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TOPOLOGY IN LATTICES

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Introduction

Various intrinsic topologies which can be introduced into a lattice have been defined by Kantorovitch (5), Birkhoff (2) and Frink (4), and it is with these that the present investigation is concerned. The thesis is divided into three chapters. The first consists, for the most part, of basic definitions and well known theorems concerning lattice theory, general topology and intrinsic topologies in lattices, while the second and third chapters contain an exposition of the results of this thesis. These results are summarized on pages 16 and 17 after adequate terminology has been introduced. Mainly they deal with the Hausdorff charadter of the intrinsic topologies.

The various numbered problems referred to are from a list which has been compiled by Garrett Birkhoff and appears in (1). This book also contains proofs of the various assertions about lattice theory which are made in Chapter I. It is the standard reference for the subject. The numbers in parentheses refer to the bibliography at the end. An expression of the type $\{x \mid P(x)\}$, where x is an element of some given set E and P(x) is a property of x, stands for the subset of E consisting of all elements having property P.

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Chapter I Preliminary Results

A. Partly Ordered Sets and Lattices

A partly ordered set is a collection of elements and a binary relation defined on these elements which is reflexive, asymmetric and transitive. If we denote the relation by $\leq$ then the three axioms for a partly ordered set are:

(i) $x \leq x$

(ii) $x \leq y$ and $y \leq x$ imply $x=y$

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$.

We may write $y \geq x$ instead of $x \leq y$ and in this case we may say y is over x or x is under y. If $x \leq y$ and $x \neq y$ we write $x < y$ and say that x is properly under y. Given a partly ordered set P we can construct another P', called the dual of P, by saying that $x \leq y$ in P' if and only if $y \leq x$ in P. If $x \leq y$, $x=y$ or $y \leq x$, we say that x and y are comparable. Otherwise they are called incomparable.

Most partly ordered sets which have mathematical significance satisfy certain other axioms. The least restrictive of the axioms we shall consider is the compositive axiom of E. H. Moore. A partly ordered set is compositive or directed, as we say nowadays, if given x and y there exists z such that $x \leq z$ and $y \leq z$. If a directed set D is mapped into another set S by a mapping f, we call the resulting pair (D,f) a directed system taking values in S. A directed system is thus a generalized sequence, and in fact the use of diredted systems is essential if the usual convergence statements concerning sequences on the real line are to be carried over to fairly general topological spaces. A <u>terminal</u> <u>subset</u> T of a directed set D is a collection of the type $\{x \mid x \geq a\}$ for some $a \in D$.

If $\{a_\iota\}$ is any collection of elements of a partly ordered set P we say that u is an <u>upper</u> <u>bound</u> of $\{a_\iota\}$ if $a_\iota \leq u$ for all a_ι. An element u is a <u>least</u> <u>upper</u> <u>bound</u> (or lub) of $\{a_\iota\}$ if u is an upper bound and is under any other upper bound. A lub is clearly unique. In a similar manner we define <u>lower</u> <u>bound</u> and <u>greatest</u> <u>lower</u> <u>bound</u> (glb). If a partly ordered set is such that any pair of elements has a glb and a lub it is called a <u>lattice</u>. In this case the glb and lub of x and y are denoted respectively by $x \cap y$ and $x \cup y$. Clearly the operations $\cap$ and $\cup$ are idempotent, commutative, associative and satisfy the absorption laws: $x \cup (x \cap y)=x$ and $x \cap (x \cup y)=x$. Also $x \geq y$ if and only if $x \cup y=x$ or $x \cap y=y$. A one to one mapping of one lattice onto another is called an <u>isomorphism</u> if it preserves lub's and glb's. A lattice is called <u>complete</u> if any collection $\{a_\iota\}$ of elements has a glb and a lub. These are denoted respectively by $\bigwedge a_\iota$ and $\bigvee a_\iota$. In particular a complete lattice has a least element, usually denoted by 0, and a greatest element, usually denoted by I. In a lattice with 0, x and y are called _disjoint_ if $x \cap y = 0$. A lattice for which every set having an upper bound has lub and similarly for lower bounds is called _conditionally complete_. The real line under the usual ordering is conditionally complete but not complete. Any conditionally complete lattice can be made complete by adjoining an 0 or an I or both.

Just as on the real line, we can define open and closed intervals in a partly ordered set. The _open interval_ (x,y) where $x < y$ is $\{z \mid x < z < y\}$ and the _closed interval_ $[x,y]$ is $\{z \mid x \leq z \leq y\}$. Also we can define semi-infinite intervals in the usual way. $[x,\infty]$ is $\{z \mid z \geq x\}$ and $[-\infty,x]$ is $\{z \mid z \leq x\}$. In a lattice we can show that the intersection of two closed intervals is a closed interval. In fact if $x \in [a,b]$ and $x \in [c,d]$, then $a \leq x \leq b$ and $c \leq x \leq d$. Hence $a \cup c \leq x \leq b \cap d$ or $x \in [a \cup c, b \cap d]$ and conversely this interval is contained in each of the first two. It should be noted that in general the intersection of two open intervals is not an open interval and that in a partly ordered set the intersection of two closed intervals need not be a closed interval. In the future, unless otherwise stated, all intervals will be assumed to be closed.

A lattice having the property that any two elements are comparable is a _chain_ and is referred to as being _simply ordered_. Any subset of a partly ordered set which is a chain in the induced ordering is called a chain of the partly ordered set. An infinite chain not isomorphic to the positive integers is called a <u>transfinite sequence</u>. To see that a lattice is conditionally complete it is not necessary to test every collection of elements for lub and glb. Rennie has shown (6; p 387) that a lattice is conditionally complete if any chain having lower bound has a glb. An <u>atom</u> in a lattice having an 0 is an element x such that $0 < y \leq x$ implies $y=x$. In this case we say that x <u>covers</u> 0. In general if $z < y \leq x$ implies $y=x$, we say that x <u>covers</u> z.

Any lattice which satisfies the <u>distributive laws</u>:
$x \cup (y \cap z)=(x \cup y) \cap (x \cup z)$ and $x \cap (y \cup z)=(x \cap y) \cup (x \cap z)$
is called <u>distributive</u>. It should be noted that either of these laws implies the other. Any collection of sets satisfies these laws and in fact it can be shown that this example includes all distributive lattices in the sense of isomorphism. A complete or conditionally complete lattice may satisfy either or both of the <u>infinite distributive laws</u>:

(i) $a \cup (\bigwedge b_\alpha)=\bigwedge(a \cup b_\alpha)$

(ii) $a \cap (\bigvee b_\alpha)=\bigvee(a \cap b_\alpha)$.

Each of these implies both of the finite distributive laws but a distributive lattice may satisfy either, none or both of the infinite distributive laws. In a non-conditionally complete lattice (i) and (ii) will be said to hold if the required upper and lower bounds exist and are equal. If (i) or (ii) holds, so does its extension to any finite number of terms:

$$(i)'\ (\Lambda_A a_\alpha) \cup (\Lambda_B b_\beta) = \Lambda_{AB}(a_\alpha \cup b_\beta)$$

$$(ii)'\ (V_A a_\alpha) \cap (V_B b_\beta) = V_{AB}(a_\alpha \cap b_\beta).$$

Here (α, β) ranges over the Cartesian product set (A,B). Each of these is a special case of the corresponding doubly infinite distributive law but does not imply it.

$$(i)''\ V_C\left[\Lambda_{A_\gamma}\ u_{\gamma,\alpha}\right] = \Lambda_F\left[V_C\ u_{\gamma,\varphi(\gamma)}\right]$$

$$(ii)''\ \Lambda_C\left[V_{A_\gamma}\ u_{\gamma,\alpha}\right] = V_F\left[\Lambda_C\ u_{\gamma,\varphi(\gamma)}\right]$$

Each $A_\gamma = \{\alpha \ldots\}$ is the index set of a collection of elements of the lattice and γ ranges over a set C. F is the class of all single valued functions φ, assigning to each $\gamma \in C$ a value $\varphi(\gamma) \in A_\gamma$. When C has two members, (i)'' and (ii)'' reduce to (i) and (ii) respectively. Complete chains and the lattice of all subsets of any given set are examples which satisfy (i)'' and (ii)''. Such lattices are called <u>completely distributive</u>.

In a lattice with 0 and I, an element x' is called a complement of x if $x \cap x' = 0$ and $x \cup x' = I$ and a lattice in which every element has a complement is called complemented. A complemented distributive lattice is a <u>Boolean algebra</u>. In a Boolean algebra the complements are unique and are orthocomplements ie $(a')' = a$. Any Boolean algebra is isomorphic to a Boolean algebra of subsets where finite glb and lub and complementation have their usual set-theoretic interpretation.

Next we shall define several methods for building new partly ordered sets out of given ones. The simplest process is <u>ordinal addition</u>. If L and M are (non-overlapping) partly ordered sets, then their <u>ordinal</u> sum $L \oplus M$ is constructed out of the sum of the two sets by putting the elements of L and M in their given order and then putting every element of L under every element of M. In general this sum is non-commutative and always the ordinal sum of two chains is a chain. This definition can be extended to an arbitrary number on non-overlapping summands. If $M=\{m, m', m''\ldots\}$ is a lattice and $\{L_m\}$ is a collection of lattices indexed by M we can define $\sum_M L_m$ as follows: $l'_{m'} \leq l''_{m''}$ if and only if $m' < m''$ or $m'=m''$ and $l'_{m'} \leq l''_{m''}$ in $L_{m'}$. Later we will be interested in the case where L and M are chains and in this case $\sum_M L_m$ is clearly a chain too. The ordinal product $L \circ M$ is defined on the Cartesean product set and here we say $(l',m') \leq (l'',m'')$ if and only if $l' < l''$ or $l'=l''$ and $m' \leq m''$. Intuitively for each point of L we "substitute" the partly ordered set M.

A <u>lattice ordered group</u> (l-group) is a set which is a lattice and a group in which the group translations are compatible with the order relation. In other words $x \leq y$ implies $a+x+b \leq a+y+b$ for all a and b. The group operation is usually denoted by + even though it may be non-commutative. The real numbers under addition are one example. Another is the set of all real valued continuous functions defined on the unit interval. Here $f \leq g$ means that $f(x) \leq g(x)$ for all x in the interval.

B. Filters

Filters were introduced by H.Cartan to facilitate techniques of general topology and recently J.Schmidt (8) has started a series of papers which will develop the theory of filters as a separate discipline. The results presented here without proof can be found in N.Bourbaki (3; pp 32-46) or in J.Schmidt (8). The process used here to show the equivalence of filters and directed systems is probably known but may not have appeared in print.

If E is any set, a <u>filter</u> F on E is a collection of subsets of E having the following properties:

(i) the empty set $\emptyset$ does not belong to F

(ii) any set containing a set of F belongs to F

(iii) the intersection of any two sets of F belongs to F.

Thus, for example, the collection of all subsets containing a point x is a filter. It should be observed, however, that the totality of all sets of a filter can have empty intersection. Ordinarily when working with filters it is convenient to make use of the concept of a filter base. A collection B of sets of a filter F is called a <u>base</u> of F if every member of F contains a set of B. In this case F is the collection of all subsets of E, each of which contains some set of B. In the above example the point x is a base for filter described. In order that an arbitrary collection B of sets be a base for some filter it is necessary and sufficient that the intersection of any two sets of B is non empty and contains a set of B. Under a mapping of E into any set X the image of a filter on E is in general a filter base on X. If E is any directed set, the terminal subsets clearly form a filter base, which is called the associated filter base. A collection G of subsets of E is a system of generators of a filter if the finite intersections of the sets of G form a filter base. It is easy to see that this is the case if and only if any pair of sets of G has non empty intersection.

The set of all filters on E can be partly ordered in the following manner. We say that a filter F' is finer than F" and write $F' \geq F''$ if every set of F" is a set of F'. We say that a filter base B' is finer than another B" if every set of B" contains a set of B'. Two filter bases are called equivalent if they give rise to the same filter. If $\{F_\iota\}$ is a collection of filters on E, it can be verified that those sets which appear in each F_ι form a filter and that this filter, called the intersection of $\{F_\iota\}$, is the finest filter which is coarser than each member of $\{F_\iota\}$. This filter intersection can be constructed in the following manner.

For each ι pick any set in F_ι and take the union of the sets so chosen. The totality of these unions is exactly the filter intersection. Under the above partial ordering the set of all filters on E does not form a lattice, nor even a directed set, since in general there need not be any filter finer than two given ones. If F' and F'' are two filters, they have a common refinement if and only if every set of F' has non empty intersection with every set of F'', and in fact these intersections form the coarsest filter finer than F' and F''. Two filters which do not have a common refinement are <u>disjoint</u>.

Finally we establish a correspondence between directed systems and filters which will be used in Chapter III. If E is a directed set and f is a mapping of E into a set X then the images under f of the terminal subsets of E form a filter base on X which we call the <u>associated filter base</u>. We now show that any filter base $B=\{b_\lambda\}$ can be obtained in this manner. Looking at the sets of B as abstract elements, we say $b_\iota \leq b_\lambda$ if b_ι contains b_λ as a set. Let us denote this directed set by B' and form the ordinal product $B' \circ \omega = B'' = \{(b_\lambda, j)\} = \{b_{\lambda,j}\}$ where j is an integer. As usual ω is the chain formed by the positive integers under their usual ordering. Let b''_λ stand for the subset b_λ of X simply ordered in some manner. Then in the directed set $\sum_{B''} b''_{\lambda,j}$ each element corresponds to a point of X and the terminal subsets of the resulting directed system are exactly the sets of our filter base B.

We shall call this the <u>associated directed system.</u> If we start with a transfinite sequence then the associated filter base will be nested and conversely a nested filter base has a transfinite sequence as associated directed system. By a nested collection of sets we mean one for which every set is comparable with every other set.

C. General Topology

The definitions and results used here are classical although filters and directed sets have been employed only since 1937. All of the results in this section will be found in Bourbaki (3). A <u>topological space</u> is defined to be a set E together with a collection U of its subsets, called open sets, which satisfies the following axioms:

(i) E and $\emptyset$ are open

(ii) the intersection of any two open sets is open

(iii) the union of any collection of open sets is open.

If a point is an open set it is called <u>isolated</u>. A closed set is one whose complement is open, and it is easy to see that the closed sets have the following properties:

(1) E and $\emptyset$ are closed

(2) the union of any two closed sets is closed

(3) the intersection of any collection of closed sets is closed.

We could have just as easily axiomatized the closed sets by (1), (2) and (3), and then (i),(ii), and (iii) would appear as theorems if an open set were defined to be the complement of a closed set. This duality permeates much of general topology. Since we will be concerned mainly with closed sets in the future, we state the important definitions of basis and sub-basis in terms of these. The corresponding definitions for open sets can be obtained by applying the duality principle. A collection $\{F_i\}$ of closed sets is called a <u>basis</u> if any closed set can be obtained by taking the intersection of suitable members of $\{F_i\}$. In order that $\{F_i\}$ be a basis for the closed sets of some topology it is necessary and sufficient that $\{F_i\}$ contain all finite unions of its members. Thus starting with any collection of sets we can get a topological space by first taking finite unions and then arbitrary intersections. The collection of sets we started with is called a <u>sub-basis</u> for the generated topology. Needless to say, the topology will have little mathematical significance unless there are sound a priori reasons for the sets in the sub-basis to be closed.

The various topologies on a set E can be compared in much the same manner as filters. We say that a topology T' is <u>finer</u> than T'' if every open set of T'' is an open set of T'. Under this partial ordering it can be shown that the topologies form a complete lattice wherein the top element is the discrete topology, for which every set is open, and the bottom element has only E and $\emptyset$ as open sets.

For technical reasons it has been found convenient to introduce the idea of a neighborhood of a point x. We say that U is a neighborhood (nbhd) of x if U contains an open set which contains x. It is clear that the nbhds of x have the following properties:

(i) the totality of nbhds of x forms a filter

(ii) every nbhd of x contains x.

(iii) every nbhd U of x contains a nbhd V of x such that U is a nbhd of every point of V.

Conversely if to each point of E we assign a collection of sets satisfying (i),(ii) and (iii), we can call a set "open" if it is a nbhd of each of its points and indeed the resulting "open" sets do satisfy the open set axioms. Property (iii) above assures us that if we start with a collection of nbhds, define open sets and then define nbhds in terms of these open sets, the new nbhds will be exactly those we started with.

We define convergence of a directed system in a topological space in much the same manner as we define convergence of an ordinary sequence on the real line. If $A=\{\alpha \ldots\}$ is a directed set and x_α a directed system, we say that x_α converges to x if each nbhd of x contains a terminal subset of x_α . A filter F on a topological space is said to converge to x if it is finer than the filter of nbhds of x, and a filter base B is defined to converge to x if the derived filter converges to x. This is equivalent to saying that every nbhd of x contains a set of B. It follows immediately from the definitions that a directed system converges to x if and only if its associated filter base does, and that a filter base converges to x if and only if its associated directed system does.

One of the basic properties of sequences on the real line is that a limit is unique if it exists, and furthermore it is easy to show that all convergent directed systems and all convergent filters have unique limits too. In an arbitrary topological space if we want directed systems and filters to have unique limits it is necessary and sufficient that the open sets have the following property: given any distinct points x and y, there exist disjoint open sets U and V each containing one of the points. This property is called the <u>Hausdorff axiom</u>, and a topological space in which it is satisfied is called a <u>Hausdorff space</u>. Obviously any topology finer than a Hausdorff topology is Hausdorff too. It should be noted that in general the existence of unique limits for sequences alone is not enough to insure that a space be Hausdorff. Many of the spaces encountered in the sequel will not be Hausdorff, but all of them will satisfy the following weaker separation axiom: a point is a closed set. A space satisfying this axiom is called T_1.

Just as was the case with lattices, it is possible to combine the axioms for a topological space with the group axioms in a significant way. A <u>topological group</u> is a set E which is a topological space and a group and for which the group operations are continuous in the topology. That is if $x \cdot y = z$ and W is any nbhd of z there exist nbhds U of x and V of y such that $U \cdot V$ is contained in W where $U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$. Also given a nbhd W of z there is a nbhd X of z^{-1} such that X^{-1} is contained in W. One property of topological groups we will use is that if a topological group is T_1 it is Hausdorff. This is easily shown.

D. <u>Topology in Lattices</u>

Now we shall introduce various intrinsic topologies into partly ordered sets and especially lattices. It is possible to define the ordinary topology of the real line in several ways which depend only on the ordering. First we can take the open intervals as a basis for the open sets. Second we can take the closed intervals as a subbasis for the closed sets and finally we can introduce order convergence of sequences in the following manner. A sequence x_n order converges to x ($x_n \rightarrow x$) if $\bigwedge_m \bigvee_{n \geq m} x_n = x = \bigvee_m \bigwedge_{n \geq m} x_n$. It is easily seen that on the real line this is equivalent to ordinary convergence. A set X is called closed if $x_n \in X$ and $x_n \to x$ imply $x \in X$. We could just as easily have defined order convergence to x of an arbitrary directed set x_α by requiring that $\bigwedge_\beta \bigvee_{\alpha \geq \beta} x_\alpha = x = \bigvee_\beta \bigwedge_{\alpha \geq \beta} x_\alpha$ and then we would say that a set X is closed if $x_\alpha \in X$ and $x_\alpha \to x$ imply $x \in X$. Both of these definitions can be extended immediately to lattices. The statement $x_\alpha \to x$ carries with it the tacit assertion that the various infinite upper and lower bounds in the definition exist.

The topology using ordinary sequences was first investigated by Kantorovitch (5) in the special case of Abelian l-groups. He discussed applications to convergence in measure and functions of bounded variation. Sometimes this topology is named after him. The generalization to directed sets is due to Garrett Birkhoff (2) and the topology so obtained is called the <u>order topology</u> of a lattice. Clearly the Kantorovitch topology is finer than the order topology. If we use transfinite sequences in the above we get a third type of convergence topology, finer than the order topology but coarser than the Kantorovitch topology, which we call the <u>sequential order topology</u>. Some of its properties have been determined by Rennie in (6) and (7).

As for the topologies based on open and closed intervals, the former is discrete on the cardinal product ($(x,y) \leq (x',y')$ if $x \leq x'$ and $y \leq y'$) of the real line with itself and apparently it has not been considered worth investigating. The topology using closed intervals was introduced by Frink (4) and is known as the <u>interval topology</u>. This topology has the rather interesting property of being compact on complete lattices and furthermore it agrees with the order topology on the product of a closed line segment with itself and in other cases. It is easy to see that a closed interval of a lattice is closed in the order topology, which means that the order topology is always finer than the interval topology. We see at once that all of the topologies defined so far are always T_1.

We are now in a position to describe the results of the present investigation. Birkhoff's Problem 76 asks if the order and interval topologies agree in a complete Boolean algebra. This is answered in the negative by an example of a Boolean algebra which is Hausdorff in its order topology but not in the interval topology. This problem has already been solved by Rennie (7). Our approach first of all leads to a necessary condition (Theorem 2.2) that the interval topology be Hausdorff and this condition gives an easy (negative) answer to Problem 104: "Is any l-group a topological group and a topological lattice in its interval topology ?" Also in Theorem 2.5 we find a necessary and sufficient condition that the interval topology of a Boolean algebra be Hausdorff. These ideas further lead to a solution to part of Problem 21. We find a necessary and sufficient condition that an element of a lattice be isolated in the interval topology (Theorem 2.7). This completes the main results of Chapter II.

Chapter III deals, for the most part with the Hausdorff character of the order topology. Theorem 13 of Birkhoff (1; p 60) asserts that the order topology is always Hausdorff, but Professor Birkhoff has recently agreed that the proof given is inadequate and suggests that the problem is an interesting open question. By the introduction of order convergent filters we are led to a lattice in which the sequential order topology is not Hausdorff. Since this lattice is complete, the L_- topology of Rennie is non-Hausdorff too. This answers Problem 2. of Rennie (6). We are able to show that at least in a complete, completely distributive lattice the order topology is always Hausdorff. Finally as a side result we answer the other part of Birkhoff's Problem 21, which is to find a necessary and sufficient condition that an element be isolated in the order topology.

Chapter II The Interval Topology

First we shall investigate the Hausdorff character of the interval topology in a general partly ordered set. It follows at once from the definition of a basis for open sets that if x and y are any two points in a Hausdorff space and B is any basis for the open sets, then x and y can be separated by open sets from B, say, U and V. Looking at the complements of U and V we obtain the dual requirement that given any two distinct points, the space can be covered by two closed sets each of which contains exactly one of the points, and in addition we may select these sets from any given basis for the closed sets. In particular:

Theorem 2.1. *The interval topology of a partly ordered set is Hausdorff if and only if given any two distinct points there is a covering of the set by means of a finite number of closed intervals such that no interval contains both points.*

To obtain a necessary condition that the interval topology of a lattice be Hausdorff we look at any pair of comparable elements, $x < y$, for which by Theorem 2.1 there is a covering of the lattice by a finite number of closed intervals such that no interval contains $[x,y]$. Taking the trace (intersection) on $[x,y]$ of each member of the covering we obtain a covering of $[x,y]$ by a finite number of closed subintervals, no one of which is $[x,y]$ itself. In other words, if we exclude x and y, each point of $[x, y]$ is comparable with at least one of the

remaining end points of the subintervals. The same is true if either x or y is infinite. Let us say that a collection of elements $\{a_i\}$ is a separating set of the interval $[x,y]$ if $x < a_i < y$ for each a_i and every element of $[x,y]$ is comparable with at least one of the a_i. If y covers x we will agree that the empty set separates $[x,y]$. Summarizing we have:

Theorem 2.2. <u>A necessary condition for the interval topology of a lattice to be Hausdorff is that every closed interval have a finite separating set (fss).</u>

We are now in a position to prove:

Theorem 2.3. <u>In a Boolean algebra without atoms, the interval 0,I has no fss.</u>

Proof. If $\{a_1, a_2, \dots a_n\}$ is a fss, adjoin the complements a_i' of the a_i obtaining a new set B. For each subset of B form the meet of its elements and from this collection of meets let $\{c_1, c_2, \dots c_k\}$ be the non null minimal ones. It is convenient to think in terms of sets in which case the c_i are a collection of disjoints sets whose union intersects each a_k and its complement. Now for each c_i choose d_i so that $0<d_i < c_i$ and let $d = d_1 \cup d_2 \dots \cup d_k$. Then since $a_i > d \cap a_i$ we have $d \not\leq a_i$ and since $d \cap a_i' > 0$, $d \not\leq a_i$. In other words d is not comparable with any a_i.

Remark. The lattice of all measurable subsets (modulo sets of measure zero) of the unit interval is a complete Boolean algebra without atoms, so its interval topology is, by the preceeding theorem, not Hausdorff whereas the order topology is (1; p 169 and p 80). Theorem 2.3 may be applied to the solution of problem 76 of (1). This problem has already been solved by B.C. Rennie (7) using a different method.

We might observe further, that an examination of the proof of Theorem 2.3 shows that the following somewhat more general result may be established.

Theorem 2.4 . A distributive lattice without atoms, in which each element (except I) has a non null disjoint element, is not Hausdorff in its interval topology.

Theorem 2.5 . The interval topology of a Boolean algebra is Hausdorff if and only if every element is over an atom.

Proof. If some element x is over no atom, then the interval [0,x] is a Boolean algebra without atoms. Hence, by Theorem 2.3, it has no fss and thus from Theorem 2.2 the topology is not Hausdorff. Assume then that every element is over an atom and let x and y be any pair of distinct elements. Since $x \cap y'$ and $y \cap x'$ cannot both be null there must be an atom a under, say, x but not y. It follows at once that the intervals [a,I] and [0,a'] are disjoint closed intervals which cover the algebra, and the topology is Hausdorff (Theorem 2.1).

Next we apply Theorem 2.2 to Problem 104 of (1), which should read: "Is any l-group a topological group and a topological lattice in its interval topology ?"

Since the interval topology is T_1 it must be Hausdorff if the l-group is to be a topological group. Now the additive group of all continuous real valued functions defined on the closed unit interval is an l-group using the natural ordering (1; p 216). If f_0 denotes the function $f(x)=0$ and f_1 denotes the function $f(x)=1$, we show that the interval $[f_0, f_1]$ has no fss. If $\{a_1 \dots a_n\}$ were such a set, choose for each a_i some point x_i where $a_i(x_i)<1$. Define a continuous function $a(x)$ to be 1 at each of the x_i and elsewhere to take on values between 0 and 1 so that its integral over the interval is less than that of any a_i. Clearly $a(x)$ is not comparable with any of the a_i. It is interesting to note that the set of all real-valued functions does have a fss for any interval and in fact (2) the interval topology is Hausdorff. We have shown:

<u>Theorem 2.6</u>. <u>An l-group need not be a topological group in its interval topology.</u>

Finally we find a necessary and sufficient condition for a point x to be isolated in the interval topology of a lattice L. This is part of Problem 21 of (1). First suppose that $0 < x < 1$. If x is isolated then $L-x$ is a closed set and in fact must be the union of a finite number of closed intervals $I_1 \dots I_k$. Let P denote the set of elements of L under x and take the trace of each I_k on P, which is a closed interval. From the set of upper endpoints of the traces select the maximal ones. These form a non-empty finite set $\{x_1 \dots x_n\}$, each x_i is covered by x and any element under x is under some x_i. The same argument can be applied to the set of elements over x. Looking at the lower endpoint of each I_k let us replace it by $-\infty(x)$ if it is under, (over) x. Then if an upper endpoint is under (over) x replace it by $x(\infty)$. Having done this we have a covering of L by a finite number of closed intervals for which none of the endpoints (except possibly x, 0, or I) is comparable with x. In other words x belongs to a fss of L in which no other member is comparable with x, and we have shown the necessity of the conditions in the following

<u>Theorem 2.7</u>: <u>The following conditions are necessary and sufficient for an element x to be isolated in the interval topology of a lattice L.</u>

(a) x <u>covers a finite number of elements and every element under</u> x <u>is under an element covered by</u> x.

(b) x <u>is covered by a finite number of elements and every element over</u> x <u>is over an element which covers</u> x.

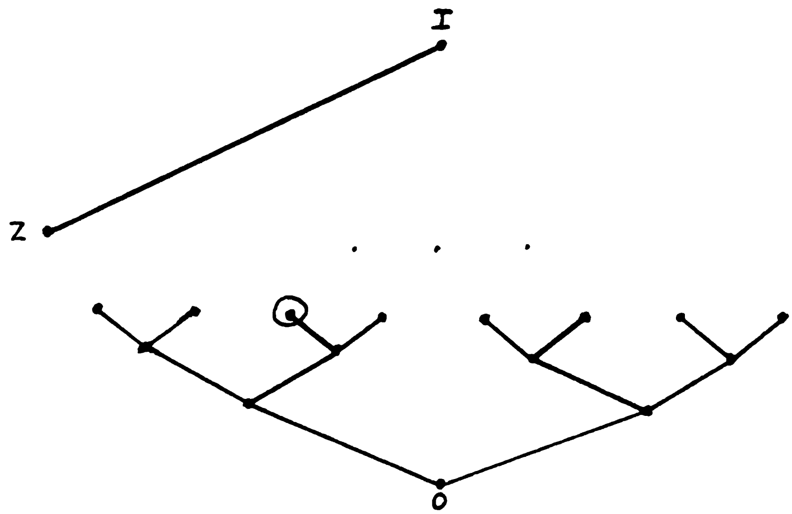
(c) x <u>belongs to a fss of L in which no other member is comparable with</u> x.

It is easy to see that the above conditions are sufficient. If the fss is $\{x, a_1 \ldots a_k\}$ and if x covers $\{b_1 \ldots b_m\}$ and if x is covered by $\{c_1 \ldots c_n\}$ then L-x is the union of the following intervals:

$[-\infty, a_i]$ $[a_i, \infty]$ $[-\infty, b_i]$ $[c_i, \infty]$ for all permissible values of i.

If x is 0 or I then clearly (b) or (a) is necessary and sufficient for x to be isolated.

We shall conclude our discussion of the interval topology with an example which shows that the necessary condition of Theorem 2.2 is not sufficient and that condition 3 of Theorem 2.7 is not a consequence of the first two.



The lattice is formed by all finite and infinite sequences (s_n) which take values in a two element set, say $\{x,y\}$, and a top element I. We say that $(s_n) \leq (t_n)$ if (t_n) is a continuation of (s_n). The diagram has been arranged so that at any given term of a sequence an x means "take the left branch" and a y means " take the right branch". Thus the circled point stands for the finite sequence (xyx). Let us call the points corresponding to finite sequences,finite points, and those corresponding to infinite sequences, infinite points. It is obvious that there are an uncountable humber of infinite points over any finite point, hence any finite collection of intervals whose upper endpoint is I contains all infinite points or does not contain an uncountable number of them. Any other type of interval can contain only a countable number of elements. In other words if the lattice is covered by a finite number of intervals in any way, each infinite point is contained in an interval whose upper end point is I. By Theorem 2.1 this means that no infinite point can be separated from I by open sets, so the interval topology of this lattice is not Hausdorff.

If we insert an element u between some infinite point z, say (x, x, ...), and I, we get a lattice in which u satisfies the first two conditions of Theorem 2.7 but not the third. Consider the elements not on the chain $[0,z]$ but which cover members of it. There are an infinite number of these, $\{z_n\}$, none of which are comparable with z. Furthermore the intervals $[z_n, I]$ are disjoint except for I (which cannot belong to a fss), and any two intervals $[0,z_n]$ and $[0,z_m]$ have intersection contained in the chain $[0,z]$. This means that the z_n are comparable with no finite collection of points, none of which is on the chain $[0,z]$. Thus any fss of the lattice must contain some point of $[0,z]$, hence a point comparable with u.

Chapter III The Order Topology

When solving Birkhoff's Problem 76 (1; p 166) we might have used our Theorem 2.5 together with Theorem 13 of Birkhoff (1; p 60) to get a large class of Boolean algebras for which the order and interval topologies do not agree. This theorem asserts that the order topology of any partly ordered set is Hausdorff. The argument given there proceeds as follows. First it is noted that if a directed set order converges, then the limit is unique. This follows immediately from the uniqueness of glb and lub. Secondly reference is made to the fact that in any topological space, if directed systems have unique limits, then the space is Hausdorff. Now in the order topology, in general, there are convergent directed systems which do not order converge and we must show that these have unique limits too, if we are to argue in this manner. It turns out however that the order topology can have convergent directed systems with non-unique limits, and hence need not be Hausdorff.

The main difficulty when working with the order topology is that one must take into consideration not only a large variety of directed sets but also the various ways each can be mapped into the lattice. Tukey (9) has shown that actually we need consider only very special types of directed sets, but nevertheless the direct attack on the Hausdorff character seems difficult to carry through. By the introduction of order convergent filters and use of the equivalence between directed sets and filters we shall reduce the problem to a discussion of the intervals on the lattice, and describe a process whereby the Hausdorff character may be determined in certain cases. Throughout we shall denote filters, and collections of sets in general, by script letters. Individual sets of a collection will be denoted by capital Roman letters and as usual small Roman letters will stand for elements of a lattice.

By definition, a filter base $\mathcal{B}=\{B_\alpha\}$ <u>order converges</u> to x if $\bigwedge j_\alpha = x = \bigvee m_\alpha$ where $B_\alpha=\{x_\alpha^\beta\}$, $j_\alpha = \bigvee x_\alpha^\beta$, and $m_\alpha = \bigwedge x_\alpha^\beta$. Since any two sets of a filter base have non empty intersection it is clear that every m_α is under every j_β. This implies that if a filter base order converges to x then so does any finer filter base. The following lemma is an immediate consequence of the definitions.

<u>Lemma 3.1</u> <u>A directed system order converges to x if and only if the associated filter base order converges to x. Also a filter base order converges to x if and only if the associated directed system order converges to x.</u>

Now for each memeber of a filter base $\mathcal{B}=\{B_\alpha\}$ order converging to x, we define m_α and j_α as above and then assign to B_α the interval $[m_\alpha, j_\alpha]$, which contains B_α. Each interval contains x and the collection of intervals forms a system of generators for a new filter which clearly order converges to x too. If B_α, $B_\beta \in \mathcal{B}$ then $B_\alpha \cap B_\beta \subset [m_\alpha, j_\alpha] \cap [m_\beta, j_\beta] = [m_\alpha \cup m_\beta, j_\alpha \cap j_\beta]$, which tells us that the generated filter base is actually coarser than B.

Summarizing we have

<u>Lemma 3.2</u> <u>Every filter base order converging to</u> x <u>is finer than a filter base of intervals, which likewise order converges to</u> x.

Let us call the intersection of all filters order converging to x the filter of pseudo-neighborhoods of x. Since every filter base order converging to x must converge to x in the order topology, it follows that the filter of pseudo-nbhds must converge in the order topology. If the filter of pseudo-nbhds satisfied the nbhd axioms, it would indeed be the nbhd filter, but there are cases where the nbhd filter is properly coarser than the pseudo-nbhd filter. Using Lemma 3.2 we see that in order to obtain the pseudo-nbhd filter cf x we need only consider the intersection of the filter bases of intervals which order converge to x, and it is easy to decide whether a filter base of intervals order converges to x.

<u>Lemma 3.3</u> <u>A filter base of intervals order converges to</u> x <u>if and only if the intersection of all the intervals is</u> x.

A basis for the pseudo-nbhd filter is formed than as follows. From each collection of intervals having the finite intersection property (the intersection of any two intervals of the collection is a member of the collection) whose intersection is x, select an interval and form the union of the selected intervals. The totality of all such unions is a basis for the pseudo-nbhd filter.

Now in order that the order topology of a lattice be Hausdorff it is necessary (but probably not sufficient) that for any two distinct points x and y, their pseudo-nbhd filters be disjoint. The next paragraphs will be devoted to the construction of a lattice where this necessary condition is not satisfied, but first we note

<u>Lemma 3.4</u> <u>The filter $\mathcal{U}$ of nbhds of</u> x <u>is disjoint from any filter base $\mathcal{B}$ of intervals which order converges to</u> y.

Proof: We can pick any interval G of $\mathcal{B}$ which does not contain x. Then if every set of $\mathcal{U}$ had non empty trace on G these traces would form a filter base on G which would converge in the order topology to x, but this is impossible since G is closed in the order topology.

In a lattice, to show that every pseudo-nbhd of x intersects every pseudo-nbhd of y, it is necessary and sufficient to find a collection of filter bases of intervals, each order converging to x, such that no matter how we select an interval from each base and form the union, some filter base order converging to y has non empty trace on this union. The necessity is obvious.

The sufficiency follows from the fact that each pseudo-nbhd contains one of the above unions. If the filter bases are nested, then the (transfinite) sequential order topology will be non Hausdorff. Let us denote the first uncountable ordinal number by ω_1 and let Δ be the dual of $\omega_1 + 1$. Δ is a complete chain and for the sake of convenience we denote its first element by 0. The set of all ordinary sequences taking values in Δ with the natural (componentwise) ordering is a complete lattice and we obtain a lattice with non Hausdorff sequential order topology if we restrict ourself to those sequences for which all but a finite number of values are 0 and then adjoin a top element I. This lattice L is clearly conditionally complete and since it has a top element is complete. Now we shall exhibit a collection of nested filter bases of intervals, each order converging to the sequence (0,0 ...), which we shall henceforth denote by 0'. The bases are of the type $\{[(\alpha,0,0,\ldots),0']\}$ $\{[(\beta,\beta,0,0,\ldots),0']\}$, $\{[(\gamma,\gamma,\gamma,0,0,\ldots),0']\}$ etc. Where $\{\alpha,\beta,\gamma,\ldots\}$ range over Δ and are not 0. If we pick one interval from each basis we shall have selected, as upper end points, a countable collection of elements of Δ. There is an element of Δ which is under each upper end point and yet properly over 0. Now each member of the following sequence of elements of L is in the union of the selected intervals

$(\lambda,0,0,\ldots)$, $(\lambda,\lambda,0,0,\ldots)$, $(\lambda,\lambda,\lambda,0,0,\ldots)\ldots$ and the sequence order converges to I. The associated filter base likewise order converges to I and has non empty trace on the union, hence the sequential order topology is not Hausdorff.

Theorem 3.5 The sequential order topology of a lattice need not be Hausdorff.

This example also provides an answer to a problem posed by Rennie (6; p 400) as to whether the L-topology of a lattice is always Hausdorff. He defines the L-topology of a lattice by taking as a basis for the open sets those sets which intersect each maximal chain in an open interval (of the chain) and are convex. That is if a and b belong to a set, $a < b$, then all c such that $a \leq c \leq b$ belong to the set. A maximal chain is one which is contained in no other chain. Rennie has shown (7; p 20) that in a complete lattice the L-topology is coarser than the (transfinite) sequential order topology. Thus we have the

Corollary 3.6 The L-topology of a lattice need not be Hausdorff.

The following somewhat more direct argument shows that the sequential order topology of the above example is not Hausdorff. If U is any nbhd of 0 we see by the equivalence of directed sets and filters that it must contain some interval out of every nested collection whose intersection is 0. Hence U contains a sequence converging to I and any nbhd of I must intersect U. The more roundabout argument is presented because the concept of pseudo-nbhd enables us to describe a process whereby the Hausdorff property may be established in certain cases. It is easy to verify the following lemma which relates pseudo-nbhds to open sets.

Lemma 3.7 A set is open in the order topology if and only if it is a pseudo-nbhd of each of its points.

This means that any open set containing an element x can be constructed in the following manner. First take a pseudo-nbhd of x. Then choose a pseudo-nbhd of each of its points and form the union U which will be called a p_2-nbhd of x. Having defined a p_n-nbhd V of x, we define a p_{n+1}-nbhd by selecting for each point of V a pseudo-nbhd and taking the union. The union of all the p_n-nbhds is clearly an open set.

Now we shall apply the above process to a complete, completely distributive lattice, where we have

$$(i)^* \bigvee_C \left[\bigwedge_{A_\gamma} u_{\gamma,\alpha}\right] = \bigwedge_F \left[\bigvee_C u_{\gamma,\varphi(\gamma)}\right] \qquad \text{and}$$

$$(ii)^* \bigwedge_C \left[\bigvee_{A_\gamma} u_{\gamma,\alpha}\right] = \bigvee_F \left[\bigwedge_C u_{\gamma,\varphi(\gamma)}\right]$$

as stated on page 5. If C indexes the set of filter bases of intervals order converging x and the $u_{\gamma,\alpha}$ are the upper (lower) endpoints of the filter base A_γ in $(i)^*((ii)^*)$ then the left sides of $(i)^*$ and $(ii)^*$ are

x and we see that the pseudo-nbhds order converge to x and have a basis of intervals. Since order convergent filters have unique limits, this means that the pseudo-nbhd filters are disjoint. So given any distinct elements x and y we have disjoint pseudo-nbhds $[a_1, b_1]$, $[c_1, d_1]$ of x and y respectively. Focusing attention on the former, we see that if we select a pseudo-nbhd $[u_\iota, v_\iota]$ for each point of $[a_1, b_1]$ and form the union of these sets, this union is contained in the interval $[\bigwedge u_\iota, \bigvee v_\iota]$, which is a p_2-nbhd of x containing $[a_1, b_1]$. It follows from the infinite distributive laws that the intersection all such p_2-nbhds is exactly $[a_1, b_1]$. If each of these p_2-nbhds had non empty trace on $[c_1, d_1]$, these traces would form a filter base of intervals, for which the sets would have non empty intersection since every upper end point must be over every lower end point. This is a contradiction so there must be some interval $[a_2, b_2]$ which is a p_2-nbhd of x and is disjoint from $[c_1, d_1]$. Then we apply the same argument to $[c_1, d_1]$ and get a p_2-nbhd $[c_2, d_2]$ of y disjoint from $[a_2, b_2]$. Continuing in this manner we obtain ascending sequences of intervals whose unions will be disjoint open sets containing x and y respectively. Thus we have established

<u>Theorem 3.8</u> <u>The order topology of a complete</u>

completely distributive lattice is Hausdorff.

The first part of Birkhoff's Problem 21 (1; p 62) is concerned with finding necessary and sufficient conditions that an element x be isolated in the order topology. By Lemma 3.7 this will be the case if and only if x is a pseudo-nbhd of itself. In other words $[x,x]$ belongs to any collection of intervals having finite intersection property whose intersection is x. This shows the necessity of the conditions in the following

Theorem 3.9 *In order that x be isolated in the order topology of a lattice it is necessary and sufficient that any collection of elements whose glb is x have a finite subset whose glb is x and dually that any collection of elements whose lub is x have a finite subset whose glb is x.*

Proof: To show sufficiency let $\{[a_\iota, b_\iota]\}$ be a collection of intervals whose intersection is x and suppose that M and N are finite subsets such that $\bigwedge_M x_m = x = \bigvee_N x_n$. Then the intervals whose upper endpoints are the x_m together with those whose lower endpoints are the x_n form a finite subcollection whose intersection is x.

If the lattice is complete we can use a result found in Viadyanathaswamy (10; p 39) to get much more tractable conditions for x to be isolated. An element x is called a *jump element* of a lattice if it is not the lub of any chain whose members are properly under x and dually if it is not the glb of and chain whose elements are properly over x.

<u>Theorem 3.10</u> <u>In a complete lattice the following properties of an element x are equivalent:</u>

(i) x <u>is isolated in the order topology</u>

(ii) x <u>is isolated in the sequential order topology</u>

(iii) x <u>is a jump element</u>

(iv) <u>any collection of elements whose glb is</u> x <u>has a finite subset whose glb is</u> x <u>and any collection of elements whose lub is</u> x <u>has a finite subset whose lub is</u> x.

Proof: (i) implies (ii) since the second topology is finer than the first. (ii) implies (iii) since any chain whose glb or lub is x gives rise to a transfinite sequence order converging to x. The above cited result of Viadyanathaswamy is essentially the statement that (iii) implies (iv) and it follows from Theorem 3.9 that (iv) implies (i).

One problem in this area, as yet unsolved, is to decide whether the Kantorovitch topology is always Hausdorff. This topology is finer than the others and can be Hausdorff, for instance in the example of Theorem 3.5, where the others are not. Theorem 72.3 of Vaidyanathaswamy (10; p 273) asserts that any sequential convergence scheme with unique limits gives rise to a Hausdorff topology and would settle this question if it were correct. However there exist sequential convergence schemes having unique limits which do not give rise to Hausdorff topologies. By a <u>sequential convergence scheme</u> having unique limits we mean any process for assigning limits to sequences (if x is assigned to x_n we write $x_n \to x$) such that

(i) if $x_n=x$ for all n then $x_n \to x$

(ii) if $x_n=y_n$ for all but a finite number of n then x_n and y_n have the same limit or do not converge

(iii) if $x_n \to x$ and x_{n_j} is a subsequence then $x_{n_j} \to x$

(iv) if $x_n \to x$ and $x_n \to y$ then $x=y$

The derived topology is obtained by calling a set X closed if $\{x_n\} \epsilon X$ and $x_n \to x$ imply $x \epsilon X$.

Now let us re-define convergence on the closed unit interval of the real line. As required we say that $x_n \to x$ if $x_n=x$ for all but a finite number of n. If x_n is monotone non-decreasing and does not converge by the previous requirement we say $x_n \to 1$, and if x_n is monotone non-increasing and does not converge by the first requirement we say $x_n \to 0$. It is easy to see that any open set containing 0 but not 1 must have its complement well ordered and hence countable. Similarly any open set containing 1 but not 0 must have bountable complement so 0 and 1 cannot be separated by open sets.

In conclusion we indicate various extensions of the results obtained here which might be expected. First of all there is a large gap between the example in Theorem 3.5 and the complete, completely distributive lattices, so far the only extensive class for which we have been able to verify the Hausdorff character. The lattice of Theorem 3.5 is distributive and in fact satisfies the infinite distributive law (i) but not (ii). Among the conditionally complete lattices satisfying (i) and (ii) we find the (conditionally complete) Boolean algebras and l-groups, both of which are of interest in certain applications. Theorem 18 of Birkhoff (1; p 231) implies the Hausdorff character in the latter case, but the proof seems open to the same objections discussed here on page 25. In view of the essentially negative results obtained, it seems reasonable that future studies of lattice topologies will be fruitful only if restricted to the lattices which enter in the applications. So far none of the lattice topologies has given much insight into the structure of lattices in general.

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