CONTINUOUS NEAR-HOMOGENEITY

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Hudson Van Etten Kronk

1964

This is to certify that the

thesis entitled

Continuous Near-Homogeneity

presented by

Hudson Van Etten Kronk

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

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Date July 28, 1964

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ABSTRACT

CONTINUOUS NEAR-HOMOGENEITY

by Hudson Van Etten Kronk

In a recent paper [1], P. Doyle and J. Hocking introduced the concept of continuous invertibility and investigated its application to continua. The first part of this thesis deals with the analogous but weaker concept of continuous near-homogeneity. The object here being to generalize the results in [1] to continuously near-homogeneous spaces and also to study continuously near-homogeneous plane continua as a special case. Among the main results obtained are:

(1) A compact set in E^{n+1} is an n-sphere if it is continuously near-homogeneous and contains an n-sphere.

(2) Every decomposable continuously near-homogeneous plane continuum is a simple closed curve.

(3) Every proper subcontinuum of a continuously near-homogeneous plane continuum is an arc.

(4) Every continuously near-homogeneous plane continuum separates the plane.

As a by-product of this part of the investigation, several properties of the continuous orbits in a continuously near-homogeneous indecomposable plane continuum are established. In particular, such orbits are identical with the composants

of such a continuum and each such orbit is the image of the real line under a one-to-one continuous transformation.

The second part of the thesis is concerned with the localization of continuous near-homogeneity. The principal result obtained is a characterization of those plane Peano continua which are continuously near-homogeneous at one or more points.

REFERENCE

1. P.H. Doyle and J.G. Hocking, Continuously invertible spaces, Pacific J. Math., Vol. 12 (1962) pp. 499-503.

CONTINUOUS NEAR-HOMOGENEITY

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1964

ACKNOWLEDGEMENT

I wish to express my sincere appreciation to Professor J.G. Hocking for his help and encouragement throughout the course of this research.

To Ginny

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SECTION 1 INTRODUCTION

In [10] P. Doyle and J. Hocking introduced the concept of continuous invertibility and investigated its applications to continua. The first part of this thesis deals with the analogous but weaker concept of continuous near-homogeneity which was introduced in [11]. The object here is to generalize the results in [10] to the case of a continuously near-homogeneous space and also to study continuously near-homogeneous plane continua as a special case. Some of the principal results obtained are:

Theorem 3.10: A compact set in E^{n+1} is an n-sphere if it is continuously near-homogeneous and contains an n-sphere.

Theorem 4.12: Every decomposable continuously near-homogeneous plane continuum is a simple closed curve.

Theorem 4.14: Every proper subcontinuum of a continuously near-homogeneous plane continuum is an arc.

Theorem 4.18: Every continuously near-homogeneous plane continuum separates the plane.
(Concerning this last result, it is an open question to determine precisely the number of complementary domains which a continuously near-homogeneous plane continuum can have.) As a by-product of this first part of the thesis, several properties of the continuous orbits in a continuously near-homogeneous indecomposable plane continuum are established. For instance, such orbits are identical with

the composants of such a continuum and each such orbit is an image of the real line under a one-to-one continuous transformation.

The second part of this thesis investigates the concept of local continuous near-homogeneity which was introduced in [11]. In particular, the following characterization of plane Peano continua which are continuously near-homogeneous at one or more points is obtained: Let K be a plane Peano continuum with non-empty CN(K). If K is one-dimensional, then K is the union of (at most) a countable number of simple closed curves having only one point p in common and all but a finite number of these simple closed curves have diameter less than any previously assigned positive number. Moreover, K is a simple closed curve if and only if CN(K) contains more than one point. If K is two-dimensional and CN(K) contains more than one point, then K is a closed disc. If CN(K)=p and if p is a non-cut point of K, then K is a closed singular disc and, finally, if CN(K)=p and p is a cut point of K, then K is a union of a countable number (≥ 2) of continua of the types already described.

SECTION 2 FUNDAMENTAL DEFINITIONS

In this section we present those definitions which are basic to continua theory and which are used in this thesis.

A topological space S is said to be <u>connected</u> if the only two subsets of S that are simultaneously open and closed are S itself and the empty set $\emptyset$. A subset X of S is connected if it is connected with respect to the relative topology. Each point x of S belongs to a unique maximal connected subset of S called a <u>component of x</u>. The components of a space constitute a partition of the space into maximal connected closed subsets. If X is a closed proper subset of S, then every component of S-X(the complement of X in S) is called a <u>complementary domain</u> of X. If each two points of S can be joined by an <u>arc</u> (the homeomorphic image of the unit interval I=[0,1]) in S, then S is said to be <u>arcwise connected</u>. An <u>arc component</u> of S is a maximal arcwise connected subset of S.

A compact connected set containing at least two points is called a <u>continuum</u>. There are two quite different types of continua, the decomposable and the indecomposable. A continuum is <u>decomposable</u> if it is the union of two proper subcontinua; otherwise it is <u>indecomposable</u>. If p is a point of a continuum M, then the union of all proper subcontinua of M that contain p is called a <u>composant</u> of M.

If a Hausdorff continuum is indecomposable, then its composants are equivalence classes and are uncountable in number. A continuum is *indecomposable of index n* if it is the union of n continua such that no one of them is a subset of the union of the others and it is not the union of n+1 such continua. A proper subcontinuum K of a continuum M is called a *continuum of condensation* if every point of K is a limit point of M-K. It is easy to show that a Hausdorff continuum is indecomposable if and only if each of its proper subcontinua is a continuum of condensation.

A space S is said to be *locally connected at a point* *$p \in S$* if each open set U of p contains an open connected set V of p. A space S is said to be *aposyndetic at $p \in S$* if for each point $q \in S$ distinct from p, there exists a closed connected set H and an open subset U such that $S-q \supset H \supset U \supset p$. A space is said to be *locally connected* (*aposyndetic*) if it is locally connected (aposyndetic) at each of its points. A continuum M is said to be *hereditarily locally connected* provided every subcontinuum of M is locally connected.

A locally connected metric continuum is called a *Peano continuum*. A fundamental result concerning Peano continua is the Hahn-Mazurkiewicz Theorem; which states that a necessary and sufficient condition that a space be a Peano continuum is that it be the image of the unit interval under a continuous mapping into a Hausdorff space. Every Peano continuum is arcwise connected. A Peano continuum that doesn't contain a simple closed curve (the homeomorphic image of the unit circle) is called a <u>dendrite</u>.

A <u>cut point</u> of a connected space S is a point $p \in S$ such that S-p is disconnected; otherwise p is a <u>non-cut point</u> of S. A point $p \in S$ is said to be a <u>weak cut point</u> of S if there exists two points $q, r \in S-p$ such that every closed connected subset of S that contains both q and r also contains p. In general a weak cut point is not a cut point, however, the two concepts are equivalent for Peano continua. A point p of a connected space S is a <u>local cut point</u> of S if it is a cut point of an open connected subset U of S.

A space S is said to be <u>cyclicly connected</u> provided that every two points of S lie together on some simple closed curve in S. A Peano continuum is cyclicly connected if and only if it has no cut points (Cyclic Connectivity Theorem).

A continuum M is said to be of <u>order less than or equal to n at $p \in M$</u> if for each open set V of p, there exists an open set U of p with $U \subset V$ and such that the boundary of U contains at most n points of M. If M is of order $\leq n$ at $p \in M$, but not of order $\leq n-1$, then M is said to be of <u>order n at p</u>. A <u>branch point</u> of a continuum is a point of order greater than two.

A space S is <u>homogeneous</u> if for each pair of its points p,q, there exists a homeomorphism of S onto itself which carries p into q. A space S is <u>locally homogeneous</u> if for each pair of its points p,q, there is a homeomorphism between two open subsets of S, one containing p, the other containing q, such that p is mapped into q. A space S is <u>near-homogeneous</u> (<u>invertible</u>) if for each point $p \in S$ (proper closed set C of S) and each non-empty open set U of S there exists a homeomorphism of S onto itself which carries p (C) into U.

SECTION 3 CONTINUOUSLY NEAR-HOMOGENEOUS SPACES

This section deals with the general continuously near-homogeneous space.

We use the symbol $G(S)$ to denote the group of all homeomorphisms of the topological space S onto itself.

<u>Definition 3.1</u> An <u>isotopy</u> of a topological space S is a continuous map $H:S\times I\longrightarrow S$ with the properties:

(1) If we define $h_t:S\longrightarrow S$ by setting $h_t(x)=H(x,t)$, then for all $t\varepsilon I, h_t\varepsilon G(S)$.

(2) $h_0(x)=x$ for all $x\varepsilon S$, i.e., h_0 is the identity mapping of S onto itself.

Usually, we shall use h_t to designate the isotopy.

We remark that most authors omit property (2) in defining an isotopy of S.

<u>Definition 3.2</u> A homeomorphism $h\varepsilon G(S)$ is said to be a <u>deformation</u> of S if there exists an isotopy h_t of S with $h_1=h$.

The set of all deformations of S is denoted by $H(S)$. It is easy to verify that $H(S)$ forms a normal subgroup of $G(S)$.

<u>Definition 3.3</u> The set of all images of a point $x\varepsilon S$ under deformations of S is called the <u>continuous orbit</u> of x and is denoted by $P(x)$.

The continuous orbits of a space S are invariant under deformations of S, i.e., if $h\varepsilon H(S)$, $x\varepsilon S$, then $h(P(x))=P(x)$. Also, if x and y are any two points of S, then either $P(x)=$

$P(y)$ or $P(x) \cap P(y) = \emptyset$, i.e., the continuous orbits of S are equivalence classes. These properties follow directly from the fact that $H(S)$ forms a group.

Definition 3.4 The isotopy path of a subset $X \subset S$ under an isotopy H of S is the image of $X \times I$ under H.

Definition 3.5 A topological space S is continuously near-homogeneous if, for each point $x \in S$ and each non-empty open set U of S, there is a deformation $h \in H(S)$ such that $h(x) \in U$.

Obviously, a continuously near-homogeneous space is near-homogeneous. The concepts of a continuously invertible space and a continuously homogeneous space are defined in an analogous manner. It is clear that a space S is continuously near-homogeneous if and only if each of its continuous orbits is dense in S and is continuously homogeneous if and only if it has exactly one continuous orbit.

Before stating any results, we first mention some examples. Euclidean n-space, E^n, is an example of a continuously homogeneous space. For let, $a=(a_1,a_2,\ldots,a_n)$ and $b=(b_1,b_2,\ldots,b_n)$ be any two points in E^n. Define $H: E^n \times I \to E^n$ by setting

$$H(x_1,\ldots,x_n,t)=(x_1+t(b_1-a_1),\ldots,x_n+t(b_n-a_n)).$$

The map H is easily seen to be an isotopy of E^n and $H(a_1,\ldots,a_n,1)=(b_1,b_2,\ldots,b_n)$. More generally, it can be shown that any n-manifold (a separable connected metric space each of whose points has a neighborhood homeomorphic to E^n) is continuously homogeneous. For example, if M is a 1-manifold and a and b are any two points in M, then a suitable isotopy may be constructed as follows: Since M is a manifold, there exists in M a closed 1-cell V containing a and b in its interior. Let g be a homeomorphism from V onto [0,1] and assume without loss of generality that $0<g(a)<g(b)<1$. There exists an isotopy h_t of I such that

(1) $h_1(g(a))=g(b)$

(2) $h_t(0)=0, h_t(1)=1$ for all $t \in I$.

Define h_t by setting

$$h_t(x)= \frac{(1-t)g(a)+tg(b)}{g(a)}x \text{ if } 0\leq x\leq g(a) \text{ and}$$

$$h_t(x)= \frac{[(1-t)g(a)+tg(b)]-1}{g(a)-1}(x-1)+1 \text{ if } g(a)\leq x\leq 1.$$

The desired isotopy f_t of M is obtained by setting

$$f_t(x)=g^{-1}h_t g(x) \text{ for } x\in V \text{ and}$$

$$f_t(x)=x \quad \text{if } x\in M-V.$$

We remark that the general case is treated in the same manner and uses isotopies like h_t.

The n-sphere S^n is also continuously invertible and is the only invertible n-manifold [8, Theorem 1]. The solenoid [12, Example 5, p.30] is an example of a homogeneous, continuously invertible continuum which is not continuously homogeneous. In [4], R.H. Bing describes a plane continuum which is continuously near-homogeneous, but not homogeneous. This continuum will be discussed in Section 4.

We proceed now to develop some of the basic properties of continuous orbits.

Theorem 3.1 The continuous orbit $P(x), x \epsilon S$, is a continuously homogeneous subspace of S.

Proof: Let p and q be any two points in $P(x)$. Then, by definition of $P(x)$, there exists deformations $f, g \epsilon H(S)$ such that $f(x)=p$ and $g(x)=q$. Since $H(S)$ is a group, gf^{-1} is in $H(S)$ and $gf^{-1}(p)=q$. Since $P(x)$ is invariant under deformations of $S, gf^{-1}|P(x)$, the restriction of gf^{-1} to $P(x)$, is a deformation of $P(x)$ and carries p into q.

Theorem 3.2 The continuous orbit $P(x), x \epsilon S$, is a connected subspace of S.

Proof: Let y be any point $P(x)$ and let h_t be an isotopy of S with $h_1(x)=y$. Then $H(x \times I)$, the isotopy path of x under H, is the continuous image of a continuum and hence is connected. Therefore, $P(x)$ is a union of connected sets, each containing the point x, and hence is a connected subspace of S.

Theorem 3.3 In a (non-degenerate) continuously near-homogeneous Hausdorff space S, each continuous orbit $P(x)$ is arcwise connected and each point in such an orbit is an interior point of an arc in the orbit.

Proof: Let p and q be any two distinct points in $P(x)$ and let h_t be an isotopy of S with $h_1(p)=q$. The isotopy path of p under H is the continuous image of the unit interval in a Hausdorff space. Hence, by the Hahn-Mazurkiewicz Theorem, the isotopy path is a Peano continuum. Since every

Peano continuum is arcwise connected, there is an arc in the isotopy path (and hence in S) joining p and q.

That every point in P(x) is an interior point of an arc in P(x) follows from the homogeneity of P(x).

Theorem 3.4 Every continuously near-homogeneous space is connected.

Proof: As remarked earlier, in a continuously near-homogeneous space the continuous orbits are dense. Since the closure of a connected set is connected [12, Corollary 1-36], the result follows from Theorem 3.2.

As a consequence of Theorems 3.1-3.4, it follows that every continuously near-homogeneous (non-degenerate) Hausdorff space is the union of non-degenerate, dense, disjoint, arcwise connected, continuously homogeneous subspaces. The solenoid is a good example of this. The continuous orbits of the solenoid are the composants of the solenoid and are uncountable in number (since the solenoid is indecomposable).

The next theorem implies that a continuously near-homogeneous space either has no cut points or each of its points is a cut point. The real line is an example of continuously near-homogeneous space in which every point is a cut point. It is not known, in general, if this is the only space having this property.

Theorem 3.5 If S is a continuously near-homogeneous T_1-space and has a cut point x, then P(x)=S.

Proof: Let $S-x=U\cup V$ be a separation of S. Since S is a T_1-space, both U and V are open in S. Let p be any point in V and let h_t be an isotopy of S such that $h_1(p) \in U$. Clearly, for some $t, 0<t<1$, we must have $h_t(p)=x$. But then we must have $x \in P(p)$. Since this relation is an equivalence relation, we also have $p \in P(x)$. An identical argument holds for each point $q \in U$. Therefore $P(x)=S$.

Corollary 3.6 No continuously near-homogeneous T_1-continuum has a cut point.

Proof: Every T_1-continuum always has at least two non-cut points [12, Theorem 2-18]. Thus the existence of a cut point would contradict Theorem 3.5.

Corollary 3.7 Every continuously near-homogeneous Peano continuum M contains a simple closed curve.

Proof: By Corollary 3.6, M has no cut points. Hence, by the Cyclic Connectivity Theorem every two points of M lie on a simple closed curve in M.

Theorem 3.8 Let S be a continuously near-homogeneous separable metric space. If S is locally Euclidean of dimension n at any point $p \in S$, then S is an n-manifold.

Proof: S is connected by Theorem 3.4. Let x be any point of S and let U be an open n-cell neighborhood of the point p. There is a deformation h of S such that $h(x) \in U$. Then $h^{-1}(U)$ is an open n-cell neighborhood of x. Hence S is an n-manifold.

The next theorem is a generalization of one of the principal theorems in [10]. In the proof of this theorem we need to use the following result from [15].

Lemma 3.9 Let X be a compact subset of E^{n+1} separating the points p and q. Let $H: X \times I \rightarrow E^{n+1}$ be a continuous map such that $H|(X \times t)$ is a homeomorphism of $X \times t$ into E^{n+1}, for all $t \in I$. If $H(X \times I)$ doesn't contain either p or q, then p and q are separated by $H(X \times t)$, for all $t \in I$.

Theorem 3.10 Let M be a compact continuously near-homogeneous subspace of E^{n+1}. If M contains an n-sphere S, then $M=S$.

Proof: Assume that $M-S$ is not empty. Without any loss of generality, we may assume that there exists a point p of M in the bounded component A of $E^{n+1}-S=A \cup B$. Let q be any point of S and U an open neighborhood of p such that $S \cap U$ is empty. Since M is continuously near-homogeneous, there is an isotopy h_t of M such that $h_1(q) \in U$.

Now consider the intersection V of A and the unbounded component of $E^{n+1}-h_1(S)$. Clearly V is not empty. Either V lies entirely in the isotopy path of S or there is a point x in V not covered by $H(S \times I)$. In the latter case, the point x and a point y in $B \cap (E^{n+1}-M)$ are separated by S, but they are not separated by $h_1(S)$. This contradicts Lemma 3.9 since the isotopy path of S doesn't contain either x or y. On the other hand, if V lies entirely in the isotopy path of S, then V lies in M and hence M contains an open $(n+1)$-cell. Thus by Theorem 3.8, M would be a compact $(n+1)$-manifold embedded in E^{n+1}. This is known to be impossible. Having been led to a contradiction in either case, it follows that the point p is non-existent and hence M=S.

Corollary 3.11 The only continuously near-homogeneous plane Peano continua are the simple closed curves.

Proof: By Corollary 3.7, such a space contains a simple closed curve (a 1-sphere) and hence, by Theorem 3.10, is a simple closed curve.

Corollary 3.12 Let M be a continuously near-homogeneous plane continuum other than a simple closed curve. Then every two points in a continuous orbit of M are joined by a unique arc in the orbit.

Proof: By Theorem 3.3, each two points in the same continuous orbit are joined by an arc. If there were another arc joining them, then M would contain (and hence be) a simple closed curve.

Corollary 3.11 is similar to Mazurkiewicz's result [17] that every homogeneous plane Peano continuum is a simple closed curve. This result was later generalized by H.J. Cohen [7], who showed that one need only require the homogeneous continuum to contain a simple closed curve. Cohen's result is thus similar to a special case of our Theorem 3.10. In connection with these remarks, it should be noted that there do exist near-homogeneous plane continua other than simple closed curves. A well known example of such a continuum is the universal plane curve (the continuum obtained by considering a square and successively deleting first the (open) middle-ninth of that square, second the middle-ninths of each of the eight squares remaining, third the middle-ninths of each of the sixty four squares remaining, etc.). A proof of the near-homogeneity of the universal plane curve is given in [9].

C.E. Burgess has shown [6] that, if a metric continuum is homogeneous and hereditarily locally connected, then it is a simple closed curve. The last theorem of this section shows that the corresponding theorem for continuously near-homogeneous, hereditarily locally connected metric continua is also true.

Theorem 3.13 If the metric continuum M is continuously near-homogeneous and hereditarily locally connected, then M is a simple closed curve.

Proof: Every hereditarily locally connected metric continuum is separated by some countable set [20,p. 99]. Any countable set that separates M contains a local cut point x of M [20, Corollary 9.41, p. 62]. Hence, M contains a continuous orbit P(x) of local cut points and since all but a countable number of the local cut points of any metric continuum are points of order two [20, Theorem 9.2, p. 61], every point of P(x) is of order two. This, however, implies that P(x) is the only continuous orbit of M and thus every point of M is of order two. K. Menger has shown [18] that a simple closed curve is the only metric continuum, each of whose points is of order two.

SECTION 4 CONTINUOUSLY NEAR-HOMOGENEOUS PLANE CONTINUA

In [4], R.H. Bing describes an indecomposable plane continuum K, each of whose proper subcontinua is an arc, such that K is near-homogeneous but not homogeneous. This continuum is pictured in Figure 4.1.

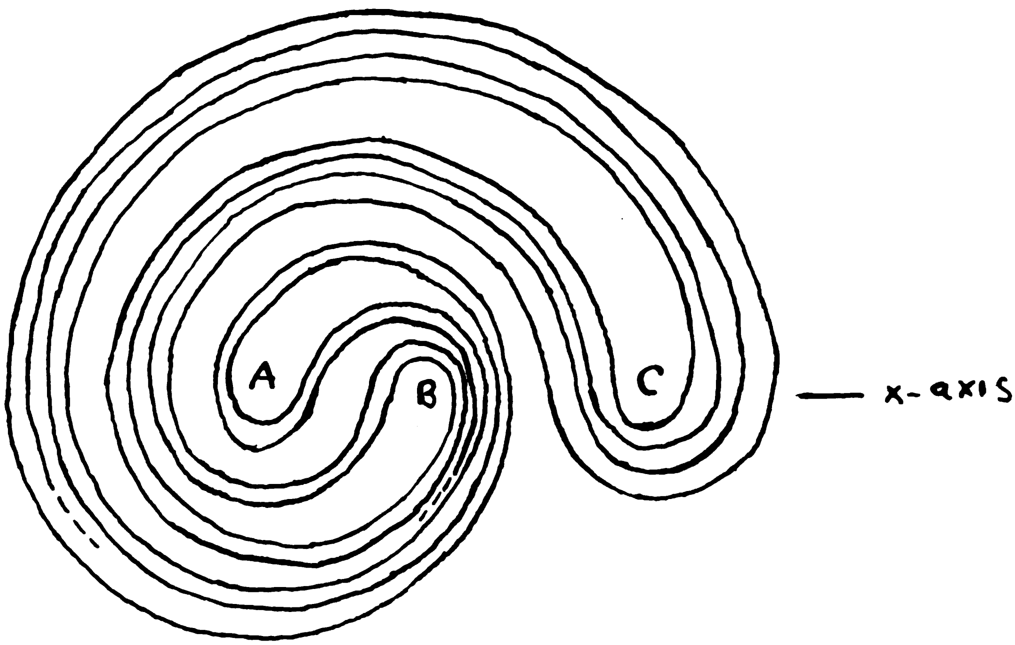

Figure 4.1

The example K intersects the x-axis in a (topological) Cantor set and is the union of semicircles with ends on this Cantor set. The continuum K is obtained by starting with a punctured disc D with three holes and digging canals (in the manner described on pages 222, 223 of [4]) into the disc from the four complementary domains of D. The continuum

K has the properties that each arc component is dense in K and each arc in K lies in an open subset homeomorphic with the Cartesian product of the Cantor set and an open interval. On the basis of these two properties, Bing noted that K is near-homogeneous. To see that these two properties also imply that K is continuously near-homogeneous, we proceed as follows: We first observe that if (a,b_1), $(a,b_2)\epsilon C\times I$, where C denotes the usual Cantor set, then there is an isotopy h_t of $C\times I$ such that

(1) $h_1(a,b_1)=(a,b_2)$

(2) $h_t(x,0)=(x,0)$, $h_t(x,1)=(x,1)$ for all $t\epsilon I$.

(h_t can be defined in the same manner as we defined the isotopy h_t of $[0,1]$ on page 9) Now let p be any point in K and let U be any non-empty open subset of K. Since the arc component of p is dense in K, there exists an arc pq in K such that $q\epsilon U$. Let V be a open neighborhood of this arc homeomorphic to the Cartesian product of the Cantor set and an open interval. Clearly V contains a closed set W homeomorphic to $C\times I$ and such that p and q are interior points of W. Let $g:W\longrightarrow C\times I$ denote this homeomorphism. As mentioned above, there exists an isotopy h_t of $C\times I$ such that

(1) $h_1(g(p))=g(q)$

(2) $h_t(x,0)=(x,0),h_t(x,1)=(x,1)$ for all $t\epsilon I$.

Finally, the desired isotopy f_t of K is obtained by setting

(1) $f_t(x)=g^{-1}h_t g(x)$ for all $x\epsilon W$ and

(2) $f_t(x)=x$ for all $x\epsilon K-W, t\epsilon I$.

It would be nice to know if the continuum K is also continuously invertible, but this appears to be a difficult open question.

Motivated by this example, we proceed now to investigate continuously near-homogeneous plane continua. We recall from Section 3 that the only such Peano continua are the simple closed curves. We shall eventually prove the much stronger result that the only decomposable continuously near-homogeneous plane continua are the simple closed curves.

R.H. Bing has shown [4, Theorem 1] that every homogeneous plane continuum that contains an arc is a simple closed curve. Since a continuously near-homogeneous plane continuum obviously contains an arc, we have the following result.

Theorem 4.1 If M is a homogeneous, continuously near-homogeneous plane continuum, then M is a simple closed curve.

We next show that no continuously near-homogeneous plane continuum can contain a simple triod. (A simple triod is a continuum formed by three arcs px, py, and pz such that each pair of these arcs have just the point p in common; the point p is called the emanation point of the triod.) Our result is similar to Cohen's result [7, Corollary 2.12] that no locally homogeneous plane continuum can contain a simple triod. There do exist, however, near-homogeneous plane continua containing simple triods, e.g., the universal plane curve.

Theorem 4.2 No continuously near-homogeneous plane continuum M contains a simple triod.

Proof: Assume that M did contain a simple triod T with emanation point p. Since a triod is obviously not near-homogeneous, M-T is not empty and is an open subset of M. Let h_t be an isotopy of M such that $h_1(p) \varepsilon M-T$. It follows from the Hahn-Mazurkiewicz Theorem that the isotopy path of T is a Peano continuum. If the isotopy path of T contained a simple closed curve, then by Theorem 3.10, M would be a simple closed curve contradicting our assumption that M contains a simple triod. Therefore, the isotopy path of T is a dendrite. But the isotopy path of the emanation point p contains an uncountable number of branch points in the isotopy path of T. This is impossible since no dendrite contains more than a countable number of branch points [20, Theorem 1.2, p. 89].

This theorem has several immediate corollaries. The first is a generalization of Corollary 3.11 and is similar to Cohen's result [7, Theorem 3] that every homogeneous arcwise connected plane continuum is a simple closed curve.

Corollary 4.3 Every arcwise connected, continuously near-homogeneous plane continuum M is a simple closed curve.

Proof: We observe that M cannot have two continuous orbits. For if it did, there would exist an arc joining these orbits. Recalling that each point in a continuous orbit is an interior point of an arc in the orbit, it follows that M would contain a simple triod contradicting Theorem 4.2. Thus M contains just one continuous orbit (and hence is homogeneous) and Theorem 4.1 applies.

Corollary 4.4 The arc componants and continuous orbits of a continuously near-homogeneous plane continuum M are the same.

Proof: Let x be any point of M and let A denote the arc component of M containing x. It follows from Theorem 3.3, that P(x) is contained in A. If there were a point p in A-P(x), then by definition of A, there exists an arc px in M. This, however, implies that M contains a simple triod. Therefore A-P(x) is empty and thus A=P(x).

Corollary 4.5 If M is a continuously near-homogeneous plane continuum, then every proper Peano subcontinuum of M is an arc.

Proof: Let N be a proper Peano subcontinuum of M. By Theorem 3.10, N doesn't contain a simple closed curve. Thus N is a dendrite and every dendrite other than an arc contains a simple triod.

We now prove the very useful result that no proper subcontinuum of a continuously near-homogeneous plane continuum separates the plane. In [10], Doyle and Hocking used the corresponding theorem for continuously invertible plane continua to show that every proper subcontinuum of a continuously invertible plane continuum is an arc; from which it followed that every continuously invertible plane continuum other than a simple closed curve is indecomposable. We shall eventually prove these same results for continuously near-homogeneous plane continua, but in the opposite order. Their proof that every subcontinuum of a continuously invertible plane continuum is an arc makes strong use of invertibility, which we aren't assuming.

In proving Theorem 4.6, we again make use of Lemma 3.9 for E^2 and the proof is analogous to the proof of Theorem 3.10.

Theorem 4.6 No proper subcontinuum of a continuously near-homogeneous plane continuum separates the plane.

Proof: Assume that X is a proper subcontinuum of M that separates the plane. Without loss of generality, we may assume that there exists a point p of M in a bounded complementary domain B of X. Let q be any point of X and let h_t be an isotopy of M such that $h_1(q) \epsilon U$, where U is an open neighborhood of p such that $U \cap X = \emptyset$. Since $h_1(X)$ is homeomorphic to X, $h_1(X)$ also separates the plane.

Consider the intersection V of B and the unbounded complementary domain of $h_1(X)$. Clearly V is not empty and, as in Theorem 3.10, there are two cases to consider. Either V lies entirely in the isotopy path of X or there is a point x in V not covered by the path. In the first case, M would contain an open 2-cell which is clearly impossible. The second case is also impossible, since then x and any point y in the unbounded complementary domain of X and not in the isotopy path of X would be separated by X, but not by $h_1(X)$. This contradicts Lemma 3.9. Hence the only subcontinuum of M that can separate the plane is M itself.

Corollary 4.7 Every continuously near-homogeneous plane continuum M is the common boundary of each of its complementary domains.

Proof: Since M is a nowhere dense subset of the plane, this is clearly true if M doesn't separate the plane. (Later we will show that this case can't occur.) Assume then that M separates the plane and let B be the boundary of a complementary domain of M. It is shown in [19] that B is a subcontinuum of M and since it is the boundary of a domain, it separates the plane. By Theorem 4.6, this is only possible if B=M.

In connection with Corollary 4.7, we remark that the universal plane curve is near-homogeneous and not the boundary of each of its complementary domains. However, it is true that every homogeneous plane continuum is the boundary of each of its complementary domains [6, Theorem 2].

K. Kuratowski has shown [16, Theorem 3] that every plane continuum which is the common boundary of three or more complementary domains is indecomposable or is indecomposable of index 2. Burgess has shown [5, Theorem 2] that the latter type can't be near-homogeneous. The following result, which will later be generalized, is then immediate.

Theorem 4.8 Every continuously near-homogeneous plane continuum which has three or more complementary domains is indecomposable.

Definition 4.1 Let M be a continuously near-homogeneous plane continuum other than a simple closed curve and let p and q be two distinct points in the same continuous orbit of M. The union of all the arcs in M that have p as an end point and contain q is called a ray starting at p.

A ray in the above sense differs from an ordinary ray of the plane in that it is neither straight nor closed. However, it has a starting point and as a consequence of the next lemma is the image of an ordinary ray under a one-to-one continuous transformation.

Lemma 4.9 Let M be a continuously near-homogeneous plane continuum other than a simple closed curve and let R be a ray in M with starting point p. Then R is the union of a countable number of arcs pp_1, pp_2, pp_3,... such that (1) for each i and j either $pp_i \subset pp_j$ or $pp_j \subset pp_i$ and (2) each arc pp_i is a proper subset of some pp_j.

Proof: Let $\{p_i\}$ be a countable dense subset of R. Then R is the union of the arcs pp_1, pp_2, pp_3,..., since if there were a point r of R not in any pp_i, then consider the arc pr. By Theorem 3.3, the arc pr may be extended to an arc ps such that r is contained in the interior of ps. Since M contains no simple triod, each p_i belongs to the arc pr. But then $\{p_i\}$ would not be dense in R, since no p_i is near s. That the sequence of arcs $\{pp_i\}$ satisfies (1) follows directly from Corollary 3.12. Property (2) follows from the denseness of the p_i.

<u>Lemma 4.10</u> Let M be a continuously near-homogeneous plane continuum other than a simple closed curve. Then for each point x of M, P(x) is the union of two rays L,R, starting at x such that $L\cap R=x$. Hence P(x) is the image of E^1 under a one-to-one continuous transformation.

<u>Proof</u>: By Theorem 3.3, x is an interior point of an arc ab in P(x). It follows from the fact that M contains no simple triods that P(x) is the union of two rays starting at x and going through a,b respectively. Since M contains no simple closed curve, these rays intersect only at x.

Since P(x) is the one-to-one continuous image of E^1, its points can be simply ordered in an obvious way. If $P(x)=L\cup R$, where L and R are rays starting at p and passing through a and b respectively, then we will always order P(x) such that $a<b$.

<u>Lemma 4.11</u> Let M be a continuously near-homogeneous plane continuum other than a simple closed curve and let a and b be two points of the same continuous orbit such that $a<b$. Then under any isotopy h_t of M, $h_1(a)<h_1(b)$.

<u>Proof</u>: If under h_t it turned out that $h_1(a)>h_1(b)$, then for some value of $\tilde{t}, 0<\tilde{t}<1$, we would have $h_{\tilde{t}}(a)=h_{\tilde{t}}(b)$. This obviously contradicts the fact that $h_{\tilde{t}}$ is a homeomorphism of M.

<u>Theorem 4.12</u> Every continuously near-homogeneous plane continuum M other than a simple closed curve is indecomposable.

<u>Proof</u>: Assume that M is decomposable. Then M contains a subcontinuum K that is not a continuum of condensation. Let $x \in K$ and let L and R be rays starting at x and passing through a and b respectively such that $P(x) = L \cup R$, $L \cap R = x$. We break our argument up into two cases:

Case (1) Either $\bar{L} = M$ or $\bar{R} = M$. Suppose without any loss of generality that $\bar{R} = M$. Let y and z be two points in $R \cap (M-K)$ such that y is interior to the arc xz. Such points exist, since M-K is open and R is dense in M. Let R_1 be the ray starting at y and going through z. Clearly $R_1 \subset R$ and $\bar{R}_1 \supset K$. Since K contains an open subset of M, R_1 contains points of K. Let q be one of them. Then the subcontinuum consisting of K together with the arcs xy and yq is a proper subcontinuum of M which separates the plane. This contradicts Theorem 4.6.

Case (2) Both $M-\bar{L}$ and $M-\bar{R}$ are non-empty. Since $P(x) = L \cup R$ and is dense in M, it follows that $\bar{L} \cup \bar{R} = M$. Hence the ray R must be dense in the open set $M-\bar{L}$. Let r and s be points of $R \cap (M-\bar{L})$ such that r is interior to the arc xs. If the ray $R_2 \subset R$ starting at r and going through s intersects $\bar{L}$, then, as in Case (1), M would contain a proper subcontinuum separating the plane. Assume then that R_2 and $\bar{L}$ have no points in common. Let t be a point of R_2 such that s is interior to the arc rt and let R_3 be the ray starting at s and passing through t. From the denseness of R in M-$\bar{L}$, it follows that there exists a sequence of points r_1, r_2, r_3,... on R_3 with $r_1 > r_2 > r_3 > \cdots$ and having r as a limit point. Now let h_t be an isotopy of M such that $h_1(r)=u$, where u is any point in $L \cap (M-\bar{R})$. Since h_1 is a homeomorphism of M, the sequence of points $h_1(r_1)$, $h_1(r_2)$, $h_1(r_3)$,... should have u as a limit point. By Lemma 4.11, the isotopy h_t preserves the order of points so that $h_1(r)=u>h_1(s)\geq h_1(r_1)>h_1(r_2)>\cdots$. This, however, implies that the set of points $h_1(r_1)$, $h_1(r_2)$,... can't have u as a limit point unless the ray R_2 returns to $\bar{L}$ which is contrary to our assumption.

Having been led to a contradiction in both cases, it follows that M must be indecomposable.

F.B. Jones has shown [13] that a homogeneous plane continuum is a simple closed curve if it is either aposyndetic at some point or contains a non-weak cut point. Since an indecomposable continuum is not aposyndetic at any of its points [14] and consists entirely of weak cut points, the analogous result for continuously near-homogeneous plane continua follows directly from Theorem 4.12.

Corollary 4.13 A continuously near-homogeneous plane continuum is a simple closed curve if it is either aposyndetic at some point or contains a non-weak cut point.

We now use Theorem 4.12 to show that every proper subcontinuum of a continuously near-homogeneous plane continuum is an arc; thus generalizing Corollary 4.5.

<u>Theorem 4.14</u> Every proper subcontinuum of a continuously near-homogeneous plane continuum M is an arc.

<u>Proof</u>: Clearly this is true if M is a simple closed curve. If M is other than a simple closed curve, then by Theorem 4.12 M is indecomposable and hence every proper subcontinuum of M is a continuum of condensation. Assume that K is a proper subcontinuum of M other than an arc. Let x be a point of K that is a limit point of points in K not in the continuous orbit of x. Such a point exists, since K is not an arc. Let h_t be an isotopy of M such that $h_1(x)=x'$ is a point of the open set M-K. It follows from the continuity of h_1 and the choice of x that there exists a point y in K-P(x) such that $h_1(y)=y'$ also belongs to M-K. Let L be an irreducible subcontinuum of M between x and y [12, Theorem 2-10] and let $h_1(L)=L'$. This subcontinuum L is unique and is contained in K. To see this, assume that N is another irreducible subcontinuum of M between x and y. Then $L\cap N$ must be connected, since otherwise $L\cup N$ would separate the plane (this follows from the well known Janiszewski-Mullikan Theorem [19, Theorem 22, p. 175]). But by Theorem 4.6, this would be possible only if $L\cup N=M$. This in turn would imply that M is decomposable, which is contrary to our assumption. Consider now the proper subcontinuum of M formed by L,L' and the arcs xx' and yy'.

This subcontinuum is the union of two subcontinua B and D, where B consists of L, xx', and yy' and D consists of L', xx', and yy'. Now $B \cap D$ must be disconnected, since otherwise it would be a proper subcontinuum of L containing x and y, which contradicts the fact that L was chosen to be irreducible between these two points. But if $B \cap D$ is disconnected, then by the Janiszewski-Mullikan Theorem $B \cup D$ would separate the plane. As we have previously observed, this is not possible by Theorem 4.6 and the fact that M is indecomposable. Having arrived at a contradiction, we can conclude that every proper subcontinuum of M is an arc.

Corollary 4.15 Let M be a continuously near-homogeneous, indecomposable plane continuum. Then the composants of M and the continuous orbits of M are identical.

Proof: Let x be any point of M and let C be the composant of M containing x. By definition C is the union of all proper subcontinua of M containing x and hence P(x) is a subset of C. If C were not a subset of P(x), then there would exist a proper subcontinuum of M joining x to a point in C-P(x). But by Theorem 4.14, this subcontinuum is an arc and hence M would contain a simple triod which is impossible.

We conclude this section by showing that every continuously near-homogeneous plane continuum has at least two complementary domains. Before proving this, however, we need a couple of lemmas.

Lemma 4.16 Let a and b be any two points in the same continuous orbit of an indecomposable, continuously near-homogeneous plane continuum M. Then there exists a continuous map $H:M\times I\longrightarrow M$ with the properties:

(1) If $h_t:M\longrightarrow M$ is defined by $h_t(x)=H(x,t)$, then, for all $t\in I$, $h_t\in G(M)$

(2) $h_0(a)=a$, $h_1(a)=b$

(3) $H(a\times I)=ab$.

Proof: Let G be an isotopy of M such that $g_1(a)=b$. Clearly it is sufficient to show that there is a subinterval $[t_1,t_2]$ of I such that $g_{t_1}(a)=a, g_{t_2}(a)=b$, and $G(a\times[t_1,t_2])=ab$. Assume that the continuous orbit of a is ordered such that $a<b$. Then for some t_1, $0\leq t_1<1$, $g_{t_1}(a)=a$ and $g_t(a)\geq a$ for all $t\in[t_1,1]$. Otherwise there would exist sequences $\{t_i\}$, $\{t'_i\}$, t_i, $t'_i\in I$, such that $\lim t_i=\lim t'_i=1$ and $\lim G(a,t_i)=a$, $\lim G(a,t'_i)=b$. This obviously contradicts the continuity of G. It now suffices to let t_2 be the smallest value of $t\in[t_1,1]$ for which $g_t(a)=b$. Then $g_t(a)\leq b$, for all $t\in[t_1,t_2]$ and hence $G(a\times[t_1,t_2])=ab$.

Definition 4.2 Let a_1b_1, a_2b_2, a_3b_3,...be a sequence of arcs in the plane converging to an arc xy. The sequence is called a folded sequence converging to xy if the sequence of points $a_1,b_1,a_2,b_2,\ldots$ converges to either x or y.

Lemma 4.17 Let M be a continuously near-homogeneous plane continuum other than a simple closed curve. Then M doesn't contain a folded sequence of arcs converging to an arc.

Proof: Assume that a_1b_1, a_2b_2,... is a folded sequence of arcs in M converging to an arc xy such that $a_1,b_1,a_2,b_2,\ldots$ converges to x. Let z be a point of M such that y is interior to the arc xz. By Lemma 4.16, there exists a continuous map $H:M\times I\longrightarrow M$ such that $h_0(y)=y$, $h_1(y)=z$, and $H(y\times I)=yz$. Then $h_0(a_1b_1)$, $h_0(a_2b_2)$,... is a folded sequence of arcs converging to the arc $h_0(xy)=h_0(x)y$ such that $h_0(a_1),h_0(b_1),h_0(a_2),h_0(b_2),\ldots$ converges to $h_0(x)$. If P(x) is ordered such that x<y, then by Lemma 4.11 $h_0(x)<y$. Let T,U, and V be open sets of M such that $y\varepsilon T, z\varepsilon U, h_0(xy)\varepsilon V, U\cap V=\emptyset$, and $h_1(T)\subset U$. Let $\{c_i\}$, $c_i\varepsilon h_0(a_ib_i)$, be a sequence of points converging to y. It is not difficult to see that there exists a positive interger N such that $n\geq N$ implies that $h_0(a_nb_n)\varepsilon V$ and also $h_1(c_1)\varepsilon U$. If $n\geq N$, then (since $U\cap V=\emptyset$) the isotopy path of c_n must contain either $h_0(a_n)$ or $h_0(b_n)$. But this is impossible, since H is continuous and $H(y\times I)=yz$.

The proof of the last theorem in this section is based upon R.H. Bing's result [3, Theorem 6] that a plane continuum K is tree-like if no subcontinuum of K separates the plane. In order to define the concept of being tree-like, it is convenient to associate with any open covering G of a subset of the plane a one-complex C(G), called the 1-nerve of G, such that there is a one-to-one correspondence between the elements of G and the vertices of C(G) and two elements of G intersect if and only if the corresponding vertices of

$C(G)$ are joined by a one-simplex in $C(G)$.

Definition 4.3 A plane continuum M is tree-like if for each positive number ϵ there is an open covering G of M, each of whose elements is of diameter no more than ϵ, such that the 1-nerve of G contains no simple closed curve. G is called a tree-chain covering of M.

Theorem 4.18 Every continuously near-homogeneous plane continuum M separates the plane.

Proof: Since a simple closed curve separates the plane, we shall assume that M is other than a simple closed curve and doesn't separate the plane. Hence by Bing's result [3, Theorem 6], M is a tree-like continuum. For each positive interger i, let D_i be a $1/i$ tree-chain covering of M. There exists in M an arc $a_i b_i$ such that both ends of $a_i b_i$ lie in the same link of D_i and

$$\text{diameter } M/4 - 1/i < \text{diameter } a_i b_i < \text{diameter } M/2.$$

Such an arc may be found as follows: Let xy be an arc in M such that diameter $xy >$ diameter $M/4 - 1/i$ and both ends of xy lie in the same link α of D_i. If diameter $xy \not<$ diameter $M/2$, then we reduce this arc by throwing away the part of it in α and consider one of the larger components of the remainder. These components are arcs and at least one of them, say $x'y'$, must have diameter greater than diameter $M/4 - 1/i$. For otherwise, it would follow that diameter $xy \leq 1/i + 2(\text{diameter } M/4 - 1/i) = \text{diameter } M/2 - 1/i$. The end points of $x'y'$ lie in the same link β of D_i, since if they didn't M wouldn't be tree-like. If the arc $x'y'$ doesn't suffice, then we reduce it by throwing away the part of it not in and consider one of the larger components of the remainder, etc. Eventually we shall either obtain an arc of the desired diameter or one having a subarc of the desired diameter. The sequence of arcs $\{a_i b_i\}$ has a convergent subsequence [20, Theorem 7.1, p. 11] and the limit of this subsequence is a closed, connected subset of M [20, Theorem 9.11, p. 15]. Clearly the limit contains at least two points and thus is a proper subcontinuum of M. By Theorem 4.14, this subcontinuum is an arc ab. However, there is a folded sequence of arcs (each in one of the $a_i b_i$'s) converging to a subarc of ab. This contradicts Lemma 4.17.

Thus a continuously near-homogeneous plane continuum must have at least two complementary domains. The exact number of complementary domains such a continuum can have remains an open question.

SECTION 5 LOCAL CONTINUOUS NEAR-HOMOGENEITY

In this section we consider the concept of a space being continuously near-homogeneous at a point and apply this concept to plane Peano continua.

Definition 5.1 A topological space S is said to be continuously near-homogeneous at a point p if for each point $q \in S$ and each open neighborhood U of p, there exists a homeomorphism $h \in H(S)$ such that $h(q) \in U$. If we only require h to be in G(S), then S is said to be near-homogeneous at p.

The set of points at which S is continuously near-homogeneous (near-homogeneous) is denoted by CN(S) (N(S)). Evidently, a space S is continuously near-homogeneous if and only if CN(S)=S and near homogeneous if and only if N(S)=S.

The first few preliminary results of this section appeared in [11]. We give proofs here for the sake of completeness. For the first five results of this section on CN(S), the corresponding result for N(S) (using G(S) in place of H(S)) is also true and is proved in an analogous manner.

Theorem 5.1 For any space S, the set CN(S) is carried onto itself by each $h \in H(S)$.

Proof: Let p be any point of CN(S) and let $h \in H(S)$. We want to show that $h(p) \in CN(S)$. Let U be any open neighborhood of h(p) and let q be any point in S. Then $h^{-1}(U)$ is an open neighborhood of p and hence there exists a deformation $g \in H(S)$ such that $g(h^{-1}(q)) \in h^{-1}(U)$. Then $hgh^{-1}(q) \in U$

and $hgh^{-1} \in H(S)$. Hence, by definition $h(p) \in CN(S)$.

Theorem 5.2 The set $CN(S)$ is a closed subset of S.

Proof: Let p be a limit point of $CN(S)$ and let U be any open neighborhood of p. By definition of limit point, there is a point $q \neq p$ in $CN(S) \cap U$. Thus for any point $x \in S$, there exists a deformation h of S such that $h(x) \in U$. It follows that $p \in CN(S)$ and hence $CN(S)$ is closed.

Theorem 5.3 The set $CN(S)$ is a continuously near-homogeneous subspace of S.

Proof: Let p and q be any two points in $CN(S)$ and let V be an open neighborhood of p in the subspace topology of $CN(S)$. By definition of the subspace topology, there is an open neighborhood U of p (open in S) such that $V=U \cap CN(S)$. Let h be a deformation of S such that $h(q) \in U$. By Theorem 5.1, $h[CN(S)]=CN(S)$ and hence $h|CN(S)$ is an element of $H(CN(S))$ which carries the point q into $U \cap CN(S)=V$. Thus $CN(S)$ is continuously near-homogeneous.

In order to have an example or two on hand, we note that, for the closed n-cube $I^n, n>1$, $CN(I^n)$ is a topological $(n-1)$-sphere. An example in which $CN(S)$ consists of exactly one point is pictured in Figure 5.1. It consists of the tangent circles $x^2+(y-1/2^n)^2=1/2^{2n}, n=1,2,\ldots,$. We will later show that every one-dimensional plane Peano continuum having exactly one point in $CN(S)$ may be homeomorphically embedded in this continuum. We remark that such Peano continua are often called _roses_.

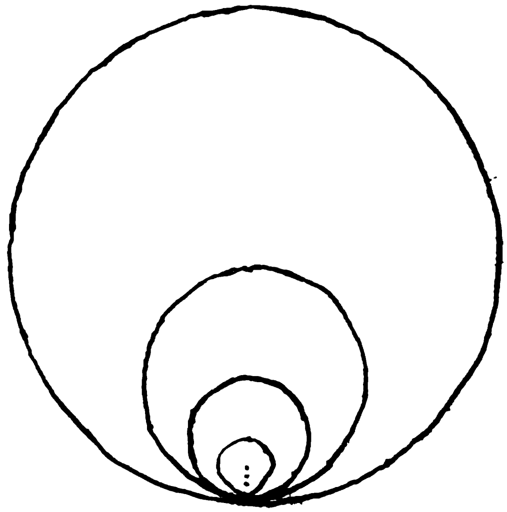

Figure 5.1

<u>Theorem 5.4</u> If CN(S) contains a non-empty open subset of S, then S is continuously near-homogeneous.

<u>Proof</u>: Suppose that CN(S) contains an open set U of S. Since U is an open neighborhood of a point in CN(S), each point $x \in S$ can be carried into U by a deformation h of S. This implies that h(x) (and hence x) is in CN(S). Therefore CN(S)=S.

Thus for any space S, the set CN(S) is either all of S or is a closed, nowhere dense subset of S. The next result is of interest in examples where CN(S) is a proper subset of S containing more than one point.

<u>Theorem 5.5</u> Let S be a space with non-empty CN(S). Then CN(Q) is also non-empty, where Q denotes the quotient space S/CN(S).

<u>Proof</u>: Let $p:S \rightarrow Q$ be the natural projection map and let p(CN(S))=w. We show that w belongs to CN(Q). Let U be

an open neighborhood of w in Q and let q be any point of Q. If $q=w$, the identity map of $CN(Q)$ carries q into U. If $q\neq w$, then $p^{-1}(q)=q\epsilon S$. By definition of the quotient topology, $p^{-1}(U)$ is an open neighborhood of $CN(S)$ in S. Hence there exists $h\epsilon H(S)$ such that $h(q)\epsilon p^{-1}(U)$. Since $CN(S)$ is invariant under elements of $H(S)$, the composition php^{-1} is a one-to-one transformation of Q onto itself and is a homeomorphism because php^{-1} and $(php^{-1})^{-1}=ph^{-1}p^{-1}$ are both closed. Then $php^{-1}\epsilon H(Q)$ and carries q into U. Hence $w\epsilon CN(Q)$.

Theorem 5.5 gives rise to an interesting unsolved question. For each positive interger $n>1$, let S_n denote the quotient space $S_{n-1}/CN(S_{n-1})$ and let $S_1=S/CN(S)$. Does there always exist an interger N such that for all $n\geq N$, the spaces S_n are all homeomorphic? An example for which $N=3$ is pictured in Figure 5.2. For this example, $CN(S)=S^1$ and $S/CN(S)$ $=D$, a (topological) 2-disc. Hence $D/CN(D)=S^1$ and $S^1/CN(S^1)$ is a single point.

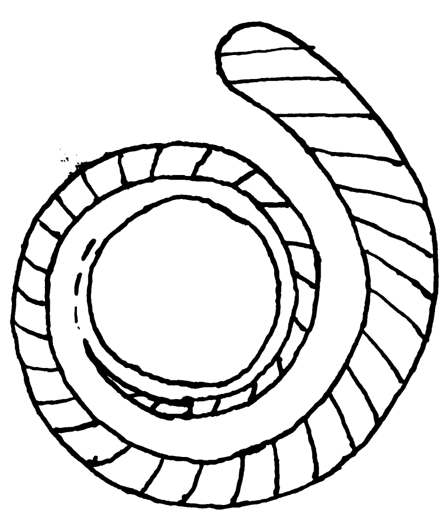

Figure 5.2

Theorem 5.6 If S is a Hausdorff space, then CN(S) has 0,1 or an uncountable number of points.

Proof: Suppose that there are two points p and q in CN(S). Let U be an open neighborhood of p not containing q. The isotopy path of q under an isotopy of S carrying q into U is a non-degenerate continuous image of the unit interval in a Hausdorff space and hence is a Peano continuum. Every non-degenerate Peano continuum contains uncountably many points. Finally, it is evident that each point in this isotopy path is in CN(S).

In analogy to Theorem 3.4, we have the following two results.

Theorem 5.7 If CN(S) is non-empty, then S is connected.

Proof: Let $p \epsilon CN(S)$. Since the continuous orbit P(x) of each point $x \epsilon S$ contains a point in every open neighborhood of p, it follows that $p \epsilon \overline{P(x)}$. Therefore $S = \bigcup \overline{P(x)}$ is a union of connected sets, each containing the point p, and hence is connected.

Theorem 5.8 For each space S, the set CN(S) is connected.

Proof: This follows immediately from Theorems 5.3 and 5.7.

The next theorem is a slight generalization of Theorem 8 of [11].

Theorem 5.9 Let S be a T_1-continuum with non-empty CN(S). Then if S has a cut point p, it is the only cut point of S and CN(S)=p.

Proof: We show that $CN(S)=p$, by showing that each point $q \in S-p$ is not in $CN(S)$. Assume that $q \neq p$ is in $CN(S)$. Since p is a cut point of S, $S-p=U \cup V$, where U and V are disjoint, non-empty open sets in S. Assume without loss of generality that $q \in U$. For each point $x \in V$, there is an isotopy h_t of S such that $h_1(x) \in U$. But then there must be some t_0, $0<t_0<1$, such that $h_{t_0}(x)=p$. This implies that x is a cut point of S too. Since every point of V is a cut point of S, it follows that the non-cut points of S must lie in U. This, however, implies that $U \cup p$ is a proper subcontinuum of S containing all the non-cut points of S. This contradicts Theorem 2-19 of [12]. Therefore $CN(S)=p$. The same type of argument applies to show that p is the only cut point of S.

Theorem 5.10 Let S be a Peano continuum with non-empty $CN(S)$. Then S contains a simple closed curve.

Proof: If S contains no cut points, then the theorem follows from the Cyclic Connectivity Theorem.. If S has a cut point p, then by Theorem 5.9, p is the only cut point of S. Since every dendrite is known to contain an uncountable number of cut points [20, Theorem 1.3, p. 89], S is not a dendrite. Therefore S contains a simple closed curve. In fact, p and each point $q \neq p$ in S lie together on a simple closed curve in S [20, Corollary 2, p. 79].

We next show that every one-dimensional plane Peano continuum K having $CN(K)=p$ is the union of a finite (≥ 2) or countably infinite number of simple closed curve, each two having only p in common, and such that only a finite number of these curves have diameter greater than any previously assigned positive number. In [2], T.C. Benton characterized such continua as being the only plane Peano continua that are homogeneous except for one point.

Theorem 5.11 Let K be a one-dimensional plane Peano continuum with $CN(K)=p$. Then K is the union of a countable number (≥ 2) of simple closed curves, each two having only p in common, and all but a finite number have diameter less than any given positive number $\epsilon>0$.

Proof: As we noted in the proof of Theorem 5.10, p and each point $q\neq p$ in K lie together on a simple closed curve in K. Since $CN(K)=p$, it follows that K contains at least two simple closed curves. Moreover, all simple closed curves in K contain the point p. For otherwise, we could apply Lemma 3.9 and the type of argument used in Theorem 3.10 to arrive at a contradiction of the fact that K is one-dimensional. Now let R and S be any two simple closed curves in K. We want to show that $R\cap S=p$. Assume that there did exist another point $q\epsilon R\cap S$. Without loss of generality, we can assume that q is the emanation point of a simple triod T in K. Let U be an open neighborhood of p not containing q and let h_t be an isotopy of K such that $h_1(q)\epsilon U$. Since the point p remains fixed throughout the isotopy, the isotopy path of T is a Peano continuum in K which doesn't contain

p. It follows that the isotopy path of T is a dendrite, since otherwise K would contain a simple closed curve not containing p. But the isotopy path of T contains an uncountable number of branch points and this is known to be impossible. Hence K is the union of simple closed curves, each two of which have only p in common. If S is any simple closed curve in K, then S-p is a component of K-p. Since an open subset of a Peano continuum contains only a countable number of components, it follows that K consists of a countable number of simple closed curves. By Theorem 3.9 of [12], only a finite number of the simple closed curves in K have diameter greater than any positive number $\epsilon>0$.

As an immediate corollary to the proof of Theorem 5.11, we get a result analogous to Theorem 4.2.

Corollary 5.12 Let M be a one-dimensional plane continuum with CN(M)=p. Then M-p contains no simple triod.

Theorem 5.13 Let K be a plane one-dimensional Peano continuum such that CN(K) contains at least two points. Then K is a simple closed curve.

Proof: Since CN(K) is closed and connected, it is a subcontinuum of K. By Theorem 5.10, K contains a simple closed curve S. Now CN(K) is a subset of S, for if there existed a point $p\epsilon$CN(K)-S, we could apply Lemma 3.9 to show that K contained an open 2-disc contrary to the assumption that K is one-dimensional. Since CN(K) is continuously near-homogeneous and is a continuum, it follows that CN(K)=S.

Clearly S is the only simple closed curve in K and thus K=S.

P. Alexandroff has shown [1] that the isotopy path of a simple closed curve S in E^3 is at least two-dimensional provided that, under the isotopy h_t, $h_1(S) \neq h_0(S)=S$. This result can be used to extend Theorems 5.11 and 5.13 to one-dimensional Peano continua in E^3. The proofs would be identical, except for the using of Alexandroff's result in place of Lemma 3.9.

<u>Theorem 5.14</u> Let K be a two-dimensional plane Peano continuum such that CN(K) contains at least two points. Then K is a closed 2-disc.

<u>Proof</u>: Since CN(K) contains at least two points, it follows from Theorem 5.9 that K has no cut points. Hence by [19, Theorem 46, p. 199], the boundary of each complementary domain of K is a simple closed curve. Let S be the simple closed curve which is the boundary of the unbounded complementary domain. We show that $CN(K) \subset S$ and hence that $CN(K)=S$. Assume there did exist a point $p \in CN(K)-S$ and let $q \in S$. We can carry q by a deformation h of S into an open neighborhood U of p, where $U \cap S = \emptyset$. The isotopy path of S under this isotopy would have to be two-dimensional by Lemma 3.9. This is a contradiction, since we would be mapping boundary points into interior points. Thus $CN(K)=S$ and hence K has only one complementary domain. Being two-dimensional, it follows that K is a closed 2-disc.

Suppose now that K is a plane Peano continuum with CN(K)

=p, where p is a non-cut point of K. It follows from Theorem 5.9 that K has no cut point. As noted in the proof of Theorem 5.14, this implies that the boundary of each complementary domain of K is a simple closed curve containing p. We denote the union of these simple closed curves by L. Evidently, L contains at least two simple closed curves and each two have only p in common. If $S \in L$ is the boundary of any bounded complementary domain of K, then since p is a non-cut point of K, there are no points of K in the bounded component of E^2-S. Since p is a non-cut point of K, K does contain each point in the intersection of the bounded component of the unbounded complementary domain with the unbounded components of the bounded complementary domains. If we call this set of points M, then $K=L \cup M$. We shall call a continuum of this type a <u>closed</u> <u>singular</u> <u>disc</u>. The "pinched annulus", Figure 5.3, is the simplest example of a closed singular disc.

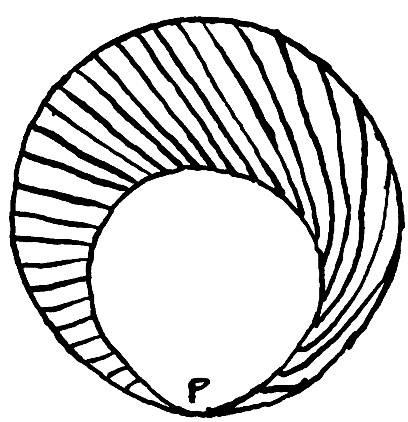

Figure 5.3

<u>Theorem 5.15</u> Let K be a plane Peano continuum with CN(K) =p, where p is a non-cut point of K. Then K is a closed singular disc.

Before completing the classification of plane Peano continua K with non-empty CN(K), we state three lemmas.

<u>Lemma 5.16</u> Let K be a T_1-continuum with CN(K)=p, where p is a cut point of K. If C is a component of K-p, then $C\cup p$ is carried onto itself under isotopies of K.

<u>Proof</u>: Let h_t be an isotopy of K. Since p is the only cut point of K, it remains fixed under h_t. Now let x be any point of C. Since p is the only cut point of K, the isotopy path of x under h_t lies in K-p. But the isotopy path of x is connected and since C is a component of K-p, the path must lie in C. Therefore $h_t(x)\varepsilon C$, for all $t\varepsilon I$.

<u>Lemma 5.17</u> Let K be as in Lemma 5.16. Then for each component C of K-p, $p\varepsilon CN(C\cup p)$.

<u>Proof</u>: Let V be an open neighborhood of p in $C\cup p$ and let $x\varepsilon C\cup p$. Then $V=U\cap(C\cup p)$, where U is an open subset of K. There exists an isotopy H of K such that $h_1(x)\varepsilon U$. By Lemma 5.16, $H|(C\cup p)$ is an isotopy of $C\cup p$ and $h_1|(C\cup p)$ carries x into V.

<u>Lemma 5.18</u> Let K be as in Lemma 5.16. If C is a component of K-p, then $C\cup p$ has no cut points.

<u>Proof</u>: By Lemma 5.17, $p\varepsilon CN(C\cup p)$ and hence the only possible cut point of $C\cup p$ is p itself. Since $(C\cup p)-p=C$,

p is a non-cut point of C p.

Theorem 5.19 Let K be a plane Peano continuum with CN(K) =p, where p is a cut point of K. Then K is the union of a countable number (≥ 2) of continua of the types characterized in Theorems 5.13, 5.14, and 5.15, each two of which have only the point p in common.

Proof: Let C be a component of K-p. Then $C \cup p$ is a Peano subcontinuum of K. By Lemmas 5.17 and 5.18, $p \in CN(C \cup p)$ and $C \cup p$ has no cut points. Since we have characterized such continua previously in Theorems 5.13, 5.14, and 5.15, it follows that K is characterized.

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