper prefix of p_{v_i} with two embeddings in v_i' that satisfies $x = p_{v_i} \setminus k$ for some k, and $C(v_i, p_{v_i})$ is a type (2) interval, no word between p_{v_i} and x can be a principal factor of a word v_k in C. Therefore, $C(v_\ell, x)$ is reduced to the type (3) interval $C[p_{v_i}, x)$ in $J(C)$, completing the reverse implication.

Now suppose that $C(v_i, pv_i)$ is followed by a type (3) interval $C[pv_i, x)$ in the set $J(C)$. Then by Theorem 38, x is a principal factor of a word v in $C(v_i, pv_i)$ and $x = pv_i \setminus k$ for some k. From the proof of Theorem 38, we know $|v| = |v_i'|$ and v has at least two embeddings in v_i'. By Theorem 38 part (3), it suffices to show that no longer prefix of pv_i containing x has two embeddings in v_i'. For a contradiction, suppose y is such a prefix. Then $y = xm_1 \dots m_\ell$ has at least two embeddings in v_i', and by the argument in the paragraph above, is a principal factor of some word in $C(v_i, pv_i)$. Thus, $C(v, y)$ would be an MSI in $I(C)$. This would be reduced to the interval $C[pv_i, y)$, which is contained in $C[pv_i, x)$, implying that $C[pv_i, x)$ could not be in $J(C)$. This is a contradiction. Thus, x is the longest proper prefix of pv_i with two embeddings in v_i', implying it is the primary prefix of the word pv_i. □

Let $\mu(u, v)$ be the normal Möbius function if u and v are both elements of F^*, or zero if either is undefined. Define the function $\nu(u, v)$ to be

$$\nu(u, v) = \sum_{i \geq 0} \mu(u, v \setminus i).$$

Notice all the terms in the summation will be zero beyond the largest value $i = k$ for which $v \setminus k$ is defined, or the smallest value $i = k$ for which $v \setminus k \leq u$.

We are now ready to state the contribution of critical chains whose first type (2) interval is a specific interval. By replacing the words $pv_i \setminus 1^k$ in the proof of Proposition 26 by $pv_i \setminus k$, it is easy to adapt that proof to work for the next proposition.

Proposition 40. *Suppose w is not rooted and $C(v_i, pv_i)$ is the first type (2) interval of some critical chain C of $[u, w]$. Then pv_i is a principal factor of w of degree i and v_i is based in C. Furthermore, the contribution to the Möbius value $\mu(u, w)$ of all critical chains in $[u, w]$ that* *have $C(v_i, pv_i)$ as the first type (2) interval is*

$$(-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right),$$

where $x(pv_i)$ is the primary prefix of pv_i. □

Our next goal is to consider the intervals $[u, w]$ for w rooted. Note that when w is rooted, the forest case reduces to the antichain case. Therefore, the formula for $\mu(u, w)$ is given by Theorem 1, which the reader may wish to refresh at this time. Thus, we must show this theorem is now a special case of our formula from Section 3. The key fact in the proof is that the primary prefix of a rooted principal factor pv_i is $pv_i \setminus 1$, which was also the key in proving Björner's formula using discrete Morse theory.

Let ρ denote the rank function in F^*, and $\rho(u, w) = \rho(w) - \rho(u)$. For simplicity, let $0 \leq t \leq 2$ be the number of critical chains in $[u, w]$ that consist entirely of strong descents, and define

$$d(u, w) = \begin{cases} t(-1)^{\rho(u,w)} & \text{if } \rho(u, w) > 1 \\ (-1)^{\rho(u,w)} & \text{if } \rho(u, w) \leq 1. \end{cases}$$

Proposition 41. *Suppose $u \leq w$ in the poset F^* and w is a rooted word. Then*

$$\mu(u, w) = d(u, w) + \sum (-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right) = d(u, w) + \mu(u, o(w)),$$

where $o(w)$ is the outer word as defined on page 1 and the sum is over all triples $v_i, pv_i, x(pv_i)$ such that pv_i is a principal factor of w of degree i with base v_i and primary prefix $x(pv_i)$.

Proof. First we show rooted words can only have principal factors of degree 0. Since any rooted strong descent v_i satisfies $v_{i+1} = mv_{i-1}n$ for some minimal elements m and n, and every principal factor of v_i has principal index $|v_i| \neq 1$, rooted words do not have bases of positive degree. Thus, they cannot have principal factors of positive degree either.

We need to show the formula in Theorem 1 agrees with the one in the statement of this proposition. Suppose $|w| - |u| \leq 2$ and $u \neq o(w)$. Then by Proposition 34 and Theorem 1,

$$\mu(u, w) = d(u, w).$$

Furthermore, by the definition of principal factor, the only word that can be a principal factor is the outer word $o(w)$. Note $|w| - |o(w)| > 1$ when $o(w)$ is not flat. So $u \not\leq o(w)$, implying the summation is 0 because $\nu(u, o(w)) = 0$ and $\mu(u, o(w)) = 0$ by definition. This completes the proof in this case.

Next, suppose $|w| - |u| > 2$, $u \leq o(w) \setminus 1$, and $o(w) \not\leq i(w)$. Note $o(w)$ is the unique principal factor of w. From the discussion following the definition of the primary prefix, we know $o(w) \setminus 1$ is the primary prefix of $o(w)$. Furthermore, by Proposition 34, $d(u, w)$=0 because $|w| - |u| > 2$. Thus,

$$\begin{aligned}
\sum (-1)^i \left(\nu(u, p_{v_i}) - \nu(u, x(p_{v_i}))\right) =& \nu(u, o(w)) - \nu(u, x(o(w))) \\
=& \mu(u, o(w)) + \nu(u, o(w) \setminus 1) - \nu(u, o(w) \setminus 1) \\
=& \mu(u, o(w)),
\end{aligned}$$

completing the proof in this case.

Suppose $|w| - |u| \geq 2$, $u \leq o(w)$, $u \not\leq o(w) \setminus 1$ and $o(w) \not\leq i(w)$. Note $o(w)$ is the unique principal factor of w, but the primary prefix of $o(w)$, $o(w) \setminus 1$, does not contain u. Furthermore, by Proposition 34, $d(u, w)$=0 because $|w| - |u| \geq 2$ and $u \neq i(w)$. Thus,

$$\begin{aligned}\sum(-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right) =&\nu(u, o(w)) - 0 \\ =&\mu(u, o(w)) + \nu(u, o(w) \setminus 1) \\ =&\mu(u, o(w)),\end{aligned}$$

completing the proof in this case.

Finally, in all other cases, $|w| - |u| > 2$ and $o(w) \leq i(w)$. Since $|w| - |u| > 2$, $d(u, w) = 0$. Since $o(w) \leq i(w)$, $o(w)$ is not a principal factor of w, meaning the summation is 0 as well. This agrees with Theorem 1, which states $\mu(u, w) = 0$ in all other cases, completing the proof. □

Using Propositions 34, 40, and 41 we are able to write down a formula for $\mu(u, w)$ for $u \leq w$ in F^*.

Theorem 42. *Suppose $u \leq w$ in the poset F^*. Then*

$$\mu(u, w) = d(u, w) + \sum(-1)^i \left(\nu(u, pv_i) - \nu(u, x(pv_i))\right),$$

where the sum is over all triples $v_i, pv_i, x(pv_i)$ such that pv_i is a principal factor of w of degree i with base v_i and primary prefix $x(pv_i)$.

Proof. If w is rooted, the result follows from Proposition 41.

If w is unrooted, the proof of the desired result is an easy adaptation of the proof of Theorem 27. □

Chapter 5

Future Research and Open Problems

5.1 Generalizing and Simplifying this Formula

As noted at the end of Section 3, many of the coefficients in our formula for generalized factor order on the integers are zero. This phenomenon is even more pronounced in the rooted forest case. Since our formula simplifies considerably in the case of rooted words, it is natural to wonder whether the general formula can be simplified as well.

Should this formula be simplified, it will likely be done in one of two ways. It may be possible to find a formula which applies to a more general class of posets ordered by generalized factor order. Sagan and McNamara are attempting to prove a formula that works for any poset ordered by generalized subword order which is simpler than the one given in [7]. Thus, it is possible a similar situation could arise here.

To investigate generalized factor order on other posets P^*, one needs to consider words which cover multiple elements, complicating the poset lexicographic order we used in this investigation. One way to resolve these complications is to place an order on the children of each element of P. This would generalize our current chain ids by including a subscript on the label indicating which child each letter is reduced to. In the context of F^*, such subscripts would always be 1 because each element has a unique child, making this new type of chain id a clear generalization of the current one. Initial data suggests it is worth pursuing this line of thought to see if a formula can be found in more general cases.

It may also be possible to simplify the formula at the level of critical simplices. In particular, it is often the case that critical simplices of dimension d and $d+1$ are present when a coefficient of $\nu(u, pv_i)$ is zero. This leads one to suspect it may be possible to use the discrete analogue of "The First Cancellation Theorem" from smooth Morse Theory to cancel critical simplices.

However, it is not clear whether reducing the number of critical simplices in such a manner would result in a simplified formula. For example, it is often the case that a principal factor p will have have unique minimal degree base b_i (of degree i) and maximal base degree base b_j (of degree j) such that if b_k (of degree k) is another base, then $i < k < j$. When this happens, a binomial sum zeros out the coefficient of $\nu(u, p)$. But this is not always the case - the smallest counterexample we found was the word $u = 2111222$ in the interval $[2111222, 2111222112221222]$, in which u occurs as a principal factor of degree 1 twice, degree 2 once, but is not a principal factor of degree 0 because 21112221111 is a longer outer factor. We were not able to reconcile the previous observation with this exception to it in a desirable manner.

5.2 The Topology of $\mathbb{P}^*$ and F^* Under Generalized Factor Order.

Besides being useful results in proving the formula given for the Möbius function of $\mathbb{P}^*$ and F^*, Theorems 24 and 38 can be used to get a detailed description of the critical simplices of intervals in $\mathbb{P}^*$ and F^*. Thus, it is a first step into investigating the homotopy type of these posets. The next step is again checking whether there are critical simplices of dimension d and $d+1$ which cancel each other out.

5.3 The Consecutive Pattern Poset and Ordinary Factor Order

In a paper submitted to the arXiv in 2011, Bernini, Ferrari and Steingrímsson calculate the Möbius function of the consecutive pattern poset [2]. Let S_d be the set of all permutations of the first d positive integers. A consecutive pattern $\sigma = a_1 a_2 \ldots a_k$ appears in a permutation $\tau = b_1 b_2 \ldots b_n$ if the letters of some subsequence $b_i b_{i+1} \ldots b_{i+k-1}$ of τ appear in the same order of size as the letters in σ. The consecutive pattern poset is $\cup_{d \geq 0} S_d$ ordered with respect to consecutive pattern containment.

One of their results is a formula for Möbius function of the consecutive pattern poset which has many similarities to Björner's formula for the Möbius function of ordinary factor order. This suggests there may be some common generalization of these posets. In particular, both posets consist of words which only cover elements obtained by reducing the first or last letter, and all Möbius values are -1, 0, or 1. By investigating this poset with discrete Morse theory, it may be possible to find a more general poset in which all critical chains have decreasing chain ids of the form used in this paper, giving at most one critical chain in every interval. This may lead to more results about the Möbius function in the case of posets ordered by permutation patterns.

APPENDIX

Appendix A

The Matching of Babson and Hersh

In [1], Babson and Hersh give the acyclic matching of simplices in $\Delta(u, w)$ based on whether the sets $I(C)$ and $J(C)$ cover C. Although knowledge of this matching is not required to apply Theorem 3, we record the matching below for completeness.

The Matching:

• If $I(C)$ does not cover $C(w, u)$, let ρ_0 be the lowest rank (that is, the last) vertex not covered by $I(C)$.

Match each new simplex h with $h \triangle \{\rho_0\}$, where $\triangle$ is the symmetric difference operator (that is, $h \setminus \{\rho_0\}$ if $\rho_0 \in h$ and $h \cup \{\rho_0\}$ if $\rho_0 \notin h$.) This matching matches all new simplices of C.

Note that $h \triangle \{\rho_0\}$ is always in $C \setminus (\bigcup_{C' < C} C')$ because the inclusion/exclusion of ρ_0 does not affect whether the simplex hits every MSI. Also, this matching works for the very first maximal chain since the empty set is consider a simplex in $\Delta(u, w)$.

• Otherwise, $I(C)$ covers C and we base the matching on $J(C) = \{J_1, \ldots, J_r\}$. Let ρ_i be the lowest rank vertex of $J_i \in J(C)$. Let $J_{r+1} =$ the set all vertices not in $J(C)$, and let ρ_{r+1} be the lowest rank vertex in J_{r+1}. Define a map τ that associates an integer with each new simplex h based on the first set J_i which h intersects in more than the lowest rank element.

That is,

$$\tau : C \setminus (\bigcup_{C' < C} C') \to [r] \cup \{\infty\}$$

$$h \mapsto \min_{1 \le i \le r} \{i | h \cap J_i \neq \{\rho_i\}\},$$

setting $\tau h = \infty$ when h intersects each J_i in exactly the lowest rank element (that is, when the set $\{i | 1 \le i \le r, h \cap J_i \neq \{\rho_i\}\}$ is empty.)

First, if $\tau h \neq \infty$, match the new simplex h with $h \triangle \{\rho_{\tau h}\}$. This matches all simplices for which $\tau h \neq \infty$.

If $J_{r+1} = \emptyset$, then $J(C)$ covers C and there is one simplex satisfying $\tau h = \infty$. This simplex contains each ρ_i. Thus, there is one simplex unmatched and it is a critical simplex of this matching.

If $J_{r+1} \neq \emptyset$, then $J(C)$ does not cover C even though $I(C)$ does. In this case, we match each simplex h satisfying $\tau h = \infty$ with $h \triangle \{\rho_{r+1}\}$. This matches all simplices for which $\tau h = \infty$.

The matching is more complicated when $I(C)$ covers C but $J(C)$ does not because ρ_{r+1} is in $I(C)$. Thus, removing it from a simplex does not guarantee every MSI is still hit, meaning that $h \setminus \{\rho_{r+1}\}$ might not be in $C \setminus (\bigcup_{C' < C} C')$. To guarantee $h \setminus \{\rho_{r+1}\}$ is a new simplex, we need the lowest rank element of each J_i, ρ_i, in h. Indeed, any interval from $I(C)$ reduced and not included in $J(C)$ intersects some interval J_i in at least the element ρ_i. This issue led to a mistake in the published version of Babson and Hersh's article, but this mistake is absent from later versions.

Please refer to Tables 3 and 4 for the context of the below examples.

- $I(C)$ does not cover C. Example: Chain $1 - 2 - 5 - 3$ in the interval $[b, bbabb]$. Match new simplices based on inclusion/exclusion of vertex abb. That is, match ab with $abb - ab$ and $babb - ab$ with $babb - abb - ab$.

- $J(C)$ covers C. Example: Chain $5 - 4 - 3 - 1$ in $[b, bbabb]$. Match 2 of 3 new simplices, based on inclusion/exclusion of vertex bba. That is, $bba - bb$ is an unmatched, critical simplex, while $bbab - bb$ is matched with $bbab - bba - bb$.

- $I(C)$ covers C, but $J(C)$ does not. Example: Chain $6 - 5 - 4 - 3 - 2$ in $[a, abbabb]$. We match all new simplices based on the matching rules above. In particular, match $abbab - abb$ with $abbab - abba - abb$, $abbab - abb - ab$ with $abbab - abba - abb - ab$, and $abba - abb$ with $abba - abb - ab$.

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